Improving Searches for Gravitational Wave Bursts.

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IMPROVING SEARCHES FOR GRAVITATIONAL WAVE BURSTS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Physics and Astronomy

by
Carroll Andrew Morse
B.S., Case Western Reserve University, 1991
M.S., Louisiana State University, 1996
May 1999
Acknowledgments

I was continuously awed by the fact that William Hamilton had converted a dingy basement at LSU into a world-renowned scientific observatory. Bill allows his students the freedom to roam and is willing to spend part of his valuable time to help them move in the right direction. Warren Johnson has the largest breadth of theoretical and practical physics knowledge of anyone I have ever met. His attention to detail forced me to become very meticulous in treating any problem.

I need to thank my friends and fellow graduate students at LSU. Erik and Tammy Young made it possible for me to have almost normal human relationships while a graduate student, Evan Mauceli actually survived being my roommate for over a year (and thoughtfully listened to every complaint about every aspect of life I ever made), and Zach Allen kept me from becoming too complacent near the end of my graduate student tenure. Ken Bernstein, Russ Clark, Donovan Hall, Eric Barnes, Rob Nichols, and Andy Kolchin all were good friends to me over the years.

In my attempt to become Mr. SQUID, I met the real Mr. SQUID, Insik Jin. He is one of the smartest, technically competent, selfless, and considerate people I have ever met. It’s rare to find that combination in anyone.
I need to thank the other members of the gravity lab during my tenure at LSU, Martin McHugh (Bayes Forever!!), Norbert Solomonson, Kenny Geng, Steve Merkowitz, Brad Price, and Lalitha Kommajosyula. I thank Professor Ravi Rau for providing a forum for people to express an occasional unorthodox idea, and Professor Samuel Finn for introducing me to Bayesian analysis (chapters 4 and 5 of this dissertation were born in a 10 minute discussion with Dr. Finn). Also from LSU, I am eternally grateful Karla Lockwood and Arnell Jackson for helping me navigate the LSU graduate school bureaucracy.

I need to thank all my friends in the Baton Rouge ballroom dancing community, especially Maurice and Carol Doyle, Cecil and Neila Phillips, Larry Pentecost and Debby Cooper, Yuri Weydling, Betty Tamas for running the Sunday night dance like clockwork, and Letizia Carroll, who I hope I did not scare away from taking on Ph.D. students as dance students.

I need to thank Phyllis Taylor for a great many things which I cannot describe fully in this acknowledgment.

Finally, I need to thank my father George Francis Morse. Where I go now, I am glad I can take myself with me. Finally, I must thank my family who has supported me all through this Ph.D. odyssey, especially my mother Catherine Morse, and my brothers and sisters John DiBenedetto Morse, Suzanne Nicole Morse, and George Anthony Morse.

This research was supported by the National Science Foundation under Grant No. PHY-9311731.
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Abstract

The first half of this dissertation discusses the details of calibrating a resonant mass gravitational wave antenna and determining its sensitivity. We dispense with the assumption of a perfectly tuned antenna and transducer and model the system using a coordinate rotation. We demonstrate that all of the important model parameters can be directly measured. We demonstrate that the signal response of the two detector modes should be equal despite any mistuning and that the mistuning parameter can be measured in two separate ways. These properties are useful for determining the degree to which a real detector's behavior parallels that of an ideal two-mode system. We compare the predictions of the model to the output of the ALLEGRO system and determine that a resonance in the hardware used to apply calibration signals is the source of an observed 15% difference in signal response. We extend the model to include this additional resonance.

The second half of this dissertation discusses the problem of comparing lists of candidate events acquired from different gravitational wave detectors in search of statistically meaningful coincidences. We demonstrate that a Bayesian approach is the most robust method of inferring if signals are present in the data. We
use a combination of multinomial and Poisson distributions to form a likelihood function describing the results of any coincidence experiment. We establish a meaningful basis for choosing a prior probability for Bayesian analyses. We show that the results of a Bayesian analysis do not depend arbitrarily on how the data is subdivided. Finally, using the results of the 1991 and 1994 ALLEGRO and EXPLORER runs, we establish an upper limit on the mean rate of detectable gravity wave bursts that is no more than 9 events per year above a dimensionless strain threshold of $2.3 \times 10^{-18}$. 
Chapter 1

Introduction

1.1 General Relativity and Gravity Waves

One of the most interesting predictions of Einstein's general theory of relativity is the existence of gravity waves. In a Newtonian universe, the universe of our everyday experience, gravity acts instantaneously, creating an attractive force directed towards the source of the gravitational field. In general relativity, the effects of gravity propagate with a finite speed. Similar to electromagnetic radiation, the acceleration of mass (which can be thought of as "gravitational charge") produces oscillating forces which propagate away from the source at the speed of light. Today, 80 years after Einstein developed his theory of general relativity, scientists are still attempting to directly observe the gravity waves predicted by relativity.

At the heart of the theory of general relativity lies the theory of special relativity. Special relativity is based on two postulates; the laws of physics are the same in every inertial reference frame and the speed of light is absolute. General relativity
extends these ideas to accelerated, or non-inertial, reference frames. Why a theory based on these postulates implies the existence of gravity waves is far from obvious. The next few paragraphs are a brief summary of the steps that lead from the postulates of relativity to existence of gravity waves.

1.1.1 From Curved Spacetime to Gravity Waves

Given the postulates of special relativity, neither space nor time is absolute. The measured length of an object depends upon the relative velocity between the object and the observer performing the measurement. The rate at which a clock runs depends upon the relative velocity between the observer and the clock being observed. Relativity does not say, however, that there are no absolutes. Quantities independent of relative velocity are defined by combining time and space into a single coordinate system called spacetime. For any inertial observer, for example, the magnitude of the interval between the origin and any spacetime event,

\[ ds^2 = ct^2 - x^2 - y^2 - z^2, \]  

(1.1)

has the same value. \( t \) is the time coordinate of the event. \( x, y, \) and \( z \) are the three spatial coordinates of the event. \( c \) is the speed of light.

General relativity describes the force of gravitation as a manifestation of the fact that spacetime is "curved", not "flat". Flat spacetimes are coordinate systems where invariant intervals are can be written in the form of Equation (1.1). The idea of flatness refers to the fact that the geometry of the spatial part of Equation (1.1) is Euclidean. In this picture, the gravitational force is a manifestation of the fact
that spacetime is not flat over arbitrarily large distances. Under the influence of gravity, particles still move in straight lines, but they are straight lines in a curved coordinate system.

As a simple example of curvature appearing to be a force, consider motion along the longitude lines of a globe. Two particles are placed side-by-side on the equator. Both particles begin moving directly north. At the start of their journey, they appear to be traveling parallel to one another. As they move further and further northward, however, they appear to be drawn towards one another. Eventually, at the north pole, their paths cross even though each particle traveled along a straight line in its coordinate system. The “force” that appeared to pull them together is a property of the coordinate system in which they exist.

The amount of curvature in a region of spacetime is related to the local density of matter and energy by Einstein's field equations. In tensor form, Einstein’s field equations are

$$G_{ik} = \frac{8\pi G}{c^4} T_{ik}. \quad (1.2)$$

$G_{ik}$, the Einstein tensor, is a measure of the amount of curvature in a region of spacetime. $G$ is the universal gravitational constant. $c$ is the speed of light. $T_{ik}$ is the energy-momentum tensor which describes the local distribution of matter and energy. Einstein constructed Equation (1.2) by assuming that there is no preferred coordinate system, and that energy and momentum are conserved. A few years after Einstein, Hilbert derived Equation (1.2) from the principle of least action [1].

$G_{ik}$ is constructed from an important second-rank tensor called the metric. The metric is needed because the interval of Equation (1.1) is not the general form of
the invariant interval for an arbitrarily accelerated observer. When moving from
the reference frame of an inertial observer to the reference frame of an accelerated
observer, the form of the interval that accommodates any reference frame is

\[ ds^2 = \sum_i \sum_k g_{ik} x_i x_k \] (1.3)

\( x \) is a vector with four components which describes a space-time coordinate. The
usual convention in general relativity is that \( x = (t, x, y, z) \). \( g_{ik} \) is the metric
tensor which describes the properties of the local spacetime. The flat spacetime of
Equation (1.1), for example, is described by a metric \( \eta_{ik} \), where

\[
\eta_{ik} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{bmatrix}.
\] (1.4)

\( \eta_{ik} \) is called the Minkowski metric.

Away from their source, gravity waves are very weak, significantly weaker than
the Earth's static gravitational field. Flat spacetime is an excellent approximation
to the conditions on Earth. Gravity waves which propagate first through empty
space and then through the Earth's gravitational field are approximated as a small
perturbation on the Minkowski metric.

\[ g_{ik} = \eta_{ik} + h_{ik} \] (1.5)
In empty space, all of the components of the energy momentum tensor are equal to zero. Written in the gravitational equivalent of the Lorentz gauge [2], the Einstein equations take the form

\[
(\nabla^2 - \frac{1}{c^2} \frac{d^2}{dt^2}) h_{ik} = 0. \tag{1.6}
\]

Equation (1.6) describes wave propagation. Gravity waves result directly from the Einstein field equations.

Far away from their source, the solutions of Equation (1.6) look like plane waves propagating at speed \(c\). If the wave is described in a frame where the wave propagates along the \(z\)-axis, the tensor \(h_{ik}\) is transverse and traceless. This implies that \(h_{ik}\) has only two independent components, corresponding to two possible polarization states.

\[
h(t) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & h_+(t) & h_\times(t) & 0 \\
0 & h_\times(t) & -h_+(t) & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \tag{1.7}
\]

With electromagnetic radiation, the leading term is dipole. In the case of gravitational radiation, there is no analog to electric dipole radiation because of conservation of momentum. Similarly, the magnetic dipole analog of gravitational radiation vanishes everywhere due to the conservation of angular momentum. The leading term of gravitational radiation is, therefore, quadrupolar [3].
The strain induced by a gravity wave is proportional to the second time-derivative of a mass-distribution's quadrupole moment. The constant of proportionality resulting from the Einstein field equations is $G/c^4 = 8 \times 10^{-45} \text{ s}^2 \text{ kg}^{-1} \text{ m}^{-1}$.

The magnitude of the induced strain falls as $1/r$ as you move away from the source. A rotating laboratory object produces gravity wave strains on the order of

$$h \sim \frac{G \bar{I}}{c^4 r}$$

(1.8)

$I$ is the quadrupole moment of the system. Using an apparatus that could (optimistically) be constructed on Earth, a long bar ($L = 10$ m) that weighed $M = 10,000$ kg could be rotated 10 revolutions per second ($\omega = 60 \text{ rad s}^{-1}$). Since
\( \ddot{I} \sim ML^2\omega^2 \), the strain produced is on the order of \( h \sim 10^{-35} \). This strain is too small to be detected. At the moment, the amount of mass required to produce a detectable gravity exceeds what can be manufactured by even the most clever experimentalist.

General relativity, however, in combination with quantum mechanics, predicts a universe rich with exotic objects that are capable of producing gravity waves. Super-dense neutron stars pack enough mass into one place to produce decent sized gravity waves. More exotic still, black holes should produce even larger gravity waves. The theorized motions of a significant population of astrophysically plausible objects allow the production of both burst and continuous sources of gravitational radiation with amplitudes large enough to be observed on Earth. Evidence for these objects has been observed. Today, it is obvious that gravity waves are not merely a theoretical curiosity; they are something that the universe is expected to produce.

Over the past two decades, a great amount of computational work has gone into predicting the amplitude and frequency of the gravity waves resulting from both periodic sources and cataclysmic astrophysical events. A catalog of these results can be found in Thorne [4]. Many of these results are stated in terms of \( \epsilon \), the "efficiency" of gravity wave production, or the percentage of a star's initial mass that is converted into gravity waves during a collapse or a coalescence. For example, in the case of a star collapsing to a black hole, the strains produced are on the order of

\[ h \sim \sqrt{\epsilon} \frac{G M}{c^2 r} \quad (1.9) \]
$M$ is the mass of the collapsing star. Because gravity wave production depends on the quadrupole moment, a spherically symmetrical stellar collapse has an efficiency of $\varepsilon = 0$ and does not produce any gravity waves. Assuming an initial mass of 10 solar masses collapses in a region of the universe that is 10 kiloparsecs away, gravity waves of $h \sim 10^{-17} \times \sqrt{\varepsilon}$ reach the surface Earth. An important reason to measure gravity waves, therefore, is the fact that they will provide a measure of the amount of asymmetry in gravitational collapse. Analogously, detection of continuous waves will offer insight into the asymmetries associated with neutron stars.

Today, particularly in light of the binary pulsar observations of Joseph Taylor and Russell Hulse, there exists little doubt that gravitational waves exist [5]. Now the challenge is to directly observe them.

1.2 Resonant Bar Gravity Wave Detectors

1.2.1 A Brief History

Joseph Weber conceived the idea of using a resonant mass to detect gravity waves. Weber’s idea is to take advantage of the fact the gravitational field of a passing wave accelerates different parts of an extended object at different rates. In the laboratory frame, these accelerations appear as a tidal force on the detector. The difference in acceleration between any two points in the antenna increases as a function of the separation between the two points. In other words, the longer the body, the larger the force. Any extended object, such as a bar, has a resonant frequency determined by its length and its mass-density. A short-duration impulse creates the largest oscillations at this resonant frequency.
Beginning in the early 1960's, Weber designed, constructed, and operated the first resonant-bar gravitational wave detector [6]. A series of piezoelectric crystals attached to the center of the bar could detect changes as small as a tenth of an atomic nucleus. Though Weber's basic design was sound, resolution on the order of a tenth of a nucleus was still too coarse for the detection of gravity-waves. Further technological advancement was necessary.

In 1969, at Stanford University, William Fairbank and William Hamilton determined that more sensitive detectors could be built using cryogenic technology [7]. The primary purpose of the cryogenics was to reduce the physical temperature of the bar and, as a result, reduce the size of the thermally driven oscillations which limit detector sensitivity. The use of cryogenic systems has another benefit; it allows the use of superconducting components which improve detector sensitivity by orders of magnitude.
Since the antenna is so massive any motion induced by a gravity wave is of very small amplitude. To increase the sensitivity of the system, a resonant transducer is mounted on one end of the antenna. The small amplitude motions of the antenna's face are mechanically transformed into larger amplitude motions of the lighter mass transducer. At present, the most sensitive transducers developed are superconducting [8]. The Meisner effect of a superconducting diaphragm at the end of the transducer modulates the magnetic field of a persistent current. The changes induced in this current are detected and amplified. The quietest amplifier presently available, a superconducting quantum interference devices (SQUID), also makes use of superconducting technology.

The cryogenic resonant bars pioneered by Fairbank and Hamilton are the most successful, continuously operating network of gravity wave detectors in the world, and a significant international effort to detect gravity waves using resonant bars has been underway for several years. At present, resonant-mass gravitational wave detectors are currently being operated by research groups at the University of Rome and the INFN [9], the University of Western Australia (UWA) [10], and Louisiana State University (LSU) [11]. Several coincidence runs have been performed, as well as a directed search for gravitational radiation form supernova 1993J, and searches for radiation from continuous wave sources.

1.2.2 The ALLEGRO Project

Under the direction of William Hamilton and Warren Johnson at Louisiana State University, resonant bar gravity wave detection has made important strides
forward. Most importantly, the overall sensitivity of the detector has been improved. ALLEGRO is, at present, capable of detecting bursts with a dimensionless strain on the order of $2 \times 10^{-18}$. Almost as important, resonant bars have been proven to be reliable observatories, capable of running with a duty cycle of greater than 95%. Their reliable operation has allowed searches for continuous wave sources as well as burst sources, and has inspired confidence that more sensitive but technologically more complex generation of detectors can be operated.

The sensitivity improvement results from reductions in the fundamental noise sources which limit any amplifier. Experiments on early generations of inductive transducer, as well as on the ALLEGRO system confirm that there are two separate sources of dissipation in the transducer that contribute to noise. According to the fluctuation-dissipation theorem, thermal noise generated in the transducer is related to its mechanical quality factor. A second, independent source of dissipation is related to the presence of electrical losses. At this time, the specific mechanism behind the electrical losses is not well understood. Using the empirical data available on both electrical and mechanical losses, Norbert Solomonson developed fabrication procedures for an inductive transducer. He then went on to develop a design procedure which balances noise contributions related to electrical losses and SQUID amplifier noise [12]. Currently, the transducer designed and built by Solomonson is still in operation on ALLEGRO.

Sensitivity can be further improved by using a 2-mode transducer. Building on the optimization calculations of Solomonson, Ziniu Geng designed and began fabrication of a 2-mode transducer that is to be eventually installed on ALLEGRO.
By introducing the use of electrical discharge machining techniques, he further refined the fabrication techniques needed to produce high mechanical and electrical quality factors [13].

As with any amplifier chain, the noise performance is limited by the first amplifier in the system. Thus, to detect one of the smallest signals ever sought, the quietest amplifier available is needed. At present, the quietest practical amplifiers are superconducting quantum interference devices. Research going on at the University of Maryland by Frederick Wellstood and Insik Jin has come close to constructing and operating amplifiers with sensitivity that is only two orders of magnitude greater than the limit imposed on amplifier operation by fundamental principles of quantum mechanics [14].

Having demonstrated the success of a bar-antenna as an observatory, the LSU group began the preliminary work on the next generation of resonant antenna, a spherical geometry. A sphere has the advantages of omnidirectionality and a larger interaction cross-section. Stephen Merkowitz supervised construction and testing of a prototype spherical antenna. He successfully demonstrated that the magnitude and location of an incident impulse can be recovered from an arrangement of six transducers placed on the surface of the sphere [15, 16].

Although designed to search for burst events, the fact that ALLEGRO could be operated for long periods of time with minimal interruption opened the possibility of another type of gravity wave search. ALLEGRO can be used to search for continuous wave signals. Continuous wave signals are weaker than burst signals, but since they occur over a long period of time, the output of the detector can be
integrated to increase signal to noise ratio. Using data acquired by the ALLEGRO detector, Evan Mauceli has shown, despite the difficulties associated with gravitational Doppler shifting created by the motion of the earth around the sun, resonant bars can also be used to search for continuous wave gravitational signals [17].

1.2.3 Do we Understand a Single Detector?

A resonant antenna gravity wave detector is remarkably simple and remarkably complex at the same time. The operating principle is very simple. Isolate a spring as best you can from the outside world. If it starts shaking, apparently on its own, you have caught a gravity wave. The complexity lies in the fact that the signals sought are so very small. It is hard to determine if a meaningful signal is present. The signal is passed through many different stages of amplification, from mechanical transformations, through two different sets of superconducting devices, and then finally to conventional lock-in detection and analog to digital conversion. Since the point where a signal will be applied, the antenna itself is locked away in cryogenic isolation, applying test signals to the system is not a trivial matter.

Particularly as we move towards multi-mode transducers and spherical antennas, a complete understanding of the dynamics of a resonant system becomes essential. In chapters 2 and 3 of this dissertation, this problem will be considered at length. After the large scale cryogenics have been constructed, the superconducting systems are switched on, and the digital signal processing algorithms are applied to the output, can we recover the mass-on-a-spring behavior that we claim lies at the heart of a gravity wave antenna?
1.3 The Future

In the near future, it is hoped that the resonant antenna groups will be joined by other detection technologies. Three ideas have been proposed for the next generation of detectors. The first is the spherical resonant antenna discussed in the previous section. Proposals to build spherical detectors have been made in the United States (TIGA), Italy (OMEGA) and the Netherlands (GRAIL).

The second type of detector is a ground-based laser interferometer. The lengths of the arms of an interferometer with arms positioned at right angles will change by different amounts in response to an incident gravity wave. The sensitivity of the detector increases with arms length. Presently, the technology exists to construct interferometers with arms of 3-4 km. Interferometric detectors are broadband, operating in the frequency range between a few hertz and 1 kHz. Their low-frequency sensitivity is ultimately limited by seismic noise. Construction on interferometers is underway in the United States (LIGO) [18] and Europe (VIRGO) [19].

Perhaps the most ambitious project is the LISA project, which involves operating a laser interferometer in space [20]. A spaced based interferometer can have much longer baseline arms than a ground based interferometer. The current proposed arm lengths for LISA are approximately $5 \times 10^6$ km. Also, a space-based interferometer is free of many of the seismic and non-stationary noise sources which plague ground-based detectors.

As these different detectors come on-line and sensitivities improve, there will be three phases of gravity wave research. We are already in the first phase, examining the data from narrowband detectors of limited sensitivity. For the observation of a
large burst in this phase to be believed, either corroboration with an independently observed astrophysical event, or new physics will be required.

As LIGO phase II comes online, bar sensitivity approaches the quantum limit, and spheres are operated, we enter a second phase. This will be the initial era of real gravity wave astronomy. In this phase, detectors should be sensitive enough to a wide enough range of objects to see things predicted by accepted astrophysical models. In this phase, we can learn details about the processes which produce gravity waves.

LISA, if built, will mark the third era. At this time, sensitivity will be so great that we are expected to see things. LISA should be able to see the gravity waves we are reasonably sure exist based on radio frequency observations of low-frequency objects. LISA should also be able to reach into the regime of stochastic gravitational radiation. If nothing is seen, our fundamental notions about cosmology and perhaps even the nature of relativity itself will have to be reconsidered.

1.3.1 Do we Understand Multiple Detectors?

After data has been acquired and analyzed for the individual detectors in a network of detectors, the problem of searching the results for evidence of a common excitation is a statistical one. Statistical data analysis problems fall into two categories, problems of detection and problems of measurement. In the detection problem, the goal is to determine if any signal is present at all. In the measurement problem, the goal is to determine the parameters of a detected signal. The choice of statistical methods is based, in part, on whether an experimenter believes that his apparatus is sensitive enough to see anything. If he thinks it is
not, the problem is to determine if the features in the data are real. Otherwise, the problem is to map features in the data onto an assumed signal form.

Unfortunately, as detection sensitivity improves, as gravitational observatory sensitivity will over the next several years, no flag appears alerting the experimenter to the fact the peaks in the data are now real, detection has been achieved, and measurement problems can be approached. There is, in fact, at least one example of a missed discovery because of an improperly statistical analysis. In 1961, R.A. Ohm had radio-telescope data available to observe the 3 K cosmic microwave background [21]. Because he approached his analysis with a strong a priori bias that no signal was present, he convinced himself that nothing was present. The discovery of this background had to wait for the analysis of Penzias and Wilson several years later [22].

This type of error can be avoided by making sure the statistics are done properly. In the field of gravity wave detection, this problem takes on an added dimension as the results of radically different technologies with varying thresholds, cross-sections, and bandwidths are compared. We are entering an era of physics where the many of experiments performed are large-scale. Data obtained from these experiments is at a premium. In the case of the ALLEGRO project, for example, the it has taken 20 years of effort to obtain approximately 3 years of data. After over 20 years of planning, construction has only recently begun on the LIGO project.

In this case, it is absolutely essential to understand exactly how much information can be obtained from the data that is acquired. We do not want to miss a signal because of an inappropriate a priori belief that nothing is present. In chapter 4
of this dissertation, before attacking the specific problem of how perform an optimum statistical search for gravity waves, we make sure that we have complete understanding of the statistical tools we plan to use. Then, in chapter 5, the question of how to search for a signal in a background that is almost entirely composed of noise is addressed.
Chapter 2

The Complete Transfer Function of a Resonant-Mass Gravitational Wave Antenna

The result of the years (even decades) of hard work required to construct a single gravity-wave antenna is a device which can produce only two pieces of information per gravity wave. The detector's only outputs are a time of occurrence and an event amplitude. State-of-the-art timekeeping allows the time of occurrence to be determined with relative ease. Determining an event's amplitude in some manner relevant to gravitational wave astronomy is, however, a more complicated problem. Since they are the experimental results reported to the world-at-large, accurate determination of event amplitudes is of the utmost importance. In this chapter, we detail how a meaningful amplitude is assigned to a gravitational wave burst-event.
2.1 Modeling Overview

2.1.1 Objectives

From the interaction of the gravity-wave with the bar, to the details of the operation of the SQUID, there already exists an extensive literature on how to model the entire system described in the introduction. What is there to be gained from a new model? The answer to this question lies in the fact that narrowband detectors, like resonant bars, produce only a few pieces of information per gravity wave. The sole outputs are a time of occurrence and an amplitude response from each detector mode. There is no waveform information or particle track available to offer information about the physics of the detector.

Theoretically, a "black box" system like a resonant bar can be calibrated by exciting the input terminal with an independently calibrated force gauge and then observing the size of the output. Unfortunately, on a system as complex as a cryogenic resonant-bar gravitational wave antenna, the input terminal is physically isolated from the outside world and cannot be reached with an independently calibrated instrument. This means that the calibration procedure depends on referring measured quantities to the assumed behavior inside of the black box. The strength assigned to a candidate event is a model dependent quantity. It is only meaningful, therefore, if the model appropriately describes reality.

Resonant bars are modeled as a system of coupled harmonic oscillators. The existing bars all have two resonant detection modes. Development of two-mode transducers/three mode detection systems is underway. The previous work in this field has taken two different approaches to the multimode nature of a detector.
One body of work numerically solves a full multimode picture with an emphasis on describing how intrinsic noise sources limit a detector's overall sensitivity [23, 24, 25]. There is little analysis of the relationship between the modes. Furthermore, these models do not provide a detailed analysis of a detector's response to a signal. The analytic analyses of sensitivity [26] and advanced calibration procedures [27] simplify the multimode nature of a detector analysis. They either treat the two mode system as two completely independent antennas, or they assume that the antenna and transducer are perfectly tuned.

Both modeling approaches described above fail to take advantage of the fact that the harmonic oscillator is one of the most studied systems in all of physics. The relationships between coupled oscillator modes in response to various stimuli are well understood. If a harmonic oscillator model truly applies, there are certain quantitative relationships which must be observed. Given the paucity of information in this system, it is important that to use all available information for the purposes of calibration and event selection. Quantitative analysis of the relationships between modes in response to an excitation provides information that should not be ignored.

Mathematically, it is not difficult to describe the motion of two oscillators. However, for such a description to be a useful physical model, each parameter used in the description must relate to some quantity that can be reliably measured. Ideally, there should only be as many parameters in our description as there are unique measurements that can be made on the system. The work presented in this dissertation helps to advance the field by deriving a description of a resonant bar.
gravitational wave antenna that is expressed in a minimum number of measurable parameters while still maintaining full rigor.

2.1.2 Starting Point

The model developed in this chapter builds upon the work of Boughn et al. [27]. They developed a calibration procedure consisting of two basic elements. First, they chose a description of signal propagation through the detector system based on the amount of energy deposited in the antenna by a signal. Second, given the system calibration hardware available to them, they determined their system's measurable responses in terms of the deposited energy.

This procedure requires accurate determination of the energy added by the calibration system. Boughn et al. pioneered the use of impedance matrix techniques for calibration of the force generator on a gravitational wave detector. The calibrator can be used to apply a force, or it can be used to detect motion. Doing both in turn and comparing the results determines the ratio of voltage applied to the calibrator to energy deposited in (or, as we will show, force applied to) the antenna. This method has the advantages that it can be performed in situ and that it requires only straightforward electrical measurements. The model developed in this chapter retains the impedance matrix formalism for describing the calibration system.

In developing their model, Boughn et al. assumed that the bare antenna and bare transducer had exactly the same resonant frequency, and treated the two mode system as two independent antennas. We present a more detailed account of the impedance matrix calibrator model, and explicitly extend its theoretical basis.
to imperfect two-mode antennas. This extension of the theory explains apparent
discrepancies in the experimental results, determines two-mode parameters that
previously were not measurable, and determines the degree to which mistuning
affects calibration.

Finally, we reconsider how the strength of burst signals is to be quantified.
Historically, the relation between incident burst and detected output has been de­
scribed in terms of the energy deposited in the antenna modes [27]. With imperfect
tuning, this description can be misleading, because the burst does not deposit equal
amounts of energy in both modes.

2.2 The Mixing-Angle Representation of
The Antenna-Resonator Interaction

The ALLEGRO detector (A Louisiana Low-temperature Experiment and Grav­
itational Radiation Observatory) is a cryogenic resonant-mass antenna designed
and constructed for the purpose of directly observing gravitational radiation. The
system is built around a right circular cylinder (‘the bar’) composed of aluminum
5056. The bar is 3.0 meters long and has a physical mass of 2296 kg. A resonant
inductive transducer (‘the resonator’), the primary sensor of the antenna motion,
is attached to one face of the cylinder. A capacitive transducer (‘the force genera­
tor’) mounted on the opposing face is used to apply known forces to the bar. The
ALLEGRO system is shown schematically in Figure 2.1.

An incident gravity wave causes a strain on the bar. In response to this strain,
the bar undergoes simple harmonic motion (see Appendix A). The oscillations of
the bar, in turn, excite the resonator. Since the resonator is also an elastic body,
Figure 2.1: A physical schematic of the ALLEGRO detector, including the primary inductive transducer and capacitive force generator.
its motion is also harmonic oscillator (although its deformation function is much more complex than the antenna's). The readout system produces a voltage that is directly proportional to the resonator's inertial displacement. To operate the detection system, the mechanical relationships between antenna, resonator, and externally applied forces must be quantified.

The model presented in this paper extends previous work by treating the coupled antenna-transducer as a single detector component possessing two degrees of freedom. In this way, we determine a transfer function describing the motion of the transducer (the observable motion) in terms of force applied to the bar (the physically interesting source). A second transfer function describes the motion of the transducer in terms of a force applied directly to the transducer (a dominant noise source). A third transfer function, describing the calibration procedure, describes the motion of the antenna face in terms of the force applied to the antenna.

The ideal detector is built from an antenna and resonator with identical resonant frequencies. In practice, perfect tuning between the components is nearly impossible to achieve. The normal mode description of detector operation combines the effects of imperfect tuning into a single parameter common to all of the transfer functions. By combining the normal mode solution for the coupled antenna-transducer with the impedance matrix calibration procedure of Boughn et al., the mistuning parameter can be experimentally determined. These transfer functions explain the reason for the different responses observed during calibration of the force generator, and allow determination of two-mode parameters that previously were not measurable.
Figure 2.2: The two-mass and two-spring model of a gravitational wave antenna and resonant transducer, including a two-port representation of the force generator.

At this point, the following conventions are adopted and used throughout this chapter and the next. $\tilde{X}$ represents the Fourier transform of $X(t)$. The Fourier transform of an external force applied to the bar is represented by $\tilde{F}_1$. In this context, external means any force not created by the antenna-transducer interaction. $\tilde{F}_2$ represents an external force applied directly to the transducer. $\tilde{a}_1$ represents the inertial displacement of the of the bar resulting from $\tilde{F}_1$ and/or $\tilde{F}_2$, and $\tilde{a}_2$ represents the inertial displacement of the transducer resulting from $\tilde{F}_1$ and/or $\tilde{F}_2$.

Also, from this point on, since all of the physically interesting motion of the system is longitudinal, the equations of motion are written in one dimension.

The coupled oscillator model of an antenna and resonator, assuming that damping is small enough to be ignored, is pictured schematically in Figure 2.2. In terms
of mass-normalized coordinates, defined by \( a = Mu \), where

\[
M = \begin{bmatrix}
\frac{1}{\sqrt{m_1}} & 0 \\
0 & \frac{1}{\sqrt{m_2}}
\end{bmatrix},
\]

(2.1)

the equations of motion of the coupled oscillators are

\[
\ddot{u} = -Ku + MF.
\]

(2.2)

The displacement of the antenna face is given by \( u_1 \). The displacement of the transducer is given by \( u_2 \). The matrix \( K \), called the mass-normalized elastic matrix, is equal to

\[
K = \begin{bmatrix}
-(\omega_1^2 + \frac{m_2}{m_1} \omega_2^2) & \sqrt{\frac{m_2}{m_1}} \omega_2^2 \\
\sqrt{\frac{m_2}{m_1}} \omega_2^2 & -\omega_2^2
\end{bmatrix},
\]

(2.3)

where \( \omega_2 = k_2/m_2 \) is the uncoupled resonant frequency of the resonator. \( F \) contains the external forces acting on the system,

\[
F = \begin{bmatrix}
F_1 \\
F_2
\end{bmatrix},
\]

(2.4)

Since the matrix \( K \) is both real and symmetric, we can rewrite Equation (2.2) in terms of normal coordinates. The transformation from mass-normalized coordinates to normal coordinates is a rotation; we denote the rotation matrix as \( A \). Mathematically, the matrix \( A \) diagonalizes the matrix \( K \) so that \( KA = AD \), where
D is the diagonal matrix of eigenfrequencies,

\[
D = \begin{bmatrix}
\omega_+^2 & 0 \\
0 & \omega_-^2
\end{bmatrix}.
\]  

(2.5)

Since this system has two degrees of freedom, A is characterized by a single parameter \( \theta \), the "mixing-angle",

\[
A = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}.
\]  

(2.6)

The value of the mixing angle depends primarily on the difference in resonant frequency between the bare antenna and the bare transducer. The coupling of two equal masses with identical resonant frequencies produces a value of \( \theta = 45^\circ \).

In terms of the mass-normalized coordinates, the normal coordinates for the coupled oscillator system are \( y = A^T u \), where \( A^T \) denotes the transpose of the matrix A. Each component of \( y \) satisfies a harmonic oscillator equation [28]. Using a Fourier transformation, we rewrite the rotated version of Equation (2.2) as an algebraic equation, \( \tilde{y} = \Delta A^T \tilde{M} \tilde{F} \), where the matrix \( \Delta \) is the diagonal matrix of harmonic oscillator response functions,

\[
\Delta = \begin{bmatrix}
\frac{1}{\omega_+^2 - \omega_-^2} & 0 \\
0 & \frac{1}{\omega_-^2 - \omega_+^2}
\end{bmatrix}.
\]  

(2.7)
Transforming back to the inertial coordinates measured by the readout system, \( \ddot{a} = M \Delta \ddot{y} \), the response of either mass to a set of externally applied forces is

\[
\ddot{a} = M \Delta A^T \tilde{F}. \tag{2.8}
\]

The utility of this representation is seen by rewriting Equation (2.8) in matrix form,

\[
\begin{bmatrix}
\ddot{a}_1 \\
\ddot{a}_2
\end{bmatrix} =
\begin{bmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{bmatrix}
\begin{bmatrix}
\tilde{F}_1 \\
\tilde{F}_2
\end{bmatrix}. \tag{2.9}
\]

Each matrix element is a transfer function from force applied to a system component to inertial displacement of a system component. If the set of forces applied to the system is known, the motion of the system is completely determined by Equation (2.9). Expressed in terms of effective masses, eigenfrequencies \( \omega_+ \) and \( \omega_- \), and \( \theta \), the specific matrix elements are

\[
G_{11} = \frac{\cos^2 \theta}{m_1(\omega_+^2 - \omega^2)} + \frac{\sin^2 \theta}{m_1(\omega_-^2 - \omega^2)}, \tag{2.10}
\]

\[
G_{12} = G_{21} = \frac{-\sin \theta \cos \theta}{\sqrt{m_1 m_2}(\omega_+^2 - \omega^2)} + \frac{\sin \theta \cos \theta}{\sqrt{m_1 m_2}(\omega_-^2 - \omega^2)}, \tag{2.11}
\]

\[
G_{22} = \frac{\sin^2 \theta}{m_2(\omega_+^2 - \omega^2)} + \frac{\cos^2 \theta}{m_2(\omega_-^2 - \omega^2)}. \tag{2.12}
\]

The response of the coupled antenna-transducer is a superposition of the responses at the mode frequencies. Since these frequencies can be measured with a high degree of accuracy, this representation is highly useful for comparison with
experiment. The next section outlines the properties of the different measurable quantities determined by these transfer functions.

2.2.1 Determining Values for Observable Quantities

The inertial displacement of a single detector component (antenna or transducer) in response to an external forces is dominated by the response at the resonant frequencies. As no damping has been built into this model, mathematically expressed by the fact that all frequencies in the partial fraction expansions of Equations (2.10), (2.11), and (2.12) are real-valued, the time domain amplitude response of a mode is determined by evaluating the residue of the appropriate transfer function at the appropriate frequency.

A signal, real or simulated, is described by the condition $\tilde{F}_1 \gg \tilde{F}_2$. In this case, the response of the transducer, $a_2$, is given by $G_{21}$. The magnitude of the residue of $G_{21}$ at $\omega_+$ differs only slightly from its residue at $\omega_-$. The sign difference implies a 180° phase shift between the modes. If the incident excitation is equal in both modes (a condition that can be made true for calibration signals, and is expected to be true for gravitational wave bursts) $P_+$ and $P_-$, the immediate post-excitation mode amplitudes (PEMAs) should be nearly equal, differing only by the ratio of mode frequencies, for any value of $\theta$.

$$\frac{P_+}{P_-} = \frac{\omega_-}{\omega_+}.$$  \hspace{1cm} (2.13)

This equal excitation criteria is important for deciding whether an observed
excitation is a viable candidate signal, as well as a useful diagnostic for determining whether the system's behavior is consistent with the model.

A stochastic impulse between antenna and transducer is represented by an action-reaction pair $\vec{F}_1 = -\vec{F}_2$. Stochastic impulses are the dominant forces acting on the antenna during normal operation, and are the limiting sources of noise observed at the resonant frequencies. Since $m_1 \gg m_2$, and because the transducer is more lossy than the antenna, the response of the transducer to a stochastic force is dominated by $G_{22}$. In this case, the mode amplitudes are equal only in the special condition that $\theta = 45^\circ$, which is only true when the antenna and transducer have different identical resonant frequencies. Since the noise amplitudes of oscillation are too small to directly measure, the properties of the noise are measured with autocorrelation functions, which go as the square of the mode amplitude. In section 2.7, we derive a relationship between the mixing angle and the autocorrelation function ratio,

$$\frac{\langle a(t)^2 \rangle}{\langle a(t)^2 \rangle} = \frac{\sin^4 \theta \omega^4}{\cos^4 \theta \omega^4} \quad (2.14)$$

Finally, $G_{11}$ describes the response of the antenna's face to a force applied to the antenna's face. Again, in this case, the mode amplitudes are equal only when $\theta = 45^\circ$. This response is important because the force generator applies signals directly to the face of the antenna during the calibration process. The data obtained from this process determines $\theta$. In section 2.4, the calibration process and its relation to $\theta$ are described in detail.
2.3 The Force Generator

Having described the dynamics of the coupled antenna-transducer, we now turn our attention to the force generator used to apply calibration signals. The force generator is an electrostatic transducer that converts between electrical and mechanical quantities. It is a reciprocal device, meaning that an applied voltage/current combination can be used to produce a force and velocity, or that an applied force/velocity combination can be used to produce a voltage and current. Its description includes electrostatic quantities as well as mechanical quantities. The amount of force produced by a given voltage depends on the geometry of the device. In this section, the force generator is described in a manner that does not rely on detailed knowledge of the device.

2.3.1 Physical Description of the Force Generator

Physically, the force generator is a parallel plate capacitor mechanically bolted to one end of the bar. The simplest possible model of a capacitive force generator, a parallel plate capacitor whose plates are free to move under the force of the electric field between the plates, is shown in Figure 2.3. The first transducer relation is derived from the equation relating the force on either plate to the magnitude of the electric field,

$$F_e = \frac{1}{2} \varepsilon_0 E^2 A,$$  \hspace{1cm} (2.15)

where $\varepsilon_0$ is the permittivity of free space, $E$ is the magnitude of the electric field between the plates, and $A$ is the surface area of the plates. $E$ depends on both the voltage across the plates, and the separation between the plates. The voltage
Figure 2.3: Schematic of the simplest possible configuration for a capacitive force generator.
is composed of a large DC value, $V_{dc}$, plus a small AC component, $V_{ac}$. The separation, or "gap", between the plates is $g_0$, the size of the gap when only the DC field is present, plus the difference in displacements of the plates from their DC-equilibrium positions,

$$E = \frac{V_{dc} + V_{ac}}{g_0 + (a_3 - a_1)}.$$  \hfill (2.16)

Equation (2.16) can be inserted into Equation (2.15) and Taylor series expanded. Keeping only first-order terms, the time-varying electrical force on the plate in the frequency domain is

$$\tilde{F}_e = C_0 E_0 \tilde{V}_{ac} - C_0 E_0^2 (\tilde{a}_3 - \tilde{a}_1),$$  \hfill (2.17)

where $E_0 = V_{dc}/g_0$ and $C_0 = \epsilon_0 A/g_0$, the strength of the electric field when only the DC field is present and the capacitance when only the DC field is present, respectively.

The second transducer relation is derived from the definition of capacitance. For capacitor plates which are free to move, the equation for the current passing through the plates is

$$I = C \dot{V} + \dot{CV}. \hfill (2.18)$$

The time dependence of the capacitance results from the fact that the width of the gap changes as a function of time

$$C = \frac{\epsilon_0 A}{g_0 + (a_3 - a_1)}. \hfill (2.19)$$

33
Again, Taylor series expanding and keeping only first-order terms, the frequency domain equation for the voltage across the plates is

$$\tilde{V}_{ac} = \frac{I_{ac}}{i\omega C_0} + E_0(\bar{a}_3 - \bar{a}_1). \quad (2.20)$$

### 2.3.2 The Impedance Matrix Representation

The most compact way to combine the description of the antenna-transducer given by Equations (2.9) to (2.12) with the description of the electromechanical transducer given by Equations (2.17) and (2.20) is with an impedance matrix representation [29]. In this representation, the force generator is described in terms of the currents and voltages passing through an electrical port, and the forces and velocities at a mechanical port. The antenna-transducer system is a load which terminates the mechanical port. The impedance matrix relates voltage \( \tilde{V}_p \) and current \( \tilde{I}_p \) to force \( \tilde{F}_p \) and velocity \( i\omega \bar{a}_p \) in the Fourier domain.

$$\begin{bmatrix} \tilde{F}_p \\ \tilde{V}_p \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} i\omega \bar{a}_p \\ \tilde{I}_p \end{bmatrix}. \quad (2.21)$$

These quantities are chosen since, at either end, their product is power passing through the port. When operated as described above, with a small AC voltage applied in addition to a constant DC voltage, the calibrator becomes a linear device.

To use the impedance matrix, it is necessary to identify \( a_p \), the position of the mechanical port, and \( F_p \), the force "sent" or "received" from the mechanical port.
The mechanical port, an imaginary boundary which separates the inside of the transducer from the outside world, is an infinitesimal layer at the inside edge of \( m_1 \). \( a_p \) is merely the position of \( m_1 \). Since an infinitesimal layer has zero mass, any force applied on it from the internal dynamics of the transducer must be balanced by an equal and opposite force; if this were not the case, the layer would undergo an infinite acceleration. This balancing force is \( F_p \). From these definitions, the force and position of the mechanical port are

\[
\tilde{a}_p = \tilde{a}_1, \quad (2.22)
\]

\[
\tilde{F}_p = -\tilde{F}_1. \quad (2.23)
\]

With the mechanical port clearly identified, the impedance matrix for the simple force generator model of Figure 2.3 is found by eliminating the variables describing the internal dynamics of the transducer and expressing everything in terms of the set of port variables \( \tilde{F}_p, \tilde{V}_p, \tilde{I}_p, \) and \( i\omega \tilde{a}_p \). The four internal variables that must be eliminated are \( \tilde{F}_c, \tilde{a}_3, \tilde{V}_{ac}, \) and \( \tilde{I}_{ac} \). The forces resulting from the electric field are the only forces present on the two masses \( F_1 = -F_3 = F_c \). \( F_c \) and \( \tilde{a}_3 \) are eliminated using the Fourier transform of Newton's second law for \( m_3 \).

\[
\tilde{F}_1 = \tilde{F}_c. \quad (2.24)
\]

\[
-m_3\omega^2\tilde{a}_3 = -\tilde{F}_c. \quad (2.25)
\]
The voltage and current inside of the transducer are connected to the electrical port (in this simple model of a capacitor) via an ideal short circuit,

\[
\begin{bmatrix}
    \hat{V}_{ac} \\
    \hat{I}_{ac}
\end{bmatrix} =
\begin{bmatrix}
    1 & 0 \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    \hat{V}_p \\
    \hat{I}_p
\end{bmatrix}.
\]

(2.26)

The impedance matrix derived from Equations (2.17), (2.20), (2.22), (2.23), (2.24), (2.25), and (2.26) is

\[
\begin{bmatrix}
    \hat{F}_p \\
    \hat{V}_p
\end{bmatrix} =
\begin{bmatrix}
    0 & -\frac{E_0}{\omega} \\
    -\frac{E_0}{\omega} & \frac{1 + \frac{C_0 E_0^2}{\omega C_0}}{\omega C_0}
\end{bmatrix}
\begin{bmatrix}
    i\omega \bar{a}_p \\
    \bar{I}_p
\end{bmatrix}.
\]

(2.27)

Although the dynamics of a real force generator are more complex than this, this example illustrates the properties that make this representation useful. The \(Z_{ij}\)'s of Equation (2.21) are rational polynomials which can be represented in terms of poles and zeros. For practical operation of the calibrator, it is important that the impedance matrix have no poles near the resonant frequencies of the antenna-transducer system. Under this condition, within the detection bandwidth, all of the matrix elements are approximately constant. Also, many electromechanical transducers exhibit the property of Equation (2.27) that the off-diagonal elements are related by \(Z_{12} = -Z_{21}^*\). This property is very important to the calibration procedure described in the next section.
2.4 The Antenna-Force Generator Interaction

Equations (2.9) and (2.21) completely describe the dynamics of the system. The antenna-transducer system is the termination of the mechanical port of the force generator, constraining the relationship between $F_p$ and $i\omega\bar{a}_p$ at the end of the bar. A quantitative description of how a calibration pulse mimics a gravitational wave can be provided by examining the different conditions that the system is operated in. First, we describe the system's condition when a calibration pulse is applied.

1. $\bar{V}_p$ is constant, fixed by a voltage source.
2. The only external force acting on the bar comes from the force generator, $\bar{F}_1 = -\bar{F}_p$. The sign difference arises from Neubert's convention that $F_p$ and $i\omega\bar{a}_p$ are defined so that power flows into the mechanical port of a transducer.
3. The mechanical port of the force generator is rigidly attached the antenna face, $\ddot{a}_1 = \ddot{a}_p$.
4. There is no external force acting directly on the transducer, $\ddot{F}_2 = 0$. (It is assumed that the effects of Brownian noise at the bar-transducer interface can be neglected immediately after a large signal is applied). Given these 4 conditions, it is possible to solve for $\ddot{a}_1$ in terms of $\bar{V}_p$. The result is

$$\ddot{a}_1 = \left( \frac{\bar{G}_{11}}{1 + i\omega ||Z||_2 G_{11}} \right) \left( \frac{-Z_{12} \bar{V}_p}{Z_{22}} \right),$$

(2.28)

where $||Z||$ is the determinant of the electromechanical impedance matrix shown in Eq. (2.21).

When the bar is operated as a detector, these conditions are different. In this mode of operation,

1. The calibrator is short circuited, $\bar{V}_p = 0$.
2. The
force applied to the antenna is the sum of any external force on the bar (such as an incident gravity wave) plus any response from the short-circuited calibrator, \( \vec{F}_1 = -\vec{F}_p + \vec{F}_{ext} \). Conditions (3) \( \tilde{a}_1 = \tilde{a}_p \), and (4) \( \tilde{F}_2 = 0 \) remain the same. In this case, \( \tilde{a}_1 \) in terms of the external force is given by

\[
\tilde{a}_1 = \left( \frac{G_{11}}{1 + i\omega \|Z_{22}\| G_{11}} \right) \vec{F}_{ext} \tag{2.29}
\]

From these equations it is clear that the effect of applying a voltage \( V_p \) to the calibrator described by Eq.(2.28) is exactly the same as applying a force of magnitude \( \vec{F}_{ext} = -(Z_{12}/Z_{22})\vec{V}_p \) directly to the bar.

One other configuration of the detector system is of interest. After driving the antenna with the force generator, the oscillations of the antenna face will drive a current through the force generator's electric port. This current can be measured with a low-impedance current amplifier. The low-impedance property of this amplifier implies \( \vec{V}_p \approx 0 \). Again \( \tilde{a}_1 = \tilde{a}_p \), and \( \tilde{F}_2 = 0 \). \( \vec{I}_p \), in terms of \( \tilde{a}_1 \) then yields

\[
\vec{I}_p = \frac{Z_{21}}{Z_{22}} i\omega \tilde{a}_1. \tag{2.30}
\]

In the next section, we will show that the presence of the coefficient \( Z_{12}/Z_{22} \) in both the sending and receiving modes of the force generator allows a detailed measurement of the generator constant and mixing angle to be made, and the amount of force for a given voltage to be determined without detailed knowledge of the force generator geometry.
2.5 Measuring the Generator Constant

The calibration is carried out in two stages. First, the ratio of voltage applied to the force generator to force applied to the bar must be experimentally determined. Once this generator constant is known, an artificial burst signal of known strength can be applied to the antenna. The voltage response of the inductive transducer can then be determined as a function of magnitude of force applied to the bar.

The amount of force applied to the antenna for a given voltage is given by the product of $\tilde{V}_p$ and the ratio $Z_{12}/Z_{22},$

$$\tilde{a}_1 = \left( \frac{\cos^2 \theta}{m_1(\omega_+^2 - \omega^2)} - \frac{\sin^2 \theta}{m_1(\omega_-^2 - \omega^2)} \right) \left( \frac{-Z_{12}}{Z_{22}} \right) \tilde{V}_p.$$  \hspace{1cm} (2.31)

We denote $Z_{12}/Z_{22}$ (evaluated at $\omega_{\pm}$), the voltage to force constant for either the plus or minus mode, as $Z(\omega_{\pm})$. The inverse Fourier transform of the combined force generator-antenna-transducer transfer function, $R(t) = \mathcal{F}^{-1}[-G_{11}Z_{12}/Z_{22}],$

$$R(t) = \frac{1}{m_1} \left( \frac{Z(\omega_+\cos^2 \theta}{\omega_+} \sin(\omega_+t) + \frac{Z(\omega_-\sin^2 \theta}{\omega_-} \sin(\omega_-t) \right),$$  \hspace{1cm} (2.32)

is convolved with the time domain voltage input in order to determine $a_1(t)$, the time domain response of the antenna face. For a sinusoidal voltage of amplitude $V_0$, $V(t) = V_0\sin(\omega_{\pm}t)$, and duration $T$ the response is

$$a_1(t) = \int_0^T V_0 \sin(\omega_{\pm}t') R(t - t')dt'.$$  \hspace{1cm} (2.33)
Assuming that losses in the force generator are negligible, the amplitude of the antenna face in response to a wave train at $\omega_+$ is

$$a_1(t) = \frac{V_0 T Z(\omega_+) \cos^2 \theta}{m_1 \omega_+} \cos(\omega_+ t). \quad (2.34)$$

For a driving force at $\omega_-$, the response is

$$a_1(t) = \frac{V_0 T Z(\omega_-) \sin^2 \theta}{m_1 \omega_-} \cos(\omega_- t). \quad (2.35)$$

Equations (2.34) and (2.35) imply that a sinusoidal voltage applied to the force generator at either resonance frequency sends the antenna face into simple harmonic motion.

When the driving voltage is disconnected and the force generator is short circuited, the oscillations of the antenna face drive a current through the electrical port of the force generator. The driving oscillation is given by Equation (2.34) after an excitation of the plus mode, and by Equation (2.35) after an excitation of the minus mode. The current passing through the electrical port is given by solving Equation (2.21) with $\tilde{V}_p = 0$. The time-domain amplitude of this current is determined by taking the inverse Fourier transform of Equation (2.30). When the driving term $\tilde{a}_1$ is monochromatic,

$$I_{p\pm}(t) = \mathcal{Z}(\omega_{\pm}) \hat{a}_1(t). \quad (2.36)$$
Substituting Equation (2.34) into the above equation, functions for $I_{p\pm}(t)$ for each mode are obtained.

$$I_{p+}(t) = \frac{V_0 T z^2(\omega_+) \cos^2 \theta}{m_1} \cos(\omega_+ t). \quad (2.37)$$

$$I_{p-}(t) = \frac{V_0 T z^2(\omega_-) \sin^2 \theta}{m_1} \cos(\omega_- t). \quad (2.38)$$

To concisely express the relationships between the measurements and the model, the directly measured quantities ($V_0$, $T$, and the amplitude of $I_{p\pm}(t)$) are combined into a single parameter for each mode labeled $\gamma_+$ and $\gamma_-,$

$$\gamma_{\pm} = \left( -\frac{I_{0\pm}}{V_0 T} \right). \quad (2.39)$$

By substituting $\gamma_+$ and $\gamma_-$ into Equations (2.37) and (2.38) the relationships between the measured responses of ALLEGRO and the transfer function parameters are

$$\gamma_+ = \frac{z^2(\omega_+) \cos^2 \theta}{m_1}, \quad (2.40)$$

$$\gamma_- = \frac{z^2(\omega_-) \sin^2 \theta}{m_1}. \quad (2.41)$$

The differences between $\gamma_+$ and $\gamma_-$ observed in both the Stanford detector and ALLEGRO are thus explained as a function of the mechanical mistuning. Since $z(\omega_+)$ and $z(\omega_-)$ should be equal, the mode amplitudes are equal only when $\theta \approx 45^\circ$. If the antenna and transducer have different uncoupled resonant frequencies, this condition is not true.
Assuming that \( Z(\omega_+) = Z(\omega_-) \), \( Z(\omega_\pm) \) and \( \theta \) are simple functions of \( \gamma_+ \) and \( \gamma_- \),

\[
\tan \theta = \sqrt{\frac{\gamma_-}{\gamma_+}}, \tag{2.42}
\]

\[
Z(\omega_\pm) = \sqrt{m_1(\gamma_+ + \gamma_-)}. \tag{2.43}
\]

Assuming that this two-mode picture is valid, \( Z(\omega_\pm) \) and \( \theta \) are now experimentally determined quantities.

With known values for \( \omega_+ \), \( \omega_- \), and \( \theta \), values for uncoupled parameters \( \omega_1 \), \( \omega_2 \), and \( m_2 \) can be determined from the equation \( K = ADA^{-1} \). In matrix form, this equation is

\[
\begin{bmatrix}
-(\omega_1^2 + \frac{m_2}{m_1}\omega_2^2) & \sqrt{\frac{m_2}{m_1}\omega_2^2} \\
\sqrt{\frac{m_2}{m_1}\omega_2^2} & -\omega_2^2
\end{bmatrix}
= \begin{bmatrix}
-w_+^2 \cos^2 \theta - w_-^2 \sin^2 \theta & (w_1^2 - w_2^2) \sin \theta \cos \theta \\
(w_1^2 - w_2^2) \sin \theta \cos \theta & -w_1^2 \sin^2 \theta - w_-^2 \cos^2 \theta
\end{bmatrix}.
\]

The left side of Equation (2.44) is the model of the system's elastic behavior. The right side is a matrix of experimentally determined numbers. The value of \( m_1 \) has been independently measured, leaving three equations for three unknowns.

### 2.6 Inferring the Strength of a Gravity Wave

With the generator constant measured, we can return to the original problem of determining the detector's response to a gravitational wave. Operationally, this is done by measuring the detector's response to short duration impulses applied with the force generator. A nearly instantaneous impulse causes a sudden change in the amplitude and phase of the Brownian-driven oscillations of the bar. This
is reflected in a sudden change in the amplitude and phase of the motion of the transducer. These short duration impulses, referred to as “calibration pulses”, can be considered artificial gravity waves.

The strength of an applied impulse is measured using the demodulation scheme shown in Figure 2.4. The voltage output of the detector is mixed with a reference signal between frequencies $\omega_+$ or $\omega_-$, moving the information contained in the resonant modes into a frequency band centered at DC. After being passed through an anti-aliasing filter, the voltage is then sampled every 8 milliseconds. The now digitized signal is mixed with a reference signal that moves one of the resonant
Figure 2.5: Comparison of the raw voltage output and digitally filtered output from ALLEGRO.

modes to zero frequency. This DC signal is digitally filtered to optimize the signal to noise ratio [30]. The same procedure is repeated for the other mode. The digital filter is optimized for identifying sudden changes in the sampled output voltage.

Using the force generator, single cycle calibration pulses with voltages of 1.0V, 2.0V, and 3.0V and frequency 907.53 Hz were applied approximately 10 seconds apart from one another (this is equal to applying dimensionless strains of $3.3 \times 10^{-16}$, $6.5 \times 10^{-16}$, and $9.8 \times 10^{-16}$ to the bar). The top half of Figure 2.5 shows the samples of the in-phase channel of demodulated transducer voltage. The bottom half of Figure 2.5 shows the corresponding filtered output of the minus mode. The
filtered output is a maximum at the samples corresponding the times when pulses were applied, and the heights of the peaks are proportional to the applied voltage. Figure 2.5 demonstrates that the filter responds to a sudden change in mode amplitude, not simply to the size of the voltage amplitude. The complex amplitude corresponding to the signals applied in Figure 2.5 is graphed in Figure 2.6.

Figure 2.6: A plot of the in-phase component versus the quadrature component of the minus mode for the data shown in 2.5. The height of the peak produced by the digital filter is proportional to the change in mode amplitude.
This procedure is best understood by noting that a short-duration impulse causes a sudden change in the amplitude and phase of the Brownian-driven oscillations of the bar. This is reflected in a sudden change in the amplitude and phase of the motion of the transducer. The demodulation scheme divides the real-valued oscillations of the antenna into in-phase and quadrature components measured with respect to the reference signal. The amplitudes of the in-phase and quadrature channels contain all of the magnitude and phase information about the original time-domain signal [31]. Between any two samples, the change in amplitude of the in-phase component is proportional to the Fourier sine transform of the driving force on the oscillator at that instant, while the change in amplitude of the quadrature component is equal to the Fourier cosine transform of the driving force. We define the complex sum of the components as the complex amplitude. By looking for sudden changes in the complex voltage amplitude of the transducer’s oscillations with the digital filter, both the magnitude and the phase of the transducer’s time-domain oscillations are used in detecting a signal.

The filtered output is recorded in digital units. The relation between transducer voltage and digital units depends on the analog to digital conversion, as well as any gains introduced during demodulation and filtering. The calibration pulses are used to determine the force implied by a sudden change in mode amplitude and its related quantity, the height of its filtered output peak. With this information, the digital units of the output can be normalized in terms of the strength of a gravity wave.
The force applied to the antenna by an interaction with a gravity wave is

\[ \tilde{F}(\omega) = -\frac{1}{2} m_1 l_1 \omega^2 \tilde{h}(\omega), \]  

(2.45)

where \( l_1 \) is the effective length of the antenna and \( \tilde{h}(\omega) \) is the Fourier transform of the dimensionless strain of a gravity wave. For the purpose of calibration, it is convenient to define a standard reference signal. A candidate event is assigned the amplitude that would be produced by a single cycle of a sine wave at the frequency of the detector. The Fourier components of this reference signal are

\[ \tilde{h}(\omega) = \pi h_G / \omega \pm, \] 

(2.46)

where \( h_G \) is the amplitude of the reference signal.

The amount of force generator voltage needed to simulate a burst of amplitude \( h_G \) is found from \( V_p = F_p / Z(\omega) \). The relationship between the magnitude of an applied voltage and its filtered output peak height is linear. The comparison between the voltage of the calibration pulse and the size of the burst filter output is used to determine the constant of proportionally between the applied voltage and the peak height in digital units. Once it is known that a calibration pulse of \( V_{\text{cal}} \) volts (zero-to-peak) produces a peak of \( D_{\text{cal}} \) digital units, a filtered output sample of \( D_{\text{obs}} \) digital units is assigned a burst amplitude of

\[ h_G = \sqrt{\frac{2Z(\omega) V_{\text{cal}}}{\omega^2 m_1 l_1}} \left( \frac{D_{\text{obs}}}{D_{\text{cal}}} \right). \] 

(2.47)
This convention for inferring the strength of a gravity wave differs from the convention usually used in this field. Historically, the strength of an incident gravity wave has been reported in terms of "burst energy" or "energy innovation" [32]. Burst energy calculations assume that a gravitational wave interacting with a resonant antenna deposits equal amounts of energy in each detector mode. This assumption is not correct. For a gravity wave burst of finite duration, the energy deposited in a two-mass, two-spring system, is given by

\[ E = \frac{1}{8} m_1 I_1 (|\tilde{h}(\omega_\text{+})|^2 \omega_\text{+}^4 \cos^2 \theta + |\tilde{h}(\omega_\text{-})|^2 \omega_\text{-}^4 \sin^2 \theta). \]  

(2.48)

For \( \theta \neq 45^\circ \), the amount of energy deposited in each mode is not equal. This difference in mode energies is not directly observed in the output of the detector. The inertial displacement of the diaphragm is given by

\[ a_2(t) - a_1(t) = \frac{1}{m_1 \omega_\text{+}} \left( -\sqrt{\frac{m_1}{m_2}} \sin \theta \cos \theta - \sin^2 \theta \right) \Im \left( \tilde{F}_\text{ext} \times e^{i \omega_\text{+} t} \right) \]  

(2.49)

\[ + \frac{1}{m_1 \omega_\text{-}} \left( \sqrt{\frac{m_1}{m_2}} \sin \theta \cos \theta - \cos^2 \theta \right) \Im \left( \tilde{F}_\text{ext} \times e^{i \omega_\text{-} t} \right), \]

where \( \Im \) denotes imaginary part. For this generation of antennas and transducers, \( \sqrt{m_1/m_2} \approx 40 \). For the broadband signal cause by a short duration burst, \( \tilde{F}(\omega_\text{+}) = \tilde{F}(\omega_\text{-}) \). Under these conditions the amplitudes of the modes are approximately equal even when different amounts of energy are deposited in each mode.

For these reasons, a convention based on energy becomes unnecessarily confusing. Different values of energy should be assigned to the two modes in response to a burst excitation despite the fact that they are responding to the same impulse.
The energy difference is a property of the observing antenna, and does not contain any information about the nature of an incident gravitational wave. By using Equation (2.47), the strength of a gravity wave can be expressed in terms of the observed mode amplitudes without any reference to antenna dependent mistuning.

2.7 Noise Characterization

A large portion of the effort to detect gravity waves is concerned with the design and construction of the lowest noise amplifiers possible. It is not inaccurate, in fact, to describe a resonant bar detection system as a low-noise amplifier of gravitational radiation. Optimizing the noise performance of the system requires that the different sources of noise be well understood. In this section, we demonstrate how the model developed in this chapter provides a useful diagnostic for characterizing one of a resonant antenna’s limiting sources of noise.

Experiments on early generations of inductive transducers, as well as on the ALLEGRO system, confirm that there are two separate sources of dissipation in the transducer that contribute to noise. According to the fluctuation-dissipation theorem, thermal noise generated in the transducer is related to losses inversely proportional to its mechanical quality factor. A second source of dissipation is results from electrical losses that occur in the transducer (in spite of the fact that the system is superconducting). The specific mechanism behind the electrical losses is not well understood. These two sources of noise, the result of forces acting directly on the transducer, are called transducer force noise. At the detector’s resonant frequencies, the transducer force noise is much greater than the noise.

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originating in the DC SQUID amplifier. The SQUID white noise, added in series to the output of the transducer, is the dominant noise source off-resonance.

The transducer force noise can be modeled as a series of noise impulses acting directly on the transducer, \( F_2(t) = F_n(t) \). The impulses are stochastic with a spectral density, \( S_{nn}(\omega) \), that is white. Stochastic forces on the antenna can be neglected because the inertial motion of the antenna is smaller due to its larger mass, and because it has been experimentally determined that the transducer is significantly more lossy than the antenna. Since transducer force noise is both applied and observed at the transducer, the Fourier-domain response of the transducer is given by multiplying the transfer function \( G_{zz} \) by the noise force.

Since the decay time of either mode is on the order of tens or hundreds of seconds, the effects of damping cannot be neglected when considering noise. The effects of damping are considered by including an imaginary damping term in the denominator of \( G_{zz} \).

\[
\tilde{a}_2 = \left( \frac{\sin^2 \theta}{m_2(\omega_+^2 + \omega^2)} + \frac{\cos^2 \theta}{m_2(\omega_-^2 + \omega^2)} \right) \tilde{F}_n. \tag{2.50}
\]

The size of the damping term is inversely proportional to the measured decay time of the mode. The decay times can be experimentally determined by observing the mode amplitudes for several minutes after they are excited with a large impulse. For ALLEGRO, the plus mode has a decay time of \( \tau_+ \approx 80 \) seconds, and the minus mode has a decay time of \( \tau_- \approx 50 \) seconds. The spectral density of the noise driven
oscillations of the transducer is given by

\[ S_{22}(\omega) = \frac{\sin^4 \theta}{m_2^2} \frac{S_{nn}(\omega)}{(\omega^2 - \omega_2^2)^2 + \frac{\omega_2^4}{\tau^2}} + \frac{\cos^4 \theta}{m_2^2} \frac{S_{nn}(\omega)}{(\omega^2 - \omega_3^2)^2 + \frac{\omega_3^4}{\tau^2}}. \]  

(2.51)

The properties of the noise are measured by determining the autocorrelation functions of the output of the individual modes. The autocorrelation function of the transducer output is defined as the expectation value of the product \( a_2(t)a_2(t - t') \).

For a noise driven harmonic oscillator, the autocorrelation function is a decaying exponential function of \( t' \) [33]. The mean-squared value of the transducer’s displacement is given by the autocorrelation function evaluated when \( t' = 0 \). By the Wiener-Khintchin theorem, the Fourier transform of the autocorrelation function is equal to the power spectrum of the process. Therefore,

\[ \langle a^2(t) \rangle_+ = \frac{S_{nn}(\omega)\tau_+ \sin^4 \theta}{2m_2^2\omega_2^2} \]  

(2.52)

\[ \langle a^2(t) \rangle_- = \frac{S_{nn}(\omega)\tau_- \cos^4 \theta}{2m_2^2\omega_-^2} \]  

(2.53)

If both modes are allowed to freely decay, the magnitude of the mean squared displacement of the transducer can be calculated using the equipartition theorem. However, in actual operation, detectors using inductive transducers are run with constant feedback from the SQUID applied to the antenna face via the force generator to avoid instabilities. The feedback causes the decay time in each mode is different, and the conditions for the equipartition theorem do not hold.
We treat the change caused by the feedback as a change in the effective masses of the modes. Although both modes are driven by the same stochastic force, we expect to observe different mean-squared displacements of the modes in the absence of any large impulses. Since the amplitude of the signal response of both modes is equal, this implies that one mode is more sensitive than the other in a mistuned system.

Equations (2.52) and (2.53) add to our diagnostic tools. The ratio of the two autocorrelation functions provides a second measure of the mixing angle, providing a test of the consistency of the model. Knowing the mixing angle, the magnitudes of the autocorrelation functions provides another measure of how close our system is to being limited by the fundamental noise sources predicted by the fluctuation-dissipation theorem.

2.8 Summary of Results

Modifying the previously used models of resonant antenna calibration proves enlightening for a number of reasons. Reducing the dynamics of the antenna-transducer interaction to a single mixing angle refines our understanding of the mechanics of a coupled resonant system. The propagation of both signals and the noise sources can be followed end-to-end. The result in either mode is described directly in terms of the incident force. Observed differences between $\gamma_+$ and $\gamma_-$ observed in both the Stanford and LSU antennas are explained. The model makes clear that different values for $\gamma$ in each mode do not contradict the equal-amplitude response from the transducer motion in response to signals.
Two important diagnostic tools result from this model. First, both mode amplitudes measured at the transducer should respond equally to an impulse applied to the bar. This result is true for any degree of mistuning between the bar and transducer. Second, the ratio of noise autocorrelation functions should be equal to the fourth-power of the measured ratio between $\gamma_+$ and $\gamma_-$. The magnitude of the autocorrelation function yields information on how close the system is to being limited by transducer force noise.

Finally, this model improves the understanding of the relationship between a resonant antenna and incident gravitational wave. The intuitive starting point that an incident gravity wave deposits a certain amount of energy in a bar need not be altered. How that energy is observed with use of a transducer is not necessarily obvious. The observed mode amplitudes are equal, even when the energy is not equally divided between the two modes.
Chapter 3

Observing and Extending the Two-Mode Antenna Model

In Chapter 2, a model describing the operation of an idealized, two-mode resonant antenna was developed. In this chapter, we test the model against reality. When the experimental results match our predictions, we gain some measure of confidence that the model accurately describes reality. Results which deviate from our predictions, however, also have value. They provide the first clues on where the model can be refined and extended. Both of these cases require a model that makes unambiguous predictions that can be compared to experiment. In this chapter, the normal-mode model of a resonant-bar gravitational antenna is compared to the real behavior of the ALLEGRO system.

3.1 The Mystery Mode

ALLEGRO cannot be satisfactorily modeled as a 2-mode system. Figure 3.1 shows spectra of 20 second intervals of ALLEGRO’s output before and after the force generator was used to apply a large impulse to the bar. These plots show that ALLEGRO has a third mode with a resonant frequency of 887.742 Hz. Figure 3.2
Figure 3.1: The output spectrum of ALLEGRO, before and after a large calibration pulse is applied.

shows several cycles of ALLEGRO's demodulated output acquired immediately after the impulse was applied. The data has been shifted in frequency to produce a graph of the real-time response of the detector. Three sine-waves, shown in Figure 3.3, are required to reconstruct this signal. The frequencies, amplitudes, and decay constants of the fitted waves were extracted from the data using a lock-in amplifier. The only free parameter is the initial starting time. Subtracting these three sine waves from the actual data shows that these three frequency components account for most of ALLEGRO's response to a burst in the data that is taken.

Beyond the presence of an unwanted mode, ALLEGRO's designed modes display behavior inconsistent with the criteria developed in the previous chapter for a two-mode system. For example, when a large impulse is applied to the antenna,
Figure 3.2: The solid line is the real-time response of ALLEGRO to an applied impulse. The circles are the result of a fit of three sine-waves (shown in the next plot) to the real data.
Figure 3.3: The individual components of ALLEGRO's real-time response to an impulse. The upper three plots are ideal sine waves generated using amplitudes, frequencies, and decay constants determined by fitting the real data of Figure 3.2. The fourth plot is the result of subtracting the three ideal sine waves from the actual data.
the transfer function of the detector is dominated by $G_{21}$, and the mode amplitudes should be equal. This is not observed. After a calibration pulse is applied, the amplitude of the minus mode is consistently larger than the amplitude of the plus mode. Repeated measurements made immediately after exciting the antenna show that the plus mode has only $82 \pm .03$ percent of the amplitude of the minus mode. The presence of a "mystery mode" in such close proximity to the detector's fundamental modes is the likely source of these discrepancies.

This mystery mode has a measurable $\gamma$-value. Figure 3.4 shows a plot of the current measured from the force generator immediately after 25 second wave trains of equivalent voltage amplitude were applied to the plus, minus, and mystery modes. Clearly, the mystery mode shows the largest response, nearly an order of magnitude larger than the response of the minus mode.

The location of the unwanted resonance creating these discrepancies is believed to be the force generator. There are two reasons for this belief. First, an additional resonance added to the model of the antenna-transducer lowers the amplitude of the minus mode relative to the amplitude of the plus mode. The observed fact is that the amplitude of the minus mode is always higher than the amplitude of the plus mode in response to an excitation. Second, during the original assembly of ALLEGRO, the mounting of the force generator produced a resonance at approximately 865 Hz at room temperature. Since resonant frequencies change as a function of temperature, it is possible that this resonance moved upward towards the designed modes of the detector.
The effect of this additional mode on the calibration procedure must be determined. The models developed in Chapter 2 are robust enough to handle this question. In the next 2 sections, some of the assumptions made to simplify the quantitative application of the model are reexamined in the presence of an additional mode in the force generator.
3.1.1 Where do the discrepancies fit?

When solving the antenna-force generator interaction in section 2.4, the term depending on $|Z|/Z_{22}$ in Equations (2.28) and (2.29) was neglected. This approximation must be reexamined. In this chapter, the physical meaning and experimental effect of this correction is considered.

To begin this analysis, the transfer function matrix including the $|Z|/Z_{22}$ term is written out. The conditions used in this case are (1) The force generator is short circuited, $V_p = 0$. (2) The force applied to the antenna is the sum of any external force on the bar plus the response from the short-circuited force generator, $\tilde{F}_1 = -\tilde{F}_p + \tilde{F}_{ext}$. (3) The force generator is rigidly attached to the face of the antenna. $\tilde{a}_1 = \tilde{a}_p$. In this case, no assumption about $\tilde{F}_2$ is made; it is be explicitly retained in the expression. The modified response matrix is

$$\begin{bmatrix}
\tilde{a}_1 \\
\tilde{a}_2
\end{bmatrix} = \frac{1}{1 + i\omega ||Z|| G_{11}} \begin{bmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22} + i\omega ||Z|| |G|
\end{bmatrix} \begin{bmatrix}
\tilde{F}_1 \\
\tilde{F}_2
\end{bmatrix}. \tag{3.1}
$$

The denominator depending upon $|Z|/Z_{22}$, neglected in the previous chapter, is included in the analysis that follows. To make the notation through this section and the next more compact, the denominator of coefficient of the matrix is represented by $\Delta = 1 + i\omega ||Z|| G_{11}/Z_{22}$.

3.2 The General Electrostatic Transducer

Determining the corrections implied by using Equation (3.1) in place of Equation (2.9) requires a more realistic model of the force generator than was developed in section 2.3. This model should, if possible, include the effect of the resonance
in a manner that does not depend on detailed knowledge of the force generator geometry. Such a model can be built by combining the transfer-function picture of a system of harmonic oscillators developed in section 2.2 with the electrostatic transducer relations of section 2.3.

A capacitive force generator is a set of coupled harmonic oscillators with a few additional properties. Two of the masses, instead of being connected by springs, are connected by the transducer relations of Chapter 2. This most general model of a force generator is shown in Figure 3.5. The "gap" in this model depends on the separation between \( m_3 \) and \( m_4 \). (The convention that \( m_1 \) is the antenna and \( m_2 \) is the inductive transducer is retained through this chapter). The mass that is attached to the antenna is defined to be the mechanical port of the force generator, and is labeled \( m_0 \).

Physically, a capacitive force generator is a set of coupled harmonic oscillators (illustrated in Figure 3.5) with two masses connected by an electric field instead of springs. Its mechanical dynamics are described with an oscillator response matrix analogous to the oscillator response matrix of Equation (2.9). The electrostatic forces enter the problem as a pair of equal and opposite external forces acting on masses \( m_3 \) and \( m_4 \). There is also an external force on the mechanical port, \( \vec{F}_0 \), resulting from the force generator's interaction with the antenna. For a system
Figure 3.5: Schematic of the model of the general electrostatic transducer. Any number of additional resonances can be handled by this model.
composed of $N$ masses,

\[
\begin{bmatrix}
\ddot{a}_0 \\
\ddot{a}_3 \\
\ddot{a}_4 \\
\vdots \\
\ddot{a}_N
\end{bmatrix} = \begin{bmatrix}
G_{00} & G_{03} & G_{04} & \cdots & G_{0N} \\
G_{30} & G_{33} & G_{34} & \cdots & G_{3N} \\
G_{40} & G_{43} & G_{44} & \cdots & G_{4N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
G_{N0} & G_{N3} & G_{N4} & \cdots & G_{NN}
\end{bmatrix} \begin{bmatrix}
\ddot{F}_0 \\
\ddot{F}_e \\
\ddot{F}_e \\
\ddots \\
0
\end{bmatrix}.
\]  

(3.2)

Equations (2.17), (2.20), and (3.2) completely describe the motion of the force generator resulting from both electrostatic and mechanical forces. In order to form the impedance matrix for this system, the variables describing motion inside of the force generator ($\ddot{F}_e$, $a_3$, and $a_4$) must be eliminated. The force and velocity of the capacitor plates are concisely described in terms of the voltage and current at the electrical port by rewriting Equations (2.17) and (2.20) in their transfer matrix representation,

\[
\begin{bmatrix}
\ddot{F}_e \\
i\omega \dddot{q}
\end{bmatrix} = \begin{bmatrix}
0 & \frac{E_0}{i\omega} \\
i\omega & \frac{1}{C_0 E_0}
\end{bmatrix} \begin{bmatrix}
\dddot{V}_p \\
\ddot{I}_p
\end{bmatrix},
\]  

(3.3)

\[q = a_4 - a_3.\]  

(3.4)

Equation (3.3) assumes that the voltage and current at the port are connected to the capacitor plates by an ideal short circuit, $V_p = V_{ac}$, $I_p = I_{ac}$. Equation (3.2) can also be recast in a transfer matrix form relating the force and velocity of the
mechanical port to the force and velocity of the capacitor plates,

\[
\begin{bmatrix}
\vec{F}_p \\
i\omega \vec{\alpha}_p
\end{bmatrix} = \frac{1}{G_2} \begin{bmatrix}
G_1 & \frac{-1}{i\omega} \\
i\omega (G_2^2 - G_{00}G_1) & G_{00}
\end{bmatrix} \begin{bmatrix}
\vec{F}_e \\
i\omega \vec{q}
\end{bmatrix},
\tag{3.5}
\]

where

\[
G_1 = G_{43} + G_{34} - G_{44} - G_{33},
\tag{3.6}
\]

\[
G_2 = G_{30} - G_{40}.
\tag{3.7}
\]

This assumes that \( m_0 \) has been defined as the mechanical port of the force generator, so that \( \vec{a}_p = \vec{a}_0 \) and \( \vec{F}_p = -\vec{F}_0 \). The right side of Equation (3.3) is used to eliminate \( \vec{F}_e \) and \( \vec{q} \) from Equation (3.5). Rearranging the result into an impedance matrix form, the impedance matrix for a general electrostatic transducer is

\[
\begin{bmatrix}
\vec{F}_p \\
\vec{V}_p
\end{bmatrix} = \begin{bmatrix}
\frac{-1}{i\omega G_0} & \frac{E_0 G_2}{i\omega G_{00}} - \frac{E_0^2 (G_2^2 - G_{11}G_{00})}{i\omega G_{00}} \\
\frac{E_0 G_2}{i\omega G_{00}} & \frac{1}{i\omega C_0} - \frac{E_0^2 (G_2^2 - G_{11}G_{00})}{i\omega G_{00}}
\end{bmatrix} \begin{bmatrix}
i\omega \vec{\alpha}_p \\
i\omega \vec{\alpha}_p
\end{bmatrix}.
\tag{3.8}
\]

Note that the necessary symmetry property between the off-diagonal elements is present.

Using Equation (3.8), we can determine the frequency dependence of \( ||Z||/Z_{22} \) and \( Z(\omega) \) without relying on detailed knowledge of the force generator geometry. From linear network theory, it is known that the inverse of the driving point response function \( G_{00} \) is also a driving point response function [34]. This implies that \( 1/G_{00} \) can be written as a partial fraction expansion. Since there is only a single additional resonance in ALLEGRO, all but one of the terms of this expansion can
be combined into a single, frequency-independent constant. The remaining term is written as a constant coefficient over a resonant denominator.

The first quantity needed is $||Z||/Z_{22}$. The determinant of the general electrostatic transducer matrix is

$$||Z|| = \frac{1}{\omega^2 C_0 G_{00}} + \frac{E_0^2}{\omega^2 G_{00}}. \quad (3.9)$$

Since $C_0$ is small, and $Z_{22}$ is nearly constant, $Z_{22} \approx 1/i\omega C_0$. Under this assumption,

$$\frac{||Z||}{Z_{22}} = \frac{1}{i\omega} \left( \beta_0 + \frac{\beta_1}{(\omega^2 - \omega_f^2)} \right), \quad (3.10)$$

where $\omega_f$ is the frequency of the extra resonance in the force generator. The form of the generator constant in the resonant case is also a partial fraction expansion with a single frequency-dependent term of interest,

$$\frac{Z_{12}}{Z_{22}} = \beta_2 + \frac{\beta_3}{(\omega^2 - \omega_f^2)}. \quad (3.11)$$

All of the $\beta$'s are real-valued constants. There is no obvious relationship between the different $\beta$'s in this representation. The denominator frequency is the same in both expressions.

### 3.3 Effect of Loading by the Force Generator

Before the effects of an additional resonance are considered, it is useful to extend the non-resonant force-generator model in order to illustrate how additional elements can be added to the system without the distraction of the full mathematical
complexity required to add another resonance. The fact that the force generator alters the mass of the system has, so far, been neglected in the description of the electromechanical dynamics of its operation. The effective antenna mass is now the bare bar's effective mass plus the force generator’s effective mass. The perturbations produced by this loading of the antenna is the effect considered in this section.

Physically, adding additional mass to the antenna changes its resonant frequency and, as a result, changes the coupled antenna-transducer mode frequencies. The frequencies $\omega_+$ and $\omega_-$ measured at the output of the detector are not, strictly speaking, the frequencies produced by coupling the bare antenna to the transducer. $\omega_+$ and $\omega_-$ are the result of coupling the antenna, transducer, and force generator together. Though this additional frequency shift is physically uninteresting, the fully coupled mode frequencies are known with great accuracy, and, for comparison between model and experiment, are still the obvious choice for describing the system. For each of the mode frequencies,

$$\omega'_\pm = \omega_\pm + \delta_\pm. \quad (3.12)$$

The primed frequencies are the result of coupling the bare bar to the transducer. They are not observable once the force generator has been mounted. The unprimed frequencies are the coupled frequencies in the presence of the force generator. These frequencies are directly measurable. $\delta_\pm$ is the frequency shift of the normal modes of the detector created when the force generator is added to the system. Experimentally it is known that these frequency shifts are small.
The magnitudes of $\delta_\pm$ can be determined from the formalism developed in Chapter 2. Since no resonances are yet being assumed, $|Z|/Z_{22}$ can be represented by $\beta_0/\omega$ at both resonant frequencies of the detector. The voltage-to-force constant of the force generator is still given by $Z_{12}/Z_{22}$, and can also still be represented by a single number, $Z(\omega_\pm)$.

Consider the measurement of $\gamma_\pm$. In this case, the transfer function from force generator voltage to antenna face motion (the function needed to describe the system's response to a calibration wave train) is

$$
\frac{G_{11}}{\Delta} \left( \frac{-Z_{12}}{Z_{22}} \right) = \frac{Z(\omega_\pm)(\cos^2 \theta (\omega_\pm^2 - \omega^2) + \sin^2 \theta (\omega_\pm^2 - \omega^2))}{m_1(\omega_1^2 - \omega^2)(\omega_\pm^2 - \omega^2) - \beta_0(\cos^2 \theta (\omega_\pm^2 - \omega^2) + \sin^2 \theta (\omega_\pm^2 - \omega^2))}.
$$

(3.13)

Mathematically, including the correction term in the denominator changes the positions of the poles of the transfer functions without changing the position of the zeros. Using Equation (3.12) to eliminate the unobservable primed quantities, Equation (3.13) becomes

$$
\frac{G_{11}}{\Delta} \left( \frac{-Z_{12}}{Z_{22}} \right) = \frac{Z(\omega_\pm)(\cos^2 \theta (\omega_\pm^2 - 2\delta_\omega_\pm - \omega^2) + \sin^2 \theta (\omega_\pm^2 - 2\delta_\omega_\pm - \omega^2))}{m_1(\omega_1^2 - \omega^2)(\omega_\pm^2 - \omega^2)}.
$$

(3.14)

At frequencies $\omega_+$ and $\omega_-$, the residues are

$$
\text{Res}_{\omega_+} \left( \frac{G_{11}}{\Delta} \frac{-Z_{12}}{Z_{22}} \right) = \frac{Z(\omega_\pm)}{2m_1\omega_+} (\cos^2 \theta + \epsilon_1),
$$

(3.15)

$$
\text{Res}_{\omega_-} \left( \frac{G_{11}}{\Delta} \frac{-Z_{12}}{Z_{22}} \right) = \frac{Z(\omega_\pm)}{2m_1\omega_-} (\sin^2 \theta - \epsilon_1),
$$

(3.16)
\[ \epsilon_1 = \frac{\delta_+ \omega_+ \sin^2 \theta + \delta_- \omega_- \cos^2 \theta}{\omega_+^2 - \omega_-^2}. \] (3.17)

The time-domain amplitude of \( \tilde{a}_1 \) is twice the residue evaluated at the appropriate frequency. Using a set of masses and spring constants that approximately describe the force generator mounted on ALLEGRO, these frequency shifts are estimated to be less than 1 Hz. This implies a maximum correction of \( \epsilon_1 \approx 0.02 \).

Ultimately, even this small value for \( \epsilon_1 \) is invisible to the calibration procedure. The mass-loading of the antenna produces no additive correction to the displacement to current transfer function defined in Equation (2.30). The value of \( \gamma_{\pm} \) is thus proportional to the product of \( Z(\omega_{\pm}) \) and the amplitude of \( a_1 \),

\[
\gamma_+ = \frac{Z^2(\omega_{\pm})(\cos^2 \theta - \epsilon_1)}{m_1}, \quad (3.18)
\]

\[
\gamma_- = \frac{Z^2(\omega_{\pm})(\sin^2 \theta + \epsilon_1)}{m_1}. \quad (3.19)
\]

Comparing Equations (3.18) and (3.19) to Equations (2.40) and (2.41) shows that the only change from the unloaded case is the addition of equal and opposite \( \epsilon_1 \) terms. Since the correction terms are equal and opposite, summing \( \gamma_+ \) and \( \gamma_- \) still yields the value of the force-to-voltage constant.

There is a similar cancellation in the case of the PEMAs. In this case, the frequency shifts arising from the mass loading of the antenna change the force generator voltage/transducer motion transfer function to

\[
\frac{G_{21}}{\Delta} \left( \frac{-Z_{12}}{Z_{22}} \right) = \frac{Z(\omega_{\pm}) \sin \theta \cos \theta (\omega_+^2 - \omega_-^2 + 2(\delta_- \omega_- - \delta_+ \omega_+))}{\sqrt{m_1 m_2}(\omega_+^2 - \omega_-^2)(\omega_+^2 - \omega_-^2)}. \quad (3.20)
\]
The residues of Equation (3.20) are

\[
\operatorname{Res}_{\omega_+} \left( \frac{G_{21} - Z_{12}}{\Delta Z_{22}} \right) = -\frac{Z(\omega_\pm) \sin \theta \cos \theta}{2\sqrt{m_1 m_2 \omega_\pm}} (1 + \epsilon_2), \tag{3.21}
\]

\[
\operatorname{Res}_{\omega_-} \left( \frac{G_{21} - Z_{12}}{\Delta Z_{22}} \right) = \frac{Z(\omega_\pm) \sin \theta \cos \theta}{2\sqrt{m_1 m_2 \omega_-}} (1 + \epsilon_2), \tag{3.22}
\]

\[
\epsilon_2 = \frac{2(\delta_- \omega_- - \delta_+ \omega_+)}{\omega_+^2 - \omega_-^2}. \tag{3.23}
\]

Again, assuming frequency shifts of approximately 1 Hz, \( \epsilon_2 \) will be on the order of \( 5 \times 10^{-4} \). The measured post-excitation mode amplitudes are directly proportional to these residues,

\[
P_+ \propto \frac{Z(\omega_+) \sin \theta \cos \theta}{\sqrt{m_1 m_2 \omega_+}} (1 + \epsilon_2), \tag{3.24}
\]

\[
P_- \propto \frac{Z(\omega_-) \sin \theta \cos \theta}{\sqrt{m_1 m_2 \omega_-}} (1 + \epsilon_2). \tag{3.25}
\]

Although the absolute values of the amplitudes have changed slightly, their ratios should only differ by the small amount accounted for by the difference in frequency. Both modes should show nearly the same response to a calibration pulse. More than just the presence of the calibrator is needed to explain the fact that the minus mode is consistently 15 percent larger than the plus mode immediately after a calibration pulse is applied.

Finally, all of the changes described here for the case of an artificial gravity wave carry through to the case of an externally applied force. In this case, the \( Z_{12} \tilde{V}_c / Z_{22} \) term of Equation (3.1) is replaced by \( \tilde{F}_{\text{ext}} \). The same correction, \( \epsilon_2 \),
is present for both real and artificial gravity wave bursts, and $Z(\omega_{\pm})$ is still the constant of proportionally between voltage and force.

### 3.4 Effect of a Resonant Force Generator

If the resonance is in the force generator, there is a strong possibility that the force generator produces different forces at different frequencies, and the assumption that $Z(\omega_{+}) = Z(\omega_{-})$ must be abandoned. A function that describes the change in calibration constant as a function of frequency must be determined. As in the case of non-resonant mass loading, the various observable amplitudes can be calculated by evaluating the residues of the appropriate combination of Equations (3.1), (3.10), and (3.11). Under these conditions, the force generator voltage/antenna face motion transfer function which determines $\gamma_{\pm}$ now has the form

$$G_{11} \left( \frac{-Z_{12}}{Z_{22}} \right) = \frac{(\beta_2(\omega^2 - \omega_+^2 - 2\delta_\omega \omega_\tau) + \beta_3)(\cos^2 \theta(\omega_+^2 + 2\delta_\omega \omega_- - \omega^2) + \sin^2 \theta(\omega_+^2 + 2\delta_\omega \omega_+ - \omega^2))}{m_1(\omega^2 - \omega_+^2)(\omega_-^2 - \omega^2)(\omega_+^2 - \omega^2)}.$$  

(3.26)

The residues of Equation (3.26) at $\omega_+$ and $\omega_-$ are

$$\text{Res} \left( \frac{G_{11} - Z_{12}}{\Delta \frac{Z_{22}}{Z_{22}}} \right)_{\omega_+} = \left( \cos^2 \theta + \epsilon_1 \right) \left( \beta_2(1 - \epsilon_{3+}) + \frac{\beta_3}{(\omega_+^2 - \omega_+^2)} \right),$$  

(3.27)

$$\text{Res} \left( \frac{G_{11} - Z_{12}}{\Delta \frac{Z_{22}}{Z_{22}}} \right)_{\omega_-} = \left( \sin^2 \theta - \epsilon_1 \right) \left( \beta_2(1 - \epsilon_{3-}) + \frac{\beta_3}{(\omega_-^2 - \omega_-^2)} \right).$$  

(3.28)

Note that Equations (3.27) and (3.28) differ from Equations (3.15) and (3.16) only in the fact that $Z(\omega)$ has different values for different frequencies in the vicinity of the mystery mode frequency. Qualitatively, the motion of the antenna is no
different than in the non-resonant case. Driving the face of the antenna with a long wave train still sets the bar into simple harmonic oscillations. Only the fact that the ratio of force generated to voltage applied is no longer constant within the detection bandwidth is changed.

In the presence of the resonance, the new expressions for $\gamma_+$ and $\gamma_-$ are

$$\gamma_+ = \frac{(\cos^2 \theta + \epsilon_1)}{m_1} \left( \beta_2(1 - \epsilon_{3+}) + \frac{\beta_3}{(\omega_+^2 - \omega_i^2)} \right)^2 (1 + \epsilon_{3+}),$$  

$$\gamma_- = \frac{(\sin^2 \theta - \epsilon_1)}{m_1} \left( \beta_2(1 - \epsilon_{3-}) + \frac{\beta_3}{(\omega_-^2 - \omega_i^2)} \right)^2 (1 + \epsilon_{3-}),$$  

$$\epsilon_{3\pm} = \frac{2\delta_4 \omega_i}{\omega_{\pm}^2 - \omega_i^2}. \quad (3.31)$$

In terms of the residues, we now define the voltage-to-force constant to be

$$Z(U_{\pm}) = \beta_2(1 - \epsilon_{3\pm}) + \frac{\beta_3}{(\omega_{\pm}^2 - \omega_i^2)}. \quad (3.32)$$

Since $Z(\omega_+) \neq Z(\omega_-)$, the voltage-to-force constant is no longer directly proportional to the square root of the sum of $\gamma_+$ and $\gamma_-$. Since there are now three unknowns involved in these two measurements, additional information is required to complete the calibration.

The required information is obtained from the PEMAs. In the case of a single force generator resonance, the force generator voltage/transducer motion transfer function is
Table 3.1: Measured ALLEGRO parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Antenna mass</td>
<td>$m_1 = 1148 \text{ kg}$</td>
</tr>
<tr>
<td>plus mode frequency</td>
<td>$\omega_+ = 919.659 \text{ Hz}$</td>
</tr>
<tr>
<td>minus mode frequency</td>
<td>$\omega_- = 896.414 \text{ Hz}$</td>
</tr>
<tr>
<td>mystery mode frequency</td>
<td>$\omega_\gamma = 887.742 \text{ Hz}$</td>
</tr>
<tr>
<td>$\gamma_+ = (2.22 \pm 0.09) \times 10^{-14} \text{ mho/s}$</td>
<td></td>
</tr>
<tr>
<td>$\gamma_- = (1.17 \pm 0.05) \times 10^{-14} \text{ mho/s}$</td>
<td></td>
</tr>
<tr>
<td>plus/minus PEMA ratio</td>
<td>$P_+/P_- = 0.82 \pm 0.03$</td>
</tr>
<tr>
<td>minus/mystery PEMA ratio</td>
<td>$P_-/P_\gamma = 4.4 \pm 0.2$</td>
</tr>
<tr>
<td>plus mode generator constant</td>
<td>$Z(\omega_+) = (5.8 \pm 0.1) \times 10^{-6} \text{ N/V}$</td>
</tr>
<tr>
<td>minus mode generator constant</td>
<td>$Z(\omega_-) = (7.2 \pm 0.2) \times 10^{-6} \text{ N/V}$</td>
</tr>
<tr>
<td>2-mode mixing angle</td>
<td>$\theta = (30.8 \pm 2.2)^\circ$</td>
</tr>
<tr>
<td>uncoupled antenna frequency</td>
<td>$\omega_1 = (913.83 \pm 0.76) \text{ Hz}$</td>
</tr>
<tr>
<td>uncoupled transducer frequency</td>
<td>$\omega_2 = (902.58 \pm 0.79) \text{ Hz}$</td>
</tr>
<tr>
<td>transducer effective mass</td>
<td>$m_2 = (0.64 \pm 0.05) \text{ kg}$</td>
</tr>
</tbody>
</table>

\[
\frac{G_{21}(-Z_{12})}{\Delta} = \frac{\sin \theta \cos \theta (\beta_2(\omega^2 - \omega_+^2 - 2\delta_+\omega_\gamma) + \beta_3(\omega_+^2 - \omega^2 + 2(\delta_+\omega_+ - \delta_-\omega_-)))}{\sqrt{m_1m_2(\omega^2 - \omega_+^2)(\omega^2 - \omega^2)(\omega_+^2 - \omega^2)}},
\]

(3.33)

Again, as in the unloaded and the loaded-but-non-resonant cases, the residues are directly proportional to the PEMAs.

\[
P_+ \propto \frac{-\sin \theta \cos \theta}{\sqrt{m_1m_2\omega_+}} \left( \beta_2(1 - \epsilon_+) + \frac{\beta_3}{(\omega_+^2 - \omega_+^2)} \right) (1 + \epsilon_2),
\]

(3.34)

\[
P_- \propto \frac{\sin \theta \cos \theta}{\sqrt{m_1m_2\omega_-}} \left( \beta_2(1 - \epsilon_-) + \frac{\beta_3}{(\omega_-^2 - \omega_-^2)} \right) (1 + \epsilon_2).
\]

(3.35)

The values measured for the ALLEGRO system are shown in Table 3.1. All of the quantities need for calibration are determined. Note that there is no longer single voltage-to-force constant for the bar. Each mode has a separate voltage-to-force constant. This fact, however, is a property of the specific method used to apply
calibration signals. External forces not applied through the force generator still excite both modes approximately equally.

Due to the presence of the resonance in the force generator, the modified expressions for voltage-to-force constants and mixing angle are

\[ Z(\omega_+) = \sqrt{m_1 \left( \gamma_+ + \left( \frac{\omega_+ P_+}{\omega_- P_-} \right)^2 \gamma_- \right)}, \quad (3.36) \]

\[ Z(\omega_-) = \frac{\omega_- P_-}{\omega_+ P_+} Z(\omega_+), \quad (3.37) \]

\[ \tan \theta = \frac{\omega_+ P_+}{\omega_- P_-} \sqrt{\gamma_- / \gamma_+}. \quad (3.38) \]

The values of the parameters for ALLEGRO are shown in Table 3.1.

### 3.5 Noise Behavior

Figure 3.6 shows measured autocorrelation functions for the plus and minus modes. These plots were generated using one day’s worth of data (day 011 of 1998), decimated to a rate of one sample per second. There were no large burst events on this day. The absolute values on the y-axis of both graphs are accurate to approximately 10% due to uncertainties in the parameters needed to find the digital-units to transducer amplitude coefficient. The peak of each of these graphs is equal to the mean-squared displacement of the transducer in the absence of any large impulses applied to the bar.

The ratio of the mean-squared displacements is approximately 6.5, consistent with a mixing angle of 29.4°, within the experimental uncertainty of the driving
Figure 3.6: Autocorrelation functions of ALLEGRO’s noise in the plus and minus modes over day 004 of 1998. This day was chosen because no large bursts are present.
point measurement of $\theta$. Assuming that the noise force is given by the fluctuation-
dissipation theorem, this driving force has a spectral density of

\[ S_{nn}(\omega) = 4k_bT m_2 \omega / Q_2, \quad (3.39) \]

where $Q_2$ is the mechanical quality factor of the transducer, equal to approximately $1.5 \times 10^6$, and $T$ is the physical temperature of the bar, 4.2K. Given the set of antenna parameters determined from the three-mode model, Equations (2.52) and (2.53) predict values of $1.2 \times 10^{-31}$ m$^2$ in the plus mode and $6.0 \times 10^{-31}$ m$^2$ in the minus mode. The measured values are within the experimental uncertainty determined by uncertainties in the mixing-angle and the quality factor of the transducer.
Chapter 4

Establishing a Foundation of Bayesian Theory for Physical Scientists

In the physical sciences, the goal of an experiment is often to determine the value an unknown quantity with maximum accuracy. Quantities that can be directly measured, or "raw data", are used learn about quantities whose direct measurement is infeasible. In the field of gravitational wave astronomy, for example, the measured output from a network of resonant antennas can be used to learn about the flux of gravity-wave bursts incident upon the earth. After the experimental apparatus has been operated and the data has been purged of systematic errors, the problem of inferring an unknown quantity from the data is a statistical one. In this chapter a rigorous framework for a particular method of statistical inference, Bayesian inference, is outlined. Though this method has existed for almost 300 years, it has only slowly gained acceptance amongst physical scientists for reasons that are detailed below. This method proves to have many advantages for the analysis of gravitational wave burst data.
4.1 Bayes' Theorem

4.1.1 The Mathematical Expression

Let $\sigma$ represent some parameter that cannot be directly measured. Anything learned about $\sigma$ must be inferred from measurements of other quantities. Let $s$ represent some quantity that can be directly observed. $s$ may be the result of a single measurement, or the result of a series of measurements. Let quantities in the form $A(B|C)$ denote conditional probabilities, where the function $A$ is the probability of observing $B$ given that $C$ is true. Given these definitions, $P(\sigma|s)$ is "the probability that $\sigma$ is the true value of an unknown parameter given that $s$ is observed".

In the 18th century, the Reverend Thomas Bayes used the multiplicative law of probability to rewrite $P(\sigma|s)$ as [35]

$$P(\sigma|s) = \frac{Y(\sigma)L(s|\sigma)}{R(s)}.$$  \hspace{1cm} (4.1)

Equation (4.1) is called Bayes' theorem. To make calculations with Bayes' Theorem, there must be a known relationship between the measured quantity and the unknown parameter. This relationship, $L(s|\sigma)$, is called the likelihood function. The likelihood function is the probability that, given the parameter $\sigma$, a process described by $L(s|\sigma)$ produces a value of $s$. $P(\sigma|s)$ is called the posterior probability. Given an observation $s$, $P(\sigma|s)$ is the probability that any particular value of $\sigma$ is the true value of the unknown parameter. $R(s)$ is simply a normalization constant, which normalizes $P(\sigma|s)$ to a value of one [36].

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A posterior probability cannot be determined from a likelihood function without knowing the form of $Y(\sigma)$. It is the meaning and form of $Y(\sigma)$ that have kept Bayesian methods from gaining universal acceptance. Unlike the likelihood, there is no obvious prescription for determining $Y(\sigma)$. At this point, the philosophical questions regarding the meaning of $Y(\sigma)$ will be avoided; they are revisited in section 4.4. For the moment, we accept as an operational fact that performing a Bayesian analysis requires us to choose some function for $Y(\sigma)$. In the next several sections, we attempt to make the best possible choice for $Y(\sigma)$.

### 4.1.2 The Bayesian Interpretation

There are cases where the choice of $Y(\sigma)$ is obvious. Consider a radioactive counting experiment; given observation of a total number of counts over a period of time, what has been learned about the decay constant of the isotope? If no other information is provided, the observed number of counts is converted, by some prescription, into upper and lower limits for the decay constant. If, however, the experimenter knows beforehand that the isotope is one of a few possible choices, he reaches a different conclusion. Instead of providing limits, the additional information leads the experimenter to specify a single value for the decay constant that he thinks is most likely. A proper choice of $Y(\sigma)$ allows the experimenter to quantify the additional information and formally include it in his analysis.

The form of $Y(\sigma)$ which allows a discrete set of conclusions is a set of delta functions located at the appropriate number of counts. When multiplied by the likelihood, the relative strengths of the delta functions change. As illustrated in Figure 4.1, the delta-function closest to the peak of the likelihood function ends
Figure 4.1: Prior, likelihood, and posterior probabilities for a radioactive counting experiment where the final result can be only one of a few choices.

up with the largest posterior value. This gives formal expression to the intuitive notion that the option nearest to the observed number of counts should be chosen as the likely outcome of this experiment.

This example illustrates of the Bayesian interpretation of $Y(\sigma)$ as a prior probability. $Y(\sigma)$ is everything we know before we measure $L(s|\sigma)$. The prior is taken to contain everything known, from whatever source, about $\sigma$ independent of the likelihood. This interpretation of $Y(\sigma)$ gives Bayes’ Theorem an intuitively appealing form. What is known after performing an experiment (the posterior probability) is the result of combining what was known before (the prior probability) with what was learned from the data (the likelihood function).

An important idea in Bayesian analysis is the idea that probability represents a “state of mind” or “degree of certainty”. Sir Harold Jeffreys has rigorously
developed a set of postulates which quantitatively define these ideas [37]. The
important concepts are as follows. Before an event occurs, how certain we are of
its outcome can be described by a number between 0 and 1. 1 is certainty. 0 is
impossibility. A higher value for the probability then means we are more certain
of that event than an event with a lower value for probability. If we are absolutely
certain of the value of a parameter, then the prior is a delta function at the value,
and no amount of data can change our conclusion. This is shown in the counting
experiment where no amount of data can lead to a conclusion that the answer is
not one of the original choices.

Of course, in the physical sciences, cases where the results of an experiment
point to a few, discrete choices are rare. It is much more common that all quantities
can take a continuous range of values. In the continuous case, since our final state
of knowledge is equal to the product of the likelihood and the prior, the likelihood
must be sharper than the prior in order to add anything of significance to our
knowledge. In this case, the resulting posterior covers a narrower range than the
prior; analyzing the experiment has narrowed the possible range of values, and
what is known after the experiment is due largely to the data. As long as the
prior does not change quickly over the range where the likelihood is significant,
it does not exert much influence on the final result. This example represents the
case where data is readily available. Eventually, with lots of data, the likelihood
converges to some sharply peaked function. In this case, the results do not differ
from “maximum likelihood” methods, which completely ignore the prior.
There is also the possibility, however, that the prior is sharper than the likelihood. In this case, the results are more strongly dependent on the prior. Clearly, in this case, the form of the prior cannot be ignored. In this case, the conclusion reached is that doing the experiment did not tell us much more than was already known. This represents a case where data is hard to obtain.

The search for gravity waves is a case where data is hard to obtain. There exists a strong possibility that the data from any given run will not lead to any clear conclusion. Clearly, this is a case where the form of \( Y(\sigma) \) cannot be ignored. This makes significant the problem of choosing a function \( Y(\sigma) \) when little prior information is available. Bayesians refer to this as the problem of choosing a non-informative prior. In the next section, a criteria for judging the utility of a prior is specified.

4.2 The Meaning of a Confidence Interval

In general, a particular set of observations, \( s_0 \), does not fix a unique value of \( \sigma \). Instead, there is a range of \( \sigma \)-values that are consistent with the data. This range is usually expressed in terms of a confidence interval constructed according to some rule chosen by the experimenter. There is no limit on the number of possible rules to that could be chosen to narrow down the value of \( \sigma \) given \( s_0 \). For example, a simple order of magnitude estimate would be to declare that a result of \( s_0 \) means that the true value of \( \sigma \) is no more than \( 2s_0 \), and no less than \( s_0/2 \). Is this rule obviously inferior to a confidence interval constructed from a Bayesian posterior or a maximum-likelihood formalism? To answer this question, some criteria for selection of a meaningful confidence interval must be established.
The first step in evaluating the utility of a confidence interval is to define its "confidence level". For a normalized probability function, a confidence level of $A_1$ should mean that there is a $A$ percent chance that the true value of $\sigma$ lies between $\sigma_{lo}$ and $\sigma_{hi}$. (The difference between Bayesians and frequentists arises in defining exactly what an "$A_1$ percent chance" is). In a Bayesian analysis, confidence intervals and confidence levels are defined in terms of the posterior probability. For uni-modal posteriors, the interval with a confidence level of $A_1$ is usually defined by

$$A_1 = \int_{\sigma_{lo}}^{\sigma_{hi}} P(\sigma | s_0) d\sigma,$$

$$P(\sigma_{lo} | s_0) = P(\sigma_{hi} | s_0).$$

In a Bayesian formalism, where $P(\sigma | s_0)$ is read as "the probability that $\sigma$ is true given that $s_0$ is observed. Equations (4.2) and (4.3) are a direct, mathematical expression of this definition.

Figure 4.2 illustrates the quantities in Equations (4.2) and (4.3). For uni-modal posteriors, Equations (4.2) and (4.3) define unique set of $\sigma_{lo}$ and $\sigma_{hi}$ for each possible outcome $s_0$. A maximum likelihood formalism would use the same mathematical formula to define a confidence interval; the function $P(\sigma | s_0)$ would simply be replaced with the maximum likelihood estimator.

Given these definitions for confidence level and confidence interval, Equations (4.2) and (4.3) are not the only definition of the confidence interval that is available. Figure 4.3 shows $P(s | \sigma)$ curves for 5 different possible outcomes $s$. The dashed line shows the true value of $\sigma$, $\sigma_{true}$. If the measured value of $s$ is smaller than $s_2$, 82
or larger than $s_4$, the true value of $\sigma$ is outside of the confidence interval. Again, assuming that the posterior is uni-modal, the true value of $\sigma$ is contained within the confidence interval only when the measured value of $s$ lies between $s_2$ and $s_4$. That probability is given by the integral of the likelihood function at $\sigma_{\text{tru}}$ between $s_2$ and $s_4$, which will now be labeled $s_{lo}$ and $s_{hi}$, respectively.

$$A_2 = \int_{s_{lo}}^{s_{hi}} L(s|\sigma_{\text{tru}})ds. \quad (4.4)$$

Equation (4.4) is also an exact mathematical statement of the definition of a confidence interval. As defined in Equation (4.4), however, the value of $A_2$ is, in
Figure 4.3: Each possible observation $s$ is associated with a confidence interval. The dashed line represents the true value of the parameter being measured. Only observations between $s_2$ and $s_4$ will produce confidence intervals which contain the parameter.
general, different for different values of $\sigma_{\text{true}}$. Since the value of $\sigma_{\text{true}}$ is unknown when a statistical inference problem begins, $P(\sigma|s)$ must be carefully chosen so that the system of inference produces results independent of the particular value $\sigma_{\text{true}}$.

This criterion has a direct bearing on the choice of $Y(\sigma)$. $Y(\sigma)$ must be chosen to satisfy the condition

$$
\int_{\sigma_{\text{true}}}^{\sigma_{\text{hi}}} P(\sigma|s)d\sigma = \int_{\sigma_{\text{true}}}^{\sigma_{\text{hi}}} L(s|\sigma_{\text{true}})d\sigma,
$$

(4.5)

$$
\int_{\sigma_{\text{true}}}^{\sigma_{\text{lo}}} P(\sigma|s)d\sigma = \int_{\sigma_{\text{true}}}^{\sigma_{\text{lo}}} L(s|\sigma_{\text{true}})d\sigma,
$$

for any value of $\sigma_{\text{true}}$. In the next section, it is shown that, under certain conditions, there is a choice of $Y(\sigma)$ which does just this.

4.3 Jeffreys’ Prior and Fisher’s Information

4.3.1 Historical Part

Both Bayes and Laplace suggested the use of uniform distributions for $Y(\sigma)$ when no prior knowledge was available for an interval estimation problem. In 1922, the statistician R.A. Fisher offered a critique of this choice [38]. He pointed out that a Bayesian inference gives different results if, instead of determining $\sigma$, the problem is re-expressed in terms of some function of $\sigma$. He suggested that the result should be independent of reparameterization. In the same paper, however, he defined the quantity $(I(\sigma))$ that would come to be known as the Fisher Information Function. Given that $\langle F(s|\sigma) \rangle$ denotes the expectation value of the function $F$ over $s$,

$$
I(\sigma) = \left\langle \frac{\partial^2 \log L(s|\sigma)}{\partial \sigma^2} \right\rangle.
$$

(4.6)
In 1939 Sir Harold Jeffreys answered Fisher's critique. He seized on this idea of invariance. He suggested that a non-informative prior should not change when reparameterized [37, 39]. He pointed out that a probability function which satisfies this criteria is the square root of the Fisher information function, $Y(\sigma) \propto \sqrt{I(\sigma)}$. With the exception of the uniform prior, Jeffreys' prior is the most commonly used prior in Bayesian analysis when no prior information about $\sigma$ is available.

Kendall and Stuart also used a likelihood multiplied by the square root of Fisher's information function in describing interval estimation in their *Advanced Theory of Statistics* [40]. Their rationale is exactly the confidence-interval definition expressed in section 4.2. They, however, very clearly state that their system of inference is done "without reference to Bayes' postulate", and do not consider $\sqrt{I(\sigma)}$ to be a prior probability. Despite their declaration that they are not Bayesians, however, their analysis procedures are identical to using Bayes's theorem with Jeffreys' prior.

Box and Tiao are Bayesians, and do make use of the properties of the Fisher information function. They describe, in great detail, how using Jeffreys' prior leads to equal sized confidence intervals for any value of $\sigma_{\text{true}}$ [41]. They do not, however, offer any version of Equation (4.5) as a further rationale for the use of Jeffreys' prior.

Finally, though they will not be discussed in great detail in this work, there have been resolving power arguments made [42]. In brief, these arguments point out that the ability to distinguish between certain kinds of parameters depends on their scale. For example, it is much easier to distinguish a Poisson distribution of
mean 1 from one of mean 2 than it is to distinguish a Poisson distribution of mean 1001 from one of mean 1002. An appropriate quantitative statement of this can be used to derive Fisher's information function.

4.3.2 Computational Part

A set of Monte Carlo simulations was performed in order to demonstrate that Jeffreys' prior satisfies Equation (4.5). The computer randomly selects a value for an unknown parameter \( \sigma_{\text{tru}} \), and then selects a set of \( N \) random numbers from some distribution parameterized by \( \sigma_{\text{tru}} \). The \( N \) random numbers are used to determine a confidence interval with a confidence level of \( A_1 \). Many sets of \( N \) random numbers, all chosen from the distribution parameterized by the same value of \( \sigma_{\text{tru}} \) are generated, and the number of times the confidence interval contains \( \sigma_{\text{tru}} \) is tallied. If the mathematical definition of the confidence interval is equal to the conceptual definition, the confidence intervals should contain \( \sigma_{\text{tru}} \) in \( A_1 \) percent of the trials.

In this example, we examine the problem of inferring the standard deviation of a Gaussian distribution of known mean. The computer selected a standard deviation of \( \sigma_{\text{tru}} \) from a set of numbers distributed uniformly between 1 and \( 10^6 \). The computer then selected a set of \( N \), Gaussian distributed, random numbers with a standard deviation of \( \sigma_{\text{tru}} \) and a mean of zero. The likelihood of drawing a sample of \( N \) numbers, \( (n_1, n_2, \ldots, n_N) \), with a standard deviation of \( s \) from a Gaussian distribution with a mean of zero and standard deviation of \( \sigma_{\text{tru}} \) is

\[
L(s|\sigma_{\text{tru}}) = \frac{1}{(\sqrt{2\pi}\sigma_{\text{tru}})^N} \exp\left(-\frac{Ns^2}{2\sigma_{\text{tru}}^2}\right),
\]

(4.7)
where $s^2$ is the sample variance,

$$s^2 = \frac{\sum(n_i)^2}{N}.$$  \hspace{1cm} (4.8)

The simulation was performed with two different priors, a uniform prior and Jeffreys' prior. For the uniform prior, $Y(\sigma)$ is equal to a constant for all values of $\sigma$. Jeffreys' prior for a Gaussian likelihood, calculated from Equation (4.6), is $Y(\sigma) \propto \sigma^{-1}$. A small data sample of $N = 5$ was initially chosen to simulate the case where the likelihood is not so sharp as to overwhelm the prior. 1000 trials were performed under these conditions.

For each trial, the shortest confidence interval which included $\sigma_{\text{tru}}$ was calculated, yielding $A_{\text{min}}$, the smallest value of $A_1$ that would contain $\sigma_{\text{tru}}$. Given the conceptual definition of the confidence interval, for a given confidence level of $A_1$ (the confidence level 'quoted' as the reliability of the result), $A_{\text{min}}$ should be less than $A_1$ in $A_1$ percent of the results. In other words, if a set of 95% confidence intervals are constructed, 5% of the results should produce values of $A_{\text{min}}$ greater than 95%.

The results are shown in the form of a cumulative histogram in Figure 4.4. The dashed line shows the ideal case, where the assigned confidence interval exactly matches the simulated confidence interval. The result is somewhat surprising. Even though we know the initial distribution of $\sigma_{\text{tru}}$ is uniform, that is not the best choice for a prior. The prior that yields a result consistent with the definition of the confidence interval is Jeffreys' prior. This result points to the resolving power argument as being important.
Figure 4.4: Cumulative histogram of the results of the Gaussian variance inference problem. In this case, Jeffreys’ prior produces results consistent with the definition of the confidence interval. The uniform prior does not.

Figure 4.5 is for a simulation identical to Figure 4.4, except that, in this case, $N = 25$. At this point, the results of Jeffreys’ Prior and the uniform prior are nearly identical. This is because the likelihood for a sample size of 25 yields a much sharper likelihood. The information obtained in the likelihood is far more informative than the prior information.

Intuitively, one may be tempted to think that the correct answer may be to multiply Jeffreys’ prior by the initial distribution. Caution must be exercised in generalizing this result. The next plot, Figure 4.6, shows that this is not a good idea. In this case, instead of drawing standard deviations from a uniform distribution, the standard deviations themselves are gaussian distributed. This time, results are shown for 3 cases, the distribution prior, the Jeffreys’ prior, and
Figure 4.5: If enough data is used, Jeffreys' prior and the uniform prior converge to the same result.

the product of the two. The product prior gives the worse result. Both Jeffreys' prior alone and the distribution prior give confidence-definition consistent intervals. In this case, the distributional prior is superior because it consistently provides a shorter confidence interval.

It is obvious that the theoretical questions involving the prior are far from settled. It seems that $Y(\sigma)$ must contain information about the sampling distribution, as well as the resolving power of the statistic. Despite the lack of perfect theoretical understanding, we do have enough understanding of how our statistical tools should behave in order to determine priors in individual cases.
4.4 Avoiding Philosophy in Favor of Physics

To this point, references to the philosophical differences between Bayesians and frequentists have been kept to a minimum. Since, however, neither a pure Bayesian nor a pure frequentist will be satisfied with the previous section's justification for Jeffreys' prior, something must be said.

Physicists work very hard to express themselves in such a manner that avoids confusion arising in the nuances of language. This is why mathematics is the chosen language of physics. Unfortunately, even when using Jeffreys' postulates, the term probability cannot be defined without some reference to a subjective term such as "degree of certainty", or "state of mind". This appearance of subjectiveness has
propagated further into the Bayesian lingo, in the form of terms such as "information", "prior knowledge", and "non-informative prior". Quoting experimental results in terms of such non-physical concepts has rightly given physicists pause, and has been a primary cause of the slow acceptance of Bayesian methods in the physical sciences.

Physical scientists tend to be more comfortable with the operational definition of probability provided by frequentist theory. The frequentist says that the only meaningful definition of probability is a count of the frequency of the possible outcomes of repeated experiments. Even if an experiment can only be done one time, it must be imagined as one trial of many identical experiments. Although this provides a very concrete definition for probability, frequentists should remain aware that this is not always an accurate description of reality. In a large-scale experiment that may take several years to perform, there may never be runs that can be considered even approximately identical. The reality that there are two general classes of scientific experiment, those which are repeatable, and those which are not, cannot be avoided.

The greatest strength in frequentist interval estimation methods lies in the fact that there is no ambiguity in judging if their results are sensible. Given the definition of the confidence interval in section 4.2, repeated trials of an experiment can be simulated and checked to see that resulting confidence intervals contain the unknown parameter the expected amount of times. As shown by the simulations in section 4.3, frequentists must concede that multiplying the likelihood function by some other function is a valid way to define a confidence interval. Even though a
frequentist is loathe to call this other quantity a “prior probability”, his inferential process will follow exactly the same steps as a Bayesian analysis using Jeffreys’ prior.

Unfortunately, the pure Bayesian rejects the idea that multiple trials be used to justify the method. To a Bayesian, the posterior probability is a state of mind about single event, not the outcome of a set of fictional experiments. By this definition, the outcome cannot necessarily be simulated. The Bayesian, however, should not overlook the fact that there are cases simple enough to be simulated. In these cases, the frequentist version of the confidence interval is an acceptable, quantitative definition of a “state of mind”. The question now becomes should the same definition of state of mind apply to experiments which are not repeatable? Until the analysis differences between repeatable and non-repeatable experiments are somehow quantified, the same priors should be used for both. Thus, a new assumption has been added; procedures for a single trial of a repeatable experiment should be quantitatively identical to procedures for a single trial of a non-repeatable experiment.

There is probably less difference between the Bayesian and the frequentist outlook than either side would care to admit. Whether you choose to call the prescriptions which follow a Bayesian analysis with Jeffreys’ prior, or a maximum likelihood analysis modified by Fisher’s information function, the procedure and the results are the same. Indeed, arguments about the resolving power of a statistical distribution could probably provide the best framework about such counter-intuitive
notions as prior information having a specific form that depends upon the likelihood function used in analyzing a particular problem.
Chapter 5

A Bayesian Treatment of Coincidence Searches for Gravitational Wave Bursts

5.1 Introduction

Gravity waves are difficult to observe directly. Optimistic estimates predict that burst sources produce dimensionless strains on the order of $10^{-21}$ lasting for approximately a millisecond. In one detector, a single event of this nature is indistinguishable from the many non-gravitational excitations which dominate a real detector's output. For this reason, definitive detection of a burst event can only be achieved through the simultaneous excitation of multiple detectors. Previous coincidence searches involving the most recent generation of cryogenic resonant bar detectors have applied statistical methods to determine the expected behavior of detectors if no signal is present. These searches have taken two forms, tests that are sensitive to detection of more coincidences than are statistically probable [43, 44],

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and tests that are sensitive to an event or events coincident with an astrophysical
observation [45].

In the absence of signal, a certain number of coincidences occur between the random noise present in the detectors. These coincidences are called accidental coincidences. If enough gravity waves are present in the data, the observed number of coincidences will exceed the expected number of accidentals. In order to limit the number of accidental coincidences, arbitrary thresholding is often applied before a total coincidence search is undertaken. Statistically, it is not clear how to combine the results of searches performed with different thresholds.

In this chapter, we show that it is possible to design a more robust statistical test that is sensitive to a small number of true gravitational events against a background of noise. This test takes into account the distribution of observed coincidence burst energies in a rigorous statistical manner. A test incorporating this information can identify high-energy events that are unlikely to have occurred in uncorrelated detectors while still maintaining sensitivity to a larger excess of lower energy coincidences. The need to perform different analyses with varying thresholds is obviated.

The previous coincidence searches have relied primarily upon techniques of classical (or "frequentist") hypothesis testing. Beyond the difficulty of combining results from different thresholds, classical hypothesis testing suffers from a more fundamental limitation. Classical methods compare the observed data to a model based solely on detector noise characteristics. The form of a gravity wave signal is not quantitatively included in the analysis. Recently, in considering the detection
and measurement problems of interferometric gravitational wave detectors, Finn has demonstrated that a Bayesian formalism allows direct posing of questions about the presence of gravity waves in the data [46]. Ultimately, these are the questions of primary interest to the scientists involved in the search for gravity waves.

Dickson and Schutz have suggested several guidelines that should be followed in searches for coincident events between gravitational wave antennas [47]. A properly done Bayesian analysis implicitly incorporates three of these principles. Dickson and Schutz state that “the analysis methods used should be standard where possible, and that in any case, the statistics of the analysis methods should be well understood or explained, and clear enough to be questioned easily.” For this particular problem, all of the information can be reduced into a set of Poisson distributions, a commonly used and well understood statistical distribution. They state that “a clear model should be given and tested.” In a Bayesian analysis, quantitative models for both signal and noise must be explicitly chosen. Finally, they state that “once a new model has been postulated on the basis of a given data set, any new data should be analyzed in the same way as the original data were.” A Bayesian framework provides an obvious way for combining new data with the results of old experiments.
5.2 Defining the probable outcome of a coincidence experiment

5.2.1 Inhomogeneities in Time

Assume that two independent sources each generate a series of pulses randomly distributed in time. A coincidence occurs when one source emits a pulse within time $\Delta t$ of a pulse emission from the other source. If repeated trials of this experiment are performed, the probability of observing $C$ coincidences in a length of time $T$ is given by the Poisson distribution,

$$P(C) = \frac{A^C}{C!} e^{-A}, \quad (5.1)$$

where $A$ is the total number of coincidences expected in the experiment, given by,

$$A = \frac{MN \Delta t}{T}. \quad (5.2)$$

where $M$ events are generated by source 1, $N$ events are generated by source 2, and $T$ is the total observation time [48]. This assumption has been used in the previous searches for excess numbers of coincidences.

Using the totals $M$ and $N$ assumes that event rates in both detectors remain reasonably constant over the period of observation. If this condition does not hold, Equation (5.1) is not a rigorously accurate predictor of the number of coincidences [48]. In real detectors, event rates do vary with time, and Equation (5.1) must be modified. The expected number of coincidences should be found by multiplying event rates from each detector together, and then integrating the product.
Numerically, this integral can be approximated by dividing the data into a set of time bins, determining the expected number of coincidences in each bin, and summing the result. As long as the event rate within each time-bin is approximately uniform, Equation (5.2), with values of $M$, $N$, and $T$ calculated for each bin, yields an accurate prediction of the expected number of coincidences.

As an example of this, consider a simple two-bin case. For the first three months of operation, two antennas each detect approximately 1000 events per month. For the final nine months of the year, improved digital filtering techniques improve both detectors so that they produce only 500 events per month. Using Equation (5.2) to estimate the expected number of coincidences with total numbers of events $M = N = 7,500$ and a $\Delta t = 2$ second coincidence window yields an expected number of $A = 3.6$ coincidences. This underestimates the true number that should be expected. Taking into consideration the non-uniform event rate, 2.3 coincidences are expected in the first three months, and 1.7 coincidences are expected in the final nine months, for a total of $A = 4.0$ coincidences over the year.

5.2.2 Inhomogeneities in Energy

Event rates are not only inhomogeneous in time; they also vary as a function of energy. Unlike their distribution in time, the distribution of events as a function of energy is expected to vary. In the energy range dominated by thermal noise, the rate of accidental coincidences is high, perhaps exceeding one per day. If only those events whose burst energy lies well above the thermal noise are considered, the expected number of accidentals is very nearly zero. If the two detectors are
Figure 5.1: The results from a coincidence experiment can be sorted into a 3-dimensional bin structure. The x-axis is time, the y-axis is the energy deposited in the 1st detector, and the z-axis is the energy deposited in the 2nd detector.

truly uncorrelated, an excess of 1 or 2 events, though unlikely, still has a well-defined probability of occurrence. This fact can be utilized in order to design a test which is sensitive to a just a few high-energy coincidences while making use of all of the available data.

In a 2-detector experiment, each coincidence is specified by 3 parameters; the time of occurrence, the energy measured in detector 1, and the energy measured in detector 2. To develop a statistically robust procedure, any possible coincidence experiment result must be described in terms of this information. Each coincidence can be plotted as a point on the 3-dimensional graph shown in Figure 5.1. The x-axis is time of occurrence, the y-axis is energy deposited in detector 1, and the z-axis
is energy deposited in detector 2. Bin edges along each axis are chosen. In this way, the parameter space is completely divided up into 3-dimensional volumes. Each volume is an analysis bin containing coincidences which share similar properties. Each coincidence must fall into one and only one bin. The number of coincidences that fall into the ith bin is labeled \( C_i \). Given a total number of coincidences \( C \), the multinomial distribution describes the conditional probability, \( P(c_1, c_2, \ldots, c_k \mid C) \), of a particular configuration of \( c_i \)'s occurring in uncorrelated detectors [49],

\[
P(c_1, c_2, \ldots, c_k \mid C) = \frac{C!}{c_1!c_2!\cdots c_k!} p_1^{c_1} p_2^{c_2} \cdots p_k^{c_k}.
\]  

(5.3)

The \( p_i \)'s are the fraction of accidental coincidences expected to appear in bin \( i \). The method of calculating the \( p_i \)'s is discussed in section 5.3. Since the multinomial probability \( P(c_1, c_2, \ldots, c_k \mid C) \) is conditional upon the value of \( C \), the probability, \( P(c_1, c_2, \ldots, c_k) \), of any possible experimental output is given by the product of \( P(c_1, c_2, \ldots, c_k \mid C) \) and \( P(C) \),

\[
P(c_1, c_2, \ldots, c_k) = \frac{A^C e^{-A}}{c_1!c_2!\cdots c_k!} p_1^{c_1} p_2^{c_2} \cdots p_k^{c_k}.
\]  

(5.4)

An expected number of coincidences can be defined for bin \( i \) by multiplying the value of \( p_i \) by \( A \), the total number of expected coincidences. This expected value is labeled \( a_i \). The \( p_i \)'s from Equation (5.4) are eliminated using the \( a_i \)'s,

\[
a_i = Ap_i,
\]  

(5.5)
For any observed configuration of coincidences, the summation of the \( c_i \)'s over all the bins must be equal to \( C \). This fact is used to eliminate \( C \) from Equation (5.6).

\[
P(c_1, c_2, ..., c_k) = \frac{A^C e^{-A}}{c_1! c_2! ... c_k!} \frac{a_1^{c_1} a_2^{c_2} ... a_k^{c_k}}{A^{c_1} A^{c_2} ... A^{c_k}}. \tag{5.6}
\]

Finally, the sum of the expectations for each bin must equal the total expected number of coincidences. This fact is used to eliminate \( A \) from Equation (5.8),

\[
A = \sum_{i=1}^{k} a_i, \tag{5.9}
\]

\[
P(c_1, c_2, ..., c_k) = \left( \frac{a_1^{c_1}}{c_1!} e^{-a_1} \right) \left( \frac{a_2^{c_2}}{c_2!} e^{-a_2} \right) ... \left( \frac{a_k^{c_k}}{c_k!} e^{-a_k} \right). \tag{5.10}
\]

Equation (5.10) says that the probability associated with any particular configuration of coincidences is the product of the Poisson probabilities for each bin. Thus, if the expected number of coincidences can be determined for each bin, Equation (5.10) gives a statistically rigorous value for probability of any possible detector output.
5.3 Determining the Expected Configuration of Accidental Coincidences

Calculation of $p_i$ for each bin begins by determining of the expected time-distribution of accidental coincidences. As illustrated in Figure 5.2, the event lists from both detectors are divided into a set of $S$ time bins, each labeled with an index $\tau$. Time bin $\tau$ will contain $N_\tau$ events in detector 1 and $M_\tau$ events in detector 2. The expected number of coincidences in each bin $\tau$ can be calculated using Equation (5.2). If both data sets consist purely of random noise, the fraction of events, $f(\tau)$, expected to be detected in time-bin $\tau$ is determined by normalizing...
Figure 5.3: Each time bin is subdivided into a set of energy bins.

the sum of the expectation values to 1,

\[ f(\tau) = \frac{A_\tau}{\sum_{i=1}^{S} A_i}. \]  \hspace{1cm} (5.11)

The energies of the observed coincidences are incorporated into the analysis by subdividing the data in each time-bin according to energy, as shown in Figure 5.3. There are \( n_{\tau u} \) events in energy bin \( u \) of time-bin \( \tau \), and \( m_{\tau v} \) events in energy bin \( v \) of time-bin \( \tau \). If the data consists only of randomly distributed noise events, a first-detector single event of any energy is equally likely to pair with a second-detector single event and form a coincidence. Within time-bin \( \tau \), the fraction of accidental coincidences expected to be detected in energy-bin \( u \) is \( n_{\tau u} \) divided by
\( N_r \), the total number of events in time bin \( \tau \),

\[
e_1(u \mid \tau) = \frac{n_{ru}}{N_r}.
\]  

(5.12)

Similarly, for detector 2,

\[
e_2(v \mid \tau) = \frac{m_{rv}}{M_r}.
\]  

(5.13)

For an accidental coincidence, its detector 1 energy, and its detector 2 energy are statistically independent. The probability that an accidental coincidence from time-bin \( \tau \) has a detector 1 energy within energy-bin \( u \) and a detector 2 energy within energy-bin \( v \) is, therefore, simply the product of \( e_1(u \mid \tau) \) and \( e_2(v \mid \tau) \). After performing a coincidence search, the fraction of coincidences expected to lie in the bin labeled \( \tau \), \( u \), and \( v \) is

\[
P_i(\tau, u, v) = e_1(u \mid \tau)e_2(v \mid \tau)f(\tau).
\]  

(5.14)

In Equation (5.14), the dependence of \( i \) on \( \tau \), \( u \), and \( v \) has been explicitly written out. The only information needed to calculate the \( P_i(\tau, u, v) \)'s is the singles rate in each detector.

As an example of this, consider dividing the output of the 2 detectors described in the example of section 5.2 into 2 energy bins. In the each of the first three months, 750 of the 1000 events per month in the first detector are in the lower energy bin, and 600 of the 1000 events per month in the second detector are in the lower energy bin. This partitioning of the data defines a bin structure totalling 8 bins. There are two time bins, each of which bins is divided into 4 energy
bins. A coincidence may lie in the high energy bin of both detectors, the low energy bin of both detectors, the high energy bin of detector 1 and the low bin of detector 2, or the low bin of detector 1 and the high bin of detector 2. The fraction of observed coincidences expected to appear in the first nine months is 

\[ f(τ_1) = \frac{3000}{7500} = .4. \]

The probability that that coincidence had a low energy in detector 1 is 

\[ e_1(u_{lo}|τ_1) = \frac{2250}{3000} = .75. \]

The probability that the coincidence had a low energy in detector 2 is 

\[ e_2(v_{lo}|τ_1) = \frac{1800}{3000} = .6. \]

The probability that the sampled coincidence occurred in bin \( τ_1 u_{lo} v_{lo} \) is 

\[ p_{i(τ_1,u_{lo},v_{lo})} = .18. \]

Many different configurations of coincidences may be spread out over the bin structure. In the next two sections, two different ways relating the presence of gravity waves to the observed configuration, classical hypothesis testing and Bayesian interval estimation, are considered.

5.4 Classical Statistical Procedure

A single coincidence experiment produces a single value of \( P(c_1, c_2, ..., c_k) \). Knowledge of more than just the single, observed value of \( P(c_1, c_2, ..., c_k) \) is necessary to determine if gravity waves are present in an observed data set. One possible method of interpreting an experiment which produces a single value of \( P(c_1, c_2, ..., c_k) \) is to determine the parent distribution from which null results are drawn. An integral probability test is then used to assign a quantitative value to the probability that the observed configuration of coincidences resulted from pure noise with no signal present. A small value for this integrated probability implies one of two possibilities; either the statistical assumptions about the noise are not accurate, or physical coincidences are present in the data.

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The parent distribution of \( P(c_1, c_2, \ldots, c_k) \) in the absence of gravity waves can be determined from a Monte Carlo simulation of the experimental results. A random number is drawn from a Poisson distribution whose mean is equal to the expected number of coincidences. This number, \( C_{\text{sim}} \), simulates the possible total number of coincidences observed in a real experiment. \( C_{\text{sim}} \) coincidences are then randomly placed one-by-one into the bin structure described in section 5.2, according to the probabilities that were calculated in section 5.3. After each coincidence has been placed, \( P(c_1, c_2, \ldots, c_k) \) is determined for this Monte Carloed configuration using Equation (5.10) with values for \( a_i \) identical to the observed case. This procedure is repeated until a distribution of likely outcomes of \( P(c_1, c_2, \ldots, c_k) \) is determined.

The results of this procedure strongly depend on the particular choice of bin structure. Data from the 1991 and 1994 ALLEGRO-EXPLORER runs were used to generate distributions of \( P(c_1, c_2, \ldots, c_k) \) for two different bin structures. In the first case, a total of 63 bins were used, seven time bins, and three energy bins per detector. In the second case, 425 bins were used, 17 time bins, and 5 energy bins per detector. Distributions of \( P(c_1, c_2, \ldots, c_k) \) were generated, initially assuming uncorrelated detectors. In successive simulations, 1, 2, 3, and 4 additional high-energy coincidences were added to the data. In the simulations for both bin structures, additional coincidences were added to bins where the expected value was much less than one. In the 63-bin case, the coincidences were added to a bin where approximately .06 coincidences would be expected. In the 425 bin case, an identical 4 coincidences landed in bins with .005, .004, .002, and .001 expected coincidences. Figures 5.4 and 5.5 compare the resulting distributions. The +3 coincidence case
Figure 5.4: This figure shows the result of a frequentist analysis of the 1991-1994 ALLEGRO-EXPLORER data with 63 bins. The dashed plot shows the result of Monte Carlos assuming no real coincidences were present. The solid plots show the results when 1, 2, 3, and 4 coincidences were added to the highest energy bin.
Figure 5.5: This is a result of an analysis identical to that of Figure 5.4, except that a total of 425 bins was used. Note the +4 coincidence case. In the 63 bin case, there was almost no chance the noise alone could produce a logarithmic probability that would overlap. In this case, the +4 result may overlap the null result.
yields a particularly interesting result; in the 425 bin case, the mean value of of $P(c_1, c_2, ..., c_k)$ still has approximately a 90% chance of occurring in uncorrelated detectors, a result not statistically significant enough to yield a conclusive answer in a detection problem. In the 63 bin case, however, the integral probability is less than one hundredth of a percent. Yet, there is no clear indication that one bin structure is superior to the other. The analysis is, therefore, ambiguous.

5.5 Bayesian Procedure

5.5.1 Bayes’ Theorem

For a gravitational wave coincidence search, Bayes’ theorem is

$$P(G_0 \mid C) = \frac{Y(G_0)L(C \mid G_0)}{\int_0^\infty Y(G_0)L(C \mid G_0)dG_0}.$$  \hspace{1cm} (5.15)

$C$ refers to the set of observations $c_1, c_2, ..., c_k$. $P(G_0 \mid C)$ is the posterior probability, the probability that the incident rate of detectable gravity waves is $G_0$, given that a configuration of $C$ coincidences is observed. $Y(G_0)$ is the prior probability. $L(C \mid G_0)$ is the likelihood. Equation (5.15) states that the probability that $G_0$ is the true rate of detectable gravity waves is proportional to the product of the prior probability that the true rate is $G_0$ times the likelihood of observing $C$ given a rate of $G_0$. The resulting function of $G_0$ determines a Bayesian confidence interval [50], a range of possible $G_0$ values consistent with the observations.

To apply Equation (5.15), it is necessary to determine the likelihood function $L(C \mid G_0)$. Assuming that gravity wave bursts are single pulses that occur independently of one another, the total number of detectable events in a given period
of time is Poisson distributed. Assuming that the astrophysical processes responsible for gravity wave production do not change over the period of observation, the appearance of a gravity wave in any analysis bin has some fixed probability. The probability of any configuration of gravity waves is described with the multinomial distribution. As was shown in section 5.2, the product of a Poisson distributed total number times a multinomially distributed configuration is a set of independent Poisson distributions.

As in the case of noise, the probability of any signal configuration is given by a product of Poisson distributions. As the numbers of both signal and noise coincidences appearing in any bin are independent of the numbers appearing in any other bin, the likelihood $L(C \mid G_0)$ is given by

$$L(C \mid G_0) = \prod_{i=1}^{k} \lambda(c_i \mid g_i a_i), \quad (5.16)$$

where $g_i$ is the expected number of gravity waves in bin $i$. Since the processes which create signals are independent of the processes which create noise, the probability that $c_i$ coincidences appear in bin $i$ is given by

$$\lambda(c_i \mid g_i a_i) = \sum_{m=0}^{c_i} \frac{g_i^m}{m!} e^{-g_i} \frac{g_i^{c_i-m}}{(c_i-m)!} e^{-g_i}. \quad (5.17)$$

This equation states, for example, that 3 coincidences observed in bin $i$ could be the result of 3 accidentals, 2 accidentals and 1 gravity wave, 1 accidental and 2 gravity waves, or 3 gravity waves.
5.5.2 The Signal Configuration Likelihood

To use Equation (5.17), it is necessary to determine how a total of $G_0$ gravity waves would be divided amongst the bins into a set of $g_1, g_2, ..., g_k$. The number in each bin is proportional to $G_0$. The choice of a distribution in time is obvious; the probability of appearing in time bin $\tau$ is proportional to the duration of the bin, $\delta T_\tau$. In other words, longer intervals should contain more coincidences, and shorter intervals should contain fewer.

For energy, we assume a uniform and isotropic distribution of sources around the detector. We assume there is a maximum distance, $r_{\text{max}}$ that is observable from the network, and that $G_0$ gravity waves are produced within that distance. This implies that the number of sources within a distance $r$ of the detector is $\nu = G_0(r/r_{\text{max}})^3$. The function which describes the number of sources at a distance $r$ from the detector is

$$\frac{d\nu}{dr} = \frac{3G_0}{r_{\text{max}}^3}r^2. \quad (5.18)$$

Assuming a standard source strength, $E_S$, gravity waves from sources a distance of $r$ away deposit an energy of

$$E_G = \frac{E_S \sigma}{4\pi r^2} \quad (5.19)$$

in the detector, where $\sigma$ is a detector cross section that specifies the percentage of gravity wave energy deposited in the detector. Combining these two estimates, the function $d\nu/dE_G$ which describes the number of sources with an energy of $E_G$ is

$$N(E_G) = \frac{d\nu}{dE_G} = \frac{3G_0E_{\text{min}}^{3/2}}{2}E_G^{-5/2}. \quad (5.20)$$
The normalization implies that there are $G_0$ detectable sources in the range of the detector which produce signals which have an energy greater than $E_{\text{min}}$ when they reach the detector.

A gravity wave deposits equal amounts of energy in detectors of identical orientation. Naively, this implies that the detected burst energies in two antennas would be equal in the case of a true signal. Experiments and theory agree that because of the noise present in each detector, equal excitation of both detectors does not imply equal sized outputs. The probability that energies of $E_1$ and $E_2$ are simultaneously observed is the probability that an excitation of energy $E_G$ occurs times the probability that the excitation simultaneously produces $E_1$ in the first detector and $E_2$ in the second detector, integrated over all possible values of $E_G$,

$$S(E_1, E_2) = \int_0^\infty \chi_1(E_1|E_G)\chi_2(E_2|E_G)N(E_G)\,dE_G. \quad (5.21)$$

The endpoints of energy bin $u$ are labeled $E_{u-}$ and $E_{u+}$. The endpoints of energy bin $v$ are labeled $E_{v-}$ and $E_{v+}$. The probability that a gravity wave appears in analysis bins denoted by $u$ and $v$ is calculated by integrating Equation (5.21) over the appropriate energy ranges,

$$B(u, v) = \int_{E_{u-}}^{E_{u+}} \int_{E_{v-}}^{E_{v+}} S(E_1, E_2)\,dE_1\,dE_2. \quad (5.22)$$

Thus, in terms of $E_G$, the strength of the common excitation, the probability that a coincidence is observed residing in bin $u$ of detector 1 and bin $v$ of detector 2 is
given by the triple integral

$$B(u, v) = \int_0^\infty \int_{E_{u-}}^{E_{u+}} \int_{E_{v-}}^{E_{v+}} \chi_1(E_1|E_G)\chi_2(E_2|E_G)N(E_G)dE_2dE_1dE_G. \quad (5.23)$$

### 5.5.3 Details of the Signal Probability Integral

The energy detected at an antenna’s output, $E_{out}$, is distributed about the energy of the excitation, $E_{in}$, according to [30]

$$\chi(E_{out}|E_{in}) = \frac{1}{T_n} \exp \left( -\frac{E_{in} + E_{out}}{T_n} \right) I_1 \left( \frac{2\sqrt{E_{in}E_{out}}}{T_n} \right) \sqrt{\frac{E_{out}}{E_{in}}}, \quad (5.24)$$

where $T_n$ is the noise temperature of the detector, and $I_1$ is a modified Bessel function of first order. Since only bursts that are approximately 10 times greater than the noise temperature are declared to be candidates, $\sqrt{E_{in}E_{out}} > T_n/2$, and the asymptotic expansion of the Bessel function can be used,

$$I_1(x) \approx \frac{\exp(x)}{\sqrt{2\pi x}}, \quad (5.25)$$

$$\chi(E_{out}|E_{in}) \approx \frac{1}{\sqrt{4\pi T_n}} \exp \left[ -\frac{(\sqrt{E_{out}} - \sqrt{E_{in}})^2}{T_n} \right] \frac{E_{out}^{1/4}}{E_{in}^{3/4}}. \quad (5.26)$$

The energy “in” is $E_G$, the energy of an incident gravity wave. The energy “out” is $E_i$, the energy read out at detector $i$ as the result of the excitation $E_G$. Using this approximation, Equation (5.23) can be expressed as a single integral over $E_G$, which can be easily numerically integrated,
\[
B(u, v) = \frac{1}{4\pi \sqrt{T_1 T_2}} \int_0^\infty \frac{N(E_G)}{E_G^{3/2}} J_1(u, E_G) J_2(v, E_G) dE_G, \quad (5.27)
\]

\[
J_1(u, E_G) = \int_{E_{u-}}^{E_{u+}} E_1^{1/4} \exp \left[ -\frac{(\sqrt{E_1} - \sqrt{E_G})^2}{T} \right] dE_1. \quad (5.28)
\]

Defining \( \sigma_i^2 = T_i/2 \) and making the change of variables \( \xi_i = \sqrt{E_i} \), the integral over \( E_i \) from bin edges \( E_{u-} \) to \( E_{u+} \) is written as

\[
J_1(u, E_G) = 2 \int_{\sqrt{E_{u-}}}^{\sqrt{E_{u+}}} \xi_i^{3/2} \exp \left[ -\frac{(\xi_i - \sqrt{E_G})^2}{2\sigma_i^2} \right] d\xi_i. \quad (5.29)
\]

This integral can be done in closed form by approximating \( \xi_i^{3/2} \) with its Taylor series expansion about \( \sqrt{E_G} \). This point is chosen because it is the region where most of the contribution from the exponential term in Equation (5.29) comes from.

\[
\xi_i^{3/2} = E_G^{3/4} + \frac{3}{2} E_G^{1/4} (\xi_i - \sqrt{E_G}) + \frac{3}{8} E_G^{-1/4} (\xi_i - \sqrt{E_G})^2 + \ldots \quad (5.30)
\]

Inserting this expression into Equation (5.29), the resulting integral can be expressed in terms of error functions, exponential functions, and incomplete gamma functions,

\[
Q_{i\pm} = \frac{\sqrt{E_{i\pm}} - \sqrt{E_G}}{\sqrt{2}\sigma_i}, \quad (5.31)
\]
\[ J_1(u, E_G) = \sqrt{2\pi\sigma_1 E_G^{3/4}} \left( \text{erf}(Q_{u+}) - \text{erf}(Q_{u-}) \right) \]

\[ + 3\sigma_1^2 E_G^{1/4} \left( \exp(-Q_{u-}^2) - \exp(-Q_{u+}^2) \right) \]

\[ + \frac{3\sqrt{2}}{4} \sigma_1^3 E_G^{-1/4} \left( \gamma\left(\frac{3}{2}, Q_{u+}^2\right) - \gamma\left(\frac{3}{2}, Q_{u-}^2\right) \right) + \ldots, \]

where \( \gamma(\frac{3}{2}, f) \) is the incomplete gamma function of order \( 3/2 \) and argument \( f \). The remaining terms in the series are functions of incomplete gamma functions of increasing order.

In each bin, the expected number of gravity waves is now determined except for \( G_0 \), the overall rate of gravity wave production. For any rate of \( G_0 \) that is hypothesized, \( g_i(\tau, u, v) \), the number of coincidences expected to fall in the bin labeled with \( \tau, u \), and \( v \) is

\[ g_i(\tau, u, v) = G_0 \frac{\delta T_\tau}{T} B(u, v). \]

### 5.5.4 Choice of Prior

To apply the Bayesian formalism, it is necessary to choose a prior probability. The prior contains everything known about the true value of \( G_0 \) before the set of observations \( C \) is made. In searches for gravity wave bursts there is no quantitative reason to favor one value of \( G_0 \) over another before the data is actually analyzed. This total lack of knowledge about the true value of \( G_0 \) must be expressed in the form of a prior probability that contains no information about the parameter. Using the criteria established in the previous chapter, simulations using a bin structure and expected numbers of accidentals identical to the real data showed that the
uniform prior gave results consistent with the definition of confidence intervals for this likelihood.

5.6 Results and Discussion

Using the 1991 and 1994 data, \( P(G_0 \mid C) \) is calculated for values of \( G_0 \) ranging from \( 10^{-3} \) to \( 10^2 \). Equations (5.16) and (5.17) are the likelihood. The prior chosen is uniform. The same 63-bin structure used in the frequentist analysis in section 5.4 is used. Posterior probabilities plotted against trial values for \( G_0 \) are shown in Figure 5.6. The integral of the posterior probability must equal one, since some value must be the true one. The concentration of probability density determines a range where the true value of \( G_0 \) is most likely located. This particular plot indicates that there is nothing in the data beyond accidental coincidences. As \( G_0 \) approaches zero, the probability that smaller and smaller values of \( G_0 \) are correct increases.

The 95% confidence level for a dimensionless strain threshold of \( 2.3 \times 10^{-18} \) is 9 events per year. This means that if the mean rate of gravity wave production was greater than 9 per year, it would be extremely unlikely that this few coincidences would have been observed.

To demonstrate how gravity waves would manifest themselves in this analysis, the analysis was redone with additional coincidences in the data. The placement of additional coincidences was identical to what was done in the analysis in section 5.4. Figure 5.7 shows the results. For each coincidence that is added, the probability that \( G_0 = 0 \) becomes smaller and smaller, eventually becoming negligible. A useful way to quantify this effect is to compare the value of the posterior probability
Figure 5.6: $P(G_0|C)$, the probability that $G_0$ is the true rate of gravity waves using the data acquired from the ALLEGRO and EXPLORER detectors in 1991 and 1994. The 95% confidence upper limit set by this posterior distribution is 9 events per year.
Figure 5.7: A plot of the posterior probability $P(G_0|C)$ for the combined 1991 and 1994 ALLEGRO-EXPLORER with added coincidences data sets using the same 63 bins employed in the frequentist analysis.
function at the peak of \( P(G_0 \mid C) \) to its value at the smallest value of \( G_0 \) tested. In the case of +1 coincidence, the ratio of the peak value to the value at \( G_0 = 10^{-3} \) is only 2:1. By the time 4 coincidences have been added, the peak-to-zero ratio is approximately 25,000 : 1. This would indicate that zero is not a good choice for the true value of \( G_0 \), and that a physically significant set of coincidences are present in the data.

The greatest advantage of a Bayesian analysis becomes obvious when the analysis is redone using a different bin structure. Figure 5.8 shows the result when 425 bins are used instead of 63. In sharp contrast to the frequentist procedure, the results do not change substantially when the number of bins is varied. Furthermore, the changes that do occur have a well-defined cause; the changes reflect the degree to which different bin-structures are consistent with the assumption that the Poisson means can be determined by counting the total number of events within a bin. The fact that a many bin analysis can be done without altering the resolution of the test is very important, as the practice of throwing away large amounts of data can be replaced by selecting bins in a manner that minimizes the effects of spurious effects created by noisy periods in the data.

Finally, it should be noted that Bayesian analysis provides a natural way of combining data from different runs. The posterior probability of one experiment can be used as the prior probability of the next experiment. Since the result expressed solely as a function of \( G_0 \), the details of the detection technology do not matter as long as the same model of the signal is maintained from experiment-to-experiment.
Figure 5.8: Posterior probabilities for the same analysis as in the previous plot, but using 425 bins instead of 63. Note that the results are much less dependent on bin structure than in the frequentist case.
5.7 Conclusion

The probability of any configuration of accidental coincidences occurring between two detectors can rigorously be shown to be the product of a set of Poisson distributions. The means of the Poisson distributions can easily be determined from the data. Both classical and Bayesian methods were used to interpret this probability. Because the results of a Bayesian analysis do not strongly depend on the details of how the data is binned, the Bayesian method proves to be a more robust form of analysis. This result, rather than any philosophical argument regarding the nature of probability, is the most important reason for giving Bayesian methods serious consideration.

It is not suggested that the particular models plugged into the Bayesian formalism in this chapter are the final word in gravitational wave data analysis. What is important is the fact that the Bayesian approach allows unambiguous expression of all assumptions that have been made once models for signal and noise are selected. Both stationary and non-stationary noise sources are quantitatively treated in this analysis. Finally, the result is not merely a statement about whether the observed data was anomalous. The Bayesian confidence interval provides a direct expression of the probability that gravity waves are present in the data which, ultimately, is the quantity of interest to researchers working to detect gravitational radiation.

Gravity wave detection will always be a difficult endeavor. The construction of experiments in this field is measured in terms of decades. It is thus of the utmost importance that the data analysis techniques make use of all of the data which is available. This means comparing data from radically different technologies, from
giant aluminum spheres to space-based interferometers. Bayesian methods, which report their results in quantities independent of the technology used to acquire the data, are well suited to this task.
References


[34] Shlomo Karni, *Network Theory: Analysis and Synthesis*, Chapter 3 (Allyn and Bacon, Boston, 1966)


Appendix

The General Elastic Body

There is a "standard model" widely used to describe the principles of operation of gravitational wave detectors. Though we are trying to measure a result of relativistic effects, the motion of the antenna itself is described by classical mechanics. In the laboratory frame the interaction of a resonant antenna with the weak field of a gravitational wave is equivalent to a system of coupled harmonic oscillators driven by a contact force. The purpose of presenting a detailed outline of this model is twofold. First, we wish to present the theoretical basis for treating the effect of a gravity wave on an extended object as a contact force in the laboratory frame. Second, we wish to offer a brief demonstration of the validity of the harmonic oscillator models of the detector system's response. This argument is a restatement of the work most recently articulated by Merkowitz and Johnson [15].
A.1 Interaction of a Gravity Wave with an Extended Object

The theory of general relativity predicts the existence of gravitational waves. More rigorously, general relativity allows the propagation of space-time curvature as traveling waves, mathematically analogous to the propagation electromagnetic waves. The physical effects of such a passing disturbance are described by the Riemann curvature tensor $R_{\kappa\lambda\mu\nu}$.

The simplest conceptual detector is two massive particles in free fall with a very small relative initial velocity, and a sensor to repeatedly measure the physical distance $x_i(t)$ between them. $x_i(t)$ changes linearly with time due to the ordinary inertial motion, but only has a second derivative, or geodesic deviation, if the curvature is non-zero. This deviation behaves like the acceleration of Newtonian mechanics, obeying the equation

$$\ddot{x}_i(t) = -\sum_j R_{\kappa\alpha j}x_j = \sum_j \frac{1}{2}\dddot{h}_{ij}(t)x_j,$$

where the second equality follows from the use of the tensor function $h_{ij}(t)$ (technically the metric deviation in the transverse traceless gauge) to describe the wave field. The double dots indicate second time derivative. We assume that particle separation is small compared to the gravitational wavelength.

This equation has a natural physical interpretation. If we integrate the time dependence, we find that the change in separation $\Delta l(t)$ between the particles is
given by
\[ \frac{\Delta l(t)}{l} = \sum_j \frac{1}{2} h_{ij}(t) e_j, \]  
(A.2)

where \( e \) is a unit vector pointing from one particle to another, and \( l \) is the initial separation. The function \( h_{ij}/2 \) is exactly equivalent to the strain tensor of ordinary mechanics. For conventional reasons, \( h \) is referred to as the (gravitational) strain.

If we interpret \( x_i(t) \) as the coordinate of test particles, in a system where the coordinates measure the physical distances, the equation of geodesic deviation can be interpreted as showing that the wave field produces an effective force on a particle of mass \( m \) given by
\[ F_i^G = m \sum_j \frac{1}{2} h_{ij}(t)x_j, \]  
(A.3)

so that we have a Newtonian equation of motion for each particle,
\[ m\ddot{x}_i(t) = F_i^G, \]  
(A.4)

and \( F^G \) has the linear variation with position of a "tidal" force.

**A.2 Response of an Elastic Object to a Force**

Non-gravitational physical effects also cause accelerations of test particles, and we must specify how the two effects are to be combined. It is universally assumed that gravitational and non-gravitational forces \( F_i^{NG} \) can be superposed in this coordinate system, so that the mass elements of any detector obey superposition
of gravitational and non-gravitational forces,

\[ m\dddot{x}_i(t) = F_i^G + F_i^{NG}. \quad (A.5) \]

The dominant non-gravitational force on the particles of a solid body are the molecular forces described by elastic mechanics. The second derivative of the particle displacement \( u \equiv x(t) - x(0) \) is equal to \( \ddot{x}(t) \), and the net force elastic force on the mass element \( m = \rho dV \) by its neighbors is found to be \( (\lambda + \mu) \nabla (\nabla \cdot u) + \mu \nabla^2 u \), where \( \lambda \) and \( \mu \) are the Lame form of the elastic moduli of the material. Therefore the complete equation of motion for an elastic body, under the action of a gravitational force density \( f_i^{GW} \),

\[ f_i^{GW} = \frac{\rho}{2} \sum_j \ddot{h}_{ij} x_i, \quad (A.6) \]

plus any other external force density \( f_i^{X} \) is

\[ \ddot{u}_i = (\lambda + \mu) \nabla_i (\nabla \cdot u) + \mu \nabla^2 u_i + f_i^{GW} + f_i^{X}. \quad (A.7) \]

It is well known that this equation has an exact solution, for arbitrary forces, in terms of separable functions, the eigenfunctions \( \Psi_m(x) \), and the eigenmode amplitudes \( a_m(t) \),

\[ u = \sum_m a_m(t) \Psi_m(x). \quad (A.8) \]
The eigenfunctions are determined by the shape of the body. The mode amplitudes $a_m(t)$ must solve a driven harmonic oscillator equation with the driving forces given by the "overlap" of the total force density function with the eigenfunction.

If consider the bar alone (without resonator), and we find ourselves restricted to a frequency band near the 1st mode frequency, then it should be a good approximation to keep only this one term in the expansion for $u$. In other words, the displacement of the end face of the bar $u(x = L/2, t)$ will be equal to $a_1(t)$ to a good approximation (assuming we normalize its eigenfunction so that $\Psi_1(x = L/2) = 1$). The mode amplitude $a_1$ is a "collective coordinate", representing coherent motion of the entire bar with spatial variation given by its eigenfunction.

The fundamental equation for a simple resonant antenna (e.g., a bar without resonator) becomes the harmonic oscillator equation for the first mode. It is

$$a_1(t) + \omega_1^2 a_1(t) = \frac{1}{M} \frac{1}{N_1} \int (f^GW + f^X) \cdot \Psi_1 d^2 x,$$

(A.9)

where $\omega_1$ is the mode angular frequency, $M$ is the physical mass of the body, and $N_1$ is the mode normalization constant $N_1 \equiv \int \Psi_1 \cdot \Psi_1 d^2 x$. The normalization constant is fixed by our desire to identify $a_1$ with the displacement of the end face in an inertial coordinate system, and found for a cylinder to be approximately one half of the volume. The effective mass $m_1$ is therefore defined to be $M/2$ so that the effective force $F_1(t)$ on the mode is equal numerically to a real force applied to the end of the bar.
Integrating the force densities over the spatial coordinates, we obtain the equation of motion of a forced harmonic oscillator,

\[ a_1(t) + \omega_1^2 a_1(t) \equiv \frac{1}{m_1} F_1(t). \quad \text{(A.10)} \]

An equivalent description is found by Fourier transformation,

\[ \tilde{a}_1(\omega) = \frac{1}{m_1(\omega_1^2 - \omega^2)} \tilde{F}_1(\omega), \quad \text{(A.11)} \]

where tildes indicate the Fourier transform.
Vita

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