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Jordan Algebras and Lie Semigroups.

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JORDAN ALGEBRAS AND LIE SEMIGROUPS

A Dissertation

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in
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Dedication

To my wife, Hooju
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Abstract

For a Euclidean Jordan algebra $V$ with the corresponding symmetric cone $\Omega$, we consider the semigroup $\Gamma_\Omega$ of elements in the automorphism group $G(T_n)$ of the tube domain $V + i\Omega$ which can be extended to $\Omega$ and maps $\Omega$ into itself. A study of this semigroup was first worked out by Koufany in connection to Jordan algebra theory and Lie theory of semigroups. In this work we give a new proof of Koufany's results and generalize up to infinite dimensional Jordan algebras, so called $JB$-algebras.

One of the nice examples of the semigroup $\Gamma_\Omega$ is from the Jordan algebra $\text{Sym}(n, \mathbb{R})$ of symmetric matrices. However, $V_\sigma$, the set of all self-adjoint operators on $\mathbb{R}^n$ with respect to a non-degenerated symmetric bilinear form $\sigma$, is a non-Euclidean Jordan algebra with a cone $\Omega_\sigma$ which is isomorphic to $\Omega$ of the symmetric cone of $\text{Sym}(n, \mathbb{R})$. We get an isomorphism of the automorphism groups between two tube domains which also induces an isomorphism between two Lie semigroups.

The Lorentzian cone, which is one of the irreducible symmetric cones, is an essential tool in the study of semigroups in Möbius and Lorentzian geometry. J.D.Lawson studied the Möbius and Lorentzian semigroups with an Ol'shanskii decomposition even in the infinite dimensional cases. We study these semigroups via a Jordan algebra theory.
Introduction

The aim of this dissertation is to study Lie semigroups arising in Jordan algebra theory. The Lie theory of semigroups has been developed recently in many areas. In particular, we are interested in decompositions of Lie semigroups which are mostly compression semigroups in Lie transformation groups.

One of important types of non-associative algebras consists of the Lie algebras which are widely known in many contexts. However the Jordan algebras form an important type of the non-associative algebra. They were first studied by Jordan, von Neumann and Wigner in the mid-1930s with the aim of providing a suitable setting for axiomatic quantum mechanics. In the mathematical foundations of quantum physics one of the natural axioms is that the observables form a Jordan algebra. It was not until the mid-1960s that Jordan algebras were systematically studied from the point of view of functional analysis. From then on there has developed a theory which closely resembles that of $C^*$ and von Neumann algebras, and which is concerned with the infinite dimensional analogues of the original algebras of Jordan, von Neumann and Wigner (for the historical background of Jordan algebras and their applications, one can refer to McCrimmon’s paper [35]).

The relationship of Jordan algebras to holomorphic functions in several variables was noted by Koecher [25],[26] (cf, [17],[18],[27],[32],[33],[41],[48]). The main result in this area states that certain symmetric domains in $C^*$ can be completely characterized in terms of formally real Jordan algebras. By the results of E.Catán and Harish-Chandra, every hermitian symmetric space of non-compact type can be realized as a bounded domain in a complex vector space. These spaces generalize the upper half-plane realization of the unit.
open ball. The irreducible hermitian symmetric spaces which did not appear in Koecher's theory, called by Siegel domains of type II, were studied by I.I.Pjateckii-Sapiro. Koranyi and Wolf [27] constructed a general Cayley transform which carries the bounded domain of Harish-Chandra realization into a generalized half-plane. Their Cayley transform yields the domains considered by Koecher and Siegel domains of type II as special cases. Later on, Kaup and Upmeier extended the theory to an infinite number of variables ([19],[20],[21],[22],[23],[24],[45],[46]) by introducing Jordan $C^*$-algebras, which we call $JB^*$-algebras. The notion of a $JB^*$-algebra has been introduced by Kaplansky and Wright [49]. Every $C^*$-algebra and every closed self-adjoint Jordan subalgebra are nice examples of $JB^*$-algebras. The real analogues of $JB^*$-algebras are the $JB$-algebras introduced and throughly studied by Alfsen, Shultz and Stomer [2]. They and Wright gave the one-to-one correspondence between $JB$ and $JB^*$-algebras in [49]. In finite dimensions $JB$-algebras are precisely the formally real Jordan algebras and $JB^*$-algebras are the semisimple complex Jordan algebras with respect to a suitable norm.

The basic example of a Lie semigroup in this dissertation is the closed semigroup $SL(2, \mathbb{R})^+$ of all 2 by 2 matrices of determinant 1 with non-negative entries. The group $SL(2, \mathbb{R})$ acts on the upper half-plane $\mathbb{R} + i\mathbb{R}^+$ by fractional linear transformations. The group $PSL(2, \mathbb{R})$ is the automorphism group of the upper half-plane. Then the semigroup $SL(2, \mathbb{R})^+$ can be decomposed in the following way. Let
\[ \Gamma^+ = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mid a \geq 0 \right\}, \quad \Gamma^- = \left\{ \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \mid b \geq 0 \right\}, \quad H = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \mid a > 0 \right\}. \]
Then $SL(2, \mathbb{R})^+ = \Gamma^+ H \Gamma^-$. However, the set of all elements $g \in SL(2, \mathbb{R})$ such that $g$ can be extended to $\mathbb{R}^+$ and $g(\mathbb{R}^+) \subset \mathbb{R}^+$ is a subsemigroup of $SL(2, \mathbb{R})$. But this is exactly the semigroup $\pm SL(2, \mathbb{R})^+$ (cf. III.6). If we consider $V = \mathbb{R}$
as the Jordan algebra with usual multiplication, then the symmetric cone corresponding to $V$ is $\Omega = \mathbb{R}^+$. In general, let $V$ be a JB-algebra with the symmetric cone $\Omega$ and let $T_\Omega$ be the tube domain. Define a subsemigroup $\Gamma_\Omega$ in the identity component $G(T_\Omega)$ of the automorphism group $T_\Omega$ of all elements $g$ such that $g$ can be extended on $\Omega$ and $g(\Omega) \subset \Omega$. This semigroup has been considered by Koufany [28] who has obtained decompositions of this semigroup in finite dimensional formally real Jordan algebras. In chapter III, we will study this semigroup in any JB-algebra with a more direct proof of Koufany's result.

On the other hand, the group $SL(2, \mathbb{R})$ can be identified with the proper Möbius group on the unit circle. That is, it is the smallest subgroup of Möbius type [31]. The Möbius semigroup is the compression semigroup of half-ray in positive direction. Via the ray bundle, the study of Möbius geometry is the same as the study of Lorentzian geometry [31],[42]. Using the Lorentzian coordinates, the Lorentzian cone $\Omega$ can be decomposed as (cf. II.2.4)

$$\Omega = \Omega^+ \cup \Omega^0 \cup \Omega^-.$$  

The Möbius semigroup corresponds to the Lorentzian semigroup: the set of all pseudo-orthogonal transformations which carries the boundary of $\Omega^+$ into itself. The Lorenztian semigroup has the Ol'shanskii decomposition and is a maximal subsemigroup of the group [31]. However, the cone $\Omega$ is the symmetric cone of the spin factor which is one of the simple formally real Jordan algebras. In chapter II, we analyze the spin factor, in particular the automorphism group of the cone. Motivated by the study the semigroup, using the decomposition of the cone, we will study the compression semigroup $S$ of $\Omega^+$ in the automorphism group $G(\Omega)$. In chapter IV, we will see the relation between $S$ and the Möbius semigroup. The group of units of $S$ will be the
automorphism group of the symmetric cone $\Omega^0$. Furthermore, the compression semigroup $S = G(\Omega^0) \cdot \exp C$ has the Ol'shanskii decomposition.
In this chapter, we recall some basic and important properties of symmetric cones and Jordan algebras. In particular, we will concentrate on the Euclidean Jordan algebras. The classical background of studying symmetric cones and Euclidean Jordan algebras is the relationship between formally real Jordan algebras, Euclidean Jordan algebras, self-dual homogeneous cones and symmetric upper half planes due to Koecher in finite dimensional cases [25],[26]. A nice book concerning the classification of the symmetric cones via Jordan algebras and general Jordan algebra theories is Faraut and Koranyi, *Analysis on Symmetric Cones* [5]. We will refer to this book as the main reference of this chapter. In section I.5, we summarize the results of finite dimensional analogues in infinite-dimensional Jordan algebras and state the link between bounded symmetric domains and Jordan algebras in infinite-dimensional cases.

### 1.1. Symmetric cones

In this section, we will give the definition of a symmetric cone in a Euclidean vector space and its automorphism group. The classification of irreducible symmetric cones is also stated.

Let $V$ be a finite dimensional real Euclidean space with the inner product $\langle x|y \rangle$. A subset $C$ of $V$ is said to be a cone if $\lambda x \in C$ for any $x \in C$ and $\lambda \in \mathbb{R}^+ = (0, \infty)$. And a cone $C$ is said to be proper or pointed if $\overline{C} \cap -(\overline{C}) = \{0\}$. In the following we assume that $\Omega$ is an open convex cone in $V$. 

Chapter I
Symmetric Cones, Tube Domains
and Jordan Algebras

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The open dual cone is defined by

\[ \Omega^* = \{ y \in V \mid \langle x|y \rangle > 0, \forall x \in \Omega^* - \{0\} \}. \]

An open convex cone \( \Omega \) is called a self-dual if \( \Omega^* = \Omega \).

The automorphism group of an open convex cone \( \Omega \) is defined by

\[ G(\Omega) = \{ g \in GL(V) \mid g\Omega = \Omega \}. \]

Then \( G(\Omega) \) is a closed subgroup of \( GL(V) \) and hence a Lie group. The open convex cone is said to be a homogenous if \( G(\Omega) \) acts on it transitively. The open cone \( \Omega \) is said to be symmetric if it is homogenous and self-dual. It is known [3],[5] that if \( \Omega \) is a proper open convex cone, then for any element \( a \in \Omega \), the stabilizer of \( a \) is compact and maximal compact if \( \Omega \) is homogeneous. In this case, all stabilizer groups are conjugate. Furthermore, for a symmetric cone \( \Omega \), we define

\[ K = G \cap O(V), \]

where \( G \) is the connected component of the identity in \( G(\Omega) \) and \( O(V) \) is the orthogonal group of \( V \). We write \( g \) for the Lie algebra of \( G(\Omega) \) and \( \mathfrak{t} \) for the Lie algebra of \( K \). Then

\[ \mathfrak{t} = \{ X \in g \mid X^* = -X \}. \]

Here \( X^* \) is the usual adjoint operator of \( X \) for the inner product on \( V \).

We define

\[ \mathfrak{p} = \{ X \in g \mid X^* = X \}. \]

Then \( g = \mathfrak{t} \oplus \mathfrak{p} \) with the following relation

\[ [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{t}, \quad [\mathfrak{t}, \mathfrak{p}] \subset \mathfrak{p}. \]
Furthermore, there exists a point $e$ in $\Omega$ such that

$$G_e := \{g \in G \mid ge = e\} = K.$$  

For such element $e$, an element $X \in g$ belongs to $\mathfrak{t}$ if and only if $X \cdot e = 0$ [5].

**Theorem I.1.1.** The mapping

$$p \mapsto V, \quad X \mapsto X \cdot e$$

is a bijection.

**Proof.** (cf. [5]). □

A symmetric cone $\Omega$ in a Euclidean space $V$ is said to be **irreducible** if there do not exist non-trivial subspaces $E_1, E_2$ and symmetric cones $\Omega_i \subset E_i$ such that $E$ is the direct sum of $E_i$ and $\Omega = \Omega_1 + \Omega_2$. In fact ([5], [25]), any symmetric cone is, in a unique way, the direct product of irreducible symmetric cones.

Now the irreducible symmetric cones have been completely classified [25], [26], and [3]. They are

1. The cone of positive-definite symmetric matrices.
2. The cone of positive-definite Hermitian complex matrices.
3. The cone of positive-definite Hermitian quaternion matrices.
4. The spherical cone in $\mathbb{R}^{n+1}$ of $(x_0, \cdots, x_n)$ with

   $$x_0 > (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}.$$  

5. The 27-dimensional cone of positive-definite Hermitian octavic matrices of third order.

Here (1) – (3) are classical, (4) is semi-classical and (5) is exceptional.
1.2. Euclidean Jordan algebras

In this section we recall some basic notions of Jordan algebras that will be needed in the sequel. Jordan algebras were first introduced by Jordan, von Neuman, and Wigner [15]. In this paper, they classified the simple formally real Jordan algebras which turn out to be the simple Euclidean Jordan algebras. In the previous section, we have seen the classification of irreducible symmetric cones. In this section we will see a one-one correspondence between these two categories.

1.2.1. Definitions and basic properties

Let \( F \) be the field \( \mathbb{R} \) or \( \mathbb{C} \). A commutative algebra \( V \) over \( F \) with product \( xy \) is said to be a Jordan algebra if, for all elements \( x \) and \( y \) in \( V \):

\[
x(x^2y) = x^2(xy).
\]

This identity is called the Jordan identity. For \( x \in V \), we denote

\[
L(x)y := xy,
\]

the multiplication operator representation. Then the Jordan identity can be written

\[
[L(x), L(x^2)] = 0,
\]

where the bracket is usual Lie bracket on \( gl(V) \) the set of all bounded linear operators on the vector space \( V \).

For \( x \in V \), we define

\[
P(x) = 2L(x)^2 - L(x^2).
\]

The map \( P \) is called the quadratic representation of \( V \).
Every associative algebra \( V \) with product \( x \circ y \) becomes a Jordan algebra with the anticommutator product:
\[
xy = \frac{1}{2}(x \circ y + y \circ x).
\]
This kind of Jordan algebra is called a \textit{special Jordan algebra}. In a special Jordan algebra,
\[
P(x)y = x \circ y \circ x
\]
\[
P(x \circ y \circ x) = P(x)P(y)P(x).
\]
A Jordan algebra \( V \) is said to be \textit{formally real} if
\[
x^2 + y^2 = 0 \Rightarrow x = y = 0.
\]
An element \( x \) of a Jordan algebra \( V \) with identity \( e \) is called \textit{invertible} with inverse \( y \) if \( xy = e \) and \( x^2y = x \). One can see that an element \( x \) in a Jordan algebra \( V \) is invertible if and only if \( P(x) \) is invertible. In this case,
\[
P(x)x^{-1} = x.
\]
\[
P(x)^{-1} = P(x^{-1}).
\]
The following formula is called the \textit{fundamental formula}:
\[
P(P(x)y) = P(y)P(x)P(y).
\]
An element \( c \in V \) is said to be an \textit{idempotent} if \( c^2 = c \). We say that an idempotent is \textit{primitive} if it is non-zero and cannot be written as the sum of two non-zero idempotents. Usually we denote \( \mathcal{P} \) by the set of all primitive elements of a Jordan algebra \( V \).

Now for every idempotent \( c \in V \) we have the \textit{Pierce decomposition} of \( V \) relative to \( c \): for \( k = 0, 1, \frac{1}{2} \), let
\[
V(c,k) := \{ x \in V \mid cx = kx \}.
\]
Then $V = V(c, 0) \oplus V(c, \frac{1}{2}) \oplus V(c, 1)$, with the multiplication rule:

- $V(c, 0)V(c, 0) \subseteq V(c, 0)$,
- $V(c, 0)V(c, 1) = \{0\}$,
- $V(c, 1)V(c, 1) \subseteq V(c, 1)$,
- $V(c, 0)V(c, \frac{1}{2}) \subseteq V(c, \frac{1}{2})$,
- $V(c, 1)V(c, \frac{1}{2}) \subseteq V(c, \frac{1}{2})$,
- $V(c, \frac{1}{2})V(c, \frac{1}{2}) \subseteq V(c, 0) + V(c, 1)$.

In particular, $V(c, 0), V(c, 1)$ are subalgebras of $V$.

Let $V$ be a finite dimensional Jordan algebra and let $\tau(x, y) = TrL(xy)$. Then $\tau(x, y)$ is an associative symmetric bilinear form on $V$. That is,

$$\tau(xy, z) = \tau(y, xz),$$

for all $x, y, z$ in $V$. A Jordan algebra is said to be semi-simple if the bilinear form $\tau(x, y)$ is non-degenerate on it. A semi-simple Jordan algebra is called simple if it has no non-trivial ideal. A semi-simple Jordan algebra over $\mathbb{R}$ or $\mathbb{C}$ is, in unique way, a direct sum of simple ideals [5].

Using the Pierce Decomposition, we have the following useful characterization of simple Jordan algebras [5]:

**Lemma 1.2.1.** Let $V$ be a semisimple Jordan algebra. Then $V$ is simple if and only if $V(c, \frac{1}{2}) \neq 0$ for every non-trivial idempotent $c$.

**Proof.** If $V(c, \frac{1}{2}) = 0$, then $V = V(c, 0) \oplus V(c, 1)$. By the multiplication rule on the Pierce decomposition, $V(c, 0)$ and $V(c, 1)$ are ideals on $V$. Hence $V(c, 0) = V$ or $V(c, 1) = V$. Then $c$ is zero or the identity. Hence $V(c, \frac{1}{2}) \neq 0$ for a non-trivial idempotent $c$. Conversely, suppose that $I$ is a non-trivial ideal.
in $V$. Then the orthogonal complement

$$I^L = \{ x \in V \mid \tau(x, y) = TrL(xy) = 0, \forall y \in I \}$$

is an ideal. Since $V$ is semisimple, the ideal $I$ is also semisimple. Hence $I$ has an identity $c$. Since $I$ is non-zero, $c$ is non-trivial. By the Pierce decomposition, $V = V(c, 0) \oplus V(c, \frac{1}{2}) \oplus V(c, 1)$. Now let $x \in V(c, \frac{1}{2})$. Since $c \in I$ and $c$ is the identity for $I$, we have $x = 2c x$, $cx = x$. This implies that $x = 0$. So $V(c, \frac{1}{2}) = 0$, which gives a contradiction. □

I.2.2. Euclidean Jordan algebras

One generalization of a (finite-dimensional) formally real Jordan algebra is a Jordan-Hilbert algebra admitting an associative inner product. This concept was considered by Nomura [37].

A real Jordan algebra $V$ is called a Jordan-Hilbert algebra if $V$ is a real Hilbert space with inner product $\langle x | y \rangle$ such that

$$\langle xy | z \rangle = \langle y | xz \rangle,$$

for all $x, y, z \in V$. If $V$ is finite dimensional, then it is called a Euclidean Jordan algebra. In general a Jordan-Hilbert algebra does not contain a unit element. To avoid the trivial algebra, we assume that the linear map $L : V \rightarrow L(V)$ is injective. That is,

$$L(x) = 0 \implies x = 0.$$

In finite dimensional cases, this condition is equivalent to the existence of units. Clearly if $V$ has unit $e$ then $L(x) = 0$ implies that $xe = x = 0$. We will show the converse part.

It was proved by Koecher that every (finite-dimensional) semisimple Jordan algebra has a unit. We will see that every Euclidean Jordan algebra has
a unit using Koecher's method. With this result, one can see in [5] that
every Euclidean Jordan algebra is semi-simple and the associative bilinear
form \( r(x, y) = TrL(xy) \) is positive definite. In particular, it is easy to see that
every Jordan-Hilbert algebra with unit is formally real.

The following two lemmas are in [37]:

**Lemma 1.2.2.** Let \( V \) be a Jordan-Hilbert algebra. Then

1. There exists \( x \in V \) such that \( x^2 \neq 0 \).
2. If \( x^2 \neq 0 \), then \( x^n \neq 0 \) for all \( n \in \mathbb{N} \).

**Proof.** Suppose that \( x^2 = 0 \) for all \( x \in V \). Then \( 0 = (x+y)^2 = x^2 + 2xy + y^2 = 2xy \) which implies that \( xy = 0 \) for all \( x, y \in V \). But this is impossible.

Suppose \( x^{n+1} = 0 \), for \( n > 1 \). Then

\[
0 = \langle x^{n+1}|x^{n-1} \rangle = \langle x^n|x^n \rangle \implies x^n = 0.
\]

The proof follows by induction. \( \square \)

**Lemma 1.2.3.** Let \( V \) be a Jordan-Hilbert algebra and let \( x \in V \). Then if \( x^2 = 0 \), then \( x = 0 \).

**Proof.** Since \( P(x) \) is a self-adjoint operator, for any \( y \in V \),

\[
\langle P(x)y|P(x)y \rangle = \langle P(x^2)y|y \rangle = 0.
\]

Thus \( 2L(x)^2 = P(x) + L(x^2) = 0 \) so that \( ||L(x)y|| = \langle L(x)^2y|y \rangle = 0 \). Therefore \( L(x) = 0 \) implies \( x = 0 \). \( \square \)

Now let \( \mathbb{R}[x] \) be the associative subalgebra of \( V \) generated by \( x^n, x^{n+1}, \ldots \).

**Theorem 1.2.4.** Let \( V \) be a Euclidean Jordan algebra. Then \( V \) has an idempotent.
PROOF. Choose \( x \in V \) such that \( x^2 \neq 0 \). Then by lemma I.2.2, \( \mathbb{R}_n[x] \neq 0 \), for all \( n \in \mathbb{N} \). Since \( \dim V < \infty \) and

\[ \cdots, \mathbb{R}_3[x] \subset \mathbb{R}_2[x] \subset \mathbb{R}_1[x] \subset V, \]

there exists \( n \in \mathbb{N} \) such that \( \mathbb{R}_n[x] = \mathbb{R}_{n+k}[x] \), for all \( k = 1, 2, \ldots \). Therefore \( x^n \cdot \mathbb{R}_n[x] = \mathbb{R}_{2n}[x] = \mathbb{R}_n[x] \) and the linear map

\[ L(\mathbb{R}^n)_{\mathbb{R}_n[x]} : u \to x^n u \]

of \( \mathbb{R}_n[x] \) into \( \mathbb{R}_n[x] \) is onto, hence an isomorphism. Therefore there exists \( c \in \mathbb{R}_n[x] \) such that \( x^n c = x^n \). Since \( \mathbb{R}_n[x] \) is associative, \( c \) is the identity for \( \mathbb{R}_n[x] \). In particular \( c \) is an idempotent of \( V \). \( \square \)

Let \( E(V) \) be the set of all idempotents in \( V \). Then by the theorem \( E(V) \) is non-empty if \( V \) is a Euclidean Jordan algebra, but it is still true in an arbitrary Jordan-Hilbert algebra [37]. It is known [5] that the set of primitive elements in a simple Euclidean Jordan algebra is a compact Riemannian symmetric space of rank one and that any such space arises in this way. However, in an infinite-dimensional Jordan-Hilbert algebra, it is a Riemannian Hilbert manifold [37].

**Lemma I.2.5.** Let \( V \) be a Jordan-Hilbert algebra and let \( c \in E(V) \) such that the dimension of the space \( V(c, 0) \) is minimal. Then \( E(V) \cap V(c, 0) = \emptyset \).

**Proof.** Suppose \( d \in E(V) \cap V(c, 0) \). Then \( cd = 0 \) implies \( (c+d)^2 = c^2 + 2cd + d^2 = c+d \). Hence \( c+d \in E(V) \). Since \( (c+d)d = cd+d^2 = d, d \in V(c+d, 1) \). Now \( d \in V(c, 0) \) but \( d \) is not in \( V(c+d, 0) \). Let \( x \in V(c+d, 0) \). Since \( d \in V(c+d, 1) \) and \( V(c+d, 0)V(c+d, 1) = 0 \) by the multiplication rule, \( xd = 0 \) and hence \( x \in V(c, 0) \). This implies that \( V(c+d, 0) \) is a proper subspace of \( V(c, 0) \), this gives a contradiction to the minimality of \( V(c, 0) \). \( \square \)
THEOREM I.2.6. Let $V$ be a Euclidean Jordan-algebra. Then $V$ has a unit.

**Proof.** Choose $c \in E(V)$ such that the dimension of $V(c, 0)$ is minimal.

**Step 1.** $V(c, 0) = 0$. Let $x \in V(c, 0)$. If $x^2 \neq 0$, then $\mathbb{R}[x]$ contains an idempotent from the proof of the Theorem I.2.4. Since $V(c, 0)$ is a subalgebra containing $x$ by the multiplication rule, $V(c, 0) \cap E(V) \neq \emptyset$. This is a contradiction to lemma I.2.5. Hence $x^2 = 0$. By the lemma I.2.3, $x = 0$. This shows that $V(c, 0) = 0$.

**Step 2.** $V(c, \frac{1}{2}) = 0$. Let $x \in V(c, \frac{1}{2})$. Then $cx = \frac{1}{2}x$. By step 1 and the multiplication rule,

$$V(c, \frac{1}{2})V(c, \frac{1}{2}) \subset V(c, 0) + V(c, 1) = V(c, 1)$$

and hence $x^2 \in V(c, 1)$. That is, $cx^2 = x^2$. From the Jordan identity,

$$x^3 = (cx^2)x = x^2(cx) = \frac{1}{2}x^3$$

which implies $x^3 = 0$. But $\langle x^2 | x^2 \rangle = \langle x | x^3 \rangle = 0$ and hence $x = 0$ by lemma I.2.3.

**Step 3.** $V = V(c, 0) \oplus V(c, \frac{1}{2}) \oplus V(c, 1) = V(c, 1)$ implies that $c$ is the identity for $V$. □

As in the finite-dimensional cases, if $V$ is a Jordan-Hilbert algebra, then the linear map $L : V \to End(V)$ defined by the multiplication operator is continuous and the Jordan product $V \times V \to V : (x, y) \mapsto xy \in V$ is continuous. Furthermore, idempotents $c, d$ are orthogonal if and only if $\langle c | d \rangle = 0$ and every idempotent in $V$ can be written as a finite sum of orthogonal primitive idempotents.

In the finite dimensional case, Euclidan Jordan algebras and formally real Jordan algebras are equivalent categories. That is, every formally real Jordan
algebra becomes a Euclidean Jordan algebra, and vice versa. For the proof, see [5].

**Examples 1.2.7.** (1) The algebra $\text{Sym}(n, \mathbb{R})$ of $n \times n$ real symmetric matrices with the Jordan product

$$x \circ y = \frac{1}{2}(xy + yx)$$

is a Euclidean Jordan algebra since the bilinear form $Tr(xy)$ is positive definite and associative.

(2) Let $W$ be a real vector space, and $B$ be a positive definite bilinear form on it. Let $V = \mathbb{R} \times W$ be the Jordan algebra for which the product is

$$(t, x)(s, y) = (ts + B(x, y), ty + sx),$$

then the Jordan algebra is Euclidean. Sometimes we call this Jordan algebra a *spin factor*. In chapter II, we will see that a spin factor can be constructed from a Lorentzian form.

(3) (Non-Euclidean Jordan algebra) Let $V_{1,1}$ be the space of the following $2 \times 2$-matrices of the form:

$$V_{1,1} := \{ \begin{bmatrix} x & y \\ -y & z \end{bmatrix} \mid x, y, z \in \mathbb{R} \}.$$  

Then $V_{1,1}$ is a 3-dimensional Jordan algebra with the anti-commutative product.

Let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Then $A^2 = 0$ and hence $V_{1,1}$ is not formally real, hence not a Euclidean Jordan algebra.

Finally, in a finite-dimensional Jordan algebra $V$ with unit $e$ one defines the exponential function as usual:

$$\exp x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$
One of nice properties of quadratic representation and the exponential mapping which will be used later is the following:

**Proposition 1.2.8.** \( P(\exp x) = \exp 2L(x) \).

**Proof.** (cf. [5]). \( \square \)

### I.2.3. Jordan frames and spectral theorems

Let \( V \) be a Euclidean Jordan algebra with the associative inner product \( \langle x|y \rangle \).

A set of idempotents \( \{c_i\}_{i=1}^k \) is said to be a complete system of orthogonal idempotents if

\[
c_i c_j = 0 \text{ if } i \neq j, \\
\sum_{i=1}^k c_i = e.
\]

**Theorem 1.2.9.** (Spectral theorem, first version) For \( x \in V \), there exist unique real numbers \( \lambda_1, \ldots, \lambda_k \), all distinct, and a unique complete system of orthogonal idempotents \( c_1, \ldots, c_k \) such that

\[
x = \sum_{i=1}^k \lambda_i c_i.
\]

**Proof.** (cf. [5], III.1.1). \( \square \)

We say that \( \{c_i\}_{i=1}^n \) is a Jordan frame if \( \{c_i\}_{i=1}^n \) is a complete system of orthogonal and each \( c_i \) is primitive.

**Theorem 1.2.10.** (Spectral theorem, second version) For every element \( x \) in \( V \), there exists a Jordan frame \( c_1, \ldots, c_r \) (\( r \) is fixed) and real numbers \( \lambda_1, \ldots, \lambda_r \) such that

\[
x = \sum_{i=1}^r \lambda_i c_i.
\]
By the first version of spectral theorem, one can see that an idempotent $c$ is primitive if and only if $\dim V(c, 1) = 1$.

Let $\{c_1, \cdots, c_r\}$ be a Jordan frame on a Euclidean Jordan algebra $V$. Let us consider the following subspaces of $V$

\[ V_i = V(c_i, 1) = R c_i, \]
\[ V_{ij} = V(c_i, \frac{1}{2}) \cap V(c_j, \frac{1}{2}). \]

**Theorem 1.2.11.**

1. $V = \bigoplus_{i < j} V_{ij}$.
2. $V_{ij} \cdot V_{ij} \subseteq V_{ij} + V_{jj}$, $V_{ij} \cdot V_{jk} \subseteq V_{ik}$, $i \neq j$, $V_{ij} \cdot V_{kl} = \{0\}$, $\{i, j\} \cap \{k, l\} = \emptyset$.

**Proof.** (cf. [5], IV.2.1). $\square$

**I.2.4. Symmetric cones of Euclidean Jordan algebras**

Let $V$ be a finite dimensional Jordan algebra over $\mathbb{R}$ or $\mathbb{C}$. A derivation of $V$ is a linear transformation $D \in gl(V)$ such that

\[ D(xy) = Dx \cdot y + x \cdot Dy, \]

for all $x, y$ in $V$. The set $\text{Der}(V)$ of all derivations of $V$ is a Lie algebra with respect to the usual bracket, $[D_1, D_2] = D_1 D_2 - D_2 D_1$. For any $x, y$ in $V$ and $D \in \text{Der}(V)$, we have

\[ [L(x), L(y)] \in \text{Der}(V), \quad [D, L(x)] = L(Dx). \]
One characterization of the $\text{Der}(V)$ is given by: $D \in \text{Der}(V)$ if and only if $D e = 0$ ([5], VIII.2.6).

An automorphism is an invertible linear transformation $g$ of $V$ such that
\[ g(xy) = g(x)g(y), \]
for all $x, y$ in $V$. The set $\text{Aut}(V)$ of all automorphisms of $V$ is a closed Lie subgroup of $\text{GL}(V)$ with Lie algebra $\text{Der}(V)$. Using the fundamental formula, one can show that if $w \in V$ with $w^2 = e$, then the quadratic representation $P(w)$ is an automorphism of $V$ and $P(w)^2 = I$. Let $\text{Aut}(V)_0$ be the identity component of $\text{Aut}(V)$. Then

**Proposition I.2.12.** If $V$ is a simple Euclidean Jordan algebra, then $\text{Aut}(V)_0$ acts transitively on the set of primitive idempotents, and also on the set of Jordan frames.

**Proof.** (cf. [5], IV.2.7).

Suppose that $\{c_1, \cdots, c_r\}$ is fixed Jordan frame in a simple Euclidean Jordan algebra $V$. By the spectral theorem and the previous proposition, any element $x$ in $V$ can be expressed as
\[ x = k(\sum_{i=1}^{r} \lambda_i c_i), \]
for some $k \in \text{Aut}(V)_0$ and real numbers $\lambda_i$.

Let $V$ be a Euclidean Jordan algebra with the associated bilinear form $\langle x|y \rangle$. Let $Q$ be the set of squares:
\[ Q = \{ x^2 \mid x \in V \}. \]
Then the set $Q$ is a self dual cone and is
\[ Q = \{ y \in V \mid L(y) \geq 0 \}. \]
Let $\Omega$ be the interior of $Q$. Then it is a symmetric cone. That is, $\Omega$ is a self-dual cone and the group

$$G(\Omega) := \{ g \in GL(V) \mid g\Omega = \Omega \}$$

acts on it transitively. Furthermore,

**Theorem 1.2.13.**

$$\Omega = \exp V,$$

$$= \text{int}\{ u^2 \mid u \in V \},$$

$$= \text{the identity component of } V^{-1},$$

$$= \{ u^2 \mid u \in V^{-1} \},$$

$$= \{ P(u)e \mid u \in V^{-1} \}$$

$$= \{ u \in V \mid L(u) \text{ positive definite} \}.$$

*Here $L(u)$ is positive definite means that $\langle L(u)v|v \rangle > 0,$ for all non-zero element $v \in V.$*

**Proof.** (cf. [5], III.2.1.). □

For an invertible element $x$ in $V$, $P(x) \in G(\Omega)$. If $k \in Aut(V)$, then it carries a square element to a square element hence $k(Q) = Q$. In particular, $k(\Omega) = \Omega$.

**Theorem 1.2.14.** (Polar decomposition) $G(\Omega) = P(\Omega)Aut(V)$.

**Proof.** Let $g \in G(\Omega)$. Then $g(e) \in \Omega$. Hence by theorem 1.2.13, we can find $w \in \Omega$ such that $g(e) = w^2 = P(w)e$. Put $h = P(w)^{-1}g$, then $h(e) = e$, hence $h \in Aut(V)$. So $g = P(w)h \in P(\Omega)Aut(V)$. □
Lemma 1.2.15. The map
\[ \Omega \rightarrow \Omega, \ w \rightarrow w^2 \]
is bijective.

Proof. Suppose \( w^2 = w_1^2 \) for \( w, w_1 \in \Omega \). Then \( L(w + w_1)(w - w_1) = 0 \).
From theorem 1.2.13, \( w - w_1 = 0 \). Obviously, this map is surjective. \( \square \)

The Lie algebra of \( G(\Omega) \) is given by [5]:
\[ \text{Lie}(G(\Omega)) = \text{Der}(V) \oplus L(V), \]
where \( L(V) = \{ L(u) \mid u \in V \} \).

Now we assume that the Jordan algebra \( V \) is semisimple. For \( g \in GL(V) \), we denote \( g^t \) by the adjoint operator with respect to the bilinear form \( \text{Tr}_L(xy) \).
The structure group of \( V \) is the set of all invertible operators \( g \in GL(V) \) such that
\[ P(gx) = gP(x)g^t. \]
It is known [5] that every \( x \in V^{-1}, P(x) \in \text{Str}(V) \) and \( P(x)^t = P(x) \). Furthermore, the automorphism group \( \text{Aut}(V) \) is a subgroup of \( \text{Str}(V) \) and an element \( g \) in \( \text{Str}(V) \) belongs to \( \text{Aut}(V) \) if and only if \( ge = e \). In particular, \( g^t = g^{-1} \) for \( g \in \text{Aut}(V) \).

Theorem 1.2.16. If \( V \) is a simple Euclidean Jordan algebra, then \( \text{Str}(V) = \{ \pm I \}G(\Omega) \). In particular, if \( g \in \text{Str}(V) \), then \( g(\Omega) = \Omega \) or \( g(\Omega) = -\Omega \).

Proof. (cf. [5], VIII.2.8.). \( \square \)

There is a complete classification of simple Euclidean Jordan algebras. This work is given in Jordan, von Neumann, and Wigner [15] (cf. [5]):
\[ \text{Sym}(n, \mathbb{R}), \text{Herm}(n, \mathbb{C}), \text{Herm}(n, \mathbb{H}), \text{Herm}(3, \mathbb{Q}), \mathbb{R} \times \mathbb{R}^n. \]
Here \( \mathbb{H} \) is the algebra of quaternions and \( \mathbb{Q} \) is the algebra of octonions.
We have seen that a Euclidean Jordan algebra $V$ gives a symmetric cone. Conversely, if $\Omega$ is a symmetric cone, then the mapping

$$p \rightarrow V, \quad X \rightarrow X \cdot e$$

is bijection by theorem I.1.1. Define on $V$ the product

$$xy = L(x)y,$$

where $L(x)$ is the unique element in $p$ such that $L(x)e = x$. Then $V$ is a Euclidean Jordan algebra with identity $e$ and the corresponding symmetric cone $\Omega$. This gives a one-one correspondence between symmetric cones and Euclidean Jordan algebras, or equivalently finite dimensional formally real Jordan algebras. Furthermore, the isotropy group $G_e$ in the identity component $G$ of $G(\Omega)$ is exactly the identity component $Aut(V)_0$ of the automorphism group $Aut(V)$ of $V$ and the Lie algebra corresponding to $Aut(V)$ is the set of derivations ([5], III.5.1).

For $x \in \Omega, \, u, v \in \Omega$ we set

$$\gamma_x(u, v) = (P(x)^{-1}u|v).$$

Then the family of bilinear form $\gamma_x$ defines a $G$-invariant Riemannian metric on $\Omega$. And the map

$$x \rightarrow x^{-1}$$

is an involutive isometry with the unique fixed point $e$. Therefore a symmetric cone $\Omega$ is a symmetric Riemannian space [5].

1.2.5. Complex Jordan algebras

The complexification $V^C$ of a vector space $V$ can be defined as the set $V \times V$ with usual multiplication by complex numbers. The elements of $V^C$ can be

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written \( x + iy \) with \( x \) and \( y \) in \( V \). If \( V \) is an algebra, then \( V^C \) also becomes an algebra by defining the product \((x+iy)(x'+iy')\) in obvious way. In particular, if \( V \) is a Jordan algebra, then \( V^C \) becomes a Jordan algebra. In [25], it has shown that \( V^C \) is semisimple if and only if \( V \) is semisimple.

For \( z = x + iy, w = u + iv \in V^C \), we have \( L(z) = L(x) + iL(y) \) and \( TrL(zw) = TrL(xu) - TrL(yv) + i(Tr(xv) + TrL(yu)) \). Furthermore, \( \overline{zw} = \overline{z} \overline{w} \), where \( \overline{z} = x - iy \).

A real form of complex Jordan algebra \( V \) is a real Jordan subalgebra \( W \) of \( V \) such that \( V \) is a complexification of \( W \). In this case the map \( u + iv \to u - iv \) is an involutive automorphism of \( V \) regarded as a real Jordan algebra, and is complex antilinear. Conversely, every complex antilinear involutive automorphism of \( V \) has as its fixed pointed set a real form, i.e. an algebra of which \( V \) is the complexification.

**THEOREM I.2.17.** Every semi-simple complex Jordan algebra is the complexification of a Euclidean Jordan algebra.

**PROOF.** (cf. [5], VIII.5.2.). \( \square \)

From the classification of simple Euclidean Jordan algebras, one deduces that

\[
\text{Sym}(n, \mathbb{C}), M(n, \mathbb{C}), \text{Skew}(2n, \mathbb{C}), \mathbb{C} \times \mathbb{C}^n, \text{Her}(3, \mathbb{Q}) \otimes \mathbb{C}
\]

are the complexification of the simple Euclidean Jordan algebras which were classified in [5].

**REMARK I.2.18.** This theorem does not contain the uniqueness of the complexification of the given complex Jordan algebra. That is, there is a possibility of a non-Euclidean Jordan algebra which is a real form of a semi-simple complex Jordan algebra. The Jordan algebra \( M(n, \mathbb{C}) \) of all complex \( n \times n \) matrices has two real forms \( M(n, \mathbb{R}) \) and \( \text{Herm}(n, \mathbb{R}) \). But \( M(n, \mathbb{R}) \) is not Euclidean for...
Let us consider the Jordan algebra

\[ V_{1,1} = \{ \begin{bmatrix} x & y \\ -y & z \end{bmatrix} | x, y, z \in \mathbb{R} \}. \]

Then

\[ V_{1,1}^\mathbb{C} = \{ \begin{bmatrix} z & v \\ -v & w \end{bmatrix} | z, v, w \in \mathbb{C} \}. \]

However, \( V_{1,1}^\mathbb{C} \) is the complexification of the Jordan algebra

\[ W = \{ \begin{bmatrix} x & iy \\ -iy & z \end{bmatrix} | x, y, z \in \mathbb{R} \}. \]

One can show that the Jordan algebra \( W \) is formally real, hence a Euclidean Jordan algebra.

### I.3. Symmetric tube domains

Let \( D \) and \( D' \) be two domains (connected open subset) in \( \mathbb{C}^n \). We say that a map \( f : D \rightarrow D' \) is a holomorphic isomorphism if it is bijective holomorphic map with holomorphic inverse. A biholomorphic map \( f : D \rightarrow D \) is called a holomorphic automorphism of \( D \). The group of all biholomorphic automorphisms is denoted by \( G(D) \). A domain \( D \) is homogenous if \( G(D) \) is transitive on \( D \). And we say that \( D \) is symmetric if it is homogenous and if there exists \( z_0 \in D \) and \( j \in G(D) \) such that \( j^2 = id \) and \( z_0 \) is an isolated fixed point of \( j \).

Let \( \Omega \) be a proper convex cone in a real vector space \( V \). If \( \Omega \) is homogeneous, then the tube domain \( T_{\Omega} = V + i\Omega \) over \( \Omega \) in \( V^\mathbb{C} \) is homogeneous. In the case of symmetric cone \( \Omega \), the tube domain \( T_{\Omega} \) is symmetric domain [5]. In fact, via the construction of Euclidean Jordan algebras from symmetric cones, we may assume that \( V \) is a simple Euclidean Jordan algebra and \( \Omega \) the associated symmetric cone. Then the map \( z \mapsto -z^{-1} \) is an involutive automorphism of \( T_{\Omega} \) having \( i\Omega \) as its unique fixed point, and hence \( T_{\Omega} \) is a symmetric domain.
We define
\[ D(p) = \{ z \in \mathbb{C} \mid z + ie \text{ is invertible} \} \]
\[ D(c) = \{ z \in \mathbb{C} \mid e - z \text{ is invertible} \}, \]
and for all \( z \in D(p), w \in D(c) \),
\[ p(z) = (z - ie)(z + ie)^{-1} \]
\[ c(w) = i(e + w)(e - w)^{-1}. \]

Then \( p \) is a holomorphic bijection with its inverse \( c \), which is called the Cayley transform.

Set
\[ \Sigma = \{ z \in \mathbb{C} \mid z^{-1} = \bar{z} \}. \]

Then it is known (cf. [5]) that \( \Sigma \) is the Shilov boundary of \( D = p(T_0) \) with the following characterization:

**Proposition 1.3.1.** For \( z \in \mathbb{C} \) the following properties are equivalent:

(1) \( z \in \Sigma \),
(2) \( [L(z), L(\bar{z})] = 0 \) and \( zz = e \),
(3) \( z = x + iy(x, y \in V), [L(x), L(y)] = 0 \) and \( x^2 + y^2 = e \),
(4) \( z \in \exp(iV) \),
(5) \( z \in \overline{p(V)} \).

**Proof.** (cf. [5], X.2.3.) \( \square \)

The automorphism group of the Shilov boundary \( \Sigma \) is defined by:
\[ G(\Sigma) = \{ g \in GL(\mathbb{C}) \mid g\Sigma = \Sigma \}. \]
Then this group is a Lie subgroup of $GL(V^C)$ and is equal to

$$G(\Sigma) = \text{Str}(V^C) \cap U(V^C)$$

$$= P(\Sigma) \text{Aut}(V),$$

with the Lie algebra $\mathfrak{t} + ip$.

The domain $\mathcal{D} = p(T_0)$ can be identified with the unit open ball of $V^C$ with respect to the spectral norm [5].

Via the Cayley transform, the automorphism groups of $T_0$ and the open unit ball $\mathcal{D}$ are conjugate, i.e. $cG(D)c^{-1} = G(T_0)$. The isotropy group

$$G(D)_0 = \{g \in G(D) \mid g(0) = 0\}$$

is exactly the same as the group $G(\Sigma)$.

We will now describe the group $G(T_0)$. An element in $G(\Omega)$ acts on the tube domain $T_0$ by

$$z = x + iy \rightarrow g(z) = g(x) + ig(y).$$

For $u$ in $V$, the transformation

$$t_u : z \rightarrow z + u$$

is a holomorphic automorphism of $T_0$ and the group of all real transformations $t_u$ is an abelian group $N^+$ isomorphic to the vector space $V$. The map

$$j : z \rightarrow -z^{-1}$$

is in $G(T_0)$. We set

$$\tilde{t}_u = j \circ t_u \circ j$$

and

$$N^- = j \circ N^+ \circ j.$$
Finally, the group $cG(\Sigma)c^{-1}$ is the isotopy subgroup of $G(T_n)$ at $\text{id}$. The following results are in [5] and [33].

**Proposition 1.3.2.** $G(T_n) = N^+G(\Omega)G(T_n)\text{id} = N^+G(\Omega)N^-N^+$.

**Theorem 1.3.3.** The subgroups $G(\Omega)$ and $N^+$, together with the element $j$, generate $G(T_n)$.

The Lie algebra of $G(T_n)$ can be described in the following way: Let $g_t$ be a one-parameter subgroup of $G(T_n)$, then

$$
\dot{X}f(z) = \frac{d}{dt}f(g_t(z))|_{t=0}, \quad f \in C^1(T_n),
$$

defines a vector on $T_n$, and the set of vector fields obtained in this way is a real Lie subalgebra for usual Lie bracket.

**Proposition 1.3.4.** The Lie algebra of $G(T_n)$ is the set of vector fields of the form

$$
X(z) = u + Tz + P(z)v,
$$

where $u, v \in V$ and $T \in \text{Lie}(G(\Omega))$.

**Proof.** (cf. [5]). \[ \square \]

A vector field $X$ in $g(T_n)$:

$$
X = u + Tz + P(z)v,
$$

can be identified with $(u, T, v) \in V \times g(\Omega) \times V$. Let

$$
g_{-1}(T_n) = \{(u, 0, 0) \mid u \in V\} \cong V,
g_0(T_n) = \{(0, T, 0) \mid T \in g(\Omega)\} \cong g(\Omega),
g_1(T_n) = \{(0, 0, v) \mid v \in V\} \cong V.
$$

If $X = (u, T, v), X' = (u', T', v')$, and $u \square v = L(uv) + [L(u), L(v)]$, then

$$
[X, X'] = (Tu' - T'u, 2u' \square v + [T, T'] - 2u \square v', T^tv' - T'tv).
$$

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Let $\theta = \text{Ad}(j)$. Then $\theta(u, T, v) = (v, -T^*, u)$. The Cartan decomposition corresponding to $\theta$ is $t \oplus p$, where

\[
t = \{(u, T, u) \mid T^* = -T\} = \text{Lie}(G(T_n)_t),
\]
\[
p = \{(u, T, -u) \mid T^* = T\}.
\]

Define $\eta(u, T, v) = (-u, T, -v)$. Then

\[
g_0 = \{X \in g(T_n) \mid \eta X = X\},
\]

and

\[
g_{-1} \oplus g_1 = \{X \in g(T_n) \mid \eta X = -X\}.
\]

Set $\mathfrak{h} = g_0, q = g_{-1} \oplus g_1$. Then $g(T_n) = t \oplus p = \mathfrak{h} \oplus q$. If we define $\tau(z) = -z$, then $\eta = \text{Ad}(\tau)$ and $G(\Omega) \subset G(T_n)^\tau = \{g \in G(T_n) \mid \tau g \tau = g\}$.

### 1.4. Hermitian symmetric spaces of tube type

The tube domain over a symmetric cone (which is equivalent to a bounded symmetric domain) is the realization of a hermitian symmetric space of non-compact type using the construction of a Cayley transform due to [27] (cf, [38],[39],[18]). In this section, we summarize their work.

Let $D$ be a hermitian symmetric space of non-compact type, $G^0$ its connected group of isometries, $K$ the isotropy group. So $\mathcal{D} = G^0/K$. Then $G^0$ is a centerless semisimple Lie group, $K$ its maximal compact subgroup. We denote the Lie algebra of $G^0$ ($K$ resp.) by $g^0$ ($t$ resp). Let $g^C$ be the complexification of $g^0$. Let $G^C = \text{Ad}(g^C)$ be the adjoint group of $g^C$. Let

\[
g^0 = t \oplus p^0
\]

Cartan decomposition of $g^0$. With $p = ip^0, g = t \oplus p$ is a compact form of $g^C$. 

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Let $\mathfrak{t}^c$ complexification of $\mathfrak{t}$ and let $K^c$ the corresponding analytic subgroup of $G^c$. Let $G$ denote the analytic subgroup of $G^c$ corresponding to $g$. There exists $Z$ in the center of $\mathfrak{t}$ such that $(\text{ad}Z)^2 = -1$ on $p^c$. Now let $p^+$ be the $i$-eigenspace of $\text{ad}Z$ and $p^-$ the $-i$-eigenspace. Let $P^\pm$ be the analytic subgroups of $G^c$ corresponding $p^\pm$. Then the exponential mapping from $p^\pm$ to $P^\pm$ is one-to-one. Furthermore,

1. The group $K^cP^-$ is the normalizer of $P^-$ in $G^c$.

2. $P^+K^cP^-$ is a dense open subset of $G^c$.

3. $G \subset P^+K^cP^-$.

4. $P^+ \cap K^cP^- = \{e\}$.

5. The mapping induced by the exponential mapping

\[ \xi : p^+ \to M = G^c/K^cP^- \]

is a one-to-one holomorphic map onto a dense open subset of $M$.

6. For $u \in p^+$ and $g \in G^c$ such that $g \exp u \in P^+K^cP^-$, define $g \cdot u \in p^+$ by $\xi(g \cdot u) \in g \exp uK^cP^-$. 

7. $M$ is compact and

\[ G^0 \cdot 0 \subset \xi(p^+) \subset G \cdot 0 \]

is the Borel embedding, where $0$ is the identity coset in $M$.

Let $D_b = \xi^{-1}(G^0 \cdot 0)$. Then it is a bounded domain in the complex vector space $p^+$: this is the Harish-Chandra realization of $D$ as bounded domain.

REMARK 1.4.1 The $\xi$-equivariant action of $G^0$ on $D_b$ is just the action of the connected group of holomorphic automorphisms of $D_b$. 

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It is always possible to construct the Cayley transform of the domain $D_b$ embedded in $p^+$. It turns out [27] that this is always a tube domain over a symmetric cone or a Siegel domain of type II. The hermitian symmetric space $D$ of non-compact type is of tube type if it is holomorphically equivalent to the tube domain over a self-dual cone. It is equivalent to say that the Cayley transform of the domain $D_b$ is tube domain. In this case, there is $c \in G^c$ which is also called the Cayley transform satisfying $Adc^4 = 1$ and $Adc^2$ is an automorphism of $t$. The map $\xi^{-1}c\xi$ is a holomorphic homeomorphism of $D_b$ onto the tube domain $T_\Omega$ which is called the Cayley transform of $D_b$. Set $cG = cG^0c^{-1}, cK = cKc^{-1}, c_0 = Ad(c)g^0$. Define $n^\pm = p^\pm \cap c_0$. Then there is a formally real Jordan algebra structure on $n^+$ with the symmetric cone $\Omega$. Koecher's map $p : T_\Omega \to p^+, p(z) = (z - ie)(z + ie)^{-1}$. The inverse Cayley transformation $q = \xi^{-1}c^{-1}\xi$ defined on $T_\Omega$ is exactly $q = ip$ [27].

I.5. Infinite dimensional Jordan algebras

I.5.1. $JB$-algebras

A real Jordan algebra $V$ with unit $e$ is called a $JB$-algebra if $V$ carries a Banach space norm $\| \cdot \|$ satisfying

\[ \|xy\| \leq \|x\| \|y\|, \]
\[ \|x\|^2 \leq \|x^2 + y^2\|. \]

From the second condition, every $JB$-algebra is a formally real Jordan algebra. Hence a finite dimensional real Jordan algebra $V$ is a $JB$-algebra if and only if $V$ is fomally real if and only if $V$ is an Euclidean algebra.

EXAMPLES I.5.1.

(1) Let $E$ be a complex Hilbert space and let $\mathcal{L}(E)$ be the algebra of bounded
operators on $E$. Then a closed unital subalgebra $V$ of $\mathcal{H}(E) := \{x \in \mathcal{L}(E) : x^* = x\}$, is called a $JC$-algebra. Every $JC$-algebra is a special $JB$-algebra. If we interpreted $\mathcal{H}(E)$ as the bounded observables of the quantum mechanical system, then unlike the associative product $xy$, the anticommutator product preserves observables and therefore has a physical meaning.

(2) $\mathcal{H}_p(\mathbb{K})$ is a $JB$-algebra for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

(3) $\mathcal{H}_3(\mathbb{Q})$ is the only $JB$-algebra which cannot be realized as algebra of self adjoint operators on a complex Hilbert space.

(4) Every spin factor $V = E \times \mathbb{R}$ is a $JB$-algebra under the norm $$||(x,t)|| := ||x|| + |t|.$$ For a $JB$-algebra $V$, the set $Q$ of square elements of $V$

$$Q := \{x^2 | x \in V\}$$

is a closed convex cone satisfying $V = Q - Q$ and $Q \cap -Q = 0$. The $JB$-norm $|| \cdot ||$ on $V$ is uniquely determined by the algebraic structure. In fact, it coincides with the order unit norm$$||x|| = \inf \{t \in \mathbb{R}^+ : te \pm x \in Q\}.$$ is equal to$$\inf \{t > 0 : -te \leq x \leq te\},$$i.e. the closed unit ball in $V$ is given by $(e - V^2) \cap (V^2 - e)$.

For given $a \in V$ the polynomials in $a$ from an associative subalgebra and the closure of this algebra is a commutative Banach algebra: the Banach algebra $C(a)$ generated by $a$ and $e$. Like finite dimensional cases, we can define
the exponential of an element of $V$. For every compact topological space $S$ and every $JB$-algebra $V$ also the algebra $C(S, V)$ of all continuous functions $f : S \to V$ is a $JB$-algebra. In particular $C(S, \mathbb{R})$ is an associative $JB$-algebra. The converse is also true: Every associative $JB$-algebra is isometrically isomorphic to $C(S, \mathbb{R})$ for some compact topological space $S$. Hence $C(a)$ is isometrically isomorphic to $C(S, \mathbb{R})$ for some compact Hausdorff space $S$.

**Theorem I.5.2.** [Gelfand-Neumark Theorem for $JB$-algebras] Suppose $V$ is a $JB$-algebra. Then there exists a complex Hilbert space $E$ and a compact topological space $S$ such that $V$ is isometrically isomorphic to a closed subalgebra of

$$\mathcal{H}(E) \oplus C(S, \mathcal{H}_3(\mathbb{Q})).$$

**Proof.** (cf. [4],[7]).

Like Euclidean Jordan algebras, we define the symmetric cone $\Omega$ of a $JB$-algebra $V$ to be the interior of the set of square elements. Then the characterization of the symmetric cone of a Euclidean Jordan algebra ([5]) holds for $JB$-algebras. That is, $\Omega = \exp V = V^{-1} \cap Q$ and $\Omega$ is the $e$-connected component of $V^{-1}$ [22].

For the automorphism group $G(\Omega)$ of $JB$-algebras, it is true that the automorphism group $Aut(V)$ is the isotropy subgroup of $G(\Omega)$ at $e \in \Omega$. Furthermore, the group $G(\Omega)$ has the polar decomposition.

**I.5.2. $JB^*$-algebras**

The complexification of a $JB$-algebra is a $JB^*$-algebra and the definition of $JB^*$-algebra is the following:
A complex Jordan algebra $U$ with unit $e$ and involution $\ast$ is called a $JB^*$-algebra if $U$ carries a Banach space norm $\| \cdot \|$ satisfying

\[
\|zw\| \leq \|z\|\|w\|,
\]
\[
\|\{zz^*z\}\| = \|z\|^3 = \|P(z)z^*\|.
\]

Here $\{uv^*w\} := u(v^*w) - v^*(wu) + w(uv^*)$ denotes the Jordan triple product of $u, v, w \in U$.

**Examples 1.5.3.**

1. Let $H$ be a complex Hilbert space and $\mathcal{L}(H)$ be the complex Banach algebra of bounded linear operator on $H$ with usual adjoint $\ast$ as involution. Then $\mathcal{L}(H)$ is a $JB^*$-algebra in the Jordan product $ab := \frac{1}{2}(a \circ b + b \circ a)$, since every $z \in \mathcal{L}(H)$ satisfies

\[
\|\{zz^*z\}\|^2 = \|z \circ z^* \circ z\|^2 = \|((z^*z)^3)\| = \|z^*z\|^3 = \|z\|^6.
\]

Therefore $\mathcal{H}(H) := \{x \in L(H) \mid x^* = x\}$ is a $JB$-algebra.

2. Every unital $C^*$-algebra is $JB^*$-algebra under the anticommutator product. In this case

\[
\{uv^*w\} = \frac{1}{2}(uv^*w + wv^*u).
\]

3. A closed unital Jordan $\ast$-subalgebra of a $C^*$-algebra is called a $JC^*$-algebra.

There is a one-to-one correspondence between $JB$-algebras and $JB^*$-algebras by the following facts: For every $JB^*$-algebra $U$ the hermitian part $V := \{x \in U \mid x^* = x\}$ is a $JB$-algebra under the restricted norm. Conversely, for every $JB$-algebra $V$, the complexified algebra $U := V^C = V + iV$ has a
unique norm making $U$ a $JB^*$-algebra in the canonical involution. This norm extends the norm on $V$ [4].

Under the one-to-one correspondence between $JB$ and $JB^*$-algebras, $JC$-algebras correspond to $JC^*$-algebras whereas finite dimensional formally real Jordan algebras correspond to semisimple complex Jordan algebras.

I.5.3. $JB^*$-algebras and symmetric tube domains

Let $U$ be a $JB^*$-algebra with unit $e$ and hermitian part $V$. Let $\Omega$ be the symmetric cone corresponding to the $JB$-algebra $V$. Then the domain

$$T_\Omega = V + i\Omega$$

is called the upper half-plane. In fact, $T_\Omega$ is a symmetric tube domain from the following result [4].

**Lemma 1.5.4.** Every $z \in T_\Omega$ is invertible and

$$s(z) := -z^{-1}$$

defines a symmetry of $T_\Omega$ at the point $ie \in i\Omega$. Furthermore,

$$p(z) = i(z - ie)(z + ie)^{-1}$$

defines a biholomorphic mapping of $T_\Omega$ onto the open unit ball $D$ for a suitable equivalent norm.

To obtain the $JB^*$-algebras from a symmetric tube domain, we need the concept of hermitian Jordan triple system. But we just mention the results [4], [22]. Suppose that $U$ is a complex Banach space with conjugation $*$ and $V := \{z \in U \mid z^* = z\}$ is the corresponding real form of $U$. Then $V$ is a real Banach space with $U = V^C = V + iV$. Let $\Omega$ be an open convex cone in $V$. Then $T_\Omega = \{z = x + iy \in U \mid y \in \Omega\}$ is called a tube domain if $T_\Omega$
is biholomorphically equivalent to a bounded domain in $U$. Note that in this case $\Omega$ does not contain any affine real line, that is, it is a proper convex cone. In the following we denote $\text{Aut}(T_\Omega)$ the group of all biholomorphic automorphisms of the tube domain $T_\Omega$ and by $\mathfrak{g}(T_\Omega)$ the cone of all complete holomorphic vector fields on $T_\Omega$. Then the group $\text{Aut}(T_\Omega)$ is a real Banach Lie group acting analytically on $T_\Omega$ and $\mathfrak{g}(T_\Omega)$ is a real Banach Lie algebra, the Lie algebra of $\text{Aut}(T_\Omega)$. Furthermore, the Lie algebra $\mathfrak{g}(T_\Omega)$ can be decomposed into $\mathfrak{g}(T_\Omega)_{-1} \oplus \mathfrak{g}(T_\Omega)_0 \oplus \mathfrak{g}(T_\Omega)_1$.

Now suppose that $s : T_\Omega \rightarrow T_\Omega$ is a symmetry at $i.e.$ Then it is known [4] that the cone $\Omega$ is a symmetric cone and there is a unique Jordan algebra structure on $U$ with unit $e$ such that $s(z) = -z^{-1}$ and there exists a unique equivalent norm on $U$ such that $U$ is a $JB^*$-algebra with hermitian part $V$. In particular, $V$ is a real subalgebra with $\overline{\Omega} = \{x^2 \mid x \in V\}$ and there exists the Cayley transformation $\sigma : T_\Omega \rightarrow U$ defined by

$$\sigma(z) = i(z - a)(z + a)^{-1}$$

which maps $T_\Omega$ biholomorphically onto the open unit ball $D$.

**Theorem 1.5.5.** Suppose that $V$ be a $JB$-algebra with the corresponding symmetric cone $\Omega$. Then there is a connected complex manifold $M$ containing $U$ as a dense open subset such that every vector field in $\mathfrak{g}_C = \mathfrak{g} + i\mathfrak{g}$, where $\mathfrak{g} = \mathfrak{g}(T_\Omega)$, can be holomorphically continued to a complete holomorphic vector field on $M$. In the case $\dim(U) < \infty$, then $M$ is the compact dual of the tube domain $T_\Omega$.

**Proof.** (cf. [22]). \qed

**Remark 1.5.6.** If $M$ is defined as in this theorem, then $\text{Aut}(D)$ and $\text{Aut}(T_\Omega)$ can be regarded as subgroups of $\text{Aut}(M)$.
We denote $N^+ := \exp(g_{-1})$ by all translations on $U$ by members of $V$ and $N^- := \exp(g_1) = s \cdot N^+ \cdot s$.

Let

$$G(T_h)_0 := \{ g \in Aut(T_h) \mid g(0) = 0 \}.$$  

Let $\infty = s(0) \in M$. Then

**Proposition I.5.7.**

1. $G(T_h)_0 = G(\Omega) \cdot N^-.$

2. $G(T_h)_\infty = N^+ G(\Omega).$

**Proof.** (cf. [22]). □

**Corollary I.5.8.** If $g(0) \in V$, then $g \in N^+ G(\Omega) N^-.$

**Proof.** Let $g(0) = x \in V.$ Then $t_{-x} g(0) = 0$, hence $t_{-x} g \in G(T_h)_0.$ Thus $g \in N^+ G(\Omega) N^-.$ □
Chapter II
Jordan Algebras Associated
with Bilinear Forms

In this section, we investigate two types of Jordan algebras associated with non-degenerate symmetric bilinear forms on real or complex vector spaces. One of these Jordan algebras can be constructed by using the Lorentzian form which is a one factor of simple Euclidean Jordan algebras, named spin factors. We have seen the characterization of simple Euclidean Jordan algebras in chapter I which were classified by Jordan, von Neumann, and Wigner [15]. The first type is $\text{Sym}(n, \mathbb{R})$ the space of symmetric $n \times n$-matrices under the anti-commutative product. If we change the usual inner product into any symmetric bilinear form, then the space of all self-adjoint transformations is still a Jordan algebra with the anti-commutative product.

II.1. Bilinear and quadratic forms

Let $F$ be the fields $\mathbb{R}$ or $\mathbb{C}$. Let $V$ be a finite-dimensional vector space over $F$.

A bilinear form on $V$ is a function $\sigma : V \times V \to F$ satisfying

1. $\sigma(r_1 x_1 + r_2 x_2, y) = r_1 \sigma(x_1, y) + r_2 \sigma(x_2, y)$

2. $\sigma(x, r_1 y_1 + r_2 y_2) = r_1 \sigma(x, y_1) + r_2 \sigma(x, y_2)$

for all $x, y \in V$, and $r_i \in F, i = 1, 2$.

A sesquilinear form on a complex vector space $V$ is a function $\sigma : V \times V \to \mathbb{C}$ satisfying (1) and

3. $\sigma(x, r_1 y_1 + r_2 y_2) = \overline{r_1} \sigma(x, y_1) + \overline{r_2} \sigma(x, y_2)$.
Let $B = \{v_1, \cdots, v_n\}$ be a basis on $V$ and $\sigma$ be a bilinear (sesquilinear) form on $V$. Define the matrix of $\sigma$ with respect to the basis $B$, denoted $A_B$ by

$$\text{ent}_{ij}(A_B) := a_{ij} = \sigma(v_i, v_j).$$

Let $x = \sum_{i=1}^n r_i v_i$, $y = \sum_{i=1}^n s_i v_i$. Then

$$\sigma(x, y) = \sum_{i,j} r_i s_j a_{ij}.$$

If $\sigma$ is sesquilinear, then

$$\sigma(x, y) = \sum_{i,j} r_i \overline{s_j} a_{ij}.$$

Let $P$ be the change of basis matrix from a basis $B$ to a $C$, then

$$A_B = P^t A_C P.$$

A bilinear form $\sigma$ is said to be symmetric if $\sigma(x, y) = \sigma(y, x)$. It is equivalent to the matrix $A$ corresponding to $\sigma$ is symmetric. $\sigma$ is skew symmetric if $\sigma(x, y) = -\sigma(y, x)$ if and only if $A^t = -A$. A sesquilinear form $\sigma$ is said to be Hermitian if $\sigma(x, y) = \overline{\sigma(y, x)}$ if and only if $A \neq \overline{A}$ and $A^t = \overline{A}$; and $\sigma$ is skew-Hermitian if $\sigma(x, y) = -\overline{\sigma(y, x)}$ if and only if $A \neq \overline{A}$ and $A^t = -\overline{A}$.

A symmetric bi(sesqui)-linear form $\sigma$ is called if $\sigma(x, y) = 0$ for all $y \in V$ implies $x = 0$. Equivalently, the matrix $A_B$ is invertible for some (hence every) basis $B$ on $V$.

Let $\sigma$ a fixed non-degenerate symmetric bilinear form of $V$ and $a \in V$. By the definition

$$\sigma^*(x) = \sigma(a, x)$$

we get a linear form $\sigma^*$ of $V$. Obviously is the map $a \rightarrow \sigma^*$ a homomorphism $\phi : V \rightarrow V^*$. The kernel of $\phi$ contains the $a$ with $\sigma(a, x) = 0$ for all $x \in V$. But since $\sigma$ is non-degenerate it follows $a = 0$. Hence $\phi$ is bijective. Consequently,
for each linear form \( l \in \mathcal{V}^* \), there exists a unique vector \( l_a \in \mathcal{V} \) such that

\[
l(x) = \sigma(l_a, x).
\]

Let \( T \) be a linear transformation of \( \mathcal{V} \) and \( a \in \mathcal{V} \). The map

\[
V \rightarrow \mathbb{F}, \ x \rightarrow \sigma(a, Tx)
\]

is a linear form. Hence there exists a unique vector \( \tilde{a} \in \mathcal{V} \) such that \( \sigma(\tilde{a}, x) = \sigma(a, Tx) \). Obviously the map \( a \rightarrow \tilde{a} \) is a linear transformation of \( \mathcal{V} \) which is denoted by \( T^* \). \( T^* \) is called the adjoint transformation to \( T \) with respect to \( \sigma \), in formular:

\[
\sigma(y, Tx) = \sigma(T^* y, x)
\]

\[
(rT + r'S)^* = rT^* + r'S^*
\]

\[
(TS)^* = S^* T^*
\]

\[
(T^*)^* = T^*
\]

\[
T^* = A_B^{-1} T^t A_B.
\]

**Example II.1.1.** Let \( x = \sum_{i=1}^n r_i v_i, \ y = \sum_{i=1}^n s_i v_i \). For \( p, q \in \mathbb{N} \) with \( p+q = n \),

\[
J_{p,q}(x, y) = \sum_{i=1}^p r_i s_i - \sum_{i=p+1}^n r_i s_i,
\]

\[
J_{p,q}^C(x, y) = \sum_{i=1}^p r_i \overline{s_i} - \sum_{i=p+1}^n r_i \overline{s_i}
\]

are non-degenerate symmetric and Hermitian forms, respectively. If we set

\[
J_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix},
\]

then \( J_{p,q}(x, y) = (x|J_{p,q}y) \), where \((x|y)\) be the usual inner product on \( \mathbb{R}^n, \mathbb{C}^n \).

Let \( \sigma \) and \( \tau \) be bi(sesqui)-linear forms on vector spaces \( \mathcal{V} \) and \( \mathcal{W} \) respectively. Then \( \sigma \) and \( \tau \) are isomorphic if there is an isomorphism \( P : \mathcal{V} \rightarrow \mathcal{W} \).
with
\[ \tau(P(x), P(y)) = \sigma(x, y) \]
for all \( x, y \) in \( V \).

**THEOREM II.1.2. (Sylvester's law of inertia)** Let \( \sigma \) be a non-degenerate symmetric bilinear form over \( \mathbb{R} \), or Hermitian form over \( \mathbb{C} \) on a finite dimensional real (complex) vector space \( V \). Then \( \sigma \) is isomorphic to \( J_{p,q} \) or \( J_{p,q}^c \).

**PROOF.** (cf. [1]). \( \square \)

Let \( \sigma \) be a non-degenerate symmetric bilinear form on \( V \) and let \( P : V \to V \) be an isomorphism such that
\[ \sigma(x, y) = J_{p,q}(P(x), P(y)) \]
for all \( x, y \in V \). And let \( T \in \mathfrak{gl}(V) \). Because \( J_{p,q}(x, y) = \langle x | J_{p,q} y \rangle \),
\[
\begin{align*}
\sigma(x, Ty) &= J_{p,q}(P(x), PT(y)) \\
&= \langle P(x) | J_{p,q} PT(y) \rangle \\
&= \langle x | P^t J_{p,q} PT(y) \rangle \\
&= \sigma(T^*(x), y) \\
&= J_{p,q}(PT^*(x), P(y)) \\
&= \langle PT^*(x) | J_{p,q} P(y) \rangle \\
&= \langle x | (T^*)^t P^t J_{p,q} P(y) \rangle.
\end{align*}
\]
Therefore \((T^*)^t P^t J_{p,q} P = P^t J_{p,q} PT \) and so
\[ T^* = P^{-1} J_{p,q} (P^{-1})^t T^t P^t J_{p,q} P. \]

A self-adjoint linear transformation \( T \) of \( V \) is called **positive-definite** if
\[ \sigma(Tx, x) > 0 \]
for all $x \neq 0$. $T$ is said to be \textit{positive-semidefinite} if

$$\sigma(Tx, x) \geq 0$$

for all $x \in V$.

A \textit{quadratic form} on $V$ is a function $\Phi : V \to \mathbb{F}$ satisfying

1. $\Phi(rx) = r^2\Phi(x)$ for any $r \in \mathbb{F}, x \in V$; and

2. the function $\phi : V \times V \to \mathbb{F}$ defined by

$$\phi(x, y) = \Phi(x + y) - \Phi(x) - \Phi(y)$$

is a symmetric bilinear form on $V$.

The quadratic form $\Phi$ is called \textit{non-degenerate} if the symmetric bilinear form $\phi$ is. Let $\psi$ be any bilinear form on $V$ and define the symmetric bilinear form $\phi$ on $V$ by

$$\phi(x, y) = \psi(x, y) + \psi(y, x).$$

The function

$$\Phi : V \to \mathbb{F}$$

denoted by

$$\Phi(x) = \psi(x, x)$$

is a quadratic form on $V$ with associated bilinear form $\phi$.

A non-degenerate, skew-symmetric bilinear form is called a \textit{symplectic form}.

\textsc{Example II.1.3}. The standard symplectic form on $\mathbb{F}^{2n}$ is given by

$$\mathcal{U}(p, q) = p_1 q_2 - p_2 q_1, \quad p = (p_1, p_2), q = (q_1, q_2) \in \mathbb{F}^{2n}.$$ 

Let $J_n \in gl(2n, \mathbb{F})$ be defined in the block form by

$$J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$
Then
\[ J_n^t = -J_n, \quad (J_n) J_n = I = J_n J_n^t. \]
The standard symplectic form \( \mathcal{U}(p, q) \) can be written by
\[ \mathcal{U}(p, q) = p^t J_n q. \]
For \( T \in \text{gl}(2n, \mathbb{F}) \),
\[ \mathcal{U}(p, T^* q) = \mathcal{U}(T p, q) = p^t T^* J_n q = p^t J_n J_n^t T^* J_n q = \mathcal{U}(p, J_n^t T^* J_n q). \]
Hence \( T^* = J_n^t T^* J_n = -J_n T^* J_n \).

**Example II.1.4.** Let \( E \) be a Hilbert space with the inner product \( \langle p | q \rangle \). For \( p = (p_1, p_2) \in V = E \times E \),
\[ \Phi(u) = \langle p_1 | p_2 \rangle \]
is a quadratic form on \( V \) with the associated symmetric bilinear form
\[ \phi(p, q) = \frac{1}{2} \left( \langle p_1 | q_1 \rangle + \langle p_2 | q_2 \rangle \right), \]
for \( q = (q_1, q_2) \in V \). Set
\[ Q = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \]
Then \( \phi(p, q) = \frac{1}{2} \langle p | Q q \rangle_V \), where \( \langle \cdot | \cdot \rangle_V \) is the extended inner product on \( V \). In this case the adjoint operator \( T^* \) with respect to \( \phi(p, q) \) is
\[ T^* = QT^* Q \]
because
\[ \phi(T p, q) = \langle T p | Q q \rangle = \langle p | T^* Q q \rangle = \phi(p, T^* q) = \langle p | QT^* q \rangle. \]

By Sylvester's law of inertia, the bilinear form \( \phi(p, q) \) on \( V = E \times E \) is isomorphic to \( J_{r,s}(p, q) \) with \( r + s = \text{dim}(V) \). It can easily seen that \( \phi(p, q) = J_{n,n}(p, J q) = J_{n,n}(P p, P q) \), where
\[ P = \begin{bmatrix} I & I \\ I & -I \end{bmatrix}. \]
II.2. Spin factors

In this section, we are going to study a spin factor which is closely related to a Lorentzian geometry. A spin factor is a JB-algebra of rank 2 and has a Lorentzian cone as a symmetric cone. Using the matrix representation, one may find the automorphism group $G(\Omega)$ even infinite-dimensional cases.

The following construction of a Jordan algebra with a symmetric bilinear form was shown by Koecher ([25]): we start with a real or complex vector space $V$, an element $e \in V$, a linear form $\lambda(u)$ and a symmetric bilinear form $\mu(u, v)$ of $V$. Define an algebra structure on $V$ by:

$$u \circ v = \lambda(u)v + \lambda(v)u - \mu(u, v)e.$$  

It is obvious that $e$ is the unit element of $V$ if and only if

$$\lambda(u) = \mu(e, u), \quad \lambda(e) = 1.$$  

From $u^2 = 2\lambda(u)u - \mu(u, u)e$ follows that and $L(u^2)$ is a linear combination of $L(u)$ and $I$ the identity transformation on $V$. Hence $[L(u), L(u^2)] = 0$ and it is equivalent to $(V, \circ)$ is a Jordan algebra. It is known that the Jordan algebra $V$ constructed in above is semi-simple if and only if the bilinear form $\mu(u, v)$ is non-singular. Furthermore,

**Theorem II.2.1.** Let $V$ be a vector space of the dimension $n$ over $\mathbb{R}$ or $\mathbb{C}$, $\tau$ a non-singular symmetric bilinear form of $V$ and $e \in V$, such that $\tau(e, e) = n$. Then the definitions

$$u \circ v : = \lambda(u)v + \lambda(v)u - \mu(u, u)e,$$

$$\lambda(u) : = \frac{1}{n} \tau(e, u),$$

$$\mu(u, v) : = 2\lambda(u)\lambda(v) - \frac{1}{n} \tau(u, v)$$
gives rise to a semi-simple Jordan algebra \((V, o)\) with unit element \(e\). The bilinear form \(TrL(u \circ v)\) associated with \(V\) is the given form \(\tau(u, v)\).

**Proposition II.2.2.** \(\mu(u, u) \neq 0\) if and only if the inverse of \(u\) exists.

Let \(E\) be a Euclidean vector space with inner product \(\langle \cdot, \cdot \rangle\). Set \(V := E \oplus \mathbb{R}^2\).

Let \(\mu(u, v)\) be the Lorenztian form on \(V\) defined by

\[
\mu(u, v) = -(x|y) - x_1 y_1 + x_2 y_2,
\]

\(u = (x, x_1, x_2), v = (y, y_1, y_2) \in V\). Let \(e = (0, 0, 1) \in V\). Then \(\mu(e, e) = 1\), and \(\lambda(u) := \mu(u, e)\) is a linear functional on \(V\). Hence \((V, o)\) is a Jordan algebra with \(e\) as unit:

\[
u \circ v = \lambda(u)v + \lambda(v)u - \mu(u, v)e
= (y_2 x + x_2 y, x_1 y_2 + y_1 x_2, (x|y) + x_1 y_1 + x_2 y_2).
\]

Now suppose that \(u^2 + v^2 = 0\), for \(u = (x, x_1, x_2), v = (y, y_1, y_2)\). Then it becomes

\[
2\lambda(u)u - \mu(u, u)e + 2\lambda(v)v - \mu(v, v)e = 0.
\]

Arranging in the last variable, we get

\[
x_2^2 + (x|x) + x_1^2 + y_2^2 + (y|y) + y_1^2 = 0.
\]

So we conclude that \(u = v = 0\). This means that \((V, o)\) is a formally real Jordan algebra which is known as one factor of simple Jordan algebras, called a *spin factor*.

If we define a positive definite bilinear form \(B\) on \(E \oplus \mathbb{R}\) by: for \(u = (x, s), v = (y, t)\),

\[
B(u, v) = (x|y) + ts,
\]

then the Jordan algebra multiplication is exactly the same as the case of Example 1.2.7.
Remark II.2.3. $E \oplus \{0\} \oplus \mathbb{R}$ and $\{0\} \oplus \mathbb{R} \oplus \mathbb{R}$ are Jordan subalgebras of $V$ and hence are Euclidean Jordan algebras. The subalgebra $\{0\} \oplus \mathbb{R} \oplus \mathbb{R}$ has the symmetric cone

$$\{(x, y) \in \mathbb{R}^2 \mid y > 0, \ |y| > |x|\}.$$  

Denote by $V^0 = E \oplus \{0\} \oplus \mathbb{R}$. Then the corresponding symmetric cone will be

Now suppose that $\text{Dim}(E) = 1$. Then the spin factor on $V = \mathbb{R}^3$ is isomorphic to the Jordan algebra $\text{Sym}(2, \mathbb{R})$: In fact, if we let 

$$c_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, c_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, d = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

then $\text{Sym}(2, \mathbb{R}) = \mathbb{R}(c_1 + c_2) + \mathbb{R}(c_1 - c_2) + \mathbb{R}d.$ Set

$$W = \mathbb{R}(c_1 - c_2) + \mathbb{R}d$$

and define a bilinear form on $W$ by: for $u = \alpha(c_1 - c_2) + x d, v = \beta(c_1 - c_2) + y d,$

$$B(u, v) = \alpha + xy.$$  

Then $\text{Sym}(2, \mathbb{R})$ is isomorphic to the spin factor $\mathbb{R} \oplus \mathbb{R}^2$. In general, if $c_1, c_2$ are two orthogonal primitive idempotents in a Euclidean Jordan algebra, then the Jordan subalgebra generated by $c_i$ has dimension three and is isomorphic with $\text{Sym}(2, \mathbb{R})$ [5].

II.2.1. Invertible and primitive elements

Let $\mathcal{P}$ be the set of non-trivial idempotents in $V$. If $u = (x, x_1, x_2) \in \mathcal{P}$, then

$$u^2 = (2x_2 x, 2x_2 x_1, ||x||^2 + x_1^2 + x_2^2) = (x, x_1, x_2)$$

which implies $x_2 \neq 0$. If $x = 0$, then $x_2 = \frac{1}{2}, x_1^2 = \frac{1}{4}$. If $x \neq 0$, then $x_2 = \frac{1}{2}$ and $||x||^2 + x_1^2 = \frac{1}{4}$. Therefore

$$\mathcal{P} = \{(x, x_1, x_2) \mid x_2 = \frac{1}{2}, ||x||^2 + x_1^2 = \frac{1}{4}\}$$
which is diffeomorphic to the sphere in $E \times \mathbb{R}$. With the inner product on $E \oplus \mathbb{R}$ given by:

$$\|(x, x_1)\| = \sqrt{\|x\|^2 + x_1^2},$$

$$P = \{(x, x_1, \frac{1}{2}) \in E \oplus \mathbb{R}^2 \mid \|(x, x_1)\|^2 = \frac{1}{4}\}.$$  

**PROPOSITION II.2.4.** Every element in $P$ is primitive. More precisely, $P$ is the set of all non-trivial idempotents and is also the set of all primitive idempotents.

**PROOF.** Because the third coordinates of any non-trivial idempotents are $\frac{1}{2}$, it could not be written as two non-trivial idempotents. Hence every element in $P$ is primitive. $\square$

Using the second coordinate, the set $P$ can be decomposed as the following way: Set

$$P^+ = \{(x, x_1, x_2) \in P \mid x_1 > 0\}$$
$$P^- = \{(x, x_1, x_2) \in P \mid x_1 < 0\}$$
$$P^0 = \{(x, x_1, x_2) \in P \mid x_1 = 0\}$$
$$P^\pm = P^+ \cup P^0.$$ 

Then $P$ is the disjoint union of $P^\pm$ and $P^-$. The Jordan subalgebra $\{0\} \oplus \mathbb{R} \oplus \mathbb{R}$ has exactly two primitive idempotents.

**LEMMA II.2.5.** $u = (x, x_1, x_2), v = (y, y_1, y_2)$ are orthogonal primitive idempotents if and only if $(x, x_1) = -(y, y_1)$.

**PROOF.** $u \circ v = \frac{1}{2}u + \frac{1}{2}v - \mu(u, v)e = 0$ implies $x = -y, x_1 = -y_1$. Conversely, if $(x, x_1) = -(y, y_1)$, then $\mu(u, v) = -(x|y) - x_1y_1 + \frac{1}{4} = \|(x, x_1)\|^2 + \frac{1}{4} = \frac{1}{2}$. Therefore $u \circ v = \frac{1}{2}u + \frac{1}{2}v - \mu(u, v)e = \frac{1}{2}(x, x_1, \frac{1}{2}) + \frac{1}{2}(y, y_1, \frac{1}{2}) - \frac{1}{2}e = 0$. $\square$
REMARK II.2.6. From the previous lemma, if $u = (x, x_1, x_2) \in \mathcal{P}^+$, then $v = (-x, -x_1, x_2) \in \mathcal{P}^-$ and $u, v$ are orthogonal primitive idempotents.

Define a linear transformation $\sigma : V \to V$ by

$$\sigma(x, x_1, x_2) = (-x, -x_1, x_2).$$

Then $\sigma$ sends $\mathcal{P}^+$ to $\mathcal{P}^-$ and $\mathcal{P}^-$ to $\mathcal{P}^+$ fixing $\mathcal{P}_0$. By lemma II.2.5, for each $c \in \mathcal{P}$, $\{c, \sigma(c)\}$ is a Jordan frame, i.e,

$$c \circ \sigma(c) = 0, c + \sigma(c) = e.$$

THEOREM II.2.7. [Spectral Decomposition]

$$V = E \oplus \mathbb{R}^2 = \{\lambda_1 c + \lambda_2 \sigma(c) \mid \lambda_i \in \mathbb{R}, c \in \mathcal{P}_-\}.$$ 

PROOF. Let $u = (x, x_1, x_2)$. If $(x, x_1) = 0$, then

$$u = x_2(y, y_1, \frac{1}{2}) + x_2(-y, -y_2, \frac{1}{2})$$

for any $(y, y_1) \in E \oplus \mathbb{R}$ with $||(y, y_1)|| = \frac{1}{2}$. If $k = ||(x, x_1)|| \neq 0$, then

$$u = (x_2 + k)(\frac{x}{2k}, \frac{x_1}{2k}, \frac{1}{2}) + (x_2 - k)(\frac{-x}{2k}, \frac{-x_1}{2k}, \frac{1}{2}).$$

For $(x, x_1) \neq 0$, we set

$$k_u = ||(x, x_1)|| = \sqrt{||x||^2 + x_1^2},$$

$$\alpha_u = x_2 + k_u,$$

$$\beta_u = x_2 - k_u,$$

$$c_u = (\frac{x}{2k_u}, \frac{x_1}{2k_u}, \frac{1}{2}).$$

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For \( u = (x, x_1, x_2) \in V \), if \( x = 0 \) and \( x_1 = 0 \), then \( u = x_2 e \). In this case,

\[
x_2 = \alpha_u = \beta_u, \quad c_u = \sigma_u = (0, 0, \frac{1}{2}).
\]

Then the spectral decomposition theorem states:

\[
u = \alpha_u c_u + \beta_u \sigma(c_u).
\]

Let \( V^{-1} \) be the set of all invertible elements in the Jordan algebra \( V \). Note that

\[
\begin{align*}
(0, 0, x_2)^{-1} &= (0, 0, \frac{1}{x_2}), \\
(x, x_1, 0)^{-1} &= \frac{1}{K_x^u}(x, x_1, 0), \\
(x, x_1, x_2)^{-1} &= \frac{1}{\alpha_u \beta_u} \sigma(u), \text{ otherwise,}
\end{align*}
\]

PROPOSITION II.2.8. \( V^{-1} = \{(x, x_1, x_2) \mid |(0, x_1)| \neq |x_2|\} \). The inverse of \( u = (x, x_1, x_2) \in V \) is given by:

\[
u^{-1} = \frac{1}{\mu(u, u)} \{-u + 2\lambda(u)e\}.
\]

REMARK II.2.9. The Lorentzian cone

\[
K_E := \{(x, x_1, x_2) \in V \mid \mu(u, u) = 0\}
\]

is exactly the complement of \( V^{-1} \). The open cone which is bounded by the positive forward cone is special interest of spin factors. In the case of \( \{0\} \oplus \mathbb{R} \oplus \mathbb{R} \), the corresponding Lorentzian cone is exactly two rays generated by the two primitive idempotents. The interior of the positive forward cone is the corresponding symmetric cone of this Jordan subalgebra.
II.2.2. Quadratic representation of $V$

Recall that for $u \in V$, let $L(u)$ be the linear map of $V$ defined by

$$L(u)v = u \circ v.$$ 

And

$$P(u) = 2L(u)^2 - L(u^2).$$

**Proposition II.2.10.** Let $u = (x, x_1, x_2), \ u = (y, y_1, y_2) \in V$. Then

$$P(u)(v) = \begin{bmatrix}
\mu(u, u)y + 2((x|y) + x_1y_1 + x_2y_2)x \\
(x_1^2 + x_2^2 - ||x||^2)y_1 + 2((x|y) + x_2y_2)x_1 \\
(||x||^2 + x_1^2 + x_2^2)y_2 + 2(x_1y_1 + (x|y))x_2
\end{bmatrix}.$$ 

**Proof.** Since $P(u)(v) = 2L(u)^2(v) - L(u^2)(v)$, we need to compute $L(u)^2$ and $L(u^2)$. $u^2 = (2x_2x, 2x_1x_2, ||x||^2 + x_1^2 + x_2^2)$ implies that

$$L(u^2)(v) = \begin{bmatrix}
2x_2y_2x + ||x||^2y + x_1^2y + x_2^2y \\
x_1x_2y_2 + ||x||^2y_1 + x_1^2y_1 + x_2y_1 \\
2(x|y)x_2 + 2x_1x_2y_1 + ||x||^2y_2 + x_1^2y_2 + x_2^2y_2
\end{bmatrix}.$$ 

And

$$2L(u)^2(v) = \begin{bmatrix}
2x_2y_2x + 2x_1^2y + 2(x|y)x + 2x_1y_1x + 2x_2y_2x \\
2x_1x_2y_2 + 2y_1x_2^2 + 2(x|y)x_1 + 2x_1^2y_1 + 2x_1x_2y_2 \\
2y_2||x||^2 + 2(x|y)x_2 + 2x_1^2y_2 + 4x_1x_2y_1 + 2(x|y)x_2 + 2x_2^2y_2
\end{bmatrix}.$$ 

Therefore

$$P(u)(v) = \begin{bmatrix}
x_2^2y + 2(x|y)x + 2x_1y_1x + 2x_2y_2x - ||x||^2y - x_1^2y \\
y_1x_2^2 + 2x_1^2y_1 + 2x_1x_2y_2 - ||x||^2y_1 \\
(||x||^2 + x_1^2 + x_2^2)y_2 + 2x_1x_2y_1 + 2(x|y)x_2
\end{bmatrix}.$$ 

□

II.2.3. Matrix representations

Let $T \in gl(V)$ be a linear (bounded) transformation on $V$. Define $A = A_T : E \to E$ to be the composition

$$A = \phi_1 \circ T \circ i,$$
where \( i : E \to V \) the inclusion and \( \phi_1 : V \to \mathbb{R} \) is the orthogonal projection with kernel \( \mathbb{R}^2 \). For \( i = 2, 3 \), let \( \phi_i \) be the orthogonal projection onto the \( i \)-th coordinate. Define \( a_{i1} \in E \) so that for all \( x \in E \),

\[
\phi_1(T(x)) = \langle a_{i1}|x \rangle;
\]

such an \( a_{i1} \) exists uniquely since the mapping on the left is a linear functional on \( E \). For \( i = 2, 3 \), define \( a_{ij} \in E \) by

\[
\phi(T(e_i)) = a_{ij},
\]

where \( e_2 = (0,1,0) \) and \( e_3 = (0,0,1) = e \). Finally for \( 2 \leq i, j \leq 3 \), define \( a_{ij} \in \mathbb{R} \) so that

\[
\phi_j(T(e_i)) = a_{ij}e_j.
\]

Then \( T \) has a block matrix representation of the form

\[
T = \begin{bmatrix}
A & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
2 & a_{31} & a_{32} & a_{33}
\end{bmatrix}.
\]

For \( u = (x, x_1, x_2) \in V \), let \( A_u y = \mu(u, u)y + 2\langle x|y \rangle x \). Then \( A_u \) is a linear transformation on \( E \).

**Proposition II.2.11.**

1. \( P(u) = \begin{bmatrix} A_u & 2x_1x & 2x_2x \\
x_2 + x_1^2 - ||x||^2 & 2x_1x_2 \\
2x_2^2 & 2x_1x_2 & ||x||^2 + x_1^2 + x_2^2\end{bmatrix} \)

2. \( L(u) = \begin{bmatrix} x_2I & 0 & x \\
0 & x_2 & x_1 \end{bmatrix} \)

**Corollary II.2.12.**

1. \( P(0, 0, x_2) = x_2^2I \).

2. \( \text{Tr} L(u \circ v) = (\text{Dim}(E) + 2)(u|v) \) if \( E \) is finite dimensional.
II.2.4. The symmetric cone $\Omega$ of $V$

Let $(u|v) = TrL(u \circ v) = (\text{Dim}(E) + 2)(u|v)$. Then $(\cdot|\cdot)$ is a positive definite symmetric bilinear form on $V$ such that

$$(L(u)v|w) = (v|L(u)w)$$

for all $u, v, w \in V$.

Let $\Omega$ be the symmetric cone of the spin factor $V = E \oplus \mathbb{R}^2$. By proposition II.2.8, the set of invertible elements $V^{-1}$ is given by

$$V^{-1} = \{(x, x_1, x_2) \mid ||(x, x_1)|| \neq |x_2|\}.$$  

Using the characterization of the symmetric cone in a Euclidean Jordan algebra (Theorem 1.2.13),

$$\Omega = \text{the identity component of } V^{-1},$$

$$= \{(x, x_1, x_2) \mid ||(x, x_1)|| < x_2\}.$$  

Like the set of primitive idempotents $\mathcal{P}$, the symmetric cone $\Omega$ can be decomposed by using the Lorentzian coordinate.

Set

$$\Omega^+ = \{(x, x_1, x_2) \in \Omega \mid x_1 > 0\}$$
$$\Omega^- = \{(x, x_1, x_2) \in \Omega \mid x_1 < 0\}$$
$$\Omega^0 = \{(x, x_1, x_2) \in \Omega \mid x_1 = 0\}$$
$$\Omega^0 = \Omega^+ \cup \Omega^0.$$  

Then $\Omega$ is the disjoint union of $\Omega^0$ and $\Omega^-$. And its boundary contains $\mathcal{P}$ as generators, i.e., $\partial \Omega = \mathbb{R}^+\mathcal{P}$.  

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REMARK II.2.13. $\Omega^0$ is the symmetric cone of the Jordan subalgebra $E \oplus \{0\} \oplus \mathbb{R}$. Let $u \in \Omega$ and let

$$u = \alpha_u c_u + \beta_u \sigma(c_u)$$

be the spectral decomposition of $u$. Then $\alpha_u$ and $\beta_u$ are positive real numbers.

In [5], we have seen that $g \in Aut(V)$ if and only if $g(e) = e$. In the Jordan algebra $V$ case, there is another characterization for $Aut(V)$ using Lorentzian coordinate.

PROPOSITION II.2.14. Every automorphism leaves the third coordinate fixed.

PROOF. Let $g \in Aut(V)$ and let $u = (x, x_1, x_2) \in V$. Consider the spectral decomposition of $u$:

$$u = \alpha_u c_u + \beta_u \sigma(c_u).$$

Since $g(c_u) \in P$ and

$$g(u) = \alpha_u g(c_u) + \beta_u g(\sigma(c_u)),$$

the third coordinate is

$$\frac{1}{2}(\alpha_u + \beta_u) = \frac{1}{2}(x_2 + ||(x, x_1)|| + x_2 - ||(x, x_1)||) = x_2.$$

\[\square\]

COROLLARY II.2.15 If $h \in Aut(V)$, then $h(\mathbb{R} \oplus \mathbb{R} \oplus \{0\}) = \mathbb{R} \oplus \mathbb{R} \oplus \{0\}$.

PROOF. By proposition II.2.14, $h(\mathbb{R} \oplus \mathbb{R} \oplus \{0\}) \subset \mathbb{R} \oplus \mathbb{R} \oplus \{0\}$. Since $h^{-1} \in Aut(V)$, $h^{-1}(\mathbb{R} \oplus \mathbb{R} \oplus \{0\}) \subset \mathbb{R} \oplus \mathbb{R} \oplus \{0\}$. Therefore $h(\mathbb{R} \oplus \mathbb{R} \oplus \{0\}) = h(\mathbb{R} \oplus \mathbb{R} \oplus \{0\})$. \[\square\]

THEOREM II.2.16 $Aut(V)$ is the set of all $g \in G(\Omega)$ which leaves the third coordinate fixed.

PROOF. Suppose that $g \in G(\Omega)$ such that $g$ leaves the third coordinate fixed. Let $g = P(w)h$ be the polar decomposition of $g$ with $w = (x, x_1, x_2) \in \Omega$ and
$h \in \text{Aut}(V)$. Then by the previous corollary, $h(E \oplus \mathbb{R} \oplus \{0\}) = E \oplus \mathbb{R} \oplus \{0\}$.

Choose $(y, y_1, 0) \in V$ such that $h((y, y_1, 0)) = (x, x_1, 0)$. Then

$$g((y, y_1, 0)) = P(w) \circ h((y, y_1, 0)) = P(w)(x, x_1, 0)$$

has the third coordinate by proposition II.2.10:

$$2x^2_1x_2 + 2||x||^2x_2$$

and it must be zero. Since $w \in \Omega$, $x_2 \neq 0$ and hence $x = 0, x_1 = 0$. Therefore $w = (0, 0, x_2) = x_2e$. But $P(x_2e) = x_2^2I$ from corollary II.2.11, hence $g = x_2^2h$.

Since $g$ leaves the third coordinate fixed, by evaluating $e$ at both sides, we get $1 = x_2^2$. Since $x_2 > 0$, $x_2 = 1$ and $g = h \in \text{Aut}(V)$. □

**II.2.5. Involutive automorphisms of $V$**

**Proposition II.2.17.** \{ $w \in V$ | $w^2 = e$ \} = \{(x, x_1, 0) | ||(x, x_1)|| = 1 \} $\cup$ \{(0, 0, \pm 1)\}.

**Proof.** Let $w = (x, x_1, x_2)$ with $w^2 = e$. Then by solving the equation

$$w^2 = (2x_2x, 2x_2x_1, ||x||^2 + x_1^2 + x_2^2) = (0, 0, 1),$$

we have if $x_2 = 0, x_1^2 + ||x||^2 = 1$. If $x_2 \neq 0, x = 0, x_1 = 0$ with $x_2^2 = 1$. The converse part is obvious. □

**Proposition II.2.18.** If $w = (0, \pm 1, 0)$, then $P(w)(x, x_1, x_2) = (-x, x_1, x_2)$. If $w = (x, x_1, 0)$ with $w^2 = e$, then

$$P(w)(y, y_1, y_2) = (2((x|y) + x_1y_1)x - y, 2x_1((x|y) + x_1y_1) - y_1, y_2).$$

Define

$$\tau(x, x_1, x_2) = (x, -x_1, x_2),$$
$\tau$ is an involutive linear transformation. The involution $\tau$ also sends $P^+$ to
$P^-$ and fixing $P^0$. Note that $\mu(\tau(u), \tau(u)) = \mu(u, u)$, for all $u \in V$.

**Lemma II.2.19.** $\sigma$ and $\tau$ are automorphisms of $V$. If $w = (0, \pm 1, 0)$, then
$P(w) \in Aut(V)$. In particular, $\sigma = \tau \circ P(w)$.

**Proof.** By proposition II.2.18 and theorem II.2.16, $P(w) \in Aut(V)$ for
$w = (0, \pm 1, 0)$. It is clear that $\sigma = \tau \circ P(w)$. To complete the proof, we need
to see that $\tau \in Aut(V)$. Let $u = (x, x_1, x_2) \in V$. Then $\lambda(\tau(u)) = \mu(\tau(u), e) =
\lambda(u)$, hence

$$
\tau(u^2) = 2\lambda(u)\tau(u) - \mu(u, u)e = 2\lambda(\tau(u))\tau(u) - \mu(\tau(u), \tau(u))e = \tau(u)^2.
$$

Therefore $\tau$ is an Jordan algebra homomorphism of $V$. Hence $\tau \in Aut(V)$. □

Set $j = P(w)$, for $w = (0, \pm 1, 0)$.

**Lemma II.2.20.**

1. $j \circ P(u) = P(j(u)) \circ j$.
2. $\tau \circ P(u) = P(\tau(u)) \circ \tau$.
3. $\sigma \circ P(u) = P(\sigma(u)) \circ \sigma$.

**Proof.** Use Proposition II.2.10. □

**Corollary II.2.21.**

1. $j \circ P(\Omega^\pm) \circ j = P(\Omega^\mp), j \circ P(\Omega^0) \circ j = P(\Omega^0)$.
2. $\sigma \circ P(\Omega^\pm) \circ \sigma = P(\Omega^\mp), \sigma \circ P(\Omega^0) \circ \sigma = P(\Omega^0)$. 

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Set

\[ G(\Omega)_\sigma = \{ g \in G(\Omega) \mid \sigma \circ g = g \circ \sigma \} \]
\[ G(\Omega)_\tau = \{ g \in G(\Omega) \mid \tau \circ g = g \circ \tau \}. \]

**Theorem II.2.22.**

\[ G(\Omega)_\tau = P(\Omega^0) Aut(V)_\tau = P(\Omega^0) Aut(V^0) = G(\Omega^0) \]
\[ G(\Omega)_j = P(\Omega \cap \{0\} \oplus \mathbb{R} \oplus \mathbb{R}) Aut(V)_j. \]

**Proof.** Let \( g \in G(\Omega)_\tau \) and let \( g = P(w) \circ h \) be the polar decomposition of \( g \). Then by lemma II.2.20,

\[ \tau \circ g = \tau \circ P(w) \circ h \]
\[ = P(\tau(w)) \circ \tau \circ h, \]
\[ = g \circ \tau \]
\[ = P(w) \circ h \circ \tau. \]

Thus

\[ P(w^{-1}) \circ P(\tau(w)) = h \circ \tau \circ h^{-1} \circ \tau \in Aut(V). \]

So \( P(w^{-1})P(\tau(w))(e) = e \). This implies that \( \tau(w)^2 = w^2 \) and hence \( \tau(w) = w \) by lemma I.2.15. Therefore \( h \circ \tau = \tau \circ h \) and \( h \in Aut(V)_\tau \). Since \( \tau(w) = w \) implies \( w \in \Omega^0 \), we conclude that \( G(\Omega)_\tau \subseteq P(\Omega^0) Aut(V)_\tau \). Conversely, if \( w \in \Omega^0 \), then by lemma II.2.20, \( \tau \circ P(w) = P(\tau(w)) \circ \tau \). Because \( \tau(w) = w \) and \( Aut(V)_\tau \subseteq G(\Omega)_\tau \), we conclude that \( G(\Omega)_\tau = P(\Omega^0) Aut(V) \). For \( G(\Omega)_j \), we need to check that \( j(w) = w \) if and only if \( w \in \Omega \cap \{0\} \oplus \mathbb{R} \oplus \mathbb{R} \). □

**Remark II.2.23.** For each non-zero real number \( t \), \( P(te) = t^2 I \). Hence \( P(te) \in G(\Omega)_\sigma \).
PROPOSITION II.2.24. $\text{Aut}(V) \subset G(\Omega)_o$.

PROOF. Let $h \in \text{Aut}(V)$ and let $u = (x, x_1, x_2) \in V$. Set $u' = (x, x_1, 0)$. Then $u = u' + x_2 e$. Let $h(u) = (y, y_1, y_2)$. Then by theorem II.2.16, $y_2 = x_2$. So

$$h(u) = h(u') + x_2 e = (y, y_1, 0) + x_2 e,$$

$$h \circ \sigma(u) = h(-u' + x_2 e) = -h(u') + x_2 e,$$

$$\sigma \circ h(u) = \sigma(h(u') + x_2 e) = \sigma(h(u')) + x_2 e.$$

Since $h(u') = (y, y_2, 0), \sigma \circ h(u') = \sigma((y, y_1, 0)) = -(y, y_1, 0) = -h(u')$. Hence $h \circ \sigma = \sigma \circ h$. □

PROPOSITION II.2.25. For $w \in \Omega, P(w) \in G(\Omega)_o$ if and only if $w \in \mathbb{R}^+ e$.

PROOF. Let $w = (x, x_1, x_2) \in \Omega$. If $w \in \mathbb{R}^+ e$, then $w = x_2 e$ and $P(w) = x_2^2 I \in G(\Omega)_o$. Conversely, suppose $P(w) \in G(\Omega)_o$ and $(x, x_1) \neq 0$. Then $\sigma \circ P(w) = P(w) \circ \sigma$ implies that

$$P(w)^{-1} \circ \sigma \circ P(w) = P(w^{-1}) \circ \sigma \circ P(w) = \sigma \in \text{Aut}(V).$$

So $[P(w^{-1}) \circ \sigma \circ P(w)](e) = e$. But $P(w)(e) = w^2$ and $w^{-1} = \frac{1}{t} \sigma(w)$ by Proposition II.2.6, where $t = \alpha_w \beta_w > 0$. Hence $e = P\left(\frac{1}{t} \sigma(w)\right)(\sigma(w)^2) = \frac{1}{t^2} \sigma(w^4)$ which implies $(\frac{w^2}{t})^2 = e$. From the Proposition II.2.6, $\frac{w^2}{t} \in E \oplus \mathbb{R} \oplus \{0\}$ or $\frac{w^2}{t} = (0, 0, \pm 1)$. But $w \in \Omega$ implies that $w^2 \in \Omega$ and hence $\frac{w^2}{t} = e$. Therefore $w^2 = (x, x_1, x_2)^2 = (2x, 2x_1, \|(x, x_1)^2 + x_2^2\|) = (0, 0, t)$. Hence $x = 0, x_1 = 0$ which gives a contradiction. Therefore $w \in \mathbb{R}^+ e$. □

THEOREM II.2.26. $G(\Omega)_o = \mathbb{R}^+ \text{Aut}(V)$.

PROOF. Suppose $g \in G(\Omega)_o$ and let $g = P(w)h$ be the polar decomposition with $w \in \Omega$ and $h \in \text{Aut}(V)$. Note that

$$\sigma \circ g = \sigma \circ P(w) \circ h.$$
is equal to \( g \circ \sigma = P(w) \circ \sigma = P(w) \circ h \). Hence we get \( P(w) \circ \sigma = \sigma \circ P(w) \).

By proposition II.2.25, \( w \) must be in \( R^+ e \). Therefore \( g \in P(R^+ e) \cdot \text{Aut}(V) \). From proposition II.2.24 and remark II.2.23, the converse inclusion is obvious. □

**Proposition II.2.27.** \( \text{Der}(V) \oplus R L(e) \) is a Lie subalgebra of \( \text{Lie}(G(\Omega)) \).

**Proof.** Let \( D_1, D_2 \in \text{Der}(V) \). Then

\[
[D_1 + tL(e), D_2 + sL(e)] = [D_1, D_2] + t[L(e), D_2] + s[D_1, L(e)].
\]

Since \( D_i e = 0 \) for \( i = 1, 2 \), \([D_i, L(e)] = L(D_i e) = 0\). Thus \( \text{Der}(V) \oplus R L(e) \) is a Lie subalgebra of \( \text{Lie}(GL(\Omega)) \). □

**Corollary II.2.28.** \( \text{Lie}(G(\Omega)_\sigma) = \text{Der}(V) \oplus R L(e) \).

**II.2.6. Pseudo-orthogonal group of \( V \)**

Let \( O_1(V) \) the group of all pseudo-orthogonal transformations of \( V \), i.e., \( T \in O_1(V) \) if and only if

\[
\mu(Tu, Tv) = \mu(u, v)
\]

for all \( u, v \in V \), or equivalently,

\[
\mu(Tu, Tu) = \mu(u, u)
\]

for all \( u \in V \).

Each member of \( O_1(V) \) sends \( K_E^+ \) into \( K_E^+ \) or \(-K_E^+ \), where

\[
K_E = \{ u = (x, x_1, x_2) \in V \mid \mu(u, u) = 0 \}
\]

\[
K_E^+ = \{ u = (x, x_1, x_2) \in K_E \mid x_2 > 0 \}.
\]

Note that \( K_E^+ \) is the boundary of the symmetric cone \( \Omega \). Therefore \( K_E^+ \) can be decomposed into \( P^2 \) and \( P^0 \). Set \( O_1^+(V) \) be the subgroup of elements in \( O_1(V) \) which carry \( K_E^+ \) into itself:

\[
O_1^+(V) = \{ g \in O_1(V) \mid g(K_E^+) = K_E^+ \}.
\]
This group is called the Lorentzian group and members of the Lorentzian group are called Lorentzian transformations.

**Proposition II.2.29.** $O_1^+(V) \subset G(\Omega)$. In particular, $O_1(V) \subset \text{Str}(V)$.

**Proof.** Suppose that $g \in O_1^+(V)$. Let $u = (x, x_1, x_2) \in \Omega$ and let $g(u) = v = (y, y_1, y_2)$. Consider the spectral decomposition of $u$:

$$u = \alpha_u c_u + \beta_u \sigma(c_u).$$

Because $u \in \Omega$, $\alpha_u$ and $\beta_u$ are positive real numbers by remark II.2.13. Furthermore, $c_u$ and $\sigma(c_u)$ are in $K^+_E$. Therefore $g(c_u)$ and $g(\sigma(c_u))$ are in $K^+_E$ since $g \in O_1^+(V)$. Hence

$$g(u) = \alpha_u g(c_u) + \beta_u g(\sigma(c_u))$$

has the positive third coordinate. So $y_2 > 0$. Now $\mu(u, u) = \mu(v, v)$, so we have

$$y_2^2 - \|\langle y, y_1 \rangle\|^2 = x_2^2 - \|\langle x, x_1 \rangle\|^2 > 0.$$ 

Thus $g(u) = v \in \Omega$ and hence $g(\Omega) \subset \Omega$. Doing this job for $g^{-1}$, we get $g(\Omega) = \Omega$. That is, $g \in G(\Omega)$. Furthermore, $O_1(V) \subset \text{Str}(V)$ by theorem I.2.16. □

The Lie algebra of $O_1(V)$ is given by ([31]):

$$\text{Lie}(O_1(V)) = \{ \begin{bmatrix} A_{11} & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ a_{13} & -a_{23} & 0 \end{bmatrix} | A_{11}^* = -A_{11} \}.$$ 

Here $A^*$ is the adjoint of $A$ with respect to the inner product $\langle \cdot, \cdot \rangle$ on $E$.

Let

$$A = \begin{bmatrix} A_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \text{gl}(V).$$

Set

$$A^\sigma := \sigma \circ A \circ \sigma.$$
Then

\[
A^* = \begin{bmatrix}
A_{11} & a_{12} & -a_{13} \\
-a_{21} & a_{22} & -a_{23} \\
-a_{31} & -a_{32} & a_{33}
\end{bmatrix}.
\]

Let

\[
t = \begin{cases}
A_{11} & a & 0 \\
-a^* & 0 & 0 \\
0 & 0 & 0
\end{cases} \quad | \quad A_{11}^* = -A_{11},
\]

\[
p = \begin{cases}
0 & 0 & a \\
0 & 0 & b \\
a^* & b & 0
\end{cases} \quad | \quad a \in E, b \in R.
\]

Then

\[
t = \{ A \in \text{Lie}(O_1(V)) \mid A^* = A \},
\]

\[
p = \{ A \in \text{Lie}(O_1(V)) \mid A^* = -A \},
\]

and \(\text{Lie}(O_1(V)) = t \oplus p\).

**Proposition II.2.30.**

\[
t \subset \text{Der}(V)
\]

\[
p \subset L(V) = \{L(u) \mid u \in V\}.
\]

**Proof.** By proposition II.2.11, \[
\begin{bmatrix}
0 & 0 & a \\
0 & 0 & b \\
a^* & b & 0
\end{bmatrix} = L((a, b, 0)) \in L(V),
\]

hence \(p \subset L(V)\). Let

\[
D = \begin{bmatrix}
A_{11} & a & 0 \\
-a^* & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \in t.
\]

Then we will show that \(D\) is a derivation. For \(u = (x, x_1, x_2), v = (y, y_1, y_2) \in V\),

\[
Du = (A_{11}x + x_1a, -(a|x), 0),
\]

\[
Dv = (A_{11}y + y_1a, -(a|y), 0),
\]

\[
Du \circ v = (y_2(A_{11}x + x_1a), -y_2(a|x), (A_{11}x + x_1a|y) - y_1(a|x))
\]
\[ u \circ Dv = (x_2(A_{11}y + y_1a), -x_2(a|y), (A_{11}y + y_1a|x) - x_1(a|y)) \]

\[ Du \circ v + u \circ Du = (A_{11}(y_2x + x_2y) + (x_1y_2 + x_2y_1)a, -(a|y_2x + x_2y), 0). \]

Note

\[
(A_{11}x + x_1a|y) - y_1(a|x) + (A_{11}y + y_1a|x) - x_1(a|y)
\]

\[
= (A_{11}x|y) + (A_{11}y|x)
\]

\[
= (x|A_{11}y) + (A_{11}y|x)
\]

\[
= (x|(A_{11}^* + A_{11})y)
\]

\[
= 0.
\]

But

\[
D(u \circ v) = \begin{bmatrix} A_{11}(x_2y + y_2x) + (x_1y_2 + y_1x_2) \\ -(a|x_2y + y_2x) \\ 0 \end{bmatrix}.
\]

Hence \( D(u \circ v) = Du \circ v + u \circ Du \). Therefore \( D \) is a deverivation of \( V \). \( \square \)

**Proposition II.2.31.** \( \text{Der}(V) = t \).

**Proof.** Let

\[
D = \begin{bmatrix} A & a_{12} & a_{13} \\ a_{21}^* & a_{22} & a_{23} \\ a_{31}^* & a_{32} & a_{33} \end{bmatrix} \in \text{Der}(V).
\]

Since \( De = 0 \), we have \( a_{13} = a_{23} = a_{33} = 0 \). Let \( u = (x, x_1, x_2) \) and let \( v = (y, y_1, y_2) \). Then

\[
D(u \circ v) = \begin{bmatrix} x_2Ay + y_2Ax + (x_2y_1 + x_1y_2)a_{12} \\ (a_{31}|x_2y + y_2x) + (x_2y_1 + x_1y_2)a_{22} \\ (a_{31}|x_2y + y_2x) + (x_2y_1 + x_1y_2)a_{32} \end{bmatrix}.
\]

\[
u \circ Du = \begin{bmatrix} (a_{31}|y) + a_{32}y_2x + x_2Ay + x_2y_1a_{12} \\ x_2(a_{21}|y) + x_2a_{22}y_1 + x_1(a_{31}|y) + x_1a_{32}y_2 \\ (x|Ay) + y_1(a_{12}|x) + x_1(a_{31}|y) + x_1a_{22}y_1 + x_2(a_{31}|y) + x_2a_{32}y_2 \end{bmatrix}.
\]

\[
Du \circ v = \begin{bmatrix} (a_{31}|x)y + a_{32}x_1y + y_2Ax + x_1y_2a_{12} \\ * * * * * * * * * * * \\ (y|Ax) + x_1(a_{12}|y) + y_1(a_{21}|y) + y_2(a_{31}|x) + y_2a_{32}x_1 \end{bmatrix}.
\]
(1) \((a_{31}|y)x + a_{32}y_2x + (a_{31}|x) + a_{32}x_1y = 0\) for all \(u, v \in V\) implies that \(a_{32} = a_{31} = 0\).

(2)

\[
(x|Ay) + (a_{12}|x)y_1 + x_1(a_{21}|y) + x_1a_{22}y_1 \\
+ (y|Ax) + x_1(a_{12}|y) + y_1(a_{21}|x) + y_1a_{22}x_1 = 0
\]

implies that \(A^* = -A\) and \(a_{12} = -a_{21}, a_{22} = 0\). Therefore \(D \in k\). □

**Corollary II.2.32.** \(\text{Lie}(G(\Omega)) = \text{Lie}(O_1(V)) \oplus L(\{0\} \oplus \{0\} \oplus \mathbb{R})\).

**Corollary II.2.33.** \(G(\Omega_0) = O_1^+(V) \cdot \mathbb{R}^+\).

**Proof.** For \(u = (0, 0, x_2)\), \(\exp L(u) = P(\exp \frac{u}{2})\). Furthermore, \(\exp \frac{u}{2} = e^{\frac{x_2}{2}} e\) and hence

\[
\exp L(u) = P(e^{\frac{x_2}{2}} e) = e^{x_2 I}.
\]

Hence via the exponential map we can identify \(P(\mathbb{R}^+ e)\) with \(\mathbb{R}^+\). □

Note that the Lie algebra of \(\text{Aut}(V)\) is contained in the Lie algebra of pseudo-orthogonal group and hence the identity component \(\text{Aut}(V)_0\) of \(\text{Aut}(V)\) is a subgroup of \(O_1(V)\). But the following proposition shows that \(\text{Aut}(V) \subset O_1^+(V)\).

**Proposition II.2.34.** \(\text{Aut}(V) \subset O_1^+(V)\).

**Proof.** Let \(g \in \text{Aut}(V)\) and let \(u = (x, x_1, x_2) = \alpha_u c_u + \beta_u \sigma(c_u)\) be the spectral decomposition of \(u\). Then

\[
g(u) = \alpha_u g(c_u) + \beta_u g(\sigma(c_u)) \]

\[
= \alpha_u g(c_u) + \beta_u \sigma g(c_u)
\]

since \(g \in G(\Omega)\) by proposition II.2.24. Now since \(c_u \in P\) and \(g \in \text{Aut}(V)\), we may assume that

\[
g(c_u) = (y, y_1, \frac{1}{2}), \quad ||(y, y_1)|| = \frac{1}{2}.
\]
So
\[ g(u) = \alpha_u(y, y_1, \frac{1}{2}) + \beta_u(-y, -y_1, \frac{1}{2}) \]
\[ = ((\alpha_u - \beta_u)y, (\alpha_u - \beta_u)y_1, \frac{1}{2}(\alpha_u + \beta_u)). \]

Since \( \alpha_u = x_2 + \| (x, x_1) \| \) and \( \beta_u = x_2 - \| (x, x_1) \| \),
\[ \mu(g(u), g(v)) = -(\alpha_v - \beta_v)^2 \| y \|^2 - (\alpha_u - \beta_u)^2 y_1^2 + \frac{1}{4}(\alpha_u + \beta_u)^2 \]
\[ = -(\alpha_v - \beta_v)^2 (\| y \|^2 + y_1^2) + \frac{1}{4}(\alpha_u + \beta_u)^2 \]
\[ = \frac{1}{4}[(\alpha_u + \beta_u)^2 - (\alpha_u - \beta_u)^2] \]
\[ = x_2^2 - \| x \|^2 - x_1^2 \]
\[ = \mu(u, u). \]

So \( g \in O_1(V) \). Since \( g \in G(\Omega) \), \( g \) must be in \( O^+_1(V) \). □

By theorem II.2.16 and proposition II.2.34, the automorphism group of \( V \) is exactly the orthogonal group \( O(E \oplus \mathbb{R}) \).

**Corollary II.2.35.** \( \text{Aut}(V) \cong O(E \oplus \mathbb{R}) \).

**Theorem II.2.36.** \( G(\Omega) = O^+_1(V) \cdot \mathbb{R}^+ \cong O^+_1(V) \times \mathbb{R}^+ \), the direct product of \( O^+_1(V) \) and \( \mathbb{R}^+ \).

**Proof.** Clearly, \( O^+_1(V) \cdot \mathbb{R}^+ \subseteq G(\Omega) \). Let \( g \in G(\Omega) \) and let \( g = P(w) \circ h \) be the polar decomposition of \( g \) with \( w \in \Omega \). Let \( \alpha_w c_w + \beta_w \sigma(c_w) \) be the spectral decomposition of \( w \). Then \( w = \exp w' \), where
\[ w' = (\log \alpha_w c_w + \log \beta_w \sigma(c_w)) \]
and hence \( P(w) = P(\exp w') = \exp 2L(w') = \exp L(2w') \). Note that we can decompose \( L(2w') \) into \( L(v) + tI \) for some \( v \in E \oplus \mathbb{R} \oplus \{0\} \) and \( t \in \mathbb{R} \). From corollary II.2.27, \( L(v) \in \text{Lie}(O^+_1(V)) \). Hence we have \( P(w) = \exp(L(v) + tI) = \exp L(v) \cdot e^{tI} \in O^+_1(V) \cdot \mathbb{R}^+ \). By proposition II.2.34, \( \text{Aut}(V) \subset O^+_1(V) \).

Therefore, \( g = P(w) \circ h \in O^+_1(V) \cdot \mathbb{R}^+ \) which completes the proof. □
II.2.7. The Gauss decomposition of $G(\Omega)$

We have seen that the vector space $V = E \oplus \mathbb{R}^2$ has a natural Jordan algebra structure, named, spin factor. For $u = (x, x_1, x_2), v = (y, y_1, y_2),$

$$u \circ v = (x_2y + y_2x, x_2y_1 + y_2x_1, (x|y) + x_1y_1 + x_2y_2).$$

Then $(V, \circ)$ is a simple Euclidean Jordan algebra with the associative symmetric bilinear form:

$$(u|v) = TrL(u \circ v).$$

It is not hard to see that $(u|v) = (\dim V)(x|y)$, if $E$ is finite dimensional. Furthermore, $V$ has a symmetric cone

$$\Omega := \{(x, x_1, x_2) \in V \mid x_2 > 0, ||x||^2 + x_1^2 < x_2^2\}.$$

In this case $e := (0, 0, 1)$ is the identity for $V$ and if we let $c = (0, 1, 1)$, then \{c, \sigma(c)\} is a Jordan frame, where

$$\sigma(x, x_1, x_2) = (-x, -x_1, x_2).$$

Then $V$ can be decomposed as

$$V = Rc \oplus V_{12} \oplus R\sigma(c),$$

where $V_{12} = V(c, \frac{1}{2}) \cap V(\sigma(c), \frac{1}{2}).$

We let

$$R = \{ac + b\sigma(c) \mid a, b \in \mathbb{R}\}.$$ 

Then

$$R \cap \Omega = \{(0, a - b, a + b) \mid a, b \in \mathbb{R}^+\}.$$ 

Denoting by $A$ the set of the elements $P(w)$, with $w \in R \cap \Omega$. Then using the fundamental formular, we have
PROPOSITION II.2.37. A is a subgroup of $G(\Omega)_o$, the identity component of $G(\Omega)$.

Set

$$\alpha = \{L(u) \mid u \in R\}.$$ 

Then $\alpha$ is a maximal abelian subalgebra of $p$, where

$$p = L(V) = \{L(u) \mid u \in V\}.$$ 

For $Y = L(u) = L(ac + b\sigma(c)) \in \alpha$, we define $\alpha(Y) = \frac{k-a}{2}$ and

$$\gamma = \{X \in \text{Lie}(G(\Omega)) \mid [Y, X] = \alpha(Y)X, \text{ for all } Y \in \alpha\}.$$ 

Then $\gamma$ is a nilpotent subalgebra of $\text{Lie}(G(\Omega))$.

The triangular subgroup $T$ of $G$ is the set of elements $g$ in $G$ such that, for all $u \in V$,

$$(gx_{ki})_{ij} = 0, \text{ if } (i, j) < (k, l)$$

for the lexicographic order, and

$$(gx_{ij})_{ij} = \lambda_{ij}x_{ij}, \forall i, j,$$

with positive numbers $\lambda_{ij}$, which do not depend on $x$. The strict triangular subgroup of $G$ is the set $N$ such that $\lambda_{ij} = 1$. The Lie algebra of $N$ is $\gamma$.

THEOREM II.2.38. [Gauss Decomposition]

$$T = NA = AN, \quad G(\Omega)_o = TK = NAK.$$ 

REMARK II.2.39. The Gauss decomposition of $G(\Omega)_o$ is a special case of the Iwasawa decomposition. This decomposition occurs in any simple Euclidean Jordan algebras.
II.3. Jordan algebras of symmetric operators

Let $V$ be a finite-dimensional vector space over $\mathbb{R}$ and let $\sigma(x, y)$ be a symmetric bilinear form on $V$. We also assume that $\sigma$ is non-degenerate. Then the bilinear form $\sigma$ is represented as follows by a symmetric matrix $S = (a_{ij})$ for a basis on $V$:

$$\sigma(x, y) = \sum_{i,j} a_{ij} x_i y_j.$$  

We have seen in section 1 that for $T \in gl(V)$, the adjoint operator $T^*$ is given by $T^* = S^{-1} T S$.

Let $\mathcal{V}_\sigma$ be the set of all self-adjoint operators with respect to the fixed non-degenerate, symmetric bilinear form $\sigma$ on $V$. Then $\mathcal{V}_\sigma$ is a Jordan algebra with the product

$$A \circ B = \frac{1}{2}(AB + BA).$$

**Lemma II.3.1.** If $\sigma$ is a positive definite or negative definite, then $\mathcal{V}_\sigma$ is a Euclidean Jordan algebra.

**Proof.** Since $\mathcal{V}_\sigma$ is finite dimensional, we want to show that it is formally real. Suppose $A^2 + B^2 = 0$. Then

$$\sigma(Ax, Ax) + \sigma(Bx, Bx) = \sigma(x, A^2 x) + \sigma(x, B^2 x) = \sigma(x, A^2 x + B^2 x) = 0$$

implies $Ax = Bx = 0$ for all $x \in X$. Hence $A = B = 0$. □

**Examples II.3.2.** (1) Let $X = \mathbb{R}^4$ and let $\sigma(x, y) = x_1 y_1 + x_2 y_2 - x_3 y_3 - x_4 y_4$.

Set

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Then $\sigma(x, y) = (Sx|y)$ and $A^* = SA^4 S$. The corresponding Jordan algebra
$V_\sigma$ is the set of all the following matrices

$$\begin{bmatrix} A & C \\ -C^t & B \end{bmatrix}$$

where $A, B \in \text{Sym}(2, \mathbb{R}), C \in M_2(\mathbb{R})$.

Since $\begin{bmatrix} A & A \\ -A & -A \end{bmatrix}^2 = 0$, for any $A \in \text{Sym}(2, \mathbb{R})$, $V_\sigma$ is not formally real, hence is not a Euclidean Jordan algebra.

(2) Let $\sigma(x, y) = x_1y_1 + x_2y_2 - x_3y_3$. And let

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Then

$$V_\sigma = \{ \begin{bmatrix} A & b \\ -b^t & d \end{bmatrix} | A \in \text{Sym}(2, \mathbb{R}), b \in M_{2,1}(\mathbb{R}), d \in \mathbb{R} \}.$$  

Then $V_\sigma$ is not a Euclidean Jordan algebra since if $x \neq 0$ and

$$A = \begin{bmatrix} x & 0 & x \\ 0 & 0 & 0 \\ -x & 0 & -x \end{bmatrix}$$

then $A \in V_\sigma$ and $A^2 = 0$. Therefore $V_\sigma$ is not formally real, hence it is not a Euclidean Jordan algebra.

(3) Let $\sigma(x, y) = -(2x_1y_1 + x_2y_2)$ and let

$$S = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

be the corresponding matrix representation of $\sigma$. Then the corresponding Jordan algebra is the following

$$V_\sigma = \{ \begin{bmatrix} a & b \\ 2b & d \end{bmatrix} | a, b, d \in \mathbb{R} \}.$$  

Elementary calculation shows that this is a Euclidean Jordan algebra which is isomorphic to $\text{Sym}(2, \mathbb{R})$.

The Jordan algebra $V_\sigma$ is non-Euclidean but has similar Jordan structure to the Euclidean Jordan algebra $\text{Sym}(n, \mathbb{R})$.
From now on, we denote $V_{p,q}$ be the Jordan algebra of all self-adjoint operators on $V$ with respect to the bilinear form

$$J_{p,q}(x,y) = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{n} x_i y_i = \langle x|J_{p,q}y \rangle.$$ 

Using the Sylvester’s law of inertia, we have the following theorem:

**Theorem II.3.3.** Let $\sigma$ be a non-degenerate symmetric bilinear form on a finite dimensional vector space $V$. Then the Jordan algebra $V_{\sigma}$ is isomorphic to $V_{p,q}$ for some integers $p, q$ with $p + q = \text{dim}V$.

**Proof.** Let $T \in \mathfrak{gl}(V)$ and let $T^*, T^\#$ be the adjoint operator with respect to $\sigma$ and $J_{p,q}$ respectively. By the Sylvester’s law of inertia, there is an isomorphism $P : V \to V$ such that

$$\sigma(x, y) = J_{p,q}(Px, Py).$$

Note that

$$J_{p,q}(Tx, y) = \langle Tx|J_{p,q}y \rangle = \langle x|T^t J_{p,q}y \rangle = J_{p,q}(x, T^\#(y)) = \langle x|J_{p,q}T^\#(y) \rangle.$$ 

Hence

$$T^\# = J_{p,q}T^t J_{p,q}.$$ 

But $T^*$ is given in section II.1 by

$$T^* = P^{-1}J_{p,q}(P^{-1})^t T^t P^t J_{p,q} P.$$
Define a mapping $\hat{P} : V_\sigma \to V_{p,q}$ by

$$\hat{P}(T) = PTP^{-1}.$$  

Then for $T \in V_\sigma$,

$$PTP^{-1} = P(P^{-1}J_{p,q}(P^{-1})^tT^tP^tJ_{p,q}P)P^{-1} = J_{p,q}(P^{-1})^tT^tP^tJ_{p,q} = J_{p,q}[PTP^{-1}]^tJ_{p,q} = (PTP^{-1})^\#.$$  

Therefore the mapping $\hat{P}$ is well-defined. Since the Jordan structures of $J_\sigma$ and $V_{p,q}$ are from the associative algebra $gl(V)$, it is clear that $\hat{P}$ is an isomorphism.  

Let $J_{p,q} = \text{diag} [\lambda_1, \lambda_2, \cdots, \lambda_n]$.  

The Jordan algebra $V_{p,q}$ is the space of all the following matrices:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \tilde{\lambda}_{12}a_{12} & a_{22} & a_{23} & \cdots & a_{2n} \\ \tilde{\lambda}_{13}a_{13} & \tilde{\lambda}_{23}a_{23} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{\lambda}_{1n}a_{1n} & \tilde{\lambda}_{2n}a_{2n} & \tilde{\lambda}_{3n}a_{3n} & \cdots & a_{nn} \end{bmatrix}$$

where $\tilde{\lambda}_{ij} = \lambda_i\lambda_j^{-1}, i < j$, and $a_{ij}$'s are arbitrary real numbers.

**Theorem II.3.4.** $V_\sigma$ is a Euclidean Jordan algebra if and only if $\sigma$ is positive definite or negative definite if and only if $V_{p,q}$ is isomorphic to $V_{n,0}$ or $V_{0,n}$.

**Proof.** By the lemma II.3.1, if $\sigma$ is a positive definite or negative definite, then $V_\sigma$ is a Euclidean Jordan algebra. Conversely, suppose $V_\sigma$ is a Euclidean Jordan algebra. By theorem, $V_\sigma$ is isomorphic to $V_{p,q}$ for some $p, q$. If $p = 0$ or $q = 0$, then $\sigma$ is positive or negative definite. Suppose that $p \neq 0$ and $q \neq 0$. 

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Let

\[ A = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
-1 & 0 & 0 & \cdots & 0 & -1
\end{bmatrix} \]

Then \( A \in V_\sigma \) by the remark. But \( A^2 = 0 \), hence \( V_\sigma \) is not formally real which gives a contradiction. \( \square \)

**Corollary II.3.5.** If \( \dim V = 2 \), then \( V_\sigma \) is isomorphic to \( \text{Sym}(2, \mathbb{R}) \) or \( W \), where \( W = \{ [x \ y \ z] | x, y, z \in \mathbb{R} \} \) for every symmetric non-degenerate bilinear form \( \sigma \).

Let \( E_{ii} \) the diagonal matrix with one \( ii \)-th entry. For \( i < j \), let

\[ E_{ij} = [e_{ki}], e_{ij} = 1, e_{ji} = \text{sgn}(\tilde{\lambda}_{ij}). \]

**Proposition II.3.6.**

1. \( \{E_{ii}, E_{ij} | i, j = 1, 2, \ldots, n\} \) is a basis of \( V_{p,q} \).
2. \( E_{ii}^2 = E_{ii} \).
3. \( E_{ii}E_{ij} = \frac{1}{2} E_{ij} = E_{ij}E_{ij} \).
4. \( E_{ij}E_{il} = \text{sgn}(\tilde{\lambda}_{ij})E_{jl} \).
5. \( E_{ij}E_{ji} = E_{ii} \).
6. \( E_{ij}E_{kl} = 0 \) if \( i, j, k, l \) all distinct.
7. \( E_{il}E_{jl} = \text{sgn}(\tilde{\lambda}_{ji})E_{ij} \).

**Corollary II.3.7.**

1. \( \mathbb{R} \cdot E_{ii} \oplus \mathbb{R} \cdot E_{ij} \oplus \mathbb{R} \cdot E_{ij} \) is a 3-dimensional subalgebra of \( V_{p,q} \) for each \( i < j \).

In particular, if \( \tilde{\lambda}_{ij} > 0 \), then this is a Euclidean Jordan subalgebra.
(2) $\mathbb{R} \cdot E_{ii} \oplus \mathbb{R} \cdot E_{jj} \oplus \mathbb{R} \cdot E_{tt} \oplus \mathbb{R} \cdot E_{ij} \oplus \mathbb{R} \cdot E_{ii} \oplus \mathbb{R} \cdot E_{jj}$ is a subalgebra of $V_{p,q}$ for each $i < j < l$. In particular, if $\lambda_{ij} > 0$, $\lambda_{ii} > 0$ and $\lambda_{jj} > 0$, then this is a 6-dimensional Euclidean Jordan subalgebra.

PROPOSITION II.3.8.

1. $V(E_{ii}, \frac{1}{2}) \cap V(E_{jj}, \frac{1}{2}) = \mathbb{R} \cdot E_{ij}$ for $i < j$.

2. $P(E_{ii})$ and $4L(E_{ij})L(E_{jj})$ are orthogonal projections onto $\mathbb{R} \cdot E_{ii}$ and $\mathbb{R} \cdot E_{ij}$ respectively.

PROOF. Let

$$X = \sum_{k=1}^{n} a_{kk} E_{kk} + \sum_{k<l} a_{kl} E_{kl} \in V(E_{ii}, \frac{1}{2}) \cap V(E_{jj}, \frac{1}{2}).$$

Then

$$E_{ii}X = a_{ii} + \sum_{k=1}^{i-1} a_{ki} E_{ii} E_{ki} + \sum_{k=i+1}^{n} a_{ik} E_{ii} E_{ik}$$

$$= a_{ii} E_{ii} + \frac{1}{2} \left( \sum_{k=1}^{i-1} a_{ki} E_{ki} + \sum_{k=i+1}^{n} a_{ik} E_{ik} \right)$$

$$= \frac{1}{2} X$$

implies that

$$X \in \mathbb{R} \cdot E_{1i} \oplus \mathbb{R} \cdot E_{2i} \oplus \cdots \mathbb{R} \cdot E_{i-1i} \oplus \mathbb{R} \cdot E_{ii+1} \oplus \cdots \mathbb{R} \cdot E_{in}.$$ 

Similarly we get

$$X \in \mathbb{R} \cdot E_{1j} \oplus \mathbb{R} \cdot E_{2j} \oplus \cdots \mathbb{R} \cdot E_{j-1j} \oplus \mathbb{R} \cdot E_{jj+1} \oplus \cdots \mathbb{R} \cdot E_{jn}.$$ 

Therefore $X \in \mathbb{R} \cdot E_{ij}$. The converse is obvious.

The following proposition shows that the Jordan algebra $V_{p,q}$ is a semisimple. After this section, we will show that it is a simple Jordan algebra.
**Proposition II.3.9.** Let

\[ A = \sum_{i=1}^{n} a_{ii} E_{ii} + \sum_{i<j} a_{ij} E_{ij}, \quad B = \sum_{i=1}^{n} b_{ii} E_{ii} + \sum_{i<j} b_{ij} E_{ij} \in V_{p,q}. \]

Then

\[ \text{Tr} L(AB) = \frac{n+1}{2} \sum_{j=1}^{n} l_j, \]

where

\[ l_j = \sum_{i=1}^{j-1} \text{sgn}(\lambda_{ij}) a_{ij} b_{ij} + \sum_{i=j}^{n} \text{sgn}(\lambda_{ji}) a_{ji} b_{ji}. \]

**Proof.**

\[ AB = \left( \sum_{i=1}^{n} a_{ii} E_{ii} \right) \left( \sum_{i=1}^{n} b_{ii} E_{ii} \right) + \left( \sum_{i=1}^{n} a_{ii} E_{ii} \right) \left( \sum_{i<j} b_{ij} E_{ij} \right) \]
\[ + \left( \sum_{i=1}^{n} b_{ii} E_{ii} \right) \left( \sum_{i<j} a_{ij} E_{ij} \right) + \left( \sum_{i=1}^{n} a_{ii} E_{ii} \right) \left( \sum_{i<j} b_{ij} E_{ij} \right) \]
\[ = \sum_{i=1}^{n} a_{ii} b_{ii} E_{ii} + \sum_{i<j} a_{ij} b_{ij} E_{ij}^2 + \sum_{i<j} c_{ij} E_{ij}, \text{ for some } c_{ij}. \]
\[ = \sum_{j=1}^{n} \sum_{i=1}^{j-1} \text{sgn}(\lambda_{ij}) a_{ij} b_{ij} + \sum_{i=j}^{n} \text{sgn}(\lambda_{ji}) a_{ji} b_{ji} + \sum_{i<j} d_{ij} E_{ij}, \]

for some \( d_{ij}. \) \( L(AB) \) has the following diagonal elements;

The component of \( E_{ii} \) of \( L(AB) E_{ii} \) is \( l_i. \)

The components of \( E_{ij} \) of \( L(AB) E_{ij} \) is \( \frac{1}{2} l_i + \frac{1}{2} l_j. \) Therefore

\[ \text{Tr}(L(AB)) = \sum_{i=1}^{n} l_i + \frac{1}{2} \sum_{i<j} (l_i + l_j) \]
\[ = \sum_{i=1}^{n} l_i + \frac{n-1}{2} \sum_{i=1}^{n} l_i \]
\[ = \frac{n+1}{2} \sum_{i=1}^{n} l_i. \]

\[ \square \]

**Corollary II.3.10.** \( V_{p,q} \) is semisimple.

**Proof.** \( \sum_{i=1}^{n} l_i = 0 \) for all \( b_{ij} \) implies that \( a_{ij} = 0. \)

\[ \square \]
II.3.1. The Jordan algebra $V_{1,1}$

Consider the Jordan algebra

$$W = V_{1,1} := \{ \begin{bmatrix} x & y \\ -y & z \end{bmatrix} \mid x, y, z \in \mathbb{R} \}$$

induced from the symmetric non-degenerate bilinear form $J_{1,1}$. Let $c_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $c_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $c_1, c_2$ are orthogonal primitive idempotents generating the Euclidean Jordan algebra of all diagonal matrices $\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \mid x, y, \in \mathbb{R} \}$.

For $y \neq 0$, let $a = \sqrt{1 + 4y^2}$. Set

$$c_y := \begin{bmatrix} \frac{1}{2}(1 + a) & y \\ -y & \frac{1}{2}(1 - a) \end{bmatrix}, \quad c_{-y} := \begin{bmatrix} \frac{1}{2}(1 - a) & -y \\ y & \frac{1}{2}(1 + a) \end{bmatrix}.$$

**Remark II.3.11.**

1. $c_y, c_{-y}$ are orthogonal primitive idempotents.
2. $E(V_{1,1}) = \{ e, c_1, c_2, c_y, c_{-y} : y \in \mathbb{R}, y \neq 0 \}$.

Now let's look at the Pierce decompositions on $V_{1,1}$.

**Proposition II.3.12.**

1. $V(c_i, 0) = \mathbb{R} \cdot c_j, i, j = 1, 2$.
2. $V(c_i, 1) = \mathbb{R} \cdot c_i, i = 1, 2$.
3. $V(c_i, \frac{1}{2}) = \{ \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix} \mid y \in \mathbb{R} \}, i = 1, 2$.
4. $V(c_y, 0) = \{ \begin{bmatrix} \frac{2y}{(1+a)} & b \\ -b & \frac{2y}{(1-a)} \end{bmatrix} \mid b \in \mathbb{R} \}$.
5. $V(c_y, 1) = \{ \begin{bmatrix} \frac{2y}{(a-1)} & b \\ -b & \frac{-2y}{(1+a)} \end{bmatrix} \mid b \in \mathbb{R} \}$.
6. $V(c_y, \frac{1}{2}) = V(c_{-y}, \frac{1}{2}) = \{ \begin{bmatrix} \frac{2yb}{a} & -b \\ a & \frac{-2yb}{a} \end{bmatrix} \mid b \in \mathbb{R} \}$. 

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(7) \( V(c, y, 0) = \left\{ \begin{bmatrix} \frac{2y}{(1+a)} & b \\ -b & \frac{2y}{(a-1)} \end{bmatrix} \right\} \mid b \in \mathbb{R} \). 

(8) \( V(c, y, 1) = \left\{ \begin{bmatrix} \frac{2y}{(a+1)} & b \\ -b & \frac{2y}{(1-a)} \end{bmatrix} \right\} \mid b \in \mathbb{R} \).

Corollary II.3.13. \( V_{1,1} \) is a simple Jordan algebra.

Proof. By the Proposition II.3.12, \( V(c, \frac{1}{2}) \neq 0 \) for all idempotents \( c \). Thus \( V_{1,1} \) is simple Jordan algebra, by the lemma I.2.1. \( \Box \)

II.3.2. The Jordan algebra \( V_{2,2} \)

Let \( J_{2,2} \) be the matrix representation of the non-degenerate symmetric bilinear form \( \sigma(x, y) = (x|y) := x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4 \) on \( \mathbb{R}^4 \). Let

\[
X = \sum_{i=1}^{4} a_{ii}E_{ii} + \sum_{i<j} a_{ij}E_{ij}
\]

be an idempotent of \( V_{2,2} \). Then

\[
X \quad = \quad \sum_{i=1}^{4} a_{ii}^2E_{ii}^2 + \sum_{i<j} a_{ij}^2E_{ij}^2 + 2 \sum_{i=1}^{4} a_{ii}E_{ii} \sum_{i<j} a_{ij}E_{ij}
\quad = \quad (a_{11}^2 + a_{12}^2 - a_{13}^2 - a_{14}^2)E_{11}
\quad + \quad (a_{22}^2 + a_{12}^2 - a_{23}^2 - a_{24}^2)E_{22}
\quad + \quad (a_{33}^2 - a_{13}^2 - a_{23}^2 + a_{34}^2)E_{33}
\quad + \quad (a_{44}^2 - a_{14}^2 - a_{24}^2 + a_{34}^2)E_{44}
\quad + \quad (a_{11}a_{12} + a_{22}a_{12} - a_{13}a_{23} - a_{14}a_{24})E_{12}
\quad + \quad (a_{11}a_{13} + a_{33}a_{13} + a_{12}a_{23} + a_{14}a_{34})E_{13}
\quad + \quad (a_{11}a_{14} + a_{44}a_{14} + a_{12}a_{24} + a_{13}a_{34})E_{14}
\quad + \quad (a_{22}a_{23} + a_{33}a_{23} + a_{12}a_{13} + a_{24}a_{34})E_{23}
\quad + \quad (a_{22}a_{24} + a_{44}a_{24} + a_{12}a_{14} + a_{23}a_{34})E_{24}
\quad + \quad (a_{33}a_{34} + a_{44}a_{34} - a_{13}a_{14} - a_{24}a_{23})E_{34}.
\]
Proposition II.3.14. Let

\[ A = \sum_{i=1}^{4} a_{ii}E_{ii} + \sum_{i<j} a_{ij}E_{ij} \in V_{2,2} \]

be an idempotent and let \( A_i = (a_{11}, a_{12}, a_{33}, a_{44}) \). Then \( A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \) and

\[
(A_1|A_1) = a_{11}, \\
(A_2|A_2) = a_{22}, \\
(A_1|A_2) = a_{12}, \\
(A_2|A_3) = -a_{33}, \\
(A_3|A_4) = -a_{44}, \\
(A_4|A_4) = -a_{44},
\]

\( i = 1,2,3, j = 3,4. \)

Let \( c_1 = E_{11} + E_{22} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, c_2 = E_{33} + E_{44} = \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix}, c_3 = E_{11} + \\
E_{22} + E_{33} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, c_4 = E_{11} + E_{33} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, c_5 = E_{11}. \) Then \( c_i \) are

idempotents in \( V_{2,2}. \) Now we look at the Pierce decompositions with respect to these idempotents. Set \( V_2 := Sym(2,\mathbb{R}). \)

Proposition II.3.15.

(1)

\[
V(c_1,0) = V(c_2,1) = \{ \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} | A \in V_2 \},
\]

\[
V(c_1,1) = V(c_2,0) = \{ \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} | A \in V_2 \},
\]

\[
V(c_1, \frac{1}{2}) = V(c_2, \frac{1}{2}) = \{ \begin{bmatrix} 0 & B \\ -B^t & 0 \end{bmatrix} | B \in M(2,\mathbb{R}) \}.
\]
\[(2)\]

\[V(c_3, 0) = \{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x \end{bmatrix} \mid x \in \mathbb{R} \},\]

\[V(c_3, 1) = \{ \begin{bmatrix} A & y & 0 \\ -y & z & 0 \\ 0 & 0 & x \end{bmatrix} \mid A \in \mathbb{V}_2, x, y, z \in \mathbb{R} \},\]

\[V(c_3, \frac{1}{2}) = \{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & y & 0 \\ -y & z & x \end{bmatrix} \mid x, y, z \in \mathbb{R} \}.\]

\[(3)\]

\[V(c_4, 0) = \{ \begin{bmatrix} 0 & 0 \\ 0 & x \\ 0 & 0 \\ 0 & -y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid A \in \mathbb{V}_2, x, y, z \in \mathbb{R} \},\]

\[V(c_4, 1) = \{ \begin{bmatrix} z & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mid A \in \mathbb{V}_2, x, y, z \in \mathbb{R} \},\]

\[V(c_4, \frac{1}{2}) = \{ \begin{bmatrix} 0 & x \\ z & 0 \\ 0 & 0 \\ -y & 0 \end{bmatrix} \mid A \in \mathbb{V}_2, x, y, z, w \in \mathbb{R} \}.\]

\[(4)\]

\[V(c_5, 0) = \{ \begin{bmatrix} 0 & 0 \\ 0 & x \\ 0 & 0 \\ 0 & -y \end{bmatrix} \mid A \in \mathbb{V}_2, x, y \in \mathbb{R} \},\]

\[V(c_5, 1) = \{ \begin{bmatrix} x & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mid x, y \in \mathbb{R} \},\]

\[V(c_5, \frac{1}{2}) = \{ \begin{bmatrix} 0 & x \\ 0 & y \\ z & 0 \\ -y & 0 \end{bmatrix} \mid x, y \in \mathbb{R} \}.\]
II.3.3. $V_{p,q}$ is simple

**Lemma II.3.16.** Let $c$ be an idempotent of a Jordan algebra $V$. If $V(c, \frac{1}{2}) = 0$, then $L(c) = L(c)^2$.

**Proof.** Suppose that $V(c, \frac{1}{2}) = 0$. Then from the Pierce decomposition with respect to the idempotent $c$, $V = V(c, 0) \oplus V(c, 1)$. Let $x \in V$. Then $x = x_0 + x_1$, where $x_0 \in V(c, 0)$ and $x_1 \in V(c, 1)$. Since $cx_0 = 0$ and $cx_1 = x_1$, $cx = x_1$. Note that the projection of $V$ onto $V(c, 1)$ is $P(c)$. Therefore $cx = x_1 = P(c)x$ for all $x \in V$. Therefore $L(c) = P(c)$. However, $P(c) = 2L(c)^2 - L(c^2) = 2L(c)^2 - L(c)$ implies that $L(c) = L(c)^2$. □

**Theorem II.3.17.** $V_{p,q}$ is simple.

**Proof.** From the lemma I.2.1, we show that if $V(c, \frac{1}{2}) = 0$, then $c$ is identity of $V_{p,q}$, i.e., the identity matrix. By the lemma II.3.16, we know that $L(c) = L(c)^2$. That is, $cx = c(cx)$, for all $x \in V_{p,q}$. Let $c = \sum_{k=1}^n a_{kk} E_{kk} + \sum_{k<l} a_{kl} E_{kl}$. Then

$$cE_{ii} = a_{ii}E_{ii} + \left( \sum_{k<i} a_{ki} E_{ki} + \sum_{i<l} a_{il} E_{il} \right) E_{ii}$$

$$= a_{ii}E_{ii} + \frac{1}{2} \left( \sum_{k<i} a_{ki} E_{ki} + \sum_{i<l} a_{il} E_{il} \right).$$

Note that in this representation, the coefficient of $E_{ii}$ is $a_{ii}$ and the coefficient of $E_{jj}$ is 0 for $i \neq j$. However,

$$c(cE_{ii}) = \left[ \sum_{k=1}^n a_{kk} E_{kk} + \sum_{k<l} a_{kl} E_{kl} \right] \left[ a_{ii}E_{ii} + \frac{1}{2} \left( \sum_{k<i} a_{ki} E_{ki} + \sum_{i<l} a_{il} E_{il} \right) \right]$$

$$= a_{ii} \left[ a_{ii}E_{ii} + \frac{1}{2} \left( \sum_{k<i} a_{ki} E_{ki} + \sum_{i<l} a_{il} E_{il} \right) \right]$$

$$+ \frac{1}{2} \left( a_{ii} E_{ii} + \sum_{k<i} a_{ki} E_{ki} + \sum_{i<l} a_{il} E_{il} \right) \left( \sum_{k<i} a_{ki} E_{ki} \right)$$

$$+ \frac{1}{2} \left( a_{ii} E_{ii} + \sum_{k<i} a_{ki} E_{ki} + \sum_{i<l} a_{il} E_{il} \right) \left( \sum_{i<l} a_{il} E_{il} \right).$$

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The coefficient of $E_{jj}$ in $c(cE_{ii})$ for $j \neq i$:

$$\frac{1}{2}a_{jj}^2 \text{sgn}(\lambda_{jj}), \text{ for } j < i.$$ 

$$\frac{1}{2}a_{ij}^2 \text{sgn}(\lambda_{ij}), \text{ for } i < j.$$ 

Therefore $a_{jj} = 0$, for all $i \neq j$, by comparing the coefficient of $E_{ij}$ in $cE_{ii}$ and $c(cE_{ii})$, respectively. Hence $c$ is a diagonal matrix. But, the coefficient of $E_{ii}$ in $c(cE_{ii})$ is $a_{ii}^2$. On the other hand, the coefficient of $E_{ii}$ in $cE_{ii}$ is $a_{ii}$. Therefore, $a_{ii} = \pm 1$. Since $c$ is an idempotent and diagonal matrix, $c$ must be the identity matrix. This completes the proof. □

II.4. The Cone $\Omega_{p,q}$

Let $V_{p,q}^{-1}$ be the set of all invertible elements in the Jordan algebra $V_{p,q}$. Note that an element in $V_{p,q}$ is invertible with respect to the Jordan algebra if and only if it is a non-singular operator. In general, an element $x$ in a Jordan algebra $V$ is invertible if and only if the quadratic representation

$$P(x) = 2L(x)^2 - L(x^2)$$

is non-singular. Note that in the Jordan algebra $V_{p,q}$, $P(A)B = ABA$.

Let

$$\Omega_n = \{A \in \text{Sym}(n, \mathbb{R}) | A > 0\}$$

be the set of positive definite operators with respect to the usual inner product $\langle \cdot | \cdot \rangle$. Then $\Omega_n$ is a symmetric cone of the Euclidean Jordan algebra $\text{Sym}(n, \mathbb{R})$.

Let $\Omega_{p,q}$ be the set of all self-adjoint positive definite operators with respect to the bilinear form $\langle \cdot | \cdot \rangle$, i.e.,

$$\Omega_{p,q} := \{A \in V_{p,q} | (Ax|x) > 0, \forall x \in V - \{0\}\}.$$
Then $\Omega_{p,q}$ is a proper convex cone invariant under the adjoint action of pseudo-orthogonal group $H$ where

$$H = \{ g \in GL(n, \mathbb{R}) \mid (gx|x) = (x|x), x \in V \}$$

$$= \{ g \in GL(n, \mathbb{R}) \mid g^* = g^{-1} \}.$$

Let $\Omega_{p,q}^1$ be the identity component of $V_{p,q}^{-1}$ and

$$\Omega_{p,q}^2 = \{ gg^* | g \in GL(V) \}.$$

From the theorem 1.2.12, for $\text{Sym}(n, \mathbb{R})$, we have

$$\exp V_n = \Omega_{n}^1 = \Omega_{n}^2.$$

Note that the Jordan algebras $V_{p,q}$ and $\text{Sym}(n, \mathbb{R})$ are subalgebras of $\text{gl}(n, \mathbb{R})$ the space all linear transformations on $\mathbb{R}^n$.

**Proposition II.4.1.** As subsets of $\text{gl}(n, \mathbb{R})$, $\Omega_{p,q} \cap \Omega_n = \emptyset.$

**Proof.** Suppose $A \in \Omega_{p,q} \cap \Omega_n$. Then $(Ax|x) = (J_{p,q}Ax|x) > 0$ and $(Ax|x) > 0$, for all $x \neq 0$ in $V$. Therefore $A \in \Omega_n$ and $J_{p,q}A \in \Omega_n$. Since $\Omega_n$ is convex, $A + J_{p,q}A = (I + J_{p,q})A \in \Omega_n$. In particular, every element of $\Omega_n$ is non-singular.

Hence $I + J_{p,q}$ is invertible which gives a contradiction. \(\square\)

**Proposition II.4.2.** $\exp V_{p,q} \subset \Omega_{p,q}^2 \subset \Omega_{p,q}^1$.

**Proof.** Let $X \in V_{p,q}$. Then $X^* = J_{p,q}X^tJ_{p,q} = X$. Hence

$$X = \frac{1}{2}(X + X^*)$$

$$= \frac{1}{2}X + \frac{1}{2}X^*$$

$$= \frac{1}{2}(X + J_{p,q}X^tJ_{p,q}).$$

Since $X$ and $X^*$ are commute,

$$\exp X = \exp(\frac{1}{2}X)\exp(J_{p,q}(\frac{1}{2}X)^tJ_{p,q})$$

$$= \exp(\frac{1}{2}X)J_{p,q}\exp(\frac{1}{2}X)^tJ_{p,q}.$$
So exp $X$ is of the form $gg^*$. Hence $\exp V_{p,q} \subset \Omega_{p,q}^2$. To show that $\Omega_{p,q}^2 \subset \Omega_{p,q}^1$, we claim that

$$\Omega_{p,q}^2 = \{hh^* | h \in GL(n, \mathbb{R})^+\}$$

since the right-hand side is the image of the following continuous map:

$$GL(n, \mathbb{R})^+ \rightarrow GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R}), h \rightarrow (h, h^*) \rightarrow hh^*.$$ 

Suppose that $q$ is odd. Let $g \in GL(n, \mathbb{R})$. Then

$$(gJ_{p,q})(gJ_{p,q})^* = (gJ_{p,q})(J_{p,q}^*g^*) = gg^*.$$ 

If $\det(g) < 0$, then $\det(gJ_{p,q}) = \det(g)\det(J_{p,q}) > 0$. Therefore,

$$\{gg^* | g \in GL(n, \mathbb{R})\} = \{hh^* | h \in GL(n, \mathbb{R})^+\}.$$ 

Now suppose that $q$ is even. Then we have two cases:

Case 1. $n$ is odd. In this case, $\det(-I) < 0$. From this, $(-g)(-g)^* = gg^*$ and $\det(-g) = \det(-I)\det(g)$. Thus any element of the form $gg^*$ can be written by $hh^*$ with $\det(h) > 0$.

Case 2. $n$ is even. Let $A = \begin{bmatrix} -1 & 0 \\ 0 & I_{n-1} \end{bmatrix}$. Then

$$(gAJ_{p,q})(gAJ_{p,q})^* = (gAJ_{p,q})(J_{p,q}^*A^*g^*)$$

$$= (gAJ_{p,q})(J_{p,q}^*J_{p,q}A^tJ_{p,q}g^*)$$

$$= gAJ_{p,q}^tA^tJ_{p,q}g^*$$

$$= gAJ_{p,q}AJ_{p,q}g^*$$

$$= gg^*.$$ 

Since $\det(A) < 0$, by the similar method, every element of the form $gg^*$ can be written by $hh^*$ with $\det(h) > 0$. This completes the proof. \hfill \Box
REMARK II.4.3.

(1) $\Omega_{p,q} \subset \Omega_{p,q}^1$, or $\Omega_{p,q} \cap \Omega_{p,q}^1 = \emptyset$. It is from the connectness of $\Omega_{p,q}$ and $\Omega_{p,q} \subset V_{p,q}^{-1}$.

(2) $\Omega_{p,q}^2 \cap \Omega_{p,q} = \emptyset$. For if $g^*g \in \Omega_{p,q}$, then

$$0 < (g^*gx|x) = (gx|gx) = (J_{p,q}gx|gx) = (g^tJ_{p,q}gx|x),$$

for all non-zero vector $x \in \mathbb{R}$. Therefore $g^tJ_{p,q}g$ is positive definite with respect to the standard inner product, hence

$$g^tJ_{p,q}g = \alpha \alpha^t$$

for some $\alpha \in GL(n, \mathbb{R})$. This implies that

$$J_{p,q} = (g^t)^{-1}\alpha \alpha^t g^{-1} = ((g^t)^{-1}\alpha)((g^t)^{-1}\alpha)^t.$$

Therefore $J_{p,q}$ is positive definite with respect to the inner product, which gives a contradiction. □

Consider the following action:

$$GL(V) \times gl(V) \to gl(V), \ g \cdot X = gXg^t.$$

Since $g(hh^*)g^* = gh(gh)^*$, $GL(V)$ acts on $\Omega_{p,q}^2$. If $X \in \Omega_{p,q}^2$, then $(gXg^*)x = (Xg^t)x|g^*x > 0$ for any $x \neq 0$. Hence $GL(V)$ acts on $\Omega_{p,q}$. Furthermore, since $(gXg^*)^* = gX^tg^* = gXg^*$ for $X \in V_{p,q}$, $GL(V)$ acts on the Jordan algebra $V_{p,q}$. Under this action, the group $H$ is embedded into the automorphism group of $V_{p,q}$.

EXAMPLE II.4.4. $V_{1,1}$: From the Proposition II.4.2, $\exp V_{1,1} \subset \Omega_{p,q}^2 \subset \Omega_{p,q}$. Note that $SO(2) \in V_{1,1}^{-1}$. Let

$$g = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \ h = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$
Then \( gg^* = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \Omega_{1,1}, hh^* = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \Omega_{1,1} \). But \( gg^* + hh^* = 0 \) is not in \( \Omega_{1,1} \). Thus \( \Omega_{1,1} \) is not convex. And \( \Omega_{1,1} \cap \Omega_{1,1} = \emptyset \), since \( \begin{bmatrix} 2 & \frac{1}{2} \\ -\frac{1}{2} & -1 \end{bmatrix} \) in \( \Omega_{1,1} \) but not in \( \Omega_{1,1} \).

Note that
\[
y^2 + xz = y^2 + \left(\frac{x + z}{2}\right)^2 - \left(\frac{x - z}{2}\right)^2
\]
hence the set of all singular elements in \( V_{1,1} \) is the boundary of the standard Lorenz cone. Furthermore, \( V_{1,1}^{-1} \) can be decomposed into \( \Omega_{1,1} \) and \( \Omega_{1,1}^1 \).

**Proposition II.4.5.** \( J_{p,q}V_{p,q} = \text{Sym}(n, \mathbb{R}) \). In particular, \( J_{p,q}\Omega_{p,q} = \Omega_n \), where \( \Omega_n \) is the symmetric cone of \( \text{Sym}(n, \mathbb{R}) \).

**Proof.** First, note that if \( A \in V_{p,q} \), then \( J_{p,q}A \in \text{Sym}(n, \mathbb{R}) = V_{n,0} \). For
\[
J_{p,q}A = J_{p,q}A^* = J_{p,q}J_{p,q}A^tJ_{p,q} = A^tJ_{p,q}
\]
which implies that \((J_{p,q}A)^t = J_{p,q}A\). Hence \( J_{p,q}A \) is a symmetric operator. Conversely, if \( A \in \text{Sym}(n, \mathbb{R}) \), then \((J_{p,q}A)^* = J_{p,q}X^t = J_{p,q}A\). Hence \( J_{p,q}A \in V_{p,q} \).

Therefore \( J_{p,q}V_{p,q} = \text{Sym}(n, \mathbb{R}) \). Now suppose that \( A \in \Omega_{p,q} \). Then \( (Ax|x) > 0 \), for all non-zero element \( x \) in \( V \). Since \((Ax|x) = (J_{p,q}A|J_{p,q}A)\), \( J_{p,q}A \) is a symmetric positive definite operator. So \( J_{p,q}\Omega_{p,q} \subset \Omega_n \). The converse argument is similar. \( \square \)

Now we denote \( J_{p,q}^R \) by the right multiplication:
\[
J_{p,q}^R : gl(n, \mathbb{R}) \to gl(n, \mathbb{R}), A \in gl(n, \mathbb{R}) \to AJ_{p,q}.
\]
And let \( J_{p,q}^L \) be the left multiplication:
\[
J_{p,q}^L : gl(n, \mathbb{R}) \to gl(n, \mathbb{R}), A \in gl(n, \mathbb{R}) \to J_{p,q}A.
\]

From the Proposition II.4.5, \( J_{p,q}^L \) are non-singular (involutive) linear transformation sending \( V_{p,q} \) onto \( \text{Sym}(n, \mathbb{R}) \). Let
\[
G(\Omega_n) := \{ g \in GL(V_n) \mid g\Omega_n = \Omega_n \}, \quad G(\Omega_{p,q}) := \{ g \in GL(V_{p,q}) \mid g\Omega_{p,q} = \Omega_{p,q} \}.
\]
PROPPOSITION II.4.6. $G(\Omega_{p,q}) = J^L_{p,q} \circ G(\Omega_n) \circ J^L_{p,q}$.

PROOF. For $\alpha \in G(\Omega_n)$, the mapping

$$\alpha \rightarrow J^L_{p,q} \circ \alpha \circ J^L_{p,q}$$

is an element of $G(\Omega_{p,q})$. □

PROPPOSITION II.4.7. $P(J_{p,q}w) = J^R_{p,q} \circ P(w) \circ J^L_{p,q}$, for $w \in V_{p,q}$.

PROOF. $P(J_{p,q}w)z = J_{p,q}wzJ_{p,q}w = J_{p,q}(wzJ_{p,q}w) = (J^L_{p,q} \circ P(w) \circ J^L_{p,q})(z)$. □

COROLLARY II.4.8. $P(\Omega_{p,q}) = J^L_{p,q} \circ P(\Omega_n) \circ J^R_{p,q}$.

PROOF. It is from the fact that $\Omega_{p,q} = J_{p,q} \Omega_n$ and the proposition II.4.7. □

LEMMA II.4.9. $\alpha \circ J^L_{p,q} \circ J^R_{p,q} \circ \beta \in \text{Aut}(V_n)$ for any $\alpha, \beta \in \text{Aut}(V_n)$.

PROOF. Let $x, y \in V_n$. Then $[\alpha \circ J^L_{p,q} \circ J^R_{p,q} \circ \beta](x \circ y)$

$$= [\alpha \circ J^L_{p,q} \circ J^R_{p,q}](\beta(x) \circ \beta(y))$$

$$= \frac{1}{2}(\alpha \circ J^L_{p,q})[\beta(x)\beta(y)J_{p,q} + \beta(y)\beta(x)J_{p,q}]$$

$$= \frac{1}{2}\alpha[J_{p,q}\beta(x)\beta(y)J_{p,q} + J_{p,q}\beta(y)\beta(x)J_{p,q}]$$

$$= \frac{1}{2}\alpha[J_{p,q}\beta(x)J_{p,q}\beta(y)J_{p,q} + J_{p,q}\beta(y)J_{p,q}\beta(x)J_{p,q}].$$

Since $J_{p,q}\beta(x)J_{p,q}, J_{p,q}\beta(y)J_{p,q} \in V_n$, it is equal to

$$= \alpha(J_{p,q}\beta(x)J_{p,q}) \circ \alpha(J_{p,q}\beta(y)J_{p,q})$$

$$= (\alpha \circ J^L_{p,q} \circ J^R_{p,q} \circ \beta)(x) \circ (\alpha \circ J^L_{p,q} \circ J^R_{p,q} \circ \beta)(y).$$

Therefore $\alpha \circ J^L_{p,q} \circ J^R_{p,q} \circ \beta \in \text{Aut}(V_n)$. □

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LEMMA II.4.10. $J_R^{p,q} \circ J_L^{p,q} \circ \alpha \circ J_R^{p,q} \circ J_L^{p,q} \in \text{Aut}(V_n)$, for every $\alpha \in \text{Aut}(V_n)$.

PROOF. Let $x, y \in V_n$. Then $(J_R^{p,q} \circ J_L^{p,q} \circ \alpha \circ J_R^{p,q} \circ J_L^{p,q})(x \circ y)$

\[
= \frac{1}{2}(J_R^{p,q} \circ J_L^{p,q} \circ \alpha \circ J_R^{p,q} \circ J_L^{p,q})(xy + yx)
= \frac{1}{2}(J_R^{p,q} \circ J_L^{p,q} \circ \alpha)(J_R^{p,q}xyJ_R^{p,q} + J_R^{p,q}yxJ_R^{p,q})
= \frac{1}{2}(J_R^{p,q} \circ J_L^{p,q})(\alpha(J_R^{p,q}xJ_R^{p,q}yJ_R^{p,q} + J_R^{p,q}yJ_R^{p,q}xJ_R^{p,q}))
= \frac{1}{2}(J_R^{p,q}J_R^{p,q}xJ_R^{p,q})\alpha(J_R^{p,q}yJ_R^{p,q})J_R^{p,q} + J_R^{p,q}\alpha(J_R^{p,q}yJ_R^{p,q})\alpha(J_R^{p,q}xJ_R^{p,q})J_R^{p,q}
= J_R^{p,q}J_R^{p,q}xJ_R^{p,q}J_R^{p,q} \circ J_R^{p,q}J_R^{p,q}yJ_R^{p,q}J_R^{p,q}
= [J_R^{p,q} \circ J_L^{p,q} \circ \alpha \circ J_R^{p,q} \circ J_L^{p,q}](x) \circ [J_R^{p,q} \circ J_L^{p,q} \circ \alpha \circ J_R^{p,q} \circ J_L^{p,q}](y).
\]

\[\square\]

THEOREM II.4.11. $G(\Omega_{p,q}) = P(\Omega_{p,q}) \circ J_R^{p,q} \circ \text{Aut}(V_n) \circ J_L^{p,q}$. In particular, $J_R^{p,q} \circ \text{Aut}(V_n) \circ J_L^{p,q}$ is a closed subgroup of $G(\Omega_{p,q})$.

PROOF.

\[
G(\Omega_{p,q}) = J_L^{p,q} \circ G(\Omega_n) \circ J_L^{p,q}
= J_L^{p,q} \circ P(\Omega_n) \circ \text{Aut}(V_n) \circ J_L^{p,q}
= J_L^{p,q} \circ P(\Omega_n) \circ J_R^{p,q} \circ \text{Aut}(V_n) \circ J_L^{p,q}
= P(\Omega_{p,q}) \circ J_R^{p,q} \circ \text{Aut}(V_n) \circ J_L^{p,q},
\]

from the corollary II.4.8.

To show that $J_R^{p,q} \circ \text{Aut}(V_n) \circ J_L^{p,q}$ is a closed subgroup of $G(\Omega_{p,q})$, let $\alpha, \beta \in \text{Aut}(V_n)$. Then

\[
(J_R^{p,q} \circ \alpha \circ J_L^{p,q}) \circ (J_R^{p,q} \circ \beta \circ J_L^{p,q})
\]

is in $J_R^{p,q} \circ \text{Aut}(V_n) \circ J_L^{p,q}$, since $\alpha \circ J_R^{p,q} \circ \beta \circ J_L^{p,q} \in \text{Aut}(V_n)$ by the lemma II.4.10.
Finally, since the inverse of \( J_{pq}^R \circ \alpha \circ J_{pq}^L \) is

\[
J_{pq}^L \circ \alpha^{-1} \circ J_{pq}^R = J_{pq}^R \circ \alpha \circ J_{pq}^L \circ \alpha^{-1} \circ J_{pq}^R \circ J_{pq}^L,
\]

by the lemma II.4.10, which is in \( J_{pq}^R \circ \text{Aut}(V_n) \circ J_{pq}^L \).

\[ \square \]

**Remark II.4.12.** \( J_{pq}^R \circ \text{Aut}(V_n) \circ J_{pq}^L \neq \text{Aut}(V_{pq}). \)

**Example II.4.13.** First, note that for \( \alpha \in \text{Aut}(V_n) \), \( J_{pq}^R \circ \alpha \circ J_{pq}^L \) fixes \( J_{pq} \), since \( [J_{pq}^R \circ \alpha \circ J_{pq}^L](J_{pq}) = (J_{pq}^R \circ \alpha)(I) = J_{pq} \).

Consider the following mapping:

\[
V_{1,1} \rightarrow V_{1,1}, \quad \begin{bmatrix} a & b \\ -b & c \end{bmatrix} \rightarrow \begin{bmatrix} c & b \\ -b & a \end{bmatrix}
\]

Then it is an automorphism of \( V_{1,1} \) but it does not fix \( J_{1,1} \).

\[ \square \]

**Remark II.4.14.** Let \( E \) be a real or complex Hilbert space with the inner product \( \langle \cdot | \cdot \rangle \). For \( X \in \mathcal{L}(E) \) of bounded operators, \( X^* \) is denoted by the adjoint operator of \( X \). Then

\[
V = \mathcal{H}(E) = \{ X \in \mathcal{L}(E) \mid X^* = X \}
\]

is a \( JB \)-algebra under the anti-commutator product

\[
X \circ Y = \frac{1}{2}(XY + YX).
\]

Now let \( J \in \mathcal{H}(E) \) be an involution. Then \( JVJ = V \). The subspace \( JV \) is a Jordan algebra of \( V \) under the anti-commutator product since if \( JX, JY \in JV \), then \( JX \circ JY = \frac{1}{2}(JXJY + JYJX) = J\left(\frac{1}{2}(XJY + YJX)\right) \) and \( (XJY + YJX) \in V \). We denote this Jordan subalgebra by \( V_J \).

However, the space \( \mathcal{H}(E) \) is a Jordan algebra under the new product

\[
X \cdot Y = \frac{1}{2}(XJY + YJX).
\]

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We denote $V^0$ by the Jordan algebra under this new product. Note that $V = V^0$ as underling set. Also the subspace $JV$ is a Jordan subalgebra of $V^0$. Similiary, we use $V_j^0$ as the Jordan subalgebra of $V^0$ which underling set is $JV = V_j$.

The mapping

$$X \in V_j^0 \rightarrow JX \in V$$

is an Jordan algebra isomorphism.

Let $\sigma(u,v) = (Ju|v)$. Then $\sigma(u,v)$ is a non-degenerate symmetric (Hermitian) bilinear form. For $X \in \mathcal{L}(E), X^\#$ denote the adjoint operator of $X$ with respect to the bilinear form $\sigma(u,v)$. Then the Jordan algebra

$$V_J = \{ X \in \mathcal{L}(E) \mid X^\# = X \}$$

is isomorphic to the Jordan algebra $V_J$. By proposition II.4.6., $\Omega_{p,q}$ is a symmetric cone and it comes from the Euclidean Jordan algebra $V_{p,q}^0$. 

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Chapter III
Lie Semigroups Associated with Symmetric Cones

III.1. Compression semigroups of cones

Let \( G \) be a Lie group with Lie algebra \( L(G) \) and \( S \) be a closed subsemigroup. The tangent wedge of \( S \) is defined by

\[
L(S) = \{ X \in L(G) \mid \exp \mathbb{R}^+ X \subset S \}.
\]

Then it is a Lie wedge, i.e.,

\[
e^{adX} L(S) = L(S)
\]

for all \( X \in L(S) \cap -L(S) \). A closed subsemigroup of a Lie group is called a Lie semigroup if

\[
S = \langle \exp L(S) \rangle.
\]

The systematic groundwork for a Lie theory of semigroups was worked out by K.H.Hofmann, J.Hilgert and J.D.Lawson [16] (cf. [36]).

In a real or complex vector space \( V \) with a cone \( C \), we are very interested in a semigroup associated the cone \( C \), namely the compression semigroup

\[
\text{Compr}(C) = \{ T \in GL(V) \mid T(C) \subset C \}.
\]

This semigroup is always closed in \( GL(V) \).

Let \( V \) be a finite dimensional real ( complex ) vector space endowed with a non-degenerate symmetric (Hermitian ) bilinear form \( (u|v) \). Then one of cones which is canonically related to the form is the following:

\[
C^+ = \{ v \in V \mid (v|v) \geq 0 \}.
\]
Let $C^- = \{ v \in V \mid (v|v) \leq 0 \}$ and $C^0 = C^+ \cap C^-$. Then

$$C^0 = \{ v \in V \mid (v|v) = 0 \}$$

and $V = C^+ \cup C^-$. 

There are two different semigroups which are canonically related to the form. The first semigroup is the contraction semigroup

$$S^\leq = \{ T \in GL(V) \mid (Tv|^Tv) \leq (v|v), \forall v \in V \}$$

and the expansion semigroup

$$S^\geq = \{ T \in GL(V) \mid (Tv|^Tv) \geq (v|v), \forall v \in V \}.$$

Remark III.1.1.

$$S^\leq \subset \text{Compr}(C^-), \quad S^\geq \subset \text{Compr}(C^+).$$

For $T \in g = gl(V)$, let $T^*$ be the adjoint operator of $T$ associated the bilinear form $(\cdot | \cdot)$. Then $(Tv|^u) = (u|^T^*v)$ for all $u, v \in V$. Let

$$g_+ = \{ T \in g \mid T^* = -T \},$$

$$g_- = \{ T \in g \mid T^* = T \}.$$

Then $g = g_+ \oplus g_-$. 

Remark III.1.2. $g_+$ is a Lie subalgebra of $g$ and $g_-$ is a Jordan algebra. We have studied the Jordan structure of $g_-$ in chapter II.

Now we may assume that the bilinear form $(u|v)$ is

$$J_{p,q}(u,v) = \sum_{i=1}^{p} r_i \bar{s}_i - \sum_{i=p+1}^{n} r_i \bar{s}_i,$$

$$J_{p,q}(x,y) := \sum_{i=1}^{p} r_i \bar{s}_i - \sum_{i=p+1}^{n} r_i \bar{s}_i$$

by Sylvester's law of inertia. Let $\Omega_{p,q}$ be the open cone of self adjoint positive definite operators with respect to $(\cdot | \cdot)$ and $W_{p,q}$ be the closure of $\Omega_{p,q}$. Then
THEOREM III.1.3. [Ol’shanskii Decomposition]

(1) Real case:

\[ S^\leq = O(p, q) \exp(W_{pq}), \]
\[ S^\geq = O(p, q) \exp(-W_{pq}) \]

where \( O(p, q) \) is the pseudo-orthogonal group of the bilinear form \( J_{p,q} \).

(2) Complex case:

\[ S^\leq = U(p, q) \exp(W_{pq}), \]
\[ S^\geq = U(p, q) \exp(-W_{pq}) \]

where \( U(p, q) \) is the unitary group of the bilinear form \( J^C_{p,q} \).

PROOF. (cf. [29], [30], [36]). \( \square \)

Let \( E \) be a real or complex Hilbert space with inner product \( \langle x|y \rangle \). Let \( \mathcal{L}(E) \) be the Banach algebra of bounded operators on \( E \). For \( M \in \mathcal{L}(E) \), we denote \( M^t \) (\( M^* \), complex case) be the adjoint operator of \( M \). If \( M = M^t \) (\( M = M^* \)), we say \( M \) is symmetric (Hermitian, complex case). A symmetric or Hermitian operator \( M \) is positive definite, written \( M > 0 \), if \( \langle Mx|x \rangle > 0 \) whenever \( x \neq 0 \).

Members \( M \in \mathcal{L}(E \times E) \) have a block decomposition

\[ M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, A, B, C, D \in \mathcal{L}(E). \]

Let \( J \in \mathcal{L}(E) \) be defined in block form by

\[ J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}. \]

Note that \( J^2 = -I \) and hence \( J^{-1} = -J = J^* \).
We define the skew-symmetric (skew-Hermitian, in complex case) form on $E \times E$ by

$$(x|y) = (Jx|y), \quad x, y \in E \times E.$$  

We denote by $M^*$ for $M \in \mathcal{L}(E \oplus E)$ the unique linear operator such that $(Mx|y) = (x|M^*y)$ for all $x, y \in E \times E$. Since $(Jx|y) = -(x|y)$ and $(x|J^#y) = \langle x|J^*J^#y \rangle = -(x|JJ^#y)$, $J^# = J^{-1} = -J$. So

$$(x|M^*y) = (Mx|y) = (JMx|y) = -(x|M^*Jy) = (Jx|M^*Jy) = (x|J^#M^*Jy)$$

implies that $M^* = -JM^tJ(-JM^*J)$.

We set

$$G = \{ M \in \mathcal{L}(E \oplus E) \mid (Mu|Mv) = (u|v) \}.$$  

**Proposition III.1.4.** Let $E$ be a Hilbert space and let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{L}(E \times E)$. Then the following are equivalent:

1. $M \in G$, i.e., $M$ preserves $(\cdot|\cdot)$.
2. $M^tJM = J$ ($M^*JM = J$, complex case).
3. $A^tC, B^tD$ ($A^*D, B^*D$) are symmetric (Hermitian, complex case) and $A^tD - C^tB = I$ ($A^*D - C^*B = I$).
4. $M$ is invertible with inverse $M^*$.

**Definition III.1.5.** If $E = \mathbb{R}^n$, then the group $G$ is called the symplectic group which is denoted by $\text{Sp}(2n, \mathbb{R})$.

Now let $Q(u) = \langle x|y \rangle$ be the quadratic form on $V := E \times E$, where $u = (x, y)$. Note that the quadratic form $Q$ gives a non-degenerate symmetric (Hermitian) form which is defined by

$$\sigma(u, v) = Q(u).$$  

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Then the cone $C$ corresponding to the bilinear form $\sigma$ is

$$C = \{ u \in V \mid Q(u) \geq 0 \} = \{ u = (x, y) \in V \mid \langle x | y \rangle \geq 0 \}.$$ 

Set

$$C^0 = \{ u \in V \mid Q(u) = 0 \}.$$

Then $J(C) \cap C = C^0$. In next section, we study the compression semigroup of the cone $C$ on the group $G$:

$$S := \text{Comp}(C) \cap G.$$

Note that this semigroup $S$ is closed in $G$.

### III.2. Decomposition of $S$

In the finite dimensional real Hilbert space case, the semigroup $S$ is completely characterized by Wojtkowski [47] (cf. [28]). We give the proof in more detail on finite dimensional complex Hilbert space.

**Theorem III.2.1.** [Wojtkowski] For $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp(2n, \mathbb{R})$, The following properties are equivalent:

1. $Q(\gamma u) \geq Q(u), \forall u \in E \oplus E.$
2. $g \in S,$
3. $A \in GL(n, \mathbb{R})$ and $A^tC \geq 0, BA^t \geq 0.$
4. $D \in GL(n, \mathbb{R})$ and $CD^t \geq 0, D'B \geq 0.$

Let

$$H = \{ \begin{bmatrix} A^{-1} & 0 \\ 0 & A^* \end{bmatrix} \mid A \in GL(E) \}.$$
**Lemma III.2.2.** $H$ is the group of units of $S$.

**Theorem III.2.3.** Let $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G$. Then the following are equivalent:

(a) $Q(gu) \geq Q(u)$, for all $u \in E \times E$.

(b) $g \in S$.

(c) $A$ is invertible and $A^*C \geq 0$ and $BA^* \geq 0$.

(d) $D$ is invertible and $CD^* \geq 0$ and $D^*B \geq 0$.

**Corollary III.2.4.** $S = S^g \cap G$, where $S^g$ is the expansion semigroup associated with the cone $C$.

**Remark III.2.5.** By definition, (a) implies (b).

The proof of the theorem is from the following lemmas.

**Lemma III.2.6.** Let $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G$. Then if $g \in S$, then $A$ and $D$ are invertible.

**Proof.** Suppose $g \in S$ and $Ax_0 = 0$. Since $D^*A - B^*C = I$, $B^*C(x_0) = -(x_0)$. Let $y = sx_0$. Then $(x_0|y) = s(x_0|x_0) \geq 0$, for all $s \geq 0$. Hence $v = (x_0, y) \in C$ so $gv \in C$. But $gv = (By, Cx_0 + Dy) \in C$ implies that

$$
\langle By|Cx_0 + Dy \rangle = \langle By|Cx_0 \rangle + \langle By|Dy \rangle
$$

$$
= \langle y|B^*Cx_0 \rangle + \langle y|B^*Dy \rangle
$$

$$
= -(y|x_0) + \langle y|B^*Dy \rangle \geq 0.
$$

Hence $(x_0|x_0) \leq s\langle x_0|B^*Dx_0 \rangle \to 0$, as $s \to 0$. Hence $x_0 = 0$. Therefore $A$ is invertible. If $Dx_0 = 0$, then $x_0 = -C^*Bx_0$. Let $y = sx_0$. Then $u = (sx_0, x_0) \in C$ for any positive real number $s$ and by using the same method of the case $A$, we obtain a contradiction. \(\square\)
LEMMA III.2.7. If $g \in S$, then

(1) $A^*C \geq 0, A^{-1}B \geq 0$ and hence $BA^* \geq 0$.

(2) $D^*B \geq 0, D^{-1}C \geq 0$ and hence $CD^* \geq 0$.

(3) $BD^{-1} \geq 0$.

PROOF. (1) $A^*C \geq 0$. By definition, $A^*C$ is Hermitian. Set

$$g_0 = \begin{bmatrix} A^{-1} & 0 \\ 0 & A^* \end{bmatrix}.$$ 

By lemma III.2.6, $g_0 \in H \subset S$. Since $A^*D = I + C^*jB$,

$$g_1 = g_0 g = \begin{bmatrix} I & R \\ P & I + P^*R \end{bmatrix} \in S,$$

where

$$R = A^{-1}B, P = A^*C.$$ 

For $u = (x, 0)$,

$$Q(g_1u) = (x|A^*Cx) = (x|A^*Cx) \geq 0.$$ 

Hence $P = A^*C \geq 0$.

(2) $R = A^{-1}B \geq 0$. Since $g_1 = g_0 g = \begin{bmatrix} I & R \\ P & I + P^*R \end{bmatrix} \in S \subset G$,

$$(A^{-1}B)[I + (A^*C)^*A^{-1}B] = [I + (A^{-1}B)^*(A^*C)](A^{-1}B).$$

This implies that

$$B^*A^{-1^*} + B^*A^{-1^*}C^*AA^{-1}B = A^{-1}B + B^*A^{-1^*}A^*CA^{-1}B.$$ 

But $A^*C = C^*A$ and $A$ is invertible by lemma III.2.6, $CA^{-1} = A^{-1^*}C^*$. Therefore, $B^*A^{-1^*}$ and hence $A^{-1}B$ is Hermitian. To show that $A^{-1}B \geq 0$, 

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suppose that \((A^{-1}B)y_0|y_0\) \(<\ 0\). Choose \(x_0 \in E\) such that \((x_0|y_0)\ <\ 0\). Let \(v = (sx_0 - Ry_0, y_0)\). Then

\[
(sx_0 - Ry_0) = s(x_0|y_0) - (Ry_0|y_0) > 0
\]

for sufficiently small \(s > 0\). So \(v \in C\) for sufficiently small \(s > 0\). Since \(g_1 \in S\), \(g_1v \in C\). But \(g_1v = (sx_0, sA^*Cx_0 + y_0)\) and

\[
Q(g_1v) = (sx_0|sP x_0 + y_0)
\]

\[
= s^2(x_0|P x_0) + s(x_0|y_0)
\]

\[
= s^2(x_0|P x_0) + s(x_0|y_0) < 0
\]

for sufficiently small \(s > 0\). which leads a contradiction. Therefore \(A^{-1}B \geq 0\).

(3) \(BA^* \geq 0\). \(BA^* = AA^{-1}BA^* \geq 0\).

(4) \(D^*B \geq 0\). By definition, \(D^*B\) is Hermitian. Let \(h_0 = \left[\begin{array}{cc} D^* & 0 \\ 0 & D^{-1} \end{array}\right] \in H \subset S\).

Then \(h_1 = h_0g \in S\) and applying with \((0, y)\) to \(g_1\) we get \(D^*B \geq 0\).

(5) \(D^{-1}C\) is Hermitian. Using the facts that \(h \in G\) and \(B^*D = D^*B\), one can show it via the same method of (2).

(6) \(D^{-1}C \geq 0\). Suppose that \((D^{-1}C)y_0|y_0\) \(<\ 0\). Choose \(x_0 \in E\) such that \((x_0|y_0)\ <\ 0\). Then \(v = (y_0|sx_0 - D^{-1}C y_0)\) \(\in C\) for sufficiently small \(s > 0\). Since \(h_1 \in S, h_1v \in C\). But \(Q(h_1v) < 0\) for small \(s > 0\) which gives a contradiction. Therefore, \(D^{-1}C \geq 0\).

(7) \(CD^* \geq 0\). \(CD^* = DD^{-1}CD^* \geq 0\).

(8) \(BD^{-1} \geq 0\). Since \(D^*B = B^*D, BD^{-1} = (D^{-1})^*B^*\) and hence \(BD^{-1}\) is Hermitian. Since by (1), \(D^*B \geq 0\) implies that \(BD^{-1} = (-1)^*D^*BD^{-1} \geq 0\). □

**Lemma III.2.8.** If \(A\) is invertible and \(A^*C \geq 0, BA^* \geq 0\) then \(Q(gu) \geq Q(u), \forall u \in E \oplus E\).
Clearly \( Q(gu) = Q(g_1 u) = (x + Ry|P^*x + P^*Ry) \) since \( P^* = P \).

And it is equal to

\[
= (x + Ry|y) + (x + Ry|P^*(x + Ry)) = (x|y) + (Ry|y) + (P(x + Ry)|x + Ry) \geq (x|y) = Q(u).
\]

\[\square\]

**Lemma III.2.9.** If \( D \) is invertible and \( CD^* \geq 0, D^*B \geq 0 \), then \( Q(gu) \geq Q(u) \), for all \( u \in E \oplus E \).

**Proof.**\( h_1 = h_0 g = [I + P'R' \begin{array}{cc} P' \\ R' \end{array} I], \) where \( P' = D^*B \geq 0, R' = D^{-1}C. \)

Note that \( P' \geq 0 \) implies that \( R' \geq 0 \) for \( R' = D^{-1}CD^*D^{-1} \geq 0 \).

\[
Q(gu) = Q(h_1 u) = (x + P'(R'x + y)|R'x + y) = (x|R'x) + (x|y) + (P(R'x + y)|(R'x + y)) \geq (x|y) = Q(u).
\]

\[\square\]

Set

\[
N^+ = \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} | A \in \mathcal{H}(E),
\]

\[
N^- = \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} | A \in \mathcal{H}(E),
\]

\[
H = \begin{bmatrix} (A')^{-1} & 0 \\ 0 & A \end{bmatrix} | A \in GL(E),
\]

\[
\Gamma^+ = \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} | A \geq 0,
\]

\[
\Gamma^- = \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} | A \geq 0.
\]

Then \( N^\pm \) are abelian subgroups of \( G \).
Theorem III.2.10. \( S = \Gamma^+HT^{-} \).

Proof. Obviously, \( \Gamma^+HT^{-} \subset S \). Suppose that \( M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in S \). Then by lemma III.2.6 and III.2.7, \( D \) is invertible and \( BD^{-1} \geq 0, D^{-1}C \geq 0 \). Hence

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} (D^*)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix} \in \Gamma^+HT^{-}.
\]

\[\square\]

III.3. The semigroup \( S_j \)

Let \( E \) be a \( n \)-dimensional Euclidean space with inner product \( \langle x, y \rangle \). Let \( \langle x, y \rangle \) be a non-degenerate symmetric bilinear form on \( E \) and let \( j \) be an involution on \( E \) such that

\[
\langle x, y \rangle = \langle x, jy \rangle,
\]

for all \( x, y \) in \( E \). We extend the inner product on \( E \) into \( E \times E \) by: For \( u_i = (x_i, y_i) \in E \times E \),

\[
\langle u_1 | u_2 \rangle = \langle x_1 | x_2 \rangle + \langle y_1 | y_2 \rangle.
\]

Set

\[
J = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix}.
\]

Now define a non-degenerate skew-symmetric bilinear form on \( E \times E \) by

\[
\langle u_1 | u_2 \rangle = \langle x_1 | y_2 \rangle - \langle x_2 | y_1 \rangle = \langle Ju_1 | u_2 \rangle.
\]

For \( g \in \text{gl}(E \times E) \), we denote \( g^f \) by the adjoint operator with respect to the inner product \( \langle u | v \rangle \). We define the sympletic group with respect to the bilinear form \( j \).
DEFINITION III.3.1. $\text{Sp}^j(E) = \{g \in GL(E \times E) \mid (gu|gv) = (u|v)\}$.

Note that if $g \in \text{Sp}^j(E)$, then

$$
\langle g^t J gu \mid v \rangle = \langle J gu \mid gu \rangle \\
= (gu|gv) \\
= (u|v) \\
= \langle Ju|v \rangle.
$$

The converse argument is also true. Hence the symplectic group with respect to $j$ is

$$
\text{Sp}^j(E) = \{g \in GL(E \times E) \mid (gu|gv) = (u|v)\} \\
= \{g \in GL(E \times E) \mid g^t J g = J\} \\
= \{g \in GL(E \times E) \mid g^* = g^{-1}\}
$$

Here $g^*$ is the adjoint operator of $g$ with respect to the symplectic form $(u|v)$. Furthermore, note that every element $g$ in $GL(E \times E)$ can be written as a block matrix:

$$
g = \begin{bmatrix} A & B \\ C & D \end{bmatrix},
$$

where $A, B, C, D \in GL(E)$. So by solving the equation $g^t J g = J$, we have

$$
\text{Sp}^j(E) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mid A^t j C = C^t j A, D^t j B = B^t j D, D^t j A - B^t j C = j \right\}.
$$

Let $\Phi(u) = (x|y)$ be the quadratic form on $V := E \times E$, where $u = (x, y)$. And let

$$
C = \{u \in V \mid \Phi(u) \geq 0\}.
$$

Then $C$ is a cone in $V$.

Now we consider the compression semigroup of the cone $C$ in the symplectic group $\text{Sp}^j(E)$.
**Definition III.3.2.** \( S_j = \{ g \in \text{Sp}^j(E) \mid gC \subset C \} \).

Since \( C \) is closed, the compression semigroup is a closed set in \( S_p^j(E) \). The group of unit \( H_j \) of \( S_j \) is given by:

\[
H_j = \{ g \in \text{Sp}^j(E) \mid gC = C \}.
\]

For \( j = 1 \), the symplectic group \( \text{Sp}^j(E) \) is the standard symplectic group \( \text{Sp}(E) \). The semigroup \( S_j \) can be characterized as the Wojtkowski theorem. The proof is the same as the theorem III.2.3.

**Theorem III.3.3.** Let \( g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Sp}^j(E) \). Then the following are equivalent:

(a) \( \Phi(gu) \geq \Phi(u) \), for all \( u \in E \times E \).

(b) \( g \in S_j \).

(c) \( A \) is invertible and \( A^tjC \geq 0 \) and \( BjA^t \geq 0 \).

(d) \( D \) is invertible and \( CjD^t \geq 0 \) and \( D^tjB \geq 0 \).

**Proposition III.3.4.** Suppose that \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in S_j \). Then \( BD^{-1}j \) and \( jD^{-1}C \) are positive and symmetric.

**Proposition III.3.5.** Suppose that \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in S_j \). Then

1. \( \begin{bmatrix} j & BD^{-1}j \\ 0 & j \end{bmatrix} \in S_j \).
2. \( \begin{bmatrix} j & 0 \\ jD^{-1}C & j \end{bmatrix} \in S_j \).
3. \( \begin{bmatrix} (D^t)^{-1} & 0 \\ 0 & jDj \end{bmatrix} \in S_j \).
**Theorem III.3.6.** Let

\[ \Gamma_j^+ = \left\{ \begin{bmatrix} j & A \\ 0 & j \end{bmatrix} \mid A \in \text{Sym}(n, \mathbb{R}), A \geq 0 \right\}, \]

\[ \Gamma_j^- = \left\{ \begin{bmatrix} j & 0 \\ A & j \end{bmatrix} \mid A \in \text{Sym}(n, \mathbb{R}), A \geq 0 \right\}, \]

\[ H_j = S_j \cap S_j^{-1} = \left\{ \begin{bmatrix} A^{-1} & 0 \\ 0 & jA \end{bmatrix} \mid A \in \text{GL}(n, \mathbb{R}) \right\}. \]

Then

\[ S_j = \Gamma_j^+ H_j \Gamma_j^-. \]

**Proof.** \[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} j & BD^{-1}j \\ 0 & j \end{bmatrix} \begin{bmatrix} (D^{-1})^t & 0 \\ jD_j & jD^{-1}C \end{bmatrix} \begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix}. \]

Let

\[ \alpha = \begin{bmatrix} j & 0 \\ 0 & I \end{bmatrix}, \quad \beta = \begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix}. \]

Then \( \alpha, \beta \in \text{GL}(2n, \mathbb{R}) \) are involutions and \( \alpha \beta = \gamma = \begin{bmatrix} I & 0 \\ 0 & j \end{bmatrix} \). For \( g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Sp}^j(2n, \mathbb{R}) \),

\[ a\alpha = \begin{bmatrix} jA & jB \\ C_j & D \end{bmatrix}. \]

**Theorem III.3.7.** The mapping

\[ g \in \text{Sp}^j(2n, \mathbb{R}) \to a\alpha \in \text{Sp}(2n, \mathbb{R}) \]

gives an isomorphism between \( \text{Sp}^j(2n, \mathbb{R}) \) and \( \text{Sp}(2n, \mathbb{R}) \).

**Proof.** Since \( A^jC = C^jA \),

\[ (jA)^tC_j = jA^tjC_j = jC^tjA_j = (Cj)^t(jA). \]

And \( D^tjB = B^tjD = (jB)^tD \). Now

\[ D^t(jA) - (jB)^tC_j = (D^tjA - B^tjC)j = j^2 = I. \]
Hence the mapping is well-defined. Because $\alpha$ is an involution, it is not hard to see that $\alpha$ is an isomorphism. \hfill \Box

**Corollary III.3.8.** $\gamma S_j \gamma = S$.

**Proof.** Note that

\[
a \Gamma_j^\dagger \alpha = \beta \Gamma^+, \\
a \Gamma_j^\dagger \alpha = \Gamma^- \beta, \\
a H_j \alpha = \beta H \beta = H.
\]

Therefore,

\[
a S_j \alpha = a \Gamma_j^\dagger \alpha \cdot a H \alpha \cdot a \Gamma_j^- \alpha \\
= \beta \Gamma^+ \cdot \beta H \beta \cdot \Gamma^- \beta \\
= \beta \Gamma^+ \cdot H \cdot \Gamma^- \beta.
\]

Hence $\beta a S_j \alpha \beta = S$. \hfill \Box

Set

\[
S_j^+ = \{ \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} | A j \geq 0 \}, \\
S_j^- = \{ \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} | j A \geq 0 \}.
\]

Then by proposition III.3.4, $S_j^\pm \subset S_j$.

**Theorem III.3.9.**

\[ S_j = S_j^+ H_j S_j^- . \]

**Proof.**

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & B D^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} j (D^{-1})^t j & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1} C & I \end{bmatrix} . \hfill \Box
\]
III.4. The semigroup $\Gamma_\Omega$

Let $V = \text{Sym}(n, \mathbb{R})$ and $\Omega_n$ be the open convex cone of positive definite $n$ by $n$ symmetric matrices. Then $V$ is a simple Euclidean Jordan algebra with the symmetric cone $\Omega_n$. It is well-known that any biholomorphic automorphisms on the tube domain $T_\Omega^n = V + i\Omega_n$ is the following form

$$Z \in T_\Omega^n \mapsto \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}$$

for some $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Sp}(2n, \mathbb{R})$. Hence the sympletic group

$$\text{Sp}(2n, \mathbb{R}) = \{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mid A^t C = C^t A, D^t B = B^t D, D^t A - B^t C = I \}$$

acts on the tube domain $T_\Omega^n = V + i\Omega_n$ by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}.$$

For complex Hilbert space $E$, the $JB$-algebra $\mathcal{H}(E)$ of the space of Hermitian operators has the symmetric cone $\Omega = \{ T \in \mathcal{H}(E) \mid T > 0 \}$ and the tube domain $T_\Omega = \mathcal{H}(E) + i\Omega$ is a symmetric domain. But the group $\text{Aut}(T_\Omega)$ of biholomorphic automorphisms on $T_\Omega$ has two components and every element in the identity component of $\text{Aut}(T_\Omega)$ is of the form

$$Z \in T_\Omega^n \mapsto \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}$$

for some $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G$ (cf. [23], Example 5.5). Hence the group $G$

$$G = \{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mid A^* C = C^* A, D^* B = B^* D, D^* A - B^* C = I \}$$

acts on the tube domain $T_\Omega = V + i\Omega$ via fractional transformation:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}.$$
For a real or complex Hilbert space $E$, let $\Omega$ be the open convex cone of positive definite operators in JB-algebra $V = \mathcal{H}(E)$. Define a subsemigroup $\Gamma_n$ by the set of $G$ which is defined on $\Omega \subseteq V^\sigma$ and $g \cdot \Omega \subseteq \Omega$. Simply

$$\Gamma_n = \{ g \in G \mid g \cdot \Omega \subseteq \Omega \}.$$

In this section, we will show that the following result for a finite dimensional Hilbert space $E$:

**Theorem III.4.1.** $\Gamma_n = S = \Gamma^+ HT^-.$

**Remark III.4.2.** Note that $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G$ and $D \in GL(E)$ implies that

$$A = (D^*)^{-1} + BD^{-1}C.$$ (*)

In this case, $g$ can be decomposed as

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} (D^*)^{-1} & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}.$$

**Lemma III.4.3.** For $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G$, The following properties are equivalent:

1. $g \in N^+ HN^-$,
2. $g$ can be defined at $0 \in V^\sigma$ and $g \cdot 0 \in V$,
3. $D \in GL(E)$.

**Proof.** Obviously, (1) $\implies$ (2) $\implies$ (3). Suppose that $D$ is invertible. Then $D^*B = B^*D$ implies that $BD^{-1} = (D^{-1})^*B^* = (BD^{-1})^*$. Therefore $BD^{-1}$ is symmetric. By (*), $A^* = D^{-1} + C^*BD^{-1}$. Since $D^*B = B^*D$, $C^*BD^{-1}C = C^*(D^{-1})^*B^*C$ is symmetric. But $C^*BD^{-1}C = C^*(D^{-1})^*B^*C$ is

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symmetric. Therefore $D^*C + C^*BD^{-1}C$ is symmetric. So

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} (D^*)^{-1} & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix} \in N^+HN^-.$$

\[\square\]

**Remark III.4.4.** Obviously $\Gamma^+G(\Omega)\Gamma^- \subseteq \Gamma_n$. Hence by theorem III.2.10, $S = \Gamma^+HT^- \subseteq \Gamma_n$.

**Lemma III.4.5.** $N^+HN^- \cap \Gamma_n = S = \Gamma^+HT^-.$

**Proof.** Suppose that $g = n+hn^- = \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \begin{bmatrix} (D^*)^{-1} & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ B & I \end{bmatrix} \in \Gamma_n.$ Then for $X \in \Omega$, $X(BX + I)^{-1} = (B + X^{-1})$ is an invertible element in $V = H(E)$. This implies that

$$B + \Omega \subseteq V^{-1}.$$

By the argument in linear algebra or by theorem I.2.12, $B + \Omega \subseteq \Omega$. In particular, $B \in \overline{\Omega}$. To show $A \in \overline{\Omega}$, choose $Z \in B + \Omega \subseteq \Omega$. Then $nZ \in B + \Omega$, for all natural number $n$. This is from the induction argument: Let $Z = B + X \in B + \Omega$. Then

$$nZ = nB + nX = B + (n - 1)B + nX \in B + \Omega + \Omega \subseteq B + \Omega.$$

But

$$A + \frac{1}{n}h(Z^{-1}) \in n^+h((B + \Omega)^{-1}) = n^+hn^-((\Omega) \subseteq \Omega.$$

Thus $A \in \overline{\Omega}$. \[\square\]

**Theorem III.4.6.** $S = \Gamma^+HT^- = \Gamma_n$.

**Proof.** Suppose that $g \in \Gamma_n$. Let $t_n = \begin{bmatrix} I & \frac{1}{n}I \\ 0 & I \end{bmatrix}$. Then $gt_n \in \Gamma_n$ and $gt_n(0) = g(\frac{1}{n}I) \in \Omega$. therefore, by the lemma, $gt_n \in \Gamma^+HT^- = S$. Since $S$ is closed, $g \in S$. \[\square\]
Now we return to the Jordan algebra $V_{p,q}$ and $\text{Sym}(n,\mathbb{R})$. Set

$$
\Gamma_n = \Gamma_{\Omega_n},
$$

$$
\Gamma^+_n = \left\{ \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \mid A \in \text{Sym}(n,\mathbb{R}), A \geq 0 \right\},
$$

$$
\Gamma^-_n = \left\{ \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \mid A \in \text{Sym}(n,\mathbb{R}), A \geq 0 \right\}.
$$

From theorem III.4.6,

**Corollary III.4.7.**

$$
\Gamma_n = \Gamma^+_n G(\Omega_n) \Gamma^-_n.
$$

Set

$$
\Gamma_{p,q}^\pm = J_{p,q} \circ \Gamma_{\Omega_p}^\pm \circ J_{p,q}.
$$

The involutions $J_{p,q}^L$ can be extended from $V_{p,q}^\mathbb{C} := V_{p,q} + iV_{p,q}$ into $V_n^\mathbb{C} := V_n + iV_n$ as follows:

$$
J_{p,q}^L(x + iy) = J_{p,q} x + iJ_{p,q} y.
$$

Let $G_{p,q}$ be the group of all biholomorphisms of $V_{p,q} + i\Omega_{p,q}$. Then

**Theorem III.4.8.** $G_n$ and $G_{p,q}$ are isomorphic. In particular,

$$
G_{p,q} = J_{p,q}^L \circ G_n \circ J_{p,q}^L.
$$

**Proof.** The mapping

$$
G_n \longrightarrow G_{p,q}, g \rightarrow J_{p,q}^L \circ g \circ J_{p,q}^L
$$

is an isomorphism. It is clear that this mapping is a homomorphism. Suppose

$$
[J_{p,q}^L \circ g \circ J_{p,q}^L](x + iy) = x + iy \text{ for all } x \in V_{p,q}, y \in \Omega_{p,q}.
$$

If we let

$$
g(J_{p,q} x + iJ_{p,q} y) = u + iv,
$$

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then $x + iy = J_{p,q}u + iJ_{p,q}v$. So $x = J_{p,q}u, y = J_{p,q}v$. Therefore

$$g(J_{p,q}x + iJ_{p,q}y) = u + iv = J_{p,q}x + iJ_{p,q}y$$

which implies that the map is injective. \qed

**Remark III.4.9.** This isomorphism is the same as the isomorphism of

$$\text{Sp}^j(2n, \mathbb{R}) \to \text{Sp}(2n, \mathbb{R})$$

in theorem III.3.7 if $j = J_{p,q}$.

**Proposition III.4.10.** $\Gamma_{p,q} = J_{p,q}^L \circ \Gamma_n \circ J_{p,q}^L$.

**Proof.** Let $\alpha : V_n + i\Omega_n \to V_n + i\Omega_n$ be a biholomorphism such that $\alpha(\Omega_n) \subset \Omega_n$. Then

$$J_{p,q} \circ \alpha \circ J_{p,q}$$

is a biholomorphism of $V_{p,q} + i\Omega_{p,q}$ which sends $\Omega_{p,q}$ into itself. Hence $\Gamma_{p,q} \subset J_{p,q} \Gamma_n J_{p,q}$. Similarly we have the converse inclusion. \qed

**Theorem III.4.11.** $\Gamma_{p,q} = \Gamma_{p,q}^+ G(\Omega_{p,q}) \Gamma_{p,q}^-.$

**Proof.**

$$\begin{align*}
\Gamma_{p,q} &= J_{p,q}^L \circ \Gamma_n \circ J_{p,q}^L \\
&= J_{p,q}^L \circ \Gamma^+ G(\Omega_n) \Gamma^- \circ J_{p,q}^L \\
&= J_{p,q}^L \circ \Gamma^+ \circ J_{p,q}^L \circ J_{p,q}^L \circ G(\Omega_n) \circ J_{p,q}^L \circ J_{p,q}^L \circ \Gamma^- \circ J_{p,q} \\
&= \Gamma_{p,q}^+ G(\Omega_{p,q}) \Gamma_{p,q}^-.
\end{align*}$$

The isomorphism of the groups between $G_{p,q}$ and $G_n$ induces the isomorphism of $\text{Sp}^j(2n, \mathbb{R})$ and $\text{Sp}(2n, \mathbb{R})$ in theorem III.3.5. Furthermore, this isomorphism gives the isomorphism of the compression semigroups $S_j$ and $S$.}

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III.5. Generalization

Let $V$ be a $JB$-algebra with the symmetric cone $\Omega$. Define the semigroup $\Gamma_n$ by the set of $g \in G(T_n)$ such that $g$ can be extended to $\Omega$ and $g \cdot \Omega \subset \Omega$. Note that every element in $\Aut(T_n)$ can be extended on the complex Banach manifold $M$ which we constructed in chapter I. This complex manifold $M$ construction is due to the existance of $JB^*$-triple and a complex Banach Lie algebra $\mathfrak{a} = a_1 \oplus a_0 \oplus a_1$ containing $g(T_n)$. $\mathfrak{a}$ has no center and every continuous derivation of $\mathfrak{a}$ is inner. Therefore $\mathfrak{a}$ can be identified with the Lie algebra of the complex Lie group $\Aut(\mathfrak{a})$. If $L$ is the connected component of $\Aut(\mathfrak{a})$ and $H := \{g \in L \mid g(\mathfrak{h}) = \mathfrak{h}\}$, where $\mathfrak{h} = a_0 \oplus a_1$. The complex manifold is the homogeneous space $L/H$ which contains the Banach space $U = V^C$ as open subset via $\xi: z \in U \rightarrow \exp(z)H \in M$ [22]. Thus we can write simply

$$\Gamma_n = \{g \in G(T_n) \mid g(\Omega) \subset \Omega\}.$$

**PROPOSITION III.5.1.** The following properties are equivalent:

1. $g \in \Gamma_n$,
2. $g \exp(\Omega)G(\Omega)N^- \subset \exp(\Omega)G(\Omega)N^+$,
3. $g \exp(\Omega)H \subset \exp(\Omega)H$.

**PROOF.** Suppose that $g \in \Gamma_n$. Let $x \in \Omega$ and let $g(x) = y \in \Omega$. Then $gt_x(0) = t_y(0)$ implies that $t_{-y}gt_x(0) = 0$. Since $t_{-y}gt_x \in G(T_n)$, by proposition 2.1, $t_{-y}gt_x \in G(\Omega)N^-$. Therefore, $gt_x \in \exp(\Omega)G(\Omega)N^-$ which implies that $g \exp(\Omega)G(\Omega)N^- \subset \exp(\Omega)G(\Omega)N^+$.

$(2) \implies (3)$. It comes from the fact $G(\Omega)N^- \subset H$. 

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(3) $\Rightarrow$ (1). It comes from the fact that the group $G(T_{\Omega})$ is contained in $\text{Aut}(M)$. In finite dimensional cases, by the fact of $\xi$ equivalent action of the isometry group.

Remark III.5.2. Note that $G(T_{\Omega})/G(\Omega)N^{-}$ contains $V$ as dense open set (In finite dimensional cases, it is a compactification of $V$). Since $\Omega$ is open in $G(T_{\Omega})/G(\Omega)N^{-}$, the semigroup $\Gamma_{R}$ is closed.

Lemma III.5.3. Let $V$ be a JB-algebra with the corresponding symmetric cone $\Omega$. For $x \in V$, if $x + \Omega \subset V^{-1}$, then $x + \Omega \subset \Omega$. In particular, $x \in \overline{\Omega}$. Conversely if $x \in \overline{\Omega}$, then $x + \Omega \subset \Omega$.

Proof. Since $\Omega$ is open, we can choose $t \in R^{+}$, $w \in \Omega$ such that $tx + w \in \Omega$. Then

$$x + \frac{1}{t}w \in \Omega \cap x + \Omega.$$ 

This implies that $x + \Omega$ is a connected subset of $V^{-1}$ meeting the symmetric cone $\Omega$. Using the characterization of $\Omega$ we conclude that $x + \Omega \subset \Omega$. Hence

$$x + \overline{\Omega} \subset \overline{\Omega} \implies x \in \overline{\Omega}.$$ 

Suppose that $x \in \overline{\Omega}$. Then $x + \Omega$ is open in $\overline{\Omega}$ and hence it is a subset of $\Omega$. □

Set

$$\Gamma^{+} = \{ t_{x} | x \in \overline{\Omega} \},$$

$$\Gamma^{-} = \{ \tilde{t}_{x} | x \in -\overline{\Omega} \},$$

where $\tilde{t}_{x} = s \circ t_{x} \circ s$.

Proposition III.5.4. $\Gamma^{+}G(\Omega)\Gamma^{-} \subset \Gamma_{R}$.

Proof. Suppose that $g \in \Gamma^{+}G(\Omega)\Gamma^{-}$. Then

$$g = t_{x} \circ h \circ \tilde{t}_{-y},$$

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for some $x, y \in \overline{\Omega}$. Since the symmetry $s$ on the tube domain $T_\Omega$ can be extended to $\Omega$, we only need to see that $g\Omega \subset \Omega$. Let $w \in \Omega$. Then
\[ g(w) = [t_x \circ h \circ t_{-y}](w) \]
\[ = [t_x \circ h \circ s \circ t_{-y} \circ s](w) \]
\[ = [t_x \circ h \circ s](-y - w^{-1}). \]
Since $y \in \overline{\Omega}, w^{-1} \in \Omega$, $-y - w^{-1} \in -\Omega \subset V^{-1}$. Therefore
\[ s(-y - w^{-1}) = -(y + w^{-1})^{-1} \]
\[ = (y + w^{-1})^{-1} \in \Omega. \]
Since $h \in G(\Omega), h((y + w^{-1})^{-1}) \in \Omega$. Also $x \in \overline{\Omega}$ implies that
\[ x + h((y + w^{-1}))^{-1} \in x + \Omega \subset \Omega. \]
Hence $\Gamma^+G(\Omega)\Gamma^- \subset \Gamma$. □

**Lemma III.5.5.** $N^+G(\Omega)N^- \cap \Gamma_\Omega = \Gamma^+G(\Omega)\Gamma^-.$

**Proof.** Obviously $\Gamma_\Omega \subset N^+G(\Omega)N^- \cap S(\Omega)$. Conversely suppose $g = t_x \circ h \circ t_y \in S(\Omega)$. Then $\Omega - y \in V^{-1}$ implies that $-y \in \overline{\Omega}$ from the lemma 4.5. Let $w = -y \in \overline{\Omega}$. Then $w + \Omega \subset \Omega$. Choose $z \in w + \Omega$ and write $z = w + a \in w + \Omega$. Then
\[ 2z = 2w + 2a = w + (w + 2a) \in w + (w + \Omega) \subset w + \Omega. \]
By induction, $nz \in w + \Omega, \forall n \in N^+$. So $t_x h s(-(x + \Omega)) \subset \Omega$ implies that
\[ x + \frac{1}{n} h(z^{-1}) \in \Omega. \]
As $n \to \infty, x \in \overline{\Omega}$. □

**Theorem III.5.6.** $\Gamma^+G(\Omega)\Gamma^- \subset \Gamma_\Omega \subset \overline{\Gamma^+G(\Omega)\Gamma^-}$. In particular,
\[ \Gamma_\Omega = \overline{\Gamma^+G(\Omega)\Gamma^-}. \]
PROOF. Let \( g \in \Gamma_n \). Then \( g' := g \circ t_{-t} \in S \) and \( g'_0(0) \in \Omega \). By corollary I.4.1, \( g' \in N^+G(\Omega)N^- \). From lemma III.5.5, \( g'_0 \) must be in \( \Gamma^+G(\Omega)\Gamma^- \). As \( n \to \infty \), we get \( g \in \Gamma^+G(\Omega)\Gamma^- \). □

Recall the closed convex cone \( C = \overline{\Omega \oplus \theta(-\Omega)} \) in \( g(T_{\Omega}) \) which is invariant under the adjoint action of \( G(\Omega) \). Then \( G(\Omega) \exp C \) is a closed subsemigroup of \( G(T_{\Omega}) \) with its Lie wedge \( g(T_{\Omega})_0 \oplus C \) ([30], [39], [28]) in finite dimensional cases. Clearly the compression semigroup is contained in

\[
\Gamma_n = \overline{\Gamma^+G(\Omega)\Gamma^-} \subset G(\Omega) \exp C.
\]

Let \( D_0 \) be the open unit ball in \( V \) and \( R_\Omega = iV + \Omega \). Define

\[
p'(z) = (z - e)(z + e)^{-1},
\]

\[
c'(w) = (e + w)(e - w)^{-1}.
\]

Then \( p'(R_{\Omega}) = D \) and \( p'^{-1} = c' \). Note that \( c' = N^+G(\Omega)N^- \subset G(T_{\Omega}) \) and \( p'(\Omega) = D_0 \). Let \( S(D_0) \) be the compression semigroup of the open unit ball \( D_0 \). Then

\[
\Gamma_n = c'S(D_0)c'^{-1}.
\]

Furthermore,

\[
S(D_0) = G(T_{\Omega}) \cap S(D),
\]

where \( S(D) \) is the compression semigroup in \( G_C \) of the open unit ball [28].

For a \( JB \)-algebra, it is a one problem whether the corresponding compression semigroup \( \Gamma_n \) is closed or not. In Euclidean case, it is a closed subsemigroup using the locally compactness. Here we sujest several opne problems related to the compression semigroup \( \Gamma_n \).
Problems. 1. Suppose that $V$ is a Euclidean Jordan algebra. To show

$$
\Gamma_\Omega = \Gamma^+ G(\Omega) \Gamma^- = G(\Omega) \exp C,
$$

it is enough to show

(1) $\Gamma^+ G(\Omega) \Gamma^-$ is closed.

(2) $\Gamma_\Omega \subset N^+ G(\Omega) N^-.$

(3) $\exp C \subset N^+ G(\Omega) N^-.$

(4) For every $g \in S(D^0), e - g(-e)$ is invertible. That is, if we write $g(-e) = \sum \lambda_i c_i$ for some Jordan frame $\{c_i\}$, then since $g(-e) \in \overline{D^0}, |\lambda_i| < 1$ for all $i$.

In fact, these four properties are equivalent: If $\exp C \in N^+ G(\Omega) N^-$, then $\Gamma^+ G(\Omega) \Gamma^-$ is closed. Suppose that $t_{x_n} h_n t_{y_n} \in \Gamma^+ G(\Omega) \Gamma^-$ converges $g \in G(T_\Omega)$. Since $G(\Omega) \exp C$ is closed and it contains $\Gamma^+ G(\Omega) \Gamma^-$. Thus $g \in G(\Omega) \exp C$. By assumption and $G(\Omega)$ normalize $N^+$, we may assume that $g = t_x h t_y \in N^+ G(\Omega) N^-$. By continuity, $x_n \to x \in \overline{\Omega}, y_n \to y \in -\overline{\Omega}$ and $h_n$ converges to $h$. So $g \in \Gamma^+ G(\Omega) \Gamma^-$ and is closed. From this, (3) implies (1).

If $\Gamma^+ G(\Omega) \Gamma^-$ is closed, then by theorem III.5.5,

$$
\Gamma_\Omega = \Gamma^+ G(\Omega) \Gamma^- \subset N^+ G(\Omega) N^-.
$$

Suppose that $\Gamma_\Omega \subset N^+ G(\Omega) N^-$. Then by theorem III.5.4, $\Gamma_\Omega = \Gamma^+ G(\Omega) \Gamma^-$ and is closed by remark III.5.2. Therefore (1) and (2) are equivalent.

Finally suppose that $\Gamma^+ G(\Omega) \Gamma^-$ is closed. Then $\Gamma_\Omega = \Gamma^+ G(\Omega) \Gamma^-$ is closed semigroup. The Lie wedge of $\Gamma_\Omega$ contains $g(T_\Omega) \exp C$ which is the Lie wedge of $G(\Omega) \exp C$. Therefore

$$
G(\Omega) \exp C \subset \Gamma_\Omega = \Gamma^+ G(\Omega) \Gamma^- \subset N^+ G(\Omega) N^-.
$$
2. Is $\Gamma_\Omega$ a maximal subsemigroup of $G(T_\Omega)$? If $\Omega = \mathbb{R}^+$, then the compression semigroup $\Gamma_\Omega$ in $SL(2, \mathbb{R})$ is the subsemigroup of matrices of non-negative entries. It is well-known that this semigroup is maximal \cite{31},\cite{12}.

III.6. $SL(2, \mathbb{C})$

Let $G = SL(2, \mathbb{R}) = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, ad - bc = 1 \}$,

$g = sl(2, \mathbb{R}) = \{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a, b, c \in \mathbb{R} \}$,

$G_\mathbb{C} = SL(2, \mathbb{C})$,

$G(D) = SU(1,1) = \{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} : |a|^2 - |b|^2 = 1 \}$,

$su(1,1) = \{ \begin{bmatrix} a & b \\ b & \bar{a} \end{bmatrix} : a \in i\mathbb{R}, b \in \mathbb{C} \}$

$N^+ = \{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in \mathbb{R} \} \cong \mathbb{R}$, $\text{Lie}(N^+) = n^+ = \{ \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} : x \in \mathbb{R} \}$,

$N^- = \{ \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} : c \in \mathbb{R} \} \cong \mathbb{R}$, $\text{Lie}(N^-) = n^- = \{ \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} : x \in \mathbb{R} \}$,

$H = \{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a \in \mathbb{R} - \{0\} \} \cong \mathbb{R}^*$, $\text{Lie}(H) = \{ \begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix} : x \in \mathbb{R} \}$,

$P^+ = \{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in \mathbb{C} \} \cong \mathbb{C}$, $p^+ = \{ \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} : z \in \mathbb{C} \}$,

$P^- = \{ \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} : c \in \mathbb{C} \} \cong \mathbb{C}$, $p^- = \{ \begin{bmatrix} 0 & 0 \\ z & 0 \end{bmatrix} : z \in \mathbb{R} \}$,

$K_\mathbb{C} = \{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a \in \mathbb{C} - \{0\} \} \cong \mathbb{C}^*$

$G$ and $G(D)$ act on $T_\Omega$ and $D$ by linear fractional transformations:

$$g \cdot z = (az + b)(cz + d)^{-1},$$

where $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Furthermore, the group $G_\mathbb{C}$ acts on the Riemann sphere $M = \mathbb{C} \cup \{\infty\}$ under the linear fractional transformations.

Let $p(z) = i(z - i)(z + i)^{-1}$. Then $p = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \in G_\mathbb{C}$ and it maps the upper-half domain onto $D$ with the inverse which is called the Cayley Transform,

$$c = p^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \in G_\mathbb{C}.$$
Let 
\[ p' = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \in G, \quad \epsilon = p'^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \in G. \]

Then \( p' \) maps \( \Omega + i\mathbb{R} \) onto \( D \).

Now we have the mappings (Harish Chandra Embedding theorem):

\[ p^-_c = \text{Lie}(P^+_c) \rightarrow P^+_c \rightarrow G_c/K_cP^-_c \]

is a complex analytic diffeomorphism of \( p^-_c \) onto a dense open subset of \( G_c/K_cP^-_c \) that contains \( D \).

The Parabolic subgroup \( K_cP^- \) is the isotropy group at 0 in the action of \( G_c \) on \( M \). Let \( g = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \in G(D) \). Then

\[ g = \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & a \end{bmatrix} \in P^+K_cP^-_c. \]

Since \( K_cP^- \) is group, \( G(D)K_cP^-_c \subset P^+_cK_cP^-_c \). This leads to the Borel embedding:

\[ D \cong G(D)/K \rightarrow G_c/K_cP^-_c. \]

Under this embedding, \( D \) can be identified with

\[ \{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \mid b \in D \}K_cP^-_c \]

since

\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & g \cdot z \\ c & cz + d \end{bmatrix}, \]

for \( g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \).

However, the tube domain \( T_n \) can be identified with

\[ \{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \mid b \in T_n \}K_cP^-_c. \]
Now let $C = \begin{bmatrix} 0 & r \\ s & 0 \end{bmatrix} : r \geq 0, s \geq 0$. Then $C$ is a closed invariant cone in $gl(2,\mathbb{R})$ under the adjoint action of $H$. Then $g \cdot 0 \in C$ for every $g \in \exp C$.

Let $V = \mathbb{R}$ with usual multiplication. Then $V$ is a simple Euclidean Jordan algebra with the symmetric cone $\Omega = \mathbb{R}^+$ positive real numbers. The corresponding tube domain is the upper-half plane with the $x$-axis as the boundary. We can show that the automorphism group of this upper plane is $PSL(2,\mathbb{R})$ via the linear fractional transformation.

$$[a \ b] \begin{bmatrix} z \\ 1 \end{bmatrix} = \frac{az+b}{cz+d}.$$ 

Suppose that $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2,\mathbb{R})$ and $g$ can be extended on $\Omega := \mathbb{R}^+ \times \{0\}$ and sends $\Omega$ into itself. If $d = 0$, then $bc = -1$ and

$$g(r) = \frac{abr+b^2}{-r} > 0$$

which implies that $abr+b^2 < 0$ for all $r > 0$. But it is impossible since $Det(g) = ad-bc = 1$. Therefore $d \neq 0$. Thus we may assume that $d > 0$. Since $cr+d \neq 0$, $c$ must be positive. Furthermore if $b = 0$ then $a > 0$. On the other hand if $b \neq 0$ then $ar+b > 0$ for all $r \in \mathbb{R}^+$ hence $b > 0$ and $a > 0$. So the entries of $g$ are all non-negative. Obviously every fractional transformation of this type can be extended on $\Omega$ and sends $\Omega$ into itself. Set

$$\Gamma_n = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2,\mathbb{R}) \mid a,b,c,d \geq 0 \right\}.$$ 

Conversely if $g \in \Gamma_n$, then it decomposed into the following way: $g \in \Gamma_n$ implies that $d \neq 0$. For $r \in \mathbb{R}^+$, denote $h_r : \mathbb{R} \rightarrow \mathbb{R}$ by $h_r(x) = rx$. Let $s(z) = -z^{-1}$ be the symmetry on the upper half plane $\mathbb{R} + \mathbb{i}\mathbb{R}^+$. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(z) = [t_{\frac{1}{2}} \circ h_{d^{-1}} \circ s \circ t_{-\frac{1}{2}} \circ s](z).$$
Therefore

\[ \Gamma_n \subseteq \Gamma^+ G(\Omega) \Gamma^- . \]

Obviously \( \Gamma^+ G(\Omega) \Gamma^- \subseteq \Gamma_n \). Therefore

\[ \Gamma_n = \Gamma^+ G(\Omega) \Gamma^- = G(\Omega) \exp C . \]
Chapter IV
Möbius and Lorentzian Semigroups

There is a nice representation of Möbius geometry to Lorentzian geometry [41] via a ray bundle. The study of the compression semigroup of upper-half plane in a Riemann sphere and its representation theory in its Lorentzian coordinate was studied by Lawson [31]. In his paper, he has shown that this semigroup is a maximal subsemigroup of Möbius group and has the Ol'shanskii polar decomposition. However, we have seen that the space of Lorentzian coordinate has a Jordan algebra structure (spin factor). In this section, we work the compression semigroup in the side of Jordan algebra.

IV.1. Möbius groups

Let $E$ denote real Hilbert space with the inner product $\langle x|y \rangle$. An orthogonal transformation is a linear transformation on $E$ which preserves the inner product. The group of all orthogonal transformations is denoted $O(E)$.

For $a \in E$ and $r \in \mathbb{R}^+$, we denote by

$$t_a(x) = a + x$$

the translation by $a \in E$,

$$h_r(x) = rx$$

the dilation and

$$j(x) = \frac{x}{||x||^2}$$

the inversion or reflection in the unit sphere.
Set

\[ N^+ = \{ t_a \mid a \in E \}, \]
\[ A = \{ h_r \mid r \in \mathbb{R}^+ \}, \]
\[ N^- = \{ t_a \mid a \in E \} = j \circ N^+ \circ j, \]

where

\[ t_a := j \circ t_a \circ j. \]

Note that \( N^\pm, H \) are abelian groups under composition.

Set

\[ D = N^+ O(E) \]

the set of (rigid) motions on \( E \).

\[ AO(E) = \{ h_r \circ P \mid r \in \mathbb{R}^+, P \in O(E) \} \]

set of general orthogonal transformations.

\[ S(E) = N^+ \cdot A \cdot O(E) \]

set of all similarities of \( E \).

Denote \( \tilde{E} \) by the one-point compactification of \( E \). Write \( \tilde{E} = E \cup \{ \infty \} \).

A subset \( S \) of \( \tilde{E} \) is called a Möbius sphere if \( S = \{ x \in E \mid ||x - x_0|| = r \} \) is usual sphere in \( E \) or \( S = H \cup \{ \infty \} \), where \( H \) is a closed hyperplane in \( E \). Each of component of \( \tilde{E} \setminus S \) is called a spherical domain. The closure of either is obtained by taking its union with \( S \) is called a closed Möbius ball.

The group generated by the similarity group \( S(E) \) and the inversion \( j \) is called the Möbius group which we denote by the symbol \( M(E) \). An element of group is called a transformation. It is known \([31],[42]\) that transformations are precisely those bijections which preserve spheres and closed balls. From the definition, we get the following proposition.
**Proposition IV.1.1.** Let $P \in O(E), r \in \mathbb{R}^+, x \in E$.

1. $h_r \circ t_x = t_{rx} \circ h_r, \quad t_x \circ h_r = h_r \circ t_{hx}$.

2. $j \circ h_r \circ j = h_{\frac{1}{r}}$.

3. $h_r \circ \tilde{t}_{rx} = \tilde{t}_x \circ h_r, \quad h_r \circ \tilde{t}_x = \tilde{t}_{hx} \circ h_r$.

4. $P \circ t_x = t_{P(x)} \circ P, \quad t_x \circ P = P \circ t_{P^{-1}(x)}$.

5. $P \circ h_r = h_r \circ P$.

6. $P \circ j = j \circ P$.

**Proof.**

1. $(h_r \circ t_x)(y) = r(x + y) = rx + ry = (t_{rx} \circ h_r)(y)$.

2. $(j \circ h_r)(y) = j(ry) = \frac{y}{r} = \frac{y}{r||y||^2} = (h_{\frac{1}{r}} \circ j)(y)$.

3. From (1) and (2),

\[
\tilde{t}_x \circ h_r = j \circ t_x \circ (j \circ h_r) = j \circ t_x \circ (h_{\frac{1}{r}} \circ j)
= j \circ (t_x \circ h_{\frac{1}{r}}) \circ j = j \circ (h_{\frac{1}{r}} \circ t_x) \circ j = (j \circ h_{\frac{1}{r}}) \circ (t_x \circ j)
= (h_r \circ j) \circ (t_x \circ j) = h_r \circ \tilde{t}_{rx}.
\]

4. $P \circ t_x(z) = P(z + x) = P(x) + P(z) = t_{P(z)} \circ P(z)$.

5. $P \circ h_r(z) = P(rz) = rP(z) = h_r \circ P(z)$.

6. $P \circ j(z) = P(\frac{z}{||z||}) = \frac{1}{||z||^2}P(z) = j \circ P(z)$, since $P \in O(E)$. \hfill \Box

**Corollary IV.1.2.** The abelian group $A$ normalizes $N^\pm$. In particular, $N^+A$ and $AN^-$ are subgroups of group and $jN^+A j = AN^-$.

**Proof.**

$(h_r \circ \tilde{t}_x)(h_s \circ \tilde{t}_y) = h_r \circ (\tilde{t}_x \circ h_s) \circ \tilde{t}_y = h_r \circ h_s \circ \tilde{t}_{sx+{xy}} = h_{rs} \circ \tilde{t}_{sx+y} \in AN^-.$

Hence $AN^-$ is closed under multiplication. Since

\[(h_r \circ \tilde{t}_x)^{-1} = \tilde{t}_{-x} \circ h_{\frac{1}{r}} = h_{\frac{1}{r}} \circ \tilde{t}_{-hx} \in AN^-,
\]
$AN^-$ is a subgroup of $G$. From the definition, it is not hard to see that $jN^+Aj = AN^-$. □

The subgroup $N^+A$ is said to be a homothety group.

**Proposition IV.1.3.** $S(E)$ is a group with $D$ as normal subgroup.

**Proof.**

\[
(t_a \circ h_r \circ P) \circ (t_b \circ h_s \circ P') = t_a \circ h_r \circ t_{P_0(b)} \circ P \circ h_s \circ P' = t_{a+P_0(b)} \circ h_{rs} \circ P \circ P' \in S(E)
\]

And $(t_a \circ h_r \circ P)^{-1} = t_{P^{-1}(-r^{-1}a)} \circ h_{r^{-1}} \circ P^{-1}$. □

The multiplication on $E \times \mathbb{R}^+ \times O(E)$,

\[(a, r, P) \cdot (b, s, P') = (a + rP(b), rs, PP')\]

gives a group structure which is isomorphic to $S(E)$.

**Corollary IV.1.4.** $S(E) \cong E \times \mathbb{R}^+ \times O(E)$. The motion group $D = N^+ \cdot O(E)$ is the semi-direct product of the vector group $E$ and the orthogonal group $O(E)$. For, if $t_x, t_y \in N^+$ and $P, P' \in O(E)$, then

\[(t_x \circ P) \circ (t_y \circ P') = t_x \circ t_{P_0(y)} \circ P \circ P' = t_{x+P_0(y)} \circ P \circ P'.\]

A subgroup $G$ of group $M(E)$ is called a *subgroup of Möbius type* if $N^+A \subset G$ and $j \circ G \circ j = G$.

Let $G_1$ be the group of transformations generated by $N^\pm$ and $A$. Then from the proposition IV.1.1, we have
Corollary IV.1.5. $j \circ G_1 \circ j = G_1$.

Proof. From proposition IV.1.1,

\[
\begin{align*}
    j \circ t_x \circ j &= t_x \in N^- \subset G_1 \\
    j \circ h_r \circ j &= h_r \in A \subset G_1 \\
    j \circ t_x \circ j &= t_x \in N^+ \subset G_1
\end{align*}
\]

hence $j \circ G_1 \circ j \subset G_1$. Since $j^{-1} = j$, we get $j \circ G_1 \circ j = G_1$. □

If $G'$ is a subgroup of Möbius type, then $N^+, A \subset G'$. Since $j \circ G' \circ j = G'$, $N^- = j \circ N^+ \circ j \subset G'$. Hence the subgroup $G_1$ generated by $N^\pm$ and $A$ is contained in $G'$. Thus $G_1$ is the smallest subgroup of Möbius type. Hence we get

Theorem IV.1.6. The subgroup of the group $\mathcal{M}(\mathbb{E})$ which is generated by $N^\pm$ and $A$ the dilation group is the smallest subgroup of Möbius type.

It was proved by Lawson [31] that a subgroup of Möbius type acts transitively on the set of spherical domains. In particular, the group $G_1$, the smallest subgroup of Möbius type acts transitively on the set of spherical domains.

For $a \in \mathbb{E}$ and $r \in \mathbb{R}^+$, we denote by the inversion with respect to the sphere $S(a, r) := \{x \in \mathbb{E} \mid ||x - a|| = r\}$ by

\[
    j(a, r)(x) := a + \frac{r^2(x - a)}{||x - a||^2}.
\]

And we denote by $j(a, 1)$ be the inversion with respect to the unit sphere $S(a, 1)$.

Set

\[
\begin{align*}
    I(\mathbb{E}) &:= \{j(a, r) \mid a \in \mathbb{E}, \ r \in \mathbb{R}^2\} \\
    I_1(\mathbb{E}) &:= \{j(a, 1) \mid a \in \mathbb{E}\}.
\end{align*}
\]
Proposition IV.1.7. \( j(a, r) = (t_a \circ h_r) \circ j \circ (t_a \circ h_r)^{-1} \).

Corollary IV.1.8. \( N^+ \circ j \circ N^+ = I_1(E) \cdot N^+ \).

Proof. From the proposition IV.1.7,
\[
t_a \circ j = j(a, 1) \circ t_a \in I_1(E) \cdot N^+.
\]
Hence \( N^+ \circ j \circ N^+ \subset I_1(E) \cdot N^+ \). Conversely,
\[
j(a, 1) \circ t_z = t_a \circ j \circ t_{-a} \circ t_z = t_a \circ j \circ t_{-a} \in N^+ \circ j \circ N^+.
\]

Proposition IV.1.9. \( S(E) \circ j \circ S(E) = I_1(E) \cdot S(E) \).

Proof.
\[
(t_a \circ h_r \circ P) \circ j \circ (t_b \circ h_s \circ P') = t_a \circ h_r \circ j \circ P \circ t_b \circ h_s \circ P' \\
= t_a \circ j \circ h_r \circ t_P(b) \circ h_s \circ P \circ P' \\
= t_a \circ j \circ t_P(b) \circ h_r \circ P \circ P' \\
\in N^+ \cdot j \cdot N^+ \cdot A \cdot O(E) = I_1(E) \cdot S(E).
\]
Conversely, \( I_1(E) \subset N^+ \circ j \circ N^+ \) implies that
\[
I_1(E) \cdot S(E) \subset N^+ \circ j \circ N^+ \cdot S(E) \subset S(E) \circ j \circ S(E).
\]

\[\square\]

The isotropy group at \( \infty \) is computed in [42] using the cross ratio.

Theorem IV.1.10. For \( g \in M(E) \), \( g \in S(E) \) if and only if \( g(\infty) = \infty \).

Since \( j(0) = \infty \), we can calculate the isotropy group at 0.

Corollary IV.1.11. For \( g \in M(E) \), \( g(0) = 0 \) if and only if \( g \in AO(E)N^- \).

Proof. \( g(0) = 0 \) if and only if \( jgj \in S(E) \) by theorem IV.1.10. It is equivalent to \( g \in AO(E)N^- \), by proposition IV.1.1. \[\square\]
**Theorem IV.1.12.**

\[ M(\bar{E}) = S(E) \cup I(E) \circ D = S(E) \cap D \circ I(E). \]

**Proof.** (cf. [42]). □

Because \( j(a, r) \in S(E) \circ S(E) \) by proposition IV.1.7 and it is equal to \( I_1(E) \circ S(E) \) by proposition IV.1.9, we have

**Theorem IV.1.13.**

\[
M(\bar{E}) = S(E) \cup I(E) \circ D \\
= S(E) \cup D \circ I(E) \\
= S(E) \cup S(E) \circ j \circ S(E) \\
= S(E) \cup I_1(E) \cdot S(E).
\]

*In particular, \( I_1(E) \cdot S(E) \) is the set of all transformations which sends \( \infty \) into \( E \).*

Furthermore, using the proposition IV.1.10 and 11, we have

**Corollary IV.1.14.** \( M(\bar{E}) = N^+ AO(E)N^- \cup S(E)j \). In particular,

\[ N^+ AO(E)N^- \]

is the set of all \( g \in M(\bar{E}) \) such that \( g(0) \in E \), and \( S(E)j \) is the set of all \( g \in M(\bar{E}) \) such that \( g(0) = \infty \).

**Proof.** Suppose that \( g(0) = a \in E \). Then \( t_{-a}g(0) = 0 \) and hence by corollary IV.1.11, \( t_{-a}g \in AO(E)N^- \). So \( g \in N^+ AO(E)N^- \). If \( g(0) = \infty \), then \( gj(\infty) = \infty \). So \( gj \in S(E) \). □
IV.2. Lorentzian representation

Let $V = E \oplus \mathbb{R}^2$. In chapter II, $V$ has a Jordan algebra structure with the symmetric cone $\Omega$ and its boundary $\mathbb{R} \cdot \mathcal{P}$, where $\mathcal{P}$ is the set of all primitive idempotents in $V$. We use notions in chapter II, for example $O_1(V), K_1^+$ and so on.

An arbitrary point $u \in V = E \oplus \mathbb{R}^2$ will be understood as a triple $(x, x_1, x_2)$, where $x \in E, x_i \in \mathbb{R}$. There is a standard mapping $\tau : K_1^+ \rightarrow E$ given by:

$$\tau(x, x_1, x_2) = \begin{cases} \frac{x}{x_2 - x_1}, & \text{if } x_2 \neq x_1; \\ \infty, & \text{otherwise.} \end{cases}$$

The mapping $\tau$ is continuous, surjective and that point inverses are generators, i.e., $\mathbb{R}^+ \cdot u, u \in P$.

**Theorem IV.2.1.** For any transformation $\phi$ on $E$, there exists a unique $T_\phi \in O_1^+(V)$ such that

$$\tau \circ T_\phi = \phi \circ \tau.$$ 

Under this correspondence, $\phi \rightarrow T_\phi$ is an isomorphism between the group and the Lorentzian group.

**Proof.** ([42], Section I.3.5) and ([31], Theorem 4.1). \[\square\]

Now we want to see the representation of translations, dialations, inversion and orthogonal transformations in Lorentzian coordinates via the matrix representation which we have seen in chapter II. With the matrix representation and the aids of complete construction of $T_\phi$ in [42], we have

**Proposition IV.2.2.**

$$T_{h_r} = P(u_r) = \exp L(0, \log r, 0),$$

where $u_r = (0, \frac{r-1}{2\sqrt{r}}, \frac{r+1}{2\sqrt{r}})$.
Proof.

\[ T_{hr} = \begin{bmatrix} I & 0 & 0 \\ 0 & r^{1+1} & r^{1-1} \\ 0 & r^{1-1} & r^{1+1} \end{bmatrix}. \]

Note that \( u_r \) has the following spectral decomposition:

\[ u_r = \sqrt{r}(0, \frac{1}{2}, \frac{1}{2}) + \frac{1}{\sqrt{r}}(0, -\frac{1}{2}, \frac{1}{2}). \]

Let \( u'_r = (\frac{1}{2} \log r)c + (\frac{-1}{2} \log r)\sigma(c) \), where \( c = (0, \frac{1}{2}, \frac{1}{2}) \). So

\[ P(u_r) = P(\exp u'_r) = \exp L(2u'_r) \]

\[ = \exp L(\log rc - \log r\sigma(c)) = \exp L(0, \log r, 0). \]

By proposition II.2.8, \( T_{hr} = P(u_r) = \exp L(0, \log r, 0). \)

Remark IV.2.3. \( T_A := \{ T_{hr} : r \in \mathbb{R}^+ \} = \exp L(\{0\} \oplus \mathbb{R}^+ \oplus \{0\}). \)

Set

\[ \alpha(t) = \begin{bmatrix} I & 0 & 0 \\ 0 & c(t) & s(t) \\ 0 & s(t) & c(t) \end{bmatrix}, \]

where \( c(t) = \cosh(t), s(t) = \sinh(t) \). Then \( \alpha(t) \) is a one parameter subgroup of \( O^+_1(V) \). Furthermore.

\[ \alpha(t) = \exp L(0, t, 0) \]

\[ = \exp L(0, \log e^t, 0) \]

\[ = T_{hrt}. \]

Proposition IV.2.4.

(1)

\[ T_{N^+} = \{ T_{ia} : a \in E \} = \left\{ \begin{bmatrix} I & -a & a \\ a^* & 1 - \frac{1}{2}||a||^2 & \frac{1}{2}||a||^2 \\ a^* & \frac{1}{2}||a||^2 & 1 + \frac{1}{2}||a||^2 \end{bmatrix} : a \in E \right\}. \]

(2)

\[ T_{N^-} = \{ T_{ia} : a \in E \} = \left\{ \begin{bmatrix} I & a & a \\ -a^* & 1 - \frac{1}{2}||a||^2 & \frac{1}{2}||a||^2 \\ a^* & \frac{1}{2}||a||^2 & 1 + \frac{1}{2}||a||^2 \end{bmatrix} : a \in E \right\}. \]
(3)  
$$T_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

(4) If $X \in O(E)$, then  
$$T_X = \begin{bmatrix} X & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

**Proof.** (cf. [41]. □

Note that  
$$T_{N^+} = T_j T_{N^+} T_j = T_{N^-},$$  
$$T_{A^*} = T_j T_A T_j,$$  
$$T_{O(E)}^* = T_j T_{O(E)} T_j = T_{O(E)}.$$ 

Therefore, the adjoint operation $\ast$ is exactly the same as the conjugation of $j$. That is, $g^\ast = T_j g T_j$, for all $g \in O_1(V)$.

For $a, x, y \in E$, we set  
$$N_a = \begin{bmatrix} 0 & -a & a \\ a^\ast & 0 & 0 \\ 0 & -a & 0 \end{bmatrix},$$  
$$\tilde{N}_a = \begin{bmatrix} 0 & a & a \\ -a^\ast & 0 & 0 \\ a^\ast & 0 & 0 \end{bmatrix}.$$  

$$X_x = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ x^\ast & 0 & 0 \end{bmatrix}, Z_x = \begin{bmatrix} 0 & x & 0 \\ -x^\ast & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$  

Then $X_a - Z_a = N_a, X_a + Z_a = \tilde{N}_a$. Furthermore,  
$$[X_x, Y] = Z_x, [X_x, Z_y] = (x|y)Y, [Y, Z_x] = -X_x.$$  

And  
$$[X_x, X_y] = \begin{bmatrix} xy^\ast - yx^\ast & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, [Z_x, Z_y] = -[X_x, X_y].$$
Furthermore,

\[ N_{-\frac{1}{2}} + \hat{N}_{\frac{1}{2}} = Z_x. \]

\[ N_{\frac{z^x}{2}} + \hat{N}_{\frac{z^x}{2}} + cY = \begin{bmatrix} 0 & x & y \\ -x^* & 0 & c \\ y^* & c & 0 \end{bmatrix}. \]

**Proposition IV.2.5.** For any pair \( \{x, y\} \subset E \) with \((x|y) = 1\), the Lie subalgebra generated by \( X_x, Z_y \) and \( Y \) is isomorphic to \( sl(2, \mathbb{R}) \).

**Proof.** The following map gives an isomorphism:

\[ X_x \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Z_y \rightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad Y \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \]

\[ \square \]

By the formula, \( P(\exp x) = \exp L(2x) \) and Proposition II.2.8, we have

**Proposition IV.2.6.** \( \exp L(2x, 0, 0) = \begin{bmatrix} I + 2(\frac{sh||x||}{||x||})^2xx^* & 0 & \frac{sh^2||x||}{||x||}x \\ 0 & 1 & 0 \\ \frac{sh^2||x||}{||x||}x^* & 0 & ch2||x|| \end{bmatrix} \).

An elementary calculation shows the following result:

**Proposition IV.2.7.**

1. \( \exp(N_a) = T_x \).
2. \( \exp(\hat{N}_a) = T_{x^*} \).

**Proposition IV.2.8.** \( N_E \) and \( \hat{N}_E \) are (abelian) Lie subalgebras of \( \text{Lie}(O_1(V)) \).

In particular, \( N_E^* = T_j N_E T_j = \hat{N}_E \).

**Proposition IV.2.9.** \( \text{Lie}(O_1(V)) \) is generated by \( N_E, \hat{N}_E \) and \( \mathbb{R} \cdot L(0, 1, 0) \) if \( E \) is finite dimensional.

**Proof.** Note that

\[ \begin{bmatrix} 0 & x & 0 \\ -x^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & y & 0 \\ -y^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} yx^* - xy^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \text{Lie}(O(E)). \]
The matrix representation of $L(0,c,0)$ is given by:

$$\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & c \\
0 & c & 0
\end{bmatrix}.$$ 

The Lie algebra generated by $N_E, \tilde{N}_E$ and $RL(0,1,0)$ contains $Z_E$ since

$$N_{-\frac{E}{2}} + \tilde{N}_{\frac{E}{2}} = Z.$$ 

Since $X_a = N_a + Z_a$, it contains $X_E, Z_E$. So $\text{Lie}(O(E))$ is contained in this Lie algebra. Note that

$$\mathfrak{t} = \text{Lie}(O(E)) \oplus Z_E, \mathfrak{p} = X_E \oplus RL(0,1,0).$$ 

But we have seen that the Lie algebra $\text{Lie}(O_1^+(V))$ is $\mathfrak{t} \oplus \mathfrak{p}$ which completes the proof. □

Let $G_0$ be the subgroup of group $M(\mathbb{E})$ corresponding to the connected component $O_1(V)_0$ of $O_1(V)$. This group $G_0$ contains the subgroup $G_1$ of the smallest subgroup of Möbius group because $O_1(V)_0$ contains $T_{N^+}, T_{N^-}$ and is closed under adjoint operation. In particular, $G_0$ is a subgroup of Möbius type. But $N_E, \tilde{N}_E$ and $RL(0,1,0)$ generate $\text{Lie}(O_1(V))$ and $T_G$, if $T_G$ is a Lie subgroup which is the case when $E$ is a finite dimensional by Goto's theorem.

By theorem II.2.33,

**Theorem IV.2.10.** $G(\Omega)_0 = T_{N^-} \cdot T_A \cdot \mathbb{R}^+ \cdot \text{Aut}(V)_0$.

**Corollary IV.2.11.** $T_{G_0} = T_{N^-} \cdot T_A \cdot \text{Aut}(V)_0$.

Since $\text{Aut}(V)_0 = SO(E \oplus \mathbb{R})$ and $T_{G_0} = T_{G_0}, T_{G_0} = SO(E \oplus \mathbb{R}) \cdot T_A \cdot T_{N^+}$.

**IV.3. Compression semigroups of upper-half planes**

Let $B := E_0 \oplus \mathbb{R}^+$ be the upper-half plane with boundary $\partial B := E_0 \oplus \{0\} \cup \{\infty\}$. From now, we identify $E_0$ with $E_0 \oplus \{0\}$. Denote $\overline{B}$ by the closure of $B$ in $\overline{E}$.

Note that $\overline{B} = B \cup \partial B$ and hence the point $\infty$ is in $\overline{B}$.
Set

\[ \Gamma^+ = \{ t_a \mid a \in E_0 \oplus [0, \infty) \}, \]
\[ \Gamma^- = f \Gamma^+ f, \]
\[ N_0^+ = \{ t_a \mid a \in E_0 \}, \]
\[ N_0^- = f N_0^+ f. \]

Let \( G \) be a subgroup of Möbius type. Then \( \Gamma^\pm \) and \( N_0^\pm \) are subsemigroups of \( G \). Note \( N_0^\pm \) is a subgroup of \( \Gamma^\pm \). We define a compression semigroup for the upper-half plane \( B = E_0 \oplus (0, \infty) \):

\[ S_G(B) = \{ g \in G \mid g(B) \subset B \}. \]

And

\[ S_M(E)(B) = \{ g \in M(E) \mid g(B) \subset B \}\]

is called a Möbius semigroup. The group of unit of the compression semigroup which is denoted by \( H_G(B) \) is

\[ H_G(B) = S_G(B) \cap S_G(B)^{-1}. \]

**Proposition IV.3.1.** \( S_G(B) = S_G(B) \) and

\[ H_G(B) = \{ g \in G \mid g(B) = B \}. \]

**Proof.** (cf. [31]). \( \Box \)

**Theorem IV.3.2.** \( S_G(B) \) and \( S_M(E)(B) \) are maximal subsemigroups in \( G \) and \( M(E) \), respectively.

**Proof.** (cf. [31]). \( \Box \)
Because $G_0$ and $G_1$ are subgroups of Möbius type, $S_{G_0}(B)$ and $S_{G_1}(B)$ are maximal subsemigroups of $G_0$ and $G_1$, respectively.

**Proposition IV.3.3.** $\Gamma^+ A \Gamma^- = S_G(B) \cap N^+ A N^- = M(E) \cap N^+ A N^-.$

**Proof.** Obviously, $A \subset S_G(B)$. If $x = (x_0, s) \in B$, then for $y = (y_0, s) \in B$,

$$t_x(y) = (x_0 + y_0, r + s) \in B$$

which implies that $\Gamma^+ \subset S_G(B)$. Since $j(B) = B$, for $x \in B$,

$$(j \circ t_x \circ j)(B) = (j \circ t_x)(B) \subset j(B) = B.$$ 

Therefore $\Gamma^- \subset S_G(B)$ and hence $\Gamma^+ A \Gamma^- \subset S_G(B) \cap N^+ A N^-$. On the otherhand, suppose that $t_x \circ h_r \circ \tilde{t}_y \in S_G(B)$. Then

$$[t_x \circ h_r \circ \tilde{t}_y](B) \subset B,$$

hence $[t_x \circ h_r \circ \tilde{t}_y](0) = x \in B$. Now if $-y \in B$, then since $j(B) = B$ and

$$[t_x \circ h_r \circ \tilde{t}_y](B) = [t_x \circ h_r \circ j \circ t_y](j(B)) \subset B,$$

$$[t_x \circ h_r \circ j \circ t_y](-y) = \infty \in B.$$ But it is impossible. Therefore $y$ must be in $B$.

Therefore $S_G(B) \cap N^+ A N^- \subset \Gamma^+ A \Gamma^-$. Therefore,

$$\Gamma^+ A \Gamma^- = S_G(B) \cap N^+ A N^-.$$

Since $N^+ A N^- \subset G$, the equality is trivial. □

**Lemma IV.3.4.** Let $X \in O(E)$ and let $x \in E$. Then $t_x \circ X \in S_{M(E)}$ if and only if $x \in \overline{B}$ and $X \in O(E_0)$.

**Proof.** First, note that every bounded linear transformation $X$ can be written the following block matrix:

$$X = \begin{bmatrix} X_0 & b \\ a^* & r \end{bmatrix},$$
where $X \in gl(E_0)$ and $a, b \in E_0$, $r \in \mathbb{R}$. Now suppose that $t_a \circ X \in S_{M(\overline{E})}$. Let $x = (x_0, x_1)$ and let $y = (y_0, y_1) \in B$. Then

$$t_x \circ X(y) = (x_0 + X_0 y_0 + y_1 b, x_1 + (a|y_0) + ry_1).$$

Hence the second coordinate $x_1 + (a|y_0) + ry_1 > 0$, for every $y = (y_0, y_1) \in B$. Because $y_0$ moves freely, $a = 0$ and $r > 0$. However, since $X \in O(E)$, $b$ must be zero and $r = 1$. This implies that $X \in O(E_0)$. As $y_0$ goes to zero, $x_1$ must be non-negative. The converse direction is easy to see. □

**Corollary IV.3.5.** $S_{M(\overline{E})} \cap S(E) = \Gamma^+ \cdot A \cdot O(E_0)$.

**Proof.** Clearly, $\Gamma^+ \cdot A \cdot O(E_0) \subset S_{M(\overline{E})} \cap S(E)$. Suppose that $t_x \circ h_r \circ X \in S_{M(\overline{E})}$. Since $h_r \in S_{M(\overline{E})}$ for every $r \in \mathbb{R}^+$, we may assume that $t_x \circ A \in S_{M(\overline{E})}$. But by the lemma IV.3.4, $t_x \in \Gamma^+$ and $X \in O(E_0)$ which complete the proof.

Set

$$I_1(E_0) := \{j(a, 1) \mid a \in E_0\},$$

the set of all inversions with center in $E_0$ with radius 1. And let $S(E_0)$ be the set of all similarities of $E_0$, that is,

$$S(E_0) = N_0^+ \cdot A \cdot O(E_0).$$

Then the group of $E_0$ is

$$M(E_0) = S(E_0) \cup I_1(E_0)S(E_0)$$

by theorem IV.1.13. The following shows that the group of $E_0$ is exactly the group of units of the compression semigroup $S_{M(\overline{E})}$.

**Theorem IV.3.6.** Let $H_{M(\overline{E})}(B)$ be the group of units in the compression semigroup $S_{M(\overline{E})}$. Then $H_{M(\overline{E})} = S(E_0) \cap I_1(E_0)S(E_0)$. In particular, $H_{M(\overline{E})}$ is the group of the hyperplane $E_0$. 

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First, we will show that \( H_{M(E)} \cap S(E) = S(E_0) \). Clearly, \( A \subseteq H_{M(E)} \cap S(E) \). Suppose that

\[
t_x \circ h_r \circ X \in H_{M(E)} \cap S(E) \subseteq S_{M(E)} \cap S(E) = \Gamma^+ \cdot A \cdot O(E_0).
\]

Then \( x \in B \) and \( X \in O(E_0) \). Because every element in \( H_{M(E)} \) leaves the boundary of the upper-half plane into itself, by evaluating with \((y_0, 0) \in E_0 \) in \( t_x \circ h_r \circ X \), we have \( x \in E_0 \).

Secondly, it is obvious that \( I_1(E_0) \cdot S(E_0) \subset H_{M(E)} \cap I_1(E) \cdot S(E) \). Conversely, suppose that

\[
g = t_x \circ j \circ t_{-x} \circ t_a \circ h_r \circ X \in H_{M(E)}.
\]

Then since \( g(\infty) = x \in E_0 \), \( t_x \in \Gamma^0 \). Note that \( t_x \circ j \circ t_{-x} \in H_{M(E)} \). Hence

\[
t_a \circ X \in H_{M(E)} \cap S(E) = S(E_0),
\]

by first step. So \( g \in I_1(E_0) \cdot S(E_0) \). Furthermore, \( H_{M(E)} \) is exactly the group of the hyperplane \( E_0 \).

**Proposition IV.3.7.** \( S_{M(E)}(B) \subseteq \Gamma^+AO(E_0)\Gamma^- \cup \Gamma^+AO(E)j \). Furthermore, \( H_{M(E)}(B) = N_0^+AO(E_0)N_0^- \cup N_0^+AO(E_0)j \).

**Proof.** Let \( g \in S_{M(E)}(B) \). Since \( M(E) = N^+AO(E)N^- \cup N^+AO(E)N^- \) by corollary IV.1.14, we may assume that \( g \in N^+AO(E)N^- \) or \( g \in N^+AO(E)j \).

If \( g = t_a h_r X \bar{t}_b \in N^+AO(E)N^- \), then \( g(0) = a \in B \). If \( b \) is not in \( B \), Then since \( j(B) = B \), \( g(j(-b)) = \infty \) which gives a contradiction. Therefore \( g \in \Gamma^+AO(E)\Gamma^- \). Now suppose that \( g = t_a h_r X j \in N^+AO(E)j \). Then \( g(\infty) = a \in B \). Since \( j(B) = B \), \( t_{-a} X \in S_{M(E)}(B) \). By lemma IV.3.4, \( X \in O(E_0) \). By corollary IV.1.14 and IV.3.6, \( H_{M(E)}(B) = N_0^+AO(E_0)N_0^- \cup N_0^+AO(E_0)j \).

Let \( Q = \{ t_{(x,s)} \mid x \in E_0, s < 0 \} \) and let \( e = (0,1) \in B \). Let \( G_e = \{ g \in G \mid g(e) = e \} \) be the isotropy subgroup of \( G \).
Proposition IV.3.8.

1. For $s < 1$, $t_{(0,s)} \circ h_{1-s} \in G_e$.

2. $h_s \circ \tilde{t}_{(0,s-1)} \in G_e$, for any $s \in \mathbb{R}$.

Proposition IV.3.9. $M(\overline{E}) = QAM(\overline{E})_e \cup j \circ t_{-e} \circ M(\overline{E})_e \circ j$.

Proof. Let $g \in M(\overline{E})$ and let $g(e) = (x, s) = (x, 0) + s(0, 1)$. Choose $r > 0$ so that $r + s > 0$. Then

\[(t_{(0,r)} \circ g)(e) = (0, r) + (x, s) = (x, r + s) = (x, 0) + (r + s)e = (t_{(x,0)} \circ h_{r+s})(e)\]

Let $g' = g^{-1} \circ t_{(x,-r)} \circ h_{r+s}$. Then $g'(e) = e$, hence $g' \in M(\overline{E})_e$. This implies that $g = t_{(x_0,-r)} \circ h_s \circ g' \in QAM(\overline{E})_e$. Now suppose that $g(e) = \infty$. Then $t_e \circ j \circ g \circ j \in M(\overline{E})_e$ because $j(e) = e$. Hence $g \in j \circ t_{-e} \circ G_e \circ j$. □

Proposition IV.3.10. $S_{M(\overline{E})}(B) = \Gamma^0 A(S_{M(\overline{E})}(B) \cap M(\overline{E})_e)$.

Proof. By definition, if $g \in S_{M(\overline{E})}(B)$, then $g(e) \in B$. Let $g(e) = (x, s) \in B$. Then $s > 0$. By setting

\[g' := g^{-1} \circ t_{(x,0)} \circ h_s,\]

we have $g \in \Gamma^0 AM(\overline{E})_e$. Now let $g \in S_{M(\overline{E})}(B)$. Then $g = t_{(x_0,0)} \circ h_r \circ k$ for some $g \in M(\overline{E})_e$. Then $k = h_k \circ t_{(-x_0,0)} \circ g \in S_{M(\overline{E})}(B)$. Therefore, $g \in \Gamma^0 A(S_{M(\overline{E})}(B) \cap M(\overline{E})_e)$. Hence

\[S_{M(\overline{E})}(B) \subset \Gamma^0 H(S_{M(\overline{E})}(B) \cap M(\overline{E})_e).\]

The converse argument is clear. □
**Proposition IV.3.11.** $g \in G_e$ if and only if $T_g(l) = l$, where $l = r^{-1}(e)$.

**Proof.** Let $u = (x, x_1, x_2) \in r^{-1}(e)$. Then from the property of $K_E^+$, $x_2 > 0$ and $x_2^2 = x_1^2 + ||x||^2$. Furthermore, since $r(u) = e$, $x_1 \neq x_2$ and $\frac{x}{x_2 - x_1} = e$. Let $x = (x_0, r) \in E_0 \oplus \mathbb{R}$. Then $x_0 = 0$ and $r = x_2 - x_1$. Now

$$x_2^2 = x_1^2 + (x_2 - x_1)^2 \implies x_1 = 0.$$

Hence $u$ must be of the form $u = ((0, x_2), 0, x_2)$. Conversely, it is obvious to show that if $u = ((0, x_2), 0, x_2)$ with $x_2 \neq 0$, then $u \in l = r^{-1}(e)$. So we have seen that the fiber at $e$ is the half line passing through the point $c := ((0, \frac{1}{2}), 0, \frac{1}{2})$. In particular, $c \in P^0$ and $l = \mathbb{R}^+c$.

Now let $g \in G_e$. Then $r \circ T_g = g \circ r$ implies that $r \circ T_g(l) = g \circ r(l) = g(e) = e$. Hence $T_g(l) \subset r^{-1}(e) = l$. Since $g^{-1} \in G_e$, we get $T_g(l) = l$. Conversely, suppose that $T_g(l) = l$. Then $g(e) = g \circ r(l) = r \circ T_g(l) = \tau(l) = e$. Hence $g \in G_e$. □

**Corollary IV.3.12.** $T_{G_e}$ is the set of all pseudo-orthogonal transformations in $O_+^1$ which carry $l$ into itself.

**Proposition IV.3.13.** Let $X \in O(E)$. Then $X \in G_e$ if and only if $T_X \in T_{G_e}$ if and only if $T_X(c) = c$.

**Proof.** Suppose $T_X \in T_{G_e}$. Then $X(e) = e$ by the previous proposition. So $T_X(2c) = T_X(e, 0, 1)' = (X(e), 0, 1) = 2c$. Hence $T_X(c) = c$. Conversely, if $T_X(c) = c$, then trivially $T_X(l) = l$. Hence $T_X \in T_{G_e}$. □

**Proposition IV.3.14.** Let $w \in \Omega$. Then $P(w) \in T_{G_e}$ if and only if $w \in \{0\} \oplus \mathbb{R} \oplus \{0\} \oplus \mathbb{R}$. In particular,

$$P(w) \in \exp L(\{0\} \oplus \mathbb{R} \oplus \{0\} \oplus \{0\}).$$
**Proof.** Let \( w = ((x_0, r), x_1, x_2) \in \Omega \). Suppose \( P(w) \in T_G \). Then \( P(w)(c) \in \mathbb{R}^+c \). By proposition 3.4,

\[
\langle x \mid c \rangle + x_2 = 0 \text{ or } x_1 = 0.
\]

If \( \langle x \mid e \rangle + x_2 = 0 \), then \( x_2 = -r \). But it is impossible because \( w \in \Omega \). Hence \( x_1 = 0 \). However, the first coordinate of \( P(w)(c) \) must be in \( \{0\} \oplus \mathbb{R} \). Hence \( x \in \{0\} \oplus \mathbb{R} \). Converse argument is obvious. But \( P(w) = P(\exp w') \), where

\[
w' = \log(x_2 + r)c_w + \log(x_2 - r)\sigma(c_w) \in \{0\} \oplus \mathbb{R} \oplus \{0\} \oplus \mathbb{R}.
\]

Since \( P(w) \in O^+(V) \), \( w' \) must be in \( \{0\} \oplus \mathbb{R} \oplus \{0\} \oplus \{0\} \). \( \Box \)

Recall the definition of the group \( G_0 \): \( T_G \) is the connected component of \( O_1(V)^+ \) and is equal to \( T_N T_A SO(E \oplus \mathbb{R}) \). Furthermore, \( G_0 \) is a subgroup of Möbius type containing the smallest subgroup of Möbius type \( G_1 \). Now we will study the compression semigroup \( S_{G_0}(B) \). Denote by \( O(E)' \) the complement of \( SO(E) \) in \( O(E) \). In finite dimensional case, it is the set of all orthogonal transformations of determinant \(-1\).

**Proposition IV.3.15.** \( S_{G_0}(B) \subset \Gamma^+ASO(E)^{-} \cup \Gamma^+AO(E_0)'j \). Furthermore, \( H_{G_0}(B) = N_0^+ASO(E_0)N_0^- \cup N_0^+AO(E_0)'j \).

**Proof.** Since \( S_{G_0}(B) \subset S_{M(E)}(B) = \Gamma^+ASO(E)^{-} \cup \Gamma^+AO(E_0)'j \), we need to show that \( \Gamma^+AO(E_0)'j \cap S_{G_1}(B) \subset \Gamma^+AO(E_0)'j \). Suppose \( g = t_x h_r X_j \in S_{G_1}(B) \). Since \( T_X X_j = T_X T_j \in SO(E \oplus \mathbb{R}) \) implies that \( X \in O(E_0)' \). \( \Box \)

**Example IV.3.16.** If \( E = \mathbb{R} \), then \( G_0 = PSL(2, \mathbb{R}) \) via linear fractional transformation. Since \( O(E_0)' \) is empty, \( S_{G_1}(B) = \Gamma^+A^+ \) and is the decomposition of \( PSL(2, \mathbb{R})^+ \) which we have seen in III.6.
IV.4. Lorentzian semigroups

We revisit the setting of chapter II. Let $E$ be a real Hilbert space and let $V = E \oplus \mathbb{R}^2$. Then $V$ is a JB-algebra, which is called a spin factor, obtained from the Lorentzian form

$$\mu(u, v) = -(x|y) - x_1y_1 + x_2y_2,$$

where $u = (x, x_1, x_2), v = (y, y_1, y_2)$. Then the symmetric cone $\Omega$ is the (open) Lorentzian cone

$$\Omega = \{(x, x_1, x_2) \in V \mid \|(x, x_1)\| < x_2\}$$

The boundary of $\Omega$ which is denoted by $K^*_E$ is the forward cone of the cone $K_E = \{u \in V \mid \mu(u, u) = 0\}$.

The set $\mathcal{P}$ of all non-trivial primitive idempotents is exactly

$$\mathcal{P} = K^*_E \cap S(0, \frac{1}{2}),$$

where $S(0, \frac{1}{2})$ is the sphere of radius $\frac{1}{2}$ centered at 0. Furthermore, $\mathcal{P}$ can be decomposed using the Lorentzian coordinate:

$$\mathcal{P} = \mathcal{P}^+ \cup \mathcal{P}^0 \cup \mathcal{P}^-.$$

The canonical Lorentzian semigroup $S^+_1$ on $E \oplus \mathbb{R}^2$ is defined the compression semigroup of $\mathbb{R}^+\mathcal{P}^+$:

$$S^+_1 := \{g \in O^+_1(V) \mid g(\mathbb{R}^+\mathcal{P}^+) \subset \mathbb{R}^+\mathcal{P}^+\}.$$ 

Then $S^+_1$ is a subsemigroup of $O^+_1(V)$. This semigroup has been studied in ([31]) via polar decomposition. And also this semigroup is isomorphic semigroup on $E$ ([31]). In this section, we are going to study this semigroup via Jordan algebra structure.
IV.4.1. The compression semigroup of $\Omega^+$

We have seen that there is a decomposition of the symmetric cone $\Omega$ using the Lorentzian coordinate:

$$\Omega = \Omega^+ \cup \Omega^0 \cup \Omega^-.$$ 

First, we define the compression semigroup of $G(\Omega)$ by:

$$S^+ := \{ g \in G(\Omega) \mid g(\mathbb{R}^n P^+) \subset \mathbb{R}^n P^+ \}.$$ 

Because $O_1(V)^+ \subset G(\Omega)$, $S_1^+ \subset S^+$. Since every element $g$ of $G(\Omega)$ is linear, $g \in S^+$ is equivalent to $g(P^+) \subset \mathbb{R}^n P^+$. Note that if $g \in S^+$, then $g(P^0) \subset \mathbb{R}^n P^\geq$.

By theorem II.2.23, we showed that $G(\Omega) = O_1(V)^+ \cdot \mathbb{R}^+.$

**Theorem IV.4.1.** $S^+ = S_1^+ \cdot \mathbb{R}^+$.

**Proof.** Let $g \in S^+$. Then theorem II.2.23, $g = g_1 \circ tI$ for some $g_1 \in O_1^+(V)$ and $t \in \mathbb{R}^+$. Hence $g \circ t^{-1}I = g_1 \in O_1^+(V)$. Because $g$ and $t^{-1}I$ both carry $P^+$ into $\mathbb{R}^n P^+$, $g_1 \in S_1^+$. Therefore $g = g_1 \circ tI \in S_1^+ \cdot \mathbb{R}^+$, hence $S^+ \subset S_1^+ \cdot \mathbb{R}^+$. The converse argument is obvious. □

It is known ([31]) that there is a polar decomposition of $S_1^+$:

Set

$$C_1 = \{ \begin{bmatrix} 0 & x & 0 \\ -x^* & 0 & x_1 \\ 0 & x_1 & 0 \end{bmatrix} \mid x_1 \geq \| x \| \}.$$ 

Then $C_1$ is a closed convex cone in $\text{Lie}(O_1^+(V))$.

**Theorem IV.4.2.** [Lawson] $S_1^+ = \exp C_1 \cdot H_1$, where $H_1$ is the group of units of $S_1^+$. In particular, $S_1^+$ is a maximal subsemigroup of $O_1^+(V)$.

**Lemma IV.4.3.** Let $G$ and $H$ be groups and let $S$ be a maximal subsemigroup of $G$ with identity $e_G$. Then $S \times H$ is a maximal subsemigroup of $G \times H$. 

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PROOF. Let $S'$ be a subsemigroup of $G \times H$ containing $S \times H$. Then we show that $S' = G \times H$ or $S' = S \times H$. Note that

$$S \times \{e_H\} \subset G \times \{e_H\} \cap S \times H \subset G \times \{e_H\} \cap S'.$$

Since $G \times \{e_H\} \cap S'$ is a subsemigroup of in $G \times \{e_H\} \cong G$ and $S \times \{e_H\}$ is maximal in $G \times \{e_H\}$,

$$G \times \{e_H\} \cap S' = S \times \{e_H\}$$

or

$$G \times \{e_H\} \cap S' = G \times \{e_H\}.$$

Suppose that $G \times \{e_H\} \cap S' = G \times \{e_H\}$. Then $G \times \{e_H\} \subset S'$ and hence

$$G \times H = (G \times \{e_H\}) \cdot (\{e_G\} \times H) \subset S' \subset S' \subset S'$$

since $\{e_G\} \times H \subset S \times H \subset S'$. Therefore, $S' = G \times H$. Suppose that $G \times \{e_H\} \cap S' = S \times \{e_H\}$. Let $(g, h) \in S'$. Note that $(e_G, h) \in \{e_G\} \times H$ is in the group of units of $S'$. Thus

$$(g, h) \cdot (e_G, h)^{-1} = (g, e_H) \in S' \cap G \times \{e_H\} = S \times \{e_H\}.$$

Therefore,

$$(g, h) = (g, e_H) \cdot (e_H, h) \in (S \times \{e_H\}) \cdot (\{e_G\} \times H) = S \times H.$$

This shows that $S' \subset S \times H$ and hence $S' = S \times H$. □

By theorem IV.4.1, theorem IV.4.2 and lemma IV.4.3, we get

COROLLARY IV.4.4. $S^+ = \exp C_1 \cdot H_1 \cdot R^+$. In particular, $S^+$ is maximal in $G(\Omega)$.

PROOF. Because $G(\Omega) \cong O_1^+(V) \times R^+$ and $S_1^+$ is maximal in $O_1^+(V)$, by lemma IV.4.3, $S^+$ is maximal in $G(\Omega)$. □
Because the symmetric cone $\Omega$ has a decomposition: $\Omega = \Omega^+ \cup \Omega^0 \cup \Omega^-$, it is quite natural to consider the compression semigroup of $\Omega^+$:

$$S := \{ g \in G(\Omega) \mid g(\Omega^+) \subset \Omega^+ \}.$$ 

We are going to show that this compression semigroup is exactly the Lorentzian semigroup $S^+$.

**Lemma IV.4.5.** Let $g = P(w) \circ h \in P(\Omega)\text{Aut}(V)$ be a polar decomposition of $g \in G(\Omega)$. If $g \in S^+$, then $g(e) \in \Omega^\geq$. In particular, $w \in \Omega^\geq$.

**Proof.** First, choose $x \in E$ such that $\|x\| = 1$. Then

$$g(2e) = g(x,0,1) + g(-x,0,1).$$

Since $(x,0,1) \in P^0$ and $(-x,0,1) \in P^0$, $g(e) \in \mathbb{R}^+P^\geq + \mathbb{R}P^\geq$. Clearly, $g(e)$ has the non-negative second coordinate. Hence $g(e) \in \Omega^\geq$. Now, let $w = (y,y_1,y_2) \in \Omega$. Since $g(e) = P(w)h(e) = P(w)(e) = w^2$ has the second coordinate $2y_1y_2$ and since $y_2 > 0$, $y_1$ must be non-negative. Therefore, $w \in \Omega^\geq$. \qed

**Proposition IV.4.6.** If $g \in S^+$, then $g(\Omega^+) \subset \Omega^+.$

**Proof.** Let $u = (x,x_1,x_2) \in \Omega^+$ and let $u = \alpha_u c_u + \beta_u \sigma(c_u)$ be the spectral decomposition of $u$. Because $g \in G(\Omega)$, $g(u) \in \Omega$. Hence we claim that $g(u)$ has the positive second coordinate. Since $u \in \Omega$, $\alpha_u$ and $\beta_u$ are positive real numbers. Hence $c_u \in P^+$. Note that $\sigma(c_u) = -c_u + e$. So

$$g(u) = \alpha_u g(c_u) - \beta_u g(c_u) + \beta_u g(e) = (\alpha_u - \beta_u)g(c_u) + \beta_u g(e) = 2k_u g(c_u) + \beta_u g(e),$$

where $k_u = \| (x,x_1) \|$. Because $g \in S^+$, $2k_u g(c_u)$ the positive second coordinate. But by the previous lemma, $g(e)$ has the non-negative second coordinate.
Therefore, \( g(u) \) has the positive second coordinate, that is, \( g(y) \in \Omega^+ \). Hence \( g(\Omega^+) \subset \Omega^+ \).

**Corollary IV.4.7.** \( S^+ = S. \)

*Proof.* We have seen that \( S^+ \subset S \) in the previous proposition. From corollary IV.4.4, \( S^+ \) is a maximal subsemigroup of \( G(\Omega) \). Thus \( S^+ = S \) by the maximality of \( S^+ \). □

**Proposition IV.4.8.** For \( u \in \Omega \), \( P(u) \in S^+ \) if and only if \( u \in \Omega^2 \). In particular,

\[
P(\Omega^2) \subset S^+.
\]

*Proof.* Suppose \( u = (x, x_1, x_2) \in \Omega \) and \( P(u) \in S^+ \). Then

\[
(x_1^2 + x_2^2 - ||y||^2)y_1 + 2(x|y)x_1 + 2x_1x_2y_2 > 0
\]

for all \((y, y_1, y_2) \in \mathbb{R}^+P^+. \) Choose \( y \in E \) such that \( ||y|| = \frac{1}{2} \). Then \((y, 0, \frac{1}{2}) \in P^0. \) So we have to \((2(x|y) + 2x_2y_2)x_1 \geq 0 \). Because it is possible to choose \( y \) so that \( (x|y) \geq 0, x_1 \geq 0 \). Therefore \( u \in \Omega^2 \). So if \( P(u) \in S^+ \) for \( u \in \Omega \), then \( u \in \Omega^2 \).

Conversely, suppose that \( u = (x, x_1, x_2) \in \Omega^2. \) Then the second coordinate of \( P(u)v \) becomes

\[
(x_1^2 + x_2^2 - ||x||^2)y_1 + 2(x|y)x_1 + 2x_1x_2y_2,
\]

for \( v = (y, y_1, y_2) \). To show \( P(u) \in S^+ \), it is enough to see that the second coordinate of \( P(u)v \) is positive for all \( v \in P^+. \) Let \( v = (y, y_1, \frac{1}{2}) \in P^+. \) Then the second coordinate of \( P(u)v \) becomes

\[
(x_1^2 + x_2^2 - ||x||^2)y_1 + 2(x|y) + x_1x_2.
\]
If \( (x|y) \geq 0 \), then (IV.1) is positive. Suppose that \( (x|y) < 0 \). Then since \( ||y|| < \frac{1}{2} \) and \( ||x|| < x_2 \),

\[
2(x|y)x_1 \geq -2||y|| \cdot ||x||x_1 \geq -||x||x_1 \geq -x_1 x_2.
\]

Hence the second coordinate of \( P(u)v \) is bigger than

\[
(x_1^2 + x_2^2 - ||x||^2)y_1
\]

and it is positive. \( \square \)

Let \( u \in \Omega \). Then \( u \) is invertible and hence \( P(u) \) is invertible. Furthermore

\( u = \exp u' \) for some \( u' \in V \) since \( \Omega = \exp V \). Suppose \( u = (x, x_1, x_2) \in \Omega^2 \) and \( (x, x_1) \neq 0 \). From the spectral decomposition theorem,

\[
(x, x_1, x_2) = (x_2 + ||(x, x_1)||)(\frac{x}{2||x, x_1||}, \frac{x_1}{2||x, x_1||}, \frac{1}{2})
\]

\[
+ (x_2 - ||(x, x_1)||)(\frac{-x}{2||x, x_1||}, \frac{-x_1}{2||x, x_1||}, \frac{1}{2}).
\]

Set

\[
\alpha_u = x_2 + ||(x, x_1)||,
\]

\[
\beta_u = x_2 - ||(x, x_1)||
\]

and let

\[
c_u = (\frac{x}{2||x, x_1||}, \frac{x_1}{2||x, x_1||}, \frac{1}{2}).
\]

Then \( u = \alpha_u c_u + \beta_u c(u) \). Note that \( \alpha_u \) and \( \beta_u \) are positive real numbers. If we set

\[
u' = (\log \alpha_u)c_u + (\log \beta_u)c(u),
\]

then \( u = \exp u' \). Define \( \bar{u} = 2u' \).

If \( u = (0, 0, x_2) = x_2 e \in \Omega^0 \), then we define \( u' = (0, 0, \log x_2) \) and

\[
\bar{u} = (0, 0, \log x_2^2).
\]
Let $W := \{L(u) \mid u \in \Omega^2\}$. Using the formula $P(\exp u) = \exp 2L(u)$, we have

**Lemma IV.4.9.** $P(\Omega^2) = \exp W$.

**Proof.** $P(u) = P(\exp u') = \exp L(2u') = \exp L(\tilde{u})$. □

Set

$$V^+ := \{(x, x_1, x_2) \in V \mid x_1 \geq 0\}.$$ 

**Proposition IV.4.10.** $L(V^+) = W$. Hence $W$ is a closed convex cone of $Lie(G(\Omega))$.

**Proof.** Let $u = (x, x_1, x_2) \in \Omega^2$. If $(x, x_1) = 0$, then $\tilde{u} = (0, 0, \log x_2^2) \in V^+$. Suppose $(x, x_1) \neq 0$. Then $u' = \log \alpha_u c_u + \log \beta \sigma(c_u)$ and its second coordinate is

$$\frac{x_1}{2\| (x, x_1) \|} (\log \alpha_u \beta_u^{-1}) \geq 0.$$ 

Therefore if $u \in \Omega$, then $\tilde{u} \in V^+$ and hence $W \subset L(V^+)$. Conversely, suppose that $u = (x, x_1, x_2) \in V^+$, i.e., $x_1 \geq 0$. If $(x, x_1) = 0$, then $L(u) = L(\tilde{u})$, where $v = (0, 0, e^{\frac{x_2}{2^2}}) \in \Omega^2$. Suppose $(x, x_1) \neq 0$. Set

$$y = \frac{x}{2k} (e^{\alpha_u} - e^{\beta_u})$$

$$y_1 = \frac{x_1}{2k} (e^{\alpha_u} - e^{\beta_u})$$

$$y_2 = \frac{1}{2} (e^{\alpha_u} + e^{\beta_u}).$$

Since $\alpha_u > \beta_u$ and $x_1 \geq 0$, $y_1 \geq 0$. Clearly $y_2 > 0$.

$$\|y\|^2 + y_1^2 = \frac{(e^{\alpha_u} - e^{\beta_u})^2}{4k^2} - k^2$$

$$= \left(\frac{e^{\alpha_u} - e^{\beta_u}}{2}\right)^2$$

$$< y_2^2.$$
Hence \( v := (y, y_1, y_2) \in Q^2 \). It is easy to compute that \( c_v = c_u \) and \( \alpha_v = e^{\alpha_u}, \beta_v = e^{\beta_u} \). And hence

\[
v' = \log \alpha_v c_v + \log \beta_v \sigma(c_v) = \alpha_u c_u + \beta_u \sigma_u = u.
\]

Therefore \( L(u) = L(v') \in W \). Hence \( W = L(V^+) \). \( \square \)

**Corollary IV.4.11.** \( \text{Lie}(S^+) \cap L(V) = L(V^+) = W \).

**Proof.** Let \( u \in Q^2 \). Then

\[
\exp(tL(\bar{u})) = \exp(2L(tu')) = P(\exp tu') = P(\alpha_u c_u + \beta_u \sigma(c_u)).
\]

It is clear that the second coordinate \( \exp tu' = \alpha_u c_u + \beta_u \sigma(c_u) \) is positive. By proposition IV.4.8, \( \exp(tL(\bar{u})) \in S^+ \) for all \( t \in R^+ \) and \( u \in Q^2 \). Hence \( W \subset \text{Lie}(S^+) \cap L(V) \). Conversely, suppose that \( L(u) \in \text{Lie}(S^+) \), where \( u = (x, x_1, x_2) \). If \( (x, x_1) = 0 \), then \( u = x_2 e \in L(V^+) = W \). Suppose \( (x, x_1) \neq 0 \). Let \( u = \alpha_u c_u + \beta_u \sigma(c_u) \) be the spectral decomposition. Since \( L(u) \) is in \( \text{Lie}(S^+) \), \( \exp(tL(u)) = \exp(2L(t\frac{u}{2})) \in S^+ \) for all \( t \in R^+ \). Because

\[
P(\exp tu) = \exp(2L(t\frac{u}{2})) \in S^+
\]

and by proposition IV.4.8, \( \exp(tu) \in Q^2 \). Now

\[
\exp(tu) = e^{i\alpha_u c_u} + e^{i\beta_u} \sigma(c_u).
\]

Then the second coordinate of \( \exp(tu) \) becomes: Let \( k = ||(x, x_1)|| \neq 0 \).

\[
\frac{e^{iz_2 + ik}}{2k} x_1 - \frac{e^{iz_2 - ik}}{2k} x_1 = \frac{x_1}{2k} [e^{iz_2}(e^{ik} - e^{-ik})] \geq 0.
\]
Therefore $x_1$ must be non-negative. Therefore $u \in V^+$ and hence

$$\text{Lie}(S^+) \cap L(V) = W.$$ 

\[
\square
\]

**IV.4.2. The group of units of $S^+$**

Set $H$ be the group of units of $S^+$:

$$H = S^+ \cap (S^+)^{-1} = S \cap S^{-1}.$$ 

Then

**Theorem IV.4.12.**

$$H = \{ h \in G(\Omega) \mid g(\mathbb{R}^+ P^2) = \mathbb{R}^+ P^2 \}$$

$$= \{ h \in G(\Omega) \mid g(\Omega^2) = \Omega^2 \}$$

$$= H_1 \cdot \mathbb{R}^+,$$

where $H_1$ is the group of units in $S^+_1$.

**Proof.** It follows from the fact $S^+ = S$ in corollary 5.10. And since $S^+ = S^+_1 \cdot \mathbb{R}^+$, it is easy to check that $H = H_1 \cdot \mathbb{R}^+$. 

Set $V^0 := E \oplus \{0\} \oplus \mathbb{R}$. Then $V^0$ is a Jordan subalgebra of $V$ with the symmetric cone $\Omega^0$ (Remark II.4).

**Lemma IV.4.13.** Let $w = (x_0, x_2) \in \Omega^0$. Then $P(w) \in H$.

**Proof.** By proposition IV.4.8, $P(w) \in S^+$. Since $\Omega^0$ is closed under inversion, $w^{-1} \in \Omega^0$. Since $P(w)^{-1} = P(w^{-1})$ and again by proposition IV.4.8, $P(w)^{-1} \in S^+$. Hence $P(w) \in H$. 

\[
\square
\]
Lemma IV.4.14. If \( h \in H \cap \text{Aut}(V) \), then \( h(E \oplus \{0\} \oplus \mathbb{R}) = E \oplus \{0\} \oplus \mathbb{R} \).

**Proof.** Since \( h \in H \), \( h((x,0,||x||)) \in E \oplus \{0\} \oplus \mathbb{R} \). Hence \( h((x,0,|x|)) = h((x,0,||x||)) + h(0,0,|x| - ||x||) \) and since \( h \in \text{Aut}(V) \), it is equal to

\[
h((x,0,|x|)) + (0,0,|x| - ||x||) \in E \oplus \{0\} \oplus \mathbb{R}.
\]

Therefore \( h(E \oplus \{0\} \oplus \mathbb{R}) \subseteq E \oplus \{0\} \oplus \mathbb{R} \). Since \( h^{-1} \in H \cap \text{Aut}(V) \), \( h^{-1}(E \oplus \{0\} \oplus \mathbb{R}) \subseteq V \). □

**Proposition IV.4.15.** \( H = \mathcal{P}(\mathcal{O}^0)(H \cap \text{Aut}(V)) \).

**Proof.** Let \( g \in H \). Then by previous lemma IV.4.14, \( g(E \oplus \{0\} \oplus \mathbb{R}) = E \oplus \{0\} \oplus \mathbb{R} \). Let \( g = P(w)h \) be the polar decomposition of \( g \) with \( w \in \mathcal{O}, h \in \text{Aut}(V) \). Let \( w = (x,x_1,x_2) \in \mathcal{O} \). Then \( g(e) = P(w)h(e) = P(w)(e) = w^2 \in E \oplus \{0\} \oplus \mathbb{R} \). But the second coordinate of \( P(w)e \) is \( 2x_1x_2 \). Since \( x_2 \neq 0 \), \( x_1 \) must be zero. Therefore \( w = (x,0,x_2) \in \mathcal{O}_0 \). Now \( h = P(w)^{-1}g = P(w^{-1})g \in H^2 = H \). Thus \( h \in H \cap \text{Aut}(V) \). Again by lemma IV.4.13, the converse inclusion holds and hence we finish the proof. □

**Lemma IV.4.16.** If \( h \in H \cap \text{Aut}(V) \), then \( h(0,x_1,0) = (0,x_1,0) \). In particular, \( h(\{0\} \oplus \mathbb{R} \oplus \{0\}) = \{0\} \oplus \mathbb{R} \oplus \{0\} \).

**Proof.** Since \( h \in \text{Aut}(V) \), by theorem II.2.12, \( h \) leaves the third coordinate fixed. So we may assume that \( h((0,1,0)) = (y,y_1,0) \). By lemma IV.4.13, \( h(E\oplus\{0\}\oplus\{0\}) = E\oplus\{0\}\oplus\{0\} \). So there exists \( x \in E \) such that \( h((x,0,0)) = (y,0,0) \). Note that \( (0,1,0) \circ (x,0,0) = 0 \). Hence

\[
0 = h((0,1,0) \circ (x,0,0)) = (y,y_1,0) \circ (y,0,0) = (0,0,||y||^2)
\]

which implies \( y = 0 \). Since \( (0,1,0)^2 = e, (0,y_1,0)^2 = (0,0,y_1^2) = e \) and hence \( y_1^2 = 1 \). But \( h \in \mathcal{S}^+ \) and \( h((0,1,1)) = h((0,1,0)) + e = (0,y_1,1) \) which implies...
that \( y_1 = 1 \). Therefore \( h((0, 1, 0)) = (0, 1, 0) \). From the linearity of \( h \), we get
\[
h(\{0\} \oplus \mathbb{R} \oplus \{0\}) = \{0\} \oplus \mathbb{R} \oplus \{0\}.
\]
\[\square
\]
**Corollary IV.4.17.** If \( h \in H \cap \text{Aut}(V) \), then \( h \) leaves the 2-nd and 3-nd coordinate fixed.

**Proof.**
\[
\begin{align*}
h((y, y_1, y_2)) & = h((y, 0, y_2)) + h((0, y_1, 0)) \\
& = h((y, 0, 0)) + h((0, 0, y_2)) + (0, y_1, 0) \\
& = (y^*, 0, 0) + (0, y_1, 0) \\
& = (y^*, y_1, y_2),
\end{align*}
\]
for some \( y^* \in E \).
\[\square
\]
**Theorem IV.4.18.** \( H \cap \text{Aut}(V) \) is exactly the set of all elements in \( G(\Omega) \) which leave the second and third coordinates fixed.

**Proof.** Suppose \( g \in G(\Omega) \) such that leaves the the second and third coordinates. Since \( g(K^\perp_E) = K^\perp_E \), \( g|_{E \oplus \{0\} \oplus \{0\}} \) must be in the orthogonal group \( O(E) \) of \( E \). For if \( x \in E \), then \( (x, 0, ||x||) \in K^\perp_E \) and by the hypothesis
\[
|\tau(x, 0, ||x||)| = (x^*, 0, ||x||) \in K^\perp_E.
\]
Thus \( ||x^*|| = ||x|| \) and hence \( g|_{E \oplus \{0\} \oplus \{0\}} \in O(E) \). Therefore \( g(P^+) \subset \mathbb{R}^+P^+ \), i.e., \( g \in S^+ \cap \text{Aut}(V) \). Similarly, we apply with \( g^{-1}, g \in H \cap \text{Aut}(V) \).
\[\square
\]
**Corollary IV.4.19.** \( H \cap \text{Aut}(V) \subset \text{Aut}(V)_r \). In particular, \( H \subset G(\Omega)_r \).

**Proof.** Note that \( \tau \circ h((y, y_1, y_2)) = \tau(y^*, y_1, y_2) = (y^*, -y_1, y_2) \) and \( h \circ \tau(y, y_1, y_2) = h((y, -y_1, y_2)) = h((y, 0, y_2)) + (0, -y_1, 0) \) is equal to \( (y^*, 0, y_2) + (0, -y_1, 0) = (y^*, -y_1, y_2) \). By theorem II.2.18 and proposition IV.4.15, \( H = P(\Omega^0)(H \cap \text{Aut}(V)) \subset P(\Omega^0)\text{Aut}(V)_r = G(\Omega)_r \).
\[\square
\]
Corollary IV.4.20. \( H = P(\Omega^0)O(E) \), where \( O(E) \) is the orthogonal group of \( E \).

Theorem IV.4.21. \( H = G(\Omega^0) \).

Proof. Note that the Jordan subalgebra \( V^0 \) is a spin factor and hence it is a Euclidean Jordan algebra. From the polar decomposition theorem (Theorem I.2.14), \( G(\Omega^0) = P(\Omega^0)Aut(V^0) \). Using proposition IV.4.15, we claim that \( Aut(V^0) = H \cap Aut(V) = O(E) \). By theorem IV.4.18, every element of \( H \cap Aut(V) \) leaves the second and third coordinate fixed. Hence if \( h \in H \cap Aut(V) \), then \( h(V^0) = V^0 \) which implies that \( h \in Aut(V^0) \). So, \( H \cap Aut(V) \subset Aut(V^0) \).

Now suppose that \( h \in Aut(V^0) \). Let \( u = (x, 0, 0) \) and let \( h(x, 0, 0) = (y, 0, y_2) \). Since \( u^2 = (0, 0, ||x||^2) \),

\[
h(u^2) = ||x||^2 e = (2y_2y, 0, ||y||^2 + y_2^2).
\]

Therefore, \( y = 0 \) or \( y_2 = 0 \). If \( y = 0 \), then \( h(\frac{x}{x_2}, 0, 0) = e \) and hence \( (\frac{x}{x_2}, 0, 0) = h^{-1}(e) = e \), which gives a contradiction. Hence \( y_2 = 0 \). This implies that \( h(x, 0, x_2) = (x^*, 0, x_2) \), where \( x^* = h(x, 0, 0) \). Note that

\[
h(u^2) = ||x||^2 e = (x^*, 0, 0)^2 = ||x^*||^2.
\]

Therefore \( h|_{E} \) is norm preserving. Hence \( h \in O(E) \) which completes the proof.

\( \square \)

In the following, we identify \( E \oplus \{0\} \oplus \{0\} \) with \( E \). Then

\[
L(V^0) = L(E) \oplus \mathbb{R}J
\]

and since \( [L(E), \mathbb{R}J] = 0 \),

\[
\exp L(V^0) = \exp L(E)\mathbb{R}^+.
\]
If \( u = (x, 0, x_2) \in \Omega^0 \), then \( u' = \log \alpha_u c_u + \log \beta_u \sigma(c_u) \in V^0 \) because the second coordinate of \( c_u \) is zero. Hence

\[
P(u) = P(\exp u') = \exp 2L(u') = \exp L(2u') \in \exp L(V^0).
\]

Conversely, if \( u = (x, 0, x_2) \in V^0 \), then \( \exp L(u) = \exp 2L(\frac{1}{2}u) = P(\exp \frac{1}{2}u) \) and \( \exp \frac{1}{2}u \in \Omega^0 \) using the spectral decomposition. Therefore we have seen that

**Proposition IV.4.22.** \( P(\Omega^0) = \exp L(V^0) = \exp E \cdot \mathbb{R}^+ \).

**Corollary IV.4.23.** \( H = P(\Omega^0) \cdot O(E) = \exp L(E) \cdot O(E) \cdot \mathbb{R}^+ \). And \( \text{Lie}(H) = o(E) \oplus L(V^0) \).

**Corollary IV.4.24.** \( H_1 = \exp L(E) \cdot O(E) \) and \( \text{Lie}(H_1) = o(E) \oplus L(E) \).

We have seen that \( S^+ = S_1^+ \cdot \mathbb{R}^+ = \exp C_1 \cdot H_1 \cdot \mathbb{R}^+ \).

Set

\[
C = \left\{ \begin{bmatrix} x_2 I & x & 0 \\ -x^* & x_2 & x_1 \\ 0 & x_1 & x_2 \end{bmatrix} \mid x_1 \geq ||x|| \right\}.
\]

Then \( \exp C = \exp(C_1 + RI) = \exp C_1 \cdot \mathbb{R}^+ \). Hence

\[
S^+ = \exp C_1 \cdot H_1 \cdot \mathbb{R}^+ \subset \exp C \cdot H \subset S^+.
\]

Since \( H = \exp L(E) \cdot O(E) \cdot \mathbb{R}^+ \), \( S^+ = \exp C_1 \cdot H \). Therefore, we have proved

**Theorem IV.4.25.** \( S^+ = S = \exp C_1 \cdot H = \exp C \cdot P(\Omega^0) \cdot O(E) = \exp C \cdot \exp L(E) \cdot O(E) \).
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Vita

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Major Field: Mathematics

Title of Dissertation: Jordan Algebras and Lie Semigroups

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EXAMINING COMMITTEE:

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Oct. 1st, 1996