Developing Students' Understanding of Similar Figures: a Perceptual Approach.

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DEVELOPING STUDENTS’ UNDERSTANDING OF SIMILAR FIGURES: A PERCEPTUAL APPROACH

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

The Department of Curriculum and Instruction

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TABLE OF CONTENTS

ACKNOWLEDGMENTS ................................................................. ii
ABSTRACT .................................................................................. vi

CHAPTER

1 STATEMENT OF THE PROBLEM ........................................... 1
   Introduction: Mathematics Education Reform ...................... 1
     Constructivism ................................................................... 2
     Current Practices .............................................................. 2
     Recommended Practices .................................................... 4
     Linkages ............................................................................ 5
     Phenomenology .................................................................. 6
     Overview ........................................................................... 6
   Similarity ............................................................................. 7
     Current Curriculum for Similarity ...................................... 8
     Problems With Current Approach .................................... 10
   Algebra .................................................................................. 12
   Rational Number ................................................................... 13
   Towards a New Approach .................................................... 15
     Intuition/Perception ............................................................ 15
     Similarity and Rational Number Knowing ......................... 16
     Unitizing and Norming ....................................................... 17
     Phenomenology .................................................................. 19
   Geometric Intuitions of Similarity ........................................ 21
     Phase I: Perceptual Intuition ............................................... 21
     Phase II: Quantification of Perceptual Orientations ............. 22
       Between Figures Analysis ................................................ 22
       Within Figures Analysis .................................................. 24
     Phase III: Towards Formalization ...................................... 26
   A New Curriculum for Similarity .......................................... 28
   The Significance of the Study ............................................... 29
   Similar Figures ....................................................................... 32
   Research Questions ................................................................ 32
   Organization of the Study ..................................................... 33

2 REVIEW OF THE LITERATURE ............................................. 35
   Whole Number and Rational Number Development ............. 35
   The Unit Concept ................................................................. 38
   Unitizing and Norming ......................................................... 39
   Phenomenology ..................................................................... 41
ABSTRACT

Children encounter and recognize similar figures in their everyday experiences with such things as basketballs, soccer balls, tennis balls, ping-pong balls; or a candy bar that comes in various sizes of the same shape. Yet their school experience with the mathematics of similarity generally does not build on these perceptual intuitions. Traditional mathematics curricula bypass students' visual intuitions and their quantitative understandings, proceeding directly to set piece problems solved by formal algebraic methods. The result for many students is that the topic of similarity contributes to their evolving view of mathematics as a domain of complex procedural methods divorced from their intuitive sense of quantity and space.

The purpose of this study was to explore how to develop students' mathematical understandings of similarity by having quantitative methods evolve from students' visual intuitions about similar figures. The foundation of this curricular approach was a perceptual analysis of similarity consisting of within relationships which are the static relationships within one figure that may be recognized in another, and between relationships which stem from the dynamic perception of one figure as resulting from uniform growth of the other. The curricular strategy encouraged students to verbalize and quantify these perceptual attributes of similar figures, eventually applying quantitative methods to the standard problems encountered in traditional curricula.

The subjects were secondary school students enrolled in two separate geometry classes (one classified as college-bound) which were taught by the researcher over a
period of three weeks. Qualitative data were collected and analyzed from video recordings, students’ work, and journals.

The results indicated that the students were generally able to represent their perceptual orientations quantitatively, and to utilize quantitative methods to solve problems. However, the between representation was selected by the plurality of students even when a within strategy would have been computationally more convenient. This highlights the observations of other researchers concerning the difficulty of static representations, and suggests a developmental model in which the more accessible dynamic representations of similarity should precede the static approaches. The students in the college-bound class exhibited more overall flexibility than the students in the non-college-bound class.
CHAPTER 1

STATEMENT OF THE PROBLEM

Introduction: Mathematics Education Reform

Recent mathematics instruction has often been more concerned with what procedures students have available than in the sense that students are making of the procedures they are implementing. This reflects a curriculum informed by behaviorist theory which excludes meaning, representation, and thought in its view of the learning process (Von Glasserfeld, 1987). While there is a certain value to the procedural accomplishment of mathematics, the math reform movement strives towards a more complete vision of learning.

The NCTM Standards (1989) has three ideas central to its position on the learning of mathematics, one of which is "knowing mathematics is doing mathematics". According to the Standards, doing mathematics is an active process; gathering, discovering, and creating knowledge in the course of an activity with a particular purpose. While certainly there is some value in mastering concepts and procedures, the Standards suggest "doing" mathematics is quite different, and instruction should emphasize this active process rather than memorizing facts and procedures.

The Principle and Standards for School Mathematics (1998) suggest that mathematics must build on students' prior experience. That children have formed a fairly complex network of informal mathematical ideas from their lived experiences provides a basis for continuing their understanding of mathematics. This attitude towards learning reflects the impact that constructivists' views have had on education.
Constructivism: Constructivism, as a philosophy, is the belief that children make sense of the world through their everyday experiences. They construct their knowledge of the world through abstractive reflection of their perceptions and experiences which allows them to make sense of their experiences (Simon, 1995). Constructivists do not believe that knowledge exists independent of the observer or in objects; rather, knowledge is constructed by the learner and therefore resides in, and is a part of his or her being. Students are not "empty vessels" waiting to be filled with information from the expert; rather, they have knowledge structures (although they may be informal and limited) that they have constructed from making sense of their experiences in the world. A constructivist approach to education is one in which the teacher takes some interest and responsibility for the nature of the knowledge structures that students construct in the learning process. The constructivist argue that in order for conceptual learning to occur, educators must assist students in constructing more sophisticated structures for understanding phenomena (Confrey, 1990). While the implications for the role of the teacher are not specific, it is clear from the constructivist perspective that the teacher is a facilitator of learning, not a dispenser of information from the position of an "expert". And in order for teachers to promote a conceptual approach to learning, they must understand meaning consists of the connections among the elements that constitute students’ knowledge. In this respect, conceptual approaches reflect the constructivist concern for how students are constructing their mathematical activities.

Current Practices: Mathematics curriculum of the past has viewed learning as a process whereby students become proficient in procedures, and teaching as a process whereby teachers dispense information, model procedures to obtain solutions, and
determine proficiency through assessments which measure the students ability to apply memorized procedures on demand. Relatively short products are expected from students, teachers execute their plans and routines (checking to see if student performance falls within desirable bounds), and the level of student understanding is primarily determined by the teacher. This is characteristic of what is referred to as "direct instruction" (Confrey, 1990). The student is viewed as a passive recipient of information, and understanding resides in his ability to perform procedures and/or apply them to specific problem types.

While there has been much talk about reform in the teaching and learning of mathematics, the transition to a curriculum that reflects current thinking about the manner in which mathematics is learned has been slow. Historically, algebra has been taught by direct instruction. The teacher is viewed as the expert, and dispenses information to the students, usually in the form of explicit rules and modeling. Students consolidate the rules and procedures of the algebra domain through drill and practice. Their algebraic competency is assessed by their ability to perform specific procedural tasks, and to apply these learned procedures on specific problem types. Recent results from NAEP indicate that students' understanding of algebra is often limited to procedural recall and they possess little understanding of the structural characteristics of the discipline of algebra (Kieran, 1989). Students understanding of variable is often limited, which can block their access to algebra concepts. Leitzel (1989) attributes this to the attitude of the curriculum towards this basic concept of algebra. He argues the concept of variable is more sophisticated than we realize, and demands careful attention to how it is introduced.
Students experience a wide range of difficulties with the conventions of algebra since they are different from the conventions of arithmetic (Kieran, 1994). They often rely on methods for dealing with algebra which are not grounded in a conceptual understanding. In one such instance, students relied on the powerful visual features of algebra for simplifying expressions, rather than operating from a propositional understanding (Kirshner, 1989). In his study, the students' ability to correctly parse algebraic expressions depended on the extent of their propositional understanding. There are many aspects of algebra to consider, but it is clear that this mathematics domain is being treated as a formalistic/procedural activity. While there is certainly value to being able to perform specific procedures, the intent of algebra instruction is to affect the way that students think (Leitzel, 1989). Moving in this direction, the NCTM Standards (1989) proposes the algebra curriculum move away from a tight focus on manipulative facility to include greater emphasis on conceptual understanding.

It is important to this paper to consider the manner in which elementary algebra is understood, since the traditional manner to solve for unknown lengths between similar figures has been to create proportional forms, and solve by the rules of an algebra system. The operation is rule-based, relying on formal ideas rather than students’ informal notions of similarity. Thus, past curriculum runs counter to modern mathematics reform.

**Recommended Practices:** The prominent position of Constructivism, while not offering specific methods for how mathematics should be taught, does encourage the teacher to be sensitive and responsive to the mathematical thinking of the students. This will not happen by accident, but by design. When planning lessons, teachers must be informed by the mathematics of students to be able to harmonize teaching methods with
the nature of that mathematical knowledge (Simon, 1995; Steffe, 1991). Confrey (1990) defines decentering as being able to see a situation as perceived by another and to appreciate that construction as having integrity and being sensible within that individual's framework. The implication for the teacher is he must form an adequate model of the students' ways of viewing an idea and then assist the students in restructuring those views to be more adequate from the students' and from the teacher's perspective. The students, instead of being viewed as passive recipients of information, are viewed as active participants in the construction of their own knowledge.

In the case of similarity, current curriculum does not build on the intuitions or perceptions gained from prior experiences with similar figures; rather, only with the student being able to perform a particular procedure. The nature of the students' knowledge is of no concern, as long as they are able to solve for the right answer. This kind of activity provides the students with a specific, yet limited insight into the nature of similar figures that is formalistic and rule-driven. Generally, the informal notion of similarity the students may have had is not associated with this formality (unless a student comes to make the connections independently), thus students may interpret two different meanings for similarity: one in the classroom and another outside. Lave (1988) has shown that people often have one set of conventions for working mathematics problems in school, and a different set of conventions for working similar problems in non-classroom situations.

**Linkages:** Hiebert and LeFevre (1986) define *procedural knowledge* as composed of the formal language or symbol representation system of mathematics. It consists of the algorithms, or rules, for completing mathematical tasks. It includes a
familiarity with the symbols used to represent mathematical ideas, and an awareness of the syntactic rules for writing symbols in an acceptable form.

They define **conceptual knowledge** as knowledge that is rich in relationships. It is a network in which the linking relationships are prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network. Therefore, a unit of conceptual knowledge cannot, by definition, be an isolated piece of information.

Recent reform efforts have suggested as more beneficial for learning mathematics, a curriculum that is more sensitive to providing meaningful experiences, and builds upon students' prior experiences. In short, a curriculum that is rich in relationships.

**Phenomenology:** Although constructivism suggests building on students' intuitive base, it provides no specific techniques for determining how students' initial intuitions might be structured. Conceptual knowledge, whether formal or informal, is a network of ideas. Idhe (1986) describes the intent of phenomenology and the methods associated with the discipline through geometric examples. His description inspired the methods used in this approach to understanding students' intuitive sense of mathematics and determining the nature of that knowledge. This supports the constructivist curriculum which suggests utilizing the students' current knowledge base as a starting point for new learning situations.

**Overview:** This chapter begins with a general introduction to similarity, and then describes the current curriculum for similarity, noting that its formalistic methods do not adhere to the constructivist position on the teaching of mathematics. A new conceptual approach to similarity that utilizes students' intuitions and arithmetic skills is described as
an alternative to the methods that are currently in use. Idhe’s (1996) approach to phenomenology is then discussed as a method helpful for making sense of students' intuitions of similarity. In particular, the analysis of students’ perceptual orientations allows us to understand the specifics of similarity which are the individual components of meaning. This will allow the students to mathematize their perceptual interpretations of similarity, which encourages a conceptual understanding based in mathematical relationships rather than in perceptual cues alone. This provides the framework for proposing a new curriculum for similarity. Finally, the importance of the study is presented.

![Figure 1.1 (Slope of a Line)]

**Similarity**

Similar figures are defined in current textbooks as figures that have the same shape, but differ in size (Addison & Wesley, 1992; HBJ Geometry, 1984). This definition is not very technical, but it has a basic appeal to one's perceptual and intuitive sense of similarity.
Possessing a conceptual understanding of similarity is particularly important in order for students to be successful with other topics, such as slope. In Figure 1.1, the slope of a line is represented by similar triangles. The ratio of $\frac{3}{2}$, which represents the slope of the line in one triangle, is equivalently calculated as $\frac{6}{4}$ in another triangle. The consistency of the ratio ensures the slant of the line is not modified. While most problems require a student only to find a single ratio for the slope of a line, students are expected to realize that numerical expression applies to the entire line. Working with formulas such as $m = \frac{y_2 - y_1}{x_2 - x_1}$ does not imply that students understand the working relationships inherent in slope, only that they are proficient with an algebraic equation.

Also, slope and similarity are the foundation of differential calculus. An important interpretation of the derivative is from representing the slope of a curve at a point as the slope of the line tangent to the curve at that point. Thus, similarity concepts are instrumental to comprehending mathematical topics central to the secondary school and undergraduate curricula. Furthermore, slope is a rich topic standing at the nexus of three mathematics domains: arithmetic, geometry and algebra. Therefore, it has the potential to serve as a unifying force in the mathematics curriculum.

**Current Curriculum for Similarity:** Similarity is a topic rich in perceptual and analytical orientations, and is naturally aligned with students' intuitive and informal sense of likeness. However, in many cases, middle and secondary school curricula (Addison & Wesley, 1992; HBJ Geometry, 1984) engage only a limited form of similarity to perform a specific, routine task: to determine lengths of sides between similar figures using an algebraic/proportional configuration. Bypassed are perceptual intuitions and the familiar mathematical domain of arithmetic; instead, algebraic forms are introduced as a blind
procedure for getting answers to standard problems. The curriculum quickly embraces the ideas of proportions, variable representation, and algebraic processes in which students substitute lengths of corresponding sides to a preset proportional form, and then use algebraic cross-multiplication as a method to solve the proportion. Current instructional methods tend to be formalistic, and do not build on students' intuitions about similar figures.

Students' intuitions about similarity are grounded in perceptual orientations and lived experiences; however, in the current curriculum, the specifics of these perceptions and experiences are not a matter of interest. There is no effort to understand or further students' initial understanding of similarity (especially in a quantitative manner); rather, proportions are utilized to represent equality of ratios between corresponding sides of similar triangles (students are basically just told what corresponding sides are, and practice matching up sides from figures that vary positionally). This activity is more of an analogical activity than a mathematics enterprise. Its action is based in linguistic

![Figure 1.2 (Similar Triangles)](image-url)
parameters....this is to that as this is to that....and the proportion is a result of that activity. It is doubtful that students are creating mathematics representative of proportional relationships describing similar figures.

From Figure 1.2, given the two triangles are similar, students are encouraged to match sides and solve for x in a preset proportional form.

\[
\frac{6}{4} = \frac{8}{x} \quad \text{or} \quad \frac{6}{4} = \frac{8}{x}
\]

\[6x = 32 \quad \text{or} \quad 6x = 32\]

\[x = \frac{1}{3} \quad \text{or} \quad x = \frac{1}{3}\]

Students are usually given a number of proportions to solve using the cross-multiplication method to practice before considering similarity problems. Students are told to use this proportional format of \( \frac{a}{b} = \frac{c}{d} \) as a tool for solving for the unknown side between similar figures and it is justified by definition; that is, corresponding sides of similar figures are proportional. Then they are encouraged to substitute three values into the proportion and use cross-multiplication to find the missing value. After they have practiced matching sides and performing the prescribed mathematics, they are assigned a set of problems to work in like fashion.

Problems With Current Approach: When one considers the definition of similarity as found in textbooks (figures that have the same shape but differ in size), this definition, though not detailing formal aspects, appeals to students’ perceptual and intuitive sense of likeness. It seems it would be advantageous to approach this topic from a constructivist position in order to exploit students’ intuitions and perceptions of similarity gained from their lived experiences. Constructivists note that students play an active role in the construction of their knowledge, and research (NCTM Standards, 1989; Mack, 1993; Van...
Heile, 1986) suggests that instruction which extends students informal knowledge provides the students with a meaningful start in their quest to learn mathematics. This would incline one to hope current curriculum would treat the acquisition of similarity concepts as a process of building on students' informal knowledge and experiences. However, this is not the case.

Students' school experiences with similarity, as illustrated above, are formal and rule-oriented. Informal understanding of similar figures is grounded in perceptions; yet, the current curriculum forces students to substitute lengths into a preset algebraic form which is just given to them, and to solve by a method (cross multiplication) which is not a meaningful strategy in proportional reasoning. It is simply an efficient method for obtaining an answer. Students have no ownership in this whole process, since the situation is mediated by an expert (the teacher) and forms and methods are presented to the students to practice. Therefore, they become proficient with procedures.

In general, most geometry classes are oriented towards two column proofs, and being able to utilize given formulas to determine particular characteristics of geometric figures. Theorems, postulates, and definitions provide the basis for deductive analysis, and students are encouraged to utilize these aspects of geometry in the form of proofs. However, due to the indepth nature of this activity (this is the fourth stage of the individual's understanding of geometry according to Van Hiele, 1986), it is questionable as to how much geometry the student understands before being coerced to engage this formal activity. By geometry, I mean the understanding of the individual features of a figure, and the relationships between these features that give the figure its personality, and classify it according to its relationship with other figures.
Thus, there seems to be a rush in the geometry curriculum to have students become proficient at formal procedures, and be able to write proofs. The fact that both of these areas are grounded in perceptions is given token consideration. Thus, students' informal knowledge constructed for understanding their experiences in the world is not associated with the formalities of the classroom.

**Algebra**

From Figure 1.2, the development of the proportional configuration with emphasis on the right answer is just given to students without any reference to their intuitive or perceptual notion of similarity. Current curriculum "coaches" students to match corresponding sides in order to have the correct proportion, but the features of similar figures and their relationships are not addressed in any depth, if at all. Thus, students could be proficient in solving similarity problems using the algebraic processes presented if Figure 1.2, yet still have impoverished notions of similarity.

Conversely, students may have a fairly rich conceptual understanding of similarity gained from their lived experiences, yet not understand formal operations of algebra used to represent and solve for unknowns in similar figures. Students are encouraged to use the cross-multiplication strategy which is not a method of choice by the students when given the freedom to choose. Students do not choose this method because cross-multiplication does not assist the students in making sense of proportional situations. Assessments measure only the students' understanding of similarity as related to equations and strategies that are not necessarily connected to their perceptual orientation. Advocating that students use algebraic processes to determine parameters of similar figures circumvents the conceptual and quantitative richness inherent in similarity, and opts for...
competency in matching corresponding sides between figures, as well as competency in computing algebraically the resultant proportions.

One might also question the logic of utilizing algebra as a method for performing mathematics at this junction when research reveals the lack of conceptual understanding exhibited by students in their quest for competency in algebra. In her article, The Learning and Teaching of School Algebra, Kieran (1989) investigates students' acquisition of algebraic knowledge. She describes a procedural-structural cycle to delineate what she views as two different types of conceptual activity that occurs in the process of learning algebra.

She defines **procedural knowledge** as performing arithmetic operations on numbers to yield numbers, and **structural knowledge** as carrying out specific operations on algebraic expressions. From her investigations, she concludes that the majority of students do not acquire any real sense of the structural aspects of algebra. At best, they develop and continue to rely on procedural conceptions, and at worst, they memorize a pseudo-structural content. Kieran primarily faults textbooks that do not incorporate a procedural-structural perspective on student learning of mathematics, and the lack of research that deals with how algebra teachers interpret and deliver the content of algebra texts, for causing this structure.

**Rational Number**

Rational numbers are defined as any number that can be written in the form $\frac{a}{b}$ where $a$ and $b$ are integers. This number domain has proven to be a source of difficulty for students. National assessment tests, such as NAEP, indicate students calculate fractions by applying memorized algorithms, and demonstrate little or no conceptual
knowledge (Golding, 1994). Behr, Lesh, Post, and Silver (1983) contend the national assessment findings are the product of a curriculum that is designed to foster procedural learning.

Not only do students lack a conceptual understanding of rational numbers, but a study conducted by Post, Harel, Behr, and Lesh (1988) suggested many elementary and middle-school teachers also share the same limitations. They concluded that many teachers do not know enough mathematics to promote conceptual understandings in their students. With a curriculum that promotes procedure, and teachers that have limited mathematical knowledge, it's no wonder that students are not developing meaningful interpretations of rational numbers. Many other studies have been conducted (Behr, Wachsmuth, Post, & Lesh, 1984; Hart, 1988; Lesh, Post, & Behr, 1988; Potheir & Sawada, 1983) which reveal the various difficulties students have comprehending the rational number domain.

The context of similar figures offers an ideal situation wherein to further students' rational number development. However, the rational number relationships that are inherent in similarity are currently ignored. This seems to be typical of the current 9-12 mathematics curriculum since its objective is to promote abstract features of mathematics in contrast to the K-8 curriculum's objective, which is to master arithmetic objectives. Seldom is there any interface between the two. Thus, students who are weak in particular features of the K-8 curriculum do not have the opportunity to continue a progression of study which would strengthen those areas. Rather, students are encouraged to deal with the objectives of a new curriculum.
The Standards (1988) argue against this antiquated curriculum, and suggest a cohesive curriculum where the 9-12 curriculum builds on the 6-8 curriculum, and the 6-8 curriculum builds on the K-5 curriculum. The independence of these curriculums and their over-reliance on procedural/formal activities make them the type of mathematics curriculum that has proven not to be very successful.

Towards A New Approach

Intuition/Perception: Children's experiences in the world involve making sense of the things they encounter. They perceive things, and actively engage them in order to be able to interpret them in a sensible manner. For example, consider the experiences of a child who encounters the following events: the understanding of what a ball is when there are many sizes (basketball, soccer ball, tennis ball, ping-pong ball, etc.), yet all have the same shape, and all are classified as balls; the understanding that newborns are smaller than adults (whether humans or any other creature) yet, they exhibit the same shape, and will eventually grow larger while retaining their original shape, even though, there are some exceptions, such as tadpoles which transform into frogs when adults, (this deviation from similarity is one of the reasons the study of tadpoles is fascinating). There are many such examples that illustrate a child having to cope with making sense of the phenomena of similarity in a perceptual/intuitive sense. This is done informally, yet children everyday act out this scenario. Informal knowledge structures are created that are operable and sensible to the children since it allows them to understand the world they live in.

These informal knowledge structures provide a rich and meaningful background from which to launch mathematical investigations meant to further the individual’s
understanding of a particular phenomena. Specifically, it is my objective to capitalize on students’ perceptual orientations and intuitions about similar figures in my quest to develop similarity concepts. Although students’ articulation of similar figures may prove to be crude, it nevertheless is a part of their knowledge gained from interpreting certain events in the world, and so is essential to the investigation of similar figures.

It is easier to accept mathematics as competency in operating with formal sets of rules and procedures on routine problems than it is to reconstruct mathematics as about students actively constructing knowledge structures which enable them to meaningfully interpret particular situations. However, the constructivist perspective is a very prominent position from which to develop a curriculum.

**Similarity and Rational Number Knowing:** Kieren (1993) notes the transition from whole numbers to rational numbers is not easy. The rational numbers, at least to students, are not a natural extension of the whole number domain, yet they are often treated as such procedurally. The study of similarity offers a unique context for students to further their understanding of rational numbers. The geometry of similar figures utilizes rational numbers to express in quantity the perceptual relationships inherent in similarity.

In this respect, we will see that similarity allows for a **geometric interpretation** of rational numbers. Because of the continuous manner in which similar figures may occur (growth or shrinkage), students will recognize the need for rational numbers to represent similar figures that whole numbers are unable to represent.

Investigating similarity from an arithmetic domain has the advantage of focusing on the topic without outside interference from the rules and logic of less familiar
mathematical domains. In particular, algebra, a domain that has its own formal language and operates independently of the situation it may represent, is generally unfamiliar to students. Students are able to make sense of new situations from their knowledge base acquired from previous experiences (Mack, 1993). Students do not access domains with which they are not familiar. If students are forced to utilize algebra to represent relationships between similar figures and their algebra skills are weak, they may become frustrated with the algebra, and fail to fully consider the relationships of similarity.

The Standards (NCTM, 1991, p. 48) states that children’s spatial capabilities frequently exceed their numerical skills, and tapping these strengths fosters an interest in mathematics and improves number understanding and strength. The Standards (p. 51) suggests that measurement is a natural context in which to introduce the need to learn rational number. It suggests that high school geometry (formal representation) should build on the strong conceptual foundation students will develop in the proposed K-8 mathematics curriculum.

**Unitizing and Norming:** The new approach will utilize unitizing and norming, which while not yet in schools, is well documented in the research literature. Students’ ability to reconceptualize a situation in terms of composite units has indicated a more sophisticated approach to understanding rational situations (Behr, Harel, Post, and Lesh, 1993; Lamon, 1992). For example, \( \frac{3}{4} = 3(\frac{1}{4}) \) units where \( \frac{3}{4} \) is conceptualized in terms of the singleton unit " \( \frac{1}{4} \)". In another situation, the number 10 may be considered as ten 1-units, or it may also be considered in terms of subgroups such as two 5-units or five 2-units (the 5-unit and 2-unit would be considered composite units). Freudenthal (1983) used the term norming to describe a process of reconceptualizing a system in
relation to some fixed unit or standard. The process of unitizing and norming is discussed at length in Chapter Two.

Students' eventual familiarity with unitizing and norming as a process to develop rational number knowledge should enable them to utilize this process as a vehicle of investigation into other mathematical topics, such as similarity, without interference from unfamiliar domains of mathematics (such as algebra) that are introduced as methods of inquiry. In similar figures, the utilization of algebra to represent relationships that occur between the figures introduces a new mathematical domain that students may or may not be familiar with, and the rules whereby the logic of algebra operates unconnected to the situation it represents. The unit concept is a process whereby meaningful interpretations of similarity can be made within the confines of the students' mathematical experiential base.

Confrey and Smith (1995) insist that ratio becomes a way of describing the invariance across a proportion. Ratios are never singular instances of a relationship between magnitudes but are constructed by objectifying and naming that which is the same across a proportion. As a result, to recognize ratio is to recognize the homogeneity of ratio across more than one instance. Lamon (1993) gives the following example to illustrate ratio as a unit.

On a business trip, 9 people traveled comfortably in 2 cars. Our company plans to send 18 sales representatives to a conference next week and I need to reserve some rental cars for their trip. How many rental cars should I reserve (p. 135)?

In the ratio \( \frac{9\text{ people}}{2\text{ cars}} \), Lamon labels nine as a composite unit; that is, nine individuals are considered as a 1 nine-unit, a unit of units. Likewise, two is a composite unit. Two
individual cars become a single two-unit. Lamon suggests a new level of complexity is reached when we consider the ratio itself as a unit. For example, \( \frac{18}{4} \) is two \( \frac{9}{4} \) units.

Lamon asserts that the determination of a scale factor within what she calls a measure space entails reinterpretation of one measure in terms of the other using the process of "scalar decomposition". That is, the process of decomposing a magnitude as a linear combination of other different magnitudes: multiples and fractions of some magnitude M. The magnitude 7 is decomposed into multiples of 4 and fractions of 4.

(Example of norming as used in the process of scalar decomposition)

\[
x \cdot \frac{3}{4} = \frac{7}{4} = 1(4) + \frac{3}{4}(4)
\]

This process has a particular explanatory power that makes it a practical tool (as will be shown) to use in analyzing similar concepts. Also, it is a natural extension from developing rational number concepts (Lamon, 1994; Golding, 1994) which means that students will be familiar with the unitizing/norming process. Thus, if teachers utilize something with which they are already familiar, it will allow students to investigate new situations without interference from unfamiliar domains of inquiry.

**Phenomenology:** A students’ initial response to similarity is based in perceptual orientations. This study utilized a method of inquiry in order to realize the constituents of similarity which, collectively, give meaning to what makes figures similar. The method determined these features of similarity by analyzing students' perceptual orientations. Once the relationships inherent to perceptual orientations of similar figures were explored and
identified, the students mathematized these relationships by quantification (Phase II), and utilized them to analyze similar figures from a mathematical perspective (Phase III).

These teaching methods were part of a constructivist model informed by Idhe’s (1986) explanation of phenomenology. Idhe’s explanation of phenomenology using geometric figures provided a backdrop for the design of how to investigate students’ informal sense of similarity. The teaching model, as illustrated in Figure 1.3, performs a systematic analysis designed to articulate students’ geometric intuitions about similarity to utilizing these relationships to solve traditional similarity problems. The teaching model adheres to the constructivist position because it builds on students’ intuitions or prior

![Figure 1.3 (Teaching Model)](image-url)
knowledge to develop new knowledge. The reader should also note that this curriculum has the possibility of grounding algebra in similarity, rather than vice versa, though this proposal is not pursued within the present teaching experiment.

**Geometric Intuitions of Similarity**

Students' intuitions about similar figures are grounded in their perceptions and lived experiences. Their understanding of similarity provides a beginning for developing a conceptual understanding of similarity. I am not suggesting that this analysis will reveal the absolutes of similarity; rather, it will identify certain invariants germaine to perceptual orientations of similarity and mathematize them to allow analysis of similar figures from a mathematical orientation.

**Phase I (Perceptual Intuition):** The phenomenon of similarity involves specific relationships that are based in the geometric configuration of the figures. In Figure 1.4, the two triangles are similar if they are seen as invariant despite their differences in size. This "same shapeness" is a perceptual sense that exhibits particular characteristics.

![Figure 1.4 (Similar Triangles)](image)

An analysis of perceptions reveals alternate perceptual foundations for similarity. One perception of similarity involves a between figures analysis in which particular
attention is given to the growth or shrinkage of the second figure with respect to the initial figure. The second figure is the first figure translated in time through a process of uniform growth or shrinkage. Note that this understanding of similarity is perceptual without necessarily making connections to an explicit quantification of sides, rules of growth, etc.

In a second perceptual approach the two triangles in Figure 1.2 are understood as separate and distinct entities. The perception of similarity is grounded in a featural analysis of within figure relationships. For instance, in Figure 1.4, in both figures the horizontal arm is perceived as slightly longer then the vertical arm. The amount of turning (i.e., angle) between the two arms is the same, etc. A dynamic element of comparison does enter into this perceptual approach, but it is located in the within figures relationships. Thus, the horizontal arm may be understood as an extension over time of the vertical arm as it undergoes uniform growth. Ultimately, the perception of similarity of the two triangles resides in observing that the patterns of growth within one triangle are identical to the patterns of growth within the other triangle. As in the previous case, this perceptual sense is not vested in explicit quantifications of lengths or angles.

**Phase II: Quantification of Perceptual Orientations:** The development of an intuitively grounded understanding of similarity proceeds from perceptual intuitions to quantification to algebraic symbolism. Each perceptual approach to similarity, within & between, provides its own unique resources for quantification.

**Between Figures Analysis:** In the between analysis, this study is particularly concerned with verifying a constant growth or shrinkage. In the case of Figure 1.5, if
Triangle 2 is similar to Triangle 1 then it represents the same geometrical shape as Triangle 1 and its size is an increase from Triangle 1.

![Diagram of similar triangles](image)

**Figure 1.5 (Similar Triangles)**

The increase must be a constant rate of change. Therefore, not only is the triangle as a whole affected, but the individual lengths a, b, and c are affected in like manner to produce lengths d, f, and e such that the triangles are similar.

An obvious way to capture this quantitatively is to suggest a constant (k) to represent the growth or shrinkage from the initial to the final figure. If a, b, c and d, e, f are the respective lengths of the sides of the triangles given in Fig. 1.3, then three equations result; ak=d, bk=e, and ck=f.

The understanding of rational number concepts emerges at this point as a vehicle to move from multiplicative to ratio representations. If k is the growth factor from a to d b to e, and c to f, then the ratios of d:a, e:b, and f:c are all equal. Stated fractionally,

\[
\frac{d}{a} = \frac{e}{b} = \frac{f}{c}.
\]

To further illustrate this arithmetically, Figure 1.6a illustrates the between figure norming construct on two similar triangles. The scalar operator is 2. Each side of triangle B is a multiple by some constant of a side from triangle A; in this case, two.
Within Figures Analysis: In the within figures analysis, this study is concerned with quantifying the ratio that compares sides within the figure. The horizontal arm may be thought of as containing a certain number of the vertical arms or vice versa. By reconceptualizing the horizontal arm in terms of the vertical, the horizontal arm may then be expressed in number of units where each unit is the vertical arm. This is because the horizontal arm is a multiple of the vertical arm where it has grown relative to the vertical arm by some constant $r$. I will treat the vertical arm as a unit and determine how many units are contained in the horizontal arm. This will reveal the value of $r$. Thus, in figure 1.5, $b = r*(a)$ where $b$ is normed in terms of the unit $a$. 

<table>
<thead>
<tr>
<th>Side</th>
<th>Unit</th>
<th>Side</th>
<th>Constant * unit</th>
<th>Side</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3 unit</td>
<td>6</td>
<td>2 (3 unit)</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>4 unit</td>
<td>y</td>
<td>2 (4 unit)</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>5 unit</td>
<td>x</td>
<td>2 (5 unit)</td>
<td>10</td>
</tr>
</tbody>
</table>
Similarly, e may be normed in terms of d. If the figures are similar, then the constant whereby e has grown relative to d will be r from the first figure, that is, \( e = r^*d \).

It follows that \( \frac{b}{a} = r \) and \( \frac{e}{d} = r \), or \( \frac{b}{a} = \frac{e}{d} \). Note, by algebraic manipulation this is identical to the proportion \( \frac{a}{d} = \frac{b}{e} \) from the between analysis.

The same within analysis may be utilized to compare other pairs of sides from the similar figures; however, the constant of growth will not be identical in each comparative group. This is because c and b are not the same length. Thus the reconceptualization of c in terms of a will produce a different constant, say q, from a to b. However, the corresponding sides d and f will produce the same constant q when f is reconceptualized in terms of d. That is, \( c = q^*(a) \) and \( f = q^*(d) \), or \( \frac{c}{a} = q \) and \( \frac{f}{d} = q \). Then, \( \frac{a}{d} = \frac{c}{f} \). Again, it follows that the proportion \( \frac{a}{d} = \frac{c}{f} \) is obtained through algebraic manipulation which is identical to the between analysis. For an arithmetic example, again consider Figure 1.6; however this time we will pair sides within triangle A with corresponding pairs from triangle B.

<table>
<thead>
<tr>
<th>Triangle A</th>
<th>Triangle B</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>y</td>
</tr>
<tr>
<td>5</td>
<td>x</td>
</tr>
</tbody>
</table>

Figure 1.6b
Comparison of Sides in Units

<table>
<thead>
<tr>
<th>Triangle A (1a, 1b)</th>
<th>Triangle B (2a, 2b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a. 3 = 1 (3 unit)</td>
<td>4 = 1 ( \frac{1}{3} ) (3 unit)</td>
</tr>
<tr>
<td>2a. 6 = 1 (6 unit)</td>
<td>y = 1 ( \frac{2}{3} ) (6 unit) or y=8</td>
</tr>
<tr>
<td>1b. 3 = 1 (3 unit)</td>
<td>5 = 1 ( \frac{2}{3} ) (3 unit)</td>
</tr>
<tr>
<td>2b. 6 = 1 (6 unit)</td>
<td>x = 1 ( \frac{2}{3} ) (6 unit) or x=10</td>
</tr>
</tbody>
</table>

The scaler operator for \( \frac{3}{4} \) and \( \frac{6}{y} \) is \( \frac{1}{3} \) three units, while the scaler operator for \( \frac{3}{5} \) and \( \frac{6}{x} \) is \( \frac{2}{3} \) six units. The scaler operator obtained from a within construct differs according to the pairs of sides chosen, but is stable between figures.

The unitizing and norming process allow for quantification of one’s perceptual orientations of similarity without accessing unfamiliar mathematical domains. The students’ ability to represent similar figures using this process with whole numbers only is limited, thus providing a natural and needful manner to introduce the rational number domain as a means to further develop the analytical process.

**Phase III: Towards Formalization:** Formalization sometimes is misconstrued to be “procedural mathematics”—mathematics that consists of meaningless procedures, governed by rules, and learned through modeling and repetition (Hiebert & LeFevre, 1986). This is not the interpretation of formalization used in this study. The procedures and results the students obtained and used in this phase are a result of formalizing relationships established in earlier phases. The procedures are grounded in perceptions and quantification. From this perspective, the organization of processes into a structure more readily suggests reification.

26
Sfard and Linshevski (1994) suggest the theory of reification involves a process-object duality. They contend the operational (process-oriented) conception precedes the object (structural approach) in the majority of mathematical concepts, and mathematical objects are an outcome of reification of the processes. Although Sfard and Linshevski partially support their theory from an epistemological perspective of the growth of algebra, the results have implications for the individual learner and for this particular study.

In the formal phase of similarity development, students represent and utilize relationships of similarity for problem-solving purposes without consciously accessing the fundamental relationships that exist between similar figures. It is assumed that students' understanding of the mathematics of similarity are now well-grounded in perceptual and operational processes through participation in Phase I and Phase II. Therefore, the task of determining unknown lengths resides in the resultant mathematics. The perceptual orientations still exist, but the mathematics can now operate without explicit reference to the underlying visual justification. The multiplicative constants that exist in similar figures are efficiently obtained from division, and the division product encapsulates a process of growth. The constant can be in ratio or decimal form and becomes an operator to determine the unknown lengths of a figure. Thus, the process is striving for symbolic representation that is derived from an operation. Sfard (1994) claims the introduction of symbolic notation is necessary for reification.

Understanding the relationships inherent in similar figures and representing them with numerals is not only necessary for quantification, but also for the students to view these relationships as functional. Then they can become mathematical entities with their
own set of rules. These entities have structure and become objects that are concise and manipulable. Sfard claims this metaphor of object is resultant of higher level thinking.

**A New Curriculum for Similarity**

A new curriculum is proposed that enables students to understand the mathematics of similarity based on students' perceptual/geometric intuitions. Students already have an informal interpretation of similarity due to their experiences in the world, and a new curriculum would allow students' final interpretation of similar figures to develop from their everyday experiences. Students will articulate the qualitative process of shrinking/enlarging a figure and comparing parts/whole between two existent similar figures. This will be done in an informal manner without focusing on any quantitative analyses.

The next phase will involve quantifying the relationships that students have articulated in the first analysis, in particular, the within/between relationships that exist between similar figures. The unitizing and norming process provides a theoretical foundation for understanding how the students develop multiplicative constants consistent with shrinking/enlarging (between analysis), and numerical relationships within each similar figure will produce constants that are consistent arithmetically within the second figure.

The last phase of the new curriculum will allow students to utilize learned proportional relationships between similar figures to investigate many such instances through reification of ratios without explicit reference to the underlying visual relationships (Sfard & Linchevski, 1994). Their understanding of similarity is now based in the mathematics, rather than at the visual level. However, it is assumed that by moving
through each phase, students' mathematical understanding of similar figures will be grounded in their initial perceptual orientations.

The Significance of the Study

As has been shown, students' early experiences with rational number are quite distressed. Utilizing the unit concept as a process for understanding the quantitative nature of similar figures is consistent with the efforts of numerous researchers to develop rational number competency. Similarity is based in perception, and the unit concept will allow students to quantify their perceptions rather than ignore them, and will not rely on algebraic notation as the current curriculum does.

Also, research indicates (Kieran, 1994) that students' competency in algebra is superficial at best. Utilizing a domain of mathematics that is so poorly understood cannot link students' prior knowledge of arithmetic and perceptions of similar figures in a meaningful way. Instead, students are encouraged to use a different set of conventions that are not even related to their observations and experiences to investigate similarity.

Wheeler (1989) suggests that algebra is traditionally taught for its importance as a tool needed to handle the mathematics that is to come later, rather than as a branch of mathematics with a use and character of its own. He examines the role of algebra as governed by the two extreme positions of "universal arithmetic" and a purely "symbolic system", and concludes that algebra is not irrevocably tied to arithmetic and symbolic algebra, and symbolic algebra is semantically weak. This does not mean that teachers must change the way that students are introduced to algebra, but that we should be concerned with more than giving meaning to symbols; we should be attentive to modes of thought that are essentially algebraic.
Kaput (1989) suggests that semantics in algebra is relational. That is, meaning that is developed within or relative to particular representations or ensembles of such. He claims there are no absolute meanings for particular mathematical ideas, rather meaning is interpreted by the individual relative to physical and mental representations of that idea.

It is therefore possible that algebra can be interpreted through particular geometric concepts. This presents a more precise way of thinking about a geometric concept, and represents a reversal from the way that geometry is currently taught. It could present the case where certain aspects of algebra are grounded in similarity concepts.

Finally, limited insight into the meaning of similarity can have dire consequences for the students when studying other mathematical topics such as slope. The following Figure indicates the importance of understanding the within and between analysis of similar triangles to conceptualize slope.

![Figure 1.7 (Slope and Similar Triangles)](image)

Comparing sides in a within analysis we obtain \( \frac{4}{6} = \frac{8}{12} \) where \( 6 \times \frac{2}{3} = 4 \) and \( 12 \times \frac{2}{3} = 8 \). The growth of the height (vertical side) relative to the base (horizontal side)

30
is \( \frac{2}{3} \) to 1. Note that this is the slope of the line. Reducing each ratio yields \( \frac{2}{3} = \frac{2}{3} \) which is obviously true but does not indicate to the student any growth over time.

In a between analysis, \( \frac{4}{6} = \frac{8}{12} \) where \( 4 \times 2 = 8 \) and \( 6 \times 2 = 12 \) in effect doubling the size of the first triangle; however, the resultant within analysis of \( \frac{8}{12} \) yields \( 12 \times \frac{2}{3} = 8 \) where the multiplicative constant is the same. This always occurs as long as the new triangles are similar, which they will be when measured by the same line relative to the x-axis. Since differential calculus is a derivation of the slope of a curve, a student's limited insight into slope could impact the student's ability to understand such an important branch of mathematics.

\[
M_{ap} = \lim_{x_2 \to x_1} \frac{y_2 - y_1}{x_2 - x_1}
\]

Figure 1.8 (Slope of a Curved Line)
Similar Figures

Similarity of geometric figures is initially guided by perceptual orientations. The richness of this perceptual domain provides an arena for the development of formal considerations that enable the individual to build more powerful notions of similarity. Unfortunately, school-based mathematics is currently more concerned with students learning a predetermined representation of similarity that is narrow and based in formal rules. This study attempts to develop this domain of the mathematics curriculum from students' informal sense of similarity, which will ultimately provide the framework for a curriculum that is based in students' informal knowledge. Quantification of perceptual relationships enables a mathematical interpretation of similarity. Once the mathematics is developed, students will utilize such to solve traditional similarity problems. This approach is aligned with the constructivist position and suggests a meaningful manner in which to develop similarity concepts.

Research Questions

The following research questions demarcate the scope of this study. Each question represents an attempt to understand an aspect of the difficulties inherent in the development of similarity concepts. The first question seeks to understand to what extent students intuitive sense of "likeness" of geometric figures can provide a framework for developing a meaningful approach to solving problems of similarity currently found in geometry textbooks. Current mathematics proposes an approach that is not related to students' intuitive sense of similarity, thereby contributing to the attitude towards mathematics as a formalistic, rule-based enterprise. The success of this study was contingent on being able to develop a meaningful manner in which to solve similarity
problems. The second question is more concerned with understanding how the within and between relationships inherent in similar figures impacts the students’ conceptual development of similarity. Each relationship represents an entirely different conceptual view of the relationships that exist between and within similar figures. Finally, since the two geometry classes had very different student profiles, I took advantage of the opportunity to compare the effectiveness of the curriculum with more and less talented learners.

1. Can a perceptual analysis of similarity provide the basis for students’ conceptual approach to standard problems involving similar figures?

2. How does students’ understanding of similarity evolve through the curricular development of between and within relationships within similar figures?

3. What differences are there between more and less mathematically talented students in their development of conceptual understandings of similarity?

**Organization of the Study**

A review of the literature is provided in chapter 2. The literature reviewed includes areas critical to the development and implementation of similarity concepts, such as, rational number, unitizing and norming, geometry, and algebra. Also reviewed are phenomenology and constructivism, which influenced the design of the study. The literature review concludes with a discussion of the role of constructivism in the development of pedagogy, contrasting radical constructivism and social constructivism.

The methodology is presented in chapter 3. It provides a complete description of the subjects, the design of the study, the purpose of the study, and the methods for collection and analysis of data. The results and analysis are provided in chapter 4, organized...
sequentially according to the data collected for each lesson. The summary and conclusions are presented in chapter 5. Each research question is addressed and there are sections dealing with pedagogical implications, implications for future research, and limitations.
CHAPTER 2
REVIEW OF THE LITERATURE

This chapter is a series of brief discussions of the areas pertinent to this study. Each discussion is not meant to be exhaustive, but to give the reader a flavor of the influence of each in the development and implementation of this experiment. The topics are hierarchal correlating to the levels of mathematics that occur in the development of similarity concepts.

Whole Number and Rational Number Development

Children’s understanding of the number system is systematically developed beginning with investigation into what is called the natural or counting numbers (1, 2, 3, ...). Students learn ordinality and cardinality of number. Situations such as --- Tom had $5.00 and spent the entire amount on a new baseball. Tom now has no money left? How will we represent that amount?--- required the addition of zero to the natural numbers to form the whole numbers (0, 1, 2, 3, ...) so that students may represent a quantity that is "nothing".

Fractions are eventually investigated by students without correlating them to a particular number set; rather, they are studied only in terms of their value in relation to whole numbers. Number lines are often used to position fractions relative to whole number.

Negative numbers are added to the set of whole numbers to produce the set of integers (..., -3, -2, -1, 0, 1, 2, 3, ...). Negative numbers allow students to represent a variety of situations that whole numbers cannot. For example, suppose the temperature was 30 degrees and a cold front lowered the temperature by 40 degrees. The question,
"What is the temperature reading?" would propose a situation that could not be represented with positive numbers.

Various situations in the child's experiences require the development of numbers that represent quantities, length, weight, etc., for which integral numbers fail to account. Situations such as how to find the middle of a line that is 5 inches long, figuring how much flour will be needed when halving a cookie recipe that regularly calls for 3 cups of flour, or finding how much of an hour is 30 minutes, etc., all need numerals other than integers to represent the determined values. Rational numbers are defined as any number that can be represented by a/b where a and b are both integers and b ≠ 0.

The systematic manner in which numbers are studied would seem to indicate that students who experience success with integers would also continue that success into the rational number domain; however, such is not the case. The difficulties children encounter with rational numbers is well researched and documented (Kieren, 1976; Behr, Lesh, Post, & Silver, 1983; Hart, 1988; Heller, Post, Behr, & Lesh, 1990, Mack, 1990). Also, assessments conducted by the National Assessment of Educational Progress indicate that students calculate fractions by applying memorized rules, and demonstrate little or no knowledge of the underlying concepts (Golding, 1994). Even more problematic are results from studies such as the Rational Number Project (Post, Harel, Behr, & Lesh, 1988) which indicate that teachers do not know enough mathematics to promote conceptual understanding in their students.

This does not mean to suggest or ignore the fact that there are particular troublesome spots (i.e., the concept of zero, quantitative value of negative numbers, etc.) in whole number development that students may stumble at, but assessments suggests
their difficulties in this arena pale with the difficulties they experience in rational number
development. Rational numbers are of particular interest to this study since the arena of
similarity provides an excellent opportunity to utilize rational numbers in the quantitative
analysis of similar figures, thereby, continuing rational number learning.

Kieren (1976) suggests the transition from whole numbers to rational numbers is
not a natural occurrence; in particular, the rational numbers are not a natural extension of
the integers. The acquisition of whole number concepts and rational number concepts are
each driven by their own particular rules and procedures. The rules for operations on
whole number concepts cannot be blindly applied to operations on rational numbers. This
often leads students to an impasse. For example, students learn addition facts such as 3 +
6 = 9 and 1 + 5 = 6, then when they encounter a fraction such as \( \frac{1}{3} + \frac{5}{6} \), they extrapolate
the rules for addition and suggests a common response of \( \frac{6}{9} \). Their addition is correct,
yet their response is incorrect.

The rational number domain represents a complex array of features that involve
the coordination of several variables which can be thought of as subconstructs presented
as ratio, part/whole, measure, quotient, and operator. A complete conceptual
understanding of rational numbers requires understanding each subconstruct and how
they are related (Kieren, 1976; Post, Behr, & Lesh, 1986; Behr, Lesh, Post, & Silver,
1983). To illustrate the various subconstructs, consider the fraction \( \frac{1}{4} \)
As a

- **Ratio:** It may represent ratio as one red banner for every four green banners.
- **Part/whole:** It may represent one slice of a pie cut into four pieces.
- **Measure:** It may represent position on a number line.

37
Quotient: It may represent the number 1 divided by 4.

Operator: It may be thought of as a function. A storewide sale in which everything is marked down by \( \frac{1}{4} \) or 25%. The fraction \( \frac{1}{4} \) is operable on every item in the store even though a person may only purchase one, two, or no items.

The order of the evolutionary development of rational number in relation to these subconstructs, and how the subconstructs are related to each other and as a whole, are matters of interest. Behr, Lesh, Post, & Silver (1983) claim the part/whole relationship is fundamental in developing the other features of rational numbers. Boulet (1994) notes that the part/whole relationship is the ratio between part and whole, and concludes that ratio is the first subconstruct to be considered. Golding (1994) uses the unit fraction to understand ratio. She defines unit fraction as any fraction having a numerator of one and the denominator a natural number (e.g., \( \frac{1}{4} \), \( \frac{1}{3} \), etc.). The unit fraction provides the basis for understanding the quantitative feature of fractions (e.g., \( \frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \)); thereby suggesting rational numbers are a viable number set whose elements have ordinality and cardinality.

**The Unit Concept**

Units are already utilized in the mathematics curriculum; however, its use is restricted to whole numbers. For example when learning place value, students are often asked to partition a number such as 267 into 2 (100) + 6 (10) + 7 (1). Von Glasserfeld (1981) suggests that the formation of units is a natural occurrence for students. Students count by ones but later extend to counting by twos, fives, tens, etc. to easier represent certain quantities. The formation of the unit concept continues as students represent whole numbers as the composition of various units (the number 5 as 1 + 4, 2 + 3, 1 + 2 + 1).
This development could continue in the rational number domain (e.g., \( \frac{3}{4} \) as \( \frac{1}{4} + \frac{2}{4}, \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \), etc.) thereby correlating students learning of rational numbers with their methods for understanding whole numbers, and continuing to build on students' natural tendency to form units. Von Glaserfeld's suggestion of the intuitive nature of units is supported by a study conducted by Golding (1994). However, Golding did note "algorithm dominance" was an obstacle to students' understanding and implementing the unit concept. The next section will describe the actual process of utilizing units to make sense of various situations.

**Unitizing and Norming**

Freudenthal (1983) describes unitizing as a construction of a reference unit and norming as the process of reconceptualizing a situation in terms of the unit or a composition of units. Lamon (1994) suggests the adoption of some framework of units in which to conceptualize a situation is prevalent in mathematics thinking. To illustrate, she describes a situation in which one might imagine that the earth is the size of a pin's head (about 1 mm diameter) and then reconceptualize the solar system in terms of that definition. The analysis of many rational number processes is possible by the norming construct. Of particular interest are the within and between strategies used to solve proportions.

Vergnaud (1983) describes a reconceptualization process of a scaler operator operating upon one element of a measure space to produce another and vice versa. This scaler decomposition, illustrated below, results in the reinterpretation of one measure in terms of the other.
Four represents the unit and seven is reinterpreted in terms of whole four whole number units. Seven is then understood in terms of a scaler multiple and a unit: \[ 7 = \frac{7}{4}(4) \]. Conversely, \( 4 = \frac{4}{7}(7) \) where seven represents the unit and \( \frac{3}{4} \) is the scaler operator.

Lamon (p. 95, 96) demonstrates the within strategy or scaler method for solving proportions which involves equating two within-measure space ratios and uses the sameness of scaler operators to determine the missing term with the following problem. If I can make five team shirts with seven yards of material, how many yards of material will I need to make a team shirt for each of fifteen children on the soccer team? The following schema represents this problem with five as the unit whole and fifteen as three of those units.

\[
\begin{array}{ccc}
\text{M} & \text{M} \\
(\text{number of shirts}) & (\text{number of yards}) \\
5 & 7 \\
x3 & x3 \\
15 & X
\end{array}
\]

Three is the scaler operator that represents the sameness of change within each measure space. Norming may also be used to analyze proportions in a between strategy or a functional method. Considering the same situation, we will analyze the problem from one measure space to the other instead of within each.
The function operator \( 1 - \frac{2}{5} \) represents the coefficient of the linear function from \( M \) to \( M \), \( f(x) = 1 - \frac{2}{5}(x) \). Thus, 21 yards of material will be needed.

The research suggests the unit concept is a plausible method for acquiring rational number concepts since it is a natural occurring phenomena and plays a role (even though it is currently not emphasized) in the conceptual development of whole numbers. The norming process provides a method consistent with unit formation in investigating and determining relationships between quantities that require a number set other than integers.

**Phenomenology**

Phenomenology is inquiry into the very nature of phenomenon. It is an interest in what makes a some-"thing" what it is—and without which, it could not be what it is. The essence of phenomena can only be intuited or grasped through a study of the particulars or instances as they are encountered in lived experiences. Phenomenological research tries to explicate the possible meaning structures of our lived experiences. In contrast, it is different from natural science since the subject matter is always the structures of meaning of the lived human world, as opposed to natural objects that do not have experiences which are consciously and meaningfully lived (van Manen, 1990).

Phenomenologists argue that each of us experiences an interpreted world that is unique to the observer (Spinelli; 1989). In this interpretation, we have a tendency to organize our perceptions into "things" which constitute meaningful wholes. These

\[
\begin{align*}
M & \times 1 \quad \frac{2}{5} & M & \quad 7 = 1(5) + \frac{2}{5}(5) = 1 \quad \frac{2}{5}(5) \\
3 & \times 1 \quad \frac{2}{5} & X & \quad X = 1(15) + \frac{2}{5}(15) = 1 \quad \frac{2}{5}(15)
\end{align*}
\]
meaningful wholes represent a unification of our experiences. Spinelli argues this construction process is so pervasive and taken-for-granted that we become aware of it being a process only when it breaks down. To illustrate this point Spinelli states, "If we look at a tree, we don't see its various constituents - its trunk, branches, leaves, and so on - and from these conclude that we are seeing a tree. Rather, we see the whole "thing". We unify the constituents into a meaningful whole that we label a tree." (pg. 39).

Along similar lines, Ihde (1986) uses a series of thought experiments to illustrate phenomenology inquiry. Of particular interest, Ihde utilizes a phenomenological deconstruction and then a reconstruction of multi-stable phenomena to gain insights derived from the process itself. The phenomena used in this experiment are geometric, and may be described as multi-stable or optical illusions.

At the outset, "epoche" is assumed to exclude abstractions that may apply to the drawing, and to suspend belief in any causes of the visual effects. This allows one to positively focuses on what is and what may be seen. The main idea is to seek invariants that can be obtained through the variation process; namely, reconstruction of phenomena in terms of its multiple perceptual states. The accurate reconstructive process is dependent upon the deconstructive process which possibilizes all phenomena in seeking their structure.

Since the world we experience is a series of "mediations," we can never really know true reality. Thus, the phenomenological method seeks only to clarify the variables and invariants of interpreted reality, since to expose and explore what is truly real is not possible (Spinelli, et.al.).
Figure 2.1 (Multistable Phenomena)

<table>
<thead>
<tr>
<th>Level</th>
<th>Noetic Context</th>
<th>Noema</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group H</td>
<td>Literal-mindedness</td>
<td>Hallway</td>
</tr>
<tr>
<td>Group P</td>
<td>Literal-mindedness</td>
<td>pyramid</td>
</tr>
<tr>
<td>Group R</td>
<td>Literal-mindedness</td>
<td>robot</td>
</tr>
</tbody>
</table>

Level II

- Group A': polymorphic-mindedness alternation
  - hallway/pyramid

- Group A'': polymorphic-mindedness alternation
  - hallway/pyramid/robot/?
  - (topographical possibilities)

43

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Idhe uses Figure 2.1 to demonstrate a noematic analysis. In this illustration a
collection of people, Group H and Group P, will see only a hallway and pyramid
respectively. These perceptions are the easiest, and each group claims the validity of its
perception without seeing the other's perception. While somewhat more complicated,
Group R sees a headless robot without seeing a hallway or pyramid. In each case, a single
perception of this phenomena is what Idhe refers to as literal-mindedness.

This analysis of differing perceptual orientations is a deconstructive process that
allows for a determination of the structure of the phenomena. When the groups eventually
see the multiplicity of the phenomena, Groups A' & A", Idhe refers to them as
polymorphic-minded. He claims this last state of mind is indicative of phenomenological
inquiry.

Ihde describes a building process from an individual's mono-perception of a
figure in its initial appearance (literal-mindedness) to a multi-perception that reveals
numerous yet totally different perceptions of the figure (polymorphic-minded). He claims
to be literal-minded is the natural attitude of imputing to things, a presumed set way of
being, and to be polymorphic-minded is the phenomenological attitude of a deliberate
search for variations. This possibilizing of phenomenon allows a determination of their
genuine possibilities and the invariant inhabiting those possibilities. The shift from literal
to polymorphic-mindedness seeks a particular kind of richness within phenomena.

A phenomenological inquiry suggests a method of investigation and interpretation
that is particularly useful in determining the structural components of similarity. The
students ability to recognize various interpretations of phenomena through the
deconstructive process sets the stage for determining the invariant(s) of the phenomena. It
is the invariant(s) which give quantitative reason to students' initial perceptual orientations of similar figures. Essentially, they are the constituents of the unified experience. Thus, the value of becoming aware of the constituents is to provide the individual with a basis for understanding his or her own interpretation.

**Geometry**

The Standards (1989) states that geometry "helps students represent and make sense of the world." Geometry provides a way to model situations, and analyze and make abstract representation more easily understood. Attempts to quantify real-world objects produce ideas about number and measurement. Students should be active in constructing, measuring, visualizing, comparing, transforming, and classifying geometric figures. While synthetic geometry (deductive reasoning and proof) has been a major emphasis in the curriculum, the Standards suggest students' skills in visualization, pictorial representation, and the application of geometric ideas in problem-solving are of equal importance.

The reform document also describes geometry from an algebraic perspective showing the role of functions in transformational geometry. This interplay between geometry and algebra permits concepts in one to clarify and reinforce concepts in the other.

Van Hiele (1986) proposed a model for the development of geometric thinking that identified five differential levels of thinking ordered so that the students moved sequentially from one level of thinking to the next as their capability increased. The levels are (1) the visual level, (2) the descriptive level, (3) the theoretical level with logical relations, geometry generated according to Euclid, (4) formal logic, and (5) the
nature of logical laws. A student's Van Heile level for a topic is usually ascertained by the
correct number of answers on a written test. Gutierrez, Jaime, & Fortuny (1991) argue
that evidence gained by observing students' reasoning, rather than procedural skill,
supports the notion of students operating from more than one level at a time (prior belief
was that individuals were only capable of operating from a single level) and that
acquisition of a specific level does not happen instantaneously or very quickly; rather, it
can take several months or years.

It should be noted that Van Heile considered the visual level (recognition) as the
most basic level. At this level students judge geometric figures by their appearance. They
do not explicitly consider the components or properties of the figure which give it a
particular personality, rather, they reason informally using language such as "it looks
alike" or irrelevant attributes.

At level two (analysis), students identify the components of figures (edges,
number of corners, etc.) and their properties (parallelism, regularity, etc.). Their
description of the figures is still informal and they are not able to logically relate the
properties to each other, nor can they classify families of figures.

At level three (informal deduction), students logically classify families of figures,
definitions become meaningful for students, they can make informal arguments for their
deductions, and can follow some formal proofs. At level four (formal deduction), students
understand the role of the different elements of the axiomatic system (axioms, definitions,
theorems, etc.) and can perform formal proofs.

Gutierrez, Jaime, and Fortuny (1991) conclude from their investigation of how
students come to understand 3-dimensional figures, that the human learning process is
more complex than the simple, linear manner which the Van Heile levels suggest. They do not propose a total abandonment of the levels, rather a better adoption of the levels to the human learning process.

Current literature regarding geometry does not investigate the most primal aspects of geometry knowing (such as how do students come to know what "is the same" means; level one), nor is there much interest in coordinating the activities of learning geometry with the acquisition or continued development of number concepts. This study will investigate the acquisition of similarity concepts from the basis of students' perceptual orientations and previously acquired skills with number and their operations. Multiplicative relationships between similar figures will produce invariants that act as function operators in transformational geometry. This coincides with the suggestions made by the Standards. No official attempt will be made to formalize students thinking in relation to formal aspects of Euclidean geometry. However, the effort of this study may provide researchers with groundwork to better understand the complexities of the human learning process alluded to by Gutierrez, Jaime, and Fortuny.

**Algebra.**

The learning of elementary algebra involves grappling with the topics of variables, algebraic expressions, equations, and equation solving. Students' difficulties with these topics center on the meaning of letters, the shift to a set of conventions different from those used in arithmetic, and the recognition and use of structure (Kieren, 1989).

The discussion for the reasons these difficulties occur, centers around the features that give algebra its character. The most obvious features of algebra are its use of letters,
its introduction of a new notation and convention, and its focus on the manipulation of
terms and the simplification of expressions: its **syntax** (Booth, 1989).

Booth claims the ability to manipulate algebraic symbols successfully requires
that we first understand the structural properties of mathematical operations and relations
which distinguish allowable transformations from those that are not. These structural
properties are referred to as the **semantics** of algebra. Booth further claims that algebraic
representation and symbol manipulation should proceed from an understanding of the
semantics or referential meanings that underlie it. However, in many instances evidence
indicates that the learning of algebra becomes a problem of learning to manipulate
symbols in accordance with certain transformational rules without referring to the
meaning of the expressions or transformation.

Kirshner (1989) has shown that students who have no propositional basis for
syntactic knowledge rely heavily on the powerful visual features of the notational system
to cue them as to the syntactic structure of expressions. Students' success in parsing
algebraic expressions was dependent on their having access to sound propositional rules.
Simplifying an expression such as

\[
\frac{ax}{bx} = \frac{a}{b}
\]

can cause students to extrapolate a rule which when applied below is inappropriate.

\[
\frac{a+x}{b+x} = \frac{a}{b}
\]

The similarity of the visual features encourage students, who are not operating with a
solid propositional understanding, to retrieve and apply transformational rules
inappropriately.
Matz (1980) proposes that an individual’s problem solving behavior towards algebraic competence involves knowledge accumulated prior to a problem called the "base rules", and extrapolation techniques that specify the ways to bridge the gap between known rules and unfamiliar problems. She claims students' errors are the result of reasonable, although unsuccessful, attempts to adopt previously acquired knowledge to a new situation. This occurs when a known rule is used as is in a new situation where it is inappropriate, or by incorrectly adopting a known rule so that it can be used to solve a new problem. For example given \( \frac{x+1}{x+4} = \frac{5}{6} \) students' answers were \( x=4 \), \( x=2 \).

Algebraic expressions are structured explicitly by the use of parentheses, and implicitly by assuming conventions for the order in which we perform arithmetic operations. A study conducted by Thompson and Thompson (1987) using computer presentations of structure in algebra showed that many of students' errors in manipulating algebraic expressions are due to their inattention to the expressions structure. In particular, most errors occurred while students were first learning a field property or identity. Thereafter, the errors were less frequent.

Kaput (1989) believes the ultimate aim in algebraic instruction is to account for the building and expressing of mathematical meaning through the use of notational forms and structures. Furthermore, he claims that students are alienated to algebra by teaching algebra syntax instead of semantics. He claims students would be less inclined towards alienation if linkages to other representations that might provide informative feedback on the appropriateness of actions taken were a part of algebra instruction.

From a different perspective, Kirshner (1989) believes that students' difficulties with syntactic instruction is not because it is syntactic, but because research has not
revealed the immense complexity and intricacies required of syntactic performance in algebra. He feels the human mind is naturally predisposed to approach new, structured domains syntactically.

Acknowledging that pupils can and do solve mathematical problems without using algebraic symbolism, Sutherland (1991) asks, "Can we develop a school algebra culture in which pupils find a need for algebraic symbolism to express and explore their mathematical ideas?".

In any case, we should not assume that the transition from arithmetic to algebra is obvious and clear sailing (Wheeler, 1989). Research on the learning of algebra combined with evidence that suggests students' understanding of algebra is mostly superficial (Kieran, 1994) indicate the limited conceptual understanding students possess after undergoing the rigors of the current algebra curriculum.

The Standards suggest in the K-8 Algebra Curriculum, it is critical that students study algebra in an informal way to build a foundation for the subsequent formal study of algebra. This involves studying mathematical representations, investigating patterns and predicting from these patterns as well as representing them symbolically. The Standards argue that expanding the amount of time that students have to make the transition from informal to more formal ways of thinking increases their chance of success.

The students must understand the concept of variable as well as appreciate algebra as a language through which most of mathematics is communicated. The Standards suggests the 9-12 Algebra Curriculum should move away from emphasis on manipulative skills to include a greater emphasis on conceptual understanding of algebra as a means of representation and as a problem-solving tool.
Davis (1989) says, "Mathematics is a process of clever analysis of important problems". This review reveals the varying opinions on why students' understanding of algebra is deficient and makes some suggestions on what needs to be done to promote algebra learning in meaningful way. If it is intend for students to be able to utilize algebra as a means of representation and investigation (as opposed to being competent in formal procedures that most will never be able to or want to use again once they have completed their math requirements), then we must design a mathematics curriculum that incorporates algebra into the students’ mathematics development in such a way that it is conceived of as a viable and meaningful medium through which to understand the world we experience.

For those who understand algebra, it does provides a direct, concise method for representing particular similarity concepts. Nevermind that it fails to explore the individual’s intuitions about similar figures, because the rules for operating within the algebra domain provide an easy method for solving algebra equations that are created in the light of this phenomena. Thus, there seems to be a sense of urgency to utilize proportional/algebraic equations to represent relationships between similar Figures (Addison & Wesley, 1993; Harcourt, Brace, Javonvich, 1989; Southwestern, 1998). In this rush, there is no effort to connect formal representations to informal ideas that students have developed.

In mathematics, formal representations are the end result of intensive mathematical investigations. Formal representations of mathematics are convenient and efficient tools; however, these representations operate from a formal system whose rules
are often independent of the phenomena they represent. This is a powerful feature of algebra, yet it is questionable to investigate mathematical phenomena at this stage of the students school-based experiences by this method.

**Constructivism**

Constructivism derives from a philosophical position which asserts that human beings have no access to a reality independent of their way of knowing that reality. Constructivists believe our knowledge of the world is constructed from our perceptions and experiences, which are themselves mediated through our previous knowledge. Learning is the process by which humans adapt to their experiential world (Simon, 1995).

One of the important implications of constructivism for an educator is he/she must learn to approach a foreign or unexpected response with a genuine interest in learning its character, its origins, its story, and its implications (Confrey, 1990). They must be able to see situations as perceived by others, and if their perception differs from that of the observer, that perception should be treated as one of integrity and sensible within that individual’s framework. The individual’s constructed knowledge provides the teacher with a starting point to assist the student in constructing knowledge of a situation that is adequate within a larger society.

Constructivism asserts that the process of learning involves reflecting, which is the objectivication of a construct (Confrey, et. al.). Reflecting on an activity, we learn to develop it mentally, name it, and represent it in symbols and integers. The students’ construction becomes an object itself that can be analyzed and organized. This process stabilizes the construct and provides the position for the development of future constructs.
Constructivism has become quite popular with the mathematics community, and has rallied mathematics educators behind the idea of pedagogy that builds meaning, as opposed to the formal instruction from the past.

Even though constructivism has always been with us, perhaps under another guise, its growing acceptance as an educational tenant during the last decade or so has helped to finally oust the view of mathematical teaching as the transmission of the teacher's knowledge and mathematical learning as the reception of that knowledge and the subsequent capacity to regurgitate a copy of what was taught (Kieren, 1994).

Although the previous view of learning has been rejected as an inappropriate model for guiding mathematics learning, it did promote a particular pedagogy with which teachers are proficient. That method is commonly referred to as "direct instruction."

Implications for Pedagogy

While the constructivist position on mathematics learning has become more popular, it has not been as precise in recommending a particular strategy for mathematics teaching. Several aspects of constructivist implications for pedagogy are discussed in the following section.

The students are not repositories for adult "knowledge" but organisms which are constantly trying to make sense of their experiences (Von Glaserfeld, 1987). While their constructions are valid and sensible within their framework, the constructs are often weak and account for a limited range of phenomena. Thus, it is the responsibility of the educator to promote development of more powerful and effective constructions.
Assessments of these constructions is the students’ responsibility as well as the teachers.

As a goal of instruction, Confrey (1990) states,

An instructor should promote and encourage the development of each individual within his/her class of a repertoire for powerful mathematical constructions for posing, constructing, exploring, solving, and justifying mathematical problems and concepts and should seek to develop in students the capacity to reflect on and evaluate the quality of their constructions.

For this to occur in the classroom, teachers must understand the nature of students' mathematical understanding as a starting point from which to operate. Confrey (1989) suggest creating a "case study" of each student.

The older Piagetian perspective of constructivism suggested that mathematics teaching should be that of non-intervention. Since the students are the ones responsible for constructing their own knowledge, the role of the teacher is no more than providing an environment for the advancement of specific learning. This position on mathematics teaching has been vigorously attacked with the following argument: (1) The nature of students knowledge structures reveals a similarity that is not explained in the radical constructivist approach, and (2) to suggest that students left to their own devices will learn mathematics in a natural way, yet develop knowledge structures which are compatible with that of a wider society, is farfetched to most mathematics educators (Cobb, Yackel, & Wood, 1992).

Students do construct their own knowledge, but not in isolation. They are a part of a community that investigates and interprets their experiences. In the process of their
individual cognitive development, the students actively participate in the community's negotiation and institutionalization of mathematical meanings and practices. The teacher and students mutually construct these taken-to-be-shared mathematical interpretations. Thus, mathematics learning has a social aspect, and it is this view that allows one to escape from the solipsism inherent in a purely psychological analysis of learning (Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti & Perlwitz, 1991). This is referred to as social constructivism.

Social constructivism suggests the reason behind the similarity of students' knowledge structures (even though understanding is constructed by the individual) is that mathematics is interactive. It has a socially interactive component that accounts for particular features of students' understanding of mathematics, and how they acquire that knowledge (Kieren, 1994). The teacher's role becomes one of guiding and initiating the negotiation of mathematical meaning.

The teacher's role is a high complex activity that includes highlighting conflicts between alternative interpretations or solutions, helping students develop productive small-group collaborative relationships, facilitating mathematical dialogue between students, implicitly legitimizing selected aspects of contributions to a discussion in light of their potential fruitfulness for further mathematical constructions, redescribing students' explanations in more sophisticated terms that are none the less comprehensible to students, and guiding the development of taken-to-be-shared interpretations when particular representational systems are established (Cobb et al, 1991, pg. 7).
Vygotsky's view of social interaction in the classroom supports the social constructivist position. He argues a "zone of proximal development" is developed in social interaction. The zone is the difference between what a child could accomplish unassisted in problem solving and what he or she could accomplish with assistance. Furthermore, he argues that socially supported activity in the zone of proximal development awakens and provides paths for intellectual development (Kieren et al., 1994, pg. 603).

This chapter confirms that mathematics educators view learning as a constructive process that is socially interactive in nature. It supports the notion that meaning and the situation/phenomena are woven together such that meaningful interpretation cannot be separated from the phenomena. The evidence presented supports the contention of this study that (a) the current geometry curriculum operates from a procedural, rule-based format, (b) students need more meaningful opportunities to develop and continue rational number knowing, (c) unitizing and norming are viable methods for quantifying students' perceptual orientations, (d) students' perceptual orientations and intuitions of similarity are excellent and meaningful starting points to build formal meaning for similarity and, (e) a phenomenological analysis is the best approach for understanding students' geometric orientations.
CHAPTER 3

METHODOLOGY

Purpose of Study

As was noted in earlier chapters, current mathematics curricula (Addison Wesley 1990: HBJ Geometry, 1984) provide students with a narrow perspective on similar figures that is based on rules and procedures and does not consider students' perceptual orientations or intuitions. Students' proficiency in procedures for solving for unknowns in the context of similar figures does not indicate they possess a conceptual understanding of similarity, nor does it indicate they have connected procedures with their informal knowledge of similar figures. It only indicates their ability to perform certain procedural activities with school-based problems. Many students have at least two interpretations of mathematics; one a rule-based interpretation for the formalities of the classroom, and second an informal, yet sensible interpretation, to make sense of the world they experience. To ignore the students' informal construction is to ignore the basic premise of the constructivist model of teaching (Confrey, 1990).

Humans construct their understanding to interpret their perceived world in a meaningful manner (Von Glasserfeld, 1987). These constructions represent conceptual, if informal, knowledge structures (Confrey, 1990). Through school-based instruction, students become proficient with formal procedures that are acquired by drill and practice, but that proficiency is based in memorized procedures (Kieren, 1976). Often there are no attempts at connecting formal aspects of mathematics with students' informal knowledge structures, so students interpret school-based mathematics as useful only in the context of school activity (Lave, 1988). Their informal knowledge allows them to interpret their
world in a meaningful manner because it is based in experience. Thus, it was the purpose of this study to understand the conceptual forms that students have constructed of similarity from their lived experiences and prior mathematical experiences, and to assist students in building a formal construction of similarity based on their intuitive conceptions.

The empirical part of this study was devoted to understanding to what extent a teaching approach grounded in perceptual intuitions, could promote students’ meaningful interpretations of similarity. It was hoped that the analysis of student’s informal understanding of similarity would enable students to substantiate perceptual orientations with concrete relationships followed by quantification of those relationships. This provided a careful path of conceptual development based in the students’ informal perceptions, and in the subtleties of the structure of similar figures.

The study was particularly concerned with providing students with the means to develop quantitative notions of similarity without accessing unfamiliar mathematical domains, such as algebra. Although it is beyond the scope of this study, I was interested in the possibility that by providing students with a geometry that was an interpretation of their perceptions and intuitions, the eventual introduction of algebra in that context would mean that students’ understanding of algebra could be grounded in a meaningful situation.

The study consists of eight activities the students performed. There are three different phases which the activities are grouped. Phase I is the perceptual phase and has activities 1, 2, and 3. In this phase, the students explored the particulars of their perceptual orientations. Phase II is the quantification phase that includes activites 4, 5,
and 6. The purpose of this phase was to quantify the relationships between similar figures discovered from Phase I. Phase III utilizes the mathematics developed from Phase I and II, namely the within and between methods, in activities 7 and 8 to solve traditional problems involving similar figures.

**Research Questions**

1. Can a perceptual analysis of similarity provide the basis for students' conceptual approach to standard problems involving similar figures?

2. How does students' understanding of similarity evolve through the curricular development of between and within relationships within similar figures?

3. What differences are there between more and less mathematically talented students in their development of conceptual understandings of similarity?

**Sample**

The sample for this study was two high school geometry classes comprised of approximately 12 students in Class 1 and 8 students in Class 2. Although the students had some experience with similar figures at the junior high and high school level, their experience was limited to the usual formal treatment discussed in chapters 1 and 2. The classes were representative of students enrolled in one section of geometry for the 1998-99 school year at Ropes High School. Class 1 was considered college-bound as determined by standardized testing, past performance, interest, and motivation. Class 2 was comprised of students considered to be less capable than their peers.

This was the second high school mathematics course the students in Class 2 had been enrolled in, the previous being Algebra I. Class 1 had taken Algebra I & Algebra II. The Saxon curriculum was the text being used by the entire school mathematics program.
because the administration viewed the Saxon curriculum as a means for improving standardized test scores. The Saxon text did provide students (at the 9th grade level) with the traditional formalistic treatment of algebra and similarity.

Ropes High School is a K-12 unit classified as a I-A school in West Texas. Ropes is a rural community and the school services about 375 students. There are four mathematics teachers in grades 6-12.

Teaching Sequence

The experiment was conducted over a three week period, with six lessons, one hour and twenty minutes in duration, and three lessons fifty minutes in duration. The school was on a block schedule, which means each class met once every other day, M W F with Friday’s lesson being the shorter one. The lessons were taught in the regular mathematics classroom at the time that the students normally had geometry. The lessons were designed with three different phases in mind.

Perceptual Orientations

The initial understandings of similarity are based in perceptual orientations. We make judgments about whether figures are similar based upon invariance of shape regardless of differences of size. This is usually done quickly and without mathematizing any particular parts of the figures. While this may be suitable for informal settings, a formal investigation of similarity requires understanding the mathematical relationships that determine whether figures are similar. Knowing specifics between similar figures requires that students understand the mathematics of similarity. In the spirit of constructivism, understandings of similarity should be constructed by the students from their informal intuitions, which are grounded in perceptual cues. In order to develop a
mathematical understanding of similarity that builds upon perceptual knowledge, teachers need to provide the student with sequential activities that will provide them opportunities to utilize and articulate their perceptual orientations in determining similarity. Further activities should require students develop a mathematical approach to determine similarity. And finally, the students should have the opportunity to utilize the mathematics of similarity they have developed to determine whether figures are similar.

In the first phase, students observed similar and non-similar figures to differentiate between the perceptual differences of the two. They verbalized their perceptions of figures that are similar to share a language within the classroom community that is representative of similarity concepts. In the second phase, in order to understand how a figure maintains its shape through a size increase or reduction, the students understood what is changing and what that change is relative to corresponding parts between the figures and/or relative to parts within the figure. This type of analysis will require a quantification of lengths. In the third phase, students utilized the mathematics of similarity to determine similarity and to determine unknown lengths of similar figures.

Phase I

In this initial phase of the experiment, students were provided with opportunities to identify the specific features of similar figures that support their perceptual orientations. One may make instantaneous decisions about whether figures are similar or not from their informal, perceptual orientations. However, it was considered critical to this phase for students to recognize the between and within relationships that occur when
two figures are similar. Also, these relationships (which are operating at the perceptual level) provide the basis for introducing the mathematics of similarity.

This phase also serves a linguistic function in making sure students would come to use the (common) word “similar” in a technical sense (e.g., all rectangles are similar is wrong).

**Rationale for Activity 1** Students informal knowledge of similarity is based in their own experiences and may have forms that vary between individuals. Some may have more sophisticated interpretations while some may interpret similarity primarily through semantics. In any case, students developed a cohesive manner for determining similarity that was consistent with all of the students perceptual and linguistic clues and sensible to all. In the first activity, an exercise on the general sense of similarity was presented to provide the students the opportunity to discover that uniformity in interpreting and understanding similarity.

**Activity 1.** A chart was prepared which contains numerous geometric figures. The students determined which figures are similar and which are not. They shared their perceptions of similarity/non-similarity between the figures with the group. Students had the opportunity to discuss the perceptual specifics of similar figures, such as corresponding angles and sides and the invariance of angles, without necessarily mathematizing particular relationships. The activity was designed to ensure students can associate the term similarity with mathematically similar figures. The figures were paired in this document for comparative and discussion purposes, but the actual chart used in the experiment randomly scattered the figures so that the students had to pick out which pairs of figures were similar.
The students determined which figures were similar and which were not. They discussed the characteristics of similar figures and any discrepancies in their judgements or linguistic expressions. The figures in this chart were scattered so the students had to pick and choose those figures they thought were similar based on perceptual orientations. The figures that are similar represent growth or shrinkage form one figure to the other. Some figures have been rotated or inverted so that the students will determine what is relevant/non-relevant to similarity. These examples provided the students with a basis for determining invariance of relationships between similar figures.

In general, students are quick to determine if figures are similar based on perceptual cues alone. Although this phase is the most basic, perceptual part of the activities, the example encouraged the students to determine what relationships occur between figures such that their perceptual knowledge suggest they are similar. If there is preservation of angles and shape, first glance suggests similar figures. But the width of one of the rectangles is disproportionately longer than it should be for similarity to occur. This implies the semantics between same shape and like shape differ. By same shape we are implying preservation of angles and of ratios of lengths between and within the geometric figures. We will define like shapes as figures that look alike, but whose angles and/or ratios of sides are not preserved. Regarding the rectangles in this activity, they have like shapes, but the ratio of length to width is not preserved from the smaller or mid-size rectangles to the larger one; hence they do not have the same shape. The students performed this activity without necessarily mathematizing lengths.
Figure 3.1 (Chart of Geometric Figures)
**Verbal Representations:** To proceed from a perceptual to an analytical sense of similarity the students must first be able to express notions of similarity verbally. Similarity is often described as figures that have the same shape but differ in size. Linguistic concerns will be addressed so students fully understand the semantics of words such as "same" in the context of shape. To be able to verbalize similarity is the first step to linking students perceptual and linguistic knowledge to operations on geometric figures.

There are specific relationships that occur when two figures are deemed similar. Besides angle preservation, these relationships involve lengths. One can make judgements as to whether figures are similar, but each detail that determines similarity may not be purposefully addressed. Rather, it is the sum of these relationships that actually guides one's perceptual orientations of similarity. That "two figures look alike but are different sizes" is fairly ambiguous as far as the mathematics of similarity is concerned. It is the individual components of similarity that actually make interpretation of similarity a meaningful experience, and it is these individual components that provide the basis for introducing quantification of students' perceptual orientations.

Thus, the students need experiences which will facilitate recognizing the individual components of similarity from a perceptual basis. This phase of instruction is dedicated to that end.

**Rationale for Activity 2:** When students are given two geometric figures to determine similarity, they readily make comparisons between the two and determine similarity based on perceptual cues and possibly overlooking the relationships that occur within the figures themselves. Students need to be provided with situations that will
encourage examining the relationships that exist between individual lengths of a figures, and looking for preservation of those relationships to other figures. This is the within analysis.

Activity 2: This activity will encourage students to perform a within analysis to determine if the figures are similar. This will be accomplished by organizing the students into two groups. Each group will be provided with geometric figures. The students were not allowed to view the other group’s figure. Each group described the structure of their figure to the other group, and tried to determine if the figures are similar based on their descriptions (Pimm). Students were not allowed to use measurement. This ensured that the students articulated the relationships between the sides of the figure, such as "one side is twice as long as the side adjacent to it." The activity was performed numerous times with some figures similar and with some that are not. For example;

![Figure 3.2 (Similar Figures)](image)

These two figures provided the students with the challenge of articulating the properties of similar figures. One is the same as the other except for differences in size.
In another example the figures are similar in some respects, but after articulating the length of the vertical compared to the horizontal with these figures, students suggested that the figures are not similar because of a difference in the ratios of length to width.

Figure 3.2 (Non-Similar Rectangles)

Rationale for Activity 3: If two figures are similar, then by enlarging or shrinking one of the figures, we would eventually be able to make it the exact size as the other. Conversely, if two figures appear to be similar, but are not, the enlargement/shrinkage process will not produce two figures that could ever be the same size. There is preservation of angles and ratios of sides within the figure throughout the changing of size. This activity provided the students with experiences that build the notion that two figures are similar because one is the exact replica of the other, but magnified or shrunk by some factor.

Activity 3: A. The teacher projected a rectangle on the board from a rectangle that is placed on an overhead projector. The teacher had two other rectangles on
paper of which one is the same size as the one that is projected on the board, and the other is larger. The teacher placed the paper containing the larger rectangle on a tripod at a different location from the projected rectangle. The paper containing the smaller rectangle was placed on the rectangle on the board and taped to that position. The teacher moved the overhead away from the board until the students thought that the enlarged rectangle that was projected on the board was the same size as the rectangle on the tripod. At the moment that the students thought the rectangles were the same size, they told the teacher to stop moving the overhead. Then, the larger rectangle was placed on the now larger rectangle on the board to see if students' perceptions were correct. The activity was repeated; however, this time a larger rectangle was projected on the board and students were presented with a smaller version on the tripod. As the teacher moved the overhead towards the board, students told the teacher to stop when the larger triangle was shrunk to the size of the rectangle on the tripod. Again, the rectangle on the tripod was placed on the now smaller triangle on the board to verify the correctness of students' perceptions.

B. A pentagon was projected on the board. The vertices of the pentagon were labeled and the pentagon was traced on the board. The students focused on one side while
the teacher moved the overhead away from the board. The students were asked to tell the teacher to stop when that side was doubled in length. This length was marked on a piece of paper. Then, the students were asked to complete the larger pentagon relative to the shape of the original figure. Once they constructed the pentagon, they were asked to compare the sides of the figure they constructed with the corresponding sides of the original pentagon. This revealed that all the sides had to be doubled for the second pentagon to reflect the same shape as the original.

The activity was repeated; however, this time the pentagon will begin larger and will be shrunk to half of its original size. Students will be asked to repeat the steps and note that all of the sides will have to be shrunk to half of the original in order to preserve the same shapeness.

![Figure 3.4 (Similar Pentagons)](image)

**Phase II**

**Rationale for Activity 4:** Students were provided with experiences that enabled them to link perceptual intuitions of similarity to analytical analysis. To move into an analytical realm requires students to quantify their perceptual orientations. There are two cases to consider, the between and within analysis. In the between analysis, students express the increasing or decreasing of the original configuration by comparing sides of one triangle to the other according to the number of standard units obtained from...
partitioning the sides of the triangles equally. The quantitative analysis will quantify the length of the unit in the first figure and note the growth/shrinkage of the unit in the second figure. This process will quantify the internal growth/shrinkage of a figure.

**Activity 4:** From Activity 3, the students learned that the lengths of the sides of figures that are similar differ in size by some factor. So, given any figure, we can construct another figure similar to it by increasing or decreasing the lengths of its sides by some factor. In this activity, students used this feature of similar figures to construct figures similar to a given figure.

The students were given a triangle from which they were able to measure the length of each side. Then the overhead was moved away from the board and the students told the teacher literally *how long* before the figure is three times its original size. The students then measured the sides to compare to the original.

![Figure 3.6 (Multiplicative Constant of Three)](image-url)
They will also be given a triangle whose sides they measured. Then as the overhead was moved forward the students told how long before the figure is one half the size of the original. The students measured lengths and compared these with the lengths of the original.

![Figure 3.5 (Shrunk by 1/2)](image)

Next, the students were given a triangle with lengths of sides given from which they told how long before the sides will triple. This they did without the benefit of a second figure on which to operate. In this part of the activity, growth became implicit, represented as time. That is, there are no perceptual cues to guide the student.

How long before 12 becomes 36, 8 becomes 24, .......

![Figure 3.7 (Implicit Growth)](image)
The students were given two similar triangles of which both have their sides labeled according to length. Students were asked to determine what occurred between the two triangles so that one is similar to the other, and how did that affect the lengths of the figures.

![Figure 3.8 (Measured Triangles)](image)

The activities above reveals a multiplicative relationship between similar figure that measures growth or shrinkage. These activities provide the students with an arithmetical way of expressing one facet of their perceptual orientations of similarity. In essence, students are beginning to think about similarity quantitatively. It was intended for the students to make sense of similarity through the numerical relationships that also exist in the within feature of similar figures.

**Rationale for Activity 5:** There is a relationship that exists between corresponding lengths of similar figures that is measurable, as exhibited in Activity 4. However, there are also relationships that occur between the sides of a figure or within the figure itself. There are certain invariant mathematical relationships within a figure that must be maintained through the growth or shrinkage process. It is the objective of Activity 5 to quantify these within relationships.
Activity 5: The students performed the same kind of activity as Activity 2; however, this time students will be given lengths. Students were not allowed to communicate these lengths in determining similarity; instead, they were restricted to communicate numerical comparisons or ratios of sides in decimal form to keep from revealing the individual lengths of the sides through a ratio of $a/b$.

Figure 3.9 (Similar Figures)
Rationale for Activity 6: In the between analysis, the multiplicative relationship that occurs between the corresponding sides of two similar figures is constant. Once students know the numerical value of the relationship, they are able to calculate unknown lengths. The same is true for the within relationship. It is intended for the students to be able to use the invariance of the multiplicative relationship between similar figures to determine unknown lengths. This is the point where most textbooks begin their treatment of similar concepts. Since some sides will be labeled as unknowns, these examples began utilizing variable representation.

Activity 6: Students were given two similar figures in which one figure had its sides labeled according to length, and the second has only one side given. Students matched corresponding sides and determined the multiplicative constant between the two figures from the relationship of the side from the first figure that corresponds to the side given on the second figure. They obtained the multiplicative constant and used it to determine the length of the remaining sides of the second figure. The first few exercises will involve whole numbers, but then students progressed to more difficult numerical situations. That is, the multiplicative constant was not a whole number. From the manner in which the data is given, some of the figures suggested a within strategy, which if followed, makes the arithmetic easier.

![Figure 3.10 (Battery of Similar Figures With Unknown Lengths)](image-url)
Phase III

Rationale for Activity 7: Students have utilized the consistency that has occurred among all of the examples of similar figures to solve for unknowns. Namely, the multiplicative constant that serves as an operator between the corresponding sides of two similar figures was the same for all pairs of corresponding sides. Also, the multiplicative constant between pairs of sides from within the same triangle was consistent when compared to the corresponding pair of sides on the second triangle. Thus, given two similar triangles in which all the sides of one was given and only one side of the other was given, students are able to use the multiplicative constants to determine the unknown values. However, students still perceived the relationship between the two triangles as a multiple and not as a ratio. If the sides are whole numbers of which one is a whole number multiple of the other, then students used multiplication facts to determine the multiplicative constant. If the sides are whole numbers, but one is not a whole number multiple of the other, then students used a partitioning/norming process to understand one
side as a multiple of its corresponding side. While this process is conceptually grounded, it is not very efficient arithmetically. Thus, the activities in this phase provided students with the necessary experiences to develop the notion of ratio between corresponding sides as the efficient way to think of the multiplicative constant.

**Activity 7:** The purpose of this activity was for the students to realize that division is the most efficient way to determine the multiplicative relationship between two quantities that are not equal, especially, if one of the quantities is not a whole number multiple of the other. In this activity, the largeness of the second corresponding length between the similar figures encouraged students to think of obtaining the multiplicative constant by division.

When students were comfortable with the idea of the multiplicative constant as a ratio, then they were ready to use this knowledge to readily find unknown lengths. The ratio construct can be very advantageous when dealing with corresponding sides of similar figures that are not whole multiples of each other. This provides the basis for the activity.

Students were given situations where there are two similar figures in which one figure has all of its sides labeled in terms of length. The second figure had only one side given. The students determined the multiplicative constant between the figure by the ratio formed between the side given from the second figure and the corresponding side from the first figure.
Figure 3.11 (Battery of Similar Figures That Encourage Division)
(Figures Continued)
Rationale for Activity 8: The students used division as an efficient means for determining the multiplicative relationship between similar figures in Activity 7. However, fully understanding that ratio of corresponding sides is an entity itself, rather than just an expression for dividing to obtaining a quotient, requires that students be given specific examples wherein the actual division process is so cumbersome as to encourage students to refrain from actually dividing and using the ratio of corresponding sides as the multiplicative constant. This set the stage for students to realize that the multiplicative constant in ratio form is viable as an entity itself, and that the ratio is basically given in the problem. Then they will have reached a particular plateau of cognition. That is, they now realize that the multiplicative relationships between similar figures are the specific ratios of corresponding sides between or within the figures.

Activity 8: In this activity, students were given similar figures in which the quantity of the sides did not lend themselves to actual division. Performing a within and between analysis yielded the same ratio, which encouraged the students to utilize the ratio as the multiplicative constant.
Figure 3.12 (Similar Figures With Rational Multiplicative Constants)
Data Collection

The purpose of this study suggested a qualitative analysis. Data for this study were collected from videorecordings, written responses to instructional materials, and journals. Personal interviews were conducted only if there was some aspect of student behavior that needed to be explored. Classroom dialog and the students individual work constitutes the majority of the data.

Videorecordings. Videorecordings were made of all group teaching sessions and interviews. These tapes were analyzed daily and cumulatively. The daily analysis provided feedback relative to any modifications that needed to be made in the lessons. The videorecordings were made of the teacher and entire class.

Written Work. The written work consisted of daily activities, homework, and tests. This provided the researcher with a tangible source of student performance and progress.

Research Journal. The researcher kept two journals. One for a record of observations made during the lessons and activities, while viewing the videotapes, and while analyzing students written work. The other was for a record of modifications for lessons, developing future tasks, and various items of interest.

Interviews: The researcher mostly relied on the dialog within the classroom for analysis. Students interactions with one another, their response to questions and situations, and their emotional response are all considered as valuable data within the natural setting of the classroom. Interviews of some students was conducted at various intervals throughout the experiment as needed to understand student behavior that may warrant further inquiry. The video-recorder was used when interviewing students at a
time separate from when the lessons were given. Personal interviews provided the researcher a method for gaining insight into the individual student's progress.

**Data Analysis**

The data obtained from this study were evaluated from a qualitative perspective. The collection of data and the analysis of those data were guided by Borg & Gall (1989), by Teppo (1998), and by persons associated with the development and evaluation of this study who were considered knowledgeable in the field of qualitative analysis.

The researcher analyzed students’ general performance on the activities by organizing their performance into tables which enabled the researcher to determine the success of students on each activity, and to determine what patterns may exist by the classes as a whole.

Individual student performance was analyzed three ways. First, the students’ work was analyzed individually. Student errors were classified providing valuable information as to what contributed to the their incorrectness, including conceptual inadequacies, slips, etc. How students actually worked problems, whether correct or incorrect, was of particular interest since it revealed how the students understood that situation, in particular, in a between or within method. It also revealed the students’ adeptness at the mathematics of whichever strategy they chose. However, if the student understood the problem conceptually but performed the arithmetic incorrectly, the problem was still counted wrong.

The students’ work was organized into tables of correctness to determine if the classes as a whole were understanding the mathematics of a particular activity. There were no statistical analysis performed since the samples were small. A correct response
designates correct mathematics, even though the student may not have chosen the most efficient method to solve the problem (e.g., a student’s tendency for between relationships when a within relationship would have been more mathematically more efficient).

Finally, students response to activities that required whole group participation were of interest. For example, the students’ struggles with the tasks for developing the within relationships and the amount of time required to complete those activities suggested a degree of difficulty with students verbalizing the relationships within a figure. Also, individual student responses often garnered support or rebuttal from the class which netted a feel for the overall performance of the classes.

The information obtained from the journals did not contribute much to the study, thus there is little analysis provided. The activities were performed in whole class settings and students’ comments within the classroom environment provided more insight into their understandings than individual interviews.
CHAPTER 4
RESULTS AND ANALYSIS

In this chapter I will detail students' activities during each lesson and provide summaries of data which will form the basis for analysis. There were four lessons with two activities each. Some of the activities proved to be more difficult and required more than one class period to complete. There were two different geometry classes that did each lesson. The results of both groups will be discussed simultaneously unless one class's response was different than the other. Basically, Class 1 is comprised of students that have been labeled as "college-bound". Class 2 is comprised of students that are less adept in performing at an abstract level. Thus, they required extra reinforcement and time to complete some of the tasks.

Lesson 1

Activity 1: This lesson consisted of two activities. The first activity involved students identifying and grouping together similar figures from a chart of figures that were randomly placed. There were two students in the class which came to the front of the room and moved the figures about the chart with direction from the class until they were satisfied that groups of similar figures were correctly categorized and separated from non-similar figures on the chart. The students were successful in separating the figures into their respective groups based on their likeness except for the elongated rectangle.

At first, the students from both groups placed all of the rectangles on the chart within a single group that represented similar figures, but when the researcher asked the
students to explain how they determined which figures belonged in which group, they verbalized a rationale that caused them to reconsider placing the elongated triangle in the group with the other two rectangles. That is, the longer side of the elongated rectangle was too long when compared to its other side. This compelled the students to discount this rectangle from the group of similar rectangles. At this early phase, the students mostly used the phrase, “Figures are similar if one is the same as the other but just bigger or smaller”. The following is a reconstructed summary of the conversation.

Teacher: So how did you determine what figures were similar?

Students: The figures that are similar are the same. They are just different sizes.

Teacher: And how can you tell they are the same?

Students: All the angles are the same. Yes, and the sides get bigger equally.

Teacher: Could you illustrate this with a few of the groups?

Students: Sure. These triangles are all within a group because they are equal. That is, their sides are all bigger equally so it is the same as the smaller one, just bigger.

Student to others: I don’t think that one rectangle belongs in that group.

Students: Yes, it does. They are all rectangles.

Student to others: Only one side has gotten bigger. So, it is not the same as the others.

Students: That’s right. This rectangle does not belong in this group.
One student actually used the word "grows" when describing how one side is related to the corresponding side of the first figure (suggesting a between analysis). Another student insisted that the same number of sides be a determining factor for deciding if figures were similar.

Teacher: How do we know if figures are similar?

Daniel: If they have the same number of sides.

Teacher: Well, then why are these triangles not in the same group as the others?

Daniel: Because, they are not the same. Their shapes are different.

Teacher: But, they have the same number of sides.

Class: Yes, but you need more than that. Their sides have to be the same to each other.

Daniel: Well, yes, that true. But, they still must have the same number of sides.

It's just not enough.
Daniel decided that having the same number of sides was necessary but not sufficient to determine similarity.

**Activity 2:** In this activity, the students from each class were divided into two groups. Each group was given a geometric figure printed on a sheet of paper, and they were to determine if the figures from each group were similar based on verbal cues only. The students first began to focus on the more obvious visual features of the figures to determine similarity such as, the number of sides, the number of angles, if the figures had right angles, if their figure was a rectangle, parallelogram, etc.. While all of these features were important, they eventually realized that they needed something more to determine if their figures were similar. They began to focus on how the lengths of their figures compared to each other. I should note that it was not easy for them to reach this point in their analysis. This approach represents a more complex analysis that seemed difficult for the students to verbalize and to understand. In future activities of this sort, the students continued to choose the more visual, simplistic approach to determine similarity. When these strategies failed, they would then use comparison of lengths of sides again. This was a process which required the students to refer back to the chart from activity one and practice for mastery more than I had originally anticipated.

The three groups of figures in Figure 4.2 are representative of the types of figures the students were working with in this activity. The triangles in group I were fairly easy for the students to determine non-similarity because one has a right angle and the other does not. From Activity 1, the students knew that similar figures is one figure representing the exact replica of the other only differing in size; thus, further analysis is not needed.
The triangles in Group E proved to be much more challenging. The students would establish the fact that they were both isosceles triangles, but the more obvious visual features do not determine similarity. The only way the students were able to determine similarity with any concreteness was to compare the base of their figure with one of the sides. The students dialogue would be something like;

Student1: The base of our triangle is $\frac{3}{4}$ the length of one of the sides.
Student2: The base of our triangle is only about $\frac{1}{3}$ the length of our sides.
S1 and S2: Then the figures are not similar.

The students had already experienced rectangles similar to the ones in Group III from Activity I. So, they knew that all rectangles are not similar just because they are of the same geometric shape. They were inclined to analyze this type of problem by comparing sides and their dialogue would be similar to that above:

S1: The shorter side on our rectangle is about $\frac{1}{2}$ the length of the longer side.
S2: The shorter side on our rectangle is also about $\frac{1}{2}$ the length of our longer side.
S1 & S2: The rectangles are similar because even though they differ in size, the longer and shorter sides of both rectangles are of the same ratio.

It was interesting to note that a few of the students were comfortable using the word "ratio" in this context, and the others seemed to be willing to also use it after it had been introduced as a concise and accurate word to describe the comparison of two quantities. Throughout the experiment it was necessary to repeat this activity several times because the students first inclination was to use visual features of the figures (same number of sides, is there any right angles, etc.) to determine similarity, instead of the more accurate strategy of comparing ratios of sides. Even though they had experienced success with ratios of sides previously, upon revisiting this activity, they again would work their way through the basic visual features of the figures that are not always sufficient for determining similarity. The visual features of the figures were prominent whereas comparison of ratios of sides was a more subtle and complex task. Students opted for the easier analysis first.

Lesson 2

There are at least two ways to compare sides of figures to determine similarity. One is to compare the ratio of lengths of sides within a figure to the ratio of corresponding lengths of sides of the second figure. This we have referred to as the within analysis. A second way is to form a ratio comparing a side of one figure to the corresponding side of the second figure. This is done with at least two sets of sides so that the individual can see if the ratios are equal. This entails the students seeing one figure as a growth/shrinkage of the other. We have referred to this process as the between analysis. Lesson 2 encourages the students to see one figure as a growth/shrinkage of another. This
type of activity promotes comparison between figures and lends itself to the between analysis.

**Activity 3:** The students were to tell how long it would take to move the overhead to match a second figure that had previously been constructed on the board. The students were quite successful in guessing the number of seconds needed to move the cart. Sometimes their guesses had to be slightly modified to obtain exactness between the figures, but the students accomplished this quite readily. The students used time to control the amount of growth/shrinkage. This established the relationship between the amount of growth of a figure and the operator that controls the growth. After several runs the students became adept at guessing the amount of time needed to match the figures.

Teacher: How many seconds will I need to move this cart until this figure shrinks to match the second figure I have drawn?


**Activity 4:** This activity progressed through three different levels. The overhead again was moved away and towards the board, but this time the students were given a measured figure (the lengths were given) and they were asked to tell how long before that figure was doubled, tripled, halved, etc.. This was similar to the first activity in this lesson, but now the students were focusing on the actual length, as well as on comparing lengths in a more general fashion. Also, the lengths of the enlarged/shrunken figure were measured to determine the correctness of their guess. Again, the students proved fairly adept after several runs. If their guess was slightly large they were able to modify that guess to match lengths to the predetermined size. The same for shrinking.
Next the students were given a measured figure and asked to tell how long before a figure grew 6 times its original length. However, they did not have the benefit of seeing or measuring the new figure. This activity provided them with a view of growth that is implicit. They did not have the benefit of seeing the enlarged figure. The students performed this task with no apparent difficulty. They were able to envision the enlarged figure mentally and accurately guess the necessary time value relative to the requirements of the previous tasks.

Finally, the students were given two measured figures and were asked to determine what occurred between the figures such that one is the other translated over time. They correctly guessed that one figure was double the other.

The students were asked to respond in their journal to a few questions at this point regarding their understanding of similarity. One of the questions was, “Given two similar figures, how would you determine the growth or shrinkage that occurred between them?” At this stage the students are still operating at the perceptual level recognizing that if one figure is similar to another, it’s because it is the same figure grown/shrunk over time. Their explanations fell into the following categorical responses:

C1. By measuring them, or just by comparing the objects.

C2. I would measure the sides and see if they doubled, tripled, so on.

C3. By how much growth occurred from the time you started enlarging or shrinking the figure until the time you stop.

C4. You would determine it by how long each one of the sides are compared to the size they were before.
The responses in categories one and two suggest the students are operating at a perceptual mode, but their perceptions are justified through measurement. This suggests a gentle shift from perceptual only to quantification guided by concrete actions. The responses in category three and four indicate students are justifying their perceptual orientations by comparison analysis. They do not suggest any particular quantifiable means, but they do understand that by what rate one side increased or decreased, the other sides must increase or decrease also.

Lesson 3

Activity 5: The students were asked to determine if two figures were similar, only this time the figures were measured and the students had to determine similarity based on the relationship between the numbered lengths of their own figures since the students were not allowed to share between groups the actual length of sides. The students found this to be quite difficult. The students enjoyed varying degrees of success. Not knowing what the figure of the other group measured, placed them in a quandary trying to

![Figure 4.3 (Similar Triangles)](image-url)
determine what relationships they could share with each other to determine whether the figures were similar. That we had determined in an earlier activity similar to this one, but without measured sides, that the only way to determine if the figures were similar was by determining how the sides compared to each other multiplicatively, evidently, was short lived. The students from both groups (but especially group 2) became obsessed with an additive solution that was incorrect. Students' use of an additive strategy in place of a multiplicative strategy is not uncommon. Figure 4.3 is an example of how the students used an additive strategy for this activity.

The students would determine these two figures are similar because the base of the smaller (4) increased additively by 2 equals one of the sides (6) and increased additively by 4 yields the other side (8). The second group would claim their does the same thing, 6+2= 8 and 6+4 = 10; thus the figures were similar. After the students made this conjecture, we taped the figures to the board so both parties could examine both figures. Even seeing both figures the students were still comfortable that the figures were similar until we began to think about them using our earlier work as a basis for analysis. The students knew that figures grow over time multiplicatively from the overhead activities. So when prompted to explain how one figure grew to become the other the students would begin by saying “the four increased by two, the 6 increased by 2, and eight increased by 2, so they are similar”.

I then gave them a triangle such as the one in Figure 4.4 and asked them to double the size of the triangle. The students did so proposing that;

\[ 3 \times 2 = 6; \quad X = 6, \quad 4 \times 2 = 8; \quad Y = 8, \quad 5 \times 2 = 10; \quad Z = 10 \]
I chose doubling because it seemed easiest for the students to negotiate. Consider the following constructed summary.

Teacher: How much did each side increase by?

Students: The 3 increased by three, the four increased by four, and the 5 increased by five.

Teacher: So, are these figures similar?

Students: Yes, because one is the same as the other, only doubled.

Teacher: On the previous problem with sides 4, 6, 8, each one increased by the same amount, 2, and you said that they were similar because they all increased by the same amount.

(Silence)

Students: I guess we were wrong about that. You have to multiply, not add.

At this point, they encountered an arithmetical dilemma where using the additive feature suggested the figures were similar, but the amount of growth between the figures measured multiplicatively informed them differently. The students found respite in the...
foundations provided by the earlier activities which guided them towards determining the figures were not similar. However, the additive strategy was a powerful influence and the students had to repeat the activity numerous times over a period of a couple of days to become comfortable with a multiplicative approach to similarity in this particular activity.

Activity 6. In this activity, the students were given figures where one had all of its lengths given and the other only had one known length. The students were to compare the figures and determine the growth constant from either a within or between analysis and find the missing length of the second figure. The students from each class worked independently. There were two types of figures. One encouraged a between analysis and the other a within analysis. For example;

![Figures](image)

**Figure 4.5 (Similar Figures)**

The variable x from the set of rectangles is more easily determined by the within analysis; rectangle (1) 4*2=8, so rectangle (2) 7*2 = 14...... x = 14.
The x and y values for the set of triangles is more easily determined by the between analysis;

triangle (1) \hspace{1cm} triangle (2)
\begin{align*}
5*2 &= 10 \\
12*2 &= 24 \ldots \text{x}=24 \\
13*2 &= 26 \ldots \text{y}=26
\end{align*}

The students were given 9 sets of figures to work. Table 4.1 shows how many students from each class worked all 10 problems correct, only 9 correct, etc.. The extremely poor showing by Class 2 two prompted a discussion following the activity of

Table 4.1 (Results From Activity 6)

<table>
<thead>
<tr>
<th># correct</th>
<th>Class 1</th>
<th>Class 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
</tr>
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<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

how to determine missing values with similar problem types. The majority of students in Class 2 who did poorly did so because they used an additive strategy to determine the missing values. The remaining errors could be attributed to unworked problems or the method the students used was incoherent.
After Class 2’s poor performance, we reviewed a few like situations. When the students proposed an additive solution (which resulted in incorrect lengths), they were given a measured triangle and again asked to double, triple, and halve it. They were then asked how much each side had increased. Each side increased additively by a different amount (quite different from the additive strategy in which each side increases by an equal amount) which placed the students in conflict with their use of the additive strategy. The students quickly came to a consensus that the additive strategy was the reason for their poor showing and were somewhat frustrated with themselves for using the wrong strategy.

Since Class 2 had done so poorly, I wanted to provide them with situations that would instill the notion of multiplicative rather than additive and then allow them to rework the activity. I did not allow them to review their individual results from the activity because I wanted them to be able to try the activity again without having the benefit of insight on how to work a particular problem due to any influence by the researcher or their peers. (I only shared the number they had worked correctly) Table 4.2 details the results. One of the students who only got three problems correct evidently had outside influence in utilizing a proportional strategy with cross-multiplication. However, she did not match corresponding sides which resulted in incorrect values for the missing lengths. The other student tried to use some sort of growth value, but her work indicated that she did not understand how to obtain or use that value.

At this stage, it is also interesting to note the students flexibility (or lack of) in using either the within or between strategy to solve these problems. In particular, some problems are best suited for a particular strategy and it was of interest to determine if they
Table 4.2 (Results of Retest for Class 2)

<table>
<thead>
<tr>
<th># correct</th>
<th>1st result</th>
<th>2nd result</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Group 2</td>
<td>Group 2</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
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<td>0</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

were able to choose which strategy was best suited for the particular task. This would indicate if the student had a more sophisticated development towards solving similarity problems. Table 4.3 illustrates the number of students who utilized a within strategy only, a between strategy only, or were flexible utilizing whichever strategy was appropriate for the particular task. The majority of the students preferred the between

Table 4.3 (Various Strategies Utilized)

<table>
<thead>
<tr>
<th>Class 1</th>
<th>Between Only</th>
<th>Within Only</th>
<th>Flexible</th>
<th>Neither</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Class 2</td>
<td>4</td>
<td>0</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>
method with only one student consistently utilizing the within method. Class 2 exhibited more flexibility between the two methods.

**Lesson 4**

**Activity 7:** This lesson consisted of two activities. In the first, the students were given two similar figures with one having all of its sides measured and the other having only one known length while the other sides were represented with variables. The students were to use a within strategy to determine the numerical relationships among the sides of one figure, which would translate to the second figure, or they were to use a between strategy to determine the rate of growth or shrinkage between the known corresponding sides of the two figures and use that rate as a means for determining the unknown lengths of the second figure. The difference in this activity is the known length of the second figure will be numerically large. This encouraged the students to divide to obtain the multiplicative constant.

For example, in Figure 4.6 consider the sides of the smaller figure, 4 and 2. The student could see that the side of length 2 is doubled to become 4, thus 46 (of the second

![Figure 4.6 (Similar Figures)](image-url)
The student could also consider that the side of length 2 from the smaller figure can be multiplied by 23 to equal 46, the corresponding side of the second figure. Thus, the multiplicative constant is 23 and multiply 4 by that amount to find the value of Y.

The following table will give the results of group 1 and group 2 based on their success at finding the missing lengths.

Table 4.4 (Results From Activity 7)

<table>
<thead>
<tr>
<th># correct</th>
<th>Class 1</th>
<th>Class 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Most of the errors seemed to occur when a set of similar figures were designed such that within analysis would be most convenient; that is, the larger sides of one figure were multiples of the smaller side, but the students ignored this designated relationship and opted for a computationally more difficult strategy. This resulted in them having to use a decimal or fraction as their multiplicative constant for similarity and it ultimately contributed to arithmetic errors. For example, in Figure 4.7, the student tried to find the multiplicative relationship between the two bases of the triangles, 3 and 4. Thus her division yielded 1.3333. She incorrectly converted this to 1 3/10, which she used to multiply to 56. If she would have chosen to obtain the multiplicative relationship between
the vertical side and the base of the first figure, 56 and 4 respectively, she would have obtained a multiplicative constant of 14.

Some of the errors could be contributed to arithmetic mistakes. One student divided 147/3 and got 48 as her multiplicative constant. When she multiplied by this constant to the known sides, the resultant lengths for the second figure were wrong.

Finally, a few errors occurred when the students incorrectly matched corresponding sides. This did not happen throughout any particular paper as a pattern of misunderstanding; rather, it was random and rare.

The students work demonstrated that they did use division as the means for determining the multiplicative relationships between corresponding sides in the between or within strategy. Although the student previously mentioned divided incorrectly, she did reason that division was the best means for determining the growth rate.

Table 4.5 notes what strategies the students utilized for this activity. The data for this activity was arranged as to encourage either the within or between strategies, and as
the table reveals, the influence was powerful enough to greatly increase the use of the within method making students more flexible.

Table 4.5 (Various Strategies Utilized)

<table>
<thead>
<tr>
<th></th>
<th>Between Only</th>
<th>Within Only</th>
<th>Flexible</th>
<th>Neither</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class 1</td>
<td>8</td>
<td>2</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>Class 2</td>
<td>2</td>
<td>0</td>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>

Activity 8. In this activity, the students were given two similar figures in which one figure had its sides represented with numerical values. The numerical values in this activity did not lend itself to actual division. Performing a within and between analysis yielded the same ratio and it was hoped that students would recognize the multiplicative constant as a ratio of corresponding sides. For example; considering the two bases of the

![Similar Triangles](image)

Figure 4.8 (Similar Triangles)
triangles in Figure 4.8, $5 \times \frac{7}{5} = 7$; thus, $7 \times \frac{7}{5} = X$ and $Y$. The same is true for the within relationship. An analysis of the student's work indicates that the vast majority of both groups used $7/5$ as the multiplicative constant, although they represented the multiplicative it as 1.4 or 1 2/5. The few students who chose not to use this strategy had begun representing the relationship between the figures as a proportion and were cross-multiplying to solve; however, the format of their proportions indicated a within or between strategy, and some even determined the multiplier between the figures after they knew the length of all the sides. Student use of proportions was not particularly surprising since they were already familiar with this area and it does give a "right answer". Table 4.6 indicates the students success with this activity.

Table 4.6 (Results From Activity 8)

<table>
<thead>
<tr>
<th># correct</th>
<th>Class 1</th>
<th>Class 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

There was one particular problem from this activity which eight of the students from group 1 and all of the students from group 2 worked incorrectly. This problem, as given in Figure 4.9, requires the students to shrink a triangle if you observe the problems in a left-to-right sequence.
However, most of the students were more comfortable with trying to enlarge the smaller triangle by dividing $9/7$ rather than $7/9$ (which is what they would have needed to shrink the larger). But then the result $1 2/7$ or the decimal approximation $1.2855$ requires the student to enlarge the lengths of the smaller triangle and those lengths are unknown. Thus, there was a myriad of errors. Many of the students that worked this problem correctly utilized the proportion with cross-multiplication method. Again, this strategy was not a product of this class, but once the students realized that this method produced a correct length it became a tool for them to find an unknown length without having to figure out within or between relationships.

As Table 4.7 notes, Class 2 did not show flexibility with their strategies in Activity 7, rather they reverted back to the between strategy. Class 2 slightly increased their flexibility and use of the within strategy, and only one student from Class 2 student was not able to utilize any of the methods to solve the problems.
Table 4.7 (Various Strategies Utilized)

<table>
<thead>
<tr>
<th></th>
<th>Between Only</th>
<th>Within Only</th>
<th>Flexible</th>
<th>Neither</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class 1</td>
<td>6</td>
<td>3</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>Class 2</td>
<td>7</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
CHAPTER 5
SUMMARY AND CONCLUSIONS

This chapter presents the conclusions ensuing from this study beginning with a general overview of its purposes and designs. The results from chapter four are then discussed according to the research questions. Finally, the pedagogical implications, and the implications for future research and limitations are addressed.

Summary

Students' approach school mathematics with informal conceptual constructions, which informal, are gained from their experiences in the world. They utilize these informal intuitions to make sense of the world and to give meaning to their experiences (Van Manen, 1990). Their constructions are meaningful and sensible because they mediate their lived experiences. Constructivism suggests learning occurs when educators provide students with the opportunity to build upon their informal knowledge, thereby enabling them to make new constructions that are more powerful (Confrey, 1990).

Some of the characteristics of powerful constructions according to Confrey are internal consistency, integration across a variety of concepts, a guide for future actions, agreement with experts, an ability to be justified and defended, and a convergence among multiple forms and contexts of representation. It was the intent of this study to provide activities for students that would enable them to build more powerful constructions about similarity, and to accomplish this we began with the students' informal perceptual orientations.

Students encounter similarity of geometric figures long before they study this topic in a formal school setting. A small candy bar and a larger version at a cheaper price
per ounce, a baby...adolescent...adult, a large window in a house and a smaller version in a bathroom, a baseball...a soccer ball...a basketball, etc. are all real-world instances of similar figures encountered by children everyday. Their notion of similarity is based in perceptions and concrete representations. Children and adults operate in a world where they encounter many instances of interpreting one figure in terms of the other, and they accomplish this quite readily without formal mathematics.

The informal notion of similarity is “two figures that look alike but are different sizes”. To build on this perceptual notion of similarity is to understand how the figures are related, both perceptually and mathematically. To be able to determine what relationships constitute similarity requires the student to think of the whole figure in terms of its individual components and determine how they are related to each other and to the corresponding parts of a second figure. This particular form of analysis required the student to deconstruct the whole or to think of the figure in terms of the relationships between its individual components. This is closely aligned with Idhe’s (1986) description of the search for meaning by a “deconstructive process”.

Although Idhe’s discussion represents a narrow view of phenomenological inquiry, it does help to orient this study. Constructivism offers insights into how students develop more powerful conceptions, but it does not help to characterize the nature of students’ initial intuitions.

In Phase I, the analysis of students’ perceptions reveals two distinct visual relationships that are inherent to similar figures. One is the manner in which the sides of a figure are related to the corresponding sides of the second figure through stretching and shrinking. This dynamic transformation of the figure we refer to as the between
relationship, and if students utilized to this relationship to solve problems we refer to it as between analysis. The other was the static relationship among the sides within each similar figures which we refer to as within relationships, and if students utilized this strategy to solve problems we refer to it as within analysis. Second, they needed to quantify these relationships (Phase II), both within and between. The data would determine which analysis was most accessible, comfortable, and efficient for students. The quantification process will produce multiplicative constants that are representative of the rate of growth. Once students understood how to use division to obtain these constants, they were able to utilize them to find unknown lengths.

This whole range of methods and concerns is not indicative of the current mathematics curriculum. Current mathematics curriculum is more concerned with “right answers” and procedural correctness. In the case of similarity, students are presented with similar figures, told to match corresponding sides, to place two pairs (of which one length is unknown) in a proportion, and then to cross-multiply to find the unknown length. This routine procedure is not guided by students’ perceptual notion of similarity; rather, it is designed just to give students a convenient manner in which to obtain an unknown. The mathematical relationships inherent in similar figures are not even an issue. The emphasis is on “Is it right?” Unfortunately, this contributes to the alienation of school mathematics from the students’ informal understanding. To most students, mathematics evolves into a meaningless rule-based system where you plug-in numbers, perform certain memorized procedures, and obtain answers deemed “right” by the expert (Confrey, 1990).
Figure 5.1 (Similar Triangles)

For example, in the triangles above let’s consider the between relationship first. It is obvious the triangle on the right is a bigger version of the one on the left. So how much bigger is it? Matching corresponding sides 6 → 9, 8 → X, and 12 → Y, we could say that 6 grew 1.5 times to become 9. This enables calculations of X and Y.

The within analysis reveals the longer side of the smaller triangle to be double in length to the base. Since the larger figures similar retains these internal relations enables calculation of the value of Y as 9 * 2. Similarly, X can be calculated by noting the ratios of 6 to 8, or 8 to 12.

The more traditional approach would be as follows;

\[
\begin{align*}
\frac{6}{9} &= \frac{8}{X} & \frac{6}{9} &= \frac{12}{Y} \\
6X &= 72 & 6Y &= 108 \\
X &= 9 & Y &= 18
\end{align*}
\]
This method yields the correct answers $X = 12$ and $Y = 18$, but no where in the calculations is there any relationship to linking students’ visual perceptions of similarity and quantitative concepts. It just produces the right answers mechanically.

It was from this basis that I developed a curricular unit based on these ideas and implemented them with two geometry classes as described in chapters 3 and 4.

**Conclusions**

**Research Question 1:** Can a perceptual analysis of similarity provide the basis for students’ conceptual approach to standard problems involving similar figures?

Similarity of geometric figures is initially based in perceptual orientations. The nature of those perceptions are implicit and form an intuitive sense of space and figures for children. Constructivism presents a popular theory of learning but it does explain the nature of students’ initial intuitions. Constructivism suggests a curriculum that enables students to build more powerful constructions by using the students’ knowledge as a starting point for instruction (Confrey, 1990). To be able to do this, we must understand the nature of the students’ informal knowledge, and in this case, I utilized an inquiry informed by phenomenological methods. The phenomenological question of “what makes a something what it is—and without which it could not be what it is” (van Manen, 1990) was instrumental to determining what is the nature of children’s intuitive understanding of similar figures.

The data analysis revealed much about the process students undergo to understand similarity. The students were able to separate similar figures from the chart in the first activity with considerable ease. The students did this without any instruction, confirming they already possessed the ability to determine similarity perceptually. Two students
stood at the chart from each class and moved the figures, with the aid of their classmates, into groups that represented similar figures. The students were successful except in the case of the elongated rectangle. To understand how the students had determined which figures belonged in which groups, I asked them to explain how they knew which figures were similar.

At first, the students appeared to be operating at a visual level matching figures with one serving as a template and then seeing if the other figures could represent larger or smaller version of the original, but the discussion of why some figures were in a group and others were not led to two scenarios. One was if two figures were similar the students noted that the amount of growth was equal among all the sides. Most of the students seemed comfortable with growth between the figures. A second scenario was described when a student said he saw it as replication of relationships (within). In any case, the students at this grade level were able to utilize feature analysis to determine some of the specifics of similarity. While the students initially operated at the visual level, their discussion led them to a more sophisticated level which could be described as descriptive and analytical (Van Heile, 1986). Once they were operating at this level, they decided from the elongated rectangle did not belong in the group with the others because “one side is too long.”

A possible cause for the students to place the elongated rectangle with the others lies in their interpretation of “same shape”. The elongated rectangle was not similar to the others, but there is a common name, length vs width, and equal angles shared between the figures. The students' notion of same shape and different size was consistent in determining similarity, but the notion of “same” for the rectangles may have been
dominated by the verbal-based interpretation of “same shape”, since they were all rectangles.

The students experienced great difficulty with Activity 6. To determine if figures were similar without being able to see one of the figures and relying on verbal cues only was extremely difficult for the students. This activity required the students to utilize the within strategy to determine if the figures were similar. This activity deprived them of making comparisons at the visual level or between the figures. They had to rely solely on internal relationships. They tried to determine similarity by geometric features such as: do they have the same number of sides, the same number of angles, the same angle size (which would have worked, but they had no way to measure), all to no avail. These are necessary, but not sufficient in determining similarity. It was only after repeated trials over a number of days before they were able to utilize relationships between sides. Even then, for many students, it was shaky at best. The students were much more successful at determining if figures were similar if they were able to view both figures.

The difficulty the students exhibited in determining if figures were similar when they were divided into two groups and not permitted to see the other group’s figure, suggest that the students understanding of the perceptual basis of similarity resided in the “wholes,” or in the between strategy. That is, they were able to determine similarity based on the likeness of each figure without necessarily being aware of the individual relationships inherent in similar figures, or they relied on comparing sides of one figure to the other figure. It should be noted, however, that the task of verbalizing relationships was more difficult.
The ease with which the students performed the activities of growth and shrinkage (activities involving the overhead and that utilized a between strategy) suggest the students were much more comfortable with the between relationship. Table 4.3 shows how many students from each class used either a between analysis only, a within analysis only, or was flexible utilizing whichever strategy was more convenient for the given data. Analysis of the data from both classes indicates that at the end of the third lesson 60% of the students were using a between analysis only. Only one student from Class 1 utilized a within analysis only while the rest were flexible between the two strategies depending on the data given. Two students were not able to use either strategy successfully. It should be noted, however, that the task of having to rely on oral cues alone is a much more difficult task than the tasks dealing with the overhead. A portion of the students visual or intuitive notion of similar figures was not available to them when they could not see the other group’s figure. They had to rely on the relationships within the figure and on verbal cues. This could partly explain the trend toward the between strategy with the students.

The between relationship can be characterized as dynamic. That is, one figure seen as a growth or shrinkage of the other is an action. The within relationship is static. It refers to a single state of affairs (Nesher, 1980), in this case, interpreting one side of the figure in terms of the another, not as growth, but letting one side become a unit and deciding how many of those units are in the other side. Nesher claims for students to operate with static situations in word problems equates to operating at the pragmatic level, "the level at which the student performs manipulations in the rather abstract domain of events and configurations." Nesher also claims that studies have consistently showed that the static text is more difficult than the dynamic. This difference between the
two states could account for the students' propensity to refer to similar figures in the between relationship.

In any case, Activities 6, 7, 8 were the typical similarity problems found in most Geometry textbooks and the results from these activities, as outlined in chapter four, indicate that the students solved these problems successfully, utilizing the mathematics of this curriculum. It was an approach to solving similarity problems for the students that was based in their initial perceptual orientations.

**Research Question 2:** How does students' understanding of similarity evolve through the curricular development of between and within relationships within similar figure?

The students' utilized the notion of the within and between relationship as a viable method for expressing their perceptual orientations of similar figures. Initially, the students did this without numbers. Then, they utilized time as control for growth in Activities 2 and 3. Although time was not an accurate descriptor for growth, it did set the stage for a constant that would account for being able to determine the amount of growth. For Class 1 in Activity 6, there were 8 out of 12 students that got 70% or more correct. The additive strategy (Karplus, 1974) accounted for the majority of errors for the remaining students. Class 2 only had 2 out of 8 students that got 70% or more correct. The additive strategy accounted for the vast majority of their errors. After the students discussed this type of error, they retested and 5 out of the 8 got 70% or more correct.

The results indicate that overall the students were successful at expressing the relationships between or within figures as a multiplicative constant. Also, they were successful at utilizing the constant as an operator to generate unknown lengths. Besides
the additive strategy generating error, some students had difficulty with multiplicative
costants that were rational numbers. If the multiplicative constant was a whole number
or a rational number such as ½, 1/3, or 1.5, the students were less prone to commit
arithmetical errors. If the constant was a fractional amount the students were less familiar
with, errors sometimes resulted that reflected the difficulty students’ had in operating
with those type of rational numbers. This type of error was not indicative of the students’
inability to understand the geometric properties of similar figures, rather, their inability to
operate in the rational number domain. This type of error fell under the neither category
of the tables in chapter 4.

The fact that 12 out of the 20 students utilized the between strategy only, even
when a within strategy would have been computationally more convenient, suggests a
tendency for this relationship to be easier. Often, this one-dimensional approach
accounted for student errors. If the data for a set of figures were designed such that a
within analysis was mathematically expedient, and the student insisted on a between
analysis, the mathematics was cumbersome increasing the likelihood of errors.

The students greater success with Activity 6 indicates a “catching on” to the
mathematics of the within and between strategies. The fact that the analysis called for
division given disparately large differences between numbers guided the students from an
implicit notion of a multiplier to utilizing division as the efficient means of determining
the multiplicative relationship. Most of the students exhibited flexibility with the activity
suggesting if the placement of the larger value encouraged a between strategy or if its
placement encouraged a within, the students were influenced by the arrangement. Some
still utilized only one method, regardless of placement of values, but they were in the minority.

The data were given in Activity 8 such that it was not advantageous for either method. The results indicate that Class 1 was very successful at this final activity and half of them retained their flexibility, choosing whichever method seemed more efficient for working a particular problem. Six students utilized the between strategy and three used the within strategy. The students from Class 2 were successful, but because there was no advantage to either method, they all fell back to the between strategy evidently being more comfortable with this method.

**Research Question 3**: What differences are there between more and less mathematically talented students in their development of conceptual understandings of similarity?

According to Thompson (1990), a ratio is a multiplicative comparison between two quantities. It asks “how many times bigger” is one quantity than another, or to conceive “how many of these is in that.” This is consistent with the unitizing and norming process. His interpretation is also consistent with the literature except it does not specify comparison of like units (which some researchers utilize to denote the difference between ratio and rate), instead focusing on the mental representation of multiplication. For example the ratio of 3:2 can be expressed as in two ways. Viewing both quantities as wholes, 1 (3 unit) to 1 (2 unit), or of one quantity measured in units of the other.....1 ½ (2 units) to a 1 (2 unit). This is a comparison in their independent, static states. When the units are the same, 1 ½ becomes the multiplier.
A rate is a quantity that may be analyzed into a multiplicative comparison between two other quantities — where one quantity's value is conceived as varying in constant ratio with variations in the value of the other. A rate can be reflected abstractly, independent of the situation. For example, in the previous comparison of 3 to 2, if other quantities are to be interpreted in the same ratio, then the ratio of 3:2 becomes an independent object, $\frac{3}{2}$, that determines other values. That is, $\frac{1}{\frac{2}{3}}$ is conceived by the individual as operating independently of the situation.

The within relationship between two similar figures matches the static representation of ratio, whereas the between relationship is a dynamic representation that matches a rate. Nesher (1980) notes that students working with static representations experienced much more difficulty than with dynamic representations. It required the students to operate at the pragmatic level, or to perform manipulations in the rather abstract domain of events and configurations. This would explain the difficulty both classes experienced with the within relationship. Class 2, the non-college bound students, especially found the within relationship difficult and only utilized this strategy in

![Figure 5.2 (Similar Triangles With Variable Lengths)](image_url)

Figure 5.2 (Similar Triangles With Variable Lengths)
Activity 7 when the data encouraged so. But, they resorted back to the between strategy in Activity 8 when there was no advantage.

In Figure 5.2, given the two triangles are similar, let’s consider the within relationship first. To interpret a to b is to say “How many a’s are in b?” or vice versa. You must interpret b in terms of k (a units). Then, that same multiplicative relationship must be true for A to B.....B=k(A units). The same is true for a to c and b to c. A significant point is that k is different for each pair of sides. It is not independent from the quantities with which it operates.

Let’s consider the between relationship. The between relationship implies an amount of growth or shrinkage, so if we consider the triangle with sides a, b, c, to grow to the triangle with sides A, B, C, then we know implicitly there is an amount of growth operating equally on a, b, and c to grow them to their larger representations. Once the ..., either implicitly or through division, we can now represent the action as; a * r = A, b * r = B, and c * r = C. In other words, the lengths of the first triangle times r equals the lengths of the second triangle. The amount of growth becomes explicit and the student realizes that he can vary the amount of growth be changing r. The student is now thinking of growth as an independent object, and this is what Thompson suggests is a rate.

Thus, the within relationship is actually a ratio and the between relationship is a rate. The results indicate the students found the concept of rate easier, which would agree with Nesher that dynamic representations are easier for students to understand. The less mathematically capable students were especially fond of the rate of growth and less respondent to the ratio of quantities. To understand the within, the students must be able
to reinterpret b in terms of a in their static states. To operate at this pragmatic level requires a certain mathematical maturity from the student.

At the beginning of this study, I thought of the within and between relationships as equal and both of them as representations of growth. However, the results of this study indicated otherwise. With this in mind, the analysis of data suggests a developmental path to conceptualize similarity concepts beginning with the conceptually easier between relationship and progressing to the within relationship. Mathematically challenged students should be well grounded in the rate aspect of similarity and able to operate at the pragmatic level before they are expected to successfully operate with the within relationship.

**Pedagogical Implications**

Similarity between geometric figures is a situation where students' possess a rich intuitive sense of “likeness”. Their intuitions are grounded in perceptual orientations gained from their life's experiences with similarity. This provides the backdrop for developing curricular models that will enable the students to extend their understanding of similarity. According to the constructivist position, modern curriculum should take into consideration the wealth of knowledge that the students bring to class, realizing that it provides a starting point for developing lessons that will enable the student to build more powerful constructions. Regarding similarity of figures, curricular models that build more formal notions of similarity from the basis of the students' perceptual orientations establishes a meaningful context for the student.

As has been shown in this study, the within and between analysis are explanatory of the students' perceptual orientations, offering salient methods for developing the
mathematics of similarity. Each one represents a different perceptual orientation of how one figure is related to the other, and provides the basis for a meaningful mathematical investigation of similar figures. Thus, a curriculum that utilizes these methods would be offering the student a viable mathematical experience, contrasted to the current approach of matching corresponding sides and cross-multiplying to obtain a single right answer. The problems with the current approach have also been outlined in other parts of the study.

Proportions created from the between and within analysis are grounded in the mathematics of similarity, as opposed to proportions that students create from the type of language such as “this is to that as this is to that”. This type of analogical thinking is not expressive of the mathematics of similarity, rather, it is designed to create an expression that, with a few tricks, can produce answers that are in agreement with the experts. Therefore, the between and within strategies provide a basis from which proportions can be built that are indicative of true mathematical relationships.

Also, for students to fully understand rate of change, they must progress from the static representation of the within relationship to realizing the multiplicative relationship in one figure can be applied to the other figure. Then, the multiplier will be conceived as a rate instead of a ratio. This has implications for other branches of mathematics as well. Slope in its symbolic form, $\frac{\Delta y}{\Delta x}$, is static, but a more powerful notion of slope is to conceive of it as a rate and then it exhibits functionality.

Finally, the teacher should understand that similarity of geometric figures is a situation that is descriptive of a particular mathematics grounded in perceptual orientations. Meaningless, procedural activities designed to produce right answers cannot
represent the relationships inherent in similarity. Thus, if the desirable outcome is for the students to develop a conceptual understanding of similarity before developing skills with procedural activities, they must be given the opportunity to explore the mathematics of similarity in its natural state.

Implications for Future Research

The students' difficulties with the within relationship could stem from their inability to operate at the pragmatic level. Nesher (1980) notes that static word problems are more difficult for students than word problems that are dynamic, because it requires the student to operate at an abstract level. The difficulty the students encountered with the within analysis could be because it is a static representation. The students saw some improvement throughout the activities in utilizing the within strategy, which suggests that the students did begin to integrate the within relationship into their conceptual understanding of similar figures (even though Class 2 reverted to the between strategy in Activity 8). Therefore, further research is required to determine if this developmental path is a valid trajectory for the development of similarity concepts for children.

Limitations

There are two aspects of this study that limit the generalizability of the results. Each has a particular limiting effect to the validity of the study and warrants discussion.

Effectiveness of Instruction: The lack of success with the triangles in Figure 4.9 suggests an inability by the students to adapt the processes from the prior activities to this level of problem. The students were not successful with the between strategy for this problem because the positioning of the unknown lengths required them to determine the multiplicative relationship from larger to smaller. Evidently they were more comfortable
attempting to interpret a larger quantity in terms of the smaller. However, the multiplicative constant they obtained was useless because it represented growth from smaller to larger and the lengths of the smaller triangle were unknowns. Also, none of the students were successful with the within method. Perhaps the hierarchal order of the activities of this study may not have encouraged reification to the extent that enabled the students to adjust to varying levels of difficulty. Because reification involves interconnection between objects and processes (Sfard & Lichevski, 1994), instruction should help mediate such connections. Designing activities that are recursive would enable students to negotiate the demands of each new activity from their current knowledge base and build more powerful constructions. Their understanding of similarity would represent a reification of the processes of each activity into a notion of similarity that would be versatile and adaptable (Sfard and Lincheski, 1994), as in the case of Figure 4.9.

Relative Difficulty of Within and Between Tasks: Theory suggests that the reason for the discrepancies in the students’ performance in the within and between method resides in the differences of static versus dynamic representations (Nesher, 1980). It is suggested that static representations are more difficult for students than dynamic representations. But, perhaps some of the difficulty the students experienced with the within feature in this study was not due to its static nature, but to the difficulty of the “within figures” activities through which this particular feature of similar figures was presented. This type of activity required the students to make comparisons between geometric figures based on verbal cues only. Also, the activity encourages the students to communicate exactly what it is suggesting (the within relationship), assuming they will
be able to do so. These are difficult skills for any secondary school student – particularly in comparison to the relatively simplicity of the between activities. Thus, there is a question as to whether the difficulties of the within method are intrinsic or the result of the instructional activities used.

Was the within activity more difficult because of the idiosyncratic development of these lessons? Or does the greater complexity of the within tasks follow inherent aspects of these forms of representation? Such intriguing questions as these remain for future research.
REFERENCES


131


VITA

Danny Ray McNabb was born on December 23, 1953 in Kinder, Louisiana, the son of Richard McNabb and Marcella McKay. After graduating salutatorian from Fenton High School in 1972, he enrolled at Southeastern University with a partial basketball scholarship and the W.T. Burton Scholarship. In 1973, Danny transferred to McNeese State University where he graduated with a bachelor of science degree in mathematics education and a minor in physical education in 1976. During this time, Danny married Michele Jean Miller and they had a son, Brandon Ray McNabb.

Danny's first teaching position was at Lake Arthur High School in Lake Arthur, Louisiana in 1976, where he was hired as a mathematics teacher and coach. In 1982, Danny earned his master of education degree in administration and supervision from McNeese State University. During his tenure at Lake Arthur High School, Danny and Jean welcomed another son, Marcus Earl McNabb, to their family. Danny then accepted the position of Math Chair at Hathaway High School in 1984. He also was accepted into the doctoral program at Louisiana State University and began his pursuit of a doctoral degree in mathematics education. In 1997, after twenty years of teaching mathematics in Louisiana, Danny retired and assumed the duties of Math Chair at Ropes High School in Ropesville, Texas, where he currently works.

The McNabb's now reside in Lubbock, Texas, where they have welcomed to their family a daughter-in-law, Schane Walker, the wife of Brandon, a grandson, Treyton Ray McNabb, and another son, Colton Russ McNabb.

Danny has been very active throughout his career serving as President (2 years) and Vice-President (3 years) for the Jeff Davis Association of Teachers of Mathematics.
He presented numerous workshops at regional mathematics conferences, was selected as High School Teacher of the Year in 1993, presented workshops for professional development of local teachers, was certified as a mentor teacher, worked with the Mathematics and Engineering Department at McNeese State University on the behalf of his students, and pursued his own academic interest which will culminate with this dissertation and graduation ceremonies in May 2000 with the degree of Doctor of Philosophy in mathematics education.
DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Danny Ray McNabb

Major Field: Curriculum and Instruction

Title of Dissertation: Developing Students’ Understanding of Similar Figures: A Perceptual Approach

Approved:

Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

Date of Examination:
December 16, 1999