1997

Linear Matrix Inequality Approach to Robust Emergency Lateral Control of a Highway Vehicle With Time-Varying Uncertainties.

Robert Edward Benton Jr

Louisiana State University and Agricultural & Mechanical College

Follow this and additional works at: https://digitalcommons.lsu.edu/gradschool_disstheses

Recommended Citation

https://digitalcommons.lsu.edu/gradschool_disstheses/6383

This Dissertation is brought to you for free and open access by the Graduate School at LSU Digital Commons. It has been accepted for inclusion in LSU Historical Dissertations and Theses by an authorized administrator of LSU Digital Commons. For more information, please contact gradetd@lsu.edu.
INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI
A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor MI 48106-1346 USA
313/761-4700 800/521-0600

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
LINEAR MATRIX INEQUALITY APPROACH TO
ROBUST EMERGENCY LATERAL CONTROL OF
A HIGHWAY VEHICLE WITH TIME-VARYING
UNCERTAINTIES

A Dissertation
Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mechanical Engineering

by
Robert Edward Benton, Jr.
B.S., Louisiana State University, 1991
May 1997
Acknowledgements

Although understanding into mathematics is required for much of the material in this dissertation, dead science did not inspire its writing. Only the love of my precious wife Michelle could have lead me to such an accomplishment. Her kindness and encouragement during my graduate school career have been a gift that words cannot repay. Only a dullard could have such a friend for his wife and not spend every hour praising and rejoicing over her in song.

Obviously, my Creator God is due my thanksgiving for my very existence, but I owe a much greater debt to Him. My Heavenly Father is completely satisfied with the sacrifice of His Son, Jesus Christ, on my behalf. No wrath is left for me. God’s Jesus-based acceptance of me allows His life to flow through me and frees me to let Him love the world through me. I would like to thank the Lord for His life in me and for actively writing this dissertation through me.

I would also like to thank my parents for raising me to think for myself. If I were measuring others opinions before forming my own, I would not have been able to conceive that this research were even possible. Although every family has its difficulties, I am convinced that the Lord could have chosen no better parents for me. Their support has proven invaluable. I also wish to thank my sister Liz for all of her encouragement, especially during my freshman year at Louisiana Tech.

Next, I would like to thank my committee Dr. Dirk Smith, Dr. Kemin Zhou, Dr. Warren Waggenspack, Dr. Mehdy Sabbaghian, Dr. Michael Murphy, and Dr. John Hildebrant for taking time to serve on my committee. Specifically, I
would like to thank Dr. Smith for his guidance and inspiration during his tenure as my major professor. He has allowed me the academic freedom to discover my own solutions and given wise counsel when necessary. In addition to my committee members, many professors from whom I have had the privilege to learn are deserving of my gratitude.

Various members of the Interactive Modelling Research Lab (IMR Lab) and the Center for Advanced Technology in Medicine Lab (CATiM Lab) have helped me along the way to my graduation. I owe a debt for the advice, hints, and friendship of each of these fellow students. Among them are Henry Lamousin, Greg Dobson, Scot Jones, James Frens, and Kishore Bayyapureddy.

Finally, I would like to thank the LSU Board of Regents for the Dean's Fellowship that I received during my first four years of graduate school. The support of the Louisiana Transportation Research Council through LTRC Grant No. 96-6TIRE is also greatly appreciated.
Contents

Acknowledgements .......................................................................................... ii

List of Tables ...................................................................................................... vi

List of Figures ...................................................................................................... viii

Abstract ................................................................................................................ x

1 Introduction ................................................................................................... 1
   1.1 Vehicle and Tire Model ........................................................................... 2
   1.2 Time-Varying Uncertainties in Vehicle Models .................................. 5
   1.3 Summary ................................................................................................ 7

2 Literature Review ....................................................................................... 11
   2.1 Time-Varying Systems .......................................................................... 11
   2.2 Robustness ............................................................................................. 16
      2.2.1 Matching Conditions ........................................................................ 16
      2.2.2 Unstructured Uncertainties ......................................................... 17
      2.2.3 Interval Matrices ............................................................................ 18
      2.2.4 Polytopes of Matrices ...................................................................... 21
      2.2.5 Norm-Bounded Uncertainties ...................................................... 22
      2.2.6 Extended Systems .......................................................................... 23
   2.3 Optimal Control ................................................................................... 24
      2.3.1 $\mathcal{H}_2$ Control ......................................................................... 25
      2.3.2 $\mathcal{H}_\infty$ Control ......................................................................... 26
      2.3.3 Other Ideas of Optimality ........................................................... 28
   2.4 Eigenvalue Placement ............................................................................. 29
   2.5 Output Feedback ................................................................................... 31
   2.6 Linear Matrix Inequalities .................................................................... 33

3 Preliminary Work ........................................................................................... 35
   3.1 Nonlinear-Gain-Optimized Controller with Four-Wheel-Steering ...... 36
      3.1.1 NGO Design Technique ............................................................... 36
      3.1.2 NGO 4WS-GE Performance ....................................................... 39
      3.1.3 NGO 4WS-GE Robustness ........................................................... 41

iv
3.1.4 Benefits of NGO and Suggested Improvements .......................... 45
3.2 Numerical-Eigenvalue-Optimization Method ............................... 47
  3.2.1 NEO Design Technique ......................................................... 48
  3.2.2 Comparison of NEO to a Previous Method ............................. 51
  3.2.3 NEO Design for a Cart with an Inverted Pendulum ................. 56
  3.2.4 Benefits of NEO and Suggested Improvements ....................... 59

4 Output-Feedback Stabilization .................................................... 62
  4.1 Output-Feedback Stabilization ................................................. 64
    4.1.1 Preliminaries ..................................................................... 66
    4.1.2 A New Statement of the Problem ...................................... 68
    4.1.3 Some Useful Lemmas ....................................................... 69
    4.1.4 Definitions ....................................................................... 72
    4.1.5 Algorithm ........................................................................ 74
    4.1.6 Examples ........................................................................ 75
    4.1.7 Summary .......................................................................... 81
  4.2 Reducing Feedback Gains via Linear Matrix Inequalities .............. 82
    4.2.1 Quadratic Minimization using LMI's .................................. 84
    4.2.2 Lyapunov Inequalities and Similarity Transformations .......... 86
    4.2.3 Algorithm ....................................................................... 88
    4.2.4 Examples ....................................................................... 90
    4.2.5 Summary ....................................................................... 93

5 Robust Stabilization ...................................................................... 94
  5.1 Using Polytopes to Describe Time-Varying Systems .................... 95
  5.2 Robust State-Feedback Stabilization ........................................ 98
  5.3 Lyapunov Inequalities and Similarity Transforms, Revisited .......... 98
  5.4 Robust Output-Feedback Stabilization ...................................... 101
  5.5 Reducing Robust Output-Feedback Gains .................................. 104
  5.6 Examples ............................................................................... 106
  5.7 Summary .............................................................................. 113

6 Emergency Lateral Control .......................................................... 115
  6.1 The Vehicle Control Problem .................................................. 115
  6.2 Controller Design .................................................................... 120
  6.3 Controller Performance .......................................................... 125
  6.4 Robust Emergency Lateral Control ........................................... 134

7 Summary and Future Research ..................................................... 139
  7.1 Summary of Static-Output-Feedback Stabilization ..................... 139
  7.2 Summary of Robust Static-Output-Feedback Stabilization .......... 140
  7.3 Robust Emergency Lateral Control of A Highway Vehicle ........... 142
  7.4 Future Directions .................................................................. 143

References ................................................................................. 145
A Using the Ellipsoid Algorithm to Solve LMI problems .................................. 163
B Polytopic Vertices of the Time-Varying Vehicle ........................................ 165
C Simulation Parameters .................................................................................. 169
Vita ..................................................................................................................... 176
List of Tables

1.1 Symbols Used in Vehicle Model ............................................................ 3
1.2 Parameter Uncertainties ....................................................................... 8
2.1 Matrix Measures and Induced Norms for Various Norms .................... 15
3.1 Variables for the Cart with Inverted Pendulum ............................... 58
3.2 Parameters for the Cart with Inverted Pendulum ........................... 59
4.1 Parameters Used in Vehicle Model ..................................................... 80
5.1 Open-Loop Eigenvalues ..................................................................... 112
5.2 Eigenvalues for $K = [0.2017243 \; 9.721544]^T$ ................................. 112
5.3 Eigenvalues for $K = [-0.1010633 \; 1.967605]^T$ .......................... 112
6.1 Vehicle Model Data for a 1992 Ford Taurus ..................................... 117
6.2 Additional Vehicle Model Data ......................................................... 118
6.3 Parameter Values for Vehicle System Vertices ................................ 121
6.4 Open-Loop Eigenvalues for the Vehicle .......................................... 121
6.5 Closed-Loop Eigenvalues for the Vehicle ........................................ 124
6.6 Effects of $\mu$ on Controller Performance ...................................... 131
6.7 Effects of $C_{af}$ and $C_{ar}$ on Controller Performance ...................... 134
C.1 Vehicle Model Data for a 1992 Ford Taurus .................................... 170
C.3 Vehicle Dimensions ....................................................................... 171
C.4 Vehicle Mass and Inertia ................................................................. 171
C.5 Vehicle Engine and Transmission ................................................... 172
List of Figures

1.1 Yaw-Plane Model of Vehicle Dynamics ............................................. 2
2.1 An Interconnected Feedback Loop ..................................................... 18
2.2 Region H in the s-plane ................................................................. 31
3.1 Control Gain Equations for 4WS-GE Controller ............................... 38
3.2 Dropped-throttle Step-Lane-Change Responses at $U_0 = 15$ m/s .... 40
3.3 Dropped-throttle Step-Lane-Change Responses at $U_0 = 30$ m/s .... 42
3.4 Travel Distances for Dropped-Throttle Step-Lane-Change Maneuvers 43
3.5 Robust Performance with respect to $C_\alpha$ .................................... 44
3.6 Robust Performance with respect to $\mu$ .......................................... 45
3.7 Robust Performance with respect to Additional Vehicle Mass ........ 46
3.8 Region H in the s-plane ............................................................... 48
3.9 Comparison of Eigenvalue Placement for Shieh and NEO Methods .... 54
3.10 Initial Condition Responses for Shieh and NEO methods ............... 55
3.11 Cart with Inverted Pendulum ..................................................... 56
3.12 Response of the System to $\theta_0 = \frac{\pi}{18}$ (10°) ........................... 60
3.13 Response of the System to $x_0 = 1$ ............................................. 61
4.1 A Typical Closed-Loop Control System ........................................... 62
4.2 Yaw-Plane Model of Vehicle Dynamics ............................................ 79
6.1 Yaw-Plane Model of Vehicle Dynamics ............................................ 117
6.2 Lateral Tire Forces ................................................................. 119
6.3 Responses for $U_0 = 30$ m/s, $C_{af,e} = 30$ kN/rad, and various $\mu$ values 126

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
6.4 Responses for \( U_0 = 25 \text{ m/s}, \quad C_{af,r} = 30 \text{ kN/rad}, \) and various \( \mu \) values 128

6.5 Responses for \( U_0 = 20 \text{ m/s}, \quad C_{af,r} = 30 \text{ kN/rad}, \) and various \( \mu \) values 129

6.6 Responses for \( U_0 = 15 \text{ m/s}, \quad C_{af,r} = 30 \text{ kN/rad}, \) and various \( \mu \) values 130

6.7 Responses for \( U_0 = 30 \text{ m/s}, \) various \( C_{af,r} \) values, and \( \mu = 0.75 \ldots 132 \)

6.8 Responses for \( U_0 = 15 \text{ m/s}, \) various \( C_{af,r} \) values, and \( \mu = 0.75 \ldots 133 \)

6.9 Worst-Case Double-Lane-Change Responses for \( U_0 = 30 \text{ m/s} \ldots 135 \)

6.10 Worst-Case Double-Lane-Change Responses for \( U_0 = 15 \text{ m/s} \ldots 136 \)

\( x \)
Abstract

New linear-matrix-inequality (LMI) based methods are developed for the static-output-feedback stabilization and reduced-gain static-output-feedback stabilization of time-invariant systems. Unlike previous methods, the static-output-feedback method is non-iterative in LMI solutions. The methods are extended to design robust static-output-feedback controllers for time-varying systems using a polytopic-systems approach. Examples are given which demonstrate the use of each of the new methods. The specific problem of emergency lateral control of a highway vehicle is then addressed using the new robust static-output-feedback method. A controller is designed which robustly stabilizes the vehicle over the range of highway speeds (15 to 30 m/s) and a range of expected independent changes in front and rear lateral tire stiffness (15 to 30 kN/rad).
Chapter 1

Introduction

A significant number of traffic accidents are caused by driver error. Many of these accidents are caused by inattentive drivers, while others are caused by the inability of drivers to quickly react in emergency situations (Fenton & Selim, 1991). Advanced Vehicle Control Systems (AVCS) use automation to improve the safety of highway travel. Automated vehicles should solve many of the problems associated with inattentive drivers. In addition, automated vehicles will be able to react to emergency situations much faster than human drivers can. Other potential benefits of AVCS include increased highway capacity and decreased travel times.

An important field of AVCS research involves automated lateral control of a highway vehicle. In this field, researchers are developing systems which allow a vehicle to follow the road. Much work has been done to design controllers which perform well on vehicles during low-lateral-acceleration maneuvers, for example, see (Fenton & Selim, 1976; Cormier & Fenton, 1980; Fenton & Selim, 1988; Fenton & Selim, 1991; Peng & Tomizuka, 1993). These controllers are designed based on a yaw-plane vehicle model similar to the one described in Section 1.1. However, in an emergency situation, high-lateral-acceleration maneuvers may be required. During high-lateral-acceleration maneuvers, linear vehicle and tire models are inaccurate (Smith & Starkey, 1994; Smith & Starkey, 1995b).
1.1 Vehicle and Tire Model

Figure 1.1 describes a highway vehicle, using the symbols defined in Table 1.1. The yaw-plane model based on Figure 1.1 may be used in conjunction with a linear tire model based on the tire side-slip angles, $\alpha_f$ and $\alpha_r$, to describe the dynamics of a vehicle under low-lateral-acceleration conditions. Using a linear tire model, the tire side forces for each tire are calculated from the tire side slip angles as

$$F_{sf} = C_{\alpha_f} \alpha_f$$  \hspace{1cm} (1.1)

and

$$F_{sr} = C_{\alpha_r} \alpha_r.$$ \hspace{1cm} (1.2)

Assuming the steering actuator to be a first-order system, the differential equations that describe the system in Figure 1.1 are

$$\dot{V} = -Ur + \frac{1}{m} [2C_{\alpha_f} \alpha_f + 2C_{\alpha_r} \alpha_r],$$  \hspace{1cm} (1.3)

$$\dot{r} = \frac{1}{I_{zz}} [2aC_{\alpha_f} \alpha_f - 2bC_{\alpha_r} \alpha_r],$$ \hspace{1cm} (1.4)

and

$$\dot{\delta}_f = \frac{G_{ss} \epsilon_{in}}{\tau_{sw} + 1}.$$ \hspace{1cm} (1.5)

Assuming small angles, the values of $\alpha_f$ and $\alpha_r$ may be calculated as

$$\alpha_f = \delta_f - \frac{V + \alpha_r}{U}.$$ \hspace{1cm} (1.6)
Table 1.1: Symbols Used in Vehicle Model

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>distance from the center of gravity to the front axle</td>
</tr>
<tr>
<td>$b$</td>
<td>distance from the center of gravity to the rear axle</td>
</tr>
<tr>
<td>$m$</td>
<td>vehicle mass</td>
</tr>
<tr>
<td>$U$</td>
<td>forward velocity</td>
</tr>
<tr>
<td>$V$</td>
<td>lateral velocity</td>
</tr>
<tr>
<td>$\tau$</td>
<td>yaw rate</td>
</tr>
<tr>
<td>$\delta_f$</td>
<td>steering input</td>
</tr>
<tr>
<td>$\alpha_f$</td>
<td>front lateral tire side slip</td>
</tr>
<tr>
<td>$\alpha_r$</td>
<td>rear lateral tire side slip</td>
</tr>
<tr>
<td>$F_{sf}$</td>
<td>tire side force per front tire</td>
</tr>
<tr>
<td>$F_{sr}$</td>
<td>tire side force per rear tire</td>
</tr>
<tr>
<td>$C_{af}$</td>
<td>tire cornering stiffness per front tire</td>
</tr>
<tr>
<td>$C_{ar}$</td>
<td>tire cornering stiffness per rear tire</td>
</tr>
<tr>
<td>$I_{zz}$</td>
<td>vehicle yaw moment of inertia</td>
</tr>
<tr>
<td>$G_{ss}$</td>
<td>steady-state gain for the steering system</td>
</tr>
<tr>
<td>$\tau_{sw}$</td>
<td>time constant for the steering system</td>
</tr>
<tr>
<td>$e_{in}$</td>
<td>voltage input for the steering system</td>
</tr>
</tbody>
</table>
and
\[ \alpha = \frac{b - V}{U}, \quad (1.7) \]
respectively. Smith, et al. (Smith, 1993; Smith & Starkey, 1994; Smith & Starkey, 1995b; Smith & Starkey, 1995a; Smith et al., 1995) have used a model similar to this with a first-order model of tire lag in their research. For this research, any tire lag in the system is considered as an uncertainty, and is therefore not included in the nominal model.

The model may be represented in state space as
\[ \dot{x}(t) = Ax(t) + Bu(t) + B_w w(t), \quad (1.8) \]
where \( A \in \mathbb{R}^{n \times n} \) is the state-feedback matrix, \( x(t) \in \mathbb{R}^n \) is the state at time \( t \), \( B \in \mathbb{R}^{n \times m} \) is the state-input matrix, \( u(t) \in \mathbb{R}^m \) is the state input at time \( t \), \( B_w \in \mathbb{R}^{n \times m_w} \) is the disturbance-input matrix, and \( w(t) \in \mathbb{R}^{m_w} \) is the disturbance input at time \( t \). Specifically, for the model described using the yaw-plane vehicle and linear tires, the state vector may be chosen as
\[ x(t) = \begin{bmatrix} \delta_f & V & r & y & \psi \end{bmatrix}^T, \quad (1.9) \]
where \( y \) is the lateral offset of the vehicle's center of gravity and \( \psi \) is the heading error of the vehicle. The variables \( y \) and \( \psi \) describe the vehicle’s position and orientation relative to the road, where
\[ \begin{align*}
\dot{y} &= V + U \psi, \\
\dot{\psi} &= r - \dot{\psi}_d.
\end{align*} \quad (1.10) \]

With this state vector, the state-feedback matrix is
\[ A = \begin{bmatrix}
-\frac{1}{\tau_w} & 0 & 0 & 0 & 0 \\
\frac{2\gamma_{\alpha f}}{m} & -\frac{2}{mU}(C_{\alpha f} + C_{ar}) & -U - \frac{2}{mU}(aC_{\alpha f} - bC_{ar}) & 0 & 0 \\
\frac{2\gamma_{\alpha f}}{I_{xx}} & -\frac{2}{I_{xx}U}(aC_{\alpha f} - bC_{ar}) & -\frac{2}{I_{xx}U}(a^2C_{\alpha f} + b^2C_{ar}) & 0 & 0 \\
0 & 1 & 0 & 0 & U \\
0 & 0 & 1 & 0 & 0 
\end{bmatrix}. \quad (1.11) \]
The state input is

\[ u(t) = e_m(t), \tag{1.12} \]

and the state-input matrix is

\[ B = \begin{bmatrix} \frac{e_m}{r_m} & 0 & 0 & 0 & 0 \end{bmatrix}^T. \tag{1.13} \]

The disturbance-input matrix is

\[ B_w = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \end{bmatrix}^T, \tag{1.14} \]

and the disturbance input is the yaw rate for the road,

\[ w(t) = \dot{\psi}_{rd}(t). \tag{1.15} \]

If a controls engineer wanted to include the possibility of a side force or a moment due to cross winds, the engineer could account for such disturbances with additional signals in the disturbance input \( w(t) \) and appropriately placed and scaled elements in the disturbance-input matrix \( B_w \).

### 1.2 Time-Varying Uncertainties in Vehicle Models

Linear controllers may be designed based on the linear model described in Section 1.1. However well the controllers may be designed to perform with the model, such controllers provide no guarantee of good performance or even stability when used on a real vehicle in situations where linear models may be inaccurate, such as high-lateral-acceleration maneuvers where linear models have been shown to be inaccurate (Smith & Starkey, 1994; Smith & Starkey, 1995b).

In fact, it is impossible for any model to be an exact representation of nature because of the infinite complexity of God's creation. Only ignorance and pride could compel researchers to claim that their "text-book" theories hold true for every "real-world" problem. Whatever the motive, before making such promises, the analyst should carefully consider the words of a first-century writer: If anyone...
supposes that he knows anything, he has not yet known as he ought to know. It is in the context of humility that a study of modelling inaccuracies should be undertaken. It would be nothing less than a dangerous combination of pride and ignorance to believe that creation could be completely modeled and understood by members of it. Nature defies those who cannot fully understand it to develop control schemes which adapt to its variations.

However, control scientists are hired to use their "text-book" knowledge in the stabilization of "real-world" systems such as highway vehicles, and this research aims to present theories on the stabilization of complex nonlinear systems: not with adaptive schemes, but with so-called robust schemes. To insure that a controller is robust to the types of modeling inaccuracies which are likely to occur during high-lateral-acceleration maneuvers, the inaccuracies must be classified. The controller may then be designed as robust to all expected inaccuracies of a given class and size.

Various parameters of a vehicle, such as rotational inertia, are difficult to measure. Without exact information on vehicle parameters, any model that relies on such parameters would be inaccurate. Such inaccuracies in the model may be considered as uncertainties that do not change with time, which are referred to in the literature as a time-invariant uncertainties. A system containing only time-invariant parameters and uncertainties is called a time-invariant system.

Many other parameters, such as forward velocity, change with time. Although it may be possible at any given time to accurately measure such parameters, the changing nature of these parameters leads to inaccuracies in any time-invariant model. This class of inaccuracies in the model may be considered as uncertainties that change with time, which are referred to in the literature as time-varying uncertainties. A system containing time-varying parameters and uncertainties is called a time-varying system.

\[ ^1 \text{The Apostle Paul, I Corinthians 8:2} \]
In addition, during a high-lateral-acceleration maneuver, the vehicle-tire system exhibits nonlinear behavior. Nonlinearities in the tire-road interface, specifically in the relationship between slip angles and tire side forces (Gillespie, 1992), are major contributors to such behavior (Smith & Starkey, 1995b). Nonlinear behavior may be characterized as a time-varying uncertainty in the nominal linear model (Vidyasagar, 1993; Boyd et al., 1994b; Feron, 1994).

Because most real-world systems contain parameters which vary with time, the rate of change of a system's parameters is the best method of determining whether the system may be accurately modeled as having time-invariant uncertainties. In this research, the author considers all parameters with a rate of change more than twenty times slower than the expected system response to be time-invariant parameters. For example, the mass of a vehicle varies with time as the number of passengers change. However, except in special cases, the number of passengers is not expected to change during an emergency lane change. On the other hand, the forward speed of the vehicle is expected to change during an emergency lane change. Table 1.2 classifies the parameters used in the linear model. For the parameters in Table 1.2, the uncertainties in the time-varying parameters $U$, $C_{af}$, and $C_{ar}$ have perhaps the greatest effect on the system, and have the greatest need to be addressed by robust control techniques.

Because systems stable for a bounded time-invariant uncertainty may not be stable for a similarly bounded time-varying uncertainty (Vidyasagar, 1993), any attempt to control the system must result in a controller that is robust to time-varying uncertainties within an expected bound.

1.3 Summary

In this research, a method is developed for designing emergency lateral controllers which are robust to time-varying modeling inaccuracies that occur during high-lateral-acceleration maneuvers. The method will result in that automated vehicles
Table 1.2: Parameter Uncertainties

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Type of Uncertainty</th>
<th>Reasons for Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$, $b$, $m$, $I_{zz}$</td>
<td>time invariant</td>
<td>measurement error, passenger/payload variation</td>
</tr>
<tr>
<td>$U$</td>
<td>time varying</td>
<td>breaking during an emergency</td>
</tr>
<tr>
<td>$G_{ss}$, $\tau_{ew}$</td>
<td>possibly time varying</td>
<td>measurement error, unmodeled dynamics</td>
</tr>
<tr>
<td>$C_{af}$, $C_{ar}$</td>
<td>nonlinear, time varying</td>
<td>changes in tire/road interface, changes in longitudinal slip, changes in tire normal-force due to lateral load transfer, lag in tire force, nonlinear nature of tires for large $\alpha$ angles</td>
</tr>
</tbody>
</table>
which safely react to a given set of emergency situations. In addition, the method may be used in any field where control of time-varying systems is a goal. Thus, the usefulness of the method is greater than its specific application to the field of emergency lateral control of highway vehicles.

Chapter 2 gives an overview of several current controller design techniques available in the literature. Although a complete survey of this field would be impossible, this overview gives the appropriate information necessary to understand the novelty and usefulness of the research presented in the remaining chapters. Chapter 3 presents the author's earlier research into the design of controllers for emergency lateral control of highway vehicles. Difficulties encountered in these design methodologies are addressed, and necessary improvements are enumerated. Chapter 4 presents the main result of this dissertation: reduced-effort, static-output-feedback stabilization. Chapter 5 extends the method for the case of robust stabilization. Chapter 6 demonstrates the use of the method developed in Chapters 4 and 5 on the emergency lateral control of a highway vehicle problem discussed in this chapter. Finally, Chapter 7 presents conclusions that can be drawn from this work and comments on future work that may be done to advance the field.

When compared to controller design methods currently available in the literature, the main advances in this research are

- When possible, the method allows a return to simple output-feedback controllers, yet guarantees a given decay-rate for time-varying uncertainties. Modern control has often been dismissed by industry because of the complicated large-order controllers required to estimate states and implement state-feedback control. This method will help industry see the advantages of modern control, without needless complication of the control problem to be solved.
• The method advances the use of Linear Matrix Inequalities (LMI) in the minimization of a quadratic function. The author proves in the research that LMI's can be used in such a fashion.

• The method redefines the term optimal by placing increased emphasis on minimizing controller implementation cost (feedback gain) for a given level of uncertainty as opposed to finding a controller which maximizes the size of the uncertainty allowed. The method assumes that the designer knows the size of possible uncertainties in the system. Based on these uncertainties, the method mathematically defines the set of controllers that guarantee eigenvalue placement to the left of a given vertical line in the complex plane, if such a set of controllers exists, and finds the controller which corresponds to the smallest value of a given quadratic norm of the control gains. Previously, optimal control has been defined either as minimizing the weighted quadratic performance index (LQR, or $\mathcal{H}_2$ control) or as maximizing the robustness of a system ($\mathcal{H}_\infty$ control). The definition of optimal used in this research presents a balanced view of optimization by minimizing the weighted quadratic size of a controller while guaranteeing robustness.
Chapter 2

Literature Review

Many researchers have added to the understanding of control systems. This chapter explains the portion of their work that is relevant to the research presented here. The field of automatic control is quite broad and the author does not claim to present a complete overview of all work done in this field. Such a presentation would require as many volumes as there are researchers. Section 2.1 reviews some of the major advances in the analysis of time-varying systems. Sections 2.2 and 2.3 outline Robustness and Optimal-Control Theory. Next, Sections 2.4 and 2.5 present motivation, terminology, and recent research in the areas of Eigenvalue Placement and Output Feedback. Finally, Section 2.6 presents the current theory of linear matrix inequalities and gives examples of their abundant recent use in control theory.

2.1 Time-Varying Systems

Vidyasagar has presented a thorough overview of nonlinear system stability in the book *Nonlinear Systems Analysis* (Vidyasagar, 1993). Among other topics, Vidyasagar gives a complete presentation of Lyapunov stability theory including Lyapunov's direct method (see also (Anderson & Moore, 1990; Thompson, 1992)).
Lyapunov's direct method states that a nonlinear system

\[ \dot{x}(t) = f[t, x(t)] \]  

(2.1)
is stable if there exists a continuously-differentiable, locally-positive-definite function \( V(t, x(t)) \) and a constant \( r > 0 \) such that

\[ \dot{V}(t, x(t)) \leq 0, \ \forall t \geq 0, \ \forall x : \|x\| < r, \]  

(2.2)

where \( \dot{V} \) is evaluated along the trajectories of the system in (2.1). Any such function \( V \) is known as a Lyapunov function for the system in (2.1).

Zames, in his often-cited paper (Zames, 1966), presented several theorems on the input-output stability of nonlinear systems and sector bounds on system nonlinearity (see also (Anderson & Moore, 1990; Vidyasagar, 1993) and the discussion in Section 2.2.2). Some examples of other work in the field of nonlinear systems are Thompson (Thompson, 1992) and van der Schaft (van der Schaft, 1992). Perhaps the nonlinear systems result with the most relevance to the author's current research is the fact that a nonlinear system may be modeled as a linear time-varying system (Vidyasagar, 1993; Boyd et al., 1994b; Feron, 1994).

One important difference between linear time-varying systems and linear time-invariant systems is stability criteria. Linear time-invariant systems are stable if and only if all of the system's eigenvalues are negative (Kailath, 1980). On the other hand, linear time-varying systems may be unstable even if all of the system's "frozen-time" eigenvalues (the eigenvalues of the system at any fixed time, neglecting time variance) are negative for all time (Vidyasagar, 1993).

Several researchers have discussed various special cases of linear time-varying systems such as slowly time-varying systems (Freedman & Zames, 1968; Desoer, 1969; Sundareshan & Thathachar, 1972; Ilchmann et al., 1987; Amato et al., 1993; Guo & Rugh, 1995; Megretski, 1995), where the time-varying elements of the system have bounded derivatives with respect to time. Another special case of
linear time-varying systems which has been studied is the case of periodic time-varying systems (Vemula, 1993; Vidyasagar, 1993). A periodic system is a system

\[ \dot{x}(t) = A(t)x(t), \]  
\[ A(t + T) = A(t) \forall t, \]  

for some known period, \( T \). Although both of these special cases of time-varying systems are interesting, the current research focuses on a broader class of time-varying systems which includes both of these special cases.

Many researchers have approached the problem of the stability of a linear time-varying system with the use of quadratic Lyapunov functions (Barmish, 1983; Barmish, 1985; Zhou & Khargonekar, 1988b; Khargonekar et al., 1990; Chen & Chen, 1991; Boyd et al., 1994b; Feron, 1994; Garcia et al., 1994; Mahmoud & Al-Muthairi, 1994; Xie & Soh, 1994; Petersen, 1995). A linear time-varying system

\[ \dot{x}(t) = A(t)x(t) \]  

is said to be quadratically stable if there is a Lyapunov function that has the quadratic form

\[ V(x) = x^TPx. \]

Because

\[ \dot{V}(x) = x^T[A(t)^TP + PA(t)]x, \]  

the linear time-varying system (2.4) is quadratically stable if and only if there exists a constant matrix \( P > 0 \) such that

\[ A(t)^TP + PA(t) < 0, \forall t. \]  

For a review of quadratic Lyapunov functions for linear time-invariant systems, see (Lancaster, 1969; Kailath, 1980; D'Az zo & Houpis, 1988; Boyd et al., 1994b).

Other researchers have chosen to use a quantity known as the matrix measure (also known as the logarithmic derivative) stability criteria for linear time-varying
systems (Juang, 1991; Vidyasagar, 1993) and response bounds for time-varying systems (Lehman & Shujaee, 1993; Vidyasagar, 1993). For an \( n \)th order system, let \( || \cdot ||_i \) be an induced matrix norm on \( C^{n \times n} \). The corresponding matrix measure, \( \mu_i(\cdot) : C^{n \times n} \to \mathbb{R} \), of \( A(t) \) is defined as

\[
\mu_i[A(t)] = \lim_{\varepsilon \to 0^+} \frac{||I + \varepsilon A(t)||_i - 1}{\varepsilon}.
\]

According to (Vidyasagar, 1993), the system in (2.4) is asymptotically stable if

\[
\int_{t_0}^{t_0 + t} \mu[A(\tau)]d\tau \to -\infty \text{ as } t \to \infty, \forall t_0 > 0.
\]

Obviously, if \( \mu[A(t)] \) is negative at every instant in time, then the stability criterion in (2.8) is satisfied, and the time-varying system represented by (2.4) is asymptotically stable. Table 2.1 contains the formulas for matrix measures and induced norms based on the \( \infty \)-norm \( || \cdot ||_\infty \), the 1-norm \( || \cdot ||_1 \), and the 2-norm \( || \cdot ||_2 \).

Examples of researchers using the matrix measure to prove results for systems other than linear time-varying systems include (Jiang, 1987; Wang & Lin, 1992; Juang, 1993; Pio et al., 1993; Fang et al., 1994; Tissir & Hmamed, 1994). In addition, several researchers have focused on part of the formula used to calculate the matrix measure induced by the 2-norm, \( (A^T + A)/2 \), without mentioning that it is related to the matrix measure (Zadeh & Desoer, 1979; Yadavalli, 1985a; Yadavalli, 1985b; Yadavalli, 1986; Yadavalli & Liang, 1986; Juang et al., 1987b; Soh et al., 1987; Zhou & Khargonekar, 1987; Juang & Chen, 1989; Juang et al., 1989b; Yadavalli, 1993). Further mention of the matrix \( (A^T + A)/2 \), which is noted as the symmetric part of the matrix \( A \), can be found by following the results due to Bendixson (Bodewig, 1956; Beckenbach & Bellman, 1971; Laub, 1979; Ismail & Bandyopadhyay, 1994).

One might be tempted to use the identity matrix as \( P \) in the quadratic Lyapunov function \( V = x^T P x \) that leads to the inequality (2.6). This would result in a stability test that requires one to only check that the matrix measure
Table 2.1: Matrix Measures and Induced Norms for Various Norms

<table>
<thead>
<tr>
<th>Norm on $C^n$</th>
<th>Induced Norm on $C^{n \times n}$</th>
<th>Matrix Measure on $C^{n \times n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[|x|_\infty = \max_i</td>
<td>x_i</td>
<td>]</td>
</tr>
<tr>
<td>[|x|<em>1 = \sum</em>{i=1}^n</td>
<td>x_i</td>
<td>]</td>
</tr>
<tr>
<td>[|x|<em>2 = \sqrt{\sum</em>{i=1}^n</td>
<td>x_i</td>
<td>^2}]</td>
</tr>
</tbody>
</table>

$\lambda_{\max}(A)$ denotes the eigenvalue satisfying the inequality $\lambda_{\max}(A) \geq \lambda_i(A)$ $\forall i$, where $\lambda_i(A)$ is any eigenvalue of $A$.

$A^*$ denotes the conjugate-transpose of $A$.

with respect to the two norm is negative, $(A^T + A)/2 < 0$. In fact, Jiang (Jiang, 1987) proposed this as a stability test. However, as Soh (Soh, 1989) pointed out, the symmetric matrix $(A^T + A)/2$ cannot be negative definite if any of the diagonal elements of $A$ are positive or zero. This places an unnecessary limitation on the matrix $A$ in the case of linear time-invariant systems (for the system to be stable, the matrix $A$ must have all eigenvalues negative, and one can easily find a matrix with at least one positive diagonal element that has all eigenvalues negative). Fang (Fang et al., 1994) offers the corollary that $A$ is stable if and only if there exists a matrix measure $\mu(\cdot)$ based on some norm $\|\cdot\|$ such that $\mu(A) < 0$, which would allow one to check the stability of a system by checking every possible matrix measure until a suitable matrix measure is found. However, as Soh (Soh, 1989) points out, the matrix measure based on any particular norm may not be negative for all stable matrices. Juang (Juang, 1991) proposed the use of an invertible similarity transform, $S$, on the state-space matrix $A(t)$. Juang also showed the equivalence of checking for the existence of an invertible matrix $S$ such that

\[\mu_2(SA(t)S^{-1}) < 0, \forall t\]  

(2.9)

and checking for the existence of $P = S^*S$, where $S^*$ denotes the conjugate-transpose of $S$, such that the Lyapunov inequality (2.6) holds (to show this, find

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
and use $S$ as the Choleski factor of $P$, the result follows after pre-multiplying and post-multiplying the left side of (2.6) by $S^{-*}$ and $S^{-1}$, respectively.

2.2 Robustness

A robust control system is a control system which guarantees stability or performance in the presence of plant uncertainty. Doyle, Francis, and Tannenbaum discussed robustness in their book *Feedback Control Theory* (Doyle *et al.*, 1992). They describe the difference between stability robustness and performance robustness. A controller is said to robustly stabilize a system if the controller guarantees stability in the face of expected uncertainties. On the other hand, a control system exhibits robust performance if the controller guarantees a level of performance notwithstanding expected uncertainties.

To study system robustness, one must first characterize the uncertainty in the system. Several criteria for the classification of uncertainties exist. In the context of this research, perhaps the most important classification of system uncertainty is the time variance or time invariance of the system, which has already been discussed in Section 2.1.

2.2.1 Matching Conditions

Another characterization of an uncertainty is whether or not the uncertainty meets the “matching conditions”. The matching conditions state that any uncertainty in the system must enter through the nominal input matrix of the system. For example, if the system is described by the state-space realization

$$
\dot{x}(t) = A(t)x(t) + B(t)u(t)
$$

(2.10)

where $A(t) = A_0 + \Delta A(t)$ and $B(t) = B_0 + \Delta B(t)$, then the system is matched if there exist some matrices $D(t)$ and $E(t)$ such that

$$
\Delta A(t) = B_0D(t), \forall t
$$

(2.11)
17

and

\[ \Delta B(t) = B_0E(t), \forall t. \]

Several researchers have studied systems which meet the matching conditions (Thorp & Barmish, 1981; Galimidi & Barmish, 1986; Swei & Corless, 1989; Khar­gonekar et al., 1990; Swei & Corless, 1991; Tsay et al., 1991; Dawson et al., 1992; Corless, 1993; Leitmann, 1993; Phung & Sawan, 1993; Wang et al., 1993). Some have loosened the matching conditions to modified matching conditions (Petersen & Hollot, 1986; Wei, 1990) and generalized matching conditions (Corless, 1993), while others question the need for matching conditions (Barmish, 1983; Barmish, 1985; Stalford, 1987; Zhou & Khargonekar, 1987).

One important result in the question of matching conditions is the theorem due to Swei and Corless that a system is quadratically stabilizable with arbitrary degree of stability if and only if the system is controllable and meets the matching conditions (Swei & Corless, 1991). However, note that this theorem does not say that the matching conditions are necessary to quadratically stabilize a sys­tem. The theorem states that the matching conditions are necessary to obtain an arbitrary degree of stability. Because many real-world systems are not matched, a controls engineer must have tools available which do not rely on the matching conditions. This theorem does offer an explanation if the controls engineer is unable to quadratically stabilize a system to the desired degree of stability.

2.2.2 Unstructured Uncertainties

Another characterization of uncertainties is whether an uncertainty is structured or unstructured. Many researchers have studied systems with uncertainties similar to the small-gain theorem of (Zames, 1966) (see also (Zhou et al., 1996)), which states that if two systems are interconnected as in Figure 2.1, with both \( M_1 \) and
\( M_2 \) stable, then the interconnected system is stable if and only if

\[
\|M_1\|_\infty \| M_2 \|_\infty < 1,
\]

(2.12)

where \( \| \cdot \|_\infty \) is the \( \mathcal{H}_\infty \) norm (Doyle et al., 1992; Zhou et al., 1996). To use the small-gain theorem, one would calculate \( \mathcal{H}_\infty \) norm of a nominal system \( M_1 \) and find the bound on the \( \mathcal{H}_\infty \) norm of the feedback uncertainty \( M_2 \). Because none of the uncertainty structure is used in the small-gain theorem, the uncertainty is called an unstructured uncertainty. For robustness research using unstructured uncertainties, see (Wang et al., 1987; Becker & Grimm, 1988; Juang et al., 1989a; Doyle et al., 1992; Wang & Lin, 1992; Wang et al., 1993). The problem with research based on unstructured uncertainties is that controllers designed with unstructured uncertainties are often too conservative when connected to plants that have highly structured uncertainties. The system's performance suffers due to bad approximations.

![Figure 2.1: An Interconnected Feedback Loop](image)

2.2.3 Interval Matrices

In order to include the structure of the uncertainty into any analysis, one must study the causes of the uncertainty. Usually, the uncertainty is due to several uncertain parameters of the system. Some parameter uncertainties may not have as much effect on the system as other parameters. The theory of interval matrices
allows one to more accurately describe the uncertainty in the system. An interval matrix is any matrix with individually bounded elements, for example the set of matrices

$$A_I = \{ A = [a_{ij}] \in \mathbb{R}^{n \times n} : b_{ij} \leq a_{ij} \leq c_{ij}, \ i, j = 1, 2, \ldots, n \} \quad (2.13)$$

is an interval matrix. Interval matrices allow the uncertainty in an n-th order system to be specified in terms of specific intervals for each of the n x n elements of the system's matrix. Several researchers have investigated interval matrices (Heinen, 1984; Argoun, 1986; Juang & Shao, 1989; Ismail & Bandyopadhyay, 1993; Ismail & Bandyopadhyay, 1994) (one should be careful to note that the results of (Argoun, 1986) have been questioned in (Juang & Shao, 1989; Fang et al., 1994), see below).

Heinen (Heinen, 1984) provided a stability criterion for interval matrices: The interval matrix defined in Equation 2.13 is stable if

$$c_{ij} + \sum_{j=1}^{n} \max_{j \neq i} \{|b_{ij}|, |c_{ij}|\} < 0, \ i = 1, 2, \ldots, n \quad (2.14)$$

(note the similarity to the matrix measure $\mu_\infty$ in Table 2.1). Although Heinen's stability condition is simple, the matrices are restricted to have negative diagonal elements (Argoun, 1986) (see the discussion about the results of Jiang (Jiang, 1987) in Section 2.1). Because Ismail and Bandyopadhyay (Ismail & Bandyopadhyay, 1993; Ismail & Bandyopadhyay, 1994) used the results of Heinen to design controllers for systems described by an interval matrix, controllers designed using their technique may be unnecessarily conservative by forcing the diagonal elements of $A_I - BK$ (in (Ismail & Bandyopadhyay, 1993)) and $A_I - BK_C$ (in (Ismail & Bandyopadhyay, 1994)) to be negative. Argoun (Argoun, 1986) tried to reduce the conservatism of conditions based on Gershgorin's theorem. Gershgorin's theorem (Barnett & Storey, 1970) states that every root of the matrix A lies in at least one
of the $n$ disks with centers $a_{ij}$ and radii

$$
\rho_i = \sum_{j=1, j \neq i}^{n} |a_{ij}|. 
$$

(2.15)

The motivating concept behind the research which lead Argoun to his condition is interesting, but the concept was incorrectly implemented. Juang (Juang & Shao, 1989) corrected Argoun's condition, and presented a stability criterion based on the ability to find the center and radius of disks in which the eigenvalues of the interval system are guaranteed to lie.

Closely related to the theory of interval matrices are structured uncertainties in the form of $|\Delta A| \ll \varepsilon U_{e}$, where each element of the modulus matrix $|\Delta A|$ is the absolute value (modulus) of the corresponding element of the matrix $\Delta A$, and the inequality $\ll$ holds element by element. The value $\varepsilon$ is a measurement of the level of uncertainty in the system, and the matrix $U_{e}$ contains the structure of the uncertainty. For the system described by $\dot{x} = (A + \Delta A)x$, with uncertainty matrix $\Delta A = [E_{ij}]$, Yedavalli (Yedavalli, 1985a) uses a $U_{e}$ matrix in the form of

$$
U_{eij} = \begin{cases} 
0 & \text{if } E_{ij} = 0 \\
1 & \text{if } E_{ij} \neq 0 
\end{cases}.
$$

(2.16)

Yedavalli (Yedavalli, 1985a; Yedavalli, 1986; Yedavalli & Liang, 1986) later changed the $U_{e}$ matrix to

$$
U_{eij} = \frac{\varepsilon_{ij}}{\varepsilon},
$$

(2.17)

where $\varepsilon_{ij} \geq |E_{ij}|$, $i,j = 1,2,\ldots,n$ and $\varepsilon = \max_{i,j} \varepsilon_{ij}$. Other examples of this type of structured uncertainty are contained in (Juang, 1987; Juang et al., 1987b; Juang et al., 1987a; Juang et al., 1989a; Juang & Chen, 1989; Juang et al., 1989b; Jabbari, 1990; Rachid, 1990; Sobel et al., 1990; Wang & Lin, 1992; Juang, 1993; Tissir & Hmamed, 1994).
2.2.4 Polytopes of Matrices

Perhaps a more thorough way to account for the structure of uncertainties in a system is the use of a polytope of matrices. The concept of a polytope of matrices is also very closely related to the concept of an interval matrix. A polytope of matrices may be represented in at least two ways. One way to represent a polytope of matrices is to describe the polytope in terms of the individual uncertain parameters. For example,

\[ A = A_0 + \sum_{i=1}^{k} A_i r_i(t), \quad |r_i(t)| \leq r \quad \forall t \] (2.18)

\[ B = B_0 + \sum_{i=1}^{l} B_i s_i(t), \quad |s_i(t)| \leq s \quad \forall t, \]

where \( r_i(t) \) and \( s_i(t) \) are uncertain parameters, is used to describe uncertainties in (Kosmidou, 1990; Juang, 1991; Olas, 1994). Petersen (Petersen, 1987) restricted the matrices \( A_i \) and \( B_i \) each be rank-1 matrices. Petersen was able to show that with the rank-1 restriction, the polytope in (2.18) is a subproblem of the norm-bounded uncertainty problem discussed in Section 2.2.5 (see Petersen & Hollot, 1986; Schmitendorf, 1988; Shen et al., 1991; Zanaty et al., 1994) for similar rank restrictions on \( A_i \) and \( B_i \). In the case of time-invariant uncertainties, researchers (Zhou & Khargonekar, 1987; Keel et al., 1991; Wang et al., 1993; Yedavalli, 1993; Huang et al., 1995) have used the time-invariant counterpart to the representation in (2.18). Chen (Chen & Chen, 1991) used the representation in (2.19), which is very similar to the representation in (2.18).

\[ A = A_0 + \sum_{i=1}^{k} A_i q_i(t) \] (2.19)

\[ B = B_0 + \sum_{i=1}^{l} B_i q_i(t) \]

\[ q^- \leq q_i(t) \leq q^+ \]

The representation in (2.18) is a subset of (2.19) where some of the matrices \( A_i \) and \( B_i \) might be zero.
Juang (Juang, 1991) related the representation in (2.18) with a second way to represent a polytope of matrices, which is to describe the polytope in terms of the vertices of the polytope. For example, given the vertices \( \{ V_1, V_2, \ldots, V_N \} \), one can describe all matrices in the polytope by

\[
A = \sum_{k=1}^{N} a_k V_k, \quad \sum_{k=1}^{N} a_k = 1, \quad a_k \geq 0
\]  

(2.20)

where \( N = 2^r \) and \( r \) is the number of uncertain parameters in \( A \). This is the convex hull of the vertices. A representation similar to this was used in (Jiang, 1987; Juang, 1991; Arzelier et al., 1993; Boyd et al., 1994b; Fang et al., 1994) (note that Jiang (Jiang, 1987) did not include the restriction that \( \sum_{i=1}^{N} a_i = 1 \)). Juang (Juang, 1991) formulates the vertices for the polytope described by

\[
A = A_0 + \sum_{i=1}^{r} A_i q_i(t), \quad q_i^- \leq q_i(t) \leq q_i^+
\]  

(2.21)

as

\[
V_k = \sum_{i=1}^{r} q_i(t) A_i |_{q_i(t)=q_i^- \text{ or } q_i^+}, \quad k = 1, 2, \ldots, 2^r.
\]  

(2.22)

Obviously, as (Boyd et al., 1994b) points out, the number of vertices in the polytope increases exponentially with the number of uncertain parameters. For large systems, this may cause the computation time for designing a controller to become impractical. However, the quadratic stability condition in (2.6) for a time-varying system described by the polytope (2.20) is equivalent (Boyd et al., 1994b) to

\[
V_i^T P + PV_i < 0, \quad i = 1, 2, \ldots, N.
\]  

(2.23)

2.2.5 Norm-Bounded Uncertainties

Another paradigm for the description of uncertainties is the concept of norm-bounded uncertainties. Many researchers (Hinrichsen & Pritchard, 1986; Petersen, 1987; Petersen, 1988; Zhou & Khargonekar, 1988c; Rotea & Khargonekar, 1989; Khargonekar et al., 1990; Petersen & McFarlane, 1991; Swei & Corless, 1991;
Petersen & McFarlane, 1992; Petersen & Pickering, 1992; Gu, 1993; Garcia et al., 1994; Mahmoud & Al-Muthairi, 1994; Xie & Soh, 1994; Garcia & Bernussou, 1995) have used some form of

\[
\begin{bmatrix}
\Delta A & \Delta B \\
\Delta C & \Delta D
\end{bmatrix} =
\begin{bmatrix}
H_1 \\
H_2
\end{bmatrix} F(t) \begin{bmatrix}
E_1 & E_2
\end{bmatrix},
\]

(2.24)

where \( F(t)^T F(t) < I \), to describe the uncertainty in the system

\[
\dot{x}(t) = (A + \Delta A)(t)x(t) + (B + \Delta B)(t)u(t)
\]

(2.25)

\[
y(t) = (C + \Delta C)(t)x(t) + (D + \Delta D)(t)u(t).
\]

Petersen (Petersen, 1987) relates this formulation of uncertainty to the polytope characterization (see Section 2.2.4).

Norm-bounded uncertainties may be restricted to have the uncertainty matrix \( F(t) \) diagonal (Boyd et al., 1994b). For such cases, the uncertainties are said to be scalar uncertainties. The analysis is much simplified, however a tool is needed to handle cases when the uncertainties are not scalar. The Structured Singular Value (SSV or \( \mu \)) (Zhou et al., 1996) uses the structural information about non-diagonal uncertainty matrices to measure robustness. Many researchers have discussed use of the SSV to characterize uncertainties (Doyle et al., 1991; Fan et al., 1991; Packard et al., 1991; Shamma, 1992; Zhou & Gu, 1992; Shamma, 1995). Assuming norm-bounded uncertainties leads naturally to the use of Linear Fractional Transformations (LFT) to describe the problem. Because LFT theory is not used in this research, a review of LFT theory would be out of the scope of this literature review (for a review of LFT theory, see (Zhou et al., 1996)).

2.2.6 Extended Systems

In the development of the uncertainty characterizations above, many researchers have chosen to use the concept of an “extended system” to account for uncertainties
in the input matrix (Barmish, 1983; Zhou & Khargonekar, 1988c; Wei, 1990; Chen & Chen, 1991; Geromel et al., 1991; Garcia et al., 1994; Garcia & Bernussou, 1995). Although the necessity of the extended system technique is removed for polytopic systems, its use as an alternative method of modeling input uncertainty in a system warrants mention.

Given the system

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t)
\] (2.26)

and a controller design technique which allows uncertainties in the input matrix only, one may form the extended system of (2.25) as

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{u}(t)
\end{bmatrix} =
\begin{bmatrix}
A(t) & B(t) \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
u(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
I
\end{bmatrix} v(t),
\] (2.27)

where \(v(t)\) is the time derivative of the input vector \(u(t)\).

### 2.3 Optimal Control

Anderson and Moore, in their book *Optimal Control: Linear Quadratic Methods* (Anderson & Moore, 1990), state that a system is optimal if it is the best system of a particular type. The big question in optimal-control theory should be “What particular type of control system is best?” As with everyone else, each controls engineer has his own cost function (Every man’s way is right in his own eyes¹). This section gives an overview of various solutions to several problems that have arisen in optimal-control theory. In addition, this section serves as the foundation of this author’s underlying argument for the necessity of a new optimal-control paradigm. The purpose of this section is not to serve as another text in the field of optimal control, but rather to highlight the basics of the existing theory and hopefully to allow the reader to give informed consideration to the necessity of this research.

---

¹Solomon, Proverbs 21:2

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
2.3.1 $\mathcal{H}_2$ Control

$\mathcal{H}_2$ control theory\(^2\), also known as Linear Quadratic (LQ) control theory, is based on the definition of an optimal-control system as any control system that minimizes the following cost function, known as a quadratic performance index:

$$V = \int_{t_0}^{\infty} u^T(t)Ru(t) + x^T(t)Qx(t)dt$$  \hspace{1cm} (2.28)

where $u(t)$ is the state vector for a given system, $R$ is the symmetric positive-definite input-weighting matrix, $x(t)$ is the state vector for a given system, and $Q$ is the symmetric positive-semi-definite state-weighting matrix (Anderson & Moore, 1969; Solheim, 1972; Harvey & Stein, 1978; D'Azzo & Houpis, 1988; Anderson & Moore, 1990; Ogata, 1995; Zhou et al., 1996). The quadratic performance index can be thought of as a type of energy function.

For a given system

$$\dot{x}(t) = Ax(t) + Bu(t)$$  \hspace{1cm} (2.29)

Assuming that the system is completely stabilizable, a state feedback solution is given as $u(t) = -Kx(t)$, where $K = R^{-1}B^TP$ and $P > 0$ is the stabilizing solution to the algebraic Riccati equation

$$A^TP + PA - PBR^{-1}B^TP + Q = 0.$$  \hspace{1cm} (2.30)

As Anderson and Moore (Anderson & Moore, 1990) point out, the solution exhibits excellent robustness properties for linear time-invariant systems. Several researchers have worked on ways to increase the robustness to uncertainties (Kosmidou, 1990; Tsay et al., 1991; Phung & Sawan, 1993; Huang et al., 1995). Perhaps the most severe drawback to such a controller is the necessity for measurement of all of the state variables. In many cases, such a task is expensive if not impossible.

\(^2\)The term $\mathcal{H}_2$ control is derived from analogies made between $\mathcal{H}_2$ control and $\mathcal{H}_\infty$ control solutions in state space such as in (Doyle et al., 1989). The term LQ control may be a more familiar term for this theory.
To keep from measuring all of the state variables, one may design a state observer which estimates the state. When coupled with an LQ controller, this method is called a Linear Quadratic Gaussian (LQG) controller (Anderson & Moore, 1990). However, there are two major problems with such a technique:

- LQG controllers do not necessarily have the nice robustness properties that the state-feedback LQ controllers do (Doyle & Stein, 1979).
- LQG controllers are dynamic-output-feedback controllers, which have at least the same order as the plant. Such high-order controllers add to the complexity of systems, and may be impractical for industrial applications.

Several researchers have attempted to improve the robustness of the LQG technique (Doyle & Stein, 1979; Abedor et al., 1994; Petersen, 1995). The next section describes $\mathcal{H}_\infty$ controllers which have superior robustness properties to the $\mathcal{H}_2$ controllers.

### 2.3.2 $\mathcal{H}_\infty$ Control

$\mathcal{H}_\infty$ control theory is based on minimizing the $\mathcal{H}_\infty$ norm of a system's transfer function (Francis & Doyle, 1987; Zhou & Khargonekar, 1988a; Doyle et al., 1989; Doyle et al., 1992; Zhou, 1992b; Zhou, 1992a; Zhou et al., 1996). The $\mathcal{H}_\infty$ norm is defined as

$$\|F\|_\infty = \sup_{\omega \in \mathcal{R}} \bar{\sigma}[F(j\omega)], \quad (2.31)$$

where $\bar{\sigma}[A]$, the largest singular value of $A$, is defined as

$$\bar{\sigma}[A] = \max_{||x||=1} ||Ax||. \quad (2.32)$$

The idea of minimizing the $\mathcal{H}_\infty$ norm on a transfer function is based on Zames’s small-gain theorem (Zames, 1966) (see Section 2.2.2). An optimal controller designed using this technique allows the uncertainty in the system to have a larger
\( \mathcal{H}_\infty \) norm. By choosing this cost function, a designer is defining optimality as maximum robustness.

Doyle, Glover, Khargonekar, and Francis (Doyle et al., 1989) described the state-space solutions to the \( \mathcal{H}_\infty \) problem and exposed many similarities to \( \mathcal{H}_2 \) state-space solutions. The sub-optimal \( \mathcal{H}_\infty \) state-feedback problem, which guarantees that the \( \mathcal{H}_\infty \) norm of a given transfer function is less than a pre-specified value, \( \gamma \), may be solved with the algebraic Riccati equation similar to (2.30), but where the input weighting matrix, \( R \), is a sign-indefinite function of \( \gamma \). Because one desires to minimize the \( \mathcal{H}_\infty \) norm of a given transfer function, the solution is iterative in \( \gamma \). Therefore, the design of an \( \mathcal{H}_\infty \) controller is more computationally intensive than the design of an \( \mathcal{H}_2 \) controller, which requires only one Riccati-equation solution to optimize the cost function.

Estimators are built using \( \mathcal{H}_\infty \) control to estimate the unavailable states of the system. Just as in the LQG problem, the resulting controller is high in order and complexity. However, unlike in the LQG problem, an \( \mathcal{H}_\infty \) controller retains its robustness when interconnected with an \( \mathcal{H}_\infty \) estimator.

In order to account for the structure of the uncertainty, the Structured Singular Value (SSV, or \( \mu \)) is used. The calculation of \( \mu \) requires an iterative search. The combined process of finding the best \( \mu \) for a given controller and the best controller for a given \( \mu \) is often called the \( \mu \)-K iteration (Lin et al., 1993; Safonov et al., 1994; Zhou et al., 1996). D-K iteration (Rotea & Iwasaki, 1994; Zhou et al., 1996) involves approximating \( \mu \) with a convex function to simplify the design procedure. For other examples of \( \mathcal{H}_\infty \) control techniques, see (Khargonekar et al., 1990; McFarlane & Glover, 1992; van der Schaft, 1992; Gu, 1993; Chen & Wen, 1995).
2.3.3 Other Ideas of Optimality

Many researchers have proposed a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimality criterion (Wang et al., 1993; Doyle et al., 1994; Zhou et al., 1994; Masubuchi et al., 1995). The author does not intend to present any results in mixed $\mathcal{H}_2/\mathcal{H}_\infty$ research, but simply to acknowledge the existence of such technology. Petersen (Petersen, 1995) claims to have achieved the same goals as mixed $\mathcal{H}_2/\mathcal{H}_\infty$ theory with less computational effort. One should note that none of these control methods address the problem of high controller order.

It is more important in the context of this work to mention that some researchers have chosen to define optimality to include the minimization of some norm of the feedback matrix. Heger and Frank (Heger & Frank, 1984) cite the fact that their design technique results in lower feedback gain norms than a previous method, although they do not specifically minimize the feedback norm. Another example of researchers placing importance on keeping the feedback norm small is (Swei & Corless, 1989), where a guaranteed bound of the feedback norm is implemented. In addition, (Kouvaritakis & Cameron, 1980; Sebok et al., 1986; Cameron, 1988; Ismail & Bandyopadhyay, 1993; Karbassi & Bell, 1994; Benton & Smith, 1996) actually propose design methods which minimize the feedback norm while meeting additional constraints (see Section 3.2).

A minimization of the feedback norm might lower the cost of implementing a control system by requiring smaller and less-expensive actuators to implement the design. Although controllers based on $\mathcal{H}_2$ may result in feedback matrices with higher norms for which a particular system may have lower actuation signals due to the effects of the controller on reducing the state of the system, lack of robustness to uncertainties in systems designed based on $\mathcal{H}_2$ controller/estimator configurations (LQG) may remove the ability of the controller to reduce the state of the system. The controller designed in this research minimizes the norm of the...
feedback gain required to meet various stability and performance criteria in the presence of structured uncertainties with known bounds.

2.4 Eigenvalue Placement

The linkage between the placement of a system's eigenvalues and the performance of the system has long been established (Clark, 1962; Takahashi, 1966; D’Azzo & Houpis, 1988) ((Clark, 1962) also contains interesting results on the effects of system zeros). This relationship forms the basis for such techniques as root locus. It should not be surprising that a type of performance robustness can be obtained by methods of robust eigenvalue placement (Juang et al., 1989b).

For an $n^{th}$-order system, exact eigenvalue placement involves choosing a set of $n$ eigenvalues and finding a gain matrix that makes the system’s eigenvalues equal to the chosen eigenvalues (Solheim, 1972; Kailath, 1980; Kouvaritakis & Cameron, 1980; Shieh et al., 1983; Juang & Lee, 1984; Sebok et al., 1986; Fletcher, 1987; Fletcher & Magni, 1987; Magni, 1987; Cameron, 1988; D’Azzo & Houpis, 1988; Schmitendorf & Wilmers, 1990; Keel et al., 1991; Castelan & Hennet, 1992; Yang & Tits, 1993; Karbassi & Bell, 1994; Ravi et al., 1994; Shalaby, 1994; Ogata, 1995). Some researchers have chosen to use eigenvalue placement within a given tolerance (Chen & Hsu, 1987; Soh et al., 1987), which is similar to exact eigenvalue placement. Amin (Amin, 1985) presented a method to arbitrarily change the real parts of any system eigenvalues, while retaining the complex parts.

Regional eigenvalue placement gives the designer freedom to meet other criteria by only trying to place the eigenvalues of the system into various sub-regions of the left-half complex plane. Perhaps the most important development in the field of regional pole placement is the work of Anderson and Moore (Anderson & Moore, 1969; Anderson & Moore, 1990), where the designer is able to guarantee a prescribed degree of stability, $\alpha$, by placing the eigenvalues of the system to the left of the vertical line $x = -\alpha$ in the $s$-plane. The guaranteed degree of stability
specifies that the system decays at least as fast as $e^{-at}$. Additional results on the use of the prescribed degree of stability are available in (Medanic et al., 1988)

Various shaped regions of stability have been discussed in the literature including disks (Gutman & Jury, 1981; Furuta & Kim, 1987; Wittenmark et al., 1987; Kim & Furuta, 1988; Zhang & Shu, 1988; Juang & Chen, 1989; Rachid, 1990; Chou, 1991; Bambang et al., 1993; Sivashankar et al., 1993; Sivashankar et al., 1994; Figueroa & Romagnoli, 1994; Garcia & Bernussou, 1995; Gu, 1995; Masubuchi et al., 1995), other second-order regions (Haddad & Bernstein, 1992; Yedavalli, 1993; Bakker et al., 1995), squares (Ismail & Bandyopadhyay, 1994; Masubuchi et al., 1995), strips (Gutman & Jury, 1981; Shieh et al., 1986; Wang et al., 1993), and various other regions (Bogachev et al., 1979; Ackermann, 1980; Mazko, 1980; Gutman & Jury, 1981; Abdul-Wahab & Zohdy, 1988; Juang, 1993; Piou et al., 1993).

Perhaps the most logical region in which to place eigenvalues is a sector type region similar to region $H$ in Figure 2.2, where the system damping may also be specified. Many of the regions mentioned above are approximations for sector regions (Kawasaki & Shimemura, 1983; Zhang & Shu, 1988). Various sector-type regions are used in (Davison & Ramesh, 1970; Anderson et al., 1975; Gutman & Jury, 1981; Kawasaki & Shimemura, 1983; Heger & Frank, 1984; Zeheb & Hertz, 1984; Juang, 1987; Shieh et al., 1987; Kawasaki & Shimemura, 1988; Shieh et al., 1988; Zhang & Shu, 1988; Juang et al., 1989a; Juang et al., 1989b; Jabbari, 1990; Keel & Bhattacharyya, 1990; Shieh et al., 1990; Juang, 1991; Haddad & Bernstein, 1992; Phatak & Keerthi, 1992; Wang & Lin, 1992; Arzelier et al., 1993; Fang, 1994; Figueroa & Romagnoli, 1994; Zanaty et al., 1994; Keerthi & Phatak, 1995; Solak & Peng, 1995) For linear time-invariant systems, placing eigenvalues in the region $H$ is equivalent to prescribing relative and absolute stability as defined by Takahashi (Takahashi, 1966).
2.5 Output Feedback

Controllers using static output feedback solve the problems associated with modern control. Modern control theories such as LQG, $\mathcal{H}_\infty$ control, and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control described in Section 2.3 result in high-order controllers that may not be practical in industry. Modern control was established in order to remove some of the empiricism from the techniques of classical controller design. However, due to the resulting complicated observer-based control techniques, much of the advancement in modern control has been ignored by industry. One of the major principles of design work in any field is to “keep it simple, stupid” (KISS). Static-output-feedback control is designed to do just that.

Output-feedback stabilizability has been defined in many ways, most of which stem from the Lyapunov stability criterion in (2.6). Iwasaki (Iwasaki et al., 1994) states that a static output gain $G$ stabilizes a given system if $G$ satisfies a linear matrix inequality (LMI) of the form

$$BGC + (BGC)^T + Q < 0,$$

(2.33)

where $B$, $C$, and $Q$ are defined in (Iwasaki et al., 1994). Defining $D$ and $E$ as matrices of the highest rank such that $DB = 0$, $CE = 0$, $DD^T > 0$, and

---

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
\( E^T E > 0 \), the inequality in (2.33) has a solution for \( G \) if and only if the following two inequalities both hold (Boyd \textit{et al.}, 1994b; Iwasaki \textit{et al.}, 1994; Geromel \textit{et al.}, 1994):

\[
D Q D^T < 0
\]  
\( (2.34) \)

\[
E^T Q E < 0.
\]

In a different approach, Galimidi and Barmish (Galimidi & Barmish, 1986) used the constrained Lyapunov problem defined for the system

\[
\dot{x}(t) = A x(t) + B u(t) \\
y(t) = C x(t),
\]  
\( (2.35) \)

where \( B \) and \( C \) are full rank and given a matrix \( D \) which is a matrix of the highest rank such that \( BD = 0 \) and \( DD^T = I \). The constrained Lyapunov problem is defined as finding two matrices \( P = P^T > 0 \) and \( K \) which meet the following conditions:

\[
D^T( A^T P + PA) D < 0
\]  
\( (2.36) \)

\[
B^T P = KC \text{ or } P^{-1} C^T = BK.
\]

For similar results, see (Dawson \textit{et al.}, 1992).

Researchers have used output feedback to design controllers which place poles exactly (Graham, 1981; Sebok \textit{et al.}, 1986; Chen & Hsu, 1987; Fletcher, 1987; Fletcher & Magni, 1987; Magni, 1987; Cameron, 1988; Sobel \textit{et al.}, 1990). Other researchers have used output feedback to place poles in specific regions (Ackermann, 1980; Zeheb & Hertz, 1984; Phatak & Keerthi, 1992; Fang, 1994; Ismail & Bandyopadhyay, 1994; Keerthi & Phatak, 1995). A field closely related to static output feedback (Keel & Bhattacharyya, 1990), involves the use of reduced-order feedback controllers (David & De Moor, 1994; Iwasaki & Skelton, 1994; Ravi \textit{et al.}, 1994; Iwasaki & Skelton, 1995).
2.6 Linear Matrix Inequalities

Boyd, El Ghaoui, Feron, and Balakrishnan have written a very important book entitled *Linear Matrix Inequalities in System and Control Theory* (Boyd et al., 1994b). They explain that a linear matrix inequality (LMI) has the form

\[ F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0 \]  \hspace{1cm} (2.37)

where \( x \in \mathbb{R}^m \) is the variable. Upon further inspection, many of the stability criteria discussed in this chapter are in fact LMI problems. This is due to the fact that many of these methods are based on the foundation of the quadratic Lyapunov criterion 2.6, which is an LMI. A conference paper (Boyd et al., 1994a) contains an interesting history of LMI's as does (Boyd et al., 1994b). Also interesting is an early treatment of the subject by Bellman and Fan (Bellman & Fan, 1963).

The book by Boyd, et al. (Boyd et al., 1994b) mentions the ability to solve LMI problems that search for a matrix \( x \in \mathbb{R}^m \) such that the linear function \( c^T x \) is minimized and \( F(x) > 0 \), however the author is interested in finding an LMI for which some quadratic norm of \( x \) is minimized. For example if \( x \) were a set of feedback gains which one wanted to minimize with the constraint that some \( F(x) > 0 \), the minimization of the linear function \( c^T x \) might cause the value of \( c^T x \) to become very negative, and the size of \( x \) to increase. This is due to the nature of the linear function.

LMI problems may be solved using the ellipsoid algorithm or one of several interior-point methods (Boyd et al., 1994b) (see also (Nemirovskii & Gahinet, 1994)). In addition, several computer software implementations are available (some are available via ftp, see page 31 of (Boyd et al., 1994b)).

A very attractive property of LMI problems is that any two LMI problems \( F_1(x) > 0 \) and \( F_2(x) > 0 \) may be solved simultaneously by solving the following
inequality:
\[
\begin{bmatrix}
F_1(x) & 0 \\
0 & F_2(x)
\end{bmatrix} > 0
\] (2.38)

This allows several LMI problems to be solved at once. Although Riccati equation methods may solve an individual problem more efficiently, there may not be a Riccati equation method which solves multiple problems at the same time (such as the case of a polytope of matrices).

A great number of papers on LMI techniques have recently been written including papers on eigenvalue minimization (Fan, 1993; Fan & Nekooie, 1994), calculation of the structured singular value (Doyle et al., 1991; Packard et al., 1991; Ly et al., 1994), \(\mu-K\) iteration (Goh et al., 1994; Rotea & Iwasaki, 1994; Safonov et al., 1994), mixed \(\mathcal{H}_2/\mathcal{H}_\infty\) control (Bambang et al., 1993), positive real synthesis (Chen & Wen, 1995; Turan et al., 1995), and model predictive control (Kothare et al., 1994). Perhaps more interesting in the context of this research are the numerous papers about stabilizing a polytope of matrices (Feron, 1994) (see also (Boyd et al., 1994b)), fixed-order controllers (Iwasaki & Skelton, 1994), output-feedback controllers (David & De Moor, 1994; Geromel et al., 1994; Iwasaki et al., 1994; Iwasaki & Skelton, 1995), and multi-criterion output-feedback control (Masubuchi et al., 1995).
Chapter 3

Preliminary Work

Preliminary research related to emergency lateral control of a highway vehicle is presented in this chapter. Although the reader could skip to Chapter 4 without loss of continuity, the advances made in subsequent chapters come mainly from the experience gained as a result of the work in this chapter. Insights gained as a result of this work and problems with the resulting controllers are listed. Both of the methods proposed in this chapter are based on Linear Quadratic control theory.

Linear Quadratic Regulators (LQR) are designed by finding the optimal state-feedback control, \( u(t) = -Kx(t) \), which minimizes a given quadratic performance index

\[
V = \int_{t_0}^{\infty} (u^T(t)Ru(t) + x^T(t)Qx(t)) \, dt
\]  

(3.1)

where \( x(t) \) is the state vector for a given system, \( Q \) is the symmetric positive-semidefinite state-weighting matrix, and \( R \) is the symmetric positive-definite input-weighting matrix. Clearly, the problem is to find weighting matrices, \( Q \) and \( R \), that correspond to desired response characteristics for a given system. However, the translation of response specifications into \( Q \) and \( R \) matrices is imprecise (Anderson & Moore, 1990).
3.1 Nonlinear-Gain-Optimized Controller with Four-Wheel- Steering

A Nonlinear-Gain-Optimized (NGO) controller is a Linear Quadratic controller which has been optimized to provide the best possible performance when coupled with a given nonlinear plant. The NGO controller design technique was developed by Smith and Starkey (Smith & Starkey, 1994; Smith & Starkey, 1995a; Smith et al., 1995). The author extended the NGO method to include four-wheel-steered vehicles (Smith & Benton, 1996), which have two steering inputs. In this work, a new approach was used to optimize the performance index. An optimization routine based on Powell’s method (McPhate, 1975; Press et al., 1986; Thompson, 1992) was used to find the optimal performance index which maximizes the performance of a complicated, nonlinear system interconnected with a linear state-feedback controller. This work also demonstrated the adaptability of the NGO control method to multiple-input/multiple-output (MIMO) systems.

To account for changes in vehicle speed during a maneuver the NGO control method (Smith et al., 1995) is modified from constant gains (CG) to continuous gain equations (GE) (Smith & Starkey, 1995a). In this study, it is applied to a 4WS vehicle which will allow quantification of the potential benefits of 4WS for automated emergency maneuvers.

3.1.1 NGO Design Technique

The optimal state-feedback control, \( u(t) = -Kx(t) \), which minimizes the quadratic performance index in (3.1) depends on the state-space model and the performance index. The linear state equations were developed using a two-degree-of-freedom, yaw-plane model of an automobile, similar to the model in Section 1.1, with front-wheel and rear-wheel steering actuators. As with any optimal control problem, there is some difficulty translating desired response characteristics into a performance index. The resulting feedback laws are always optimal with respect to
the chosen performance index, choosing a suitable performance index is a difficult problem in itself.

For the NGO method, the weighting factors in $Q$ and $R$ are chosen at the design speed to optimize the response of a nonlinear eight-degree-of-freedom (8D) model to a step lane change using two opposing criteria: lane overshoot and travel distance. The lane overshoot is defined as the largest lateral distance that the center of gravity of the vehicle travels past the center of the new lane, and travel distance is the farthest longitudinal distance that any point on the vehicle travels in the original lane. The governing equations for the 8D vehicle are nonlinear and require a numerical integration technique to yield the response. For the NGO method, the criteria in selecting $Q$ and $R$ were to keep lane overshoot to less than 7 cm (approximately 2% of lane width) while minimizing the travel distance. Because this is an iterative process, an optimization routine based upon Powell’s method (McPhate, 1975; Press et al., 1986; Thompson, 1992) was used to help find the $Q$ and $R$ that minimized the travel distance.

The method repeated this nonlinear-gain-optimization for a discrete set of speeds which covers the operating range of the vehicle (3, 6, 9, ..., 30 m/s). This yielded a discrete set of feedback gains for each state variable. A 6-th order polynomial least-squares fit was used to approximate each set of feedback gains and to develop the continuous gain equations. The maximum error between the gain equations and the desired gains is 0.55%. The gain equations are presented in Figure 3.1. The controller designed with these gain equations is called the 4WS-GE controller in the sections that follow. The term 2WS-GE refers to the continuous-gain-equations controller for two-wheel-steered vehicles contained in (Smith & Starkey, 1995a).
Figure 3.1: Control Gain Equations for 4WS-GE Controller
3.1.2 NGO 4WS-GE Performance

A dropped-throttle step-lane-change maneuver is used to compare the performance of the 4WS-GE controller to that of the 2WS-GE controller. A dropped-throttle maneuver is produced by setting the desired forward speed to zero at the beginning of the maneuver. Since no brakes are applied during a dropped-throttle maneuver, it is similar to removing one's foot from the accelerator of a car. The changes in velocity of the car during such a maneuver are the motivation for the GE design technique. A step-lane-change maneuver is produced by instantaneously changing the vehicle's desired path from the center of the present lane to the center of an adjacent lane.

Figure 3.2 compares the performance of the 4WS-GE controller to that of the 2WS-GE controller at an initial velocity, $U_0$, of 15 m/s. The 4WS controller yields a quicker vehicle response, characterized by a 7.9% shorter travel distance (15.53 m versus 16.86 m). This reduction of over 1.3 m could mean the difference between safety and an accident. The response overshoots are less than the 7 cm specification for both vehicles. The yaw rate is significantly reduced when using the 4WS-GE controller. The maximum yaw rate is reduced by 88% from $52^\circ$/s to less than $6^\circ$/s. This results in less yaw motion, which is a major contributor to motion sickness. The maximum vehicle side slip increases when using the 4WS-GE controller. This results from the vehicle "sliding" into the new lane as opposed to "turning" into it. This may result in a slightly less comfortable ride. However, passenger safety in an emergency is more important than passenger comfort.

Figure 3.3 compares controller performances at an initial velocity of 30 m/s. The 4WS controller yields a quicker vehicle response, characterized by a 16.5% shorter travel distance (31.27 m versus 37.46 m). The overshoots are less than the 7 cm specification. Again, the yaw rate is reduced for the 4WS vehicle. The yaw rate for the 4WS vehicle is within $\pm 3.2^\circ$/s, which is 92.6% less than the maximum...
Figure 3.2: Dropped-throttle Step-Lane-Change Responses at $U_0 = 15$ m/s
yaw rate for the 2WS vehicle (43.2°/s). Unlike when $U_0 = 15 \text{ m/s}$, 4WS reduces the vehicle side slip when $U_0 = 30 \text{ m/s}$. The maximum side slip for the 4WS vehicle is 13% less than for the 2WS vehicle. With a 6.19 m reduction in travel distance at 30 m/s, the safety advantages of the 4WS-GE controller are clear.

Dropped-throttle step-lane-change travel distances for the two controllers are displayed as a function of initial velocity in Figure 3.4. The 2WS controller produced shorter travel distances than the 4WS controller when initial speeds are slow (3 to 10 m/s). The 4WS controller's poorer performance in the low speed range can best be attributed to geometry. If a slow moving vehicle is allowed to exhibit higher yaw errors, the vehicle will switch lanes in a shorter distance. At highway speeds (20 to 30 m/s), however, large yaw errors are undesirable, and the 4WS vehicle exhibits significantly shorter travel distances. The travel distance can be reduced up 17% by using the 4WS-GE controller.

3.1.3 NGO 4WS-GE Robustness

Controller robustness can be described in terms of robust stability and robust performance. Robust stability is concerned with maintaining stability throughout a range of possible uncertainties, and robust performance focuses on maintaining performance throughout a range of possible uncertainties. Clearly robust performance implies robust stability. Here the controller performance is considered to be robust for a given uncertainty when the overshoot is below 20 cm and the travel distance increases no more than 10% from the nominal travel distance.

All models of real systems contain uncertainties due to modeling error, model simplification, incorrect measurement of parameters, and parameter variation. In lateral vehicle control, vehicle parameter values that are likely to vary include the tire cornering stiffness, $C_a$, the tire/ground friction coefficient, $\mu$, and the mass of the vehicle, $m$ and $m_e$. These parameters refer specifically to the complicated nonlinear model and are not necessarily the same as the parameters referred to by
Figure 3.3: Dropped-throttle Step-Lane-Change Responses at $U_0 = 30 \text{ m/s}$
the simple linear model in Section 1.1 (specifically, the value of $C_a$ in Section 1.1 may depend on both of the parameters $C_a$ and $\mu$).

Changes in tire pressure would cause the tire cornering stiffness, $C_a$, to vary from the design value of 30 kN/rad. For initial speeds of 15, 20, 25, and 30 m/s, the changes in dropped-throttle step-lane-change travel distance and overshoot due to varying $C_a$ are shown in Figure 3.5. For each of the initial speeds, the travel distance increases less than 1.5% from the nominal travel distance as $C_a$ decreases 25% from the nominal value of 30 kN/rad. The travel distance, at each initial speed, decreases less than 0.7% as $C_a$ increases 25%. The response overshoots are less than 18 cm for the above range of $C_a$ values. This indicates that the 4WS-GE controller performance is robust for a ± 25% variation in $C_a$.

Changes in driving conditions such as weather and road-surface conditions can cause the tire/ground friction coefficient, $\mu$, to vary from the design value of 0.85. The changes in dropped-throttle step-lane-change travel distance and
overshoot due to varying $\mu$ are shown in Figure 3.6. For each of the initial speeds, the travel distance increases 17% from the nominal travel distance as $\mu$ decreases 35% from the nominal value of 0.85 to 0.55. The response overshoots are less than 30 cm for the above range of $\mu$ values. At $\mu = 0.65$ (24% below the nominal $\mu$), the maximum overshoot is less than 19 cm and travel distances increase less than 10%.

Given the conditions for robust performance stated earlier, the 4WS-GE controller is only robust for friction coefficients greater than 0.65.

Additional passengers and cargo would increase the sprung mass of the vehicle, $m_s$, as well as the total mass of the vehicle, $m$. Three additional 80 kg passengers would increase the mass of the vehicle by 240 kg (the nominal mass of the vehicle is assumed to include the mass of the driver). The changes in dropped-throttle step-lane-change travel distance and overshoot due to additional mass are shown in Figure 3.7. For each of the initial speeds, the travel distance increases less than 1% from the nominal travel distance as 240 kg of mass is added to the

---

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
mass of the vehicle. The response overshoots increase with additional vehicle mass. However, at all speeds, the maximum overshoot is below 13 cm.

3.1.4 Benefits of NGO and Suggested Improvements

The benefits of the NGO research include

- The NGO method produces controllers that perform well with a given nonlinear system. The method yields a measure of the best possible performance of a nonlinear time-invariant highway vehicle during an emergency lane change using Linear Quadratic methods.

- Although computationally expensive, one can easily envision the use of multiple nonlinear models to find the optimal controller for a set of systems.

- Robustness with respect to the time-varying value of vehicle forward speed has been demonstrated by using continuous gain equations.
The following issues should be addressed before implementation of the NGO controller can be recommended:

- In the field of emergency lateral control of a highway vehicle, the system must not overshoot the lane by more than a tolerance of about 5% of the lane width during an emergency lane change. The NGO method needs further modification in order to provide this high level of performance robustness.

- The function being optimized has not necessarily been proven to be smooth and convex. As such, Powell's method cannot guarantee global optimization.

- The optimal gains obtained from Powell's method for each design speed do not necessarily form a smooth function with respect to vehicle speed. In practice, manual optimization of the weighting functions is helpful in further improving the smoothness of the gain equations.


3.2 Numerical-Eigenvalue-Optimization Method

The Numerical-Eigenvalue-Optimization (NEO) method (Benton & Smith, 1996) is the result of research by the author into the relationship between a Linear Quadratic Regulator's (LQR) performance and the performance index used to design the LQR.

Many methods have been proposed to find performance indices that can be used to design controllers which result in the desired response characteristics for a system. Anderson & Moore (Anderson & Moore, 1969) provided a simple design method which would result in a prescribed degree of stability $\alpha$ by placing the system's eigenvalues to the left of a vertical line at $-\alpha$. However, this method can not be used to specify damping characteristics. Following the work of Anderson & Moore, much research was done to restrict the eigenvalues of the closed-loop system to various regions of the left-half $s$-plane, such as an open hyperbola (Kawasaki & Shimemura, 1983), a vertical strip (Shieh et al., 1986), a disk (Chou, 1991), and various other regions (Haddad & Bernstein, 1992) (see Section 2.4). Some of these are approximations of the region $H$ shown in Figure 3.8, which has also received much attention in the literature (Shieh et al., 1987; Shieh et al., 1988; Juang et al., 1989a; Shieh et al., 1990; Wang & Lin, 1992). The boundaries of the region $H$ represent the rate of decay $\alpha$ and the damping ratio $\zeta_{\text{min}} = \cos(\eta)$. If a system's eigenvalues are contained in the region $H$, the system will be well damped and the rate of decay for the system will be greater than $\alpha$. Shieh et al. (Shieh et al., 1988) developed an algorithm for designing an LQR which places the closed loop eigenvalues of an asymptotically-stable system into the region shown in Figure 3.8, with $\eta$ restricted to be either $\frac{\pi}{4}$ or $\frac{\pi}{6}$. Shieh et al. (Shieh et al., 1990) expanded the original algorithm to the case of $\eta = \frac{\pi}{2k}$ where $k \geq 2$. In addition, the algorithm of (Shieh et al., 1990) systematically stabilizes originally unstable systems. However, the new algorithm adds the assumption that the uncontrolled

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
system has no eigenvalues in the region between $\pm \frac{3\pi}{24}$ and $\pm \frac{\pi}{2}$ radians from the negative real axis.

![Figure 3.8: Region H in the s-plane](image)

### 3.2.1 NEO Design Technique

The NEO research focused on the development of an LQR design method which placed the closed loop eigenvalues into the region $H$ for any arbitrarily chosen $\alpha$ and $\eta$. The algorithm also minimizes the norm of the control gain, $\|K\|$, to reduce the cost of implementing the control system and to reduce the possibility of input saturation. In general, higher control gains lead to increased input signals which may exceed the limits of less expensive actuators. This would require the designer to obtain more expensive actuators capable of handling larger signals.

A linear time-invariant system may be represented by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (3.2)$$

where $A$ is the state-feedback matrix, $x(t)$ is the state vector, $B$ is the state-input matrix, and $u(t)$ is the input vector. Given any positive-definite input-weighting
matrix, $R$, the goal of the NEO research was to find a state-weighting matrix 

$$Q = \begin{bmatrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_n \end{bmatrix}$$ (3.3)

which minimizes a weighted norm of the state-feedback-gain matrix, $K$, while placing all of the eigenvalues of the closed loop system,

$$\dot{x}(t) = (A - BK)x(t),$$ (3.4)

in the region $H$ defined for any given values of $\alpha > 0$ and $0 \leq \eta \leq \frac{\alpha}{2}$ and for any given positive-definite input-weighting matrix, $R$.

The NEO method allows the quadratic performance index in (3.1) to be converted to a cost function which may be directly translated from the system performance requirements (minimize $\|K\|$ with constraints on damping and settling time). A search algorithm written by A. J. McPhate (McPhate, 1975; Thompson, 1992), which is based on Powell's method (Press et al., 1986), is used to find the minimum of a multi-variable cost function without the use of derivatives. Because Powell's method cannot guarantee that a global minimum is found, it may be necessary to begin from several sets of initial values to find the best approximation of the global minimum.

Powell's method is an unconstrained search method. The diagonal elements of the matrix $Q$, which are adjusted to minimize $\|K\|$, are contained in the vector $q = [q_1, q_2, q_3, \ldots, q_n]^T$. Penalty functions such as those described by McPhate (McPhate, 1975) are used to constrain the vector $q$ to values that result in LQR controllers with eigenvalues in the desired region $H$. The NEO algorithm uses Powell's method to minimize the following cost function:
\[ J(q) = \|K\| + \sum_{\forall \zeta_i < 0} w_1 q_i^2 + \sum_{\forall \zeta_i < \zeta_{\text{min}}} w_2 (\zeta_i - \zeta_{\text{min}})^2 \]
\[ + \sum_{\forall t_s > t_{s_{\text{max}}}} w_3 (t_s - t_{s_{\text{max}}})^2, \ 1 \leq i \leq n \] (3.5)

where \( w_1, w_2, \) and \( w_3 \) are large positive scalars (for example, \( \geq 10^4 \)). This cost function is equal to the sum of the norm of \( K \) and the quadratic penalty imposed on any violation of the constraints. Because the elements of \( q \) are restrained from being negative by the condition that \( Q \) must be positive semi-definite, a penalty on any negative elements of the vector \( q \) is included in the cost function. The cost function also includes a penalty on any eigenvalues whose damping, \( \zeta_i \), is less than the minimum desired damping, \( \zeta_{\text{min}} \), and a penalty on any eigenvalue whose settling time, \( t_s \), is greater than the maximum desired settling time, \( t_{s_{\text{max}}} \). The values of \( \zeta_{\text{min}} \) and \( t_{s_{\text{max}}} \) are specified directly from the definition of region \( H \) as

\[ \zeta_{\text{min}} = \left( \frac{1}{1.001} \right) \cos(\eta) \] (3.6)
\[ t_{s_{\text{max}}} = \left( \frac{1}{0.999} \right) \frac{4}{\alpha} \] (3.7)

A tolerance of 0.1\% on the values of \( \zeta_{\text{min}} \) and \( t_{s_{\text{max}}} \) is insured by the scaling factors \( \left( \frac{1}{1.001} \right) \) and \( \left( \frac{1}{0.999} \right) \) in (3.6) and (3.7), respectively. Some adjustments to the values of \( w_1, w_2, \) and \( w_3 \) may be required during the design of a specific system to balance the importance of each specification to more accurately reflect the desires of the designer. However, such adjustments are strongly related to how well each constraint on the optimization is being met. The problem of finding the weighting matrix \( Q \) has been reduced to quantifying the relative importance of the following tasks:

- minimize \( \|K\| \).
- keep \( Q \) positive definite.
• keep the system damping greater than $\zeta_{\text{min}}$.

• keep the system settling time less than $t_{\text{set}}$.

The norm, $\| \cdot \|$, used in (3.5) may be chosen by the designer as any norm (such as $\| \cdot \|_2$ or $\| \cdot \|_p$, for any $p \in [1, \infty]$). In this research, the 2-norm, $\| \cdot \|_2$, was used. The NEO method begins searching from an initial vector $q$. Therefore, the designer may find that redesigning the controller using the final value of $q$ from the previous NEO search as the initial value in a second iteration of the NEO method will produce an even better approximation of the truly optimal design. In addition, the NEO method may be easily expanded to include other design considerations such as robustness to uncertainties and nonlinearities.

**3.2.2 Comparison of NEO to a Previous Method**

Shieh et al. (Shieh et al., 1988; Shieh et al., 1990), developed an algorithm for designing an LQR which places the closed loop eigenvalues of an asymptotically-stable system into the region shown in Figure 3.8, with $\eta$ restricted to be either $\frac{5}{4}$ or $\frac{5}{6}$. The Shieh method does not change any eigenvalues of the system which are already in the region $H$. A procedure is used which is guaranteed to place at least two of the eigenvalues that were not originally in the desired region into the region when the designed state-feedback is applied. This procedure is repeated for at most $\text{Mod}(n/2) + 2$ times, where $\text{Mod}(\cdot)$ represents the largest integer $\leq (\cdot)$. All of the eigenvalues will now be in the desired region $H$. Two drawbacks to this method are that

• the definition of the region $H$ is limited by the lack of freedom to choose arbitrary values for $\eta$. Arbitrary desired degrees of damping are not allowed by the Shieh method because the values of $\eta$ are restricted to $\frac{k}{2k}$ where $k$ is any integer greater than 2. To account for the restriction on the value of $\eta$, the Shieh method suggests that a shifted sector method be used, however
this would only approximate the region formed by arbitrary values of $\eta$ (for example, when there is no $k$ such that $\eta = \frac{\pi}{2k}$).

- two versions of the method, the (Shieh et al., 1988) version and the (Shieh et al., 1990) version, are each limited in the types of systems on which they may be used. For the (Shieh et al., 1988) version of the algorithm, the system must be asymptotically stabilized before the procedure is implemented. Care must be taken not to place the eigenvalues far into the desired region during this step, because once the above procedure is begun, no eigenvalues in the region $H$ will be modified. On the other hand, for the (Shieh et al., 1990) version, the uncontrolled system cannot have any eigenvalues in the region between $\pm \frac{3\pi}{2k}$ and $\pm \frac{\pi}{2}$ radians from the negative real axis.

An advantage of the Shieh algorithm over the proposed NEO method is the number of iterations required for a solution. The Shieh algorithm will require less than $\frac{n+4}{2}$ solutions to Riccati equations or Lyapunov equations, but the NEO method may require hundreds of solutions to Riccati equations during the multivariable search. Of course in today's world of high-speed computation this may only amount to a few seconds of off-line design time.

In order to compare the performance of these design methods, a linear time-invariant system described in (3.8), (3.9), and (3.10) will be used.

$$\dot{x} = Ax + Bu,$$  \hspace{1cm} (3.8)

where

$$A = \begin{bmatrix}
-2 & 0 \\
-2 & 4 \\
-4 & -2 \\
-6 & 12 \\
0 & -12 & -6
\end{bmatrix}$$  \hspace{1cm} (3.9)
and

\[ B = [1 \ 0 \ 1 \ 0]^T. \]  \hspace{1cm} (3.10)

This system is controllable and the open loop eigenvalues are \(-2, -2\pm4j, \) and \(-6\pm12j\). This system is asymptotically stable, but is not contained in a region \( H \), where \( \eta = \frac{\pi}{6} \) and \( \alpha = 4 \) (corresponding to a damping of \( \zeta = \frac{\sqrt{2}}{2} \) and a settling time of \( t_s = 1.0 \) s).

After only four iterations, the Shieh algorithm results in the state-feedback-gain matrix

\[ K_{\text{Shieh}} = \begin{bmatrix} 9.5816 & 22.3094 & -21.6289 & 2.8740 & -17.7655 \end{bmatrix}. \]  \hspace{1cm} (3.11)

The eigenvalues of the resulting closed-loop system are \(-4.3729, -6, -13.4917 \pm 7.7894j, \) and \(-15.4087\). The 2-norm of \( K_{\text{Shieh}} \) may be used as a measure of the cost of this design, because as this value increases in size, actuators capable of handling signals with increasing amplitude must be obtained. The 2-norm of \( K_{\text{Shieh}} \) is 37.1645.

After 596 iterations (starting from \( q = [1 \ 1 \ 1 \ 1]^T \)), the NEO method results in the state-feedback-gain matrix

\[ K_{\text{NEO}} = \begin{bmatrix} 1.9347 & 5.1203 & -1.2299 & 9.9714 & 0.0374 \end{bmatrix}. \]  \hspace{1cm} (3.12)

The eigenvalues of the resulting closed-loop system are \(-4.9303, -4.9861\pm2.8676j, \) and \(-10.0620 \pm 5.7981j\). The 2-norm of \( K_{\text{NEO}} \) is 11.4413. To obtain this design, the values of \( w_1, w_2, \) and \( w_3 \) are set to \( 10^4, 10^4, \) and \( 10^4, \) respectively.

Although the Shieh algorithm is 2 orders of magnitude more efficient at computing a state-feedback-gain matrix which places the eigenvalues of the resulting closed-loop system in the region \( H \), the cost, based on \( \|K\|_2 \), of implementing the NEO controller is 69% less than the cost of implementing the Shieh controller. As the costs of computing continue to decrease, the cost of implementing the controller will become much more important than the cost of computing the state-feedback-gain matrix.
One reason that the NEO method has a reduced $\|K\|_2$ when compared to the Shieh algorithm is that the NEO method does not try to place the eigenvalues far into the desired region. As shown in Figure 3.9, the NEO algorithm places the eigenvalues very close to the border of the region $H$. The designer is allowed to precisely pick the region into which the eigenvalues are placed. If the designer desires better performance than represented by the border of region $H$, then the designer should use a region with a larger $\alpha$ and a smaller $\eta$.

![Figure 3.9: Comparison of Eigenvalue Placement for Shieh and NEO Methods](image)

The responses of the uncontrolled system and the two closed-loop systems to the initial condition $x(0) = [1 1 1 1]^T$ are shown in Figure 3.10. Figure 3.10 (a) shows the free response of the system, Figure 3.10 (b) shows response of the system with the Shieh controller, and Figure 3.10 (c) shows response of the system with the NEO controller. Based on these responses, the NEO controller achieves similar performance as the Shieh controller with a 69% smaller $\|K\|_2$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Figure 3.10: Initial Condition Responses for Shieh and NEO methods
If the objective of the designer is to find the matrix $K$ with minimum size which places the eigenvalues into the region $H$, then the NEO method is better than the Shieh method. The NEO method reduces the size of $\|K\|_2$ by 69% when compared to the Shieh method.

### 3.2.3 NEO Design for a Cart with an Inverted Pendulum

To show an application of the NEO method to unstable systems, a controller is desired for a cart with an inverted pendulum as shown in Figure 3.11.

![Cart with Inverted Pendulum](image)

Figure 3.11: Cart with Inverted Pendulum

The system equations are linearized, by assuming $\theta$ is small, and modeled by the following system of linear differential equations:

\[
\dot{x}(t) = Ax(t) + Bu(t),
\]

\[
A = \begin{bmatrix}
-\frac{R_m}{L_m} & -\frac{K_bK_G}{L_m} & 0 & 0 & 0 \\
\frac{K_bK_G}{rm_1} & 0 & 0 & \frac{m_2g}{m_1} & 0 \\
\frac{K_bK_G}{rm_2L} & 0 & 0 & \frac{g(m_1+m_2)}{m_1L} & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix},
\]

\[
x(t) = \begin{bmatrix}
i_m(t) \\
\dot{i}_m(t) \\
\dot{\theta}(t) \\
\ddot{\theta}(t) \\
\theta(t)
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
\frac{1}{L_m} & 0 & 0 & 0 & 0
\end{bmatrix}^T,
\]
and
\[ u(t) = e_{in}(t), \] (3.17)

where \( i_m \) and \( e_{in} \) are the motor's armature current and voltage, respectively. The units of the variables for this system are listed in Table 3.1. Using the values of the parameters listed in Table 3.2, the system may be represented by
\[
A = \begin{bmatrix}
-285.71 & -1227.8 & 0 & 0 & 0 \\
1.729 & 0 & 0 & 0 & 2.453 \\
4.322 & 0 & 0 & 0 & 30.66 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix} \] (3.18)

and
\[
B = [178.57 \ 0 \ 0 \ 0 \ 0]^T. \] (3.19)

This cart-with-inverted-pendulum system is unstable and non-minimum phase. The eigenvalues of the model represented by the matrix \( A \) are \(-278.08, -8.70, -4.13, 0.00, \) and \( 5.20 \). The transfer function for this system is
\[
G(s) = \frac{308.69(s+4.95)(s-4.95)}{(s+278.08)(s+8.70)(s+4.13)(s-5.20)} \times \frac{771.73s}{(s+278.08)(s+8.70)(s+4.13)(s-5.20)}. \] (3.20)

The NEO method is used to design a controller for the cart-with-an-inverted-pendulum system which minimizes the armature voltage required for the motor while placing the eigenvalues of the system into the region \( H \) with \( \eta = 0.2 \) rad and \( \alpha = 4 \).

Starting from \( q = [1 \ 1 \ 1 \ 1 \ 1]^T \), the NEO method results in the state-feedback-gain matrix
\[
K_{\text{NEO}} = \begin{bmatrix}
0.0934 & -31.29 & 18.80 & -29.85 & 95.31 
\end{bmatrix}. \] (3.21)

The eigenvalues of the resulting closed-loop system are \(-4.00, -4.00 \pm 0.76j\),
Table 3.1: Variables for the Cart with Inverted Pendulum

<table>
<thead>
<tr>
<th>Notation</th>
<th>Variable</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>e(t)</td>
<td>armature voltage</td>
<td>V</td>
</tr>
<tr>
<td>i(t)</td>
<td>armature current</td>
<td>amps</td>
</tr>
<tr>
<td>x(t)</td>
<td>speed of cart</td>
<td>m/s</td>
</tr>
<tr>
<td>\dot{\theta}(t)</td>
<td>angular speed of pendulum</td>
<td>rad/s</td>
</tr>
<tr>
<td>z(t)</td>
<td>position of cart</td>
<td>m</td>
</tr>
<tr>
<td>\theta(t)</td>
<td>angular position of pendulum</td>
<td>rad</td>
</tr>
</tbody>
</table>

$-12.25, \ -278.14$. The 2-norm of $K_{\text{NEO}}$ is 106.3. To obtain this design, the values of $w_1, w_2, \text{ and } w_3$ were set to $10^{10}, 10^5, \text{ and } 10^4$ respectively.

Figure 3.12 shows the response of the closed loop system to the initial condition $x(0) = [0 \ 0 \ 0 \ 0 \ \frac{\pi}{18}]^T (\theta(0) = 10^\circ)$. To reduce the angular displacement of the pendulum, the cart moves in the negative direction. The cart reaches a maximum displacement close to $0.15 \text{ m}$ at about 0.4 seconds and begins to exponentially approach equilibrium. The entire system reaches equilibrium by 2.5 seconds. For this initial condition, the maximum armature voltage is about 17 V.

Figure 3.13 shows the response of the closed loop system to the initial condition $x(0) = [0 \ 0 \ 0 \ 1 \ 0]^T (x(0) = 1 \text{ m})$. The fact that the system is non-minimum phase causes the cart to initially move in the positive direction, away from equilibrium ($x = 0$). At 0.2 seconds the cart begins to move towards the desired cart position of $x = 0$. After 2.5 seconds, the system has reached equilibrium. Again, this response is well damped and decays rapidly. The important thing to notice about the response of the system is that the maximum armature voltage required is about 30 V. The servo-amplifier for the motor used in the model has a peak voltage of 40 V. The NEO method is able to design a feedback-gain matrix $K$ which does not saturate the input for initial $x$ values as large as 1 m. Any system
Table 3.2: Parameters for the Cart with Inverted Pendulum

<table>
<thead>
<tr>
<th>Notation</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>mass of cart</td>
<td>4 kg</td>
</tr>
<tr>
<td>$m_2$</td>
<td>mass of pendulum</td>
<td>1 kg</td>
</tr>
<tr>
<td>$L$</td>
<td>length of pendulum</td>
<td>0.4 m</td>
</tr>
<tr>
<td>$R_m$</td>
<td>armature resistance</td>
<td>1.6 Ω</td>
</tr>
<tr>
<td>$L_m$</td>
<td>armature inductance</td>
<td>5.6 mH</td>
</tr>
<tr>
<td>$K_t$</td>
<td>torque constant</td>
<td>0.154 N m/amp</td>
</tr>
<tr>
<td>$K_b$</td>
<td>back emf constant</td>
<td>0.153 V s/rad</td>
</tr>
<tr>
<td>$K_G$</td>
<td>gear reduction</td>
<td>2.25</td>
</tr>
<tr>
<td>$r$</td>
<td>radius of wheel</td>
<td>0.05 m</td>
</tr>
<tr>
<td>$g$</td>
<td>acceleration of gravity</td>
<td>9.81 m/s²</td>
</tr>
</tbody>
</table>

achieving similar performance with a larger feedback-gain matrix $K$ would likely saturate the input if the value of $x$ were to reach 1 m.

3.2.4 Benefits of NEO and Suggested Improvements

The benefits of the NEO research include

- Significant reductions in the norm of the control gain $K$ are achieved by the NEO method when compared to previous algorithms.

- The controller results in well-damped systems which perform well.

The following issues should be addressed to improve the NEO controller:

- The NEO method requires more off-line computation time than previous algorithms (due to the nature of the optimization).

- The use of penalty functions to enforce the performance constraints may require adjustment by the designer to a set of weighting factors. In essence,
the weighting matrices have been traded for another set of weighting factors. However, the new weighting functions are strongly related to the system's performance characteristics. For example, if an implementation of the NEO method results in a system whose eigenvalues are not as well damped as desired, the designer would simply increase the importance of the damping constraint in the cost function.

- The NEO method does not address the problem of uncertainties in the linear model. However, if the method were extended to include stability robustness to uncertainties, the performance-oriented nature of the method could be used to guarantee a weakened form of performance robustness (the location

Figure 3.12: Response of the System to $\theta_0 = \frac{\pi}{18} \ (10^\circ)$
of the system's transfer function zeros may change the damping in the system to outside of the desired range (Clark, 1962)).

- As in the case of the NGO controller, the cost function being optimized has not necessarily been proven to be smooth and convex. As such, Powell's method cannot guarantee global optimization.
Chapter 4

Output-Feedback Stabilization

Figure 4.1 shows a typical closed-loop control system, where the plant is the system being controlled. The sensor on the plant measures some output \( y(t) \) of the plant (such as position, velocity, temperature, voltage, etc.). The controller compares \( y(t) \) to the desired or reference signal \( y_{\text{ref}}(t) \) and calculates an input signal \( u(t) \). The actuator responds to \( u(t) \) by providing a forcing function to the plant, which changes the dynamics of the system. Often the dynamics of the actuator, plant, and sensor are lumped together for the purpose of designing a controller.

In order to systematically design a controller, it must first be decided what dynamic qualities are desired in a controller (see Section 2.3). The most important quality of a controller is stabilization of the system. Given any set of states \( x \) that define a minimal realization

\[ u(t) \quad \text{Actuator} \rightarrow \quad \text{Plant} \rightarrow \quad \text{Sensor} \rightarrow \quad y(t) \]

\[ \text{Controller} \quad \leftarrow \quad y_{\text{ref}}(t) \]

Figure 4.1: A Typical Closed-Loop Control System
\[
\dot{x}(t) = Ax(t) + Bu(t) \tag{4.1}
\]
\[
y(t) = Cx(t)
\]

of the linear time-invariant actuator-plant-sensor dynamics, a controller stabilizes the plant if and only if the corresponding realization of the closed-loop system
\[
\dot{x}(t) = A_{cl}x(t) + Bu(t) \tag{4.2}
\]
\[
y(t) = Cx(t)
\]
is such that all of the eigenvalues of \( A_{cl} \) have negative real parts (Kailath, 1980). Section 4.1 presents an algorithm to stabilize a linear time-invariant plant with a static-output-feedback controller. A static-output-feedback controller finds a constant feedback matrix, \( K \), such that the input signal \( u(t) = Ky(t) \) stabilizes the system (or, equivalently, the eigenvalues of \( A_{cl} = A + BK C \) are negative). As explained in Section 4.1, few such linear-matrix-inequality based (LMI-based) algorithms have been presented in the literature to date. Unlike previous algorithms, the algorithm of Section 4.1 is not iterative in LMI solutions.

Another important constraint on control systems implementation is the fact that physical actuators may saturate if the value of \( u(t) \) crosses some threshold value. More expensive actuators may be used to decrease the likelihood of saturation. On the other hand, in order to reduce cost it is desired that the maximum value of the signal \( u(t) \) (the amount of control effort) be minimized. A reduction of the feedback norm might lower the cost of implementing a control system by requiring smaller and less-expensive actuators to implement the design. The controller design method developed in Section 4.2 will reduce the norm of the feedback gain required to meet internal stability criteria. Such controllers will produce reduced effort stabilization.

Other qualities desired in a controller include performance and robustness. Performance may be measured by response time, steady-state error, damping, and
various other quantities which may depend on a given class of reference signals under consideration. For this research, performance is enforced via a prescribed degree of stability (as described in Sections 2.4, 3.2, and in subsequent sections of this chapter). In the future, this may be expanded to eigenvalue placement as in Sections 2.4 and 3.2. Robustness may be achieved by addressing expected uncertainties in the plant and their effect on the stability and performance of the controller. In Chapter 5, the controller design method developed in this chapter will be extended to include robustness considerations.

4.1 Output-Feedback Stabilization

Over the past decades, many advances have been made in the field of Control Theory, such as $\mathcal{H}_2$ (or LQ) and $\mathcal{H}_\infty$ control (Doyle et al., 1989). Many of these advances rely on powerful tools of state-space theory. However, the resulting systems have for the large part been limited to state-feedback control and dynamic-output-feedback extensions of state-feedback control (Zhou et al., 1996). Even the so-called mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control described in (Doyle et al., 1994; Zhou et al., 1994) requires dynamic output feedback.

State-feedback systems require the measurement of every system state, some of which may be difficult if not impossible to measure. On the other hand, dynamic-output-feedback systems (which include systems with state observers) result in high-order controllers which may not be practical in industry. Controllers using static output feedback are less expensive to implement and more reliable because they do not require the computer processors and state estimators used to implement dynamic-output-feedback control schemes. The more complicated any system is, the greater the number of individual parts that are likely to fail. If performance specifications are met by a static-output-feedback system, the implementation of such a system would be preferred over more complex feedback systems due to its intrinsic simplicity. Several researchers have characterized the problem of finding
a stabilizing static-output-feedback controller (Levine & Athans, 1970; Bernstein, 1987; Mäkilä & Toivonen, 1987; Gu, 1990; Skelton & Xu, 1990; Trofino-Neto & Kucera, 1993), but few algorithms have been developed which solve the problem (Mäkilä & Toivonen, 1987; Geromel et al., 1994; Iwasaki et al., 1994; Geromel et al., 1996).

Recently, the theory of linear matrix inequalities (LMI's) (Boyd et al., 1994b) has placed the systematic design of static-output-feedback systems within reach. Geromel, Iwasaki, and Skelton (Geromel et al., 1994; Iwasaki et al., 1994; Iwasaki & Skelton, 1995) have presented procedures which use LMI methods to design static-output-feedback controllers based on a set of Lyapunov inequalities coupled by the constraint that one Lyapunov matrix is the inverse of another. Geromel et al. (Geromel et al., 1994) showed that although the problem of designing a static-output-feedback controller is not convex, a min/max algorithm may be used to solve inversely-coupled Lyapunov inequality problems. A modified version of the min/max algorithm presented in (Geromel et al., 1994) is used by Iwasaki et al. in (Iwasaki et al., 1994) to design static-output-feedback controllers bounded by a given linear quadratic (LQ) performance index. In (Iwasaki & Skelton, 1995), all stabilizing controllers are parameterized by a set of inversely-coupled Lyapunov inequalities similar to the inequalities solved by the min/max algorithm of (Geromel et al., 1994). Iwasaki and Skelton (Iwasaki & Skelton, 1994) showed that an LMI method may be used directly to design a low-order controller only if the order of the controller is not fixed a priori, which excludes the case of static output feedback.

Anderson and Moore (Anderson & Moore, 1969) presented a simple method for guaranteeing the response time of a state-feedback system by prescribing a degree of stability. The notion of a prescribed degree of stability is important in the field of Linear Quadratic Regulators (LQR). Prescribing a degree of stability is equivalent to placing the eigenvalues of a system to the left of an arbitrary vertical line in the left half plane. A system with prescribed stability $\alpha$ is known to decay
faster than $e^{-at}$. The notion of setting a prescribed degree of stability has been the basis for much research on eigenvalue placement for state-feedback systems (see Section 2.4) and will certainly be important for the class of static-output-feedback systems as well.

This section solves the problem of designing a static-output-feedback controller via an algorithm which is fundamentally different from the min/max algorithm of (Geromel et al., 1994). The algorithm will be used to prescribe a degree of stability, while keeping the feedback gain small. Section 4.1.1 describes theories which form the basis for previous algorithms. The problem is restated in Section 4.1.2. Section 4.1.3 presents several lemmas which are used in the formulation of the new algorithm, and Section 4.1.4 defines new terms which will aid in the understanding of the problem. Section 4.1.5 presents the new algorithm. Static-output-feedback design examples and a summary of the method follow in Sections 4.1.6 and 4.1.7.

4.1.1 Preliminaries

Let (4.1) be a state-space representation of a given linear system. Lyapunov stability theory yields the following well-known theorem (Boyd et al., 1994b):

**Theorem 1** The system represented by (4.1) is asymptotically stable if and only if there is a positive definite matrix $P$ such that

$$A^TP + PA < 0.$$  \hspace{1cm} (4.3)

**Corollary 1** The system represented by (4.1) is static-output-feedback stabilizable if and only if there is a positive definite matrix $P$ and an appropriately dimensioned matrix $K$ such that

$$(A + BKC)^TP + P(A + BKC) < 0.$$  \hspace{1cm} (4.4)

**Proof.** Corollary 1 results from applying Theorem 1 to output feedback systems.
Unfortunately, (4.4) is not jointly convex in $K$ and $P$ (Boyd et al., 1994b). However, once a $P$ is specified for which a $K$ is known to exist, (4.4) may easily be solved for a stabilizing $K$.

The following lemma is proved in (Boyd et al., 1994b; Geromel et al., 1994; Iwasaki et al., 1994):

**Lemma 1** Given $G$, $U$, and $V$, there exists an $X$ such that

$$G + UXV^T + VXTU^T > 0 \quad (4.5)$$

if and only if

$$U^\perp GU^\perp > 0 \quad \text{and} \quad V^\perp GV^\perp > 0 \quad (4.6)$$

hold where $U^\perp$ and $V^\perp$ are orthogonal complements of $U$ and $V$, respectively.

Note that the definition of $U^\perp$ and $V^\perp$ as orthogonal complements follows the definition of $\bar{U}$ and $\bar{V}$ in (Boyd et al., 1994b), which is the transpose of the definition of $U^\perp$ and $V^\perp$ contained in (Iwasaki et al., 1994).

Geromel et al. (Geromel et al., 1994) and Iwasaki et al. (Iwasaki et al., 1994) use Corollary 1 and Lemma 1 to prove Theorem 2. It is not necessary to duplicate the work in (Geromel et al., 1994; Iwasaki et al., 1994) in proving Theorem 2, however to aid the reader in understanding the theorem, the outline for an alternate proof is presented here. Sylvester's law of inertia given in Theorems 2-7-2 and 2-7-2b of (Barnett & Storey, 1970) may be interpreted to mean that if $P$ is a non-singular matrix then $H < 0$ is a Hermitian matrix if and only if $P^*HP < 0$, where $P^*$ is the conjugate-transpose of $P$. Because $P^{-1}$ is non-singular and symmetric, pre- and post-multiplication of the matrix inequality in (4.4) by $P^{-1}$ yields

$$P^{-1}(A + BKC)^T + (A + BKC)P^{-1} < 0. \quad (4.7)$$

By application of Lemma 1 to the inequality in (4.7) with $G = P^{-1}A^T + AP^{-1}$, $U = P^{-1}C^T$, $X = K^T$, and $V^T = B^T$, and by application of Lemma 1 to the
inequality in (4.4) with $G = A^TP + PA$, $U = C^T$, $X = K^T$, and $V^T = B^TP$, the following theorem may be proved:

**Theorem 2** The system represented by (4.1) is static-output-feedback stabilizable if and only if there is a positive definite matrix $P$ such that

$$B^T(P^{-1}A^T + AP^{-1})B < 0 \quad (4.8)$$

and

$$C^T(P^T(A^TP + PA)C < 0 \quad (4.9)$$

hold, where $B^\perp$ and $C^T$ are orthogonal complements of $B$ and $C^T$, respectively.

The min/max algorithm developed by Geromel et al. (Geromel et al., 1994) attempts to solve (4.8) and (4.9) simultaneously for $P$. The algorithm developed in this section uses Theorem 3, below, to restate the problem in a way that eliminates the need to simultaneously solve the inversely-coupled Lyapunov inequalities (4.8) and (4.9) for $P$.

### 4.1.2 A New Statement of the Problem

Given $G$, $U$, and $U^\perp$, where $U^\perp$ is an orthogonal complement of $U$, (Boyd et al., 1994b) uses Finsler's lemma to show that

$$U^\perp^TGU^\perp > 0 \quad (4.10)$$

if and only if there exists a $\sigma$ such that

$$G - \sigma UU^T > 0. \quad (4.11)$$

Now Theorem 2 may be restated as

**Theorem 3** The system represented by (4.1) is static-output-feedback stabilizable if and only if there is a positive definite matrix $P$ and a real scalar $\sigma$ such that

$$A^TP + PA - PBB^TP < 0 \quad (4.12)$$
and

\[ A^T P + PA - \sigma C^T C < 0 \]  \hspace{1cm} (4.13)

hold simultaneously.

Proof. Assume (4.8) and (4.9) hold for some \( \bar{P} \). Finsler's lemma may now be applied to obtain new static-output-feedback stabilizability conditions

\[ \bar{P}^{-1} A^T + A \bar{P}^{-1} - \sigma_1 B B^T < 0 \]  \hspace{1cm} (4.14)

and

\[ A^T \bar{P} + \bar{P} A - \sigma_2 C^T C < 0, \]  \hspace{1cm} (4.15)

respectively. Let \( P = \sigma_1 \bar{P} \) and \( \sigma = \sigma_1 \sigma_2 \). Dividing (4.14) by \( \sigma_1 \) and multiplying (4.15) by \( \sigma \) results in

\[ P^{-1} A^T + AP^{-1} - B B^T < 0 \]  \hspace{1cm} (4.16)

and (4.13), respectively. Pre- and Post-multiplication of (4.16) by \( P \) yields the inequality in (4.12).

The problem has been changed from the problem of simultaneously solving a set of inversely-coupled Lyapunov inequalities to the problem of simultaneously solving an algebraic Riccati inequality (ARI) and a Lyapunov inequality. Due to the relationship between Lyapunov inequalities and ARI's, the latter problem may also be viewed as a set of simultaneous ARI's.

4.1.3 Some Useful Lemmas

The following lemmas are used in the implementation of an algorithm based on the problem of simultaneously solving an ARI and a Lyapunov inequality:

**Lemma 2** Let \( A \in \mathcal{R}^{n \times n} \), \( R = B B^T \), and \( Q = 0_{n \times n} \). Let \( P_{sf} \) be the stabilizing solution of the algebraic Riccati equation (ARE)

\[ A^T P_{sf} + P_{sf} A - P_{sf} RP_{sf} + Q = 0_{n \times n}. \]  \hspace{1cm} (4.17)

Let \( \lambda(A) \) represent the set of eigenvalues of \( A \). Let \( \lambda^+(A) \) represent the set of unstable eigenvalues of \( A \), and let \( \lambda^-(A) \) represent the set of stable eigenvalues of \( A \). If \( K_{sf} = -B^T P_{sf} \) then both of the following statements are true.
• for every \( \lambda_i \) contained in \( \lambda^-(A) \), \( \lambda_i \) is also contained in \( \lambda(A + BK_{sf}) \)

• for every \( \lambda_i \) contained in \( \lambda^+(A) \), \(-\lambda_i \) is also contained in \( \lambda(A + BK_{sf}) \)

**Proof.** See (Kailath, 1980; Kawasaki & Shimemura, 1983).

**Lemma 3** Let \( n^- \) be the number of stable and marginally stable eigenvalues of \( A \). Let \( \lambda_i^- \) and \( \mathbf{v}_i^- \) represent these eigenvalues and associated eigenvectors. The stabilizing solution of the equation

\[
A^TP + PA - PBB^TP = 0_{n\times n}
\]

satisfies

\[
\text{null}(P) = \text{span}(\mathbf{v}_1^-, \mathbf{v}_2^-, \ldots, \mathbf{v}_{n^-}^-)
\]

where \( \text{span}(\mathbf{v}_1^-, \mathbf{v}_2^-, \ldots, \mathbf{v}_{n^-}^-) \) denotes the linear subspace spanned by the vectors \( \mathbf{v}_1^-, \mathbf{v}_2^-, \ldots, \mathbf{v}_{n^-}^- \). Furthermore the eigenvalues of \( A - \sigma BB^TP \) are \( \{\lambda_1^-, \lambda_2^-, \ldots, \lambda_{n^-}^- \} \) and \( n - n^- \) pure left half plane eigenvalues where \( \sigma \) is an arbitrary real number satisfying \( \sigma > \frac{1}{2} \).

**Proof.** See (Kawasaki & Shimemura, 1983).

**Lemma 4** Let \( A \in \mathbb{R}^{n\times n}, R > 0, \) and \( Q \geq 0 \). Let \( P_\alpha \) be the stabilizing solution of the algebraic Riccati equation (ARE)

\[
(A + \alpha I)^TP_\alpha + P_\alpha(A + \alpha I) - P_\alpha BB^TP_\alpha + Q = 0_{n\times n}.
\]

Let \( \lambda(A) \) represent the set of eigenvalues of \( A \). If \( K_\alpha = -B^TP_\alpha \) then \( \lambda_i < -\alpha \) for every \( \lambda_i \) contained in \( \lambda(A + BK_\alpha) \) and the controlled system \( (A + BK_\alpha) \) is said to have prescribed degree of stability \( \alpha \).

**Proof.** See (Anderson & Moore, 1969).

In fact, it can be shown via Lemma 3 that every \( K_{\alpha,\sigma} = -\sigma B^TP_\alpha \), where \( \sigma > \frac{1}{2} \) and \( P_\alpha \) is the stabilizing solution to
results in a system \((A + BK_{\alpha,s})\) with prescribed degree of stability \(\alpha\).

The following lemma is used to find a static-output-feedback matrix, \(K\), once a Lyapunov matrix has been found from which it may determined via Theorem 3 that such a \(K\) exists. As a substitute, an analyst could use parameterizations contained in (Iwasaki et al., 1994; Iwasaki & Skelton, 1995). However in this research, the LMI method in Theorem 3 is used to provide continuity between the method presented in this section and that of Section 4.2. The information contained in Lemma 5 is explained in more detail in Section 4.2.1, where it plays a central role in the algorithm of Section 4.2.

**Lemma 5** Let \(H(K) > 0\) be an LMI in \(K\), and let \(M > 0\). If \(\|\cdot\|_{M_2}\), the \(M\)-scaled 2-norm (\(M\)-norm), is defined as \(\|x\|_{M_2} = (x^T M^{-1} x)^{\frac{1}{2}}\), and the \(\text{vec}(-)\) operator is defined as

\[
\text{vec}(A) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}
\] (4.22)

where \(A = [a_1, a_2, \ldots, a_n]\) \((a_i\) is the \(i^\text{th}\) column vector of \(A\), \(i = 1, 2, \ldots, n\)), then the problem

\[
\text{minimize } \|k\|_{M_2}
\] (4.23)

subject to \(H(K) > 0\),

where \(k = \text{vec}(K)\), is equivalent to the following LMI problem:

\[
\text{minimize } \lambda
\] (4.24)

subject to

\[
\begin{bmatrix}
\lambda & k^T & 0 \\
k & M & 0 \\
0 & 0 & H(K)
\end{bmatrix} > 0.
\]
Proof. By the nature of LMI's,  

\[
\text{diag}\left( \begin{bmatrix} \lambda & k^T \\ k & M \end{bmatrix}, H(K) \right) > 0 \quad (4.25)
\]

holds if and only if  

\[
\begin{bmatrix} \lambda & k^T \\ k & M \end{bmatrix} > 0 \quad (4.26)
\]

and \( H(K) > 0 \). Schur complements may be used to show that (4.26) holds if and only if \( M > 0 \) and \( \lambda > ||k||_{M^2} \). Taking the variables as \( \lambda \) and \( K \), minimizing \( \lambda \) results in minimizing \( ||k||_{M^2} \).

4.1.4 Definitions

The system represented by (4.1) is said to be stabilizable if and only if there exists a state-feedback matrix \( K_{sf} \) such that the matrix \( (A + BK_{sf}) \) is stable (Kailath, 1980) (in this research, the subscript “sf” indicates state feedback). Rewriting (4.4) for the state-feedback case results in the condition that the pair \((A, B)\) is stabilizable if and only if there are matrices \( P > 0 \) and \( K_{sf} \) such that  

\[
(A + BK_{sf})^T P + P(A + BK_{sf}) < 0, \quad (4.27)
\]

which is equivalent to the condition that there is a \( P \) such that (4.8) holds. By duality, the pair \((A, C)\) is detectable if and only if there is a \( P \) such that (4.9) holds. The triplet \((A, B, C)\) is jointly stabilizable and detectable (JSD) if and only if \((A, B)\) is stabilizable and \((A, C)\) is detectable (Kailath, 1980).

In this research, the triplet \((A, B, C)\) is said to be simultaneously stabilizable and detectable (SSD) if and only if there is a \( P \) such that (4.8) and (4.9) both hold simultaneously. By definition, every SSD system is also JSD, but there may exist JSD systems which are not also SSD. Further work is necessary to determine the exact relationship between JSD systems and SSD systems, however the following theorem shows the importance of the class of SSD systems.

Theorem 4 The realization \((A, B, C)\) in (4.1) may be stabilized using static output feedback if and only if the realization is simultaneously stabilizable and detectable (SSD).
**Proof.** Using the definition of SSD, the conditions in Theorem 2
(and, by equivalence, Theorem 3) are automatically met.

Theorem 3 has reduced the problem of finding a stabilizing output feedback
matrix $K$ to the problem of finding a $P$ and a $\sigma$ such that (4.12) and (4.13)
hold simultaneously. The author has been lead by the ARI condition in (4.12) to
investigate the following simplification of the problem. The ARE

$$A^TP_s + P_sA - P_sBB^TP_s = -\epsilon I \quad (4.28)$$

may be solved and $K_{sf} = -B^TP_s$, chosen as a state feedback controller. Now, the
system $(A + BK_{sf})$ is stable, and there is a $P > 0$ such that

$$(A + BK_{sf})^TP + P(A + BK_{sf}) < 0. \quad (4.29)$$

Once the state feedback $K_{sf}$ has been fixed, the following LMI feasibility problem
may be solved for $P$:

$$\begin{align*}
\text{find } \sigma, P \\
H_B(P) = -(A + BK_{sf})^TP - P(A + BK_{sf}) > 0 \\
H_C(\sigma, P) = -A^TP - PA + \sigma C^TC > 0
\end{align*} \quad (4.30)$$

such that $P > I$ and, $\sigma > 0$.

If the LMI problem in (4.30) is feasible for the given stabilizing matrix, $K_{sf}$,
then the realization $(A, B, C)$ in (4.1) is said to be simultaneously $K$-stable and
detectable (SKSD). If a system is SKSD, it is also SSD, by definition. Furthermore,
any matrix $P$ which satisfies the LMI problem in (4.30) may be used in Theorem
2 or Theorem 3 to show that the $(A, B, C)$ is static-output-feedback stabilizable.

It is important to note that the definition of the class of SKSD systems de­
pend on the stabilizing matrix $K_{sf}$. For some SSD systems, there may exist a
stabilizing matrix, $K_{sf}$, for which the LMI problem in (4.30) is infeasible. Con­
ditions on a system which guarantee the existence a $K_{sf}$ matrix which may be
used to show that the system is SKSD are unknown. Furthermore, a parameterization of all $K_{sf}$ matrices for which SSD systems are guaranteed to be SKSD is also needed. However, because no such parameterization is presently known, $K_{sf}$ should be chosen based on Lemma 2 or Lemma 3.

4.1.5 Algorithm

This section contains an algorithm for stabilizing systems via output feedback using linear matrix inequalities. The algorithm is based on insights gained in the discussion of simultaneously stabilizable and detectable systems from Section 4.1.4.

To stabilize a system represented by the realization $(A, B, C)$ in (4.1):

1. Define $A_\alpha = A + \alpha I$, where $\alpha$ is the desired prescribed degree of stability.

2. Solve the algebraic Riccati equation

$$A_\alpha^TP_{sf} + P_{sf}A_\alpha - P_{sf}BB^TP_{sf} + \epsilon I = 0,$$

where $\epsilon > 0$ is arbitrarily small.

3. Set $K_{sf} = -(1 + \gamma)B^TP_{sf}$, where $\gamma > 0$ is arbitrarily small.

4. Solve for $P$ using the LMI feasibility problem

$$\text{find } \sigma, P$$

such that

$$H_B(P) = -(A_\alpha + BK_{sf})^TP - P(A_\alpha + BK_{sf}) > 0$$

$$H_C(\sigma, P) = -A_\alpha^TP - PA_\alpha + \sigma C^TC > 0$$

with $P > I$ and $\sigma > 0$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
5. Solve for $K$ using the LMI minimization problem

\begin{equation}
\text{minimize } \lambda \quad (4.33)
\end{equation}

subject to

\[ F_2(\lambda, k) = \begin{bmatrix} \lambda & k^T \\ k & M \end{bmatrix} > 0 \]

and

\[ H(K) = -(A_a + BKC)^TP - P(A_a + BKC) > 0, \]

where $k = \text{vec}(K)$, and $M$ is a specified positive definite matrix ($M = I$ results in minimizing the Frobenious norm).

Step 1 is used to set the problem up to allow a prescribed degree of stability $\alpha$. Steps 2 and 3 are used to find a state-feedback matrix which stabilizes the system. The values of $\epsilon$ and $\gamma$ may be used to address tolerance issues important to any algorithm. For the examples in Section 4.1.6, values of $\epsilon = 0$ and $\gamma = 10^{-6}$ are used. Step 4 finds a Lyapunov matrix $P$ for which a static-output-feedback matrix $K$ is known to exist. Step 5 finds the $K$ matrix with the smallest $M$-norm for which $P$ may be used to prove stability. There may be stabilizing $K$ matrices with smaller $M$-norm, however the corresponding $P$ matrix would need to be known.

4.1.6 Examples

The following examples demonstrate the use of the algorithm given in Section 4.1.5. The first two examples are simple second-order systems given in controller canonical form. A second-order system given as

\[ \begin{bmatrix} a_1 & a_2 \\ 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_c = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \quad (4.34) \]

is stable if and only if both $a_1$ and $a_2$ are negative. The stabilizability of such a system may be determined by inspection of the elements $c_1$ and $c_2$ of the output.
matrix. The system is output-feedback stabilizable if and only if there is a $k$ such that $a_1 + kc_1 < 0$ and $a_2 + kc_2 < 0$. Prescribing a degree of stability via Lemma 4 takes the matrices out of controller form, which takes much of the simplicity out of the analysis. However, prescribing a degree of stability is included in the examples to illustrate how it may be used to design a system with improved performance.

Example 1 Let $(A, B, C)$ be defined as follows:

$$A = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}. \quad (4.35)$$

Find a stabilizing output-feedback matrix $K$ for each of the following prescribed degrees of stability: (i) $\alpha = 0$, (ii) $\alpha = 0.5$, and (iii) $\alpha = 1$. By inspection of the given $A$, $B$, and $C$ matrices with $\alpha = 0$, it is known that the minimum $K$ which stabilizes the system for case (i) is $K = -2$.

Solution.

(i) The algorithm of Section 4.1.5 is used to calculate the following matrices:

$$P_{sf} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix},$$

$$K_{sf} = \begin{bmatrix} -2.000002 & -4.000004 \\ -4.000004 & -2.000002 \end{bmatrix},$$

$$\sigma = 1.837461 \times 10^7,$$

and

$$P = \begin{bmatrix} 1925217 & 1237977 \\ 1237977 & 1213902 \end{bmatrix}.$$  

The resulting output-feedback matrix is $K = -2.000175$. The closed loop system has eigenvalues at $-0.9998$ and $-0.0002$.

(ii) For the case of $\alpha = 0.5$, the algorithm is unable to solve the ARE in Step 2. In fact, the ARE is unsolvable. To overcome this difficulty, the ARE may be altered by setting $\epsilon > 0$ ($\epsilon = 10^{-15}$, for example) in Step 2. The ARE would now be solvable, however the algorithm would break down in Step 4, because there is no output-feedback matrix $K$ such that the system is asymptotically stable with prescribed degree $\alpha = 0.5$ (if no $K$ exists, then no $P$ exists either). The problem must therefore be relaxed to find an output-feedback matrix $K$ such that
the system is stable with prescribed degree $\alpha = 0.49$. The algorithm results in $K = -4.222279$ with eigenvalues at $-0.5000 \pm 1.4044j$.

(iii) Due to reasons mentioned above, there is no solution for the case of $\alpha = 1.0$, and therefore use of the algorithm yields no such solution.

Example 2 Let $(A, B, C)$ be defined as follows:

$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$. \hspace{1cm} (4.36)

Find a stabilizing output-feedback matrix $K$ for each of the following prescribed degrees of stability: (i) $\alpha = 0$, (ii) $\alpha = 0.5$, and (iii) $\alpha = 1$. By inspection of the given $A$, $B$, and $C$ matrices with $\alpha = 0$, it is known that the minimum $K$ which stabilizes the system for case (i) is $K = -3$.

Solution.

(i) The algorithm calculates the following matrices:

$$P_{sf} = \begin{bmatrix} 6.000000 & -2.085753 \times 10^{-16} \\ -2.085753 \times 10^{-16} & 12.000000 \end{bmatrix}, \quad K_{sf} = \begin{bmatrix} -6.000006 & 2.085755 \times 10^{-16} \end{bmatrix},$$

$$\sigma = 2.081254 \times 10^7,$$

and

$$P = \begin{bmatrix} 1398215 & 112408.9 \\ 112408.9 & 3573905 \end{bmatrix}.$$

The resulting output-feedback matrix is $K = -3.494809$. The closed loop system has eigenvalues at $-0.2474 \pm 1.3924j$. This system would not perform well due to the placement of two imaginary eigenvalues close to the imaginary axis, however such considerations may be addressed via the prescribed degree of stability, as in the following discussion.

(ii) For the case of $\alpha = 0.5$, the algorithm results in $K = -4.513096$ with eigenvalues at $-0.7565 \pm 1.1948j$.

(iii) For the case of $\alpha = 1.0$, the algorithm results in $K = -5.832926$ with eigenvalues at $-1.4963$ and $-1.3367$.

It is known that the system in Example 2 is solvable for values of $\alpha < \sqrt{2}$, however the algorithm of Section 4.1.5 is unable to find output feedback gains for
the system in Example 2 with $\alpha > 1$. In fact, the algorithm is unable to find a $P$ for the case of $\alpha = 1.001$. The source of this problem is unknown, but likely to be the use of a Riccati equation to approximate the quadratic inequality in 4.12. As a result of this observation and the comments mentioned in Part (ii) of Example 1, it is quite possible that a given problem which may not be solved using the algorithm for $\alpha$, may be solved for $\alpha - \epsilon$, where $0 < \epsilon \ll \alpha$ is an arbitrarily small relaxation factor. If this is the case, the system would decay faster than $e^{-(\alpha-\epsilon)t}$. Depending on the size of $\epsilon/\alpha$, this may be acceptable. Care should be taken in choosing the value of $\alpha$ to be no more than the problem requires. If $\alpha$ is chosen too large, the system may not be solvable via the algorithm, or in many cases (such as Example 1) by any algorithm.

Finally, an example of an oversteer highway vehicle is given, where the forward speed of the vehicle is such that the vehicle is unstable. Vehicles may be oversteer by design (such as mid-engine vehicle which is used in Example 3), by improper loading (shifting the center of gravity closer to the rear of the car), or by driving on tires with different properties on the front and rear axles.

**Example 3** An oversteer vehicle being driven at a speed above its critical speed is unstable. The vehicle (see Figure 4.2) may be represented by the following state-space model:

\[
A = \begin{bmatrix}
    -\frac{1}{\tau_{\omega}} & 0 & 0 \\
    \frac{2GC_{a1}}{m} & -2U\left(C_{af} + C_{ar}\right) & -U - \frac{2}{mU}\left(aC_{af} - bC_{ar}\right) \\
    \frac{2aC_{a1}}{I_{ez}} & -2\frac{aC_{af} - bC_{ar}}{I_{ezU}} & -\frac{2}{I_{ezU}}\left(a^2C_{af} + b^2C_{ar}\right)
\end{bmatrix}, \quad (4.37)
\]

\[
B = \begin{bmatrix}
    \frac{G_{a1}}{\tau_{\omega}} \\
    0 \\
    0
\end{bmatrix}, \quad C = \begin{bmatrix}
    0 & 0 & 1
\end{bmatrix}, \quad (4.38)
\]

where the state vector is

\[
x(t) = \begin{bmatrix}
    \delta_f \\
    V \\
    r
\end{bmatrix}, \quad (4.39)
\]
δf is the front steer angle generated by the actuator, V is the lateral velocity, r is the yaw rate, and the vehicle parameters are defined in Table 4.1. The input to the system is the voltage signal \( u(t) = e_m(t) = Kx(t) \). Find a stabilizing output-feedback matrix \( K \) for each of the following prescribed degrees of stability: (i) \( \alpha = 0 \) and (ii) \( \alpha = 1 \).

![Figure 4.2: Yaw-Plane Model of Vehicle Dynamics](image)

**Solution.**

(i) The algorithm calculates the following matrices:

\[
P_{sf} = \begin{bmatrix}
2.092563 \times 10^{-2} & -4.626967 \times 10^{-4} & 5.041671 \times 10^{-3} \\
-4.626967 \times 10^{-4} & 1.023091 \times 10^{-5} & -1.114788 \times 10^{-4} \\
5.041671 \times 10^{-3} & -1.114788 \times 10^{-4} & 1.214704 \times 10^{-3}
\end{bmatrix},
\]

\[
K_{sf} = \begin{bmatrix}
-0.209256 & 4.626972 \times 10^{-3} & -5.041676 \times 10^{-2}
\end{bmatrix},
\]

\( \sigma = 1.768091 \times 10^7 \),

and

\[
P = \begin{bmatrix}
4.645730 \times 10^7 & -18130.45 & 4373877 \\
-18130.45 & 9335.091 & 120992.6 \\
4373877 & 120992.6 & 3120429
\end{bmatrix}.
\]

The resulting output-feedback matrix is \( K = -0.0689857 \). The closed loop system has eigenvalues at \(-7.5324 \pm 4.6321j\) and \(-0.1500\).

(ii) For the case of \( \alpha = 1.0 \), the algorithm results in \( K = -0.2056664 \) with eigenvalues at \(-6.9838 \pm 9.1279j\) and \(-1.2471\).  

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Table 4.1: Parameters Used in Vehicle Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
<td>forward velocity</td>
<td>40 m/s</td>
</tr>
<tr>
<td>$a$</td>
<td>distance from the center of gravity to the front axle</td>
<td>1.3655 m</td>
</tr>
<tr>
<td>$b$</td>
<td>distance from the center of gravity to the rear axle</td>
<td>1.0089 m</td>
</tr>
<tr>
<td>$m$</td>
<td>vehicle mass</td>
<td>1177.217 kg</td>
</tr>
<tr>
<td>$I_{zz}$</td>
<td>vehicle yaw moment of inertia</td>
<td>1621.558 kg m$^2$</td>
</tr>
<tr>
<td>$C_{af}$</td>
<td>tire cornering stiffness per front tire</td>
<td>30 kN/rad</td>
</tr>
<tr>
<td>$C_{ar}$</td>
<td>tire cornering stiffness per rear tire</td>
<td>30 kN/rad</td>
</tr>
<tr>
<td>$G_{ss}$</td>
<td>steady-state gain for the steering system</td>
<td>1.0</td>
</tr>
<tr>
<td>$\tau_{sw}$</td>
<td>time constant for the steering system</td>
<td>0.1 rad/s</td>
</tr>
</tbody>
</table>
4.1.7 Summary

The problem of determining whether a system is output feedback stabilizable has been restated to the problem of simultaneously solving an algebraic Riccati inequality and a Lyapunov inequality. Based on the newly restated problem, two classes of systems have been defined. The set of simultaneously stabilizable and detectable (SSD) systems has been shown to be equivalent to the set of output-feedback stabilizable systems, and the set of simultaneously $K$-stable and detectable (SKSD) systems has been shown to be a subset of the set of SSD systems. An initial approach has been made at deriving an algorithm to solve the redefined problem. The algorithm is based on a procedure which uses the well known solution to the algebraic Riccati equation (ARE) to find $K_{sf}$ and two linear matrix inequalities (LMI's), based on $K_{sf}$, to find a static-output-feedback matrix, $K$, which stabilizes the system. The algorithm results in small control gains and may also be used to prescribe a degree of stability. The algorithm is fundamentally different from the min/max algorithm of (Geromel et al., 1994; Iwasaki et al., 1994), yet easy to implement using standard ARE and LMI techniques. Unlike the min/max algorithm of (Geromel et al., 1994; Iwasaki et al., 1994), the algorithm developed in this section does not iterate the solution of LMI problems. Examples have been included to demonstrate the use of the algorithm.

Future work is necessary to differentiate between jointly stabilizable and detectable (JSD) and SSD systems. In addition, parameterization of the set of all state-feedback matrices, $K_{sf}$, for which a system is guaranteed to be SKSD would be helpful, as this would clarify the relationship between SKSD systems and SSD systems. Finally, a direct simultaneous solution of two algebraic Riccati inequalities would be useful in developing future algorithms.

The next section describes a method to reduce the norm of the stabilizing static-output-feedback gains designed in this section.
4.2 Reducing Feedback Gains via Linear Matrix Inequalities

Anderson and Moore, in their book Optimal Control: Linear Quadratic Methods (Anderson & Moore, 1990), state that a system is optimal if it is the best system of a particular type. The big question in optimal-control theory should be "What particular type of control system is best?" Recently, much of the work in optimal control has focused on $H_2$ (or LQ) and $H_\infty$ control (Doyle et al., 1989). However, some researchers have chosen to define optimality to include the minimization of some norm of the feedback matrix. In many papers (Kouvaritakis & Cameron, 1980; Sebok et al., 1986; Cameron, 1988; Ismail & Bandyopadhyay, 1993; Karbassi & Bell, 1994; Benton & Smith, 1996) design methods are proposed which minimize the feedback norm while meeting additional constraints. In addition, many researchers understand the benefit of small feedback gains. For example, Geromel et al. (Geromel et al., 1996) cite the fact that the design technique in (Geromel et al., 1996) results in a lower feedback gain norm than a previous method, although the feedback norm is not specifically minimized. Another example of researchers placing importance on keeping the feedback norm small is (Swei & Corless, 1989), where a guaranteed bound of the feedback norm is implemented.

As stated in (Benton & Smith, 1996) (see also Section 3.2), a decrease in the norm of the control gain reduces the cost of implementing the control system and reduces the possibility of input saturation. In general, higher control gains will lead to increased input signals which may exceed the limits of less-expensive actuators. This would require the designer to obtain more-expensive actuators capable of handling larger signals.

Minimization of a feedback gain norm differs from linear quadratic (LQ) optimal control, in that an LQ controller minimizes a performance index which is an integral over time of a quadratic function of the state, $x(t)$, and the input, $u(t)$, of the system. Thus, the decay rate of a system is measured by LQ performance
indices. Minimization of an LQ performance index will therefore force a tradeoff between performance and control effort. Moreover, the designer may have little, if any, control over this tradeoff due to the fact that the translation of design specifications (such as limits on performance and control effort) into an LQ performance index is imprecise (Anderson & Moore, 1990).

On the other hand, the decay rate of a system has no direct effect on the norm of the control gain, and a method which reduces the feedback norm may be used to decouple design limits on control effort from more performance-oriented design specifications such as decay rate. The decay rate of a system is strongly related to its degree of stability, and Anderson and Moore (Anderson & Moore, 1969) showed that a prescribed degree of stability may be imposed on a system by stabilizing a modified system. The decay rate may be set using a prescribed degree of stability, and the problem of high gain mentioned in (Anderson & Moore, 1969) may be addressed directly by finding reduced gain controllers.

This new paradigm of controller design inverts the goals of optimal control. Instead of optimizing a particular aspect of a system, such as minimizing an LQ performance index or maximizing robustness, the cost of implementing the system may be minimized over the set of controllers which meet a given set of performance and robustness constraints.

Recent research into the systematic design of static-output-feedback controllers (Geromel et al., 1994; Iwasaki et al., 1994; Iwasaki & Skelton, 1995; Benton & Smith, 1997) has benefitted from new advances in the field of linear matrix inequalities (LMI's) (Boyd et al., 1994b). Although much work has been done to minimize, bound, or characterize the LQ performance index for static-output-feedback systems (Iwasaki et al., 1994; Levine & Athans, 1970; Bernstein, 1987; Mäkilä & Toivonen, 1987; Gu, 1990; Skelton & Xu, 1990; Trofino-Neto & Kucera, 1993), the reduction of the norm of an output feedback matrix has yet to be addressed in a systematic manner using LMI techniques.
In this section, an algorithm is introduced which when given an asymptotically stabilizing static-output-feedback matrix finds a new stabilizing feedback matrix which has a smaller norm.

### 4.2.1 Quadratic Minimization using LMIs

The Schur complements method states that the matrix inequality

\[
\begin{bmatrix}
Q(x) & S(x) \\
S(x)^T & R(x)
\end{bmatrix} > 0,
\]  \hfill (4.40)

where \( Q(x) = Q(x)^T \), \( R(x) = R(x)^T \), and \( S(x) \) depend affinely on \( x \), is equivalent (Boyd et al., 1994b) to

\[
R(x) > 0, \quad Q(x) - S(x)R(x)^{-1}S(x)^T > 0. \tag{4.41}
\]

Using Schur complements, one may guarantee that

\[
x^T x < \lambda \tag{4.42}
\]

by taking \( Q = \lambda, \ R = I, \) and \( S(x) = x^T \). If \( \lambda \) is fixed, this is a suboptimal quadratic minimization problem which attempts to find an \( x \) such that (4.42) is true. However, by including \( \lambda \) as one of the variables of the LMI, the problem may be set up to solve for the smallest \( \lambda \) for which (4.42) holds.

Suppose \( M \in \mathbb{R}^{n \times n} \) is a positive-definite matrix and \( \| \cdot \|_2 \) is the 2-norm, then the \( M \)-scaled 2-norm (\( M \)-norm), \( \| \cdot \|_{M_2} \), may be defined as follows:

\[
\| x \|_{M_2} = (x^T M^{-1} x)^{1/2} = \| W x \|_2, \tag{4.43}
\]

where \( x \in \mathbb{R}^n \) and \( M^{-1} = W^T W \). In addition, if a square matrix \( A \) has column vectors \( \{ a_1, a_2, \ldots, a_n \} \) such that \( A = [a_1, a_2, \ldots, a_n] \) then the vec(\( \cdot \)) operator may

---

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
be defined as the vector valued function (Graham, 1981) such that

\[
\text{vec}(A) = \begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{bmatrix}.
\] (4.44)

If the LMI \(H(K) > 0\) guarantees internal stability for the system, and the vector \(k\) is defined as \(k = \text{vec}(K)\) and \(K\) is the feedback matrix, then the problem of finding a smaller stabilizing feedback gain may be stated as

\[
\text{minimize } \|k\|_{M^2}
\] (4.45)

subject to \(H(K) > 0\).

Although (4.45) is not an LMI problem, it may be reformulated using the Schur Complements method to the following LMI problem:

\[
\text{minimize } \lambda
\] (4.46)

subject to \(F(\lambda, K) = \begin{bmatrix}
\lambda & k^T & 0 \\
k & M & 0 \\
0 & 0 & H(K)
\end{bmatrix} > 0.\)

The matrix \(M\) determines the type of quadratic norm being minimized. For this research, \(M\) will be taken as the identity matrix \(I\), which results in the minimization of the Frobenious norm of the feedback matrix \(K\).

**Example 4** Consider the case where \(H(x) = -x - 2\). Solve the quadratic minimization problem

\[
\text{minimize } z^2
\] (4.47)

subject to \(H(x) > 0\).
Solution.
The problem is reformulated as the following LMI problem:

\[ \begin{array}{c}
\text{minimize } \lambda \\
\text{subject to } F(\lambda, x) = \begin{bmatrix}
\lambda & x & 0 \\
x & 1 & 0 \\
0 & 0 & -x - 2
\end{bmatrix} > 0.
\end{array} \]

The solution of this problem is known to be \( \lambda = 4, \ x = -2 \). Application of LMI solution algorithms available in (Bland et al., 1981; Boyd et al., 1994b; Boyd & Wu, 1995) yields the expected result. Specifically, two LMI solution algorithms have been used to test this LMI Quadratic Minimization method. First, the sdpsol parser/solver for semi-definite programs software package developed by Boyd and Wu (Boyd & Wu, 1995) was obtained and used to test the quadratic minimization method. After 10 iterations, the sdpsol program yields a solution of \( \lambda = 4, \ x = -2 \), as expected. In addition, the ellipsoid algorithm described in (Bland et al., 1981; Boyd et al., 1994b) has been implemented as a FORTRAN program by the author. The LMI Quadratic Minimization method was also tested using this program, and although the ellipsoid algorithm is not as fast as the algorithm used in (Boyd & Wu, 1995), the ellipsoid algorithm correctly yields a solution of \( \lambda = 4, \ x = -2 \), as expected.

The feasibility of using LMI methods to quadratically minimize a function has been demonstrated, and the next section discusses the definition of an LMI \( H(K) \) in (4.45) and (4.46) which meets the condition that \( H(K) > 0 \) only if the corresponding feedback matrix \( K \) internally stabilizes the system.

4.2.2 Lyapunov Inequalities and Similarity Transformations

It is well known that a system represented by the state-space realization

\[ \dot{x}(t) = Ax(t) + Bu(t) \]  
\[ y(t) =Cx(t) \] 

(4.48)

is static-output-feedback stabilizable if and only if there is a positive-definite (symmetric) matrix \( P \) and a feedback matrix \( K \) such that the following Lyapunov
inequality is satisfied:

\[(A + BK_iC)^T P + P(A + BK_iC) < 0. \quad (4.49)\]

Although (4.49) is not jointly convex in \(P\) and \(K_i\) (Boyd et al., 1994b), fixing either \(P\) or \(K_i\) results in the convex problem of finding the remaining variable such that (4.49) holds.

The author interprets the Lyapunov matrix, \(P\), in (4.49) as a state transformation. Let \(P = S^*S\) where \(S\) is nonsingular, and \(S^*\) is the conjugate transpose of \(S\). The condition in (4.49) now becomes: find a state-transform matrix, \(S\), and an output-feedback matrix, \(K_i\), such that

\[(A + BK_iC)^T S^* S + S^* S(A + BK_iC) < 0, \quad (4.50)\]

which is equivalent to

\[\left[(SAS^{-1})^* + SAS^{-1} + [(SB)K_i(CS^{-1})]^* + (SB)K_i(CS^{-1})\right] < 0. \quad (4.51)\]

This treatment of \(P\) is similar to Juang’s results (Juang, 1991) on matrix measures.

Assuming that \(A + BK_iC\) has \(n\) distinct eigenvalues, choose \(S\) such that \(S(A+BK_iC)S^{-1}\) is diagonal (i.e. the columns of \(S^{-1}\) are eigenvectors of \(A+BK_iC\)) and define \(A_d \equiv SAS^{-1}, B_d \equiv SB, C_d \equiv CS^{-1},\) and \(Q \equiv A_d^* + A_d\). Now the inequality in (4.51) may be rewritten as

\[Q + [B_dK_iC_d]^* + B_dK_iC_d < 0. \quad (4.52)\]

In addition, the matrix \(\Lambda = Q+[B_dK_iC_d]^*+B_dK_iC_d\) is a diagonal matrix containing the real parts of the eigenvalues of \(A + BK_iC\). This choice of \(S\) allows the region in \(K\)-space which is close to \(K_i\) to be searched for other \(K\) matrices which stabilize the system. Because a choice for \(S\) has been made (which sets a value for \(P\)), the non-convex Lyapunov inequality in (4.49) has been reduced to the LMI

\[H(K) = -(A + BKC)^T P - P(A + BK C) > 0, \quad (4.53)\]

which is convex in \(K\).
This section has presented a method for choosing the function $H(K)$ mentioned in Section 4.2.1. It is important to note, however, that there may exist stabilizing static-output-feedback matrices $K$ for which $H(K)$ is indefinite. Therefore, the quadratic minimization technique presented in Section 4.2.1 will not necessarily produce the smallest stabilizing static-output-feedback matrix $K$. However, given an asymptotically stabilizing static-output-feedback matrix the next section contains an algorithm which is guaranteed to find a smaller stabilizing static-output-feedback matrix.

4.2.3 Algorithm

To find a smaller stabilizing static output-feedback matrix $K$ for a system represented by the realization $(A, B, C)$ in (4.48):

1. Define $A_\alpha = A + \alpha I$, where $\alpha$ is the desired prescribed degree of stability, as described in (Anderson & Moore, 1969).

2. Find a static-output-feedback matrix, $K_0$, that stabilizes $(A_\alpha, B, C)$ using methods described in (Trofino-Neto & Kucera, 1993; Geromel et al., 1996; Benton & Smith, 1997).

3. Set $i=0$.

4. Find (using any standard eigenvalue package) $P_i = S_i^* S_i$ such that the matrix $S_i(A_\alpha + BK_i C)S_i^{-1}$ is diagonal.

5. Fix $P = P_i$ and solve for $K_{i+1}$ using the LMI minimization problem

$$\minimize \lambda$$

subject to

$$F_2(\lambda, k) = \begin{bmatrix} \lambda & k^T \\ k & M \end{bmatrix} > 0$$
and

\[ H(K) = -(A_a + BK_C)^T P - P(A_a + BK_C) > 0, \]

where \( k = \text{vec}(K) \), and \( M \) is a specified positive definite matrix (\( M = I \) results in minimizing the Frobenious norm).

6. Iterate \( i = i + 1 \)

7. If \( ||K_i - K_{i-1}|| > \delta \), where \( \delta \) is a pre-specified tolerance, then go to Step 4.

Step 1 is used to set the problem up to allow a prescribed degree of stability \( \alpha \), which is used to specify the minimum decay rate. Step 4 defines a Lyapunov matrix \( P_i \) based on the discussion contained in Section 4.2.2. Step 5 finds the \( K \) matrix with the smallest \( M \)-norm for which \( P \) may be used to prove stability. For Step 7, \( \delta > 0 \) is arbitrarily small (in Section 4.2.4, \( \delta = 10^{-3} \) is used). The algorithm is iterated until \( ||K_i - K_{i-1}|| \leq \delta \).

The following theorem shows that if \( M = I \) and \( K_0 \) asymptotically stabilizes the system, then the algorithm will always result in controllers with reduced gain.

**Theorem 5** For each iteration of the above algorithm,

\[ ||\text{vec}(K_i)||_2 > ||\text{vec}(K_{i+1})||_2. \]  

**Proof.** Because Step 5 of the algorithm finds the \( K \) with minimum norm which satisfies \( H(K) > 0 \), it is only necessary to prove that for each iteration there is a \( K_{i+1} \) satisfying \( H(K_{i+1}) > 0 \) and (4.55). Let

\[ A_i = [S_i(A_a + BK_iC)S_i^{-1}]^* + S_i(A_a + BK_iC)S_i^{-1}. \]  

Let \( k_j \) be the \( j \)-th element of \( k = \text{vec}(K_i) \), where \( k \in \mathcal{R}^r \), \( K_i \in \mathcal{R}^{m \times p} \), and \( r = mp \). Let \( E_j \in \mathcal{R}^{m \times p} \) be the matrix such that \( \text{vec}(E_j) \) is a vector with a one as its \( j \)-th element and zeros everywhere else. Note that

\[ K_i = \sum_{j=1}^{r} k_j E_j. \]  

Define the matrices

\[ F_j = [S_i(BE_jC)S_i^{-1}]^* + S_i(BE_jC)S_i^{-1}. \]
for each \( j \), such that the equation in (4.56) may be rewritten as

\[
\Lambda_i = [S_i A_\alpha S_i^{-1}]^* + S_i A_\alpha S_i^{-1} + \sum_{j=1}^{r} k_j F_j. \tag{4.59}
\]

Let \( \delta_j > 0 \) be such that \(-\delta_j I < F_j < \delta_j I\), for all \( j \). Because \( \Lambda_i < 0 \), there exists a \( \gamma > 0 \) such that

\[
\Lambda_i \leq -\gamma I \tag{4.60}
\]

(for example, choose \( \gamma \) as the negative of the real part of the eigenvalue closest to the imaginary axis). Define

\[
\Delta k_j = \frac{\gamma}{\delta_j} \tag{4.61}
\]

for each \( j \). Now, for every \( j \) corresponding to a nonzero \( k_j \),

\[
-\gamma I < \Delta k_j F_j < \gamma I, \tag{4.62}
\]

and \( \Delta k_j > 0 \). Combining (4.60) and (4.62),

\[
\Lambda_i \pm \Delta k_j F_j < 0. \tag{4.63}
\]

Pick any \( j \) corresponding to a nonzero \( k_j \). If \( k_j < 0 \) then set

\[
K_{i+1} = K_i + \Delta k_j E_j, \tag{4.64}
\]

which according to the inequality in (4.63) results in \( H(K_{i+1}) > 0 \). Because \( k_j^2 > (k_j + \Delta k_j)^2 \), the inequality in (4.55) is satisfied. If, on the other hand, \( k_j > 0 \) then set

\[
K_{i+1} = K_i - \Delta k_j E_j, \tag{4.65}
\]

which again according to the inequality in (4.63) results in \( H(K_{i+1}) > 0 \). Because \( k_j^2 > (k_j - \Delta k_j)^2 \), the inequality in (4.55) is again satisfied. ⊡

### 4.2.4 Examples

The following example is borrowed from Section 4.1 to demonstrate the use of the algorithm in Section 4.2.3 in conjunction with the algorithm in Section 4.1.5.

**Example 5** Let \((A, B, C)\) be defined as follows:

\[
A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{4.66}
\]
Find a stabilizing output-feedback matrix $K$ for each of the following prescribed degrees of stability: (i) $\alpha = 0$, (ii) $\alpha = 0.5$, and (iii) $\alpha = 1$. By inspection of the given $A$, $B$, and $C$ matrices with $\alpha = 0$, it is known that the minimum $K$ which stabilizes the system for case (i) is $K = -3$.

**Solution.**

(i) Using $K_0 = -3.494809$ as the initial output-feedback matrix (from Example 2), the algorithm in Section 4.2.3 produces the following sequence of gains:

$$
\{K_i\} = \{-3.249327, -3.124907, -3.062484, -3.031246, -3.015623,
-3.007812, -3.003906, -3.001953, -3.000977\}.
$$

After 9 iterations, $\|K_i - K_{i-1}\| = 9.8 \times 10^{-4}$, which is less than $\delta = 10^{-3}$. The resulting feedback is $K = -3.000977$

(ii) For the case of $\alpha = 0.5$, $K_0 = -4.513096$ stabilizes the system (see Example 2). After 10 iterations, the algorithm results in $K = -4.000763$ with $\|K_i - K_{i-1}\| = 7.1 \times 10^{-4}$.

(iii) For the case of $\alpha = 1.0$, $K_0 = -5.832926$ is used (see Example 2). After 15 iterations, the algorithm results in $K = -5.001143$ with $\|K_i - K_{i-1}\| = 8.1 \times 10^{-4}$.

For Example 5, the method described in Section 4.2 finds a smaller static-output-feedback gain than originally given by the method of Section 4.1 where the system was initially stabilized. The two methods may be used together to accomplish the goal of reduced effort stabilization.

The following example consists of stabilizing the nominal linearized model of a vertical takeoff and landing (VTOL) aircraft (helicopter) borrowed from (Singh & Coelho, 1984). Several researchers (Keel et al., 1988; Geromel et al., 1994; Iwasaki et al., 1994) have found stabilizing static-output-feedback matrices which are used to initialize the algorithm in Section 4.2.3.
Example 6 Let \((A, B, C)\) be defined as follows:

\[
A = \begin{bmatrix}
-0.0366 & 0.0271 & 0.0188 & -0.4555 \\
0.0482 & -1.0100 & 0.0024 & -4.0208 \\
0.1002 & 0.3681 & -0.7070 & 1.4200 \\
0 & 0 & 1 & 0
\end{bmatrix}, \quad (4.67)
\]

\[
B = \begin{bmatrix}
0.4422 & 0.1761 \\
3.5446 & -7.5922 \\
-5.5200 & 4.4900 \\
0 & 0
\end{bmatrix}, \quad (4.68)
\]

and

\[
C = \begin{bmatrix}
0 & 1 & 0 & 0
\end{bmatrix}. \quad (4.69)
\]

Find a reduced stabilizing output-feedback matrix \(K\) for each of the following initial stabilizing matrices: (i) \(K_0 = \begin{bmatrix} -1.6352 & 1.5822 \end{bmatrix}^T\) (from (Keel et al., 1988)), (ii) \(K_0 = \begin{bmatrix} -1.6368 & 4.9210 \end{bmatrix}^T\) (from (Iwasa et al., 1994)), and (iii) \(K_0 = \begin{bmatrix} -0.4385 & 2.0334 \end{bmatrix}^T\) (from (Geromel et al., 1994)).

Solution.

(i) The algorithm results in \(K = \begin{bmatrix} -0.2712759 & 0.2999029 \end{bmatrix}^T\) after 14 iterations, with \(\|K_i - K_{i-1}\| = 6.0 \times 10^{-4}\).

(ii) The algorithm results in \(K = \begin{bmatrix} -0.1707336 & 0.3038809 \end{bmatrix}^T\) after 16 iterations with \(\|K_i - K_{i-1}\| = 6.8 \times 10^{-4}\).

(iii) The algorithm results in \(K = \begin{bmatrix} -0.1351162 & 0.3045824 \end{bmatrix}^T\) after 12 iterations with \(\|K_i - K_{i-1}\| = 5.4 \times 10^{-4}\).  

The algorithm reduces the gain for each initial stabilizing feedback matrix in Example 6. Some of the feedback matrices used to initialize the algorithm were designed with different constraints in mind, such as bounded LQ performance, and the new controllers with reduced feedback norm may or may not perform as well as the original feedback gains. However, this example shows that the algorithm in Section 4.2.3 may be used with a variety of stabilization routines, even those

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
unconcerned with reducing the feedback gain norm. Performance concerns may be addressed separately using a prescribed degree of stability as in Example 5.

4.2.5 Summary

An algorithm has been developed based on well-known eigenvalue decomposition techniques and recently developed LMI methods which finds small stabilizing static-output-feedback gains. The algorithm may be initialized with previous methods such as those described in (Trofino-Neto & Kucera, 1993; Geromel et al., 1996; Benton & Smith, 1997). The algorithm has been used to decouple the problem of reducing implementation cost (control effort) from the problem of meeting a performance-oriented design specification (decay rate). Each iteration of the algorithm has been shown to reduce the feedback gain of the system, and use of the algorithm has been demonstrated by example problems.

For each iteration of the algorithm, there may be a stabilizing $K$ matrix with a smaller $M$-norm, however the corresponding $P$ matrix would first need to be known. An interesting problem would be to find a Lyapunov matrix corresponding to the stabilizing $K$ matrix of smallest $M$-norm. This would eliminate the need to iterate the algorithm, however a method which solves for such a $P$ in the output-feedback case is unknown to the author.

In the next chapter, the algorithms in this chapter are expanded to deal with robustness to time-varying uncertainties present in actual control systems.
Chapter 5

Robust Stabilization

This chapter focuses on extending the methods of Chapter 4 to design robust controllers for systems with time-varying uncertainties, such as the highway vehicle system described in Chapter 1. As stated in Section 2.2, a robust control system is a control system which guarantees stability or performance in the presence of plant uncertainty. A controller is said to robustly stabilize a system if the controller guarantees stability in the face of expected uncertainties. On the other hand, a control system exhibits robust performance if the controller guarantees a level of performance despite expected uncertainties. In this chapter, as in previous chapters, performance is defined in terms of decay rate (via a prescribed degree of stability), and therefore robust performance may be achieved for a given system if and only if robust stability may be achieved on a modified system.

Section 2.2 lists several characterizations of system uncertainty. As should be expected, the structure of a method to guarantee robustness in the presence of uncertainty depends greatly on the characterization of uncertainty chosen by the analyst. For this research, a polytope of matrices (see Section 2.2.4) is used to characterize the uncertainty in a system. A set of vertices may be defined as in Section 5.1. Stability of the linear time-varying system is now guaranteed if the vertices are simultaneously stabilized.
In general, a polytope characterization of uncertainties results in less con­servative controller designs (Boyd et al., 1994b). However, there is a price: as the number of uncertain parameters increases, the number of vertices increases exponentially, and the design time also increases exponentially. For systems with a large number of uncertain parameters, solution time for polytope-based design algorithms may become impractical. Norm-bounded uncertainties would better address such problems. In the future, the methods presented in Chapter 4 can be expanded to include norm-bounded uncertainties. However, for the vehicle model presented in Chapter 1, the number of time-varying uncertain parameters is well suited to a polytope characterization of uncertainty.

The theory of polytopic systems is presented in Section 5.1. Section 5.2 describes a state-feedback polytope method available in the literature to robustly stabilize a polytopic system via linear matrix inequalities (LMI's), and Section 5.3 presents a method of optimizing the state-feedback polytope method. Sections 5.4 and 5.5 extend the methods of Sections 4.1 and 4.2, respectively, to guarantee robust stabilization of a polytope of matrices using enlightenment gained from the preliminary sections of this chapter.

5.1 Using Polytopes to Describe Time-Varying Systems

Let the strictly-proper linear time-varying system represented by the state-variable realization

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t) \]  
\[ y(t) = C(t)x(t) \]

have \( r \) time-varying parameters \( q_i(t) \), where

\[ q_i^- \leq q_i(t) \leq q_i^+ , \quad i = 1, \ldots, r, \]  

\[ \text{Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.} \]
and let the matrices $F_i, G_i,$ and $H_i$ be such that the equations

$$A(t) = F_0 + \sum_{i=1}^{r} F_i q_i(t), \quad (5.3)$$

$$B(t) = G_0 + \sum_{i=1}^{r} G_i q_i(t),$$

and

$$C(t) = H_0 + \sum_{i=1}^{r} H_i q_i(t)$$

hold for all time. As mentioned in Section 2.2.4, the system represented by $(A(t), B(t), C(t))$ in (5.1) is a polytope of linear systems. Another way to describe a polytope is by its vertices

$$\{(A_1, B_1, C_1), (A_2, B_2, C_2), \ldots, (A_L, B_L, C_L)\}, \quad (5.4)$$

where $L = 2^r$. For each time, there exists a set of $L$ values, $\{p_k\}$, such that

$$A = \sum_{k=1}^{L} p_k A_k, \quad (5.5)$$

$$B = \sum_{k=1}^{L} p_k B_k,$$

$$C = \sum_{k=1}^{L} p_k C_k,$$

$$\sum_{k=1}^{L} p_k = 1,$$

and $p_k \geq 0$ for $k = 1, 2, \ldots, L$. As should be clear from (5.5), the parameter space is a convex set, where the vertices in (5.4) are extreme values.

The system represented by $(A(t), B(t), C(t))$ in (5.1) is quadratically stable if and only if (Boyd et al., 1994b) there is a Lyapunov matrix $P \succ 0$ such that

$$A_i^T P + P A_i < 0, \quad i = 1, \ldots, L. \quad (5.6)$$

Consequently, the system $(A(t), B(t), C(t))$ in (5.1) is static-output-feedback stabilizable if and only if there is a Lyapunov matrix $P \succ 0$ and a feedback matrix $K$ such that

$$(A_i + B_i K C_i)^T P + P (A_i + B_i K C_i) < 0, \quad i = 1, \ldots, L. \quad (5.7)$$
For this research, if (5.7) holds for a $P > 0$ and a $K$, then the vertices of the polytope (5.4) will be said to be simultaneously stabilized by $K$. As in the time-invariant case presented in Section 4.1, the matrix inequality in (5.7) is not jointly convex in $K$ and $P$.

The vertices in (5.4) may be calculated from the parameter bounds in (5.2) and the matrices $F_i, G_i, \text{and } H_i$ in (5.3) as

$$A_k = F_0 + \sum_{i=1}^{r} F_i q_i(t) \mid_{q_i(t)=q_i^-} \text{ or } q_i^+, \quad k = 1, 2, \ldots, L,$$

$$B_k = G_0 + \sum_{i=1}^{r} G_i q_i(t) \mid_{q_i(t)=q_i^-} \text{ or } q_i^+, \quad k = 1, 2, \ldots, L,$$

and

$$C_k = H_0 + \sum_{i=1}^{r} H_i q_i(t) \mid_{q_i(t)=q_i^-} \text{ or } q_i^+, \quad k = 1, 2, \ldots, L.$$

In (5.8), each vertex is calculated for a different permutation of the $r$ variables $q_i(t)$ alternatively taken at maximum and minimum values. This results in $L = 2^r$ different vertices. As the number of uncertain time-varying parameters increases, the computational time for any method based on the vertices of the polytope increases exponentially. This may cause the implementation of any such polytope method to become impractical for systems with large numbers of uncertainties. For such systems, if the matrices $F_i, G_i, \text{and } H_i$ are restricted to be rank-1, Petersen (Petersen, 1987) has shown that the system in (5.3) may be represented by a system with norm-bounded uncertainties, such as described in Section 2.2.5 and (Boyd et al., 1994b). In the future, the method presented in Chapter 4 may be extended for the case of norm-bounded uncertainties, but for the problem of robust emergency lateral control of a highway vehicle with time-varying uncertainties, the number of uncertain time-varying parameters does not warrant such work at this time.

Before a discussion of robust output-feedback stabilization can be undertaken, an understanding of robust state-feedback controllers must first be established. This is the purpose of the next section.
5.2 Robust State-Feedback Stabilization

A complete presentation of the problem of designing a robust stabilizing state-feedback controller (including the case of a polytopic set of linear systems) is given in the book by Boyd et al. (Boyd et al., 1994b). The simultaneous state-feedback stabilizability condition for a set of systems, such as the polytope vertices in (5.4), is equivalent to finding $P_{sf} > 0$ and $K_{sf}$ in

$$
(A_i + B_i K_{sf})^T P_{sf} + P_{sf} (A_i + B_i K_{sf}) < 0, \quad i = 1, \ldots, L, \quad (5.9)
$$

where $P_{sf}$ is symmetric. As with the matrix inequality in (5.7), the matrix inequality in (5.9) is not jointly convex in $K_{sf}$ and $P_{sf}$. However for the state-feedback case, Boyd et al. (Boyd et al., 1994b) present a simple change of variables which transforms (5.9) into a linear matrix inequality (LMI). Multiplying (5.9) on both sides by $Q_{sf} = P_{sf}^{-1}$ yields the equivalent stabilizability condition

$$
Q_{sf}(A_i + B_i K_{sf})^T + (A_i + B_i K_{sf}) Q_{sf} < 0, \quad i = 1, \ldots, L. \quad (5.10)
$$

Defining $Y_{sf} = K_{sf} Q_{sf}$ and substituting into (5.10) yields the condition that the vertices in (5.4) are simultaneously stabilizable if and only if there is a $Q_{sf} > 0$ and a $Y_{sf}$ such that

$$
Q_{sf} A_i^T + A_i Q_{sf} + Y_{sf} B_i^T + B_i Y_{sf} < 0, \quad i = 1, \ldots, L. \quad (5.11)
$$

Furthermore, once a $Q_{sf} > 0$ and a $Y_{sf}$ have been found such that (5.11) holds, the state-feedback matrix $K_{sf} = Y_{sf} Q_{sf}^{-1}$ is known to stabilize the system. Unfortunately, no such change of variables is known to exist for the more general case of static-output-feedback stabilizability, which necessitates the current research.

5.3 Lyapunov Inequalities and Similarity Transforms, Revisited

As mentioned in Section 4.2.2, Juang (Juang, 1991) showed that any Lyapunov matrix $Q_{sf}$ such that the Lyapunov inequality in (5.10) holds may be interpreted
as a similarity transformation matrix $S$, where $Q_{sf}^{-1} = S^*S$, such that the matrix measure of the transformed system is negative. Along the same lines, Boyd et al. (Boyd et al., 1994b) demonstrate the equivalence of finding the Lyapunov matrix $Q_{sf}$ and finding the similarity transformation matrix $S$ such that all of the nonzero trajectories of the transformed state vector, $Sx(t)$, are always decreasing in norm as $t$ increases. Boyd et al. state that with this interpretation, it is natural to seek the similarity transformation matrix $S$ with minimum condition number such that all of the nonzero trajectories of the transformed state vector are always decreasing in norm. Furthermore, the Lyapunov matrix $Q_{sf}$ with minimum condition number corresponds to the similarity transformation matrix $S$ with minimum condition number.

Similarly, the Lyapunov matrix $Q_{sf}$ may be interpreted to define an invariant ellipsoid as in (Boyd et al., 1994b). An ellipsoid in state space is said to be invariant if for every trajectory of the system, $x(t_0)$ in the ellipsoid implies that $x(t)$ is in the ellipsoid for all time $t \geq t_0$. Minimizing the condition number of $Q_{sf}$ with $Q_{sf} > I$ therefore results in finding a given system’s smallest invariant ellipsoid containing the unit sphere.

In light of the interpretations discussed in this section, the Lyapunov matrix $Q_{sf}$ may be used as a measure of how well scaled the state space of a given system is. Minimizing the condition number of $Q_{sf}$ over the values of $Q_{sf} > I$ and $Y_{sf}$ subject to the LMI in (5.11) corresponds to finding a stabilizing state-feedback matrix, $K_{sf} = Y_{sf}Q_{sf}^{-1}$, which best scales the state space of the system. The closed-loop system with such a $K_{sf}$ is said to be optimal in the sense of state scaling.

The condition number of the matrix $Q_{sf} > I$ may be minimized subject to the LMI in (5.11) by solving the following LMI problem:
minimize \gamma \quad (5.12)

subject to

\[
\begin{bmatrix}
Q_{sf} - I & 0 \\
0 & \gamma I - Q_{sf}
\end{bmatrix} > 0
\]

and

\[
Q_{sf}A_i^T + A_iQ_{sf} + Y_{sf}^TB_i^T + B_iY_{sf} < 0, \quad i = 1, \ldots, L.
\]

If the measure of the size of the ellipsoid is viewed in terms of the sum of its radii rather than its maximum radius, the size of the ellipsoid may be minimized by solving the following LMI problem:

minimize \text{tr}Q_{sf} \quad (5.13)

subject to

\[Q_{sf} - I > 0\]

and

\[
Q_{sf}A_i^T + A_iQ_{sf} + Y_{sf}^TB_i^T + B_iY_{sf} < 0, \quad i = 1, \ldots, L,
\]

where the trace of the matrix \(Q_{sf}\), \(\text{tr}Q_{sf}\), is the sum of the diagonal elements of \(Q_{sf}\) (\(\text{tr}Q_{sf}\) is also equal to the sum of the eigenvalues of \(Q_{sf}\)). The size of the resulting state-feedback gain may be large, which would necessitate modification of the LMI problem in (5.13) before use in designing state-feedback matrices. However, because the goal of the present research is not limited to state-feedback matrices the size of the gain is only limited by the numerical precision of the machine upon which the algorithm in Section 5.4 is implemented (see Section 5.4 for details about how this numerical limitation has been handled).

Using the information contained in this section, the problem of finding a state feedback which optimizes the scaling of the closed-loop state-feedback system may be solved. The next section applies this optimal scaling and the results of Section 5.2 to the case of robust output-feedback stabilization.
5.4 Robust Output-Feedback Stabilization

The algorithm in Section 4.1 may be used to find a static-output-feedback matrix for a linear time-invariant system. This section uses the theories presented in Sections 5.2 and 5.3 to extend the work of Section 4.1 to include robust stabilization of a linear time-varying system represented using a polytope of linear systems.

The algorithm in Section 4.1 uses a simple methodology to find stabilizing static-output-feedback matrices:

- Find a stabilizing state-feedback matrix $K_{sf}$ with the property that the system will then be simultaneously K-stable and detectable (SKSD) based on $K_{af}$ (recall that the definition of a set of SKSD systems is based on a corresponding $K_{af}$ matrix).

- Find a Lyapunov matrix $P$ that proves that the system is SKSD based on $K_{af}$.

- Use $P$ to find a stabilizing output-feedback matrix $K$.

As discussed in Section 4.1, a major obstacle to the use of this methodology is the difficulty in finding a $K_{sf}$ with the special property that the system is then SKSD based on $K_{af}$. Until a complete parameterization of all $K_{sf}$ matrices for which a system is SKSD has been done, this obstacle will remain. However as Section 4.1 has shown, for linear time-invariant systems, the choice of $K_{sf}$ as the solution to some linear quadratic regulator (LQR) problem works well in many cases. Although more work is necessary to fully understand this class of state-feedback matrices, the existence of a class of state-feedback matrices for which a system is SKSD has been shown.

The algebraic Riccati equation (ARE), which solves LQR problems for a linear time-invariant system, cannot be used to simultaneously stabilize a polytope of matrices. Section 5.2 discusses the stabilization of a polytope of matrices with
state feedback. A polytope of matrices is stabilizable if there are matrices $Q_{sf} > 0$ and $Y_{sf}$ such that (5.11) holds. Furthermore, once $Q_{sf} > 0$ and $Y_{sf}$ have been found such that (5.11) holds, the state-feedback matrix $K_{sf} = Y_{sf}Q_{sf}^{-1}$ is known to stabilize the system. As discussed in Section 5.3, the trace of $Q_{sf}$ may be minimized as in (5.13) when finding $K_{sf}$. This choice of $K_{sf}$ results in a well-scaled closed-loop system, where each of the state variables will have similar magnitudes. The method discussed in Section 5.3 yields optimally scaled systems. Although this is not traditionally what is meant by optimal control, the optimality of such a system suggests some relationship to LQR methods for linear time-invariant systems. Thus, the solution to the LMI problem in (5.13) is used in place of the LQR method. The matrix $K_{sf}$ may then be used to find a $P$ that proves that each vertex of the polytopic system is SKSD based on $K_{sf}$, if such a $P$ exists. Once a suitable manner for finding $K_{sf}$ is chosen, the remaining steps of the algorithm may easily be generalized to the case of simultaneously stabilizing a set of linear systems.

The resulting LMI-based algorithm may be used to design robustly stabilizing static-output-feedback controllers for the linear time-varying system in (5.1):

1. Define the vertices of the polytopic system as described in (5.8).

2. Define $A_{\alpha,i} = A_i + \alpha I$ for $i = 1, 2, \ldots, L$, where $\alpha$ is the desired prescribed degree of stability, as described in (Anderson & Moore, 1969).

3. Solve the following LMI problem:

$$\begin{align*}
\text{minimize} & \quad \text{tr}Q_{sf} \\
\text{subject to} & \quad Q_{sf} - I > 0 \\
& \quad Q_{sf} A_{\alpha,i}^T + A_{\alpha,i}Q_{sf} + Y_{sf}^T B_i^T + B_i Y_{sf} < 0, \quad i = 1, \ldots, L.
\end{align*}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
4. Set $K_{sf} = Y_{sf}Q_{sf}^{-1}$.

5. Solve for $P$ using the LMI feasibility problem

\[
\begin{align*}
\text{find } \sigma, P \\
\text{such that}
\end{align*}
\]

\[
\begin{align*}
P > I \\
(A_{\alpha,i} + B_iK_{sf})^T P + P(A_{\alpha,i} + B_iK_{sf}) < 0, \quad i = 1, 2, \ldots, L, \\
A_{\alpha,i}^T P + PA_{\alpha,i} - \sigma C_i^T C_i < 0, \quad i = 1, 2, \ldots, L,
\end{align*}
\]

and

\[\sigma > 0.\]

6. Solve for $K$ using the LMI minimization problem

\[
\begin{align*}
\text{minimize } \lambda \\
\text{subject to}
\end{align*}
\]

\[
\begin{align*}
F(\lambda, k) = \begin{bmatrix} \lambda & k^T \\ k & M \end{bmatrix} > 0 \\
(A_{\alpha,i} + B_iKC_i)^T P + P(A_{\alpha,i} + B_iKC_i) < 0, \quad i = 1, 2, \ldots, L,
\end{align*}
\]

where $k = \text{vec}(K)$, and $M$ is a specified positive definite matrix ($M = I$ results in minimizing the Frobenious norm).

Steps 1 and 2 are used to set up the problem. Step 3 finds the Lyapunov matrix $Q_{sf}$ which best scales the states of the system as discussed in Section 5.3. The variable $Y_{sf}$ is used with $Q_{sf}$ in Step 4 to find a state-feedback matrix $K_{sf}$ for which the matrix $Q_{sf}$ is an invariant ellipsoid with small condition number. The matrix $K_{sf}$ calculated in Step 4 is likely to be large because no restriction has been
Because the author's implementation of the ellipsoid method (Bland et al., 1981; Boyd et al., 1994b) is used to solve all LMI problems in this research, the solution space for the problem in Step 3 may be limited by using an initial ellipsoid with smaller radii that usual. Because of the nature of the ellipsoid method, the solution may fall outside of the initial ellipsoid, but the size of $Y_{e_f}$ will not cause the remaining steps in the algorithm to exceed the numerical overflow limits of the computer. Similarly, the size of $P$ in Step 5 may be regulated by using a smaller initial ellipsoid. Steps 5 and 6 are straight forward conversions of the corresponding steps of the algorithm in Section 4.1 to the case of a polytopic system.

The algorithm in Section 4.1 has now been converted to a form which will simultaneously stabilize a polytopic system with a static-output-feedback controller. In some cases, the norm of the resulting feedback matrix, $\|K\|$, may exceed limits based on actuator cost. In such cases, the algorithm in the next section will be useful in finding reduced static-output-feedback gains.

### 5.5 Reducing Robust Output-Feedback Gains

This section uses the theories presented in Section 5.1 to extend the work of Section 4.2 to include robust small-gain stabilization of a linear time-varying system represented using a polytope of linear systems. The primary obstacle to such an extension is the fact that in Section 4.2, an eigenvalue method is used to form a Lyapunov matrix, $P$, based on the eigenvectors of the closed-loop system. Because a polytope of systems are used to describe the uncertainties in the system, such a method would result in as many different Lyapunov matrices, $P$, as there are vertices. It is necessary to find a single Lyapunov matrix, $P$, for all of the vertices in order to prove system stability using (5.6).

The resulting algorithm follows and may be used to find a small robustly stabilizing static-output-feedback controller for the linear time-varying system.
represented by \((A(t), B(t), C(t))\) in (5.1):

1. Define the vertices of the polytopic system as described in (5.8).

2. Define 
   \( A_{\alpha,i} = A_i + \alpha I \) for \( i = 1, 2, \ldots, L \), where \( \alpha \) is the desired prescribed degree of stability, as described in (Anderson & Moore, 1969).

3. Find a static-output-feedback matrix, \( K_0 \), that stabilizes \((A_{\alpha,i}, B_i, C_i)\) for \( i = 1, 2, \ldots, L \) using the method described in Section 5.5 above. Any robustly stabilizing algorithm could be used in this step, for example (Geromel et al., 1996) describes an alternate stabilizing algorithm.

4. Set \( j = 0 \).

5. Set \( P_j \) equal to the solution, \( P \), of the following LMI minimization problem:

\[
\text{maximize } \sigma \tag{5.17}
\]

subject to

\[
P > I, \quad \sigma > 0,
\]

and

\[
(A_{\alpha,i} + B_i K_j C_i)^T P + P(A_{\alpha,i} + B_i K_j C_i) < -\sigma I, \quad i = 1, 2, \ldots, L.
\]

6. Set \( K_{j+1} \) equal to the solution, \( K \), of the following LMI minimization problem:

\[
\text{minimize } \lambda \tag{5.18}
\]

subject to

\[
F(\lambda, k) = \begin{bmatrix} \lambda & k^T \\ k & M \end{bmatrix} > 0
\]

and

\[
(A_{\alpha,i} + B_i K C_i)^T P_j + P_j(A_{\alpha,i} + B_i K C_i) < 0, \quad i = 1, 2, \ldots, L,
\]
where $k = \text{vec}(K)$, and $M$ is a specified positive definite matrix ($M = I$ results in minimizing the Frobenious norm).

7. Iterate $j = j + 1$

8. If $||K_j - K_{j-1}|| > \delta$, where $\delta$ is a pre-specified tolerance, then go to Step 5.

5.6 Examples

The first example used to demonstrate the algorithms given in this chapter is borrowed from (Galimidi & Barmish, 1986; Geromel et al., 1996). It concerns of the stabilization of the lateral axis dynamics for an L-1011 aircraft. From the work of (Geromel et al., 1996), it is known that this system is static-output-feedback stabilizable.

Example 7 Let $(A, B, C)$ be defined as follows:

$$A = \begin{bmatrix}
-2.9800 & q_1(t) & 0 & -0.0340 \\
-0.9900 & -0.2100 & 0.0350 & -0.0011 \\
0 & 0 & 0 & 1.0000 \\
0.3900 & -5.5550 & 0 & -1.8900
\end{bmatrix}, \quad (5.19)$$

$$B = \begin{bmatrix}
-0.0320 \\
0 \\
0 \\
-1.6000
\end{bmatrix}, \quad (5.20)$$

and

$$C = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad (5.21)$$

with $-0.5700 \leq q_1(t) \leq 2.4300$, for all time. Find a stabilizing output-feedback matrix $K$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Solution. First the algorithm in Section 5.4 is implemented. The two vertices are calculated to be

\[
A_1 = \begin{bmatrix}
-2.9800 & -0.5700 & 0 & -0.0340 \\
-0.9900 & -0.2100 & 0.0350 & -0.0011 \\
0 & 0 & 0 & 1.0000 \\
0.3900 & -5.5550 & 0 & -1.8900 \\
\end{bmatrix},
\]

\[
B_1 = \begin{bmatrix}
-0.0320 \\
0 \\
0 \\
-1.6000 \\
\end{bmatrix},
\]

\[
C_1 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix},
\]

and

\[
A_2 = \begin{bmatrix}
-2.9800 & 2.4300 & 0 & -0.0340 \\
-0.9900 & -0.2100 & 0.0350 & -0.0011 \\
0 & 0 & 0 & 1.0000 \\
0.3900 & -5.5550 & 0 & -1.8900 \\
\end{bmatrix},
\]

\[
B_2 = \begin{bmatrix}
-0.0320 \\
0 \\
0 \\
-1.6000 \\
\end{bmatrix},
\]

and

\[
C_2 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix},
\]

respectively. The open-loop eigenvalues of \(A_1\) and \(A_2\) are \(0.0012 \pm 0.3125j\), \(-1.8738\), and \(-3.2086\); and \(-0.1051, -1.4811 \pm 0.6239j\), and \(-2.0127\), respectively. Using \(\alpha = 0\), the following matrices are found:

\[
Y_{sf} = \begin{bmatrix}
1318.434 & -0.2124099 & -0.8380517 & 65277.79 \\
\end{bmatrix},
\]

\[
Q_{sf} = \begin{bmatrix}
1.001807 & 0.0018574 & 0.0035029 & -0.0038271 \\
0.0018574 & 1.002514 & 0.0042548 & -0.0049011 \\
0.0035029 & 0.0042548 & 1.008082 & -0.0089668 \\
-0.0038271 & -0.0049011 & -0.0089668 & 1.010127 \\
\end{bmatrix},
\]

\[
K_{sf} = \begin{bmatrix}
1560.418 & 310.4791 & 567.3625 & 64635.81 \\
\end{bmatrix},
\]

\[
P = \begin{bmatrix}
204.9804 & -64.74084 & -1.941844 & -3.870213 \\
-64.74084 & 268.6102 & -13.20636 & 1.277450 \\
-1.941844 & -13.20636 & 177.4574 & 0.1761535 \\
-3.870213 & 1.277450 & 0.1761535 & 13.44649 \\
\end{bmatrix},
\]
and

\[ K = \begin{bmatrix} 6.856283 & 4.597547 \end{bmatrix}. \]

Using this as an output-feedback controller, the eigenvalues of \( A_1 + B_1KC \) are \(-0.0733, -1.3869, -3.1099, \) and \(-7.8660\), while the eigenvalues of \( A_2 + B_2KC_2 \) are \(-1.4242, -1.5735 \pm 0.8069j, \) and \(-7.8649\). In the event that the size of \( ||K|| \) is too large, the algorithm in Section 5.5 results in

\[ K = \begin{bmatrix} 0.0058485 & 0.0015044 \end{bmatrix} \]

after 11 iterations with \( ||K_i - K_{i-1}|| = 3.0 \times 10^{-4} < \delta = 10^{-3} \). Using this as an output-feedback controller, the eigenvalues of \( A_1 + B_1KC_1 \) are \(-0.0012 \pm 0.3131j, -1.8715, \) and \(-3.2086\), while the eigenvalues of \( A_2 + B_2KC_2 \) are \(-0.1110, -1.4807 \pm 0.6230j, -2.0101\).

Example 8 is the time-varying version of Example 6 presented in Section 4.2.4. It is used here to demonstrate the use of algorithms given in this chapter on the problem of robustly stabilizing a vertical takeoff and landing (VTOL) helicopter. From the work of (Peres et al., 1993), it is known that this system is static-output-feedback stabilizable. In (Geromel et al., 1996), the same robustness problem is intended to be solved. However, the problem is presented in (Geromel et al., 1996) with an error in the second column of the first row of the \( A \) matrix, and it is unknown how this error affects the controller design in (Geromel et al., 1996).

Example 8 Let \((A, B, C)\) be defined as follows:

\[ A = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & q_1(t) & -0.7070 & q_2(t) \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad (5.22) \]

\[ B = \begin{bmatrix} 0.4422 & 0.1761 \\ q_3(t) & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix}, \quad (5.23) \]
and

\[ C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}. \quad (5.24) \]

with parameter bounds \( -0.6319 \leq q_1(t) \leq 1.3681, \quad 1.2200 \leq q_2(t) \leq 1.4200, \) and\(2.7446 \leq q_3(t) \leq 4.3446, \) for all time. Find a stabilizing output-feedback matrix \( K. \)

**Solution.** The algorithm in Section 5.4 is implemented, and the eight vertices are calculated to be

\[
A_1 = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & -0.6319 & -0.7070 & 1.2200 \\ 0 & 0 & 1 & 0 \end{bmatrix},
\]

\[
B_1 = \begin{bmatrix} 0.4422 & 0.1761 \\ 2.7446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix},
\]

\[
C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix};
\]

\[
A_2 = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & -0.6319 & -0.7070 & 1.2200 \\ 0 & 0 & 1 & 0 \end{bmatrix},
\]

\[
B_2 = \begin{bmatrix} 0.4422 & 0.1761 \\ 4.3446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix},
\]

\[
C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix};
\]

\[
A_3 = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & -0.6319 & -0.7070 & 1.6200 \\ 0 & 0 & 1 & 0 \end{bmatrix},
\]

\[
B_3 = \begin{bmatrix} 0.4422 & 0.1761 \\ 2.7446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix},
\]

\[
C_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix};
\]
\[
A_4 = \begin{bmatrix}
-0.0366 & 0.0271 & 0.0188 & -0.4555 \\
0.0482 & -1.0100 & 0.0024 & -4.0208 \\
0.1002 & -0.6319 & -0.7070 & 1.6200 \\
0 & 0 & 1 & 0
\end{bmatrix},
\]

\[
B_4 = \begin{bmatrix}
0.4422 & 0.1761 \\
4.3446 & -7.5922 \\
-5.5200 & 4.4900 \\
0 & 0
\end{bmatrix},
\]

\[
C_4 = \begin{bmatrix}
0 & 1 & 0 & 0
\end{bmatrix};
\]

\[
A_5 = \begin{bmatrix}
-0.0366 & 0.0271 & 0.0188 & -0.4555 \\
0.0482 & -1.0100 & 0.0024 & -4.0208 \\
0.1002 & 1.3681 & -0.7070 & 1.2200 \\
0 & 0 & 1 & 0
\end{bmatrix},
\]

\[
B_5 = \begin{bmatrix}
0.4422 & 0.1761 \\
2.7446 & -7.5922 \\
-5.5200 & 4.4900 \\
0 & 0
\end{bmatrix},
\]

\[
C_5 = \begin{bmatrix}
0 & 1 & 0 & 0
\end{bmatrix};
\]

\[
A_6 = \begin{bmatrix}
-0.0366 & 0.0271 & 0.0188 & -0.4555 \\
0.0482 & -1.0100 & 0.0024 & -4.0208 \\
0.1002 & 1.3681 & -0.7070 & 1.2200 \\
0 & 0 & 1 & 0
\end{bmatrix},
\]

\[
B_6 = \begin{bmatrix}
0.4422 & 0.1761 \\
4.3446 & -7.5922 \\
-5.5200 & 4.4900 \\
0 & 0
\end{bmatrix},
\]

\[
C_6 = \begin{bmatrix}
0 & 1 & 0 & 0
\end{bmatrix};
\]

\[
A_7 = \begin{bmatrix}
-0.0366 & 0.0271 & 0.0188 & -0.4555 \\
0.0482 & -1.0100 & 0.0024 & -4.0208 \\
0.1002 & 1.3681 & -0.7070 & 1.6200 \\
0 & 0 & 1 & 0
\end{bmatrix},
\]

\[
B_7 = \begin{bmatrix}
0.4422 & 0.1761 \\
2.7446 & -7.5922 \\
-5.5200 & 4.4900 \\
0 & 0
\end{bmatrix},
\]

\[
C_7 = \begin{bmatrix}
0 & 1 & 0 & 0
\end{bmatrix};
\]
and

$$A_8 = \begin{bmatrix}
-0.0366 & 0.0271 & 0.0188 & -0.4555 \\
0.0482 & -1.0100 & 0.0024 & -4.0208 \\
0.1002 & 1.3681 & -0.7070 & 1.6200 \\
0 & 0 & 1 & 0
\end{bmatrix},$$

$$B_8 = \begin{bmatrix}
0.4422 & 0.1761 \\
4.3446 & -7.5922 \\
-5.5200 & 4.4900 \\
0 & 0
\end{bmatrix},$$

$$C_8 = \begin{bmatrix}
0 & 1 & 0 & 0
\end{bmatrix};$$

respectively. The open-loop eigenvalues of the vertices are given in Table 5.1. Using $\alpha = 0$, the following matrices are found:

$$Y_{sf} = \begin{bmatrix}
-1.303612 & 5.387033 & 5.771635 & 0.1322624 \\
-284.8052 & 12694.32 & -7560.675 & 3.101902
\end{bmatrix},$$

$$Q_{sf} = \begin{bmatrix}
1.352733 & -0.0100920 & 0.4260490 & -0.0357973 \\
-0.0100920 & 1.000373 & -0.0120775 & 0.0009254 \\
0.4260490 & -0.0120775 & 1.514938 & -0.0434321 \\
-0.0357973 & 0.0009254 & -0.0434321 & 1.003785
\end{bmatrix},$$

$$K_{sf} = \begin{bmatrix}
-2.339894 & 5.415740 & 4.517891 & 0.2388063 \\
1558.748 & 12641.08 & -5333.609 & -183.7513
\end{bmatrix},$$

$$P = \begin{bmatrix}
136.1427 & -48.21138 & -88.93130 & -11.13144 \\
-48.21138 & 80.80804 & 125.2733 & 19.77287 \\
-88.93130 & 125.2733 & 221.8917 & 34.17088 \\
-11.13144 & 19.77287 & 34.17088 & 148.2397
\end{bmatrix},$$

and

$$K = \begin{bmatrix}
0.2017243 \\
9.721544
\end{bmatrix}.$$ 

Using this as the initial $K$, the algorithm in Section 5.5 results in

$$K = \begin{bmatrix}
-0.1010633 \\
1.967605
\end{bmatrix}$$

after 6 iterations with $\|K_i - K_{i-1}\| = 2.1 \times 10^{-4} < \delta = 10^{-3}$. Closed-loop eigenvalues for the vertices using the $K$ matrices that result from the algorithms in Sections 5.4 and 5.5 are given in Tables 5.2 and 5.3, respectively.
### Table 5.1: Open-Loop Eigenvalues

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Open-Loop Eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.2129, -0.0250, -1.4708 ± 0.9575(j)</td>
</tr>
<tr>
<td>2</td>
<td>1.2129, -0.0250, -1.4708 ± 0.9575(j)</td>
</tr>
<tr>
<td>3</td>
<td>1.3209, -0.0260, -1.5242 ± 0.9020(j)</td>
</tr>
<tr>
<td>4</td>
<td>1.3209, -0.0260, -1.5242 ± 0.9020(j)</td>
</tr>
<tr>
<td>5</td>
<td>0.4346 ± 1.2230(j), -0.0587, -2.5661</td>
</tr>
<tr>
<td>6</td>
<td>0.4346 ± 1.2230(j), -0.0587, -2.5661</td>
</tr>
<tr>
<td>7</td>
<td>0.4656 ± 1.1311(j), -0.0587, -2.6260</td>
</tr>
<tr>
<td>8</td>
<td>0.4656 ± 1.1311(j), -0.0587, -2.6260</td>
</tr>
</tbody>
</table>

### Table 5.2: Eigenvalues for \(K = [0.2017243 \quad 9.721544]^T\)

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Closed-Loop Eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.1072, -0.3015 ± 0.9560(j), -74.2976</td>
</tr>
<tr>
<td>2</td>
<td>-0.1066, -0.3017 ± 0.9612(j), -73.9751</td>
</tr>
<tr>
<td>3</td>
<td>-0.1610, -0.2746 ± 0.7112(j), -74.2976</td>
</tr>
<tr>
<td>4</td>
<td>-0.1589, -0.2755 ± 0.7183(j), -73.9752</td>
</tr>
<tr>
<td>5</td>
<td>-0.1006, -0.3041 ± 1.0119(j), -74.2992</td>
</tr>
<tr>
<td>6</td>
<td>-0.1001, -0.3042 ± 1.0171(j), -73.9767</td>
</tr>
<tr>
<td>7</td>
<td>-0.1406, -0.2840 ± 0.7856(j), -74.2992</td>
</tr>
<tr>
<td>8</td>
<td>-0.1391, -0.2846 ± 0.7924(j), -73.9767</td>
</tr>
</tbody>
</table>

### Table 5.3: Eigenvalues for \(K = [-0.1010633 \quad 1.967605]^T\)

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Closed-Loop Eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.1114, -0.2459 ± 0.9186(j), -16.3661</td>
</tr>
<tr>
<td>2</td>
<td>-0.1131, -0.2465 ± 0.9065(j), -16.5251</td>
</tr>
<tr>
<td>3</td>
<td>-0.1780, -0.2126 ± 0.6646(j), -16.3664</td>
</tr>
<tr>
<td>4</td>
<td>-0.1847, -0.2105 ± 0.6477(j), -16.5253</td>
</tr>
<tr>
<td>5</td>
<td>-0.0863, -0.2429 ± 1.1600(j), -16.3973</td>
</tr>
<tr>
<td>6</td>
<td>-0.0871, -0.2442 ± 1.1484(j), -16.5557</td>
</tr>
<tr>
<td>7</td>
<td>-0.1070, -0.2324 ± 0.9730(j), -16.3976</td>
</tr>
<tr>
<td>8</td>
<td>-0.1088, -0.2332 ± 0.9591(j), -16.5559</td>
</tr>
</tbody>
</table>

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
5.7 Summary

The methods of Chapter 4 have been extended for use in designing robust controllers for systems with time-varying uncertainties by replacing the algebraic Riccati equation (ARE) used in Section 4.1 with the linear matrix inequality (LMI) minimization problem described in Sections 5.2 and 5.3. Other steps in the algorithms of Sections 4.1.5 and 4.2.3 have been converted to the case of simultaneous stabilization of a polytope of matrices described in Section 5.1. The resulting algorithms are presented in Sections 5.4 and 5.5. Robust stabilization examples have been borrowed from the literature to demonstrate the algorithms of this chapter.

One characteristic of the algorithm that warrants further improvement is the large intermediate matrices $Y_{sf}$, $K_{sf}$, and $P$ that are produced in the process of finding the static-output-feedback matrix $K$. One approach would be to modify the LMI problem in Step 3 and of the algorithm in Section 5.4 using Lemma 1 from Section 4.1.1. Then a small $K_{sf}$ could be found separately, and presumably, this would result in a small $P$ matrix.

Another advance could be made by developing a parameterization of the state-feedback matrices $K_{sf}$. This would allow a method to be developed which is guaranteed to robustly stabilize every static-output-feedback stabilizable system. Until this parameterization is developed, guarantees cannot be made about methods based on finding such $K_{sf}$ matrices. In addition, replacing the polytope characterization of uncertainty used in this research with the norm-bounded characterization of uncertainty described in Section 2.2.5 and (Boyd et al., 1994b) would decrease the design time for systems with a large number of time-varying uncertainties.

Finally, the gain reducing method of Section 5.5 tends to produce small gains at the expense of system performance. In addition, the method is iterative in LMI solutions. This may increase the design time of the system dramatically compared...
to the method of Section 5.4. Perhaps some Linear Quadratic function may be
used to optimize the system performance according to more traditional optimal
control definitions. Another approach would be to maximize performance while
placing a bound on the feedback gain. This would still allow for less expensive
actuators to be used. Once an actuator is chosen, the system may be designed to
get the best performance possible for a specified range of actuator input signals.

The main theoretical advances of this research have been presented. An
algorithm has been developed for the robust stabilization of a polytope system
via static output feedback. This method may be used to robustly stabilize linear
time-varying systems. The next chapter uses the LMI-based methods developed
in Chapters 4 and 5 to design robust emergency lateral controllers for highway
vehicles with time-varying uncertainties.
Chapter 6

Emergency Lateral Control

This chapter applies the algorithms developed in Chapter 5 to the vehicle control problem described in Chapter 1. Section 6.1 restates the problem and specifies parameter ranges to be used in the solution of the problem. Section 6.2 presents the static-output-feedback controller designed by the algorithm developed in Chapter 5, and Section 6.3 discusses the performance of the controller as implemented on a nonlinear vehicle dynamics simulation developed by Smith and Starkey (Smith, 1993; Smith & Starkey, 1994; Smith & Starkey, 1995b).

6.1 The Vehicle Control Problem

In the field of automated lateral control, researchers are developing systems which allow a vehicle to follow the road. Much work has been done to design controllers which perform well on vehicles during low-lateral-acceleration maneuvers, for example, see (Fenton & Selim, 1976; Cormier & Fenton, 1980; Fenton & Selim, 1988; Fenton & Selim, 1991; Peng & Tomizuka, 1993). These controllers are designed based on a yaw-plane vehicle model with linear tires, see Chapter 1. However, in an emergency situation, high-lateral-acceleration maneuvers may be required. During high-lateral-acceleration maneuvers, linear vehicle and tire models are inaccurate (Smith & Starkey, 1994; Smith & Starkey, 1995b).

115
Assuming a first-order steering actuator, the lateral dynamics of a highway vehicle based on Figure 6.1 with linear tires are described in state space as

\[
\dot{x}(t) = Ax(t) + Bu(t),
\]  

(6.1)

where \( A \in \mathbb{R}^{n \times n} \) is the state-feedback matrix, \( x(t) \in \mathbb{R}^n \) is the state at time \( t \), \( B \in \mathbb{R}^{n \times m} \) is the state-input matrix, and \( u(t) \in \mathbb{R}^m \) is the state input at time \( t \).

The state vector is chosen as

\[
x(t) = [\delta_f \ V \ \tau \ y \ \psi]^T,
\]  

(6.2)

where \( \delta_f \) is the steer angle of the vehicle, \( V \) is the lateral velocity, \( \tau \) is the yaw rate, \( y \) is the lateral offset of the vehicle’s center of gravity, and \( \psi \) is the heading error of the vehicle. The variables \( y \) and \( \psi \) describe the vehicle’s position and orientation relative to the road, where

\[
\dot{y} = V + U\psi,\]

(6.3)

\[
\dot{\psi} = \tau - \dot{\psi}_{rd}
\]

and \( \dot{\psi}_{rd} \) is the yaw rate for the road, which is assumed to be zero for the analysis in this chapter. With this state vector, the state-feedback matrix is

\[
A = \begin{bmatrix}
-\frac{1}{r_{sw}} & 0 & 0 & 0 & 0 \\
\frac{2C_{af}}{mU} & -\frac{2}{mU}(C_{af} + C_{ar}) & -U - \frac{2}{mU}(aC_{af} - bC_{ar}) & 0 & 0 \\
\frac{2aC_{af}}{l_{sw}} & -\frac{2}{l_{sw}U}(aC_{af} - bC_{ar}) & -\frac{2}{l_{sw}U}(a^2C_{af} + b^2C_{ar}) & 0 & 0 \\
0 & 1 & 0 & 0 & U \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}.
\]  

(6.4)

The state input is

\[
u(t) = e_{in}(t),
\]  

(6.5)

and the state-input matrix is

\[
B = \begin{bmatrix}
\frac{G_m}{r_{sw}} & 0 & 0 & 0 & 0
\end{bmatrix}^T.
\]  

(6.6)
Nominal values of the parameters in the vehicle model for a 1992 Ford Taurus are given in Table 6.1. Additional parameters involving an assumed steering actuator as well as tire properties and forward velocity are listed in Table 6.2.

Table 6.1: Vehicle Model Data for a 1992 Ford Taurus

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value$^a$</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0.9637 m</td>
<td>distance from the center of gravity to the front axle</td>
</tr>
<tr>
<td>$b$</td>
<td>1.7287 m</td>
<td>distance from the center of gravity to the rear axle</td>
</tr>
<tr>
<td>$m$</td>
<td>1419 kg</td>
<td>vehicle mass</td>
</tr>
<tr>
<td>$I_{zz}$</td>
<td>2618 kg m$^2$</td>
<td>vehicle yaw moment of inertia</td>
</tr>
</tbody>
</table>

$^a$Values have been measured by the Texas Transportation Institute for a vehicle in the possession of the LSU Department of Mechanical Engineering.

Due to difficulties in its measurement, the lateral velocity, $V$, may not be available for use by the controller because instruments to measure $V$ are expensive. As a result, an output-feedback controller must be designed for the case where

$$y(t) = [\delta_f \; y \; r \; \psi]^T$$

A robust dynamic-output-feedback method, such as $H_\infty$, could be used to estimate $V$, however this would result in complication of the control scheme. Now that the robust static-output-feedback method presented in Chapter 5 has been
Table 6.2: Additional Vehicle Model Data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{ss}$</td>
<td>1.0</td>
<td>steady-state gain for the steering system</td>
</tr>
<tr>
<td>$\tau_{sw}$</td>
<td>0.07 s</td>
<td>time constant for the steering system</td>
</tr>
<tr>
<td>$C_{af}$</td>
<td>varies</td>
<td>tire cornering stiffness per front tire</td>
</tr>
<tr>
<td>$C_{ar}$</td>
<td>varies</td>
<td>tire cornering stiffness per rear tire</td>
</tr>
<tr>
<td>$U$</td>
<td>varies</td>
<td>forward velocity</td>
</tr>
</tbody>
</table>

developed, its use for this system is beneficial as it will reduce cost and complexity of the resulting control system.

Chapter 1 introduced the concept that the parameters of forward velocity $U$, front lateral tire stiffness $C_{af}$, and rear lateral tire stiffness $C_{ar}$ are time-varying parameters. Forward velocity, $U$, is expected to vary during dropped-throttle lane-change maneuvers, where the foot is released from the accelerator as the maneuver begins. Although the value of $U$ is only likely to decrease by 5 m/s during such maneuvers, the controller is likely to be implemented at various initial velocities $U_0$. For this reason, the vehicle controller should be robust to a range of highway speeds, such as 15 m/s (33.6 mph) through 30 m/s (67.1 mph).

Variations in the values of lateral tire stiffness, $C_{af}$ and $C_{ar}$, are due to inaccuracies in the linear tire model, where the lateral tire force is assumed to equal $C_{a} \alpha$ and $\alpha$ is the slip angle between the directions of tire heading and velocity. Figure 6.2, which shows lateral tire forces as a function of slip angle $\alpha$, was generated by the nonlinear tire model presented in (Wong, 1978). This figure suggests that the range of $\alpha$ over which the linear tire model holds depends on the friction coefficient $\mu$. In addition, longitudinal slip, which may result from braking or acceleration during the maneuver, reduces the tire forces for every $\alpha$. If longitudinal slip is present, there is no range of $\alpha$ over which the linear tire
model is accurate. The value of longitudinal slip may be controlled using anti-lock braking systems and traction control systems, however a longitudinal slip of 0.2 can reduce the lateral tire force by as much as 50%, as shown in Figure 6.2.

![Figure 6.2: Lateral Tire Forces](image)

In the future, the uncertainty in the lateral tire stiffness $C_\alpha$ will need to be studied in depth to decide on a range of expected uncertainty in $C_\alpha$ during an emergency lane change. However, because such a study is beyond the scope of this research, the values of $C_{\alpha f}$ and $C_{\alpha r}$ will be assumed to vary independently between 15,000 N/rad and 30,000 N/rad, which allows the lateral stiffness to reduce to half of its nominal value of 30,000 N/rad.

Neglecting parameter uncertainties in the design of controllers to be used during an emergency-lane-change maneuver could cause the vehicle to become unstable and spin out of control. Highway vehicle controllers must be designed to be robust to expected uncertainties.
6.2 Controller Design

The first step in designing robust controllers using the algorithms of Chapter 5 is to define a polytope characterization of uncertain parameters \( q_i(t) \) as described in (5.2) and (5.3). The state matrices, \((A(t), B(t), C(t))\), of the system must be a linear function of the parameters, \( q_i(t) \). Inspection of (6.4) reveals that this is not the case for the parameters \( U, C_{af}, \) and \( C_{ar} \), because of the existence of \( C_{af}/U \) and \( C_{ar}/U \) terms. In order to meet this criterion, the following parameters have been defined: \( q_1(t) = U, q_2(t) = C_{af}, q_3(t) = C_{af}/U, \) and \( q_4(t) = C_{ar}/U \). The new state matrix, \( A(t) \), is

\[
A(t) = \begin{bmatrix}
-\frac{1}{\tau_{re}} & 0 & 0 & 0 \\
\frac{2q_1(t)}{m} & -\frac{2}{m}(q_3(t) + q_4(t)) & -q_1(t) - \frac{2}{m}(aq_3(t) - bq_4(t)) & 0 & 0 \\
\frac{2aq_1(t)}{l_{re}} & -\frac{2}{l_{re}}(aq_3(t) - bq_4(t)) & -\frac{2}{l_{re}}(a^2q_3(t) + b^2q_4(t)) & 0 & 0 \\
0 & 1 & 0 & 0 & q_1(t) \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}.
\]

Using the ranges of \( U, C_{af}, \) and \( C_{ar} \) given in Section 6.1, the parameters \( q_1(t), q_2(t), q_3(t), \) and \( q_4(t) \) have bounds as follows: \( 15 \leq q_1(t) \leq 30, 15000 \leq q_2(t) \leq 30000, 500 \leq q_3(t) \leq 2000, \) and \( 500 \leq q_4(t) \leq 2000 \).

Now, the problem is set up to begin the algorithm in Section 5.4. The vertices of the polytope of systems are found as in (5.8). The four parameters defined above result in 16 vertices of the convex set of systems, where Table 6.3 lists each vertex with its corresponding set of parameter values. The resulting vertices are contained in Appendix B, and the open-loop eigenvalues of the vertices are listed in Table 6.4. Note that vertices 5 through 8 are unstable, which indicates that the system is unstable for the expected parameter variations.

A prescribed degree of stability, \( \alpha = 0.15 \), is chosen. At present, the methodology for "prescribing" \( \alpha \) involves "trial and error". Because a set of systems is to be stabilized simultaneously via static output feedback, the range of feasible \( \alpha \)
Table 6.3: Parameter Values for Vehicle System Vertices

<table>
<thead>
<tr>
<th>Vertex</th>
<th>( v_1 ) (m/s)</th>
<th>( f_2 ) (N/rad)</th>
<th>( v_3 ) (kg/(s rad))</th>
<th>( v_4 ) (kg/(s rad))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td>15000</td>
<td>500</td>
<td>500</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>15000</td>
<td>500</td>
<td>500</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>30000</td>
<td>500</td>
<td>500</td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td>30000</td>
<td>500</td>
<td>500</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>15000</td>
<td>2000</td>
<td>500</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
<td>15000</td>
<td>2000</td>
<td>500</td>
</tr>
<tr>
<td>7</td>
<td>15</td>
<td>30000</td>
<td>2000</td>
<td>500</td>
</tr>
<tr>
<td>8</td>
<td>30</td>
<td>30000</td>
<td>2000</td>
<td>500</td>
</tr>
<tr>
<td>9</td>
<td>15</td>
<td>15000</td>
<td>500</td>
<td>2000</td>
</tr>
<tr>
<td>10</td>
<td>30</td>
<td>15000</td>
<td>500</td>
<td>2000</td>
</tr>
<tr>
<td>11</td>
<td>15</td>
<td>30000</td>
<td>500</td>
<td>2000</td>
</tr>
<tr>
<td>12</td>
<td>30</td>
<td>30000</td>
<td>500</td>
<td>2000</td>
</tr>
<tr>
<td>13</td>
<td>15</td>
<td>15000</td>
<td>2000</td>
<td>2000</td>
</tr>
<tr>
<td>14</td>
<td>30</td>
<td>15000</td>
<td>2000</td>
<td>2000</td>
</tr>
<tr>
<td>15</td>
<td>15</td>
<td>30000</td>
<td>2000</td>
<td>2000</td>
</tr>
<tr>
<td>16</td>
<td>30</td>
<td>30000</td>
<td>2000</td>
<td>2000</td>
</tr>
</tbody>
</table>

Table 6.4: Open-Loop Eigenvalues for the Vehicle

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Open-Loop Eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0000, 0.0000, -1.3966 ± 2.0149j, -14.2857</td>
</tr>
<tr>
<td>2</td>
<td>0.0000, 0.0000, -1.3966 ± 2.8758j, -14.2857</td>
</tr>
<tr>
<td>3</td>
<td>0.0000, 0.0000, -1.3966 ± 2.0149j, -14.2857</td>
</tr>
<tr>
<td>4</td>
<td>0.0000, 0.0000, -1.3966 ± 2.8758j, -14.2857</td>
</tr>
<tr>
<td>5</td>
<td>0.6369, 0.0000, 0.0000, -6.4891, -14.2857</td>
</tr>
<tr>
<td>6</td>
<td>1.9757, 0.0000, 0.0000, -7.8279, -14.2857</td>
</tr>
<tr>
<td>7</td>
<td>0.6369, 0.0000, 0.0000, -6.4891, -14.2857</td>
</tr>
<tr>
<td>8</td>
<td>1.9757, 0.0000, 0.0000, -7.8279, -14.2857</td>
</tr>
<tr>
<td>9</td>
<td>0.0000, 0.0000, -4.0568 ± 4.8087j, -14.2857</td>
</tr>
<tr>
<td>10</td>
<td>0.0000, 0.0000, -4.0568 ± 7.4502j, -14.2857</td>
</tr>
<tr>
<td>11</td>
<td>0.0000, 0.0000, -4.0568 ± 4.8087j, -14.2857</td>
</tr>
<tr>
<td>12</td>
<td>0.0000, 0.0000, -4.0568 ± 7.4502j, -14.2857</td>
</tr>
<tr>
<td>13</td>
<td>0.0000, 0.0000, -5.5864 ± 3.7988j, -14.2857</td>
</tr>
<tr>
<td>14</td>
<td>0.0000, 0.0000, -5.5864 ± 5.5919j, -14.2857</td>
</tr>
<tr>
<td>15</td>
<td>0.0000, 0.0000, -5.5864 ± 3.7988j, -14.2857</td>
</tr>
<tr>
<td>16</td>
<td>0.0000, 0.0000, -5.5864 ± 5.5919j, -14.2857</td>
</tr>
</tbody>
</table>
values may be limited. After attempting to design several controllers for a given set of $\alpha$ values, the largest feasible value of $\alpha$ has been chosen from this set. The goal of this chapter is to solve a feasible control problem, not to parameterize the set of feasible control problems. In the future, the development of a priori methods of determining the range of $\alpha$ values for which a given polytopic system is static-output-feedback stabilizable will prove helpful in practice. Once $\alpha$ has been chosen, the system to be stabilized is defined as $A_{\alpha,i} = A_i + \alpha I$ for $i = 1, 2, \ldots, L$.

The next step in the algorithm requires the solution to the following LMI optimization problem:

$$\text{minimize } \text{tr}Q_{sf}$$

subject to

$$Q_{sf} - I > 0$$

and

$$Q_{sf}A_{\alpha,i}^T + A_{\alpha,i}Q_{sf} + Y_{sf}^TB_i^T + B_iY_{sf} < 0, \quad i = 1, \ldots, L.$$  

For the vehicle model specified above, the solution is found to be

$$Q_{sf} = \begin{bmatrix}
231.4372 & -0.6174809 & -42.28825 & -9.490129 & -2.393598 \\
-0.6174809 & 988.9968 & 26.82082 & 224.7372 & -17.81185 \\
-42.28825 & 26.82082 & 37.92230 & 1.571576 & -1.329090 \\
-9.490129 & 224.7372 & 1.571576 & 1041.079 & -29.92236 \\
-2.393598 & -17.81185 & -1.329090 & -29.92236 & 2.102458
\end{bmatrix}$$  

(6.8)

and

$$Y_{sf} = \begin{bmatrix}
-121457.1 & -562.4580 & -257.3868 & 3.388887 & 2.370211
\end{bmatrix}. \quad (6.9)$$

Setting $K_{sf} = Y_{sf}Q_{sf}^{-1}$, results in the following state-feedback matrix, which causes the system states to be well-scaled:

$$K_{sf} = \begin{bmatrix}
-715.4793 & -4.645003 & -885.5174 & -75.77505 & -2491.006
\end{bmatrix}. \quad (6.10)$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Next, the following LMI feasibility problem is solved:

\[
\text{find } \sigma, P \quad (6.11)
\]

such that

\[
P > I
\]

\[
(A_{\alpha,i} + B_i K_{sf})^T P + P(A_{\alpha,i} + B_i K_{sf}) < 0, \quad i = 1, 2, \ldots, L,
\]

\[
A_{\alpha,i}^T P + P A_{\alpha,i} - \sigma C_i^T C_i < 0, \quad i = 1, 2, \ldots, L,
\]

and

\[
\sigma > 0,
\]

which results in the output-feedback Lyapunov matrix

\[
P = \begin{bmatrix}
0.0058758 & 0.0000338 & 0.0072167 & 0.0006210 & 0.020376 \\
0.0000338 & 0.0012038 & -0.0004436 & 0.0000462 & 0.010614 \\
0.0072167 & -0.0004436 & 0.0363868 & 0.0015160 & 0.049036 \\
0.0006210 & 0.0000462 & 0.0015160 & 0.0017143 & 0.026454 \\
0.020376 & 0.0106140 & 0.0490356 & 0.0264545 & 0.996254
\end{bmatrix} \quad (6.12)
\]

Finally, the Lyapunov inequality based on \( P \) is used in the following LMI minimization problem to find a robust output-feedback controller:

\[
\text{minimize } \lambda \quad (6.13)
\]

subject to

\[
F(\lambda, k) = \begin{bmatrix}
\lambda & k^T \\
k & I
\end{bmatrix} > 0
\]

and

\[
(A_{\alpha,i} + B_i K C_i)^T P + P(A_{\alpha,i} + B_i K C_i) < 0, \quad i = 1, 2, \ldots, L,
\]

where \( k = \text{vec}(K) \). The resulting controller design is

\[
K = \begin{bmatrix}
-4.366392 & -5.396281 & -0.4780914 & -17.53037
\end{bmatrix}, \quad (6.14)
\]

and the resulting closed-loop eigenvalues for each vertex are listed in Table 6.5.
Table 6.5: Closed-Loop Eigenvalues for the Vehicle

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Closed-Loop Eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-0.6850 \pm 0.3571j$, $-6.6158 \pm 3.0749j$, $-64.8543$</td>
</tr>
<tr>
<td>2</td>
<td>$-0.5411 \pm 0.8624j$, $-6.7521 \pm 3.9587j$, $-64.8693$</td>
</tr>
<tr>
<td>3</td>
<td>$-0.7277 \pm 0.3222j$, $-4.8135$, $-36.5935 \pm 3.4888j$</td>
</tr>
<tr>
<td>4</td>
<td>$-0.5941 \pm 0.8883j$, $-5.4084$, $-36.4295 \pm 2.5672j$</td>
</tr>
<tr>
<td>5</td>
<td>$-0.6393$, $-64.0516$</td>
</tr>
<tr>
<td>6</td>
<td>$-1.5600 \pm 0.9934j$, $-14.7039$, $-15.8418$</td>
</tr>
<tr>
<td>7</td>
<td>$-0.7631$, $-38.9369 \pm 12.5850j$</td>
</tr>
<tr>
<td>8</td>
<td>$-0.8200 \pm 1.1531j$, $-2.4107$, $-39.2321 \pm 12.9532j$</td>
</tr>
<tr>
<td>9</td>
<td>$-0.5527$, $-1.6393$, $-8.7476 \pm 7.9482j$, $-65.0891$</td>
</tr>
<tr>
<td>10</td>
<td>$-0.7895 \pm 0.8533j$, $-8.9948 \pm 10.2991j$, $-65.2077$</td>
</tr>
<tr>
<td>11</td>
<td>$-0.5149$, $-2.0576$, $-43.0691$</td>
</tr>
<tr>
<td>12</td>
<td>$-1.0165 \pm 0.8411j$, $-18.9458 \pm 7.9451j$, $-44.8517$</td>
</tr>
<tr>
<td>13</td>
<td>$-0.5643$, $-1.6106$, $-10.6724 \pm 5.1990j$, $-64.3157$</td>
</tr>
<tr>
<td>14</td>
<td>$-0.8585 \pm 0.8800j$, $-10.8643 \pm 7.1609j$, $-64.3898$</td>
</tr>
<tr>
<td>15</td>
<td>$-0.5190$, $-2.0620$, $-10.1578$, $-37.5483 \pm 10.2918j$</td>
</tr>
<tr>
<td>16</td>
<td>$-1.0821 \pm 0.8466j$, $-12.1652$, $-36.7530 \pm 9.1796j$</td>
</tr>
</tbody>
</table>
6.3 Controller Performance

In this section, a nonlinear eight-degree-of-freedom highway-vehicle simulation developed in (Smith, 1993; Smith & Starkey, 1994; Smith & Starkey, 1995b) is used to simulate the performance of the robust static-output-feedback controller designed in Section 6.2. After the simulation allows the vehicle to reach steady-state in the center of the right lane at an initial forward velocity of \( U_0 \), the lateral offset, \( y \), encounters a step change equal to the lane width. The throttle is immediately dropped to simulate the removal of the foot from the accelerator, and data is recorded for longitudinal distance, \( x \), as well as each state of the system: \( \delta_f, V, r, y, \) and \( \psi \). To demonstrate the robustness of the controller, the simulation has been repeated for various initial velocities, \( U_0 \), friction coefficients, \( \mu \), and lateral tire stiffnesses, \( C_{af} \) and \( C_{ar} \).

Figure 6.3 shows the state responses for the nonlinear simulation with an initial forward velocity of \( U_0 = 30 \) m/s and constant lateral stiffnesses of \( C_{af}, C_{ar} = 30 \) kN/rad. Although the value of the lateral stiffnesses do not vary for this plot, the nonlinear tire simulation uses the lateral stiffness of each tire along with the friction coefficient \( \mu \) and longitudinal slip to determine the actual tire forces. Therefore, the effective lateral tire stiffness which would be used in the linear tire model may vary with time even though the actual lateral tire stiffness remains constant. Since the controller has been designed using the linear tire model, this is a realistic test for the robustness to variations in the effective lateral tire stiffness. In addition, because the throttle is dropped at the beginning of the maneuver, the forward velocity of the vehicle decreases by as much as 5 m/s during each simulation. This demonstrates the robustness of the controller to variations in the forward velocity.

In Figure 6.3, several different simulations are shown for a range of \( \mu \) values which correspond to various tire/road interface surfaces, including dry pavement...
Figure 6.3: Responses for \( U_0 = 30 \text{ m/s} \), \( C_{a_f} = 30 \text{ kN/ rad} \), and various \( \mu \) values
(μ = 0.75), wet pavement (μ = 0.45), and packed snow (μ = 0.15). As should be expected, performance decreases with the coefficient of friction, but for all cases, Figure 6.3 shows that the system is stable. Similar plots for \( U_0 = 25 \text{ m/s} \), \( U_0 = 20 \text{ m/s} \), and \( U_0 = 15 \text{ m/s} \) are shown in Figures 6.4, 6.5, and 6.6, respectively. Travel distance and overshoot provide a quantification of system performance. The travel distance is defined as the farthest longitudinal distance that any point on the vehicle travels in the original lane, and the overshoot is the farthest lateral distance that the vehicle center of gravity moves past the center of the new lane. Table 6.6 lists travel distances and overshoots for various values of \( \mu \). For all cases, the travel distance is less than 73.0 m, and the overshoot is less than 0.132 m. For each case where \( \mu \geq 0.35 \), the travel distance is less than 58.8 m, and the overshoot is less than 0.111 m.

Figure 6.7 shows the state responses for the nonlinear simulation with an initial forward velocity of \( U_0 = 30 \text{ m/s} \) and constant coefficient of friction \( \mu = 0.75 \). Four cases are shown where \( C_{af} \) and \( C_{ar} \) are set to 30 kN/rad, 30 kN/rad; 15 kN/rad, 30 kN/rad; 30 kN/rad, 15 kN/rad; and 15 kN/rad, 15 kN/rad, respectively. For each case, Figure 6.7 shows that the system is stable. A similar plot for \( U_0 = 15 \text{ m/s} \) is shown in Figure 6.8. Table 6.6 lists travel distances and overshoots for various values of \( C_{af} \) and \( C_{ar} \). The greatest travel distance for all cases is less than 61.0 m, and the greatest overshoot is less than 0.398 m.

A double-lane-change maneuver occurs when a lane-change maneuver is interrupted by the command to return to the original lane. Double-lane-change maneuvers have been used to test the stability of vehicle controllers (Smith et al., 1995). In Figures 6.9 and 6.10 a double-lane-change maneuver is performed on a vehicle with \( C_{af} = 30 \text{ kN/rad} \) and \( C_{ar} = 15 \text{ kN/rad} \). The vehicle with lowest rear lateral tire stiffness and highest front lateral tire stiffness is more likely to be an oversteer vehicle than any other configuration. Although the case where both \( C_{af} \) and \( C_{ar} \) are set to 15 kN/rad has greater travel distance and overshoot in
Figure 6.4: Responses for $U_0 = 25$ m/s, $C_{\alpha f_r} = 30$ kN/\(\text{rad}\), and various $\mu$ values

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Figure 6.5: Responses for \( U_0 = 20 \text{ m/s}, C_{\alpha f,r} = 30 \text{ kN/\text{rad}}, \) and various \( \mu \) values
Figure 6.6: Responses for $U_0 = 15 \text{ m/s}$, $C_{af,v} = 30 \text{ kN/rad}$, and various $\mu$ values
Table 6.6: Effects of $\mu$ on Controller Performance

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>Travel Distance (m)</th>
<th>Overshoot (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$U_0 = 30 \text{ m/s}$</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>55.14</td>
<td>0.111</td>
</tr>
<tr>
<td>0.65</td>
<td>55.49</td>
<td>0.111</td>
</tr>
<tr>
<td>0.55</td>
<td>56.01</td>
<td>0.111</td>
</tr>
<tr>
<td>0.45</td>
<td>56.93</td>
<td>0.110</td>
</tr>
<tr>
<td>0.35</td>
<td>58.74</td>
<td>0.110</td>
</tr>
<tr>
<td>0.25</td>
<td>62.75</td>
<td>0.107</td>
</tr>
<tr>
<td>0.15</td>
<td>72.99</td>
<td>0.131</td>
</tr>
<tr>
<td></td>
<td>$U_0 = 25 \text{ m/s}$</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>52.39</td>
<td>0.006</td>
</tr>
<tr>
<td>0.65</td>
<td>52.63</td>
<td>0.006</td>
</tr>
<tr>
<td>0.55</td>
<td>52.99</td>
<td>0.006</td>
</tr>
<tr>
<td>0.45</td>
<td>53.62</td>
<td>0.006</td>
</tr>
<tr>
<td>0.35</td>
<td>54.81</td>
<td>0.006</td>
</tr>
<tr>
<td>0.25</td>
<td>57.47</td>
<td>0.006</td>
</tr>
<tr>
<td>0.15</td>
<td>64.66</td>
<td>0.005</td>
</tr>
<tr>
<td></td>
<td>$U_0 = 20 \text{ m/s}$</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>50.96</td>
<td>--</td>
</tr>
<tr>
<td>0.65</td>
<td>51.14</td>
<td>--</td>
</tr>
<tr>
<td>0.55</td>
<td>51.39</td>
<td>--</td>
</tr>
<tr>
<td>0.45</td>
<td>51.80</td>
<td>--</td>
</tr>
<tr>
<td>0.35</td>
<td>52.54</td>
<td>--</td>
</tr>
<tr>
<td>0.25</td>
<td>54.16</td>
<td>--</td>
</tr>
<tr>
<td>0.15</td>
<td>58.78</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>$U_0 = 15 \text{ m/s}$</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>50.84</td>
<td>--</td>
</tr>
<tr>
<td>0.65</td>
<td>50.97</td>
<td>--</td>
</tr>
<tr>
<td>0.55</td>
<td>51.14</td>
<td>--</td>
</tr>
<tr>
<td>0.45</td>
<td>51.40</td>
<td>--</td>
</tr>
<tr>
<td>0.35</td>
<td>51.85</td>
<td>--</td>
</tr>
<tr>
<td>0.25</td>
<td>52.76</td>
<td>--</td>
</tr>
<tr>
<td>0.15</td>
<td>55.31</td>
<td>--</td>
</tr>
</tbody>
</table>

*No overshoot present in system response.
Figure 6.7: Responses for $U_0 = 30$ m/s, various $C_{af,r}$ values, and $\mu = 0.75$
Figure 6.8: Responses for $U_0 = 15$ m/s, various $C_{af,r}$ values, and $\mu = 0.75$
Table 6.7: Effects of $C_{af}$ and $C_{ar}$ on Controller Performance

<table>
<thead>
<tr>
<th>$C_{af}$ (kN/rad)</th>
<th>$C_{ar}$ (kN/rad)</th>
<th>Travel Distance (m)</th>
<th>Overshoot (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>30</td>
<td>55.14</td>
<td>0.111</td>
</tr>
<tr>
<td>15</td>
<td>30</td>
<td>57.34</td>
<td>0.225</td>
</tr>
<tr>
<td>30</td>
<td>15</td>
<td>58.86</td>
<td>0.296</td>
</tr>
<tr>
<td>15</td>
<td>15</td>
<td>60.98</td>
<td>0.397</td>
</tr>
</tbody>
</table>

$U_0 = 30 \text{ m/s}$

| 30                | 30                | 50.84               | —$^{a}$       |
| 15                | 30                | 50.49               | —             |
| 30                | 15                | 50.67               | —             |
| 15                | 15                | 50.46               | —             |

$U_0 = 15 \text{ m/s}$

$^{a}$No overshoot present in system response.

the 30 m/s case, Figures 6.7 and 6.8 both show that the maximum yaw rates are experienced by the vehicle with $C_{af} = 30 \text{ kN/rad}$ and $C_{ar} = 15 \text{ kN/rad}$. This is due to oversteer/understeer characteristics of the four vehicle configurations. For this reason the vehicle with $C_{af} = 30 \text{ kN/rad}$ and $C_{ar} = 15 \text{ kN/rad}$ is the most likely vehicle to be unstable and is called the worst-case vehicle. Figure 6.9 shows simulation results with $U_0 = 30 \text{ m/s}$ for cases of $\mu = 0.15$ and $\mu = 0.75$, where the command to return to the original lane is issued at 60 m. The vehicle safely returns to the original lane. Figure 6.10 shows simulation results for $U_0 = 15 \text{ m/s}$. The responses in Figures 6.9 and 6.10 show that the controller maintains stability for the worst-case vehicle during a double-lane-change maneuver.

6.4 Robust Emergency Lateral Control

In Section 6.1, the vehicle control problem has been presented as a robust control problem. Next, Section 6.2 presents the use of an algorithm developed in Chapter 5 to solve the robust control problem. From Section 6.3, it is apparent that robust emergency lateral control has been achieved for a highway vehicle with
Figure 6.9: Worst-Case Double-Lane-Change Responses for \( U_0 = 30 \, \text{m/s} \)

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Figure 6.10: Worst-Case Double-Lane-Change Responses for $U_0 = 15 \text{ m/s}$
time-varying parameters using a linear-matrix-inequality (LMI) approach. The performance is robust to changes in friction coefficient $\mu$ ranging from 0.15 to 0.75 where travel distance remains smaller than 73 m and overshoot less than 0.132 m. When $C_{af}$ and $C_{ar}$ are allowed to vary independently over the range of 15 kN.rad to 30 kN.rad, the travel distance remains smaller than 61 m and overshoot less than 0.398 m. In every case where the parameters were allowed to vary within pre-specified bounds, the vehicle remains stable. Therefore, the robust control problem posed in Chapter 1 has been solved using a novel LMI approach to robust static-output-feedback control.

Future research is needed into the development of a more accurate characterization of the variations in $C_{af}$ and $C_{ar}$. The values should continue to vary independently over a large range of values, however, the difference between $C_{af}$ and $C_{ar}$ is not likely to approach the size of the entire range. Because the oversteer/understeer characteristics, and therefore the stability of the vehicle, depend heavily on the difference between the values, a more accurate characterization would allow the designer to achieve better performance. One possible characterization would be to set $C_{ar} = C_\alpha$ and $C_{af} = C_\alpha + \Delta C_\alpha$, where $C_\alpha$ and $\Delta C_\alpha$ vary with time. The range of $C_\alpha$ should be large enough to account for all of the effective $C_{ar}$ values predicted by a nonlinear model of the tire/ground interface due to changes in friction coefficient, longitudinal slip, changes in normal tire forces, and nonlinearities due to large slip angle. The range of $\Delta C_\alpha$, on the other hand, would only need to account for the possible differences in longitudinal slip and normal tire forces likely to occur between the front and rear axles.

In addition, performance could be improved by the use of gain scheduling techniques for parameters such as forward velocity, $U$, which are easily measured. This would allow performance to be increased at each speed by removing the constraint that the system be robust to changes in forward velocity. Robust techniques are most powerful in cases where parameter variations are truly uncertain.
because they are unmeasurable. For the case of variations in the effective lateral
tire stiffnesses $C_{af}$ and $C_{ar}$, it would be difficult to measure variations in $C_{af}$
and $C_{ar}$. Forward velocity, on the other hand, would not be difficult to mea-
sure. Unless significant cost savings are generated by the elimination of need for
sensors to measure parameter variations when assuming a known variation is an
uncertainty, the price of decreased performance paid when using robust techniques
where nonlinear techniques would be better suited may result in a substandard sys-
tem. Balance is needed when attempting to solve any difficult real-world problem
such as emergency lateral control of a highway vehicle.
Chapter 7

Summary and Future Research

The research presented in this dissertation includes required advances in the use of linear matrix inequalities for robust static-output-feedback control of systems with time-varying uncertainties as well as advances in the use of the resulting algorithm for the specific case of emergency lateral control of a highway vehicle. A method of stabilizing time-invariant systems with static output feedback is presented in Chapter 4, and the method has been extended to stabilize time-varying systems as presented in Chapter 5. Finally, Chapter 6 discusses the use of the algorithm for the specific case of emergency lateral control of a highway vehicle. Major conclusions and recommendations for future research from each chapter are restated here for convenience.

7.1 Summary of Static-Output-Feedback Stabilization

In Section 4.1, the problem of determining whether a system is output-feedback stabilizable has been restated to the problem of simultaneously solving an algebraic Riccati inequality and a Lyapunov inequality. An initial approach has been made at deriving an algorithm, based on the redefined problem, which designs a stabilizing static-output-feedback controller, if possible. The algorithm is based on a procedure which uses the well known solution to the algebraic Riccati equation.
(ARE) to find $K_{sf}$ and two linear matrix inequalities (LMI's), based on $K_{sf}$, to find a static-output-feedback matrix, $K$, which stabilizes the system. The algorithm results in small control gains and may also be used to prescribe a degree of stability. The algorithm is fundamentally different from the min/max algorithm of (Geromel et al., 1994; Iwasaki et al., 1994), yet easy to implement using standard ARE and LMI techniques. Unlike the min/max algorithm of (Geromel et al., 1994; Iwasaki et al., 1994), the algorithm developed in Section 4.1 does not iterate the solution of LMI problems. Examples have been included to demonstrate the use of the algorithm.

Future work is necessary to parameterize the set of all state-feedback matrices, $K_{sf}$, for which the LMI's in the algorithm of Section 4.1 are feasible if and only if the system is static-output-feedback stabilizable. In addition, a direct simultaneous solution of two algebraic Riccati inequalities would be useful in developing future algorithms.

In Section 4.2, an algorithm has been developed based on well-known eigenvalue decomposition techniques and recently developed LMI methods which finds small stabilizing static-output-feedback gains. The algorithm may be initialized with previous methods such as those described in (Trofino-Neto & Kucera, 1993; Geromel et al., 1996; Benton & Smith, 1997) and Section 4.1. The algorithm has been used to decouple the problem of reducing implementation cost (control effort) from the problem of meeting a performance-oriented design specification (decay rate). Each iteration of the algorithm has been shown to reduce the feedback gain of the system, and use of the algorithm has been demonstrated by example problems.

7.2 Summary of Robust Static-Output-Feedback Stabilization

In Chapter 5, the methods of Chapter 4 have been extended for use in designing robust controllers for systems with time-varying uncertainties by replacing the
algebraic Riccati equation (ARE) used in Section 4.1 with the linear matrix inequality (LMI) minimization problem described in Sections 5.2 and 5.3. Other steps in the algorithms of Sections 4.1.5 and 4.2.3 have been converted to the case of simultaneous stabilization of a polytope of matrices described in Section 5.1. The resulting algorithms are presented in Sections 5.4 and 5.5. Robust stabilization examples have been borrowed from the literature to demonstrate the algorithms of Chapter 5.

Once again, the development of a parameterization of the state-feedback matrices $K_{sf}$ would allow a method to be developed which is guaranteed to robustly stabilize every static-output-feedback stabilizable system. Until this parameterization is developed, guarantees cannot be made about methods based on finding such $K_{sf}$ matrices. In addition, replacing the polytope characterization of uncertainty used in this research with the norm-bounded characterization of uncertainty described in Section 2.2.5 and (Boyd et al., 1994b) would decrease the design time for systems with a large number of time-varying uncertainties.

In addition, the gain reducing method of Section 5.5 tends to produce small gains at the expense of system performance. In addition, the method is iterative in LMI solutions. This may increase the design time of the system dramatically compared to the method of Section 5.4, which is non-iterative in LMI solutions. Perhaps some Linear Quadratic function may be used to optimize the system performance according to more traditional optimal control definitions. Another approach would be to maximize performance while placing a bound on the feedback gain. This would still allow for less expensive actuators to be used. Once an actuator is chosen, the system may be designed to get the best performance possible for a specified range of actuator input signals.
7.3 Robust Emergency Lateral Control of A Highway Vehicle

From Chapter 6, it is apparent that robust emergency lateral control has been achieved for a highway vehicle with time-varying parameters using a linear-matrix-inequality (LMI) approach. The performance is robust to changes in friction coefficient \( \mu \) ranging from 0.15 to 0.75 where travel distance remains smaller than 73 m and overshoot less than 0.132 m. When \( C_{af} \) and \( C_{ar} \) are allowed to vary independently over the range of 15 kN/rad to 30 kN/rad, the travel distance remains smaller than 61 m and overshoot less than 0.398 m. In every case where the parameters were allowed to vary within pre-specified bounds, the vehicle remains stable. Therefore, the robust control problem posed in Chapter 1 has been solved using a novel LMI approach to robust static-output-feedback control.

Future research is needed into the development of a more accurate characterization of the variations in \( C_{af} \) and \( C_{ar} \). The values should continue to vary independently over a large range of values, however, the difference between \( C_{af} \) and \( C_{ar} \) is not likely to approach the size of the entire range. Because the oversteer/understeer characteristics, and therefore the stability of the vehicle, depend heavily on the difference between the values, a more accurate characterization would allow the designer to achieve better performance. One possible characterization would be to set \( C_{ar} = C_{a} \) and \( C_{af} = C_{a} + \Delta C_{a} \), where \( C_{a} \) and \( \Delta C_{a} \) vary with time. The range of \( C_{a} \) should be large enough to account for all of the effective \( C_{ar} \) values predicted by a nonlinear model of the tire/ground interface due to changes in friction coefficient, longitudinal slip, changes in normal tire forces, and nonlinearities due to large slip angle. The range of \( \Delta C_{a} \), on the other hand, would only need to account for the possible differences in longitudinal slip and normal tire forces likely to occur between the front and rear axles.

In addition, performance could be improved by the use of gain scheduling techniques for parameters such as forward velocity, \( U \), which are easily measured.
This would allow performance to be increased at each speed by removing the constraint that the system be robust to changes in forward velocity. Robust techniques are most powerful in cases where parameter variations are truly uncertain because they are unmeasurable. For the case of variations in the effective lateral tire stiffnesses $C_{af}$ and $C_{ar}$, it would be difficult to measure variations in $C_{af}$ and $C_{ar}$. Forward velocity, on the other hand, would not be difficult to measure. Unless significant cost savings are generated by the elimination of need for sensors to measure parameter variations when assuming a known variation is an uncertainty, the price of decreased performance paid when using robust techniques where nonlinear techniques would be better suited may result in a substandard system. Balance is needed when attempting to solve any difficult real-world problem such as emergency lateral control of a highway vehicle.

### 7.4 Future Directions

In addition to the future research suggested above, investigation into optimal methods for static-output-feedback controllers based on linear matrix inequality theory is needed. Robustness to unstructured uncertainties and disturbance inputs would also prove beneficial, and counterparts to other advances in the design of state-feedback controllers should be pursued for the case of static-output-feedback controllers.

The author also envisions an integration of mechanism design with controller design using the output-feedback methodologies developed in this dissertation. For this, the author would have to establish connections between mechanism design and controller design. Each feedback element is equivalent to a mechanical element. State-feedback and dynamic-output-feedback controllers could only be replaced by complex and unimplementable mechanisms. However, because the number of outputs is less than the order of the system (assuming feasibility), static-output-feedback controllers may have simple mechanical counterparts. This
leads the author to hypothesize an analogy between mechanism design and static-output-feedback controller design. If mechanism design may be accomplished via the methods of this dissertation, then the integration of mechanism design and controller design is at hand.
References


Desoer, Charles A. 1969. Slowly Varying System \( \dot{x} = A(t)x \). *IEEE Transactions on Automatic Control, AC-14* (December), 780–781.


Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.


Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.


Zhou, Kemin. 1992a. Comparison Between $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Controllers. IEEE Transactions on Automatic Control, 37(8), 1261–1265.


Appendix A

Using the Ellipsoid Algorithm to Solve LMI problems

Several algorithms for solving linear-matrix-inequality (LMI) problems are listed in (Boyd et al., 1994b). Although the ellipsoid algorithm (Bland et al., 1981; Boyd et al., 1994b) is not as efficient as newer algorithms, its simplicity has allowed it to be programmed into FORTRAN code by the author. This appendix contains a simple ellipsoid algorithm given in (Boyd et al., 1994b).

To solve the LMI minimization problem

\[ \text{minimize } c^T x \]
\[ \text{subject to } F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0, \]

the following algorithm may be used:

1. Let the matrix \( A^{(0)} \) define an initial ellipsoid about an initial point \( x^{(0)} \) such that the ellipsoid contains the optimal solution, if it exists.

2. Set \( k = 0 \).

3. Define a cutting plane \( g^{(k)} \) as follows:
   - If \( x^{(k)} \) does not satisfy the LMI \( F(x^{(k)}) > 0 \), then there exists a nonzero \( u \) such that
     \[ u^T F(x^{(k)}) u \leq 0. \]
Define $g^{(k)} = [g_1, \ldots, g_m]^T$ by $g_i = -u^TF_iu$, $i = 1, \ldots, m$. Because $u^TF(z)u \leq 0$ for any $z$ satisfying $g^{(k)^T}(z - x^{(k)}) \geq 0$, the vector $g^{(k)}$ defines a cutting plane. All points on one side of the plane may be discarded because they cannot satisfy the LMI (such points are said to be infeasible).

- If $x^{(k)}$ satisfies the LMI $F(x^{(k)}) > 0$, then $g^{(k)} = c$ defines a cutting plane, where for all points on one side of the plane $c^Tx$ is larger than $c^Tx^{(k)}$.

4. Find smallest ellipsoid containing half-ellipsoid defined by the cutting plane:

$$
\tilde{g} = \left(g^{(k)^T}A^{(k)}g^{(k)}\right)^{-1/2}g^{(k)},
$$

$$
x^{(k+1)} = x^{(k)} - \frac{1}{m+1}A^{(k)}\tilde{g},
$$

$$
A^{(k+1)} = \frac{m^2}{m^2-1} \left( A^{(k)} - \frac{2}{m+1}A^{(k)}\tilde{g}\tilde{g}^TA^{(k)} \right).
$$

5. Set $k = k + 1$.

6. Check for convergence. If convergence criteria not met, go to Step 3.

In the above algorithm, each ellipsoid contains the solution to the LMI problem. A cutting plane is defined based on the feasibility of the point at the center of the ellipsoid. The algorithm then finds the smallest ellipsoid that contains the remaining half of the original ellipsoid. The algorithm is repeated until convergence is reached. For feasibility problems, such as

$$
\text{find } x
$$

such that $F(x) = F_0 + \sum_{i=1}^m x_iF_i > 0$,

the algorithm is stopped when the center of the ellipsoid is found to be feasible.
Appendix B

Polytopic Vertices of the Time-Varying Vehicle

The following vertices result from inserting the values contained in Table 6.3 into the polytopic system described in Section 6.2.

\[
A_1 = \begin{bmatrix}
-14.2857 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
20.6947 & -1.3796 & -14.4647 & 0.0000 & 0.0000 \\
10.3617 & 0.2807 & -1.4135 & 0.0000 & 0.0000 \\
0.0000 & 1.0000 & 0.0000 & 0.0000 & 15.0000 \\
0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000
\end{bmatrix}
\]

\[
A_2 = \begin{bmatrix}
-14.2857 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
20.6947 & -1.3796 & -29.4647 & 0.0000 & 0.0000 \\
10.3617 & 0.2807 & -1.4135 & 0.0000 & 0.0000 \\
0.0000 & 1.0000 & 0.0000 & 0.0000 & 30.0000 \\
0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000
\end{bmatrix}
\]

\[
A_3 = \begin{bmatrix}
-14.2857 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
41.3895 & -1.3796 & -14.4647 & 0.0000 & 0.0000 \\
20.7233 & 0.2807 & -1.4135 & 0.0000 & 0.0000 \\
0.0000 & 1.0000 & 0.0000 & 0.0000 & 15.0000 \\
0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000
\end{bmatrix}
\]

165

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
\[ A_4 = \begin{bmatrix} -14.2857 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 41.3895 & -1.3796 & -29.4647 & 0.0000 & 0.0000 \\ 20.7233 & 0.2807 & -1.4135 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 & 30.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \end{bmatrix} \]

\[ A_5 = \begin{bmatrix} -14.2857 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 20.6947 & -3.4491 & -16.4410 & 0.0000 & 0.0000 \\ 10.3617 & -0.7555 & -2.4031 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 & 15.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \end{bmatrix} \]

\[ A_6 = \begin{bmatrix} -14.2857 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 20.6947 & -3.4491 & -31.4410 & 0.0000 & 0.0000 \\ 10.3617 & -0.7555 & -2.4031 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 & 30.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \end{bmatrix} \]

\[ A_7 = \begin{bmatrix} -14.2857 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 41.3895 & -3.4491 & -16.4410 & 0.0000 & 0.0000 \\ 20.7233 & -0.7555 & -2.4031 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 & 15.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \end{bmatrix} \]

\[ A_8 = \begin{bmatrix} -14.2857 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 41.3895 & -3.4491 & -31.4410 & 0.0000 & 0.0000 \\ 20.7233 & -0.7555 & -2.4031 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 & 30.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \end{bmatrix} \]
$A_9 = \begin{bmatrix}
-14.2857 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
20.6947 & -3.4491 & -10.8824 & 0.0000 & 0.0000 \\
10.3617 & 2.1588 & -4.6645 & 0.0000 & 0.0000 \\
0.0000 & 1.0000 & 0.0000 & 0.0000 & 15.0000 \\
0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\
\end{bmatrix}$

$A_{10} = \begin{bmatrix}
-14.2857 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
20.6947 & -3.4491 & -25.8824 & 0.0000 & 0.0000 \\
10.3617 & 2.1588 & -4.6645 & 0.0000 & 0.0000 \\
0.0000 & 1.0000 & 0.0000 & 0.0000 & 30.0000 \\
0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\
\end{bmatrix}$

$A_{11} = \begin{bmatrix}
-14.2857 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
41.3895 & -3.4491 & -10.8824 & 0.0000 & 0.0000 \\
20.7233 & 2.1588 & -4.6645 & 0.0000 & 0.0000 \\
0.0000 & 1.0000 & 0.0000 & 0.0000 & 15.0000 \\
0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\
\end{bmatrix}$

$A_{12} = \begin{bmatrix}
-14.2857 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
41.3895 & -3.4491 & -25.8824 & 0.0000 & 0.0000 \\
20.7233 & 2.1588 & -4.6645 & 0.0000 & 0.0000 \\
0.0000 & 1.0000 & 0.0000 & 0.0000 & 30.0000 \\
0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\
\end{bmatrix}$

$A_{13} = \begin{bmatrix}
-14.2857 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
20.6947 & -5.5186 & -12.8588 & 0.0000 & 0.0000 \\
10.3617 & 1.1226 & -5.6541 & 0.0000 & 0.0000 \\
0.0000 & 1.0000 & 0.0000 & 0.0000 & 15.0000 \\
0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\
\end{bmatrix}$
\[
A_{14} = \begin{bmatrix}
-14.2857 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
20.6947 & -5.5186 & -27.8588 & 0.0000 & 0.0000 \\
10.3617 & 1.1226 & -5.6541 & 0.0000 & 0.0000 \\
0.0000 & 1.0000 & 0.0000 & 0.0000 & 30.0000 \\
0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\
\end{bmatrix}
\]

\[
A_{15} = \begin{bmatrix}
-14.2857 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
41.3895 & -5.5186 & -12.8588 & 0.0000 & 0.0000 \\
20.7233 & 1.1226 & -5.6541 & 0.0000 & 0.0000 \\
0.0000 & 1.0000 & 0.0000 & 0.0000 & 15.0000 \\
0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\
\end{bmatrix}
\]

\[
A_{16} = \begin{bmatrix}
-14.2857 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
41.3895 & -5.5186 & -27.8588 & 0.0000 & 0.0000 \\
20.7233 & 1.1226 & -5.6541 & 0.0000 & 0.0000 \\
0.0000 & 1.0000 & 0.0000 & 0.0000 & 30.0000 \\
0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\
\end{bmatrix}
\]

\[
B_1 = B_2 = \cdots B_{16} = \begin{bmatrix} -14.2857 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \end{bmatrix}
\]

\[
C_1 = C_2 = \cdots C_{16} = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix}
\]
Appendix C

Simulation Parameters

Tables C.1 and C.2 contain vehicle data for a 1992 Ford Taurus, which was used in the nonlinear simulation of Chapter 6. However, the simulation program requires additional data for the vehicle which has not been measured at this time. In such cases, data from (Smith, 1993) has been substituted. Tables C.3 through C.12 show a complete listing of parameters used in the simulation, organized as they would be entered into the graphical user interface to the simulation. Using the parameters listed, the simulation has been run for a rear-wheel-drive vehicle using a nonlinear tire model with a first-order tire-sideforce lag. Although each parameter would need to be correct before validation of the simulation is complete, the absence of exact values for each parameter does not affect the validity of the test for robustness because this level of detail is unknown during the controller design. The controller is shown to work on a vehicle which is similar to a 1992 Ford Taurus. The large number of detailed parameters required for the simulation have helped motivate the present work which aims to design simple controllers that are robust to such "real world" complexities.
Table C.1: Vehicle Model Data for a 1992 Ford Taurus

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0.9637 m</td>
<td>distance from the center of gravity to the front axle</td>
</tr>
<tr>
<td>$b$</td>
<td>1.7287 m</td>
<td>distance from the center of gravity to the rear axle</td>
</tr>
<tr>
<td>$T_w$</td>
<td>1.5748 m</td>
<td>track width</td>
</tr>
<tr>
<td>$h_{cg}$</td>
<td>0.5493 m</td>
<td>height of center of gravity above ground</td>
</tr>
<tr>
<td>$m$</td>
<td>1419 kg</td>
<td>vehicle mass</td>
</tr>
<tr>
<td>$I_{zz}$</td>
<td>2618 kg m²</td>
<td>vehicle yaw moment of inertia</td>
</tr>
<tr>
<td>$I_{xx}$</td>
<td>519 kg m²</td>
<td>vehicle roll moment of inertia</td>
</tr>
</tbody>
</table>

*Values have been measured by the Texas Transportation Institute for a vehicle in the possession of the LSU Department of Mechanical Engineering.*

Table C.2: Vehicle Body Dimensions for a 1992 Ford Taurus

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_f$</td>
<td>1.0 m</td>
<td>distance from front axle to front of vehicle</td>
</tr>
<tr>
<td>$b_r$</td>
<td>1.2 m</td>
<td>distance from rear axle to rear of vehicle</td>
</tr>
<tr>
<td>$d_{tw}$</td>
<td>0.15 m</td>
<td>distance from tread center to side of vehicle</td>
</tr>
</tbody>
</table>

*Values have been measured by author for a vehicle in the possession of the LSU Department of Mechanical Engineering.*
Table C.3: Vehicle Dimensions

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>hcg</td>
<td>0.5493 m</td>
<td>height of cg above ground</td>
</tr>
<tr>
<td>a</td>
<td>0.9637 m</td>
<td>distance from cg to front axle</td>
</tr>
<tr>
<td>b</td>
<td>1.7287 m</td>
<td>distance from cg to rear axle</td>
</tr>
<tr>
<td>tw</td>
<td>1.5748 m</td>
<td>tread width (width between center of tires)</td>
</tr>
<tr>
<td>e</td>
<td>0.2 m</td>
<td>height of cg above roll axis</td>
</tr>
<tr>
<td>arf</td>
<td>2.1 m²</td>
<td>frontal area of vehicle</td>
</tr>
<tr>
<td>af</td>
<td>1.0 m</td>
<td>distance from cg to front edge of car</td>
</tr>
<tr>
<td>br</td>
<td>1.2 m</td>
<td>distance from cg to rear edge of car</td>
</tr>
<tr>
<td>dtw</td>
<td>0.15 m</td>
<td>distance from center of tire to outside edge of car</td>
</tr>
</tbody>
</table>

Table C.4: Vehicle Mass and Inertia

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>1419.26 kg</td>
<td>vehicle total mass</td>
</tr>
<tr>
<td>ms</td>
<td>1299.26 kg</td>
<td>vehicle sprung mass</td>
</tr>
<tr>
<td>mizz</td>
<td>2618.08 kg m²</td>
<td>moment of inertia of car about Z axis</td>
</tr>
<tr>
<td>mixx</td>
<td>519.278 kg m²</td>
<td>moment of inertia of car about X axis</td>
</tr>
<tr>
<td>mixz</td>
<td>0.00 kg m⁴</td>
<td>product of inertia of car about X-Z axis</td>
</tr>
</tbody>
</table>
Table C.5: Vehicle Engine and Transmission

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>mieng</td>
<td>0.13565 kg m²</td>
<td>rotational inertia of engine and drivetrain</td>
</tr>
<tr>
<td>sfts spd(1)</td>
<td>3700 rpm</td>
<td>engine speed at which to up-shift</td>
</tr>
<tr>
<td>sfts spd(2)</td>
<td>1800 rpm</td>
<td>engine speed at which to down-shift</td>
</tr>
<tr>
<td>sfttime</td>
<td>0.5 s</td>
<td>time for shifting gears</td>
</tr>
<tr>
<td>numgr</td>
<td>5</td>
<td>number of gears</td>
</tr>
<tr>
<td>gear(1)</td>
<td>13.56</td>
<td>gear ratio for 1st gear</td>
</tr>
<tr>
<td>gear(2)</td>
<td>7.50</td>
<td>gear ratio for 2nd gear</td>
</tr>
<tr>
<td>gear(3)</td>
<td>5.37</td>
<td>gear ratio for 3rd gear</td>
</tr>
<tr>
<td>gear(4)</td>
<td>4.22</td>
<td>gear ratio for 4th gear</td>
</tr>
<tr>
<td>gear(5)</td>
<td>3.28</td>
<td>gear ratio for 5th gear</td>
</tr>
<tr>
<td>geff(i)</td>
<td>0.85</td>
<td>gear efficiency for each gear</td>
</tr>
<tr>
<td>kpvot</td>
<td>2.1</td>
<td>proportional constant for throttle control</td>
</tr>
</tbody>
</table>

Table C.6: Vehicle Roll

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>front.roll.k</td>
<td>20000 N m/rad</td>
<td>front roll stiffness</td>
</tr>
<tr>
<td>rear.roll.k</td>
<td>25000 N m/rad</td>
<td>rear roll stiffness</td>
</tr>
<tr>
<td>roll.damp</td>
<td>2600 N m/rad/sec</td>
<td>total roll damping</td>
</tr>
<tr>
<td>crsf</td>
<td>-0.05</td>
<td>front roll steer coef.</td>
</tr>
<tr>
<td>crsr</td>
<td>0.1</td>
<td>rear roll steer coef.</td>
</tr>
</tbody>
</table>

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
### Table C.7: Vehicle Properties

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$cd$</td>
<td>0.32</td>
<td>drag coefficient</td>
</tr>
<tr>
<td>$\text{fric}$</td>
<td>varies</td>
<td>coef. of friction between the tires and ground</td>
</tr>
<tr>
<td>$er$</td>
<td>0.015</td>
<td>road adhesion reduction factor (Duggof model)</td>
</tr>
<tr>
<td>$kbf$</td>
<td>0.6</td>
<td>front brake proportioning constant</td>
</tr>
<tr>
<td>$kpbrk$</td>
<td>100</td>
<td>proportional constant for brake control</td>
</tr>
<tr>
<td>$\text{brake_max}$</td>
<td>10 N m</td>
<td>maximum total brake torque</td>
</tr>
<tr>
<td>$\text{steer_maxf}$</td>
<td>25 deg</td>
<td>maximum front steering angle</td>
</tr>
<tr>
<td>$\text{steer_maxr}$</td>
<td>25 deg</td>
<td>maximum rear steering angle</td>
</tr>
</tbody>
</table>

### Table C.8: Tire Stiffness

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ctz$</td>
<td>$7 \times 10^{-6}$ m/N</td>
<td>vertical tire stiffness</td>
</tr>
<tr>
<td>$\text{caf}$</td>
<td>varies</td>
<td>cornering stiffness of one front tire (N/rad)</td>
</tr>
<tr>
<td>$\text{car}$</td>
<td>varies</td>
<td>cornering stiffness of one rear tire (N/rad)</td>
</tr>
<tr>
<td>$csf$</td>
<td>50000 N/unitslip</td>
<td>long. stiffness of one front tire</td>
</tr>
<tr>
<td>$csr$</td>
<td>50000 N/unitslip</td>
<td>long. stiffness of one rear tire</td>
</tr>
</tbody>
</table>
Table C.9: Tire Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>rn</td>
<td>0.3 m</td>
<td>nominal tire radius</td>
</tr>
<tr>
<td>crf</td>
<td>0.0135</td>
<td>Coef. of rolling resistance front tire</td>
</tr>
<tr>
<td>crr</td>
<td>0.0135</td>
<td>Coef. of rolling resistance rear tire</td>
</tr>
<tr>
<td>ctdamp</td>
<td>0.20</td>
<td>tire damping value</td>
</tr>
<tr>
<td>mift</td>
<td>2.3 kg m²</td>
<td>rotational inertia of one front tire</td>
</tr>
<tr>
<td>mirt</td>
<td>2.1 kg m²</td>
<td>rotational inertia of one rear tire</td>
</tr>
</tbody>
</table>

Table C.10: Control Actuators

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>wot_tc</td>
<td>0.07 s</td>
<td>time constant for throttle actuator</td>
</tr>
<tr>
<td>sv_tc</td>
<td>0.07 s</td>
<td>time constant for steering actuator</td>
</tr>
<tr>
<td>brk_tc</td>
<td>0.04 s</td>
<td>time constant for brake actuator</td>
</tr>
<tr>
<td>gssf</td>
<td>1.0</td>
<td>steady state gain of front steering actuator</td>
</tr>
<tr>
<td>gssr</td>
<td>1.0</td>
<td>steady state gain of rear steering actuator</td>
</tr>
</tbody>
</table>

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
### Table C.11: Path Definition

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>x1</td>
<td>0.0 m</td>
<td>path definition variable</td>
</tr>
<tr>
<td>x2</td>
<td>0.0 m</td>
<td>path definition variable</td>
</tr>
<tr>
<td>x3</td>
<td>1000.0 m</td>
<td>path definition variable</td>
</tr>
<tr>
<td>x4</td>
<td>0.0 m</td>
<td>path definition variable</td>
</tr>
<tr>
<td>x5</td>
<td>2000.0 m</td>
<td>path definition variable</td>
</tr>
<tr>
<td>y1</td>
<td>3.6576 m</td>
<td>path definition variable</td>
</tr>
<tr>
<td>y2</td>
<td>300.0 m</td>
<td>path definition variable</td>
</tr>
<tr>
<td>r1</td>
<td>920.0 m</td>
<td>path definition variable</td>
</tr>
<tr>
<td>r2</td>
<td>916.3424 m</td>
<td>path definition variable</td>
</tr>
</tbody>
</table>

### Table C.12: Simulation Time and Speed

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>endt</td>
<td>varies</td>
<td>total time of simulation (sec)</td>
</tr>
<tr>
<td>delt</td>
<td>0.001 s</td>
<td>integration time step (0.001 sec is best)</td>
</tr>
<tr>
<td>numplt</td>
<td>10000000</td>
<td>no. of int. steps to skip between data output</td>
</tr>
<tr>
<td>sensor_t</td>
<td>0.025 s</td>
<td>time between sensor updates (sec)</td>
</tr>
<tr>
<td>path_t</td>
<td>1.5 s</td>
<td>time to start path</td>
</tr>
<tr>
<td>step_t</td>
<td>5.0 s</td>
<td>time of step during a turn</td>
</tr>
<tr>
<td>plot_t</td>
<td>1.5 s</td>
<td>time to start plotting</td>
</tr>
<tr>
<td>start_speed</td>
<td>varies</td>
<td>vehicle starting speed (m/s)</td>
</tr>
<tr>
<td>uxdes</td>
<td>varies</td>
<td>desired forward speed (m/s)</td>
</tr>
</tbody>
</table>
Vita

The younger of two children, Robert was born in Baton Rouge on July 3, 1969, to Robert and Dorothy Benton. While attending the Scotlandville Magnet High School for the Engineering Professions, Robert began to play the guitar. He and three friends formed a band known as Elliterate Serial. The summer after his senior year he and band member Beaux LaCoste recorded several songs using a makeshift studio. The collection of songs was reproduced as Elliterate Serial’s first album Material and sold to friends who were kind enough to pay two dollars.

Robert’s freshman year was spent at Louisiana Tech University, where he became involved in Campus Crusade for Christ. As a result, Robert became convinced that he finally understood the message of the gospel. God loves us and created us to know Him personally (John 3:16; John 17:3), but because of sin, man is separated from God and cannot know him personally or experience His love (Romans 3:23; Romans 6:23). Jesus Christ is God’s only provision for man’s sin. Through Him alone, we can know and experience God’s love (Romans 5:8; John 14:6). We must each individually receive Jesus Christ as our personal Savior and Lord; then we can know God personally and experience His love (John 1:12; Ephesians 2:8,9; 1 John 5:12,13). Robert transferred to Louisiana State University following his freshman year. During the next few years Robert and Beaux recorded their second album Betwixt the Twa.

When a senior, Robert decided to become involved in the L.S.U. chapter of Campus Crusade for Christ, where he eventually led music for the weekly meet-
ing. He spent the summer before his last undergraduate semester in Inner City Albuquerque as a missionary with the Baptist Student Union. When he returned to school, he decided to lead a Bible study with Campus Crusade for Christ, where he learned much about the infinite and unconditional love that God has for him.

After graduating with a bachelor of science degree in mechanical engineering from L.S.U., Robert accepted a Dean’s Fellowship at L.S.U. and studied under the direction of two professors who left L.S.U. before he could finish. During this difficult time, he met the lovely Michelle Robertson at a Campus Crusade for Christ event. Robert and Michelle were married on August 7, 1993, at Michelle’s parent’s home in Little Rock, Arkansas. Soon after the wedding, Robert agreed to work with Dr. Dirk Smith who encouraged Robert to study automatic control with an emphasis on emergency lateral control of highway vehicles. Robert finished writing his dissertation in November 1996, and is scheduled to graduate with a doctor of philosophy degree in mechanical engineering from L.S.U. in May 1997. He plans to pursue an academic career and continue his research into automatic control of mechanical systems.
DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Robert Edward Benton, Jr.

Major Field: Mechanical Engineering

Title of Dissertation: Linear Matrix Inequality Approach to Robust Emergency Lateral Control of a Highway Vehicle with Time-Varying Uncertainties

Approved:

[Signatures]

Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

[Signatures]

Date of Examination:

11/26/96