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Three essays on the interest rate forward-futures differential 1. Empirical investigation of the size and the nature of the Eurodollar futures-foward differential 2. Decomposition of the interest rate forward-futures price differential 3. How much premium is there for interest rate futures?

Andrei Shynkevich
Louisiana State University and Agricultural and Mechanical College

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THREE ESSAYS ON THE INTEREST RATE FORWARD-FUTURES DIFFERENTIAL
1. EMPIRICAL INVESTIGATION OF THE SIZE AND THE NATURE OF THE EURODOLLAR FUTURES-FORWARD DIFFERENTIAL
2. DECOMPOSITION OF THE INTEREST RATE FORWARD-FUTURES PRICE DIFFERENTIAL
3. HOW MUCH PREMIUM IS THERE FOR INTEREST RATE FUTURES?

A Dissertation

submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in
The Interdepartmental Program in Business Administration
(Finance)

by
Andrei Shynkevich
B.S., Belarusian State Economic University, 2001
M.A., Economics Education and Research Consortium, 2003
M.A., State University of New York at Stony Brook, 2005
M.S., Louisiana State University, 2008
August, 2009

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I dedicate this work to my parents who raised me by having emphasized the continued pursuit of knowledge as a mean of achieving a greater success.

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Abstract

This dissertation analyzes a series of issues that surround both the theoretical modeling and the empirical estimation of the forward-futures differential, commonly known as the convexity adjustment. Opposite to theoretical implication, I find that the magnitude of the forward-futures rate differential is much smaller than what was expected, and that its sign is negative on many occasions. Neither asynchronicity bias, nor the unconventional feature of the Eurodollar futures pricing can explain the observed phenomena. The term structure interpolation error and the two business day lag between the fixing (settlement) date and the transaction (value) date to which the implied forward rates and prices are applied cannot be attributed to the observed abnormality either. I further show that the difference between the implied forward price obtained from the spot rate term structure and the original Eurodollar futures price at any point of time before maturity is composed of two parts: the element due to marking-to-market and the element arisen from the unconventional settlement of the Eurocurrency futures. It is also demonstrated that the discrepancy between the forward price and the futures price arisen from the unconventional settlement of the Eurocurrency futures can be hedged using a specific basket of caplets. This paper also performs the analysis for the three most traded interest rate futures contracts in Europe: EURIBOR futures, short sterling futures and Euroswiss franc futures. I show that the futures premium is barely detectible for the contracts with maturities below one year. The futures premium for maturities above twelve months varies across the models and is a subject to model assumptions regarding the volatility input and its evolution. Finally, I show that in the presence of the limits to arbitrage the rate on a forward rate agreement (FRA) contract and the respective implied forward rate derived from the spot yield curve would differ and their difference increases with the maturity. This finding allows to challenge the results in recently published works that argue that the convexity adjustment is not priced in by the FRA market makers.

Chapter 1 Introduction

Since the seminal work of Cox, Ingersoll and Ross (1981) it has been known that forward contracts and closely related futures contracts must be priced differently as long as the future interest rates are not known. This assertion is equally well applied to the contracts with different underlying assets and/or financial benchmarks. The published research suggests that the difference between the futures price and that of a respective forward, frequently referred as a convexity adjustment, would be more pronounced if the underlying instrument of the two contracts is an interest rate or a Treasure bond. A number of works has investigated the matter and so far, the evidence has been mixed and subject to the underlying assumptions and model construction.

This dissertation intends to tackle a number of issues that surround both the theoretical modeling of the convexity adjustment and its empirical estimation and that have not been fully addressed in the existing research on this topic. Among the subjects that are considered and incorporated in (or eliminated from, depending on the context) the analysis are the asynchronicity error, the unusual pricing feature of the interest rate futures, the extended range of maturities, the volatility input computation and its evolution specification, the frequency of marking to market, the implied nature of the forward contracts and the use of actually traded instruments instead, the affect of the limits to arbitrage and the use of international data among others. Each element has its unique implications on the results and thus must be treated with considerable care.

My dissertation consists of three essays. The first one is the empirical work, the second one is the theoretical piece and the third one is a synthesis of both theory and empirical analysis. Such construction of the study allows to legitimately tackle a series of related issues one at a time and subsequently combine the early findings into an independent yet related work that is based on solid theoretical background and demonstrated empirical evidence.

In the first essay I investigate the magnitude of the forward-futures differential, also known as the convexity adjustment, for Eurodollar interest rate instruments and attempt to identify factors affecting its size using an extensive sample for the 1988-2007 period. The innovative feature of the employed analysis is that the construction of the differential is extended from using rates for maturities up to twelve months as in previous published research to rates for maturities up to three years through the use of highly liquid swap rates. To our knowledge, this is the first empirical study that attempts to evaluate the interest rate forward-futures differential for maturities longer than a year. I also check for potential data skews and other imperfections that may be behind the obtained results.

The second essay takes a closer look at the relationship between Eurocurrency interest rate futures prices and forward prices by focusing on the way the Eurocurrency futures settlement procedure affects the forward-futures differential analysis. The phenomenon that Eurocurrency interest rate futures are settled in the way different from that of forward contracts has not been formally incorporated into early empirical works. Sundaresan (1991) argues that the implied forward price from the spot LIBOR term structure is inappropriate for the purposes of comparison with the Eurodollar futures price due to the differences in settlement procedures and introduces a hypothetical forward contract in order to eliminate the presence of the settlement factor. The essay investigates whether a more straightforward approach can be applied for the comparison of futures and respective forwards. The essay further looks at how changing frequency of the marking-to-market may affect the size of its component within the HJM term structure framework. Four popular volatility specifications are utilized to check for the robustness of the obtained results.

Until recently, the literature on the convexity adjustment in interest rate futures concentrated almost exclusively on the Eurodollar contracts. The third essay performs the analysis for the three most traded interest rate futures contracts in Europe: EURIBOR futures, short sterling futures and Euroswiss franc futures. The innovative feature of this work is the usage of the approach that allows to construct matched pairs of the empirical as well as the predicted convexity adjustment. I also extend the statement that can be met in a number of textbooks on derivatives and fixed income that the rate on a forward rate agreement (FRA) contract is a function of the current term structure and is equal to the implied forward rate. The essay investigates whether the presence of the limits to arbitrage can make the FRA rate diverge from the implied forward rate.

The rest of the dissertation is organized as follows. Chapter two presents the first essay, chapter three contains the second essay and chapter four is the third essay. Chapter five gives the summary and conclusions.

Chapter 2 Empirical Investigation of the Size and the Nature of the Eurodollar Futures-Forward Differential

2.1 Introduction

In this paper we intend to take a closer look at the difference arising from the pricing of interest rate forwards and futures known as the convexity adjustment. Futures contracts are marked-to-market daily while forward contracts are fully settled at expiration. Theoretical literature starting with Cox et al. (1981) argues that under certain conditions different cash flow patterns of the two contracts must result in different prices and rates implied by those contracts. In the stochastic interest rate environment, futures rates will exceed their forward counterparts. If futures prices are positively correlated with interest rates, then futures prices will exceed forward prices as well. Previous empirical literature concentrated on the relationship between forward and futures rates and prices and considered data limited to LIBOR rates where the latter have maturity up to 12 months[1]. This paper extends the maturity horizon up to three years by using swap rates reported by Bloomberg to construct a LIBOR/swap yield curve using an exponential interpolation technique. To our knowledge, this is the first empirical study that attempts to evaluate the interest rate forward-futures differential for maturities longer than a year. The use of reported swap rates in order to extend the spot yield curve is fully justified since they are for a highly liquid market and the bid-ask spread rarely exceeds one basis point.

Using derivatives rather than bonds for hedging purposes is generally considered more efficient since the interest rate futures market is highly liquid and short positions may be easily taken. The enormous volume of trading in interest rate futures is a reflection of the magnitude of the widespread use of such techniques. The three-month Eurodollar futures are contracts with a three-month US LIBOR as the underlying. Such contracts are traded on the Chicago Mercantile Exchange. Each contract has a face value of $1,000,000 and represents the offer interest rate on an interbank three-month deposit. The major difference between futures and over-the-counter traded forward contracts is that futures are subject to margining. Margining is a process of closing an existing contract at a closing (settled) price at the end of the trading day and writing a new but identical contract at a new (closing) price. The difference between the closing price of the contract on the previous trading day and its closing price today is paid or received at the closing time today using a margin account. The difference in the pattern of the stream of cash

[1] There do not exist LIBOR rates for longer maturities.

flows between forward and futures contracts arising from margining is believed to add extra value to a futures contract since profits obtained from margining can be reinvested daily at a higher rate, while the losses from it can be financed at a lower rate. The difference between the futures and forward prices or rates is often referred as the convexity adjustment.

Empirical investigation of the convexity adjustment may be prone to a series of measurement imperfections making the results subject to a bit of skepticism. Indeed, when the value of the differential is often expressed in single digit basis points[2], even a seemingly small skew in a data sample used or a measurement methodology applied can substantially affect the results and alter the ultimate conclusions. Interpolation error, asynchronicity bias and the way maturity length is measured are only few of such examples. Another issue that has taken a central stage recently but has not been addressed in early empirical studies is the fact that interest rate futures are priced and settled in an unusual manner resulting in the existence of the difference between a forward price and a respective futures price even on the day of expiration. This observation would lead one to suggest that the results of the convexity adjustment analysis and its conclusions may very well depend on whether such analysis is conducted rate-wise or price-wise. The purpose of this paper is to demonstrate how different such results would be provided that we account for most significant measurement imperfections in order to eliminate their influence on our conclusions.

We find that for the extended range of maturities, the magnitude of the *futures-forward rate* differential remains small and on many occasions it is negative, opposite to theoretical implications. Changes in average differential across maturities are negligible. Asynchronicity error cannot explain the observed phenomena, nor can the unusual pricing feature of Eurodollar futures or the two business day lag between the settlement date and the value date of the forward contract. If measured as the *forward-futures price* differential, the average convexity adjustment is still of tiny value but the number of negative occurrences in the sample drops significantly suggesting that the unconventional pricing characteristic of the Eurodollar futures is positively related to the value of the price differential. Regression analysis performed for the rate differential provides dubious results but if the latter is conducted for the price differential, it results in much better goodness-of-fit. Although time to maturity, level of rates and its volatility

[2] This is very much true for short maturities while the outcome for longer maturities is expected to be somewhat larger but it has not been thoroughly investigated in the existing literature.

are positively related to the value of the price differential, the default factor expressed by the TED spread is unable to capture the negative nature of the convexity adjustment which remains largely unexplained.

The rest of the paper is structured as following. Section 2.2 contains the literature review, section 2.3 describes how the convexity adjustment is derived from the obtained zero coupon rates and futures quotes. Section 2.4 is devoted to the description of the yield curve interpolation methods utilized in the paper while section 2.5 addresses several important issues that arise when interpolation techniques are to be implemented. Section 2.6 provides data description and analysis, section 2.7 contains estimation results and section 2.8 concludes.

2.2 Literature Review

Black (1976) was first to show that in a world of constant interest rates, forward prices are equal to futures prices. Cox et al. (1981) and Jarrow and Oldfield (1981) discover independently that due to the difference in the form of payments between forward and futures contracts, forward and futures prices may be quite different if interest rates are stochastic. The same is true about forward rates and futures rates. These papers introduce the layout of the theoretical background for pricing futures and forward contracts under the default-free conditions. In reality, forward contracts are subject to a higher degree of default risk by a second party than the futures. Empirical literature finds some evidence of the significant presence of the default factor in the difference between futures and forward pricing. Meulbroek (1992) focuses on contractual distinctions as an explanation for the price divergence between interest rate futures and forward contracts. She argues that market inefficiencies and imperfections are not the only explanation for the differences in futures and forward prices. According to her paper, if the covariance between fluctuations in futures prices and riskless bond prices is large, marking-to-market may be an important contributor to the futures-forward price spread.

The presence of the default risk is not the only distinctive feature between interest rate forwards and futures though. The other major distinction lies in the way the futures are prices and settled. The Eurodollar futures contract settles to the 90-day London Interbank Offered Rate (LIBOR), the yield implied by the underlying asset in the form of the 90-day Eurodollar time deposit. Sundaresan (1991) argues that since Eurodollar futures contract settles to yield as opposed to prices, interest rate forward prices should differ from the futures prices even in the absence of marking-to-market. He modifies principles developed by Cox et al. (1981) for futures contracts on yields and shows that differences between implied forward prices and futures prices arising from the unique settlement feature of Eurodollar futures contracts are much larger than differences caused by the marking-to-market effect. Using the term structure of Cox et al. (1985) to derive forward and futures prices, Sundaresan finds that, first, in all cases, futures prices are lower than forward prices and futures rates are lower than implied forward rates, and second, the differences between futures rates and implied forward rates are much greater than the differences between futures prices and forward prices. To examine the extent to which his results are supported by data, Sundaresan uses daily quotes on Eurodollar futures prices and LIBOR rates for the period 1985-1988. By choosing to work with three- and six-month LIBOR only, his sample contains only 13 matched 90-day forward and futures prices producing the mean difference between futures and implied forward prices of 50 basis points.

Such conclusions would make one think that results of the convexity adjustment analysis in interest rate futures and forwards would depend on whether the analysis is conducted rate-wise or price-wise. Grinblatt and Jegadeesh (1996) derive a closed form solutions for the futures-forward yield differential and show that, theoretically, the difference should be small. Using data on Eurodollar futures prices and LIBOR over the 1982-1992 period, they observe significant differences between futures and forward prices and argue that these differences are likely to have been caused by mispricing of futures contracts and lack of arbitrage activity, and that this mispricing was gradually eliminated over time. Neither default and liquidity risk, nor differential taxation can explain the differences between the futures and forward Eurodollar rates, while mark-to-market effect on the theoretical differences is small. Two interpolation methods are used in the paper: a cubic spline fit to the futures rates and another cubic spline applied to the spot LIBOR curve. The first approach produced larger mean values of the rate differential although a notable proportion of occurrences of the rate differential for maturities of three and six months in the second part of the sample was negative. Their assertion about market inefficiency is supported by the results of the analysis of the timeliness of information flow across the markets: while there is no delay in the flow of information from the forward market to the futures market, there exists a delay of information flow from the futures to the forward market.

Gupta and Subrahmanyam (2000) examine whether a convexity correction, arising from the negative convexity exhibited by interest rate swaps, has been incorporated efficiently into interest rate swap pricing over time in four major currencies. They consider swaps with maturities up to five years for US dollar currency and swaps with maturities of only two years for three other currencies[3]. Their evidence is based on small (on average, less than one basis point) values of the introduced swap-futures rate differential and it suggests that during 1987-1990, swaps were priced ignoring the convexity correction but after that market swap rates drifted below the rates implied by futures prices. Such results are interpreted as the evidence of mispricing of swap contracts during the early years which was subsequently eliminated over time.

Combining the results of Gupta and Subrahmanyan with those of Grinblatt and Jegadeesh, one gets an interesting picture. First, futures contracts were mispriced off the spot LIBOR curve, and after that swap contracts were mispriced off the futures curve. Although the two results do not necessarily contradict each other, it is somewhat surprising to see that during 1982-1987 futures were overpriced relative to forward contracts with shorter maturities while during 1987-1990 swap contracts were overpriced relative to futures with longer maturities.

Using the term structure model developed by Heath et al. (1992), Chance (2003) conducts an alternative test that estimates the evolution of the term structure that yields arbitrage-free futures prices. The differences between Eurodollar futures and forward prices over the period 1987-2000 are found to be much smaller than in earlier works, essentially, they are not different from zero but are consistently negative, as predicted by the theory. The model considered, however, assumes default-free environment. Nevertheless, even without the default risk present, such results allow us to infer that the forward-futures price differences attributable to marking-to-market would be even closer to zero.

Empirical research shows that differences between forward prices and futures prices are small for contracts with short maturities but there is no conventionally accepted evidence about what happens to the spread when longer maturities are considered. The aforementioned Gupta and Subrahmanyam's paper mentions that the spread between futures and forward yields due to the marking-to-market feature is expected to be more pronounced in longer-term (over a year maturity) contracts, something that is not supported by the evidence of our results. In our paper we want to make a unique attempt to conduct an investigation of the convexity adjustment within a combined market consisting of LIBOR rates for short maturities and swap rates for longer maturities on one side, and Eurodollar futures market on the other side.

[3] Due to poor liquidity of respective Eurocurrency futures contracts for longer maturities

According to several papers that employ various stochastic interest models, the difference between the Eurodollar futures rate and the respective forward rate should increase as a function of interest rate volatility, contract maturity and correlation between forward rates and spot rates (see Gupta and Subrahmanyam [2000], Hull [2006] and Meulbroek [1992]). Ho and Lee (1986) develop a model where the continuously compounded forward rate is normally distributed with a constant variance. Based on their model, the convexity adjustment can be expressed as[4]

$$\text{Eurodollar forward rate} - \text{Eurodollar futures rate} = 0.5\sigma^2 mn,$$

where σ is a standard deviation of the change in the continuously compounded forward rate in one year, m is the time until the maturity of the futures contract (in years) and n is the number of years (from the current date) until the maturity of the Eurodollar deposit that serves as the underlying for the respective Eurodollar futures. If we consider the three-month Eurodollar futures then $n = m + 0.25$.

Not every paper agrees on such conclusions though. Based on the sample period of 1982-1992, Grinblatt and Jegadeesh (1996) use the Cox-Ingersoll-Ross and the Vasicek term structure models to compute closed-form solutions for the Eurodollar futures-forward rate differences and their results say that rate differences do not appear to be monotonic functions of interest rate volatility. Recent research discovers new factors that may affect the magnitude of convexity adjustment. Piterbarg and Renedo (2004) investigate how the volatility smile can effect the convexity adjustment and how the smile can be incorporated in a model to value Eurodollar futures. They employ the reduced-form stochastic volatility model of LIBOR rates to derive a closed-form solution for the forward-futures rate differential. Their results show a pronounced impact on the convexity adjustment made by the volatility smile parameters such as the blending parameter and the volatility of *variance* for futures with long maturities. In their work the forward-futures rate differential is a monotonically increasing function of each of the two parameters but their effects are not significant when futures expiring in less than two years are considered. In the empirical results section we explore whether the magnitude of the forward-futures differential is related to the level of interest rates, time to maturity, interest rate volatility and volatility of its volatility.

[4] For derivation see Hull (2006) and Technical Note #1 on www.rotman.utoronto.ca/~hull

2.3 How Swap Rates, Forward and Futures Rates and Prices Are Measured

A swap rate for a fixed side of the swap with semi-annual payments is defined as its coupon rate (expressed as a fraction):

$$P(t,M) = \frac{c}{2}\sum_{m=1}^{2M} \exp(-r(t,0.5m)0.5m) + F\exp(-r(t,M)M), \tag{2.1}$$

where *P(t, M)* is the price of a newly issued swap at time *t* that matures at *M*, *r(t, m)* is a spot rate for maturity *m* observed at time *t* and *F* is the swap face value. Note that swap rates in (2.1) are for swaps where fixed payments are made semi-annually[5]. Since the price of a newly issued swap, if we also consider the face value paid at maturity, is assumed to be equal to its face value, then assuming without loss of generality a unit face value, we can express the swap rate as a function of discount factors:

$$c(t,M) = 2\frac{1 - S(t,M)}{\sum_{m=1}^{2M} S(t,0.5m)}, \tag{2.2}$$

where *S(t, 0.5m)* is the discount factor at time *t* applied for maturity *0.5m* and it is equal to *exp(-r(t, 0.5m)0.5m)*.

We, however, want to extract spot rates or discount factors given the swap rates. The problem is that if, for example, we consider a five years horizon then normally we know swap rates for maturities of 1, 2, 3, 4 and 5 years only. What's missing are the swap rates for maturities 0.5, 1.5, 2.5, 3.5 and 4.5 years. Swap rate for maturity 0.5 years can be constructed using a hypothetical swap with maturity of six months by employing a six-month LIBOR to compute the six-month discount factor. In such case we get

$$c(t,0.5) = 2\frac{1 - S(t,0.5)}{S(t,0.5)} = 2[\exp(r(t,0.5)0.5) - 1], \tag{2.3}$$

where *r(t, 0.5)* is the continuously compounded six-month LIBOR. To get the swap rates for other four maturities we need to utilize one of the aforementioned interpolation techniques. Once we obtain estimates of *c(t, 1.5)*, *c(t, 2.5)*, *c(t, 3.5)* and *c(t, 4.5)*, we can extract the respective discount functions iteratively. By rearranging terms in the equation (2.2) where a swap rate is a function of the discount factors, we get

[5] We use example with two semi-annual coupon payments but the general idea of swap pricing is not affected by the chosen number of payments.

$$S(t,0.5m) = \frac{2 - c(t,0.5m)\sum_{n=1}^{m-1} S(t,0.5n)}{2 + c(t,0.5m)} \quad \text{for } m = 3,...,10. \tag{2.4}$$

Now we can compute spot rates for maturities ranging from 1.5 to five years with six-month intervals. By adding reported LIBOR rates with maturities up to twelve months to the derived spot rates for longer maturities, we obtain our sample of spot rates that will be used to estimate the spot rate term structure using the exponential interpolation method. The reported LIBOR rates must be converted into continuously compounded rates first. Forward rates are obtained using estimated spot rates from the applied model as following:

$$F^R(t, m_1, m_2) = \frac{r(t, m_2)m_2 - r(t, m_1)m_1}{m_2 - m_1}, \tag{2.5}$$

where $F^R(t, m_1, m_2)$ is an implied continuously compounded forward rate from m_1 to m_2 observed at time t.

An investigation of the convexity adjustment can be performed using either forward and futures rates, or their respective prices. In order to calculate the difference between the derived three-month Eurodollar futures rate and the implied forward rate for the same period, we need to convert futures rates, which are computed as 100 minus the quoted futures prices, into the rates with continuous compounding. We also must take into account the actual/360 day count convention used for Eurodollar futures. This yields

$$f^R(t, m, m+90) = \frac{365}{90} \ln\left(1 + \frac{90}{365}\frac{365}{360} f_q^R(t, m, m+90)\right), \tag{2.6}$$

where $f^R(t, m, m+90)$ is a continuously compounded three-month Eurodollar futures rate and $f_q^R(t, m, m+90)$ is the quoted futures rate based on annual compounding and actual/360 day count (= [100 – quoted futures price] / 100).

The inverse relationship between bond prices and interest rates implies that Eurodollar futures prices are expected to exceed respective forward prices. Computation of the difference between forward and futures prices requires calculation of the implied forward price. The three-month Eurodollar futures price is obtained from the quoted futures price adjusted by the day count factor:

$$f^P(t, m, m+90) = 1 - \frac{100 - f_q^P(t, m, m+90)}{100}\frac{90}{360}, \tag{2.7}$$

where $f_q^P(t, m, m+90)$ is the quoted futures price. The respective forward price is computed from the formula below:

$$F^P(t,m,m+90) = \frac{\exp(r(t,m)m)}{\exp(r(t,m+90)(m+90))}. \tag{2.8}$$

Chance (2003) shows that due to the unusual settlement feature of the Eurodollar futures, the futures price at expiration is not equal to the respective forward price. The critical characteristic of the futures market is that the futures contract is priced as if the underlying were a discount instrument and it causes non-convergence of the futures price and the spot price at the expiration. The forward contract, however, is priced with its underlying as an add-on instrument and convergence is achieved in this case. The difference of forward minus futures price at the futures expiration date is always positive given a positive LIBOR and is a monotonically increasing function of LIBOR at expiration. These two relationships arise from the fact that, at expiration,

$$F^P - f^P = \frac{1}{1+rk} - (1-rk) = \frac{r^2k^2}{1+rk} > 0, \text{ and} \tag{2.9}$$

$$\frac{d(F^P - f^P)}{dr} = \frac{rk^2(2+rk)}{(1+rk)^2} > 0, \tag{2.10}$$

where r is the annual three-month LIBOR at expiration and k is the adjustment factor that is equal to 90 / 360.

By comparing results of the analysis made using prices versus those when rates are used will show us whether the two methods provide consistent results and lead to similar conclusion. It will also be seen whether the non-conventional Eurodollar futures pricing approach contributes in some fashion to the value of convexity adjustment.

2.4 Description of Employed Interpolation Methods

Given the set of par yields, we can extract discount factors and respective spot rates. The respective procedure is known as bootstrapping. The problem, however, is that our system is under-determined since it involves off-the-run payments: we have par yields in annual increments while fixed payments are made every six months, therefore, it is not possible to calculate all discount factors unless we know par yields for swaps with maturities of 1.5, 2.5, 3.5, etc. years which is not the case. In such cases the term structure literature suggests the use of interpolation. There have been numerous methods offered to tackle the interpolation issue but there is no ideal, widely accepted technique to rely on. The traditional measures of statistical fit include the root mean squared error and mean absolute error. The former measure is used in Mansi and Phillips (2001) who compare three different yield curve smoothing models, and the latter one is employed by Bliss (1997) who compares five distinct methods for estimating the term structure. The root mean squared error (RMSE) is defined as

$$RMSE = \sqrt{\frac{1}{n}\sum_{j=1}^{n}\varepsilon_j^2}\ , \tag{2.11}$$

and the mean absolute error (MAE) is expressed as

$$MAE = \frac{1}{n}\sum_{j=1}^{n}\left|\varepsilon_j\right|. \tag{2.12}$$

Jordan and Mansi (2003) compare five different methods of estimation of the par yield curve using US Treasury bond data. They demonstrate that in terms of interpolation properties and robustness in the face of pricing errors, continuous time bootstrapping better approximates the term structure than discrete-time bootstrapping methods with particular advantage belonging to continuous time methods based on exponential functional forms such as the Nelson and Siegel and the Mansi and Phillips methods. The latter was developed by Mansi and Phillips (2001) and it was originally proposed as a model to estimate the par yield curve. The former technique was introduced by Nelson and Siegel (1987) and has become very popular among term structure practitioners due to its good fit properties, intuitive appeal and parsimony. It however was also suggested for estimation of the forward and spot term structure, while Mansi and Phillips (2001) and later Jordan and Mansi (2003) used it as an alternative model to estimate the par yield curve and the model turned out to be relatively accurate when compared to other considered counterparts. In both aforementioned papers original Nelson-Siegel model is used and it is claimed by the authors that the Mansi-Phillips model dominates the Nelson-Siegel model since for the considered sample of on-the-run Treasuries the errors produced by the Mansi and Phillips technique are typically lower than those arising from the application of the Nelson and Siegel model in all maturity ranges, although the difference is statistically significant only for a few maturities. Svensson (1994) extends the Nelson-Siegel model by including two more parameters, which allows for a better fit especially if the term structure experiences more than a single hump. It is not clear how the extended Nelson-Siegel model would do against the Mansi and Phillips one although the former method requires two more parameters to be estimated than the latter model.

After we construct the par yield term structure, we use it to estimate par yields on hypothetical swaps with off-the-run maturities (1.5 years, 2.5 years, etc.). Given these estimated par yields, we shall have the fully determined system and discount factors as well as respective spot rates can be bootstrapped in a simple fashion. Now, after we get the set of spot rates, we can estimate the spot rate term structure by applying one of the suggested techniques. According to Bank of International Settlements paper No. 25 (2005), many European central banks use either the originally proposed Nelson-Siegel method of estimating the zero-coupon yield curve, or its extended modification by Svensson[6]. This paper follows the pack and employs the extended version of Nelson-Siegel model to interpolate the spot yield curve. Below we provide a description of the two interpolation techniques employed in the paper.

The exponential functional form of the term structure can accommodate all yield curve shapes, whether it is upward, downward, or humped. Mansi and Phillips (2001) introduce a functional form to estimate the par yield curve using the observed on-the-run Treasuries. Their model is of the exponential class which is fit by non-linear least squares and it has an easy interpretation. This model is similar to those of Diament (1993) and Nelson and Siegel (1987) although it is argued by the authors that their model outperforms the other two in terms of pricing accuracy and convergence properties. The Mansi-Phillips model represents the yield, $y(m)$, by the following continuously differentiable function:

$$y(m) = \alpha_1 + \alpha_2 \exp(\alpha_4 m) + \alpha_3 \exp(2\alpha_4 m). \tag{2.13}$$

Apart from better goodness-of-fit characteristics, this model is less restrictive when compared to the Diament model which requires two different regressions, depending on the observed yield curve shape while the Mansi-Phillips model determines the term structure using a single functional expression that accommodates any shape.

The equation (2.13) consists of a constant term and a sum of two exponential functions. The sign of the coefficient (α_2 and α_3) determines whether the respective exponential term is convex (positive sign), or concave (negative sign). The last exponential term has a higher rate of decay. For instance, the humped yield curve will require $\alpha_3 < 0$ and $\alpha_2 > 0$, since it is concave to the left of its maximum and convex to the right of its maximum.

[6] Both models belong to the class of exponential techniques.

The extended Nelson-Siegel model gives the following functional form to the *instantaneous* forward rate:

$$f(m)=\beta_0+\beta_1\exp(-m/\tau_1)+\beta_2[(m/\tau_1)\exp(-m/\tau_1)]+\beta_3[(m/\tau_2)\exp(-m/\tau_2)], \quad (2.14)$$

where β_0, β_1, β_2, β_3, τ_1 and τ_2 are the parameters to be determined from the optimization procedure. The original Nelson-Siegel forward rate curve that excludes the last term of the expression above can be viewed as a constant plus a Laguerre function, a polynomial times an exponential decay term. The generating function *T(t,x)* of the Laguerre polynomials is defined as[7]

$$T(t,x)=\frac{1}{1-t}\exp\left(\frac{-xt}{1-t}\right),\ |t|<1. \quad (2.15)$$

Given the relationship between the spot rates and the instantaneous forward rates,

$$r(t,m)=\frac{1}{m}\int_0^m f(t,u)du\,, \quad (2.16)$$

where *f(t,u)* is the instantaneous forward rate at time *u* as observed at time *t*, by integrating the equation for the instantaneous forward rate over *m* and dividing both sides of it by *m*, we can get the functional form for the spot rate:

$$r(t,m)=\beta_0+\beta_1\frac{1-\exp(-m/\tau_1)}{m/\tau_1}+\beta_2[\frac{1-\exp(-m/\tau_1)}{m/\tau_1}-\exp(-m/\tau_1)]+$$
$$+\beta_3[\frac{1-\exp(-m/\tau_2)}{m/\tau_2}-\exp(-m/\tau_2)]. \quad (2.17)$$

This functional form has a relatively simple interpretation: β_0 is the level to which this function converges when time approaches infinity, β_1 determines the general slope of the function, β_2 and β_3 determine two humps in the term structure. By restricting $\beta_3 = 0$, we get the original Nelson-Siegel model which considers only one hump. The sum of β_0 and β_1 determines the spot rate at maturity zero; hence, this sum must be positive. Normally, $\beta_0 + \beta_1$ is restricted to equal to the observed overnight LIBOR or the official repo rate. The value of β_0, the spot rate with infinite maturity, must be positive as well.

The τ parameters govern the exponential decay rate: a small (large) τ_i produces a fast (slow) decay. Also τ_1 and τ_2 determine the maturities at which the loadings on β_2 and β_3 respectively achieve their maxima. For their original model and sample, Nelson and Siegel

[7] See Bayin (2006) for details

(1989) suggest that the best-fitting value of τ would be in the range of 50-100 days. They however use different values of τ in the optimization routine and the best-fitting τ-parameter for different dates in the sample ranges from 10 to 365 which are their boundaries of the range of search. At the same time they demonstrate that little precision of fit would be lost if the median value of 50 for τ is imposed for all data sets in their sample. Fabozzi et al. (2005) fix τ at the level of three (measured in years) while fitting the Nelson-Siegel model to the swap curve for a range of maturities from three months to 30 years. Diebold and Li (2003) use the original Nelson-Siegel model to estimate the yield curve and they also fix τ_1 at a prespecified level and then use ordinary least squares to estimate the betas. Diebold and Li choose τ_1 = 0.0609 which in their setup corresponds to the medium-term factor of 30 months.

If applied to spot rates data, the vector of parameters $\beta = (\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2)$ is estimated in the following way. The continuously compounded spot rate is assumed to follow the equation above. The estimated price of a zero-coupon bond $P^e(m, \beta)$ is defined as

$$P^e(m,\beta) = \exp(-\frac{r(m,\beta)}{100} m). \tag{2.18}$$

The vector of parameters is then estimated by minimizing the sum of squared zero-coupon bond price errors:

$$\min_{\beta} \sum_{j=1}^{n} [P_j - P_j^e(m,\beta)]^2. \tag{2.19}$$

The alternative to the price error minimization problem above is to minimize the sum of squared yield errors. In fact, Svensson (1995) points out that minimizing yield errors provides a substantially better fit for short maturities while the two procedures (minimization of squared price errors versus squared yield errors) tend to perform equally well for long maturities. This is because yields for short maturity bonds are much more sensitive to changes in prices of those bonds than yields for bonds with longer maturities. In some empirical works squared yield or price errors are weighted by a value which is a function of inverse duration (see Jordan and Mansi [2003], Coleman et al. [1995] and Waggoner [1997] among others). In the case of a zero-coupon bond where duration is equal to the bond's maturity, the weights m_j^**'s* are expressed as

$$m_j^* = \frac{1/m_j}{\sum_{i=1}^{n} 1/m_i}. \tag{2.20}$$

The choice of weights related to the reciprocal of the modified duration places less weight on instruments (rates) with longer maturities (higher durations). It allows to achieve a better fit for instruments with shorter maturities by giving up a portion of the approximation for those with longer maturities. However, when the number of considered maturities in the sample rises, this weighting approach may lead to significant errors for the components with high duration.

The entire optimization algorithm can be described as following. Initialize the vector of parameters β. Calculate the estimated (theoretical) spot rates for each maturity in the sample. Compute the (un-)weighted sum of squared yield errors and examine the convergence condition. If the condition does not hold, choose a new vector of initial parameters[8]. Repeat the steps above until the convergence is achieved.

In both models described above, the term structure is estimated using nonlinear least squares and, therefore, the convergence of coefficients to final values will depend on their initial estimates. Therefore, it is important to choose appropriate initial guesses for the parameters in question in order to have a better chance for the estimation procedure to achieve a true minimum of the function. In the Nelson-Siegel model, the long-term factor β_0 governs the yield curve level and it is easy to notice that $\lim_{m\to\infty} r(t,m) = \beta_0$. The short-term factor β_1 is related to the yield curve slope. It can be checked that $\lim_{m\to\infty} r(t,m) - r(t,0) = -\beta_1$. This is exactly how Frankel and Lown (1994) define the slope. Diebold and Li (2003) define the yield curve slope as the ten-year rate minus the three-month rate. The latter constraint can be replaced by $r(t,0) = \beta_0 + \beta_1$. The medium-term factor β_2 is related to the yield curve curvature which is defined by Diebold and Li as twice the two-year yield minus the sum of the ten-year and three-month yields.

As for the initial conditions of the Mansi-Phillips model, note first that $r(t,0) = \alpha_1 + \alpha_2 + \alpha_3$ and $\lim_{m\to\infty} r(t,m) = \alpha_1$ assuming negative α_4. Therefore, initial estimate of α_1 should be close to the par yield on a swap with the longest available maturity (e. g. 30 years). Mansi and Phillips suggest to set the initial condition for $\alpha_2 + \alpha_3$ to equal to the difference between the observed three-month yield and the thirty-year yield. Also given the available observations, find out the maturity, m^*, for which swap rate is the highest in the sample. Use the

[8] This new vector will be a function of the initially chosen β.

proposed equation for the par yield to plug $r(t, m^*)$ and m^* to get the third initial condition. The last initial condition is to set $dr(t,m^*)/dm = 0$ at m^*.

Ideally, the minimization of the sum of weighted errors is subject to several additional constraints:

$$r(t,0) \geq 0,\ \lim_{m\to\infty} r(t,m) \geq 0, \text{ and } \exp(\frac{r(m_{k+1})}{100} m_{k+1}) \geq \exp(\frac{r(m_k)}{100} m_k)\ \forall\ k \geq 0.$$

The last constraint assures that forward rates are non-negative.

Since the term structure is estimated for each day in the sample independently of other dates, there should not be a place for interaction between successive observations as a result of errors in estimated coefficients. If errors are autocorrelated, it may be due to either misspecification of the functional form, or to the estimation method, or it may suggest market inefficiency. It was first noticed by Nelson and Siegel (1987) that yield errors (regression residuals) are not random but rather seem to exhibit some dependence along the maturity levels. They contribute sharp rises in the average residual yield as a function of maturity to the maturities of the *bills* auctioned by the US Treasury. In Diebold and Li (2003), the residual autocorrelations also indicate that pricing errors are persistent. The Mansi-Phillips model that we use to estimate the swap par yield curve also suffers from non-randomness of pricing errors. Bliss (1997) shows that regardless of the estimation method of the term structure used, there is a persistency in errors and conclude that there are method-independent persistent factors that affect note and bond prices that cannot be captured by the pricing configuration. Hence, it is unlikely that the fitted-price errors are purely random.

2.5 Some Specific and Subtle Features of the Yield Curve Building

While constructing the term structure of interest rates, several issues must be addressed regarding the choice of appropriate building blocks. First, one should choose mostly non-overlapping instruments since even small differences in implied spot rates or discount factors may result in erratic forward rates. Second, preference should always be given to more liquid instruments with a tighter bid-ask spread since relative illiquidity may cause a problem of synchronization for the times of observation in the sample. Use of Eurodollar futures to construct the term structure is prone to overlapping maturities even if only those contracts that are parts of the regular maturity cycle are considered. The problem may arise if there is a slight degree of overlap in or a gap between the three-month Eurodollar deposit periods associated with adjacent

Eurodollar futures. Rendleman (2004) provides a few such examples. For example, the three-month Eurodollar deposit period for the June 2005 futures contract end on 9/15/05, while the three-month deposit associated with the September 2005 futures contract starts on 9/21/05, six days later.

LIBOR is the baseline for pricing interest rate derivatives such as forward rate agreements, swaps and swaptions, Eurodollar futures and futures options, and many others. The LIBOR for the US dollar is quoted as a simple annual interest rate using the actual/360 day count. The original Nelson-Siegel model and its extended version by Svensson assume that continuously compounded annualized interest rates are used for estimation. Therefore, the quoted LIBOR rates and the implied spot rates for longer maturities that are obtained by bootstrapping from the interpolated swap rates must be converted into rates with continuous compounding. The day count applied to the rates must also be changed so that it becomes the actual/365 day count. The formula used for that purpose is identical to that in (2.6) which is used to convert Eurodollar futures rates:

$$r(t,m) = \frac{365}{m} \ln\left(1 + \frac{m}{365}\frac{365}{360} r^{q}(t,m)\right), \tag{2.21}$$

where $r^q(t, m)$ is the quoted LIBOR for maturity m at time t.

The issue of the choice of the appropriate time grid (the number of yield/price and maturity observations in the sample) have not been fully addressed in the existing literature and there are reasons to believe that the choice of grid may impact the resulting estimated term structure although it is not clear whether the differences in produced yields or prices given different grids will be statistically significant. Moreover, the choice of the time grid is usually constrained by the available data. In the case of LIBOR, it is straightforward to pick all available LIBOR rates as the components of the time grid: those are LIBOR with maturities from one month up to 12 months (although Grinblatt and Jegadeesh [1996] use 1-, 3-, 6-, 9- and 12-month LIBOR quotations only in their analysis).

The Eurodollar deposit and the swap transaction have the same timing: the rate applied to the transaction is determined on the fixing date while the actual transaction takes place on the value date which is two London business days later. In other words, there is a two-business-day lag between the date when the rate is fixed and the starting date of the LIBOR deposit. If we estimate spot rate and par yield curves that way, the forward rates derived from such term structure will be the rates applied to time intervals starting two business days later than the current date. This leads to a caveat: since changes in mark-to-market futures values result in immediate cash flows and outflows, futures rates are compared to respective forward rates that are applied for time intervals that start two business days later. Therefore, the forward model based on LIBOR value dates and maturity dates happens to be two business days apart from the one based on Eurodollar futures quotes.

In general, swap payment dates and LIBOR deposit maturity dates are determined according to the *modified following business day convention*. The British Bankers' Association website says that "the modified following business day convention states that the maturity date is the first following day that is a business day in London and the principal financial centre of the currency concerned, unless that day falls in the next calendar month. In this case only, the maturity date will be the first preceding day in which both London and the principal financial centre of the currency concerned are open for business." This is why one may observe 33 days until maturity for a one-month LIBOR deposit and 367 days for a one-year LIBOR among others. Apart from the modified following business day convention, one more rule must be mentioned. It regards the maturity days applied to LIBOR rates fixed at the end of the month and is referred as the *end-end dealing*. In cases when a deposit is made on the final business day of a particular calendar month, the maturity of the deposit shall be on the final business day of the month in which it matures, not the corresponding date of the month of maturity. For instance, a one-month deposit for value date of February, 28 of a non-leaped year would mature on March, 31, not March, 28. Choosing incorrect maturity dates for the interpolation procedure may result in magnified interpolation errors making this issue of a particular importance.

Another concern is related to the potential errors arising from the non-synchronous data in the sample. Few authors augment swap rate data with short-term LIBOR. Dai and Singleton (2000), for example, use a data set of swap rates for maturities from two to ten years and augment it with the six-month LIBOR rate. The source of their swap data is Datastream and the non-synchronicity of quotes arises immediately since LIBOR are quotes as of 11 am London time, while swap rates are recorded at the end of trading day in London which is 5:30 pm London time. It was brought to attention by Rendleman (2004) that the Bloomberg system allows historical data on swap rates to be collected as of 6 am, 1 pm and 5:30 pm Eastern time of each trading day which corresponds to Tokyo, London and New York "closing" time respectively. There is, in fact, no market closing in any of those places since swaps can be traded over-the-counter 24 hours a day, but Bloomberg created these virtual time stamps for a matter of convenience.

The limitation of the available swap data is that the one-year contract represents the contract with shortest maturity. Prior to 1997, the shortest maturity swap had a tenor of two years. The problem with bootstrapping of the swap curve using swaps with semi-annual payments necessary to produce the yield curve of zero-coupon rates is the unavailability of par yields on swaps with six-month maturity intervals starting with the six-month yield for the entire range of maturity under consideration. In order to overcome this restriction, for the sample period covering time before January 6, 1997, we convert the quoted six-month and one-year LIBOR rates into par yields of hypothetical swaps carrying semi-annual payments and maturing six months and one year from a current date. For the period from January 6, 1997, we have the data for one-year swap rate available. Therefore, we need just the six-month implied swap rate which is obtained using the six-month LIBOR. Collins-Dufresne and Solnik (2001) show that LIBOR and swap curves need not agree, especially for longer maturities. Jones (2004) examines changes in yields and finds out that one-year swap rate levels are well-predicted by synchronized LIBOR rates which suggests that the maturity of one year is sufficiently short to allow one to ignore the swap/LIBOR spread.

Above all, it remains a question whether different interpolation models used for estimation of the term structure of spot and forward rates produce significantly different results. Jordan and Mansi (2003) introduce random errors into the bond prices that are used to estimate the term structure and they show that interpolation error is the major source of error in term structure estimation while the random error contributed only 7-8% of the total error. This suggests that the choice of the functional form is relevant in term structure estimation.

2.6 Data Description and Analysis

The spot rate yield curve is constructed using US dollar LIBOR rates for maturities ranging from one to twelve months and swap rates for annual maturities ranging from two years up to five years. The US dollar LIBOR data for the period from 1/04/1988 to 5/31/2007 is taken from the British Bankers' Association official website while the swap rates are downloaded from Bloomberg electronic service. By definition, the swap rate is a par yield, i. e. it is a coupon rate on the fixed side of the swap. We use swap rate data from Bloomberg for plain vanilla swaps where the fixed rate is paid semi-annually and accrues at 30/360 and the floating rate is linked to the three-month LIBOR and is paid quarterly using actual/360 day count. Mid-market swap rates (the average of bid and ask quotes) as of Tokyo "closing" time are used. The reported swap rates on Bloomberg are for highly liquid instruments and the bid-ask spread rarely exceeds a couple of basis points. The choice of Tokyo "closing" time allows us to avoid the asynchronicity problem while combining LIBOR and swap rates to build the spot yield curve. The swap rate data covers the period from 11/1/1988 to 5/31/2007. Swap quotes before 11/1/1988 were not available. Table 2.1 provides the descriptive statistics for the sample of LIBOR and swap mid-market rates. Note that LIBOR rates with longer maturities demonstrate lower daily volatility. Interestingly, the ratio of the Friday-to-Monday LIBOR change standard deviation to the calendar day rate change standard deviation ranges from 1.07 to 1.62 across the twelve maturities with an average of 1.43 (not shown in the table), which is below the square root of three (≈ 1.73) implied by the constant daily volatility hypothesis[9] suggesting that weekends do not bring extra volatility into the quoted interest rates. The volatility of mid-market swap rates is lower than that of LIBOR rates and it decreases with maturity except for the upward jump for the two-year maturity. This phenomenon can be explained by the fact that the one-year swap rates were reported less frequently than the swap rates for longer maturities in the sample (the number of observations of the one-year swap rate represents only about 60 percent of the sample of the two-year swap rate), while samples of swap rates for longer maturities are approximately equal among each other. Interestingly, the Thursday-to-Friday volatility in swap rates with expiration of two or more years is very close to and in some cases even higher than the three-calendar-day Friday-to-Monday volatility in rates (not shown in the table) which may suggest that swap market participants account a portion of risk related to weekend volatility into the Fridays' rate quotes.

Previous literature relies on conventional wisdom to use 30 for the number of days in each month although US LIBOR rates are calculated on the actual/360 basis. The actual number of days in the deposit period that LIBOR covers is not necessarily a multiplier of 30 although the difference in spot rates for short maturities computed based on the actual/360 count versus those based on the 30/360 day count is negligible but the discrepancy increases monotonically with the maturity. The impact of such discrepancy on forward rates has not been reported previously. Since our interpolation technique requires the use of continuously compounded rates, we convert

[9] See French and Roll (1986) for details

Table 2.1 Descriptive statistics of US LIBOR and mid-market swap rates for the period from 1/04/1988 to 5/31/2007

	Obs	Mean	St. Dev.	Min	Max
LIBOR for maturities					
1 month	4905	0.0494	0.02183	0.0102	0.1031
2 months	4905	0.0499	0.02186	0.0101	0.1044
3 months	4905	0.0503	0.02185	0.0100	0.1063
4 months	4905	0.0506	0.02182	0.0099	0.1075
5 months	4905	0.0509	0.02181	0.0098	0.1088
6 months	4905	0.0512	0.02178	0.0098	0.1100
7 months	4905	0.0515	0.02175	0.0098	0.1106
8 months	4905	0.0518	0.02171	0.0098	0.1113
9 months	4905	0.0522	0.02169	0.0098	0.1119
10 months	4905	0.0525	0.02165	0.0098	0.1125
11 months	4905	0.0529	0.02159	0.0099	0.1131
12 months	4905	0.0532	0.02156	0.0099	0.1138
Swap mid-market rates for maturities					
1 year	2590	0.0443	0.01803	0.0097	0.0756
2 years	4620	0.0548	0.01918	0.0126	0.1079
3 years	4600	0.0573	0.01783	0.0163	0.1062
4 years	4596	0.0593	0.01699	0.0199	0.1052
5 years	4600	0.0609	0.01639	0.0233	0.1042

the reported LIBOR rates into continuously compounded rates with the actual/365 day count that are subsequently used to build the spot yield curve and perform the analysis of the discrepancy effect that the 30/360 day count convention would produce for the continuously compounded forward rates. We find that for our sample the average absolute differences in implied continuously compounded forward rates for monthly maturities from one to nine months are relatively low, exceeding a basis point only on one occasion, for the shortest maturity, but decline monotonically afterwards. We are most interested in absolute difference for the three-month forward rate since our derivation of the convexity adjustment is based on the comparison of the observed three-month Eurodollar futures rates/prices and the respective forward rates/prices. The difference is less than a half of a basis point (0.42 of a basis point, to be exact) and we may conclude that little bias in results reported in previous literature may be accounted to the usage of approximate maturity dates when building the yield curve.

Our original sample contains one-year swap rates starting from 6/21/1996. Before that date swap rates with maturity of one year are unavailable and for that part of the sample we generate swap rates using 6-month and 12-month LIBOR rates. This surrogate swap rate is subsequently used with quoted swap rates of longer maturities to estimate the swap rate curve. For that matter it is interesting to see how the original one-year swap rate in the subsample starting from 6/21/1996 would differ from the surrogate one-year swap rate derived from the observed six- and 12-month LIBOR rates. If the difference is too large meaning that swap market assigns a sizable premium to the one-year swap rates associated with default risk and other factors, that would reveal a presence of inconsistency in our model. The average difference between the reported one-year spot rate and the rate derived from the observed 6- and 12-month LIBOR is -5.09 basis points. The standard deviation of such difference is 5.32 basis points. The minimum value of the difference is -45.11 basis points, the maximum is 46.41 basis points and the percentage of negative differences is 84.13 percent. The t-statistics for the test of the difference being equal to zero is 48.68. The 95% confidence interval of this difference is from -5.30 to -4.89. Hence, by the conventional criteria, this difference is considered to be statistically significant from zero. Overall, we can conclude that for the most of the subsample (73.4 percent of it) the difference is a negative single-digit value but there exists a statistically significant premium assigned by the market to the one-year swap rates if compared to the LIBOR data.

The Eurodollar futures price quotes for the period from 1/04/1988 to 10/01/2002 are obtained from Turtle Trader (www.turtletrader.com), while the futures quotes for the period from 10/02/2002 to 5/31/2007 are obtained from Econstats (www.econstats.com). For the first subsample of the futures data both opening and closing daily price quotes are available while second subsample contains only closing daily futures quotes. When using the futures data, the issue of liquidity must be taken into account. Rendleman (2004) mentions that even though Eurodollar futures are available for maturities up to 10 years, contracts with maturities beyond five years are comparatively illiquid. Piterbarg and Renedo (2004) consider Eurodollar futures with expiration up to four years as liquid. Brooks and Cline (2006) use futures contracts with maturity up to three years citing the drop in volume and open interest for contracts with longer maturities as reasons to ignore such contracts. Meulbroek (1992), Grinblatt and Jegadeesh (1996) and Chance (2003) use in their analysis contracts with expiration up to nine months. In all three aforementioned papers the time horizon is limited to nine months due to the fact that those papers use LIBOR rates exclusively to construct the term structure of spot rates which is used further to estimate forward rates or prices and time span of LIBOR maturities is limited by twelve months. Futures contracts that are not parts of the quarterly expiration cycle (March-June-September-December) are also deemed relatively illiquid. We choose to use futures contracts that expire not later than three years from the current date and only those futures that belong to the quarterly expiration cycle are considered.

Table 2.2 contains information related to the asynchronicity bias that arises from the fact that forward rates are derived using LIBOR and swap rates that are reported at 6:30 am Eastern time while futures rates are based on futures daily settlement prices that are reported at 2:30 pm Eastern time. This results in the time difference between the reported rates and prices that are used to compute the forward-futures differential of eight hours. The Eurodollar futures market opens at 8:20 am Eastern time. Therefore, the futures rates and prices calculated using the open market futures quotes are only two hours apart from the reported spot and swap rates that are employed to compute implied forward rates and prices. The use of open market futures quotes may substantially reduce the problem of the quotation asynchronicity. Therefore, we want to see how volatile the obtained futures rates are during a trading day. Table 2.2 presents data about daily absolute changes in continuously compounded futures rates. Note that such rates are of derived nature as described in the previous part of the paper: reported futures quotes are used to calculate the rates and necessary conversions for compounding and day count are also performed. The average daily absolute difference in futures rates ranges from 3.07 basis points in 1997 to 5.45 basis points in 1992. There are several months in the sample when the average daily volatility is relatively low (less than 2.50 basis points) and a few periods when the volatility was relatively high (above 7 basis points). The latter includes September and November of 2001 among others. The highest daily absolute change was observed in April of 1994 (9.19 basis points). Obviously, when computing the forward-futures rate or price differentials based on the futures closing prices and taking averages, the errors stemming from relying on asynchronous quotes may at least partially cancel each other. It is however imperative to admit that different timing of quotes may potentially lead to blurred results with asynchronicity error accounting on average for 3-5 basis points of the forward-futures rate differential on a particular trading day.

2.7 Estimation Results

In order to construct the swap term structure we employ swap rates with annual maturities ranging from one to five years. The Mansi-Phillips interpolation technique is used for

Table 2.2 Average daily absolute difference in basis points in continuously compounded Eurodollar futures rates between open and close of the market, 1/1988-9/2002*

	Jan	Feb	Mar	Apr	May	Jun	Jul
1988	5.95	5.94	4.00	3.38	4.22	5.78	3.18
1989	3.89	4.95	5.40	7.42	6.60	6.40	5.00
1990	4.16	4.43	3.63	3.80	4.09	3.69	3.84
1991	4.21	3.60	4.51	3.83	2.61	3.77	3.63
1992	6.36	5.67	6.04	5.16	5.08	3.55	6.04
1993	5.24	4.83	6.06	3.40	4.03	4.42	3.76
1994	3.73	4.28	4.91	9.19	7.11	5.65	6.28
1995	5.38	7.06	4.60	3.49	5.81	8.61	4.88
1996	3.43	5.31	7.64	4.47	4.59	4.49	5.24
1997	4.15	2.97	3.34	3.73	2.97	2.50	2.49
1998	4.69	2.40	2.39	3.27	2.18	2.13	1.31
1999	3.27	3.83	3.62	3.79	3.82	4.70	3.76
2000	3.83	3.79	2.53	4.64	4.32	3.77	3.55
2001	7.68	3.89	4.18	7.31	5.11	3.55	3.71
2002	6.29	3.88	4.41	4.56	4.27	2.86	3.49

	Aug	Sep	Oct	Nov	Dec	Average
1988	3.72	4.75	3.41	4.92	4.79	4.50
1989	6.84	3.94	4.78	5.20	3.90	5.38
1990	5.38	3.22	2.78	3.00	4.11	3.85
1991	5.86	3.04	3.81	3.41	3.57	3.83
1992	4.46	5.18	7.52	5.54	4.61	5.45
1993	3.09	4.70	2.97	3.73	2.07	4.02
1994	3.67	3.54	3.98	4.61	6.11	5.23
1995	4.30	4.19	2.55	3.19	3.04	4.77
1996	5.05	4.68	4.12	1.80	3.95	4.57
1997	3.68	2.69	3.85	1.92	2.38	3.07
1998	2.24	5.25	5.75	4.26	4.22	3.34
1999	5.17	4.60	3.59	2.90	2.80	3.84
2000	2.67	2.46	2.68	2.31	4.08	3.40
2001	3.24	7.61	3.83	7.85	5.03	5.19
2002	2.46	2.49				4.13

* contracts with expiration up to 1140 days were considered only

that purpose and the loss function is defined as the sum of equally weighted swap rate errors. To construct the spot term structure, we use all twelve monthly LIBOR rates and 1.5-, 2-, 2.5-, 3-, 3.5-, 4-, 4.5- and 5-year implied spot rates derived from the estimated swap curve, which sums up to 20 knot points. The extended Nelson-Siegel method is employed and the loss function used is also the sum of equally weighted squared errors of rates. The assignment of weights tied to the inverse of duration as in (2.20) would tend to produce smaller errors for rates for shorter maturities and larger errors for rates applied for longer maturities which is especially unwelcome when a large number of knot points is employed since this would result in much larger interpolation errors on the right end of the curve. In the context of the convexity adjustment analysis, this would make the results for the forward-futures differentials calculated for longer maturities prone to interpolation error and, thus, less reliable.

To mitigate the influence of interpolation error in the analysis of the forward-futures differential we introduce criteria for an interpolated yield curve that would allow it to be included in the final sample. For the swap rate curve, a yield curve on a particular business day is considered to satisfy the interpolation criteria if the sum of all five absolute fitted errors is below 15 basis points and each absolute error does not exceed five basis points. That leaves us with fitted daily swap rate curves for 4,249 business days. For the spot rate term structure, there are two types of interpolation condition depending on the number of knots employed. For that part of the sample when swap rates are unavailable and only all twelve LIBOR rates are used to build the curve, the interpolation condition to be satisfied is the sum of all twelve absolute fitted errors must be below 25 basis points and each absolute error must not exceed five basis points. For the part of the sample when implied spot rates for maturities beyond one year are added, the condition is the sum of all twenty absolute fitted errors must be below 40 basis points and each absolute error must not exceed five basis points. That leaves us with fitted daily spot rate curves for 3,991 business days. This is the final refined sample that will be used to compute forward rates and forward prices that will be subsequently matched with respective actual futures rates and futures prices.

We construct the futures-forward rate differentials in the following manner. We consider maturities (time to expiration) of futures and respective forward rates starting with that of one month (31 days). Thereafter the futures-forward differentials with maturities within 30 days are bundled into clusters. For instance, all futures-forward rate differentials where futures and respective forwards have times to expiration between 31 and 60 days are grouped together. All futures-forward rate differentials where futures and respective forwards have times to expiration between 61 and 90 days are grouped together and so on. The last group contains differentials for maturities between 1,111 and 1,140 days. There are 37 such groups in total.

Table 2.3 reports statistics for the futures-forward rate differential when closing daily futures price quotes are used to derive the futures rates. There is no clear pattern in the average differential as a function of maturity: the differential varies across maturities rising initially and making a peak for the maturity range of 601-630 days but falling afterwards. Both mean and median of the differential are below zero for maturities below 180 days. Although the median value turns and stays positive for the rest of the sample, a significant percentage of negative values of the differential is reported for longer maturities. Its dynamics across maturities is similar to that of the average value of the differential. The percentage is always above 25 percent and is between 30 and 50 percent for most of the maturities in the sample.

The results are not safe from the errors in reported data. For example, the maximum value of the differential for the shortest maturity range in the sample reported in Table 2.3 is abnormally high if compared to the rest of the results. The reason for such abnormality lies in a probable misreporting of LIBOR quotes. On May, 29, 1999, the reported five-month LIBOR is 5.0338 percent, the six-month LIBOR is 5.525 percent, and the seven-month LIBOR is 5.075 percent. Clearly, the six-month LIBOR of 5.525 must be a typo. For comparison, on the previous business day, May, 26, 1999, the reported five-month LIBOR was 5.0375 percent, the six-month LIBOR was 5.0613 percent, and the seven-month LIBOR was 5.0912 percent. On the following business day, May, 30, 1999, the respective rates were 5.0389, 5.0616 and 5.0917 percent. Another such abnormality is observed on March, 22, 1999, when reported monthly LIBOR rates for maturities from nine to 12 months went down by about 20-30 basis points compared to the respective rates reported one business day before and the following business day all rates jumped up by about the same magnitude reaching the levels similar to those reported one business day before March, 22, 1999.

Figure 2.1 depicts the dynamics of the median value of the futures-forward rate differential as a function of maturity across the four period subsamples: 1988-1991, 1992-1995, 1996-1999, 2000-2003, and 2004-2007. During 1988-1991 the value of the forward-futures rate differential was a negative function of maturity, the result that was interpreted as mispricing in early empirical studies. The pattern for the 1992-1995 period is similar to that of the entire sample: an inverse U-shape with a peak around the maturity of 600 days. The differential is a positive function of time to maturity during 1996-1999 and 2000-2003 but the relationship becomes flat during 2004-2007.

Marking-to-market is not the only distinctive feature of the Eurodollar futures when compared to the forward interest rate contracts. The other difference is embedded in how the marking-to-market is actually performed: the Eurodollar futures are priced as if the underlying were a discount instrument. The futures price is computed according to equation (2.7), while the

Table 2.3 Futures-forward rate differential in basis points when closing daily futures quotes are used to derive continuously compounded futures rates

Maturity, days	Obs.	Mean	Median	St. Dev.	Min	Max	<0, %
31-60	1335	-1.65	-1.78	8.75	-37.66	198.07	68.0
61-90	1260	-2.60	-2.21	6.56	-37.88	26.30	68.7
91-120	1265	-1.18	-1.33	7.13	-33.16	51.08	62.1
121-150	1291	-0.75	-0.69	7.70	-42.31	35.43	54.1
151-180	1267	-0.92	-0.97	8.03	-37.80	41.50	56.6
181-210	1313	0.83	0.54	8.54	-28.15	54.14	46.3
211-240	1249	0.99	0.36	8.67	-39.08	36.23	48.2
241-270	1284	1.15	0.55	9.27	-38.46	49.36	47.4
271-300	1265	1.19	0.94	8.32	-29.07	40.71	43.4
301-330	1205	1.46	0.63	8.61	-29.24	44.91	45.6
331-360	1148	1.77	1.47	9.31	-46.98	44.77	42.4
361-390	1267	2.77	2.43	8.44	-30.42	46.03	36.5
391-420	1226	2.12	1.96	7.59	-30.13	39.43	38.0
421-450	1106	2.96	2.62	8.81	-62.89	46.03	34.2
451-480	1280	3.28	2.61	8.26	-33.49	46.49	34.4
481-510	1210	3.92	3.60	8.00	-27.29	43.59	31.2
511-540	1148	4.33	3.68	8.27	-38.05	38.37	28.4
541-570	1252	3.92	3.46	8.16	-35.54	36.53	30.7
571-600	1211	4.51	3.78	8.24	-25.54	45.42	28.2
601-630	1180	4.53	4.08	8.46	-35.37	47.94	26.9
631-660	1220	3.35	3.00	8.19	-26.39	37.69	33.5
661-690	1217	3.64	3.27	8.84	-25.99	50.68	31.7
691-720	1174	3.43	3.05	9.37	-38.56	70.94	33.7
721-750	1250	2.27	1.95	8.94	-31.53	54.58	39.1
751-780	1143	1.72	1.94	9.25	-30.91	37.28	39.0
781-810	1159	1.46	1.72	9.54	-40.04	53.10	41.2
811-840	921	0.42	0.53	8.83	-35.33	31.23	47.3
841-870	647	0.27	1.28	8.94	-32.43	34.62	44.2
871-900	699	-0.09	0.87	9.76	-40.74	51.98	46.5
901-930	726	-0.11	0.57	8.87	-38.58	29.43	47.8
931-960	635	0.52	1.65	9.24	-35.46	31.90	41.1
961-990	663	0.57	1.05	9.43	-41.64	47.61	45.2
991-1020	722	0.62	1.05	8.10	-25.96	26.83	43.2
1021-1050	619	0.93	1.79	9.53	-30.25	44.91	40.5
1051-1080	636	1.64	1.69	10.41	-40.60	60.52	39.9
1081-1110	686	1.61	2.43	9.03	-42.86	38.61	36.3
1111-1140	133	2.54	3.19	8.27	-35.10	26.95	30.1

forward price is found from (2.8). The forward-futures price differential is always positive on the day of expiration of the two contracts. Given our findings about the nature of the *futures-forward rate* differential, i. e. its small magnitude and large number of negative occurrences, it remains a question though whether the unique feature of the Eurodollar futures pricing may influence the

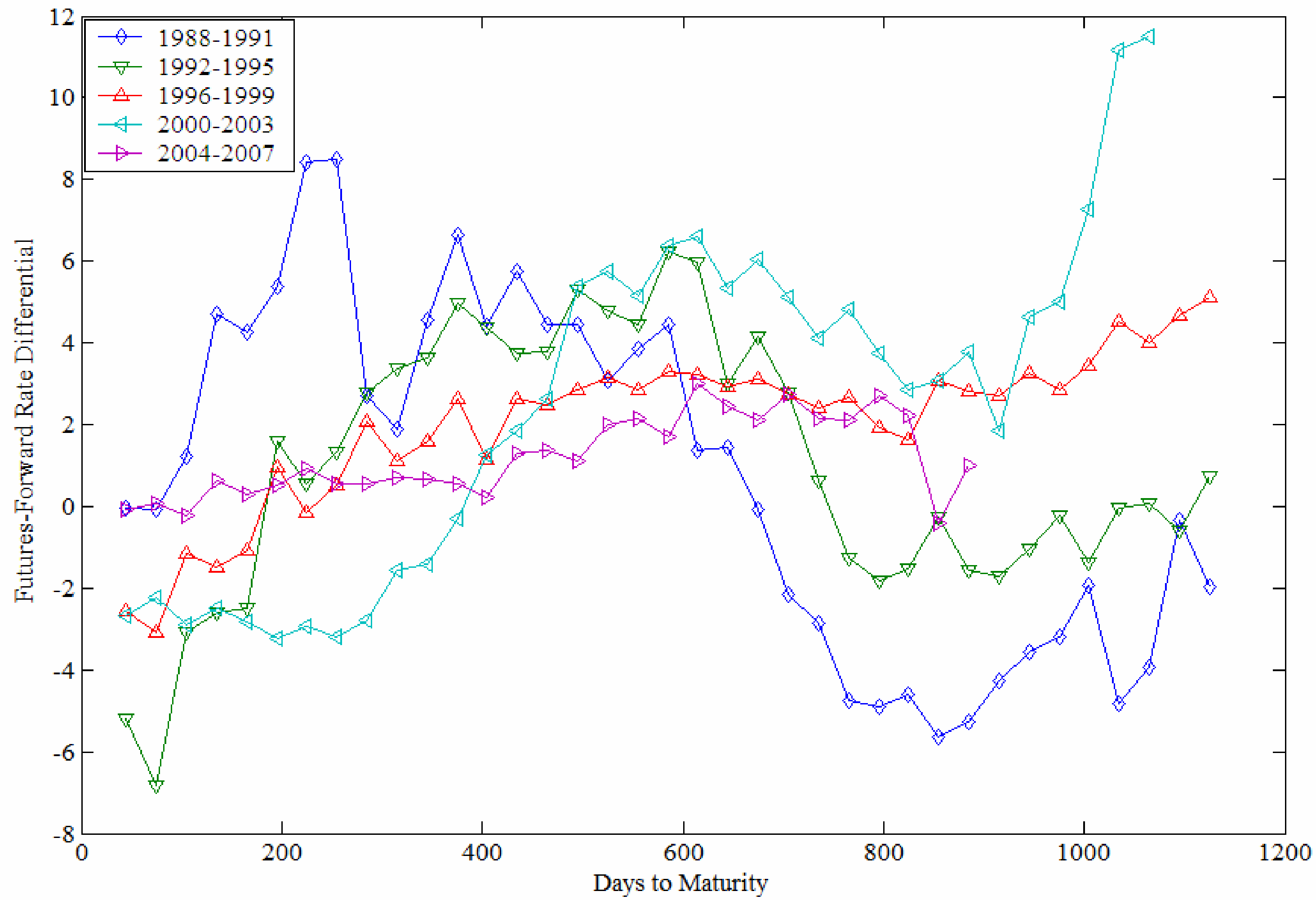

Figure 2.1 Dynamics of the futures-forward rate differential as a function of maturity over the five subsample periods, basis points

results when analysis is performed for the rate differential and whether similar analysis of the *forward-futures price* differential would lead to similar conclusions.

Table 2.4 provides information about the forward-futures price differential for the 1988-2007 period when closing market futures quotes are used to compute futures prices. Since futures and forward rates and prices are negatively related, we expect to observe the same signs for *forward-futures price* differentials as they were for respective *futures-forward rate* differentials unless the pricing feature of the Eurodollar futures makes its contribution to the results for the forward-futures price differentials. Table 2.4 has a similar format to that of Table 2.3.

The first result that attracts attention is a more pronounced positive relationship between the size of the price differential and the time to maturity: although the intermediate peak is also reached for the 601-630 maturity range, the fall for subsequent maturities is not significant and a new rise in the average value of the differential takes place starting at the 781-810 range. The maximum value of the mean is reached at the 1051-1080 range while the maximum value of the median is reached at the longest range of maturities in the sample. Both mean and median stay positive for all maturity ranges in the sample and the percentage of negative occurrences of the price differential is substantially lower if compared to results in Table 2.3. The latter is always below 31 percent and is in single digits for more than a half of the sample, predominantly, for longer maturities. These observations allow to conclude that the unique feature of the Eurodollar futures settlement attributes positively to the relationship between the size of the differential and the time to maturity. The standard deviations of the price differentials are much lower than those for the respective rate differentials suggesting that the unconventional feature of Eurodollar futures pricing attributes to the reduction of volatility in the reported results. The latter finding provides support in favor of performing the convexity adjustment analysis price-wise rather than rate-wise since the latter would be more prone to possible data imperfections and interpolation error resulting in higher deviations among the reported results. Yet, there is no support for the assertion that the unusual nature of the Eurodollar futures pricing could serve as the explanation for the observed anomalies in the behavior of the *futures-forward rate* differential.

So far, the results obtained for the futures-forward rate differential and the forward-futures price differential are somewhat unexpected in the wake of theoretical predictions and early empirical findings: the average values of the two differentials are much smaller than those reported in previous studies, the rate differential turns negative on many occasions and time to

Table 2.4 Forward-futures price differential in basis points when closing daily futures quotes are used to compute futures rates

Maturity, days	Obs.	Mean	Median	St. Dev.	Min	Max	<0, %
31-60	1335	1.30	1.02	2.60	-8.08	49.86	25.5
61-90	1260	1.09	0.93	2.25	-8.44	11.40	30.9
91-120	1265	1.43	1.22	2.44	-6.16	19.65	26.5
121-150	1291	1.58	1.38	2.67	-9.23	13.43	27.7
151-180	1267	1.57	1.40	2.80	-8.09	16.39	29.4
181-210	1313	2.03	1.80	2.87	-5.33	20.45	22.4
211-240	1249	2.09	1.79	2.95	-8.34	13.62	22.3
241-270	1284	2.18	1.76	3.20	-9.03	17.38	22.4
271-300	1265	2.10	1.95	2.64	-6.97	17.23	18.7
301-330	1205	2.17	1.97	2.67	-6.69	16.97	19.0
331-360	1148	2.26	2.07	2.82	-9.77	12.75	18.9
361-390	1267	2.58	2.41	2.55	-7.21	12.84	13.3
391-420	1226	2.41	2.30	2.32	-6.79	15.44	12.4
421-450	1106	2.65	2.43	2.52	-14.79	12.67	10.0
451-480	1280	2.81	2.52	2.39	-6.47	13.21	8.8
481-510	1210	2.95	2.69	2.31	-5.98	16.39	7.6
511-540	1148	3.09	2.89	2.22	-6.83	11.60	6.1
541-570	1252	3.06	2.91	2.24	-6.03	11.84	6.2
571-600	1211	3.18	2.94	2.30	-5.45	16.06	6.4
601-630	1180	3.22	3.08	2.20	-6.50	14.19	5.3
631-660	1220	3.00	2.94	2.13	-5.17	11.63	6.3
661-690	1217	3.05	2.90	2.29	-5.45	16.08	6.3
691-720	1174	3.03	2.86	2.27	-6.40	20.17	5.6
721-750	1250	2.80	2.75	2.18	-4.99	16.10	7.8
751-780	1143	2.67	2.63	2.26	-5.19	12.44	9.9
781-810	1159	2.65	2.61	2.27	-6.47	15.90	10.4
811-840	921	2.74	2.66	2.08	-4.76	10.49	8.7
841-870	647	2.99	3.05	2.13	-3.63	11.95	7.9
871-900	699	3.03	3.07	2.31	-6.43	15.83	7.6
901-930	726	3.04	3.11	2.08	-5.30	10.27	6.7
931-960	635	3.15	3.32	2.24	-5.31	11.46	7.6
961-990	663	3.26	3.20	2.30	-7.58	14.96	7.1
991-1020	722	3.26	3.30	1.99	-3.17	9.75	6.1
1021-1050	619	3.33	3.37	2.24	-4.62	14.90	6.0
1051-1080	636	3.56	3.37	2.50	-7.11	18.42	4.9
1081-1110	686	3.47	3.56	2.18	-6.44	12.96	5.1
1111-1140	133	3.42	3.57	2.00	-4.48	10.33	5.3

expiration does not seem to contribute much to the size of the differential even if maturity is extended to three years while previous literature had it limited up to 12 months. We further employ several robustness check techniques to verify our findings.

It was mentioned earlier in the paper that asynchronicity error arising because of the difference of 10 hours in reported rates used to construct the term structure of spot rates and the closing futures quotes used to compute futures rates and prices may reach three-to-five basis points on average. It remains a question how much such difference could contribute to the futures-forward differential. Even if the asynchronicity error does take place, its size relative to the reported results in Table 2.3 would not be able to explain such a miniscule difference in the differential across maturities or such a large number of negative occurrences of the differential.

To mitigate the asynchronicity error we turn to opening daily futures quotes instead of their closing prices. The Eurodollar futures market opens at 8:30 Eastern time which is only two and a half hours apart from the point of time when LIBOR rates are reported in London and swap rates market is "closed" in Tokyo. Our data set includes opening prices for Eurodollar futures contracts up to the end of 2002 only[10]. Futures opening quotes are converted into continuously compounded futures rates in the same fashion as were the futures closing prices and the futures-forward rate and forward-futures price differentials are calculated again. Table 2.5 shows the mean values of the asynchronicity bias contribution toward the values of the differentials across all 37 maturity ranges. The results allow to see what difference, if any, the adjustment for the asynchronicity makes in the averages of the rate and price differentials within the subsample where both opening and closing futures quotes are available. The results show that the asynchronicity error is insignificant on many occasions. However, on those few occasions when it is statistically significant, it is always positive meaning that the use of closing daily futures quotes may result in the underestimation of the value of the differential but not the overestimation of it. The size of such underestimation may reach up to 0.60 basis points for the futures-forward rate differential but only 0.15 basis points for the forward-futures price differential. This is yet another finding in favor of conducting the analysis price-wise in order to mitigate the influence of possible data imperfections.

As expected though, asynchronicity is not able to explain the small size of the average differential reported in Table 2.3 and Table 2.4, the substantial number of negative occurrences in the rate differential or the visible lack of a stable relationship between the magnitude of the differential and the time to expiration. To exclude the presence of the asynchronicity error any further, for the rest of the analysis in the paper we use opening futures quotes.

[10] Rate differential computations for 2003-2007 reported in Table 2.3 and Table 2.4 are excluded from this part, therefore.

Table 2.5 The asynchronicity error in the futures-forward rate differential and the forward-futures price differential in basis points when opening daily futures quotes are used to derive continuously compounded futures rates and respective futures prices for the period from 1988 to 2002

Maturity	Obs.	Futures-Forward Rate Differential			Forward-Futures Price Differential		
		Mean	St. Dev.	T-stat.	Mean	St. Dev.	T-stat.
31-60	938	-0.25*	4.17	-1.87	-0.06*	1.04	-1.87
61-90	891	-0.38**	4.70	-2.39	-0.09**	1.18	-2.30
91-120	901	-0.06	5.44	-0.32	-0.01	1.36	-0.31
121-150	912	-0.33*	5.70	-1.77	-0.08*	1.43	-1.76
151-180	889	-0.46**	5.92	-2.30	-0.11**	1.48	-2.29
181-210	911	0.06	6.78	0.26	0.02	1.70	0.27
211-240	866	-0.12	6.49	-0.52	-0.03	1.63	-0.52
241-270	888	-0.45**	6.50	-2.09	-0.11**	1.63	-2.08
271-300	843	0.01	7.13	0.06	-0.00	1.78	0.06
301-330	789	0.18	6.60	0.77	0.04	1.65	0.77
331-360	751	-0.59**	6.88	-2.36	-0.15**	1.72	-2.36
361-390	826	0.04	6.72	0.19	0.01	1.68	0.19
391-420	771	0.13	6.51	0.55	0.03	1.63	0.54
421-450	710	-0.50**	6.67	-2.01	-0.13**	1.67	-2.01
451-480	820	0.09	6.47	0.40	0.02	1.62	0.40
481-510	731	0.29	6.45	1.20	0.07	1.62	1.19
511-540	716	-0.51**	6.65	-2.07	-0.13**	1.67	-2.07
541-570	798	0.07	6.19	0.34	0.02	1.55	0.34
571-600	716	0.36	6.24	1.55	0.09	1.56	1.55
601-630	705	-0.43*	6.15	-1.86	-0.11*	1.54	-1.86
631-660	768	0.18	6.01	0.85	0.05	1.51	0.85
661-690	704	0.20	6.09	0.87	0.05	1.53	0.87
691-720	674	-0.35	5.90	-1.55	-0.09	1.48	-1.55
721-750	761	0.08	5.95	0.35	0.02	1.49	0.35
751-780	662	0.29	5.81	1.26	0.07	1.46	1.26
781-810	670	-0.27	5.64	-1.24	-0.07	1.42	-1.25
811-840	737	0.21	5.74	0.99	0.05	1.44	0.98
841-870	632	0.32	5.72	1.41	0.08	1.43	1.40
871-900	698	-0.28	5.61	-1.31	-0.07	1.41	-1.32
901-930	726	0.14	5.64	0.66	0.03	1.42	0.66
931-960	635	0.16	5.53	0.74	0.04	1.39	0.74
961-990	663	-0.08	5.70	-0.36	-0.02	1.43	-0.37
991-1020	722	0.24	5.53	1.17	0.06	1.39	1.17
1021-1050	619	0.15	5.40	0.70	0.04	1.36	0.70
1051-1080	636	-0.07	5.55	-0.31	-0.02	1.39	-0.31
1081-1110	686	0.29	5.49	1.41	0.07	1.38	1.41
1111-1140	133	-0.06	4.71	-0.14	-0.02	1.18	-0.15

** – significant at 5 percent level
* – significant at 10 percent level

It is reasonable to suggest that short-term volatility in interest rates may cause the differential to observe erratic variations and lead to a number of its negative values on many occasions. We already saw that asynchronicity which itself can serve as a proxy for short-term volatility did not alter the results significantly. To explore the impact of short-term volatility of interest rates further we check the correlation between the absolute values of the rate and price differentials on one side and the short-term variance of interest rates on the other. The latter is defined as the standard deviation of the three-month LIBOR calculated for five business days preceding the day when the differential is observed. The correlation coefficients are small across all maturities. For the futures-forward rate differential, the coefficient ranges from a minimum of 0.10 to a maximum of 0.33 across all 37 maturity ranges and on 21 occasions the coefficient is below 0.20. For the forward-futures price differential, the correlation goes from a mere 0.004 to a maximum of 0.32 and on 28 occasions the coefficient is below 0.20. Trimming the sample by excluding the first and the ninety ninth percentiles from the original samples of the computed differentials across each maturity range in order to eliminate possible outliers does not change the results significantly. To summarize these findings, there is no evidence that short-term volatility of interest rates could be the factor behind the high frequency of negative values of the futures-forward rate differential.

The difference in pricing and marking-to-market are not the only distinct features between Eurodollar futures and implied forwards. The timing is also different. The Eurodollar futures are marked to market daily based on closing futures quotes on that day, i. e. there is no lag present there. As for the forwards, recall that we compute implied forward rates and prices from LIBOR deposit rates and zero-coupon rates bootstrapped from the interpolated swap rate curve. Eurodollar deposit rates and swap rates are fixed on the settlement date, while the actual transaction takes place on the value date. The settlement date and the value date are two business days apart. For example, if we observe LIBOR and swap rates on June, 7, 2007 (Thursday), the deposit the rate is applied to will take place on June, 11, 2007 (Monday), two business days later. It is the same for swaps: the rate is fixed on the settlement date but the transaction is originated two business days after the settlement date. Therefore, if we compare futures rates or prices with their implied forward counterparts aligned by the dates when the rates or prices are observed, we effectively compare futures rates (prices) on that date with implied forward rates (prices) determined for the date two business days later which creates a bias for the observed differential. The higher is the volatility, the larger the bias is expected to be.

Table 2.6 shows statistics for the original sample when futures rates and prices are aligned with implied forwards by the value date of the latter. Compared to results in Table 2.3 and Table 2.4, we observe the same mixed picture: low averages and high percentage of negative occurrences of the rate differential. On many occasions average values of the futures-forward rate and forward-futures price differentials are even lower than those recorded previously with the frequency of negative observations staying roughly unchanged across all maturities. The two-sample t-test for equal means when the value dates are used versus when the settlement dates are taken for the alignment purposes shows that the hypothesis of equal means cannot be rejected at 10 percent level of significance across all maturity groups. We conclude that the two business day lag between the fixing (settlement) date of the implied forward rate and price and its transaction (value) date cannot be attributed to the observed abnormally high frequency of negative values of the rate differential or the low average magnitudes of both rate and price differential across the maturities.

As was mentioned in earlier sections, the theoretical literature provides some factors that are expected to explain the variation in the differentials. Such factors include time to expiration (maturity), the level of interest rates and its volatility, and volatility of the volatility. All aforementioned factors are believed to positively influence both the futures-forward rate differential and the forward-futures price differential. Regression analysis is to be performed to verify theoretical postulates. All respective differentials that were computed using opening futures daily quotes are bundled in the sample and independent variables are calculated for those dates that differentials in the sample are present. The futures-forward rate differential is measured in basis points, maturity is measured in years. Level is taken to be the quoted annualized three-month LIBOR, the underlying of the futures contracts, expressed as a percentage. Volatility is the standard deviation of the quoted three-month LIBOR and the volatility of the volatility is the standard deviation of the standard deviation of the three-month LIBOR. The period that volatilities are calculated for must be determined and we choose to pursue with three different models depending on the length of time the volatilities are taken for. Model 1 takes monthly volatilities by which we mean the volatility taken for 21 previous business days. Model 2 uses volatilities calculated for three months, or 63 previous business days. Model 3 utilizes the six-month volatilities, or volatilities computed using 126 previous

Table 2.6 Futures-forward rate differential and forward-futures price differential in basis points when forward and futures rates and prices are aligned by forward value dates, 1988-2002

Maturity	Obs.	Futures-Forward Rate Differential				Futures-Forward Price Differential			
		Mean	St. Dev.	<0, %	t-stat	Mean	St. Dev.	<0, %	t-stat
31-60	940	-2.45	8.38	66.8	-1.08	1.51	2.7	26.8	-0.64
61-90	903	-3.11	10.13	65.2	-0.46	1.36	3.1	32.2	-0.24
91-120	887	-0.86	10.58	55.5	0.51	1.87	3.1	25.5	0.33
121-150	916	-0.80	11.59	51.9	-1.12	1.97	3.5	27.3	-0.77
151-180	896	-0.71	12.10	53.1	-0.68	2.05	3.7	28.0	-0.45
181-210	899	2.11	12.88	43.6	0.69	2.78	3.7	20.1	0.47
211-240	869	1.96	12.43	45.7	-0.06	2.80	3.7	20.9	0.09
241-270	895	2.45	13.35	45.5	-0.68	2.98	4.0	20.8	-0.55
271-300	830	2.42	13.59	42.4	0.20	2.89	3.7	19.0	0.14
301-330	794	2.85	12.26	42.9	0.86	3.04	3.4	16.5	0.92
331-360	747	2.90	13.65	43.1	-0.90	3.04	3.7	17.5	-0.88
361-390	814	3.49	12.47	37.5	-0.48	3.29	3.3	15.1	-0.50
391-420	779	2.95	11.11	39.8	0.56	3.22	3.0	12.8	0.66
421-450	707	3.60	12.46	39.9	-0.82	3.36	3.3	11.5	-0.79
451-480	808	3.13	12.05	40.7	-0.40	3.34	3.2	12.9	-0.44
481-510	742	4.56	10.82	33.0	0.95	3.78	2.9	7.7	1.05
511-540	711	4.21	12.31	35.7	-0.99	3.66	3.1	10.0	-1.05
541-570	788	3.78	11.68	36.9	0.20	3.62	3.0	11.0	0.16
571-600	729	4.55	11.32	34.7	0.36	3.88	2.9	7.1	0.46
601-630	695	4.17	12.34	34.5	-0.71	3.78	3.1	8.5	-0.79
631-660	760	2.77	11.51	40.9	0.05	3.47	2.9	10.5	0.03
661-690	714	3.02	12.08	38.7	-0.13	3.60	3.0	8.5	-0.03
691-720	664	2.45	12.79	40.8	-0.78	3.49	3.1	9.9	-0.92
721-750	755	1.36	12.01	45.0	-0.21	3.22	2.9	12.5	-0.20
751-780	669	0.29	12.15	48.0	0.03	3.01	2.9	12.4	0.13
781-810	666	0.03	12.38	50.6	-0.45	2.99	3.0	11.6	-0.59
811-840	731	-0.35	11.78	50.9	0.31	2.89	2.8	14.0	0.31
841-870	638	-0.11	11.74	48.4	-0.09	2.96	2.8	12.9	0.07
871-900	696	-0.17	12.26	50.1	-0.61	3.00	2.9	12.6	-0.78
901-930	721	0.01	11.34	49.7	0.47	3.06	2.7	11.4	0.44
931-960	643	0.29	11.80	46.3	-0.11	3.12	2.9	12.4	0.06
961-990	655	0.39	12.08	48.1	-0.46	3.19	3.0	13.0	-0.60
991-1020	724	0.65	10.99	47.1	0.52	3.27	2.7	10.5	0.51
1021-1050	619	0.66	12.00	43.9	-0.18	3.29	2.9	11.1	-0.04
1051-1080	630	1.32	12.90	43.8	-0.62	3.46	3.1	9.4	-0.76
1081-1110	689	1.73	11.16	41.2	0.74	3.50	2.7	9.1	0.75
1111-1140	123	1.90	10.09	39.0	-0.61	3.21	2.4	6.5	-0.79

*opening daily futures quotes are used to calculate futures rates and prices business days. We also add the TED spread[11] to the set of independent regressors as a proxy for the default factor. The default variable is expected to be negatively related to the size of the futures-forward rate and the forward-futures price differential.

Panel A of Table 2.7 shows regression results for the futures-forward rate differential. As expected, the maturity factor has a positive coefficient and it is statistically significant for all three models. However, opposite to expectations, both the level and the volatility have negative

Table 2.7 Results of ordinary least squares regressions of futures-forward rate and forward-futures price differentials on a set of independent variables

Panel A

Dependent variable: futures-forward rate differential

	Model 1		Model 2		Model 3	
	coef	t-stat	coef	t-stat	coef	t-stat
cons	-0.15	-0.72	0.01	0.06	0.04	0.16
mat	0.20*	3.19	0.33*	5.35	0.38*	6.24
ted	5.18*	17.08	3.93*	12.86	3.24*	10.21
level	-0.25*	-5.02	-0.18*	-3.69	-0.14*	-2.88
vol	-6.15*	-5.44	-0.92	-1.60	-0.99*	-2.65
vol/vol	1.52	0.44	-6.49*	-3.41	-1.84	-1.40
R2 / se	0.02	8.64	0.01	8.56	0.01	8.52
obs	26821		27003		26819	

Panel B

Dependent variable: forward-futures price differential

	Model 1		Model 2		Model 3	
	coef	t-stat	coef	t-stat	coef	t-stat
cons	-1.48	-27.81	-1.61	-29.69	-1.57	-28.75
mat	0.40*	25.84	0.44*	29.25	0.46*	30.70
ted	0.56*	7.45	0.13	1.71	-0.11	-1.35
level	0.61*	49.77	0.65*	53.22	0.66*	54.25
vol	0.95*	3.39	1.09*	7.66	1.37*	14.94
vol/vol	1.61	1.88	0.69	1.46	-1.99*	-6.10
R2 / se	0.24	2.16	0.24	2.13	0.24	2.10
obs	26821		27003		26819	

"cons" stands for constant
"mat" stands for maturity
"ted" stands for TED spread
"vol" stands for volatility
"vol/vol" stands for volatility of volatility
"R2 / se" stands for R-squared and residuals standard error
R-squared and residuals standard error are adjusted for degrees of freedom
"obs" stands for observations
* – significant at 5 percent level

[11] Difference between the three-month LIBOR and the rate on a three-month T-bill

coefficients which are statistically significant for all three models with the exception of the volatility coefficient in Model 2. Also opposite to expectations, the default factor is positive and statistically significant across all three models. The finding that both the level and the volatility may stand behind the negative nature of the rate differential is puzzling as is the positive relationship between the value of the differential and the TED spread. The power of such results, however, is not significant enough due to the low values of the adjusted R-squared.

Panel B provides regression results for the forward-futures price differential. Since rates and prices are negatively related in the context of our analysis, we expect that signs of the coefficient do not change. As the table shows, however, there are quite a few changes to that. Now all three factors that are supposed to be positively related to the differential, maturity, level and volatility, have positive and statistically significant coefficients. The default factor is both positive and statistically significant for Model 1 only while large negative and statistically significant constant term points to the largely unexplained negative nature of the price differential by the employed factors. Another improved feature of the regression results in Panel B is the much higher adjusted R-squared. Apparently, the unique pricing feature of the Eurodollar futures finds more reflection if the regression analysis is performed price-wise although the negative nature of the forward-futures differential is largely unexplained or the TED factor is not a reliable proxy for the default factor in the context of the forward-futures rate differential analysis.

2.8 Conclusions

This paper constructs the spot rate term structure using the extended Nelson-Siegel exponential model and performs the analysis of the convexity adjustment for Eurodollar futures for the period from 1988 to 2007 in the form of the futures-forward rate differential and the forward-futures price differential while extending the maturity of the respective contracts under consideration to up to three years. The extension was made possible due to the employment of swap rates that allowed us to construct a LIBOR/swap curve and derive forward rates with longer maturities. To our knowledge, this is the first empirical study that attempts to evaluate the interest rate forward-futures differential for maturities longer than a year. Our findings provide unconventional results: average differentials are too small compared to the theoretical predictions and do not increase significantly with longer time to expiration while also on too many occasions the rate differential has a negative sign. Neither asynchronicity error, nor the interpolation error could explain the observed phenomena. The unusual feature of the Eurodollar pricing and the two business day lag between the settlement date and the value date of implied forward rates and prices cannot explain the observed results either.

The outcome of the regression analysis of the convexity adjustment on a set of conventional factors depends on whether the differential is measured as a rate difference or a price difference. If the latter is expressed as the rate differential, the regression coefficients have unexpected signs. If the price differential is chosen as an independent variable, all three major factors, the time to expiration, the level and the volatility, are positively related to the size of the forward-futures price differential and are statistically significant. Another finding from the regression analysis is that conducting it using the forward-futures price differential as the independent variable is beneficial to the one when the futures-forward rate is used instead since the former method results in much better goodness-of-fit. This allows to suggest that the unconventional feature of the way the Eurodollar futures are priced and settled is an important factor to consider and should be incorporated into the convexity adjustment analysis.

The forward contracts, unlike their futures counterparts, are subject to default risk. The presence of the default premium in forward prices can affect the size of the forward-futures differential. The default factor proxied by the TED spread, however, is not able to capture the negative nature of the differential which remains largely unexplained. The influence of the default factor on the size of the convexity adjustment could be studied separately. In particular, the comparison of the observed spread between the forward and futures prices with its theoretical value obtained via the utilization of a no-arbitrage term structure model would provide a quantitative estimate of the default premium in futures prices.

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Chapter 3 Decomposition of the Interest Rate Forward-Futures Price Differential

3.1 Introduction to the Problem

The differences between forward contracts and futures contracts as well as their prices were initially studied by Magrabe (1976), Cox, Ingersoll and Ross (1981), Jarrow and Oldfield (1981), and Richard and Sundaresan (1981). The major result of early studies is that in stochastic interest rate environment forward prices will not be equal to respective futures prices. The discrepancy between futures and forward prices is the product of the marking-to-market, the procedure employed by exchanges where futures settle gains and losses daily which aims to reduce the possibility of a trader's default.

Early empirical studies on the forward-futures differential, also known as the convexity adjustment, concentrated on contracts with the underlying being a commodity (French [1983]), a foreign exchange rate (Cornell and Reinganum [1981]), a stock index (Cornell and French [1983]) and a Treasury bill (Elton, Gruber and Rentzler [1984]). The first empirical study on the Eurocurrency interest rate forward-futures differential was conducted by Meulbroek (1992). She argues that the marking-to-market effect is a larger component of the interest rate forward-futures price differential than it is in the currency and commodity futures because Eurodollar deposit prices have a higher covariance with the riskless bond price. Grinblatt and Jegadeesh (1996) investigate spreads between Eurodollar futures and forward yields over the 1982-1992 period and find a vast presence of large spreads in the first half of the sample period. Their study concludes that such abnormal spreads are likely to be attributable to the mispricing of futures contracts relative to forwards. Recently Poskitt (2008) examines the pricing of Eurodollar futures contracts versus US dollar forward rate agreements (FRAs) using a high frequency data set. He finds that the median of the futures-FRA rate differential is close to zero and argues that the convexity adjustment is not priced into FRA rates for short-term contracts.

The limitation common to all of the above aforementioned studies on the Eurocurrency interest rate forward-futures differential is that their analysis is confined to short-term contracts with maturities up to nine months, whereas the spread due to the marking-to-market feature is expected to be more if not much more pronounced in longer-term contracts. Gupta and Subrahmanyam (2000) examine the convexity adjustment in the pricing of interest rate swaps versus Eurocurrency futures. The use of the swap rates allows them to extend the maturity range of contracts under consideration up to five years. They interpret their results as evidence of the underpricing of swap contracts during the late 1980s and the early 1990s which was eliminated over time. Caution must be applied, however, while considering the robustness of their results. They build the futures curve instead of the conventional spot yield curve and check whether swaps are priced off it. The use of Eurocurrency futures to construct the term structure is prone to overlapping maturities even if only those contracts that are parts of the regular maturity cycle (March-June-September-December) are considered. The problem may arise if there is a slight degree of overlap in or a gap between the three-month Eurocurrency deposit periods associated with adjacent Eurocurrency futures. Rendleman (2004) provides a few such examples. For example, the three-month Eurodollar deposit period for the June 2005 futures contract ends on 9/15/05, while the three-month deposit associated with the September 2005 futures contract starts on 9/21/05, six days later. As insignificant as this issue may look, when it comes to the computation of the forward-futures differential which is often measured in single-digit basis points, every small detail counts.

This paper utilizes a data set and methodology that allow us to avoid the two aforementioned limitations of previous studies: generalizing conclusions based on results for short maturity contracts only and error-prone yield curve building. We are able to extend the analysis for maturities up to 21 months by utilizing swap rates to build the conventional zero-coupon spot yield curve via a single interpolation procedure.

Beyond the issues of careful curve building and extending the maturity range, there are yet more important matters in the context of the interest rate forward-futures differential analysis. Sundaresan (1991) was the first to point out that since Eurodollar futures contracts settle to yields as opposed to prices[12], the implied forward prices from the LIBOR term structure should differ from the futures prices even in the absence of marking-to-market. The phenomenon that interest rate futures are settled in the way different from that of forward contracts has not been formally incorporated into early empirical works. The reason behind it is that the convexity adjustment in those studies is defined as the difference in rates. Chance (2006b) argues that it is not certain how the rates in those studies are calculated. There are several alternatives available regarding the way a "futures rate" can be calculated and, on top of that, different compounding

[12] Another way to say it is that LIBOR rates are quoted as "add-on" rates whereas the Eurodollar futures is priced as though the underlying rate were a discount instrument.

frequencies may be applied. The resulting variation in the futures-forward rate differentials computed using the same set of quotes but different approaches may be substantial and its presence would make results from different studies incompatible without knowledge of the technical details about the ways rates were calculated there. If the differential is measured as a difference in rates, the futures settlement feature is not part of it, and marking-to-market accounts for the entire difference. However, the way the "futures rate" is computed in the presence of the futures settlement feature has a few alternative applications and that is what may cause ambiguity[13]. Studying the differential measured as a price difference does not pose such problems since the price is unique no matter how rates are calculated. Recall that the convexity adjustment in its original form derived by Cox, Ingersoll and Ross (1981) was stated in terms of the price difference.

It is true that marking-to-market would still exist if both spot and futures instruments were settled in a different manner. It is also true that even if there were no marking-to-market, the convexity bias would still be present if the instruments were settled and priced differently. Therefore, it is important to understand that when one makes comparisons between Eurocurrency[14] interest rate futures and forward prices, there are two separate types of effects present which are largely independent and, hence, must not be confounded.

Sundaresan (1991) argues that due to the unique settlement feature of the Eurodollar futures contract, the implied forward price from the spot LIBOR term structure is inappropriate for the purposes of comparison with the Eurodollar futures price. In order to exclude the influence of this feature on the results of the analysis, he introduces the "Eurodollar forward price" and compares it to a given Eurodollar futures price. Chance (2006b) controls for the expiration settlement feature of Eurodollar futures by introducing the "adjusted forward price" which is the forward price whose underlying LIBOR time deposit is expressed as a discount instrument and, alternatively, by constructing a hypothetical futures contract that settles by the add-on method. In this paper we show that there is no need to come up with hypothetical contracts in order to eliminate the influence of the Eurocurrency interest rate futures unconventional settlement feature. Moreover, the latter factor can be easily incorporated into analysis. We derive the formula that shows that the Eurodollar convexity adjustment, the

[13] See Chance (2006b) for examples
[14] From here and further on terms "Eurocurrency" and "Eurodollar" are used interchangeably in general context.

difference between the implied forward price obtained from the spot rate term structure and the respective Eurodollar futures price at any point of time before maturity, is composed of two parts: the element due to marking-to-market and the element arisen from the unconventional settlement of Eurodollar futures.

The analysis of the forward-futures differential can be performed in two dimensions: empirically as well as theoretically. The latter implies the derivation of the differential in the context of the term structure modeling. By focusing on the one-factor, constant volatility version of the HJM model, Flesaker (1993a) was the first to utilize no-arbitrage term structure model in order to derive close-form solutions for both the Eurodollar futures price and the forward price. Gupta and Subrahmanyam (2000) employ the Hull and White (1990), the Black-Karasinski (1991) and the two-factor HJM model with constant volatility to compute the convexity adjustment measured as the difference in yields. The estimation of all three models was carried out by the construction of a trinomial tree of interest rates. It is not clear however what size of the tree step was chosen in those models. Chance (2006b) uses one-factor HJM model to estimate the Eurodollar forward-futures price differential[15]. His model is constructed with the size of the step of the binomial tree equal to one month. For the volatility input he uses the specification where volatility is a function of maturity time only, i. e. volatility of the forward rate N months ahead is the volatility of that same rate N-1 months ahead, N-2 months ahead, etc.

Two issues normally arise when it comes to modeling the forward-futures differential or anything else for that matter via the utilization of discrete versions of no-arbitrage term structure models. The first one is the size of the step of the binomial or the trinomial tree used. Peterson, Stapleton and Subrahmanyam (2003) caution that the value of the marking-to-market effect may be underestimated if the size of the tree step is longer than one business day, the actual interval between the two subsequent markings. They suggest that this bias is likely to be very small, probably less than a basis point, for the futures marked-to-market quarterly for maturities of a year or less, but can be far more significant for futures marked-to-market less frequently and of longer maturities. We investigate how changing frequency of the marking-to-market may affect the size of its effect on Eurocurrency futures prices. Two most common frequencies utilized in

[15] Other studies rely on equilibrium term structure models: Sundaresan (1991) utilizes the Cox-Ingersoll-Ross (1985) term structure model, while Grinblatt and Jegadeesh (1996) employ both Cox-Ingersoll-Ross and Vasicek (1977) term structure models.

previous research are used (quarterly and monthly marking-to-market) as well as the weekly size of the step is employed to determine what impact is made on the forward-futures differential and its marking-to-market component by the increasing number of the tree steps in the discrete version of the HJM model.

The second issue is the volatility specification chosen for the analysis. Flesaker (1993b) documents various biases in the fitted Eurodollar futures option prices relative to the market prices if the constant volatility version of the HJM model is used. Cakici and Zhu (2001) find that relatively high volatility can lead to a significant difference between the forward price and the Eurodollar futures price. Chance's choice of volatility specification mentioned above allows the binomial tree to recombine which avoids computational complexity but practical application of such specification may be questioned since volatility of forward rates has been shown to fluctuate over time. In order to check for the robustness of results with regard to the choice of the volatility structure, our paper utilizes four most popular volatility specifications (excluding the constant volatility case) to subsequently derive the Eurocurrency interest rate forward-futures price differential. The obtained results must answer the questions of whether and how significant the discrepancy in the value of the differential may become given the choice of the volatility specification.

Finally, the issue of major importance to the practitioners, the hedging applications of the Eurocurrency futures and its specifics, must be considered. Hedging of the market-to-market element has been addressed in the literature and is referred as tailing of the hedge (see Figlewski et al. [1991] and Kawaller [1997] among other references). Chance (2006a) brings attention to the fact that because of the way the Eurodollar futures contract is structured, the standard cash-and-carry arbitrage in which the underlying asset is purchased and the futures is sold is not risk-free since the futures price cannot converge to the value of the underlying Eurodollar time deposit. Chance shows that the perfect hedge is not possible to obtain when the borrower wants to get a fixed amount of the loan and hedge its forward interest rate exposure with Eurodollar futures. This paper demonstrates that the perfect hedge is feasible via the use of a predetermined basket of caplets. Such hedge is costly though since the caplets must be paid for at the time of the purchase.

The rest of the paper is structured as following. Section 3.2 shows a well-known derivation of the conventional Cox-Ingersoll-Ross differential and provides a new result for the

Eurocurrency forward-futures differential with an appended numerical example. Section 3.3 introduces the Heath-Jarrow-Morton (1992) term structure model and reviews previous research applications of the model to interest rate derivative pricing. Section 3.4 contains the data and methodology description. Section 3.5 provides results of the forward-futures modeling for three different marking-to-market frequencies and their extrapolation for longer out-of-sample maturities. Section 3.6 considers hedging implications and demonstrates that the nearly perfect hedge can in fact be obtained. Section 3.7 concludes.

3.2 Derivation of the Forward-Futures Differential

Cox, Ingersoll and Ross (1981) derive the closed-form solutions for forward and futures prices under a single factor square root process for the instantaneous interest rate with the additional assumption of logarithmic utility. According to their findings, the forward price is higher than the futures price given stochastic interest rate environment and both prices are decreasing convex functions of the interest rate as well as increasing functions of the time to maturity for sufficiently high interest rates. Cox, Ingersoll and Ross also show that application of the Capital Asset Pricing Model to a series of forward prices will be misdirected since forward prices do not satisfy the continuous time CAPM in consumption form as derived by Breeden (1979) unless interest rates are non-stochastic. Futures rates, however, will satisfy the consumption based CAPM.

A new series of studies reinterprets the forward price using the forward measure. Under the forward measure, the forward price is the expectation of the terminal payoff of the underlying asset price, like the futures price under the risk-neutral probability measure, but the probability measure under which this expectation is taken is different from the risk-neutral measure by an adjustment term. The adjustment term takes care of the effect of the covariance between the interest rate and the underlying asset, where the covariance term reflects the marking-to-market effect.

3.2.1 Derivation of the Cox-Ingersoll-Ross Forward-Futures Differential

Below, a formal way of derivation of the Cox-Ingersoll-Ross forward-futures differential that relies on a traditional risk-neutral probability measure is provided. These derivations can be found in many textbooks about derivatives. This section follows Jarrow (1996). Later on in the section the convexity adjustment for Eurocurrency interest rate futures is derived and this measure will differ from the Cox-Ingersoll-Ross forward-futures price differential due to the fact that interest rate futures possess a unique pricing characteristic.

Since a forward contract on a zero-coupon bond has value of zero at the time when it is initiated, it must be that

$$\tilde{E}_t\left(\frac{P(m_1,m_2)-F(t,m_1,m_2)}{B(m_1)}\right)B(t)=0, \tag{3.1}$$

where $P(m_1, m_2)$ is the price of a zero-coupon bond with maturity date m_2 at time m_1, $m_2 > m_1$,

$F(t, m_1, m_2)$ is the price of a forward contract on m_2-maturity zero-coupon bond with delivery date m_1 at time t, where t is the forward initiation date, $t < m_1$,

$B(m_1)$ is the money market account value at time m_1,

$B(t)$ is the money market account value at time t,

$\tilde{E}_t$ is the expectation under risk-neutral probability measure taken at time t.

Since interest rate cannot be negative it must be that $B(t_2) \geq B(t_1)$ for $t_2 > t_1 \geq 0$. In continuous time, money market value is written as

$$B(t)=\exp\left(\int_0^t r(u)du\right), \tag{3.2}$$

where $r(.)$ is the instantaneous interest rate.

The expression for the risk-neutral expectation (3.1) can be rewritten as

$$\tilde{E}_t\left(\frac{P(m_1,m_2)}{B(m_1)}\right)B(t)-F(t,m_1,m_2)\tilde{E}_t\left(\frac{1}{B(m_1)}\right)B(t)=0. \tag{3.3}$$

Since the arbitrage-free price of the m_2-maturity zero-coupon bond at time t is given by

$$P(t,m_2)=\tilde{E}_t\left(\frac{P(s,m_2)}{B(s)}\right)B(t) \quad \text{for} \quad t<s\leq m_2, \tag{3.4}$$

we get $P(t,m_2)-F(t,m_1,m_2)P(t,m_1)=0,$ since $P(m_1,m_1)=1$. Therefore,

$$F(t,m_1,m_2)=\frac{P(t,m_2)}{P(t,m_1)}. \tag{3.5}$$

Pricing of the futures is a little bit different. Because of the mark-to-market feature, the futures price process can be interpreted as the sum of the underlying asset price process and the dividend process where the dividend process reflects marking to market. Although in practice marking to market takes place at discrete intervals, it also can be easily expressed in continuous time framework similar to the way a continuous dividend process is modeled. The financial literature provides the result that for any price process *S* (underlying or derivative) with dividend process *D*, the normalized gains process *G(t)* in the form

$$G(t) = \frac{S(t)}{B(t)} + \int_0^t \frac{1}{B(u)} dD(u) \tag{3.6}$$

is a martingale under the risk-neutral measure. Applying this result to the futures contract, one has that

$$G(t) = \frac{\Pi(t)}{B(t)} + \int_0^t \frac{1}{B(u)} df(u, m_1, m_2) \tag{3.7}$$

is a martingale, i. e.

$$dG(t) = h(t)dW(t), \tag{3.8}$$

where *h(t)* is some adapted process and *W(t)* is a Wiener distribution under the risk-neutral measure. $\Pi(t)$ is the futures value process and $df(u, m_1, m_2) = f(u+s, m_1, m_2) - f(u, m_1, m_2)$ for a small positive *s*. Essentially, $f(u, m_1, m_2)$ is the futures price (but not its value) at time *u*.

It turns out that $\Pi(t) = 0$ for any $t \leq m_1$, since the futures contract is continuously marked to market. In other words, the value (but not the price) of the futures contract is always zero in the continuous time set up because any increase/decrease in the futures value is instantaneously captured by marking to market. Taking the derivative of both sides of (3.7) and substituting (3.8) in the resulting expression one obtains

$$\frac{1}{B(t)} df(t, m_1, m_2) = h(t)dW(t). \tag{3.9}$$

Multiply both sides by *B(t)* to get

$$df(t, m_1, m_2) = B(t)\,h(t)dW(t), \tag{3.10}$$

which implies that $f(t, m_1, m_2)$ is a martingale under the risk-neutral measure. Since at expiration after the final marking to market, the futures price is equal to the underlying, $P(m_1, m_2)$, one finally obtains expression for the futures price at any time *t*, where $0 \leq t \leq m_1$:

$$f(t, m_1, m_2) = \tilde{E}_t(f(m_1, m_1, m_2)) = \tilde{E}_t(P(m_1, m_2)). \tag{3.11}$$

From the comparison of (3.5) and (3.11) it is evident that the forward price and the futures price are not equal if interest rates are stochastic. To find the difference between the forward and the respective futures price, note from (3.4) that the forward price can be rewritten as

$$F(t,m_1,m_2) = \tilde{E}_t\left(\frac{P(m_1,m_2)}{B(m_1)}\right)\frac{B(t)}{P(t,m_1)}. \quad (3.12)$$

As Jarrow (1996) shows, one can utilize the property of the product of two random variables, $E(xy) = E(x)E(y) + cov(x,y)$ to rewrite the forward price further:

$$F(t,m_1,m_2) = \tilde{E}_t(P(m_1,m_2))\tilde{E}_t\left(\frac{1}{B(m_1)}\right)\frac{B(t)}{P(t,m_1)} + \tilde{\text{cov}}_t\left[P(m_1,m_2),\frac{1}{B(m_1)}\right]\frac{B(t)}{P(t,m_1)}. \quad (3.13)$$

Plugging in (3.11) and recalling that $\tilde{E}_t\left(\frac{1}{B(m_1)}\right)B(t) = P(t,m_1)$ finally produces

$$F(t,m_1,m_2) = f(t,m_1,m_2) + \tilde{\text{cov}}_t\left[P(m_1,m_2),\frac{1}{B(m_1)}\right]\frac{B(t)}{P(t,m_1)}, \quad (3.14)$$

that can be rewritten as

$$F(t,m_1,m_2) - f(t,m_1,m_2) = \tilde{\text{cov}}_t\left[P(m_1,m_2),\exp\left(-\int_t^{m_1} r(u)du\right)\right]\frac{1}{P(t,m_1)}, \quad (3.15)$$

since $bcov(x,y) = cov(x,by)$ for a constant b and the fact that

$$\frac{B(t_1)}{B(t_2)} = \exp\left(-\int_{t_1}^{t_2} r(u)du\right) \quad \text{for} \quad t_2 \geq t_1 \geq 0. \quad (3.16)$$

Cox, Ingersoll and Ross (1981) use no-arbitrage arguments to show that the forward-futures price differential increases with the covariance of the futures price changes with the riskless bond price changes. From (3.15) it can be seen that the forward price and the futures price will be identical only if the two random variables, $P(m_1, m_2)$ and $\exp(-\int_t^{m_1} r(u)du)$, are uncorrelated under the risk-neutral probability measure. This is true if the short rate $r(t)$ is constant or deterministic, as was shown by Cox, Ingersoll and Ross. The forward price is larger (smaller) than the futures price if the variables $P(m_1, m_2)$ and $\exp(-\int_t^{m_1} r(u)du)$ are positively (negatively) correlated under the risk-neutral probability measure. A simple intuitive argument helps to explain the aforementioned relationship. If the forward price and the futures price are equal to each other, then the total undiscounted payments from the futures contract will be equal to the terminal payment of the forward contract. Suppose that $P(m_1, m_2)$ and $\exp(-\int_t^{m_1} r(u)du)$ are positively correlated. It implies that the interest rate and the spot price of the underlying asset are negatively correlated. Then the marking-to-market of the futures tends to be negative when the interest rate is high and positive when the interest rate is low. In such case negative payments must be refinanced at a high interest rate while positive payments can be reinvested at a low interest rate. This makes the futures contract less attractive compared to its forward counterpart if both have the same price. Hence, the futures price ought to be below the respective forward price in equilibrium. Similar logic applies to the converse case when $P(m_1, m_2)$ and $\exp(-\int_t^{m_1} r(u)du)$ are negatively correlated, which implies positive correlation between the interest rate and the spot price of the underlying asset. To maintain equilibrium the futures price has to be larger than the forward price.

3.2.2 Derivation of the Eurocurrency Forward-Futures Differential

The previous section showed how forward and futures prices are obtained when the contracts are settled to the price of an underlying asset. However, this is not the case for the Eurodollar and other Eurocurrency interest rate futures. Munk (2005) mentions that the rate implied by the quoted Eurodollar futures price, equal to *1–quoted price/100*, is sometimes referred among traders and analysts as the LIBOR futures rate. As the maturity of the futures contract approaches, the LIBOR futures rate converges to the three-month spot LIBOR but the actual Eurodollar futures price (which is different from the quoted price) does not converge to the spot price of the three-month Eurodollar deposit. The reason for that lies in the method used for the pricing of Eurodollar futures.

These contracts are settled to the yield, not the price, and the Eurodollar futures price is obtained from the quoted futures price adjusted by the day count factor:

$$f(t,m,m+n) = 1 - \frac{100 - f_q(t,m,m+n)}{100}\frac{n}{360}, \tag{3.17}$$

where n is the number of actual days in the three-month LIBOR deposit that the Eurodollar futures is on and $f_q(t, m, m+n)$ is the quoted futures price.

By its design, at maturity the futures price converges to $100\times(1-LIBOR_n\times n/360)$, where $LIBOR_n$ is the n-period LIBOR in decimal form on the expiration date[16] and $n/360$ is the adjustment factor. For a generalization purpose, the above expression can be divided by 100, so that the obtained futures price is in the range between zero and one which makes it comparable to the respective forward price. In continuous time, the equilibrium Eurodollar futures price then becomes

$$f(t,m_1,m_2)=1-\tilde{E}_t\left(\frac{B(m_2)}{B(m_1)}-1\right), \tag{3.18}$$

since

$$\tilde{E}_t\left(\frac{B(m_2)}{B(m_1)}\right)=\tilde{E}_t\left[\exp\left(\int_{m_1}^{m_2} r(u)du\right)\right]. \tag{3.19}$$

Equation (3.19) represents the expectation of the amount earned at time m_2 by rolling one dollar in a sequence of instantaneous rates from time m_1 until time m_2. In discrete time, the equivalent term under the risk neutral expectation will be $LIBOR_n\times n/360$, where $n = m_2 - m_1$ and $LIBOR_n$ is the n-period rate established at time m_1. The term $n/360$ represent the adjustment factor for the interest rate applied to the holding period of length n which is less than a year. The adjustment factor is used in a discrete time set-up only, while in continuous time the integral sign implicitly takes it into account. Since expectation of a constant is the constant itself, the Eurodollar futures price can be rewritten as

$$f(t,m_1,m_2)=2-\tilde{E}_t\left(\frac{B(m_2)}{B(m_1)}\right). \tag{3.20}$$

By the virtue of (3.4) and the law of iterated expectations[17], the actual three-month Eurodollar futures price at any point of time t before the futures maturity date can also be written in discrete time in the following way[18]:

$$f(t,m,m+0.25)=2-\tilde{E}_t[P(m,m+0.25)^{-1}], \tag{3.21}$$

where

[16] This date is usually referred as the settlement date.
[17] Strictly speaking, this condition should rather be referred as the fact that the futures is a martingale and the martingale is stronger than the law of iterated expectations. We defer to the former in order to avoid possible confusion since in the previous section we show that futures on a *bond* is a martingale. Nevertheless, the Eurocurrency interest rate futures is also a martingale for all the same premises.
[18] The adjustment factor for the three-month Eurodollar futures is 90/360, or 0.25.

$$P(m,m+0.25)=\frac{1}{1+0.25\times LIBOR_{0.25}(m)}, \tag{3.22}$$

and $LIBOR_{0.25}(m)$ is the three-month LIBOR at expiration of the Eurodollar futures contract (the settlement LIBOR futures rate). Since the actual Eurodollar futures price is obtained from the quoted Eurodollar futures price through (3.17):

$$f(t,m,m+0.25)=1-0.25[1-f_q(t,m,m+0.25)]=0.75+0.25f_q(t,m,m+0.25), \tag{3.23}$$

the quoted Eurodollar futures price at any point of time t before maturity, therefore, can be written as

$$\begin{gathered} f_q(t,m,m+0.25)=4f(t,m,m+0.25)-3=5-4\tilde{E}_t[P(m,m+0.25)^{-1}]= \\ 5-4\tilde{E}_t[1+0.25\times LIBOR_{0.25}(m)]=1-\tilde{E}_t[LIBOR_{0.25}(m)]. \end{gathered} \tag{3.24}$$

From the expression above we see that the quoted Eurodollar futures price is also a martingale.

Using the previously obtained formula (3.13) for the forward price and new expression (3.20) for the actual Eurodollar futures price, their difference, Δ, can be shown to be equal to

$$\Delta=\tilde{E}_t(P(m_1,m_2))+\tilde{E}_t\left(\frac{B(m_2)}{B(m_1)}\right)-2+\tilde{\text{cov}}_t\left[P(m_1,m_2),\exp\left(-\int_t^{m_1}r(u)du\right)\right]\frac{1}{P(t,m_1)}, \text{ or}$$

$$\Delta=\tilde{E}_t\left(\exp\left(-\int_{m_1}^{m_2}r(u)du\right)\right)+\tilde{E}_t\left(\exp\left(\int_{m_1}^{m_2}r(u)du\right)\right)-2+\tilde{\text{cov}}_t\left[P(m_1,m_2),\exp\left(-\int_t^{m_1}r(u)du\right)\right]\frac{1}{P(t,m_1)} \tag{3.25}$$

Now it can be seen that the Eurodollar forward-futures price differential has three new terms in it if compared to (3.15). It can be shown further that Δ always exceeds the covariance term on the right side since e^x is a strictly convex function.

After switching to the discretely compounded rates, it is easy to see that the unique component of the Eurodollar forward-futures price differential is nothing else but the product of the Eurodollar futures settlement design feature. The forward price in (3.13), if expressed with the use of discrete rates, can be written as

$$F(t,m_1,m_2)=\tilde{E}_t\left(\frac{1}{1+x}\right)+\tilde{\text{cov}}_t\left[\frac{1}{1+x},\frac{1}{B(m_1)}\right]\frac{B(t)}{P(t,m_1)}, \tag{3.26}$$

where x is the three-month LIBOR at expiration times the adjustment factor 90/360.

The futures price in (3.20) becomes $1-\tilde{E}_t(x)$. Therefore, the Eurodollar forward-futures price differential at time *t*, given the discretely compounded rates, is

$$\Delta = \tilde{E}_t\left(\frac{1}{1+x}\right) + \tilde{E}_t(x) - 1 + \tilde{\text{cov}}_t\left[\frac{1}{1+x}, \frac{1}{B(m_1)}\right]\frac{B(t)}{P(t,m_1)}. \tag{3.27}$$

Recalling that with the discretely compounded rates, the interest rate forward-futures differential at *expiration* is equal to

$$\frac{1}{1+x} - (1-x), \tag{3.28}$$

one observes that the new component of the convexity adjustment in (3.27) is similar to the expression (3.28), except for the expectation terms. This observation allows to conclude that the new component of the convexity adjustment in the interest rate futures is a product of the pricing construction method used for Eurodollar and other Eurocurrency interest rate futures contracts. Hence, the convexity adjustment expressed as the difference between the price of the forward and the price of the respective futures contract on an interest rate consists of two parts: the element associated with the unique pricing (and settlement) feature of interest rate futures and the covariance term that reflects the marking-to-market effect. From now on the paper shall refer to them as the "settlement component" and the "marking-to-market (covariance) component" respectively[19]. The settlement component is always positive, so is the covariance component in general since interest rates for different maturities have been found to be positively correlated.

3.2.3 Numerical Example

Below, a numerical example of how the Eurocurrency convexity adjustment is calculated is provided. The input information is that from Jarrow (1996) on pages 106-107. The purpose is to calculate the convexity adjustment for the interest rate futures on a one-period spot rate. The futures matures in period 2. So, one needs to calculate the current forward price, *F(0,2,3)*, and the futures price, *f(0,2,3)*, first. The inputs consist of current bond prices and the money market account (gross short rate) process. The bond prices are: *P(0,1)* = 0.980392, *P(0,2)* = 0.961169, *P(0,3)* = 0.942322. The gross short rate process is as following: *r(0)* = 1.02, *r(1;u)* = 1.017606,

[19] Since the ultimate existence of the marking-to-market effect is due to the fact that futures are *settled* on a daily basis, some literature may use the marking-to-market and the settlement effect terms interchangeably, e. g. marking-to-market is occasionally referred as the "daily resettlement feature" of futures contracts. We distinguish the settlement component from the marking-to-market component by emphasizing that the former is an artifact of the way Eurocurrency interest rate futures are designed whereas the latter is the sole product of the expected covariance between future rates.

r(1;d) = 1.022406, *r(2;uu)* = 1.016031, *r(2;ud)* = 1.020393; *r(2;du)* = 1.019192, *r(2;dd)* = 1.024436, where *u* stands for an up move and *d* stands for a down move in the *discount factor*. The forward price is calculated using (3.5):

$$F(0,2,3) = \frac{P(0,3)}{P(0,2)} = \frac{0.942322}{0.961169} = 0.980392\,. \tag{3.29}$$

The futures price is calculated as the risk-neutral expectation of the one-period spot rate at the expiration of the contract. Jarrow derives the risk-neutral probability of an up movement in the binomial tree, *q*, for this example and it is equal to one half. Therefore,

$$\begin{aligned} f(0,2,3) &= 1 - 0.25(r(2;uu)-1) - 0.25(r(2;ud)-1) - 0.25(r(2;du)-1) \\ &- 0.25(r(2;dd)-1) = 1 - 0.020013 = 0.979987. \end{aligned} \tag{3.30}$$

Hence, the forward-futures price differential, *F(0,2,3) – f(0,2,3)*, is equal to 0.000405, or 4.05 basis points. From (3.27) one knows that the value of the differential is composed of the settlement component and the covariance component. The settlement component, $\tilde{E}_t\left(\frac{1}{1+x}\right) + \tilde{E}_t(x) - 1$, is calculated as following:

$$\begin{aligned} &0.25\left(\frac{1}{1.016031} + \frac{1}{1.020393} + \frac{1}{1.019192} + \frac{1}{1.024436}\right) + \\ &0.25(0.016031 + 0.020393 + 0.019192 + 0.024436) - 1 = 0.000401. \end{aligned} \tag{3.31}$$

The covariance element, $\tilde{\text{cov}}_t\left[\frac{1}{1+x}, \frac{1}{B(m_1)}\right]\frac{B(t)}{P(t,m_1)}$, in this example will be expressed as $\tilde{\text{cov}}_0\left[\frac{1}{r(2)}, \frac{1}{r(0)r(1)}\right]\frac{1}{P(0,2)}$. *P(0,2)* is given and the covariance is calculated as following:

$$\begin{aligned} &\tilde{\text{cov}}_0\left[\frac{1}{r(2)}, \frac{1}{r(0)r(1)}\right] = \\ &0.25\left[\frac{1}{r(2;uu)} - \tilde{E}_0\left(\frac{1}{r(2)}\right)\right]\left[\frac{1}{r(0)r(1;u)} - \frac{1}{r(0)}\tilde{E}_0\left(\frac{1}{r(1)}\right)\right] + \\ &0.25\left[\frac{1}{r(2;ud)} - \tilde{E}_0\left(\frac{1}{r(2)}\right)\right]\left[\frac{1}{r(0)r(1;u)} - \frac{1}{r(0)}\tilde{E}_0\left(\frac{1}{r(1)}\right)\right] + \\ &0.25\left[\frac{1}{r(2;du)} - \tilde{E}_0\left(\frac{1}{r(2)}\right)\right]\left[\frac{1}{r(0)r(1;d)} - \frac{1}{r(0)}\tilde{E}_0\left(\frac{1}{r(1)}\right)\right] + \end{aligned} \tag{3.32}$$

$$0.25\left[\frac{1}{r(2;dd)} - \tilde{E}_0\left(\frac{1}{r(2)}\right)\right]\left[\frac{1}{r(0)r(1;d)} - \frac{1}{r(0)}\tilde{E}_0\left(\frac{1}{r(1)}\right)\right].$$

First, one needs to find the two expectation terms:

$$\tilde{E}_0\left(\frac{1}{r(2)}\right) = \frac{1}{4}\left(\frac{1}{1.016031} + \frac{1}{1.020393} + \frac{1}{1.019192} + \frac{1}{1.024436}\right) = 0.980388\,, \tag{3.33}$$

$$\tilde{E}_0\left(\frac{1}{r(1)}\right) = \frac{1}{2}\left(\frac{1}{1.017606} + \frac{1}{1.022406}\right) = 0.961169. \tag{3.34}$$

Eventually, one obtains

$$\tilde{\mathrm{cov}}_0\left[\frac{1}{r(2)}, \frac{1}{r(0)r(1)}\right]\frac{1}{P(0,2)} = 0.000004, \tag{3.35}$$

which is exactly what one would expect since it must be equal to the difference between the obtained forward-futures price differential and the settlement component in (3.31). In summary, this example produces the convexity adjustment of 4.05 basis points, where 4.01 basis points are due to the settlement component and 0.04 basis points are due to the covariance (marking-to-market) component. Jarrow (1996) calculates the futures price as if it is settled to a zero-coupon bond price and his answer is 0.980388 (page 136). The difference between the forward price in (3.33) and 0.980388 is 0.000004, or 0.04 basis points, as expected, since such difference is represented by the last term on the right side of (3.14), namely, the covariance component, which is computed in (3.35).

It remains to be seen yet whether the settlement component indeed has much more weight in the forward-futures differential than the marking-to-market component and under what conditions. For that purpose in next section the convexity adjustment for interest rate Eurocurrency futures is derived by employing the Heath-Jarrow-Morton no-arbitrage term structure model.

3.3 The Heath-Jarrow-Morton (HJM) Term Structure Model

The model proposed by Heath, Jarrow and Morton (1992) represents a substantial contribution to how the term structure of interest rates is perceived and modeled. Their model is commonly referred as HJM. One of the main differences between HJM and earlier no-arbitrage term structure models such as the Ho-Lee model and the Black-Derman-Toy (BDT) model is that while the latter models concentrate on modeling of the short spot rate, the HJM model is built around the evolution of the instantaneous forward rate. Also, while early no-arbitrage models concentrate exclusively on modeling of the short rate, HJM models the evolution of the entire forward rate curve by developing a framework where the instantaneous forward rate curve is modeled directly.

The development of the HJM model is often referred as the most important achievement in the history of term structure modeling. Interestingly, the ideas of the paper were accepted more quickly by the practitioners than by the academic community. Hughston (2003) recalls that the very first draft of the paper and its practical implications were appreciated almost immediately by the community of interest rate modelers and traders[20].

The HJM model has several advantages over the pioneer of the no-arbitrage interest rate modeling, the Ho-Lee (1986) model. While the Ho-Lee model is a one-factor model, the HJM model can accommodate any finite number of factors although it results in a considerable increase in the complexity and computational burden associated with each added factor[21]. In addition, HJM is able to admit an extremely flexible structure of the volatility of interest rates.

3.3.1 Continuous Version of HJM

Recall that given the differentiability of the discount bond at its maturity date, the instantaneous forward rate (or forward short rate) is defined as following:

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}. \tag{3.36}$$

which comes from the fact that the continuously compounded forward rate for the time period from S to T, $S < T$, contracted at time t, $t < S$, can be written as

$$F(t,S,T) = -\frac{\ln P(t,T) - \ln P(t,S)}{T-S}. \tag{3.37}$$

The bond price can be recovered by integrating (3.36). Note also that the shortest forward rate, $f(t, t)$, is nothing but the short spot rate.

The one-factor HJM model is characterized by the following set of equations:

$$df(t,T) = \mu_f(t,T)dt + \sigma_f(t,T)dW(t), \tag{3.38}$$

$$\mu_f(t,T) = \sigma_f(t,T)\int_t^T \sigma_f(t,s)ds, \tag{3.39}$$

[20] While published in 1992, draft of the paper was already available in 1988 in the form of a technical memorandum at Cornell University.
[21] It is impractical to use more than three factors in interest rate modeling in the wake of findings of Litterman and Scheinkman (1991) and related research.

$$P(t,T) = \exp(-\int_t^T f(t,s)ds), \tag{3.40}$$

where $f(t, T)$ is the instantaneous forward rate at time T as seen at time t,

$\mu_f(t, T)$ and $\sigma_f(t, T)$ are the drift and the volatility of the stochastic process for the instantaneous forward rate respectively.

Equation (3.39) represents the no-arbitrage condition. It shows that the drift parameter must be a function of the rate's volatility. The volatility function $\sigma_f(t, T)$ is an input and can be selected arbitrarily. The exogenous specification of the instantaneous forward rate volatilities, together with the risk premium and the initial discount or yield curve, is sufficient to determine the evolution of $f(t, T)$. And by invoking the risk-neutrality assumption, all it takes to obtain the results within the HJM framework is to model the instantaneous forward rate volatilities. The essence that the volatility structure of the instantaneous forward rates can be freely specified is a blessing that comes with a curse since by the specification of volatility parameters one effectively determines the no-arbitrage conditions and the evolution of the forward rates. Developing the volatility function is central to the model building to assure an accurate pricing of various interest rate derivatives under the HJM framework. Most common volatility structures identify one or more state variables that manage to capture the entire term structure of volatility. The form restrictions are imposed on the volatility structure of the forward rate, not the spot rate.

Despite the advantages of HJM over the short rate no-arbitrage models, it was found to have several drawbacks. First, the continuous time HJM model is applicable only to a small set of volatility functions that satisfy regularity conditions such as integrability. A number of volatility term structures results in non-Markovian dynamics of the instantaneous forward rate which implies path dependence that results in impossibility to use the PDE-based computational approach for pricing derivatives and increased computational complexity. After expressing the stochastic process for the instantaneous forward rate as the equation for the short rate and differentiating it with respect to time t, the stochastic process for the short rate is obtained:

$$dr_t = \left[f'(0,t) + \frac{\partial}{\partial t}\int_0^t \sigma_f(u,t)\int_u^t \sigma_f(u,s)dsdu + \int_0^t (\sigma_f(u,t))' dW_u \right] dt + \sigma_f(t,t)dW_t, \tag{3.41}$$

where $f'(0,t) = \frac{\partial f(0,t)}{\partial t}$ and $(\sigma_f(u,t))' = \frac{\partial \sigma_f(u,t)}{\partial t}$.

As can be seen from the expression above, the risk-neutral drift for the short rate depends on stochastic variables for times earlier than *t*. Hence, the stochastic dynamics of the system is non-Markovian[22].

Second, with the exception of a few restrictive volatility structures, the HJM model does not produce a closed-form solutions. As there is generally no simple solution or method to price such derivatives as caps and swaptions, Monte Carlo methods are often called upon and the computational burden associated with them can be substantial. Third, since the distribution of modeled interest rates is normal, the HJM model permits negative interest rates. The problem of negative rates can be mitigated by the choice of an appropriate volatility specification that dampens volatility once rates move closer to zero. Examples of such volatility specifications include the exponentially dampened volatility, $\sigma(t, T) = \sigma exp[-\lambda(T-t)]$, and the proportional exponentially dampened volatility, $\sigma(t, T) = \sigma r(t)^{\gamma} exp[-\lambda(T-t)]$. Finally, if forward rates are modeled as the log-normal processes, then the HJM model can "explode"[23] (Sandmann and Sondermann [1997]). This last problem can be avoided by modeling LIBOR or swap rates as log-normal instead of the instantaneous forward rate. Models that utilize such approach are referred as LIBOR market models and swap market models respectively.

3.3.2 Discrete Version of HJM

The requirement of the existence of the continuum of instantaneous forward rates that the HJM model relies upon is incompatible with real-world markets. In practice, one or both of the two most popular approaches are used to discretize the HJM model[24]. The first method does it via the use of a binomial tree; the second one relies on Monte Carlo simulation. A major challenge has been to obtain a discretized version of the drift restriction. The first attempt was

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[22] Ritchken and Sankarasubramanian (1995) put the following restriction on the volatility function that permits the term structure to be represented by a two-state Markovian model but sacrifices some part of the generality of the HJM model:

$$\sigma_f(t,T) = \sigma_f(t,t)\exp(-\int_t^T k(x)dx),$$

where $\sigma_f(t, t)$ is the volatility of the spot rate and $k(x)$ is the exogenous deterministic function.
Bhar, Chiarella, El-Hassan and Zheng (2000) demonstrate that the specification of the forward rate volatility function within the HJM framework that depends upon time to maturity, the instantaneous spot rate of interest and a forward rate of a fixed maturity allows a two-factor Markovian representation of the stochastic dynamics of the forward rate to obtain a closed form expression for bond prices.

[23] In particular, the solution explodes if the volatility function is specified as $\sigma(t, s) = \sigma f(t, s)$, where σ is a positive constant.

[24] A good source on the pricing of Eurodollar futures and options in the continuous one-factor HJM framework with deterministic volatility is Henrard (2005).

made by Heath, Jarrow and Morton (1991) and subsequently repeated by Ritchken (1996). Their approach was to obtain the result by using a binomial model, letting the time step approach zero and demonstrating that the obtained result will be correct in the continuous time limit. Grant and Vora (1999) show however that the discrete time version of Heath, Jarrow and Morton is not correct and proceed to derive the correct discrete time formula and, eventually, the correct expression for the drift term in the discrete version of the HJM model[25]:

$$\alpha(t,T) = \frac{1}{2}\left[\sigma^2(t,T) + 2\sigma(t,T)\sum_{j=t+1}^{T-1}\sigma(t,j)\right]. \tag{3.42}$$

The value of $\alpha(t, T)$ can be easily computed from the covariance matrix of forward rate volatilities. Grant and Vora refer to the drift term above as the drift adjustment term (DAT). Numerical examples of the construction of the discrete version of the forward rate evolution via the binomial tree are demonstrated in Grant and Vora (1999), Chance (2004) and Grant and Vora (2006). Grant and Vora's implementation of the HJM model in discrete time yields more accurate results than those by Heath, Jarrow and Morton (1991) for time intervals of moderate size since it does not violate the local expectations condition, while the latter method does. See Grant and Vora (1999) for numerical illustrations.

Although HJM is flexible in accommodating the specified volatility structure, there is an important restriction: although the volatility is allowed to change over time it cannot change stochastically and independently of the level of rates. In other words, although the volatility of the period two forward rate at time one, $\sigma(1, 2)$, can be different from its volatility today, $\sigma(0, 2)$, the volatility $\sigma(1, 2)$ is deterministic and must be a function of the forward rate. The changing time series of the volatility parameter poses a computational difficulty in the context of the lattice analysis since it results in non-recombining trees. For a tree to recombine it must be that $\sigma(s_1, t) = \sigma(s_2, t)$ for $0 \leq s_1 < s_2 < t$ and for every t, which is equivalent to the assumption of constant time series volatility. The cross section of the volatilities does not have to be constant though[26]. See de Munnik (1994) for more on recombining trees within the HJM model. Li, Ritchken and Sankarasubramanian (1995) employ the changing probability technique of Nelson

[25] Equation (3.42) of the drift term can be confusing for the case when $t = 0$ and $T = 1$. Chance (2004) provides the alternative equivalent formula: $\alpha(t,T) = \sigma(t,T)\sum_{j=t+1}^{T}\sigma(t,j) - \sigma^2(t,T)/2.$

[26] By assuming constant volatility, $\sigma(t, T) = \sigma$ for any t and T, and a sole factor, the HJM model is essentially reduced to the Ho-Lee model.

and Ramaswamy (1990) to develop a lattice approach for pricing European and American interest rate claims using the HJM paradigm that focuses on a class of volatilities that permits a Markovian representation of the term structure which allows to avoid the path dependence.

The discretized version of the HJM model presented above is the one-factor model[27] but it can be easily extended to a multi-factor model although the computational intensity of such models can be substantial. Ağca and Chance (2004) generalize the Grant-Vora result by extending the single-factor discrete time Heath-Jarrow-Morton model to a multi-factor world and provide numerical examples for a two-factor HJM model. The problem of negative rates can be mitigated upon the introduction of exponential dampening to the volatility. The volatility in such case, however, becomes time-dependent which makes it impossible to construct a recombining lattice of the forward rate evolution. A large number of steps will produce an exponential number of nodes (e. g. for T steps the number of nodes at time T is 2^T) which makes it impractical to use binomial trees. Monte Carlo simulations are generally used in such circumstances instead of the binomial lattice. The trade-off between the complexity of the model and the associated computational burden is more than often faced by researchers and applies far beyond term structure modeling.

3.3.3 Previous Research on the Application of HJM to Price Eurocurrency Derivatives

Several studies investigate the pricing of interest rate futures and options on futures by utilizing the HJM framework. With one exception, those studies assume constant volatility structure. Also with one exception, the single-factor model is employed.

Using the data for Eurodollar futures and futures options from March 1985 through July 1988, Flesaker (1993b) studies the empirical performance of the one-factor, constant volatility version of the HJM model and documents the tendency for the fitted models to overvalue short-term options relative to long-term options. Using the dataset from 1985 to 1988, a range of volatilities from 1% to 2.5% and maturities of up to three years, Flesaker (1993a) employs the same version of HJM to document that the futures-forward yield difference, given the daily marking-to-market, is fairly trivial for futures contracts with maturities below a year while for longer maturities, the difference may be quite significant and it increases with the maturity and

[27] Note that in a linear one-factor model like this all rates are perfectly correlated, $corr(f(s, t_1), f(s, t_2)) = 1$ for $0 \leq s < t_1 < t_2$.

the level of interest rate volatility[28]. The study also investigates the effect of varying the frequency of marking-to-market and finds that if marking-to-market within the model takes place only twice during the life of the contract (including the final settlement), it captures a full half of the maximum effect which is reached when prices are marked-to-market continuously; if a one-year Eurodollar futures contract is marked-to-market once a month, more than 90% of the continuously evaluated marking-to-market effect is captured.

Cakici and Zhu (2001) use the algorithm developed by Ritchken and Sankarasubramanian (1995) to make comparisons among the HJM models with different volatility structures[29] in pricing the Eurodollar futures option. They demonstrate that the differences among the HJM models as well as the difference between the HJM models and Black's model are insignificant when the volatility of the forward rate is relatively small while higher volatility can lead to a significant difference between the forward price and the Eurodollar futures price. However, in the latter case, the impact on the options price remains trivial.

Using data on US LIBOR rates for the period from 1987 through 2000, Chance (2006b) shows that if the one-factor HJM model is used where volatility is constant over time but is allowed to vary cross-sectionally, the expiration settlement feature of the Eurodollar futures contract accounts for virtually all of the forward-futures price differential and, if removed from consideration, the difference between futures and forward prices is less than one basis point, which is smaller than the cost of the margin and the bid-ask spread. Using data for multiple currencies for the 1987-2000 period, Gupta and Subrahmanyam (2000) employ a two-factor HJM with constant volatility to show that the convexity adjustment, measured as a difference in yields, can be very large for long-dated contracts (up to 80-100 basis points for a ten-year futures contract).

3.4 Data Description and Methodology

This paper uses data for GBP (British pound) LIBOR that was obtained from the British Bankers' Association website. The data was acquired for the sample period from 1/23/1997 until 2/28/2007. Swap rates for the same sample period are acquired from the Bloomberg electronic service station (mid-quotes of swap rates were used). Using the GBP LIBOR quotes together

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[28] From two basis points for a two-year maturity with σ = 1% to 28.1 basis points for a three-year maturity with σ = 2.5%.

[29] Ritchken (1996) provides examples of the most popular forms of the volatility specifications.

with the data for British pound plain vanilla swaps of 18 month and two year maturities allows to extend the maturity range in the analysis of the forward-futures differential up to 21 months.

The choice of the use of interest rates for the British currency over its major counterparts, the US dollar and the euro currency, in this paper's analysis is based on two observations. First, a rate quote on a plain vanilla fixed-for-floating interest rate swap contract with a maturity of 18 months is available for the British pound but not for the US dollar on Bloomberg. The availability of the 18-month swap rate altogether with those for 12 and 24 months makes it possible to extend the maturity range in a spot yield curve up to 24 months as can be seen from equation (3.44) later in the section. That will allow to extend the maturity range for the Eurocurrency forward-futures differential up to 21 months whereas the often-cited earlier papers on the convexity adjustment limit their analysis to maturities of just nine months. Second, the euro currency has a relatively short history and the first quotes for interest rate swaps appear on 1/01/1999. Also the Bloomberg electronic service frequently reports missing swap quotes for euro rates, especially for the maturity of 18 months, up until July of 2000, while British pound swap rates for the three aforementioned maturities date back to 1/23/1997 and have a complete history of quotes since then. These two factors combined make the choice of pound data preferable to those of the dollar or the euro.

One more issue to be addressed before moving on to the description of the methodology is the potential for the presence of the asynchronicity bias that may arise when the two data subsets (LIBOR quotes and swap rates) are combined. The British Bankers' Association's LIBOR rates are established and published at 11 am London time. As for the swap rates, Rendleman (2004) mentions that the Bloomberg electronic system allows historical data on the latter to be collected as of 6 am, 1 pm and 5:30 pm Eastern time of each trading day which corresponds to Tokyo, London and New York "closing" times respectively. There is, in fact, no market closing in any of those places since swaps can be traded over-the-counter 24 hours a day, but Bloomberg created these virtual time stamps as a matter of convenience. If one chooses swap rate quotes corresponding to the Tokyo "closing" time, as this paper does, that would perfectly match the timing of the LIBOR publication (6 am in New York corresponds to 11 am in London). Hence, the results and conclusions will be clean from the errors arising from the use of non-synchronous data.

The yield curve was interpolated by employing the extended Nelson-Siegel method of Svensson (1995) and Nelson and Siegel (1988). According to Bank of International Settlements paper No. 25 (2005), this technique has been the most popular numerical optimization approach to construct the yield curve. For the interpolation of the zero-coupon yield curve, GBP LIBOR rates for all twelve monthly maturities (from one month to twelve months) and implied yields for 18 months and two years derived from the quoted swap rates were used. Respective maturity lengths were computed using the modified following business day convention and the end-end-dealing rule[30]. Interpolation is performed for rates instead of implied zero-coupon prices since the former approach provides a substantially better fit for shorter maturities (see Svensson [1995]). Since the Nelson-Siegel model implies the use of continuously compounded rates, all quoted LIBOR rates were converted into rates with actual/365 day count and continuous compounding according to the following equation:

$$r(t,m) = \frac{365}{m}\ln\left(1+\frac{m}{365}r^{q}(t,m)\right), \tag{3.43}$$

where $r^q(t, m)$ is the quoted LIBOR for maturity m at time t. Note that quoted LIBORs, as well as the swap rates, for the British pound are initially based on the actual/365 day count.

To calculate implied spot rates for maturities of 18 months and two years, swap rates must have first been used to obtain discount factors. The British pound interest rate swap rates are quoted similarly to those on US dollar interest rate swaps and both swaps have similar structure, notably, the fixed-leg payments are made semi-annually. Therefore, one needs to apply the following equation to calculate discount factors for 18 months and two years

$$S(t,0.5m) = \frac{2-c(t,0.5m)\sum_{n=1}^{m-1}S(t,0.5n)}{2+c(t,0.5m)} \quad \text{for } m=3 \text{ and } 4, \tag{3.44}$$

where $S(t, 0.5n)$ is the discount factor at time t applied for maturity $0.5n$ and it is equal to $exp(-r(t, 0.5n)0.5n)$,

$c(t, 0.5m)$ is the swap rate for maturity $0.5m$ at time t,

30 The British Bankers' Association's modified following business day convention defines the maturity date as the first following day that is a business day in London and the principal financial centre of the currency concerned, unless that day falls in the next calendar month. In this case only, the maturity date will be the first preceding day in which both London and the principal financial centre of the currency concerned are open for business. The end-end dealing rule states that in cases when a deposit is made on the final business day of a particular calendar month, the maturity of the deposit shall be on the final business day of the month in which it matures, not the corresponding date of the month of maturity.

and m of three and four correspond to maturities of 18 and 24 months respectively. Afterwards one can compute implied spot rates for those maturities using

$$r(t,m) = -[\ln S(t,m)]/m \, . \tag{3.45}$$

To mitigate the influence of interpolation error in the analysis of the forward-futures differential the appropriate criteria for an interpolated yield curve allowing it to be included in the final sample must be introduced first. A yield curve on a particular business day is considered to satisfy the interpolation criteria if the sum of all fourteen absolute fitted errors is below 25 basis points and each absolute error does not exceed five basis points. Descriptive statistics for the obtained interpolated spot rates and implied forward rates for a range of maturities is provided in Table 3.1. The annual spot rates across the defined maturities in the sample range from 3.34 to 7.83 percent with the rates for longer maturities being less volatile than the rates for shorter maturities. A similar pattern is observed for quarterly forward rates as well. After yield curves have been constructed, implied forward prices for 90-day contracts are obtained using

$$F(t,m,m+0.25) = \frac{\exp(r(t,m)m)}{\exp(r(t,m+0.25)(m+0.25))}. \tag{3.46}$$

Table 3.1 Descriptive statistics of the interpolated spot and forward rates

Panel A: Annual continuously compounded spot rates

	Maturity							
	3m	6m	9m	12m	15m	18m	21m	24m
mean	0.0527	0.0529	0.0532	0.0534	0.0537	0.0540	0.0543	0.0546
median	0.0495	0.0502	0.0509	0.0511	0.0513	0.0515	0.0518	0.0521
stdev	0.0116	0.0114	0.0113	0.0111	0.0109	0.0106	0.0104	0.0102
min	0.0338	0.0336	0.0335	0.0334	0.0336	0.0335	0.0334	0.0335
max	0.0779	0.0778	0.0774	0.0778	0.0783	0.0773	0.0758	0.0752

Panel B: Quarterly continuously compounded forward rates

	Maturity							
	spot	3m	6m	9m	12m	15m	18m	21m
mean	0.0132	0.0133	0.0134	0.0136	0.0137	0.0139	0.0140	0.0142
median	0.0124	0.0128	0.0128	0.0130	0.0132	0.0134	0.0137	0.0139
stdev	0.0029	0.0029	0.0028	0.0027	0.0026	0.0025	0.0024	0.0024
min	0.0084	0.0084	0.0082	0.0081	0.0082	0.0082	0.0083	0.0084
max	0.0195	0.0195	0.0208	0.0212	0.0201	0.0192	0.0201	0.0210

This paper follows the usual practice of setting the risk-neutral probabilities of up and down movements at one half to fit the discrete version of the HJM tree. Once the tree is established, the interest rate contingent claims can be priced using backward recursion. Futures prices are computed using (3.21)[31].

Four popular volatility specifications are considered in order to construct the lattice of interest rate evolution within the HJM framework:

a) Volatility is a function of maturity time only, i. e. $\sigma(t_1,T) = \sigma(t_2,T)$ for $0 \leq t_1 < t_2 < T$ and for every T. This is the least computationally burdensome case since it results in a recombining tree.

b) Volatility is a function of time to maturity only, i. e. $\sigma(t_1,T_1) = \sigma(t_2,T_2)$ for $T_1 - t_1 = T_2 - t_2$. This is a more realistic case compared to the one above since empirical observations demonstrate that volatility indeed varies with time to maturity. The resulting tree, however, is a non-recombining one.

c) Exponentially dampened volatility, $\sigma(t,T) = \sigma exp[-\lambda(T-t)]$. This volatility specification is similar to that in b) since it is also a function of time to maturity. The resulting tree does not recombine as well.

d) Proportional exponentially dampened volatility, $\sigma(t,T) = \sigma f(t,T)^{\gamma} exp[-\lambda(T-t)]$. The volatility is a function of both time to maturity and the forward rate of that maturity at the point of time. This kind of volatility specification is the most demanding out of all four presented in terms of required computational time and effort since future volatilities for the whole term structure of forward rates are not known at the original point of time zero but instead must be updated at each subsequent node since they depend on the resulting forward rates. The choice of the parameter γ is tricky. First, it must be positive since a negative value of γ may result in exploding forward rates. Second, the parameter is recommended to be equal to or above one since for the case when $0 < \gamma < 1$, there exists a positive probability that volatilities and subsequent forward rates will switch to complex numbers as soon as a forward rate along the tree turns negative, a common drawback of the HJM model. However, even if γ is taken to be a unit or above, the problems still may arise, as will be explained later.

[31] Note from (3.22) and (3.27), that in order to compute the futures price and the two components of the forward-futures differential, the modeled continuously compounded short rate r must be converted into the rate with periodic compounding as required by the futures pricing model. Hence, the required rate is obtained as $exp(r) - 1$.

3.5 Results for the Forward-Futures Differential and Its Two Components

3.5.1 Results for a Step Size of Three Months

It would be standard to build a tree that describes the evolution of three-month forward rates with a step length of three months. All forward rates are converted to quarterly continuously compounded rates. This paper relies on historical standard deviations to specify volatility functions. Forward rate volatilities for the first two specifications described above are computed as the daily standard deviations measured over the previous 62 business days multiplied by the square root of 62. For the exponentially dampened volatility, parameters σ and λ must be identified. Linear regression analysis is used for that purpose. Once historical quarterly volatilities of seven forward rates[32] are computed as described above, their natural logarithms are regressed on respective time to maturity intervals. The resulting slope is the negative λ, while the exponential of the intercept is equal to σ. For the proportional exponentially dampened volatility, this paper follows a number of previous studies and chooses the value of the parameter γ equal to one. Subsequently, the natural logarithms of the ratio of historical volatilities over the respective forward rates are regressed on time to maturity ranges which yields parameters λ and σ as negative of the slope and exponential of the intercept respectively. In total, the number of business day observations for each volatility specification in the sample is equal to 2,477.

The resulting statistics for the forward-futures price differential and its two components for four different volatility specifications are presented in Table 3.2. The average value of the differential increases with maturity across all volatility specifications, however, it does not exceed 3.6 basis points for the maturity of twelve months and 9.5 basis points for the maturity of 21 months. The settlement component has a dominant presence but its average percentage of the differential declines with maturity although its share still exceeds one half for the longest maturity in the sample. The average values of the marking-to-market component increase sharply with maturity but its size is miniscule: across all four panels of the table it does not exceed a mere basis point for the twelve months maturity and is below four basis points when the maturity is 21 months for three volatility specifications. Results in panel B and panel C are very close to each other since the volatility expressed as a function of time to maturity only may be treated as a special case of the exponentially dampened specification: if R-squared for the regression of the log of volatility on time to maturity is high (and in most cases in the sample it is so), the obtained

[32] Excluding the spot rate

Table 3.2 Forward-futures differential and its two components when the length of step is equal to three months, in basis points

Panel A: Volatility is a function of maturity time

	Maturity						
	3m	6m	9m	12m	15m	18m	21m
Forward-futures price differential							
mean	1.93	2.23	2.80	3.58	4.52	5.68	7.25
median	1.67	1.89	2.52	3.16	3.85	4.84	5.93
stdev	0.82	0.91	1.25	1.80	2.46	3.22	4.38
min	0.71	0.73	0.84	1.12	1.52	1.73	1.81
max	3.98	5.05	7.61	11.82	16.23	19.68	24.78
Settlement element							
mean	1.93	2.12	2.39	2.67	2.92	3.20	3.59
av. %*	100.00	95.58	87.59	78.44	69.70	61.91	54.97
median	1.67	1.81	2.18	2.52	2.70	2.93	3.20
stdev	0.82	0.85	0.93	1.01	1.09	1.24	1.63
min	0.71	0.72	0.78	0.94	1.19	1.37	1.43
max	3.98	4.71	5.69	6.32	7.03	7.89	11.01
Marking-to-market element							
mean	0.00	0.11	0.40	0.91	1.60	2.48	3.65
av. %*	0.00	4.42	12.41	21.56	30.30	38.09	45.03
median	0.00	0.06	0.20	0.49	1.00	1.73	2.69
stdev	0.00	0.13	0.44	0.94	1.53	2.16	2.96
min	0.00	0.00	0.03	0.07	0.14	0.23	0.31
max	0.00	0.69	2.47	5.49	9.19	12.62	15.00

Panel B: Volatility is a function of time to maturity

	Maturity						
	3m	6m	9m	12m	15m	18m	21m
Forward-futures price differential							
mean	1.93	2.19	2.66	3.32	4.16	5.18	6.42
median	1.67	1.86	2.39	2.94	3.49	4.16	5.18
stdev	0.82	0.90	1.16	1.65	2.30	3.08	3.97
min	0.71	0.72	0.80	0.98	1.30	1.62	1.79
max	3.98	4.74	6.79	10.33	14.82	19.76	24.70
Settlement element							
mean	1.93	2.09	2.29	2.52	2.76	3.01	3.28
av. %*	100.00	95.45	88.07	79.79	71.69	64.21	57.40
median	1.67	1.79	2.04	2.39	2.61	2.80	2.99
stdev	0.82	0.84	0.89	0.96	1.06	1.17	1.33
min	0.71	0.71	0.75	0.85	1.01	1.20	1.39
max	3.98	4.58	5.05	5.60	6.51	7.45	8.32
Marking-to-market element							
mean	0.00	0.11	0.37	0.80	1.40	2.17	3.14
av. %*	0.00	4.55	11.93	20.21	28.31	35.79	42.60
median	0.00	0.06	0.19	0.43	0.80	1.35	2.14
stdev	0.00	0.13	0.41	0.85	1.41	2.05	2.78
min	0.00	0.00	0.02	0.05	0.11	0.18	0.28
max	0.00	0.69	2.26	4.84	8.30	12.31	16.46

(Table 3.2 continued)

Panel C: Exponentially dampened volatility

	Maturity						
	3m	6m	9m	12m	15m	18m	21m
Forward-futures price differential							
mean	1.93	2.19	2.64	3.29	4.13	5.17	6.41
median	1.67	1.87	2.35	2.92	3.48	4.16	5.15
stdev	0.82	0.89	1.14	1.62	2.28	3.08	3.96
min	0.71	0.72	0.80	0.98	1.29	1.62	1.79
max	4.13	4.92	6.92	10.36	14.83	19.79	24.70
Settlement element							
mean	1.93	2.08	2.28	2.50	2.75	3.01	3.28
av. %*	100.00	95.42	88.18	79.97	71.81	64.23	57.46
median	1.67	1.79	2.02	2.38	2.61	2.80	2.99
stdev	0.82	0.83	0.87	0.95	1.05	1.18	1.32
min	0.71	0.71	0.75	0.85	1.00	1.19	1.39
max	4.13	4.59	5.01	5.54	6.52	7.47	8.31
Marking-to-market element							
mean	0.00	0.11	0.36	0.78	1.38	2.16	3.13
av. %*	0.00	4.58	11.82	20.03	28.19	35.77	42.55
median	0.00	0.06	0.19	0.42	0.79	1.32	2.12
stdev	0.00	0.13	0.41	0.84	1.39	2.05	2.78
min	0.00	0.00	0.02	0.05	0.11	0.18	0.28
max	0.00	0.68	2.31	4.84	8.31	12.34	16.46

Panel D: Proportional exponentially dampened volatility ($\gamma = 1$)

	Maturity						
	3m	6m	9m	12m	15m	18m	21m
Forward-futures price differential							
mean	1.95	2.24	2.74	3.51	4.68	6.50	9.48
median	1.70	1.89	2.47	2.95	3.55	4.38	5.71
stdev	0.83	0.92	1.27	2.01	3.33	5.58	9.93
min	0.71	0.73	0.80	0.97	1.30	1.60	1.77
max	4.16	5.03	7.80	12.55	20.70	35.65	66.37
Settlement element							
mean	1.95	2.12	2.35	2.66	3.10	3.77	4.95
av. %*	100.00	95.17	88.18	80.40	72.50	65.01	58.43
median	1.70	1.81	2.11	2.46	2.77	2.97	3.43
stdev	0.83	0.85	0.93	1.15	1.61	2.54	4.71
min	0.71	0.71	0.75	0.84	1.01	1.27	1.40
max	4.16	4.58	5.30	7.18	11.29	19.20	34.02
Marking-to-market element							
mean	0.00	0.12	0.38	0.85	1.58	2.73	4.53
av. %*	0.00	4.83	11.82	19.60	27.50	34.99	41.57
median	0.00	0.06	0.19	0.43	0.82	1.44	2.46
stdev	0.00	0.15	0.45	0.97	1.79	3.10	5.30
min	0.00	0.01	0.02	0.05	0.10	0.18	0.27
max	0.00	0.84	2.51	5.41	10.20	18.35	32.45

*av. % denotes an average share of the respective element in the forward-futures differential (percentage)

results with these two volatility specification do not differ significantly. Both values and standard deviations of the differential and its components are visibly higher for longer maturities when the volatility is specified as the proportional exponentially dampened one. This is an expected result since given this volatility specification interest rates are more prone to the explosiveness problem. Given the small number of steps (seven) in the model, the explosion of forward rates does not quite show up yet but the magnitude of variation in the produced results in panel D of the table is relatively high. If one compares minimum values of the differential and its components in panel D with those in other panels, they barely differ, while maximum values in panel D are significantly higher and the difference rises sharply with the maturity. Except for the explosiveness problem, the use of proportional exponentially dampened volatility may result in negative volatilities. This drawback can be eliminated by choosing γ equal to two. That would accelerate the explosion process of interest rates: the problem becomes more severe with a higher γ[33].

The notable feature of all four panels of Table 3.2 is that on all occasions the value of the differential and its two components is non-negative, as predicted by the theory. The value of the marking-to-market component of the three-month differential is always a zero. This is the result of the choice of the step length: since no marking-to-market takes place between now and three months from now, its value is zero. A legitimate concern about the presented results regards the possibility that the values of the differential in general and those of marking-to-market element in particular are underestimated due to the chosen length of the step of the binomial tree. The implication of such specification is that marking-to-market takes place every three months while in practice it is done daily. Theoretically, one needs to use futures prices from contracts that are marked-to-market at the same periodicity as the time interval of the model. However, only daily marked-to-market price quotes are normally available. Therefore, an appropriate form of marking-to-market adjustment must be introduced.

3.5.2 Results for a Monthly Step

To verify how length of the tree step may affect the results, one may construct a tree with a step of a smaller size; it, however, would result in an exponentially increasing number of nodes after each extra step. For instance, in the context of this paper's analysis, a tree with a monthly

[33] For a subsample of the original sample employed in this paper, γ of two results in rates above 100 percent after the third step in the tree has been reached.

step would result in 2,097,152 nodes after the last step is made. The alternative solution is to conduct Monte Carlo simulations. Note however that if a monthly step (Δt) is chosen, one no longer models the term structure of three-month forward rates, since the modeled forward rates are the rates that apply for the time period of Δt, i. e. one month. Hence, the term structure of one-month forward rates is constructed[34], while the three-month forward rates are calculated using the following no-arbitrage requirement:

$$F(t,s,s+3\Delta t) = F(t,s,s+\Delta t) + F(t,s+\Delta t,s+2\Delta t) + F(t,s+2\Delta t,s+3\Delta t) \text{ for } s \geq t. \tag{3.47}$$

Eventually, the expected future three-month spot rates are computed that allows to calculate futures prices and compare them to observed forward prices. The settlement component is computed as the sum of the first three terms on the right side of (3.27), while the marking-to-market component is taken as the difference between the obtained forward-futures price differential and its settlement element.

The author employs 1,000 draws of the vector of standard normally distributed variables for each business day observation in the sample. Also the antithetic variate technique is employed by changing signs of the values of each drawn vector of random elements and building the evolution of forward rates anew. Therefore, the total number of the forward rate paths built for each observation in the sample is 2,000. Then averages of the differential and its two elements across 2,000 different paths are taken and recorded as the forward-futures differential, the settlement element and the marking-to-market element for that particular business day in the sample for the case when marking-to-market is conducted monthly. The resulting statistics for the whole sample for four employed volatility specifications are presented in Table 3.3.

Comparing results in Table 3.3 with those in Table 3.2 panel by panel it becomes evident that reduction of the step length from three months to one month does not result in any significant differences. As expected, the marking-to-market components of the three-month differential is not zero anymore but its average value across all panels is just 0.03 basis points. For other maturities the average marking-to-market component increases slightly relative to its values in Table 3.2 for panels A, B and D and decreases slightly for longer maturities in panel C.

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[34] One-month forward rates are taken as monthly continuously compounded rates. Volatilities for the first two specifications are taken as the daily standard deviation of the respective monthly forward rate computed using data for 62 previous business days multiplied by the square root of 62/3. Parameters for exponentially and proportional exponentially dampened volatilities are found in the same manner as described above for the general case when the three-month step is used.

Table 3.3 Forward-futures differential and its two components when the length of step is equal to one month, in basis points

Panel A: Volatility is a function of maturity time

	Maturity						
	3m	6m	9m	12m	15m	18m	21m
Forward-futures price differential							
mean	1.96	2.30	2.92	3.77	4.78	5.99	7.65
median	1.71	1.95	2.60	3.27	4.01	5.06	6.23
stdev	0.84	0.96	1.36	1.98	2.68	3.48	4.68
min	0.72	0.74	0.86	1.17	1.55	1.78	1.84
max	4.05	5.30	8.38	12.93	17.68	21.30	26.19
Settlement element							
mean	1.93	2.13	2.40	2.67	2.93	3.21	3.61
av. %*	98.65	93.16	84.66	75.47	66.93	59.40	52.73
median	1.67	1.81	2.19	2.52	2.70	2.95	3.21
stdev	0.82	0.85	0.94	1.02	1.10	1.25	1.64
min	0.71	0.72	0.79	0.94	1.19	1.38	1.44
max	3.99	4.78	5.79	6.57	7.38	8.04	11.60
Marking-to-market element							
mean	0.03	0.18	0.53	1.09	1.84	2.78	4.04
av. %*	1.35	6.84	15.34	24.53	33.07	40.60	47.27
median	0.01	0.10	0.27	0.60	1.15	1.94	3.00
stdev	0.04	0.21	0.58	1.12	1.75	2.41	3.26
min	0.00	0.01	0.03	0.08	0.17	0.26	0.34
max	0.28	1.17	3.26	6.56	10.52	14.13	16.55

Panel B: Volatility is a function of time to maturity

	Maturity						
	3m	6m	9m	12m	15m	18m	21m
Forward-futures price differential							
mean	1.95	2.22	2.68	3.35	4.19	5.22	6.46
median	1.70	1.89	2.36	2.93	3.47	4.19	5.03
stdev	0.84	0.94	1.22	1.71	2.38	3.17	4.07
min	0.71	0.72	0.80	0.97	1.26	1.59	1.77
max	3.99	4.87	6.74	10.32	14.94	20.04	25.26
Settlement element							
mean	1.92	2.07	2.26	2.49	2.73	2.98	3.25
av. %*	98.70	93.92	86.79	78.76	70.83	63.48	56.83
median	1.68	1.78	1.98	2.35	2.59	2.76	2.96
stdev	0.83	0.84	0.88	0.95	1.05	1.17	1.31
min	0.71	0.70	0.74	0.83	0.97	1.16	1.36
max	3.93	4.62	4.92	5.51	6.59	7.86	8.94
Marking-to-market element							
mean	0.03	0.15	0.42	0.86	1.46	2.24	3.21
av. %*	1.30	6.08	13.21	21.25	29.17	36.52	43.17
median	0.01	0.08	0.23	0.47	0.86	1.39	2.20
stdev	0.04	0.20	0.49	0.93	1.49	2.15	2.89
min	0.00	0.01	0.02	0.05	0.10	0.17	0.27
max	0.30	1.30	2.75	5.14	8.68	12.85	17.31

(Table 3.3 continued)

Panel C: Exponentially dampened volatility

	Maturity						
	3m	6m	9m	12m	15m	18m	21m
Forward-futures price differential							
mean	1.96	2.22	2.61	3.16	3.90	4.90	6.25
median	1.72	1.89	2.27	2.79	3.31	3.98	4.99
stdev	0.84	0.93	1.17	1.57	2.14	2.88	3.85
min	0.72	0.72	0.79	0.96	1.24	1.51	1.72
max	4.16	4.86	6.57	9.26	12.93	17.72	23.84
Settlement element							
mean	1.93	2.06	2.21	2.39	2.60	2.85	3.18
av. %*	98.42	93.48	86.99	79.74	72.17	64.60	57.40
median	1.68	1.78	1.93	2.20	2.47	2.68	2.93
stdev	0.82	0.83	0.86	0.90	0.97	1.08	1.24
min	0.71	0.70	0.74	0.81	0.93	1.12	1.37
max	4.06	4.64	4.96	5.00	5.67	6.44	7.80
Marking-to-market element							
mean	0.03	0.16	0.40	0.77	1.30	2.05	3.07
av. %*	1.59	6.52	13.01	20.26	27.83	35.40	42.60
median	0.02	0.08	0.22	0.43	0.78	1.28	2.12
stdev	0.04	0.21	0.48	0.86	1.35	1.96	2.76
min	0.00	0.01	0.02	0.04	0.08	0.15	0.25
max	0.26	1.19	2.59	4.60	7.56	11.37	16.20

Panel D: Proportional exponentially dampened volatility ($\gamma = 1$)

	Maturity						
	3m	6m	9m	12m	15m	18m	21m
Forward-futures price differential							
mean	1.97	2.27	2.73	3.48	4.65	6.51	10.48
median	1.71	1.93	2.32	2.84	3.33	4.15	5.48
stdev	0.88	1.12	1.59	2.48	3.94	6.47	14.42
min	-0.16	-2.10	-0.58	-1.57	-0.28	0.18	-0.88
max	4.85	11.82	15.84	22.47	34.56	58.64	186.98
Settlement element							
mean	1.94	2.09	2.30	2.59	3.02	3.75	5.78
av. %*	99.45	95.69	91.61	84.01	76.31	69.89	62.83
median	1.69	1.79	2.02	2.35	2.65	2.90	3.34
stdev	0.83	0.84	0.93	1.15	1.69	2.89	9.00
min	0.71	0.70	0.73	0.81	0.94	1.20	1.37
max	4.11	4.67	5.84	9.98	16.55	33.14	159.09
Marking-to-market element							
mean	0.03	0.18	0.44	0.89	1.63	2.76	4.70
av. %*	0.55	4.31	8.39	15.99	23.69	30.11	37.17
median	0.01	0.05	0.16	0.37	0.76	1.33	2.25
stdev	0.21	0.56	0.95	1.57	2.44	3.80	6.23
min	-1.64	-3.99	-3.43	-3.89	-2.32	-2.07	-4.12
max	2.17	7.71	10.00	14.71	19.55	26.77	40.64

*av. % denotes an average share of the respective element in the forward-futures differential (percentage)

The difference in the respective average values of the marking-to-market component increases with maturity but never exceeds 0.4 basis points. It is the same for the average values of the forward-futures differential, except for the longest maturity in panel D where the value of the differential is above that in Table 3.2 by a full basis point. The variability of results in panel D of Table 3.3 is significant and on a number of occasions negative values of the forward-futures differential and its marking-to-market component are produced which is the result of the combination of negative rates and negative volatilities. On the opposite side, maximum values of the differential and its settlement components are well above 100 basis points, the result of rate explosiveness[35]. Overall, the results in Panel D are less reliable compared to those in other panels due to the aforementioned drawbacks of the proportional exponentially dampened volatility.

3.5.3 Results for a Weekly Step

The analysis goes further and investigates how the picture would look like if marking-to-market is conducted weekly. For that purpose weekly forward rates must be modeled[36] and thee-month forward and spot rates can be derived through the no-arbitrage condition similar to that in (3.47). Computational time rises significantly for the weekly step, hence, we limit the number of draws to 100. Along with the antithetic variate, the number of total paths for each observation is the sample totals to 200. Table 3.4 presents the results for the first 100 business day observations in the sample when simulations of weekly forward rates are conducted with 100 versus 1,000 random draws. It is evident that a sharp increase in the number of draws results in small insignificant improvements for three out of four volatility specifications. For the proportional exponentially dampened volatility the difference is significant and rises sharply with maturity which is the result of the explosive nature of forward rates and negative volatilities arising in this volatility specification. It lets one conclude that the problem of rate explosion under the proportional exponentially dampened volatility specification becomes more severe with a smaller size of the step. Also the sufficient evidence is obtained that conducting Monte Carlo simulations using a weekly step with only 100 draws along with the antithetic variate for the sake of avoiding the substantial computational burden, with the exception of the case of the proportional

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[35] See footnote 23 on page 60 that mentions a general version of the volatility specification prone to this problem. The proportional exponentially dampened volatility specification is just an another variation of it.

[36] Weekly forward rates are taken as weekly continuously compounded rates. Volatilities for the first two specifications are taken as the daily standard deviation of the respective weekly forward rate computed using data for 62 previous business days multiplied by the square root of 62/13. Parameters for exponentially and proportional exponentially dampened volatilities are found in the same manner as described above for the general case when the three-month step is used.

exponentially dampened volatility, does not result in sharply skewed results in the context of the paper's analysis. The resulting statistics for the forward-futures differential and its two components with a weekly step are presented in Table 3.5.

Table 3.4 Absolute differences in the forward-futures differential using Monte Carlo simulations with 100 draws versus 1,000 draws when the length of step is equal to one week, in basis points

Panel A: Volatility is a function of maturity time

	Maturity						
	3m	6m	9m	12m	15m	18m	21m
mean	0.01	0.01	0.01	0.02	0.04	0.07	0.11
median	0.00	0.01	0.01	0.02	0.03	0.04	0.09
stdev	0.01	0.01	0.01	0.02	0.04	0.06	0.09
min	0.00	0.00	0.00	0.00	0.00	0.00	0.00
max	0.03	0.04	0.07	0.12	0.23	0.32	0.44

Panel B: Volatility is a function of time to maturity

	Maturity						
	3m	6m	9m	12m	15m	18m	21m
mean	0.01	0.01	0.02	0.02	0.03	0.04	0.05
median	0.00	0.01	0.02	0.02	0.02	0.03	0.05
stdev	0.01	0.01	0.02	0.02	0.03	0.03	0.04
min	0.00	0.00	0.00	0.00	0.00	0.00	0.00
max	0.05	0.07	0.12	0.11	0.13	0.18	0.26

Panel C: Exponentially dampened volatility

	Maturity						
	3m	6m	9m	12m	15m	18m	21m
mean	0.01	0.01	0.02	0.03	0.03	0.04	0.06
median	0.00	0.01	0.01	0.02	0.02	0.03	0.04
stdev	0.00	0.01	0.02	0.02	0.03	0.04	0.05
min	0.00	0.00	0.00	0.00	0.00	0.00	0.00
max	0.02	0.04	0.07	0.10	0.13	0.15	0.20

Panel D: Proportional exponentially dampened volatility ($\gamma = 1$)

	Maturity						
	3m	6m	9m	12m	15m	18m	21m
mean	0.33	0.64	1.11	1.44	1.90	2.59	3.22
median	0.28	0.48	0.86	1.12	1.52	2.19	2.54
stdev	0.31	0.55	0.98	1.32	1.71	2.16	2.73
min	0.00	0.02	0.01	0.03	0.01	0.00	0.04
max	1.68	2.46	5.36	6.67	7.25	11.66	16.62

By comparing results for the volatility specification when it is a function of maturity with those in Table 3.3, no major changes are noticed. The average values of the forward-futures differential are higher than their respective values when a monthly step is employed but the maximum difference is 0.17 basis points for the longest maturity in the sample. This miniscule

Table 3.5 Forward-futures differential and its two components when the length of step is equal to one week, in basis points

Panel A: Volatility is a function of maturity time

	Maturity						
	3m	6m	9m	12m	15m	18m	21m
Forward-futures price differential							
mean	1.98	2.34	2.99	3.85	4.89	6.13	7.82
median	1.72	1.96	2.63	3.37	4.07	5.15	6.37
stdev	0.86	1.01	1.43	2.06	2.77	3.58	4.81
min	0.72	0.74	0.86	1.16	1.56	1.78	1.87
max	4.26	5.52	8.84	13.80	18.99	22.69	27.10
Settlement element							
mean	1.93	2.13	2.40	2.68	2.94	3.22	3.62
av. %*	97.81	91.88	83.25	74.12	65.69	58.26	51.74
median	1.67	1.81	2.16	2.50	2.70	2.96	3.19
stdev	0.83	0.86	0.95	1.04	1.12	1.28	1.68
min	0.71	0.71	0.77	0.91	1.14	1.36	1.41
max	4.02	4.87	6.05	7.40	8.81	8.90	12.09
Marking-to-market element							
mean	0.05	0.22	0.59	1.18	1.95	2.91	4.20
av. %*	2.19	8.12	16.75	25.88	34.31	41.74	48.26
median	0.02	0.12	0.32	0.65	1.22	2.04	3.11
stdev	0.07	0.26	0.64	1.20	1.83	2.51	3.36
min	0.00	0.01	0.03	0.08	0.17	0.26	0.34
max	0.46	1.37	3.67	7.19	11.24	15.20	17.43

Panel B: Volatility is a function of time to maturity – full sample

	Maturity						
	3m	6m	9m	12m	15m	18m	21m
Forward-futures price differential							
mean	2.78	3.21	3.86	4.73	5.77	6.98	8.42
median	1.71	1.89	2.35	2.91	3.54	4.30	5.21
stdev	4.36	4.85	5.49	6.37	7.24	8.02	9.01
min	0.71	0.72	0.80	0.97	1.25	1.59	1.76
max	34.08	36.38	38.70	42.36	50.13	59.06	66.21
Settlement element							
mean	2.55	2.71	2.89	3.14	3.39	3.63	3.91
av. %*	96.74	91.24	83.82	75.91	68.16	61.02	54.68
median	1.68	1.78	1.97	2.32	2.57	2.78	3.01
stdev	3.66	3.70	3.59	3.69	3.74	3.64	3.66
min	0.71	0.70	0.73	0.82	0.94	1.12	1.32
max	33.74	32.90	33.08	32.17	32.93	31.48	30.58
Marking-to-market element							
mean	0.23	0.50	0.97	1.59	2.38	3.35	4.52
av. %*	3.26	8.76	16.18	24.09	31.84	38.98	45.32
median	0.02	0.10	0.26	0.53	0.94	1.50	2.26
stdev	0.80	1.33	2.19	3.04	3.90	4.81	5.77
min	0.00	0.00	0.02	0.05	0.09	0.16	0.26
max	8.06	9.37	17.57	26.64	34.87	40.40	47.70

(Table 3.5 continued)

Panel C: Volatility is a function of time to maturity – subsample excluding 10 percent of data points with the highest volatility of the forward rate with maturity of one week

	Maturity						
	3m	6m	9m	12m	15m	18m	21m
Forward-futures price differential							
mean	1.81	2.07	2.50	3.14	3.96	4.97	6.18
median	1.61	1.82	2.21	2.70	3.32	3.97	4.83
stdev	0.76	0.85	1.10	1.58	2.24	3.05	3.95
min	0.71	0.72	0.80	0.97	1.25	1.59	1.76
max	3.99	4.98	6.69	10.40	14.91	20.29	25.72
Settlement element							
mean	1.78	1.92	2.11	2.33	2.58	2.84	3.11
av. %*	98.41	93.63	86.60	78.61	70.66	63.30	56.70
median	1.57	1.71	1.90	2.12	2.39	2.62	2.84
stdev	0.74	0.75	0.79	0.86	0.98	1.12	1.29
min	0.71	0.70	0.73	0.82	0.94	1.12	1.32
max	3.92	4.62	4.99	6.00	7.13	8.45	9.53
Marking-to-market element							
mean	0.03	0.15	0.39	0.80	1.38	2.14	3.07
av. %*	1.59	6.37	13.40	21.39	29.34	36.70	43.30
median	0.01	0.08	0.22	0.44	0.78	1.24	2.01
stdev	0.04	0.19	0.46	0.88	1.43	2.08	2.81
min	0.00	0.00	0.02	0.05	0.09	0.16	0.26
max	0.45	1.55	2.92	5.12	8.59	12.91	17.41

Panel D: Exponentially dampened volatility

	Maturity						
	3m	6m	9m	12m	15m	18m	21m
Forward-futures price differential							
mean	1.97	2.25	2.66	3.22	3.97	4.96	6.29
median	1.73	1.89	2.27	2.78	3.35	3.92	4.88
stdev	0.86	0.99	1.26	1.68	2.24	2.96	3.90
min	0.72	0.73	0.79	0.94	1.25	1.50	1.72
max	4.47	5.41	6.96	8.97	12.40	17.30	23.64
Settlement element							
mean	1.93	2.06	2.21	2.39	2.60	2.85	3.16
av. %*	97.86	92.69	86.14	78.92	71.44	64.01	56.94
median	1.68	1.79	1.92	2.16	2.44	2.66	2.89
stdev	0.84	0.86	0.89	0.94	1.01	1.11	1.25
min	0.71	0.71	0.73	0.79	0.90	1.10	1.37
max	4.34	4.67	5.19	5.67	6.23	7.22	8.98
Marking-to-market element							
mean	0.04	0.19	0.44	0.83	1.37	2.11	3.13
av. %*	2.14	7.31	13.87	21.08	28.56	35.99	43.06
median	0.02	0.09	0.23	0.45	0.78	1.27	2.08
stdev	0.06	0.24	0.53	0.91	1.40	2.02	2.80
min	0.00	0.01	0.02	0.04	0.07	0.13	0.22
max	0.43	1.53	3.06	4.89	7.15	10.68	15.68

*av. % denotes an average share of the respective element in the forward-futures differential (percentage)

increase is completely due to the rise in the values of the market-to-market component as the mean and median values of the settlement component stay virtually unchanged. As a result, the share of the marking-to-market component increases across all maturities, however, it remains below 50 percent for the sample's longest maturity. By comparing results of panel A with those from Table 3.2, the evidence is unchanged: reduction of the step size from three months to one week improves average values of the marking-to-market component marginally but does not affect those of the settlement component. This, nevertheless, results in a higher share of the latter component across all maturities in the sample. However, the settlement component remains the dominant one, while for the longest maturity in the sample (21 months), the two components are almost equally important.

Moving on to panel B of Table 3.5 that lists statistics for volatility defined as a function of time to maturity, and comparing the results with those in Table 3.3, quite a few discrepancies can be noticed. Average values of the differential increase by more than a basis point for six out of eight maturities, standard deviations of the differential and its settlement component are visibly higher, especially for shorter maturities. Finally, maximum values are much higher in general while in particular, striking maximum values are displayed in the section about the settlement component, being in excess of 30 basis points for all eight maturities. These results are model driven, however, and they arise due to a combination of two methods used: volatility specification and interpolation technique. Recall that LIBOR rates with maturities from one to twelve months as well as swap rates for maturities of 18 and 24 months were used to build the yield curve. With a weekly step, however, the input set of the discrete HJM model includes spot rates for maturities less than a month. Since such rates lie beyond the interval of maturities used for interpolation[37], they are less reliable and prone to considerable variation. The magnitudes of those rates are not of concern, however, since their evolution does not affect the computation of the differential with a maturity of three months or beyond, but their volatilities do enter the model and are used at every step according to the volatility specification employed. This is not an issue when the volatility is a function of *maturity time* but it is a problem when the volatility is a function of *time to maturity*. Abnormally high volatility values may enter the computations

[37] This situation could be avoided if the one-week LIBOR was included in the sample of data used for interpolation. The one-week GBP LIBOR quote, however, is available only for the period starting 12/1/1997. It is unlikely to expect that changing the interpolation interval for the subsample of data would affect the paper's findings and conclusions. Meanwhile, careful illustration of the problem deserves some attention as it represents a warning aiming to help to avoid pitfalls when modeling the term structure of interest rates.

of the futures rates with all maturities in the sample resulting in abnormally high future spot rates leading to extreme values of the differential and its two components. The good news is that just about 10-15 percent of the sample experience abnormally high volatilities for the weekly forward rates with maturities of less than a month. Figure 3.1 illustrates a scatterplot of weekly rates for the sample of shortest maturities and Table 3.6 shows values of standard deviations for ten quintiles for those weekly rates. As can be seen from the scatterplot, weekly rates for out-of-sample maturities (maturities of up to four weeks) are subject to the presence of outliers. The presence of outliers results in abnormally high volatilities for a subsample of rates. Table 3.6 presents evidence of the latter: the ninth quintiles of standard deviations of weekly rates with maturity of four weeks or below as well as their respective maximum values are substantially higher than those for rates with the nearest maturities above four weeks.

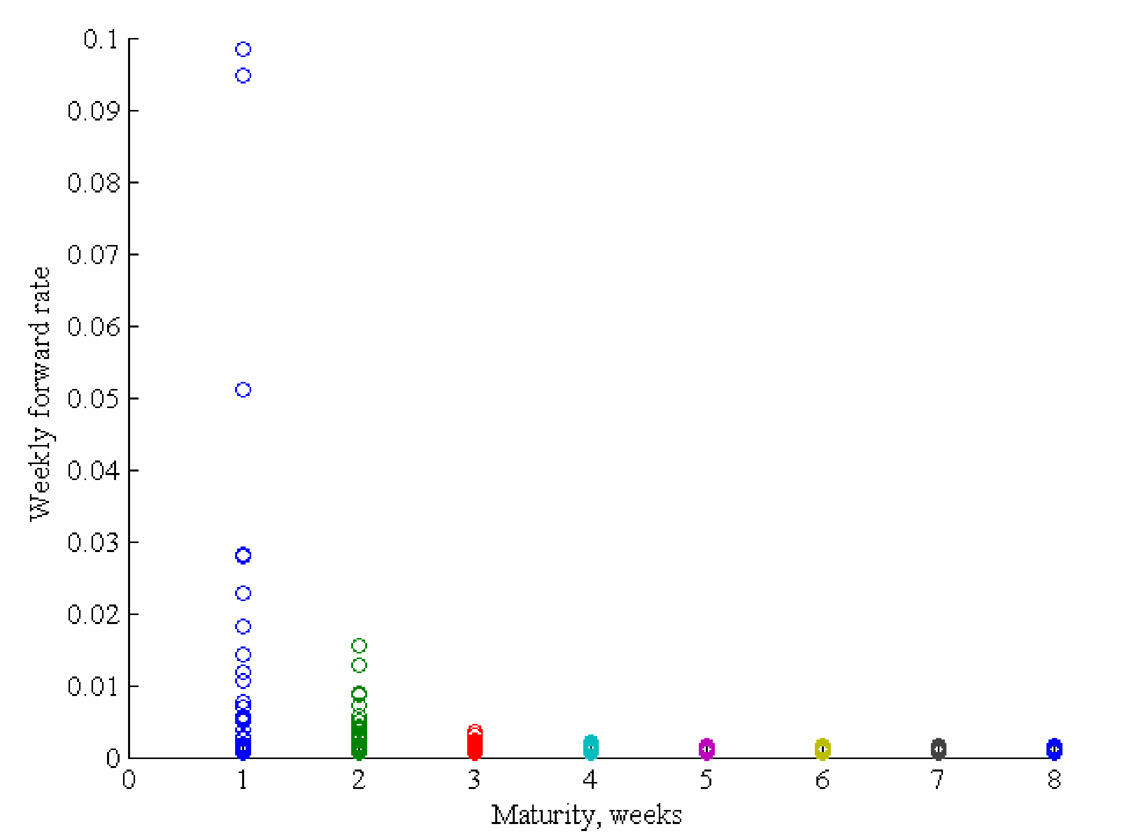

Figure 3.1 Scatterplot of weekly forward rates in the sample for maturities ranging from one week to eight weeks

The results would be more reliable if the percentage of data points with highest volatilities for rates with shortest maturities is excluded from the original sample. For that purpose all of 2,477 day-observations are sorted by the volatility of the weekly forward rate with the shortest available maturity (one week, if the spot rate is ignored) and the tenth highest quintile of the sorted data is subsequently cut out. The statistics for the resulting subsample are presented in panel C of Table 3.5. By comparing them with those in panel B of Table 3.3, a small change can be identified. The mean and the median values of the differential and its two components have become slightly lower (the artifact of excluding data points with highest volatilities) but the differences across maturities do not exceed 0.30 basis points. The shares of the two components remain virtually unchanged and they stay higher than the respective values in panel A of Table 3.5 when volatility is specified as a function of maturity only. The differences of reported statistics from those in panel B of Table 3.1 when a three-month step is used are barely noticeable either.

Table 3.6 Quintiles of standard deviation values of the forward rates with eight shortest maturities (excluding the spot rate) in the sample when the HJM is constructed with a weekly size of the step

Maturity, weeks	Quintile									
	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	max
One	$10^5 \times$ 0.96	1.70	2.67	3.46	4.43	5.10	6.95	9.70	15.98	3990
Two	$10^5 \times$ 0.85	1.49	2.32	3.31	4.37	5.13	6.64	9.62	15.79	563.5
Three	$10^5 \times$ 0.99	1.45	2.23	3.27	4.10	4.67	5.98	8.42	14.93	84.82
Four	$10^5 \times$ 0.96	1.45	2.01	2.89	3.74	4.40	5.57	7.07	12.13	32.53
Five	$10^5 \times$ 0.91	1.39	1.91	2.56	3.38	4.15	4.87	6.36	9.79	21.19
Six	$10^5 \times$ 0.90	1.39	2.01	2.57	3.53	4.22	4.95	6.39	9.55	21.34
Seven	$10^5 \times$ 0.91	1.43	2.09	2.63	3.51	4.39	5.28	6.55	9.61	21.34
Eight	$10^5 \times$ 0.99	1.47	2.10	2.67	3.45	4.59	5.57	6.79	9.67	21.21

Panel D of Table 3.5 reports statistics when volatility is defined as the exponentially dampened one. The reported results do not differ significantly from those in Table 3.3 and the pattern is similar to that observed for volatility as a function of maturity time, i. e. the average values of the marking-to-market component increase slightly (by 0.01-0.07 basis points) while the average values of the settlement component stay unchanged. These results do not differ significantly from those reported in Table 3.1 where the three-month step is used either. Since the exponentially dampened volatility specification is also defined as a function of time to maturity, the question of why it manages to avoid the drawback of using high volatilities for rates with short maturities observed in panel B of Table 3.5 arises. The answer is that the resulting shape of the volatility structure as a function of time to maturity for the regarded subsample looks more like a smile which results in a poor goodness-of-fit of the linear regression employed to identify the parameters σ and λ. This alleviates the problem of the presence of abnormally high volatilities of the shortest forward rates since the abnormal parts are consumed by the regression error terms.

Finally, note that results for the proportional exponentially dampened volatility are not reported due to the fact that the presence of negative volatilities and explosively high rates under this volatility specification increases sharply when a weekly step is used. For comparison purposes, average values of the settlement component and the marking-to-market element for the three volatility specifications excluding the proportional exponentially dampened one and for the three step sizes used are compiled in Table 3.7. The results say that changing the step size of the model has little effect on the values of the differential's components and, hence, the size of the forward-futures differential itself. The reduction of the step size 13-fold (from three months to one week) does not lead to significant changes in the marking-to-market component values. The largest increase in the latter due to such step size reduction is 0.55 basis points for the longest maturity in the sample when volatility is specified as a function of maturity time only. It will not be unreasonable to suggest that the further 5-fold reduction of the step size from one week to one business day would result in even smaller changes. It is fair to conclude that the approximation error of the tree building procedure stemming from the choice of the step size is not as large as the phenomenon studied in the paper. Moreover, the results of this section may serve as a justification for implementing simulation models employed for interest rate derivative pricing when the step size chosen is larger than one business day.

3.5.4 Extrapolation of Results for Longer Maturities

The original sample is limited to maturities of up to 21 months since the yield curve has been built using spot rates for maturities up to two years. The obtained results suggest that the marking-to-market components would have the dominant share of the forward-futures differential for maturities beyond 24 months since its in-sample rate of growth exceeds that of the settlement component when longer maturities are considered. The separate regressions of the logarithm of the average values of each component on the maturity result in remarkably high

Table 3.7 Comparison of average values of the two components of the forward-futures differential when different step sizes are used, in basis points

	Maturity						
	3m	6m	9m	12m	15m	18m	21m
Volatility is a function of maturity time							
Settlement Component							
3-month	1.93	2.12	2.39	2.67	2.92	3.20	3.59
1-month	1.93	2.13	2.40	2.67	2.93	3.21	3.61
1-week	1.93	2.13	2.40	2.68	2.94	3.22	3.62
Marking-to-market Component							
3-month	0.00	0.11	0.40	0.91	1.60	2.48	3.65
1-month	0.03	0.18	0.53	1.09	1.84	2.78	4.04
1-week	0.05	0.22	0.59	1.18	1.95	2.91	4.20
Volatility is a function of time to maturity*							
Settlement Component							
3-month	1.79	1.95	2.15	2.38	2.63	2.89	3.16
1-month	1.79	1.93	2.12	2.35	2.59	2.85	3.12
1-week	1.78	1.92	2.11	2.33	2.58	2.84	3.11
Marking-to-market Component							
3-month	0.00	0.10	0.34	0.75	1.32	2.08	3.01
1-month	0.02	0.13	0.38	0.79	1.37	2.12	3.05
1-week	0.03	0.15	0.39	0.80	1.38	2.14	3.07
Exponentially dampened volatility							
Settlement Component							
3-month	1.93	2.08	2.28	2.50	2.75	3.01	3.28
1-month	1.93	2.06	2.21	2.39	2.60	2.85	3.18
1-week	1.93	2.06	2.21	2.39	2.60	2.85	3.16
Marking-to-market Component							
3-month	0.00	0.11	0.36	0.78	1.38	2.16	3.13
1-month	0.03	0.16	0.40	0.77	1.30	2.05	3.07
1-week	0.04	0.19	0.44	0.83	1.37	2.11	3.13

* For this volatility specification results are shown for the subsample of data that excludes observations with abnormally high volatilities of weekly rates. The subsample selection is performed in the following way. All of 2,477 day-observations are sorted by the volatility of the weekly forward rate with the shortest available maturity (one week, if the spot rate is ignored) and the tenth highest quintile of the sorted data is subsequently cut out. This makes results for different step sizes comparable to each other.

values of *R*-squared. The obtained intercept and the slope coefficients can be used further to extrapolate average values of the two components for maturities beyond 21 months. The results for all four volatility specifications when the average values from Table 3.3 are used to obtain regression coefficients are presented in Table 3.8.

Table 3.8 Extrapolation of average values of the two components of the forward-futures differential for maturities of 2 to 10 years, in basis points

	Maturity, years								
Component	2	3	4	5	6	7	8	9	10
Volatility is a function of maturity time									
Settlement component	4.00	6.06	9.17	13.89	21.04	31.87	48.27	73.12	110.74
Marking-to-market component	6.07	17.20	36.01	63.85	101.97	151.47	213.41	288.76	378.45
Volatility is a function of time to maturity									
Settlement component	3.55	5.07	7.24	10.33	14.75	21.06	30.07	42.93	61.28
Marking-to-market component	4.62	12.57	25.58	44.38	69.62	101.87	141.66	189.49	245.79
Exponentially dampened volatility									
Settlement component	3.38	4.70	6.55	9.12	12.69	17.67	24.60	34.25	47.69
Marking-to-market component	4.01	10.34	20.26	34.11	52.23	74.86	102.27	134.66	172.24
Proportional exponentially dampened volatility ($\gamma = 1$)									
Settlement component	5.63	11.07	21.75	42.77	84.08	165.30	324.97	638.88	1256.00
Marking-to-market component	5.56	15.24	31.17	54.29	85.44	125.35	174.73	234.19	304.35

Two different regression specifications were considered for each component: *log (component)* $= \alpha_1 + \beta_1 \times$ *maturity* + $error_1$ and *log (component)* $= \alpha_2 + \beta_2 \times$ *log (maturity)* + $error_2$. Across all four volatility specification considered in the paper, the first regression specification resulted in higher *R*-squared for the settlement component, while the second regression specification yielded higher *R*-squared for the marking-to-market component. Results presented in Table 3.8 are based on the usage of regression specifications where the logarithm of the settlement component is regressed on the maturity, whereas the logarithm of the marking-to-market component is regressed on the logarithm of maturity. The resulting *R*-squared values exceed 0.993 in seven cases out of eight[38].

For three out of four volatility specifications with the exception of the proportional exponentially dampened volatility, extrapolated average values of the marking-to-market component exceed those of the settlement component and this difference reaches fourfold for maturities of four years and beyond suggesting that the marking-to-market component should become an increasingly dominant part of the forward-futures differential when longer maturities are considered. The proportional exponentially dampened volatility yields opposite results which is the product of the rate explosiveness problem and, hence, such results lack reliance. Average values of the settlement component for maturities beyond four years may even seem overestimated since they imply large expected values of the future three-month LIBOR rates. If that is the case, the marking-to-marking component would become an even more powerful element of the differential for maturities beyond 24 months. Some confirmation of this assertion can be found in Table 3.9. This table presents results for the average, minimum and maximum values of the two components of the differential for maturities ranging from two to ten years for the case of a flat forward rate term structure combined with constant volatility when a one-month binomial tree step is used. The constant continuously compounded monthly forward rate was chosen in the interval from 0.4 percent to 0.5 percent with an increment of 0.01 percent which

Table 3.9 Selected statistics for the two components of the forward-futures differential for maturities of 2 to 10 years when HJM is run for a range of flat term structures of forward rates and constant volatilities, in basis points

	Maturity								
	2	3	4	5	6	7	8	9	10
Settlement Component									
mean	2.74	3.27	3.84	4.48	5.19	5.97	6.84	7.82	8.92
min	1.82	2.04	2.27	2.53	2.80	3.10	3.42	3.76	4.14
max	3.82	4.74	5.76	6.91	8.19	9.65	11.31	13.20	15.36
Marking-to-market Component									
mean	3.07	7.01	12.54	19.68	28.40	38.71	50.60	64.06	79.09
min	1.31	2.98	5.33	8.37	12.08	16.47	21.54	27.28	33.70
max	5.23	11.94	21.38	33.52	48.37	65.91	86.13	109.01	134.53

[38] *R*-squared for the settlement component regression for the proportional exponentially dampened volatility is 0.907.

roughly corresponds to the observed average values of the forward rates in the data sample used. The chosen values of the constant volatility of the monthly forward rate range from 0.04 to 0.08 percent with an increment of 0.004 percent, also in line with the data sample statistics. All possible combinations of the forward rate and the volatility are considered (121 in total) and the evolution of the forward rate is run using the HJM setup via the construction of the binomial tree. The obtained results for the average values of the two components are lower than those in Table 3.8, especially for the settlement component, resulting in a much higher share of the marking-to-market component in the forward-futures differential, reaching 90 percent for the 10-year maturity. Overall, the results of this subsection provide evidence that for the out-of-sample range of maturities the marking-to-market component would have a significant presence in the forward-futures differential and it must be taken into account while pricing respective interest rate derivatives.

3.6 Hedging Implications

The hedging of the marking-to-market component has been addressed in the literature and is known as the tailing of the hedge (see Figlewski et al. [1991] and Kawaller [1997] among other references). In order to fully hedge the existing or future exposure to interest rate risk with Eurocurrency futures, hedging of the settlement component must also be addressed. Chance (2006a) demonstrates that since the futures price does not converge to the value of the underlying time deposit, the standard cash and carry arbitrage is not risk free and the interest rate hedges constructed using the Eurocurrency futures contracts are subject to errors. These errors arise from the way the futures are designed, namely, the existence of the settlement component in the convexity adjustment. It should be noted that there are other factors related to the basis risk that may cause hedge discrepancies. For instance, when the forward contract and the respective futures contract do not expire on the same day or when their underlying rates differ. Such factors are not considered in the analysis and are not addressed in the proposed solution.

Chance (2006a) shows the perfect hedge is not possible to obtain when the borrower wants to get a fixed amount of the loan. This section argues that there is a way to hedge the settlement component of the convexity adjustment but this hedge comes at a cost. The hedge is perfect in a binomial or a trinomial tree model. Since any such model is an approximation, in reality, the proposed hedge, albeit approximate, is highly efficient and substantially reduces the hedging error.

According to (3.28), the settlement component of the convexity adjustment at expiration is equal to

$$\frac{\alpha^2 r^2}{1+\alpha r}, \tag{3.48}$$

where $\alpha = t_{i+1} - t_i$ is the tenor, i. e. length of the time interval that the underlying rate is for, and r is the underlying rate at the expiration. Suppose, at the future date when the forward and the respective futures expire, the underlying spot rate can take a value from the following set of rates: $r_1 < r_2 < r_3 < \ldots < r_N$. The caplet is defined as a call option on the interest rate whose payoff at expiration is

$$\alpha \max[r - K, 0], \tag{3.49}$$

where K is the predetermined strike rate. The caplet's value is determined at time t_i, when the rate is revealed, and if the value is positive it is paid at time t_{i+1}, when the underlying time deposit will mature. Although the caplet's payoff takes place at t_{i+1}, it can be sold or settled at time t_i for its present value which is equal to

$$\frac{\alpha \max[r - K, 0]}{1+\alpha r}. \tag{3.50}$$

Without the loss of generality, for the rest of the section it can be assumed that $\alpha = 1$. Suppose that the realized underlying rate at the contracts' expiration is the lowest possible one in the set, i. e. r_1. In order to have the settlement component hedged given the realization of this state, one would have to have an amount of r_1 caplets with the strike of zero. The value of such hedge in state 1 ($r = r_1$) at time t_i will be equal to

$$\frac{r_1^2}{1+r_1}, \tag{3.51}$$

which is equivalent to the value of the settlement component of the convexity adjustment in state 1.

Suppose now that the realized state at time t_i is 2, i. e. $r = r_2$. In order to have the settlement component fully hedged, the party must hold another caplet, the one with the strike rate equal to r_1. The number of such caplets is found recursively given the holdings of the caplet with the strike rate of zero that has been determined above.

Let the holding of the caplet with strike rate r_1 be equal to x_1. Then in order to have the settlement component fully hedged in *both* state 1 and state 2, it must be that

$$\frac{x_1(r_2-r_1)}{1+r_2}+\frac{r_1r_2}{1+r_2}=\frac{r_2^2}{1+r_2}. \tag{3.52}$$

By solving the above equation for x_1, one obtains that it must be equal to r_2. Hence, in order to have the settlement components fully hedged given either state 1, or state 2, the investor must hold r_2 caplets with strike rate of r_1 and r_1 caplets with strike rate of zero.

Moving on to state 3 when the realized rate is equal to r_3, in order to have the full hedge, the party must also possess a certain amount of caplets with the strike rate of r_2. Ignoring the common denominator, the number of such caplets denoted by x_2 is determined by the following equation:

$$x_2(r_3-r_2)+r_2(r_3-r_1)+r_1r_3=r_3^2. \tag{3.53}$$

The answer for x_2 is $r_3 - r_1$. Hence, the basket of caplets must include $r_3 - r_1$ caplets with the strike rate of r_2.

Continuing in the same manner, the complete basket of caplets necessary to fully hedge the settlement component of the convexity adjustment can be determined. Given the fact that

$$\left[\sum_{i=1}^{N-2}(r_{N+1-i}-r_{N-1-i})(r_N-r_{N-i})\right]+r_2(r_N-r_1)+r_1r_N=r_N^2, \tag{3.54}$$

the solution that allows to fully hedge the settlement component of the convexity adjustment consists of the following: given the set of the possible realized rates at the expiration of the futures contract $r_1 < r_2 < r_3 < \ldots < r_N$ and denoting $r_0 = r_{-1} = 0$, the investor must hold the basket of caplets on the underlying rate with the set of strike rates $r_0, r_1, r_2, \ldots, r_{N-1}$, where the number of caplets with the strike r_i is equal to $r_{i+1} - r_{i-1}$.

Such basket of caplets shall produce a perfect hedge of the settlement component of the convexity adjustment if the realized underlying rate is one of those in the original set. If the realized rate does not belong to that set, the hedge will result in a minor error. The proposed hedge comes at a cost since the basket of caplets must be paid for at the time of the purchase.

3.7 Conclusions

This paper contributes to the existing derivatives literature in several ways. It derives the convexity adjustment for Eurocurrency interest rate futures and shows that the latter will differ from the forward-futures price differential of Cox, Ingersoll and Ross (1981). The paper shows that interest rate forward-futures price differential at any point of time until maturity can be expressed as a sum of two components: the settlement component and the marking-to-market component. The settlement component arises from the unconventional way interest rate futures are priced and settled, notably, the fact that they are settled to yield opposite to that of price as some textbooks may suggest. A numerical example that demonstrates how the two components are computed is presented. The discrete time version of the one-factor HJM model is further utilized and the values of the differential and its two components for the sample of British pound interest rate forwards and futures for the 1997-2007 period are estimated by using four most popular volatility specifications. The settlement component dominates its marking-to-market counterpart in terms of the share of the forward-futures differential for all seven maturities in the sample but its share declines with maturity. If the results are extrapolated for longer maturities (> 21 months), the size of the forward-futures differential would grow exponentially, mainly due to the sharply increasing magnitude of its marking-to-market component. It is also shown that the three-month step in the discrete HJM model is quite appropriate since the results with monthly and weekly steps do not lead to significant improvements. Of particular note, the small values of the marking-to-market component for maturities in the sample (3-21 months) cannot be attributed to the chosen length of the step in the term structure model. Finally, hedging implications of the settlement component of the interest rate convexity adjustment are considered and it is demonstrated that the perfect hedge is feasible via the use of the predetermined basket of caplets. Such hedge comes at a cost however since the caplets must be paid for at the time of the purchase.

It is imperative to note that the results presented in this paper are clear of any market frictions like tax considerations, transaction cost differences or the possibility of default by the counter-party. The latter would be of special interest since forward contracts, unlike their futures counterparts, are subject to default risk. The presence of the default premium in forward prices can affect the size of the forward-futures differential. A possible line of future work would be to investigate how large the default premium can be. The existing literature defines the presence of the default premium in interest rate forward prices by regressing them or the empirical differential values on the default proxy variables and judges upon the statistical significance of the regression coefficients whether the premium is robust. Such an approach provides *qualitative* evidence of the existence or lack of thereof of the default premium, but the *quantitative* estimation of the size of the premium has not been attempted. The first logical step in that direction would be to compare the theoretical findings of the forward-futures differential, similar to those shown in this paper, with empirical results and see how much of the difference is there and what factors, if any, could explain it. This objective is left for future research.

Another line of related work may be built around the implementation of the multi-factor term structure model in order to investigate what factors contribute the most to the value of the forward-futures differential. Principal component analysis may be used to define the level, the slope, and the curvature factors of the term structure. The computational burden of using multi-factor term structure models, however, may be substantial and the introduction of appropriate simplifications and/or model adjustments may be necessary.

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Chapter 4 How Much Premium Is There for Interest Rate Futures?

4.1 Introduction

The difference in cash flows of the forward contract and the futures contract was initially explained by Black (1976). His discussion, however, is based on the assumption of a constant interest rate. Margrabe (1976) was the first to show that in the stochastic interest rate environment equality of forward and futures prices would result in arbitrage opportunities. Jarrow and Oldfield (1981) point out that even though forward contracts and futures contracts are very similar in nature, since the two contracts' cash flows differ, it is not generally true that their prices must be the same if default-free rates are stochastic. Cox, Ingersoll and Ross (1981) consolidate results of the early works on the relation between forward prices and futures prices and develop a number of propositions characterizing this relationship. One of their findings has received a broad application in subsequent empirical and theoretical literature and it is referred as the convexity adjustment. The convexity adjustment is the difference between the forward price and the futures price on the same underlying that is attributed to the daily settlement feature of the futures market also known as marking-to-market. In a stochastic interest rate environment, if futures prices are positively correlated with interest rates, then the futures prices will exceed the respective forward prices.

Until recently, the literature on the convexity adjustment in interest rate futures concentrated almost exclusively on the Eurodollar contracts (Sundaresan [1991], Meulbroek [1992], Grinblatt and Jegadeesh [1996]). Gupta and Subrahmanyam (2000) examine whether interest rate *swaps* are priced off the Eurocurrency futures curve. For that purpose they calculate the observed swap-futures rate differential for four major currencies – US dollar, Japanese yen, Deutsche Mark (relegated) and British pound. The swap-futures rate differential is defined in their paper as the difference between the observed swap rate and a swap rate implied by zero-coupon rates calculated from the observed Eurocurrency futures quotations. This is an interesting innovation but obviously a deviation from the classic convexity adjustment defined by Cox, Ingersoll and Ross. The findings of the early empirical works are that the futures were mispriced relative to swaps/forwards suggesting the presence of a futures discount in 1982-1987 (Grinblatt and Jegadeesh [1996]) and a futures premium in 1987-1990 (Gupta and Subrahmanyam [2000]).

While having come up with the empirical estimates of the convexity adjustment, the previous literature has not been able to say what the theoretically predicted value of the convexity adjustment should be on a particular day for a particular pair of a futures contract and a respective forward and, hence, there has not been a clear answer available on whether the futures is deemed to be underpriced today unless the sign of the observed futures-forward differential is opposite to the expected. This paper utilizes a straightforward technique that allows one to identify with a certain level of confidence not only whether the futures is underpriced but whether it happens to be traded at a premium as well. The paper relies on an approach that constructs matched pairs of an observed and a predicted convexity adjustment. This method creates the theoretically predicted estimate of the convexity adjustment for each observation in the sample where estimates are obtained for a futures and a forward contract with identical characteristics, such as time to maturity and the implied maturity of the contract's underlying. This technique helps to identify whether the observed futures prices tend to be overestimated which would suggest the presence of a robust futures premium. The futures premium is defined in this paper as the difference between the observed convexity adjustment in the forward and futures prices and its theoretically predicted estimate. The analysis is performed for the three most traded interest rate futures contracts in Europe: EURIBOR futures, short sterling futures and Euroswiss futures.

The early empirical literature on the convexity adjustment in Eurodollar interest rate futures relies on the implied forward rates derived from the interpolated term structure of spot rates and compared those to the observed futures rates. The problem with this approach is that such forward rates are not the rates on actual traded forward contracts. Therefore, the mispricing of the futures, if there is any, does not necessarily imply the existence of profitable opportunities in which futures are traded against forwards. The statement about mispricing per se rather becomes the acknowledgement that the futures are not priced off the implied forward curve. Recently, the search for the convexity adjustment in the interest rate futures market has been extended by replacing the synthetic forward by a forward rate agreement (FRA), which is an over-the-counter tradable financial instrument that has all features of a forward contract. By comparing two actually traded contracts, the interest rate futures and the FRA, one can obtain a better test for the presence of a convexity adjustment or the existence of mispricing that would allow an arbitrageur to exploit profitable opportunities.

FRA contracts are typically traded with standardized times to maturity whereas futures contracts usually have standardized maturity dates. In such circumstances corresponding forward and futures contracts exist simultaneously only when the standardized time to maturity in the FRA market implies the maturity date that coincides with that in the futures markets. Such occasions are rare and provide only a small and insufficient sample for an extensive study. Poskitt (2008c) collects data on futures and FRA quotes over a stretch of one year on days when the time to expiration of a futures contract matches that of a respective FRA contract. There are only four days during a one-year period that satisfy such condition creating a substantial limitation for the sample size. His study examines the pricing of the Eurodollar futures and the US dollar FRA contracts using a high frequency data set and finds that the average futures/FRA rate differential for contracts with maturity of less than one year is negligible. Such evidence, Poskitt claims, should imply the availability of arbitrage opportunities since the convexity adjustment is not priced in. Poskitt (2008b) is a similar work as that of Poskitt (2008c) with an identical approach but based on data for sterling futures and FRA markets that produces similar results and conclusions.

It is imperative to note, however, that the arbitrage is *not* available per se since an open futures position (whether long or short) requires a trader to post a margin. Therefore, the classic definition of arbitrage cannot be applied in this case. Given that, a more appropriate argument could be posed. As long as the convexity adjustment is not priced into the futures, the latter is overpriced[39]. Hence, a trader must short the futures. In order to hedge the interest rate exposure, a short position in the respective FRA is also required[40]. The new argument would be that the position consisting of short futures and short respective FRA will consistently yield an excess positive return. This alternative argument, however, is also false since the historical backtesting of the profitability of the position consisting of short futures and short FRA assuming the futures rate has been equal to the rate on the respective FRA (meaning that the convexity adjustment is not present, i. e. it is not priced into the quotes) shows that on many occasions when the interest rates have consistently gone down this position would have produced a loss since the short futures alone would have resulted in a string of losses which must have been financed over the

[39] Interest rate futures price is a negative function of the implied futures rate. See Table 4.1 for details.

[40] Opposite to the interest rate futures, a short position in FRA is a bet on the rise in the underlying interest rate. This is a somewhat unique case that stems from the design of the interest rate futures when one has to take two long or two short positions in order to hedge the underlying risk exposure.

period of time left till maturity and such cumulative loss would have exceeded the one-time gain realized by shorting the respective FRA. Therefore, if the convexity adjustment is not priced by the market, this mispricing does not imply the availability of arbitrage, nor the abnormal profitability.

The search for the convexity adjustment can be extended using a sample of so called IMM FRA rates. The expiration dates of the Eurocurrency interest rate futures on the quarterly expiration cycle (March – June – September – December) are referred as IMM dates, where "IMM" stands for International Monetary Market, division of the Chicago Mercantile Exchange (CME). The FRA contracts with the IMM settlement dates are referred as IMM FRAs. The advantage of using the IMM FRA contracts for the analysis of convexity adjustment is that such contracts have settlement dates that match those of the respective Eurocurrency futures. Use of such data allows to avoid possible pitfalls of previous studies in the area: presence of the interpolation error and/or limited sample size. Poskitt (2008a) examines daily and intraday data on sterling interest rate futures and IMM FRA contracts and finds that the sterling futures/FRA rate differential is marginally negative, contrary to the theoretical predictions. After the regression analysis fails to find any support for the predicted positive relationship between the differential and the time to maturity and volatility of interest rates, he concludes that the convexity adjustment has not been priced into quotes in the sterling FRA market. Similarly, Poskitt (2008c) also suggests that the convexity adjustment is not priced into the US dollar FRA quotes.

The issue that has been overlooked in Poskitt's works is that the futures/FRA rate differential is supposed to differ from the futures/implied forward rate one if there are limits to arbitrage. In that case, the former is always smaller than the latter and this difference increases with time to maturity of the futures contract. A large number of textbooks on derivatives and fixed income use a no-arbitrage argument to show that the rate on a forward rate agreement has to be a function of the current term structure and it is equal to the implied forward rate. This paper demonstrates that in the presence of the limits to arbitrage the two rates would differ and the difference will increase with the maturity. If the standard arbitrage set of transactions is not possible to execute, the IMM FRA rate has to be a function of the *evolution* of the current term structure and its quote is supposed to be much closer to the respective Eurocurrency futures rate than to the implied forward rate. The latter observation allows to explain why the recent research on the convexity adjustment in the IMM FRA market has not been able to detect the presence of it in the futures quotes relative to IMM FRA rates. The whole idea of detecting the empirical presence of the convexity adjustment in the IMM FRA market or lack of thereof is questionable to implement due to the illusory nature of the subject under consideration whose size in this paper is shown to be within the limits of the bid-ask spread. Therefore, the claims that the convexity adjustment has not been priced into IMM FRA quotes lack substance and are subject to revision.

The rest of the paper consists of the following. Section 4.2 discusses the role that the interest rate futures play in the European market. Section 4.3 lays down the methodology of the utilized approach employed to detect the presence of the futures premium or lack of thereof. Section 4.4 describes the data used for the analysis and section 4.5 provides results of the futures premium search. Section 4.6 introduces the forward rate agreement and reviews its pricing while section 4.7 explains why the use of the IMM FRA quotes instead of the implied forward rates would not make the search for the convexity adjustment any easier. Section 4.8 concludes.

4.2 The Exchange Traded Interest Rate Derivatives Market in Europe

Interest rate derivatives are traded with a very high volume at several international exchanges, e. g. Chicago Mercantile Exchange (CME), London International Financial Futures and Options Exchange (LIFFE), Marché à Terme International de France (MATIF), OM Stockholm AB and Tokyo Financial Exchange (TFX, former TIFFE) among others. London International Financial Futures and Options Exchange is one market in the world that provides opportunity to trade three-month interest rate futures and options on futures for a number of major world currencies. LIFFE is the oldest European derivatives exchange. It was founded in 1982 and started trading financial futures in the same year. In 2001 LIFFE became a part of the Euronext Group and now it is referred as Euronext.Liffe. Euronext[41] was born in 2000 through the merger of Amsterdam Exchanges (AEX), Brussels Exchanges (BXS), and ParisBourse. Euronext.Liffe has quickly become one of only two major players in the bond and short term interest rate futures and options market in Europe. The other major player is EUREX, the derivatives market set up through a joint venture between the DTB Deutsche Terminbörse, the German Options and Futures Exchange, and SOFFEX, the Swiss Options and Financial Futures

[41] In April 2007 Euronext merged with New York Stock Exchange (NYSE) forming NYSE Euronext. MATIF, mentioned earlier, was absorbed in the merger of the Paris Bourse with Euronext to form Euronext Paris.

Exchange. It is fair to say that Euronext.Liffe dominates the short term interest rate market in Europe while EUREX dominates the medium and long term interest rate market, although Euronext.Liffe has made some impact with the introduction of the Swapnote[42] product range. A good source that provides a historical review and a glance at the statistics of the European derivative markets is Young (2004).

Euronext.Liffe's interest rate portfolio of products offers a wide range of liquid short term interest rate (STIR) derivative contracts, providing exposure to US, European, UK, Swiss and Japanese short term interest rates. Table 4.1 contains a summary of the description of the three-month interest rate futures contracts for four currencies other than the US dollar that are traded on the exchange. The information for the table is taken from the Euronext.Liffe documents listed in the reference section and the Euronext website. Since their respective introductions, the exchange's three-month interest rate futures contracts have developed into highly liquid hedging and trading instruments. Average daily volume for the exchange's STIR portfolio was over 1.7 million contracts in May 2007. Its flagship contract suite – the three-month EURIBOR[43] interest rate futures and options on futures contracts – had an average daily volume close to 1.1 million contracts in May 2007[44].

Table 4.2 provides statistics about futures and options trading volume and open interest on Euronext.Liffe during 2000-2007. The volume of trading of all futures traded on the exchange more than doubled during the period whereas the volume of short term interest rate futures quadrupled. A different situation is observed if one looks at the trading volume of options. The growth of trading volume in all options traded on the exchange was flat or even negative during 2003-2006, whereas the volume of trading of short term interest rate options went up tenfold since 2000 and exceeded annual volume of 100 million contracts in September of 2007. Open interest for options in the first quarter of 2008 doubled compared to that in 2004 demonstrating a non-satiated interest in interest rate option trading among the European market participants. Despite lagging futures in the volume of trading, open interest on STIR options has been above that on STIR futures since 2001 with the former being more than twice as much as the latter in 2003, 2007 and the first quarter of 2008.

[42] See more on Swapnote at http://www.euronext.com/landing/landingInfo-2114-EN.html
[43] Euro Interbank Offered Rate
[44] Source: http://www.euronext.com/trader/trader-2099-EN.html

Table 4.1 Description of three month interest rate contracts traded on Euronext.Liffe for currencies other than US dollar

	Three Month Euro (EURIBOR) Interest Rate Futures	Three Month Sterling (Short Sterling) Interest Rate Futures
Unit of trading	€1,000,000	£500,000
Delivery months	March, June, September, December, and four serial months, such that 25 delivery months are available for trading, with the nearest six delivery months being consecutive calendar months	March, June, September, December, and two serial months, such that 23 delivery months are available for trading, with the nearest three delivery months being consecutive calendar months.
Quotation	100.00 minus rate of interest	100.00 minus rate of interest
Minimum price movement (tick size and value)	0.005 (€12.50)	0.01 (£12.50)
Last Trading Day	10.00 - Two business days prior to the third Wednesday of the delivery month	11:00 - Third Wednesday of the delivery month.
Delivery Day	First business day after the Last Trading Day	First business day after the Last Trading Day.
Trading Hours	01:00 – 21:00 London time	07:30 - 18:00 London time
Exchange Delivery Settlement Price	Based on the European Bankers Federations' Euribor Offered Rate (EBF Euribor) for three month Euro deposits at 11.00 Brussels time (10:00 London time) on the Last Trading Day.	Based on the British Bankers' Association London Interbank Offered Rate (BBA LIBOR) for three month sterling deposits at 11:00 on the Last Trading Day.

	Three Month Swiss Franc (Euroswiss) Interest Rate Futures	Three Month Euroyen (TIBOR) Interest Rate Futures
Unit of trading	SFr 1,000,000	¥100,000,000
Delivery months	March, June, September, December, such that eight quarterly delivery months are always available for trading	March, June, September, December, such that 12 delivery months are available for trading
Quotation	100.000 minus rate of interest	100.000 minus rate of interest
Minimum price movement (tick size and value)	0.01(SFr25)	0.005(¥1,250)
Last Trading Day	11:00 - Two business days prior to the third Wednesday of the delivery month	16:00 - Two LIFFE business days prior to the Tokyo Financial Exchange (TFX) Last Trading Day for the equivalent TFX Euroyen contract month
Delivery Day	First business day after the Last Trading Day	N/A*
Trading Hours	07:30 - 18:00 London time	07:00 - 16:00 GMT / 08:00 - 16:00 British Summer Time**
Exchange Delivery Settlement Price	Based on the British Bankers' Association London Interbank Offered Rate (BBA LIBOR) for three month Euroswiss Franc deposits at 11:00 on the Last Trading Day.	Subject to TFX Rules, TFX will calculate their Final Settlement Price using the Zenginkyo TIBOR for three month Yen deposits at 11.00 (Tokyo time) on the TFX Last Trading Day.

* – The delivery day for the TFX contract is the next business day following the Last Trading Day. The TFX Last Trading Day is the second business day immediately preceding the third Wednesday of the delivery month.

** – Euronext.liffe will trade the Euroyen contract from 16:00 (Tokyo time). During British Summer Time, the Euronext.liffe Euroyen contract will start trading at 08:00 but will continue to close at 16:00.

Table 4.2 Volume, open interest and value of volume of trading on Euronext.Liffe in 2000-2007

Volume of Trading by Product Group, contracts (in millions)

	Exchange			All short term interest rate products					
	Futures	Options	Total	Futures	%	Options	%	Total	%
2000	206.18	256.05	462.24	85.46	41.4	12.70	5.0	98.16	21.2
2001	214.86	404.29	619.15	130.73	60.8	30.81	7.6	161.54	26.1
2002	221.94	475.06	697.01	145.05	65.4	42.69	9.0	187.74	26.9
2003	267.88	427.17	695.05	185.08	69.1	77.84	18.2	262.92	37.8
2004	310.70	477.09	787.79	221.07	71.2	76.33	16.0	297.40	37.8
2005	344.05	415.27	759.33	248.66	72.3	79.48	19.1	328.14	43.2
2006	430.04	300.28	730.32	296.01	68.8	92.99	31.0	388.99	53.3
2007	562.44	386.58	949.03	353.59	62.9	135.54	35.1	489.14	51.5

Open Interest by Product Group, contracts (in millions)

	Exchange			All short term interest rate products					
	Futures	Options	Total	Futures	%	Options	%	Total	%
2000	4.56	42.61	47.17	2.36	51.7	1.71	4.0	4.07	8.6
2001	5.74	77.19	82.93	2.60	45.2	3.67	4.8	6.27	7.6
2002	4.87	82.36	87.24	3.00	61.6	4.97	6.0	7.97	9.1
2003	5.43	73.80	79.22	3.36	62.0	8.65	11.7	12.02	15.2
2004	6.36	82.08	88.44	4.34	68.3	7.63	9.3	11.98	13.5
2005	7.45	71.50	78.95	5.24	70.4	9.59	13.4	14.83	18.8
2006	9.56	62.02	71.58	6.09	63.7	10.37	16.7	16.46	23.0
2007	13.83	58.25	72.08	6.23	45.0	14.33	24.6	20.56	28.5

Value of Volume by Product Group, euros (in trillions)

	Exchange			All short term interest rate products					
	Futures	Options	Total	Futures	%	Options	%	Total	%
2000	80.24	12.36	92.60	78.06	97.3	11.64	94.2	89.70	96.9
2001	128.43	31.50	159.93	122.32	95.2	29.19	92.7	151.51	94.7
2002	140.87	43.08	183.95	136.48	96.9	41.06	95.3	177.54	96.5
2003	176.06	75.46	251.52	171.76	97.6	73.66	97.6	245.43	97.6
2004	209.27	74.05	283.32	204.13	97.5	71.68	96.8	275.80	97.3
2005	232.51	74.62	307.13	226.24	97.3	71.86	96.3	298.10	97.1
2006	278.58	86.46	365.04	269.91	96.9	82.95	95.9	352.86	96.7
2007	328.42	125.66	454.08	316.63	96.4	120.62	96.0	437.25	96.3

Short term interest rate derivatives have been the dominant instrument in trading on Euronext.Liffe since the beginning of the century. STIR futures account for 65-70 percent of the futures trading volume on the exchange while STIR options now compose more than a third of the option trading volume after having accounted for just five percent of it in 2000. It is due to the explosive growth in STIR option trading that now short term interest rate derivatives compose more than a half of the entire trading volume on the exchange. If measured by the notional value of contracts traded, the annual count goes into trillions of euros and, due to high nominal values of STIR derivative contracts, they alone account for an incredible 96-97 percent of the value of the entire volume traded on the exchange and this percentage has not changed since 2000.

Table 4.3 shows volume of trading measured in contracts as well as in euros, open interest numbers and average size of the transaction for five different contracts within the group of STIR futures[45]. The EURIBOR three-month interest rate futures account for 65-70 percent of the contact volume and 70-80 percent of euro value of the volume. The second most traded contract is the three-month short sterling interest rate futures[46]. Numbers for 2007 demonstrate an increased interest in trading of sterling futures that amounted to roughly five percent of the market volume share gained by sterling futures from its EURIBOR counterpart. Both EURIBOR and sterling futures have shown a consistent growth in the volume of trading and its value since 2000. Another contract that showed steady year by year volume growth is the three-month Euroswiss franc interest rate futures which historically has accounted for about 2-4 percent of the market volume.

Two remaining contracts lag behind in terms of the occupied market share although they have observed different trends recently. The three-month Eurodollar interest rate futures was introduced by the exchange on March 18, 2004 and it had a good start back then delivering volume numbers just slightly below those of the Euroswiss franc contract. But trading of Eurodollar futures weakened dramatically in 2006 and became non-existent through 2007. The lack of success of the Eurodollar futures on Euronext.Liffe is attributed to the sharp response by the Chicago Mercantile Exchange, the original platform for the Eurodollar trading, who introduced several policy changes that meant to transfer its trading volume in Eurodollar futures from open outcry to the electronic trading platform, Globex, thereby reducing the transaction costs for traders while retaining its market share. Tse and Bandyopadhyay (2006) mention that about 27 firms in Europe signed on to use Globex through an incentive scheme that was started

[45] There are two more STIR futures contracts that were traded on Euronext.Liffe during the featured time period that are not included in our analysis due to their limited trading volume. The three-month Euro LIBOR (ECU) interest rate contract ceased trading in 2001 and its volume in 2000 and 2001 was very low, e. g. 21 and 7 contracts traded per day on average respectively. The one-month EONIA (Euro Overnight Index Average) contract was traded during 2003-2006 but did not have much success, e. g. seven transactions in 2005 and only one transaction in 2006. Numbers in Table 4.2 (totals for the exchange and totals for STIR products) include those of Euro LIBOR and EONIA whereas percentages in Table 4.3 are based on totals within the group of STIR futures that exclude figures for these two contracts.

[46] "Short" in "short sterling" refers to short term, not short position. The definition and construction of the short sterling contract is not different from those for interest rates of other currencies as can be seen from Table 4.1.

Table 4.3 Volume, open interest, value of volume of trading and average size of a transaction of STIR futures on Euronext.Liffe in 2000-2007

Average Daily Volume, contracts

	Euribor	%	Euro-dollar	%	Euroswiss Franc	%	Euroyen	%	Sterling	%
2000	229,136	68.1	-	-	18,195	5.4	56	0.0	89,043	26.5
2001	355,805	69.7	-	-	18,337	3.6	0	0.0	136,504	26.7
2002	413,135	72.9	-	-	19,438	3.4	7	0.0	134,015	23.7
2003	537,860	74.4	-	-	19,568	2.7	0	0.0	165,325	22.9
2004	609,061	71.0	22,763	2.7	28,173	3.3	3	0.0	198,163	23.1
2005	648,569	66.7	21,949	2.3	32,242	3.3	0	0.0	269,957	27.8
2006	792,516	68.1	274	0.0	42,133	3.6	390	0.0	328,078	28.2
2007	868,280	62.5	62	0.0	47,921	3.4	1,064	0.1	473,027	34.0

Open Interest, contracts (in thousands)

	Euribor	%	Euro-dollar	%	Euroswiss Franc	%	Euroyen	%	Sterling	%
2000	1,391.6	60.1	-	-	143.8	6.2	0.0	0.0	779.6	33.7
2001	1,717.9	66.2	-	-	142.0	5.5	0.0	0.0	733.7	28.3
2002	2,030.5	67.7	-	-	158.1	5.3	0.0	0.0	811.9	27.1
2003	2,236.5	66.5	-	-	211.6	6.3	0.0	0.0	916.0	27.2
2004	2,683.4	61.8	150.8	3.5	242.4	5.6	0.0	0.0	1,267.1	29.2
2005	3,119.8	59.5	153.9	2.9	269.6	5.1	0.0	0.0	1,699.1	32.4
2006	3,420.8	56.2	65.3	1.1	367.5	6.0	0.0	0.0	2,238.5	36.7
2007	3,335.0	53.6	36.6	0.6	212.7	3.4	0.0	0.0	2,643.5	42.4

Value of Volume by Product Group, euros (in billions)

	Euribor	%	Euro-dollar	%	Euroswiss Franc	%	Euroyen	%	Sterling	%
2000	58,016.9	74.3	-	-	2,765.7	3.5	13.0	0.0	17,261.2	22.1
2001	91,086.2	74.5	-	-	3,112.5	2.5	0.0	0.0	28,121.9	23.0
2002	105,763.8	77.5	-	-	3,392.8	2.5	1.5	0.0	27,321.3	20.0
2003	137,692.2	80.2	-	-	3,318.3	1.9	0.0	0.0	30,579.4	17.8
2004	157,746.7	77.3	3,714.7	1.8	4,727.7	2.3	0.5	0.0	37,831.2	18.5
2005	166,682.1	73.7	4,414.6	2.0	5,344.5	2.4	0.0	0.0	49,786.9	22.0
2006	202,091.6	74.9	55.8	0.0	6,823.2	2.5	65.7	0.0	60,877.5	22.6
2007	221,411.5	69.9	11.9	0.0	7,443.8	2.4	169.1	0.1	87,588.8	27.7

Average number of contracts per transaction, contracts

	Euribor	Eurodollar	Euroswiss Franc	Euroyen	Sterling
2000	12.7	-	31.2	148.5	12.8
2001	12.6	-	33.6	0.0	15.5
2002	10.5	-	28.2	294.8	13.7
2003	9.5	-	28.9	0.0	13.7
2004	8.5	19.2	25.0	147.2	12.9
2005	8.7	20.7	19.3	22.0	11.9
2006	8.6	38.6	14.0	77.8	10.3
2007	9.4	51.4	14.3	91.8	11.3

in March 2004 when LIFFE introduced its Eurodollar contract. Singapore International Monetary Exchange (SIMEX) is another exchange that offers trading in the Eurodollar futures and its market share has also diminished considerably after the introduction of Globex[47]. The further sharp decrease in the Eurodollar futures trading on Euronext.Liffe over the 2006-2007 period reflects the continued success of the CME's Globex and casts doubt on the future prospects of the Eurodollar segment on Euronext.Liffe.

The three-month Euroyen interest rate futures contract, on the other side, has been around since 2000 but volume had not been observed until 2006. In 2007 its average daily volume of trading exceeded 1,000 contracts. The major global competitor for this contract is the Tokyo Financial Exchange but there is not much if any competition within the European market borders. It remains to be seen where the contact goes from there in terms of its trading volumes and how the TFX can respond to the possible threat of losing its market share in the segment.

The observation of the average size of a typical transaction on three major STIR futures traded on Euronext.Liffe, the EURIBOR, the sterling and the franc, says that the average size of the transaction went down steadily between 2001 and 2006 but picked up slightly in 2007. It may indicate that the growth in STIR futures trading on Euronext.Liffe during 2001-2007 was spurred by the influx of smaller players who were not primarily engaged in the financial industry by the very nature of their activities but have been taking opportunity of an easy access to standardized interest rate hedging instruments.

4.3 Methodology of the Analysis

In order to identify the existence and to estimate the size of the interest rate futures premium, the empirical estimate of the convexity adjustment must be compared with its theoretically predicted magnitude. In order to compute its empirical estimate, the spot rate curve must be interpolated first using one or more of the conventional methods. After the yield curve has been constructed, the implied forward prices for 90-day contracts are obtained as

$$F^{EM}(t,m,m+0.25) = \frac{\exp(r(t,m)m)}{\exp(r(t,m+0.25)(m+0.25))}, \tag{4.1}$$

where $F^{EM}(t, m, m+0.25)$ is the implied empirical forward price of the contract for the rate applied to the period from m to $m+0.25$ (m is expressed in years) observed at time t and $r(t, m)$ is

[47] See Tse and Bandyopadhyay (2006) for more on the "Institutional war over Eurodollar market share"

the continuously compounded rate for maturity m at time t. The observed price of the three-month futures contract, $f^{EM}(t, m, m+0.25)$, is obtained using

$$f^{EM}(t,m,m+0.25) = 1 - \frac{100 - f_q(t,m,m+0.25)}{100}\frac{90}{360}, \tag{4.2}$$

where $f_q(t, m, m+0.25)$ is the quoted futures price. The empirical convexity adjustment is subsequently obtained as the difference between $F^{EM}(t, m, m+0.25)$ and the respective $f^{EM}(t, m, m+0.25)$.

In order to compute the theoretically predicted value of the convexity adjustment, the evolution of the term structure of the forward rates must be build first. This paper relies on the discrete version of the one-factor HJM model. The implementation of the HJM model in discrete time is described in Grant and Vora (1999), Chance (2004) and Grant and Vora (2006). The implied forward price is independent of the term structure model used and it is computed using (4.1). The predicted three-month interest rate futures price at point of time t before the futures maturity date is computed as

$$f^{TH}(t,m,m+0.25) = 1 - 0.25\tilde{E}_t[r(m,m+0.25)], \tag{4.3}$$

where $r(m, m+0.25)$ is the three-month LIBOR at the expiration of the futures contract (the settlement LIBOR futures rate) and the expected value is taken under risk neutral probability distribution.

In order to construct the evolution of the term structure, three important inputs must be identified first: the initial term structure of interest rates, their volatilities and volatilities' evolution in time, and the step size of the model. The initial term structure of interest rates is obtained from the interpolated spot curve that is also used to compute the implied forward rates. The choice of the volatility function can be sensitive and, therefore, three popular volatility specifications are considered for robustness purposes[48]:

Type 1: Volatility is a function of maturity time only, i. e. $\sigma(t_1, T) = \sigma(t_2, T)$ for $0 \leq t_1 < t_2 < T$, where $\sigma(t, T)$ is the standard deviation of the one-period forward rate that matures at T as observed at time t. This is the least computationally burdensome case if implemented using a binomial tree approach since it results in a recombining tree.

Type 2: Volatility is a function of time to maturity only, i. e. $\sigma(t_1, T_1) = \sigma(t_2, T_2)$ for $T_1 - t_1 = T_2 - t_2$. This is a more realistic case compared to the one above since empirical observations

[48] The trivial case of the constant volatility both in time and cross-sectionally is not considered.

demonstrate that volatility indeed varies with time left till maturity of the implied forward contract. The resulting binomial tree, if implemented however, will be a non-recombining one.

Type 3: The exponentially dampened volatility, $\sigma(t, T) = \sigma exp[-\lambda(T-t)]$. This volatility specification is similar to that in b) since it is also a function of time to maturity. The binomial tree resulting from this application would not recombine as well.

The first volatility specification puts more weight on the volatility at the back end of the yield curve where the estimated value of the convexity adjustment for longer maturities depends on the current volatilities of the forward rates with farther expiration dates. The second specification spreads the weight uniformly across volatilities of a number of current forward rates and the predicted convexity adjustment becomes a function of the current volatilities of the rates for the whole range of maturities. The third volatility specification is a variation of the second one and its advantage is that it allows to eliminate the influence of possible outlier products in the computed volatilities of the current yield curve. The choice of these three specifications, though does not cover the whole universe of all possible options, includes the often mentioned and applied ones and should serve as a mean of validating that the results of the paper are not contingent upon the choice of the volatility function. Other choices may include specification of the volatility as a (G)ARCH process. By far, the proposed analysis is more comprehensive in this matter than those used in previous related works where the volatility input and its specification have more restrictive forms and fewer considerations.

It is believed that the length of the step in the term structure evolution model should be chosen as small as possible. If implemented using a binomial tree approach, it, however, would result in an exponentially increasing number of nodes after each extra step unless the resulting binomial tree recombines. Two out of three volatility specification described earlier would bear a significant computation burden if the binomial tree modeling is performed. The alternative solution is to conduct Monte Carlo simulations. I construct a model with a weekly step. Note however that if the weekly step (Δt) is chosen, one no longer models the term structure of three-month forward rates, since the modeled forward rates are the rates that apply to the time period of Δt, i. e. one week. Hence, the term structure of one-week forward rates is constructed[49], while the three-month forward rates, assuming that three months are equivalent in length to 13 weeks, are calculated using the following no-arbitrage requirement:

[49] One-week forward rates are taken as weekly continuously compounded rates.

$$f(t,m,m+13) = \sum_{j=0}^{12} f(t,m+j,m+1+j)\,. \tag{4.4}$$

Eventually, the expected future three-month spot rates are computed and they are subsequently used to calculate the theoretical futures prices according to (4.3). The theoretically predicted futures prices are further subtracted from the implied forward prices in order to obtain the predicted value of the convexity adjustment.

In order to calculate the initial volatility estimates, all forward rates are converted into weekly continuously compounded rates and their historical standard deviations are calculated. The weekly forward rate volatilities under the first two specifications described above are computed as the daily standard deviations measured using the range of rates for the last N business days (including the current business day) multiplied by the square root of five. For the exponentially dampened volatility, parameters σ and λ must be identified. Regression analysis is used for that purpose. Once the weekly volatilities of all weekly forward rates[50] are computed as described above, their natural logarithms are regressed on respective expiration periods. The resulting slope is the negative λ, while the exponential of the intercept is the σ parameter.

In order to perform Monte Carlo simulation of the evolution of the term structure of forward rates, 100 draws of the vector of standard normally distributed variable are employed for each business day observation in the sample[51]. Also the antithetic variate technique is utilized by changing signs of the values of each drawn vector of random elements and building the evolution of forward rates a new. Therefore, the total number of the forward rate paths built for each business day observation in the sample is 200. The averages of the futures prices across 200 different paths are taken and recorded as the theoretical, or predicted, futures prices.

In order to calculate the futures premium defined as the difference between the observed (empirical) and the predicted (theoretical) convexity adjustments, the two estimates must be for the contracts on the same underlying three-month interest rate that also have same expiration time. Since the term structure is modeled using the weekly step, the maturities of the predicted forward-futures differential are restricted to those of one week, two weeks, three weeks and so on, while the observed convexity adjustment can be applied to maturities that may include a

[50] Excluding the spot rate

[51] To verify that 100 is a large enough number of draws, for the first 100 business day observations in the sample simulations were also conducted using 1,000 random draws. The evidence is that the sharp increase in the number of draws resulted in small insignificant improvements.

certain number of weeks and its fraction. The best if not the only way to get over this limitation is to interpolate the predicted values of the differential to odd maturities, i. e. the maturities that include a fraction of a week. For that purpose if, for example, the available predicted values are for maturities of 21 and 28 days but the observed value is for 25 days, the predicted values for the two adjacent weekly maturities are used to obtain the value of the in-between maturity via interpolation. The employed interpolation method can be chosen to be a linear one since in most cases the difference between the predicted values for the two adjacent weekly maturities is quite insignificant, especially for shorter maturities, and even for longer maturities it rarely exceeds one third of a basis point, and therefore this will not result in the introduction of a significant interpolation bias[52]. Numerically, the procedure looks as following:

$$\Delta(7j+i) = \frac{7-i}{7}\Delta(7j) + \frac{i}{7}\Delta(7(j+1)) \quad \text{for} \quad 0 \le i \le 7, \tag{4.5}$$

where $\Delta(t)$ is the predicted convexity adjustment for maturity of t days, j is the number of full weeks in the maturity length and i is the number of days in the maturity above that of j weeks. For the aforementioned example above where one needs to obtain the predicted value for the maturity of 25 days, j is equal to three and i is equal to four.

4.4 Data Description

Data for CHF (Swiss franc) LIBOR were obtained from the British Bankers' Association website. The data were acquired for the sample period from 10/2/2002 until 2/28/2007. In total, there are 1,116 business day observations for Swiss franc LIBOR in the sample. The Euroswiss franc futures quotes are obtained for the same sample period from Econstats (www.econstats.com). The first contract in the sample matures in December 2002 and the last contract in the sample expires in June 2008. In total, there are 3,359 quote observations for the Euroswiss franc futures. Quotes for up to three futures with consecutive quarterly maturities are considered only since the implied forwards' maturities are limited to those below or equal to nine months.

Data for GBP (British pound) LIBOR were obtained from the British Bankers' Association website. The data were acquired for the sample period from 12/01/1997 until 2/28/2007. In total, there are 2,335 day observations for GBP LIBOR in the sample. The short

[52] Shynkevich (2008) shows that the two components of the convexity adjustment are both exponential functions of maturity with different specifications. The choice of linear interpolation instead shall not affect this paper's results due to the insignificant nature of the introduced bias.

sterling interest rate futures quotes are obtained for the same sample period from Turtle Trader (www.turtletrader.com) and Econstats[53]. The first contract in the sample matures in December 1997 and the last contract in the sample expires in December 2007. Data for two contracts in the sample range (June 2000 and March 2001) are missing. In total, there are 10,664 quote observations for short sterling futures. Quotes for up to seven futures with consecutive quarterly maturities are considered. All such contracts possess sufficient liquidity and if LIBOR data are used together with that for swaps of one year, 18 month and two year maturities, it allows to extend the maturity range in the analysis of the convexity adjustment for British currency interest rate futures up to two years.

Data for EURIBOR quoted spot rates were obtained from www.euribor.org. The data were acquired for the sample period from 12/30/1998 until 9/25/2007. In total, there are 2,243 day observations for EURIBOR in the sample. The EURIBOR interest rate futures quotes are acquired for the same sample period from CRB Trader (www.crbtrader.com). The first contract in the sample matures in June 1999 and the last contract in the sample expires in June 2009. In total, there are 15,700 quote observations for EURIBOR futures. As with the short sterling futures, for the euro currency, quotes for up to seven futures with consecutive quarterly maturities are considered. All such contracts possess sufficient liquidity, as do euro swap contracts with maturities of one year, 18 months and two years[54].

The spot yield curves of rates for all three currencies were interpolated by employing the extended Nelson-Siegel method. According to the Bank for International Settlements paper No. 25 (2005), this method that was originated by Nelson and Siegel (1988) and later extended by Svensson (1995) is the most popular numerical optimization approach to construct the yield curve and has been utilized by a number of world central banks. For the Swiss franc term structure interpolation, CHF LIBOR rates for the one-week period and for all twelve monthly maturities were used to fit the curve. For EURIBOR (British pound), the EURIBOR (GBP LIBOR) rates for the one-week expiration and for all twelve monthly maturities were used as

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[53] Turtle Trader sample covers period from 12/01/1997 to 10/01/2002 while Econstats' is for the 10/02/2002 – 2/28/2007 interval.

[54] The liquidity of euro-denominated interest rate swaps improved significantly following the development of the European Monetary Union. Average daily turnover of over-the-counter (OTC) interest rate contracts reached €231 billion in 2001. By the very same year the turnover of Euro swaps had exceeded that of all interest rate products other than US Treasuries (Wooldridge [2004]). It (liquidity) was enhanced by the rapid integration of markets in the Eurozone. Wooldridge (2004) states that, unlike the euro government bond yield curve, a single euro swap curve emerged almost overnight. Trading in the euro swap market was further boosted by increased hedging activity.

well as the implied spot rates for maturities of 18 months and two years. The last two were obtained by using the mid-quotes of euro (sterling) swap rates for respective maturities. The expiration periods (maturity lengths) were computed using the modified following business day convention and the end-end dealing rule[55]. The interpolation was done for rates instead of that for implied zero-coupon prices[56] and, as the Nelson-Siegel model implies, for the interpolation procedure all quoted and derived spot rates were converted into continuously compounded rates with the actual/365 day count as following[57]:

$$r(t,m) = \frac{365}{m} \ln\left(1 + \frac{m}{365}\frac{365}{360} r^{q}(t,m)\right), \tag{4.6}$$

where $r^q(t, m)$ is the quoted / implied spot rate based on annual compounding and the actual/360 day count for maturity *m* as observed at time *t*.

To calculate the implied spot rates for euro and sterling currencies for maturities of 18 months and two years, swap rates must be used to obtain discount factors first. The sterling interest rate swap rates are quoted similarly to those on the US dollar interest rate swaps since both instruments imply semi-annual fixed payments. Therefore, one needs to apply the following equation in order to compute the discount factors for 18 months and two years:

$$S(t,0.5m) = \frac{2 - c(t,0.5m)\sum_{n=1}^{m-1} S(t,0.5n)}{2 + c(t,0.5m)} \quad \text{for } m = 3 \text{ and } 4, \tag{4.7}$$

where $S(t, 0.5m)$ is the discount factor at time *t* applied for maturity *0.5m* and it is equal to $exp(-r(t, 0.5m)0.5m)$,

$c(t, 0.5m)$ is the swap rate for maturity *0.5m* at time *t*.

And then the implies spot rates for those maturities are obtained using

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[55] The British Bankers' Association's modified following business day convention defines the maturity date as the first following day that is a business day in London and the principal financial centre of the currency concerned, unless that day falls in the next calendar month. In this case only, the maturity date will be the first preceding day in which both London and the principal financial centre of the currency concerned are open for business. The end-end dealing rule states that in cases when a deposit is made on the final business day of a particular calendar month, the maturity of the deposit shall be on the final business day of the month in which it matures, not the corresponding date of the month of maturity.

[56] Svensson (1995) points out that minimizing yield errors provides a substantially better fit for short maturities while the two procedures (minimization of squared price errors versus squared yield errors) tend to perform equally well for long maturities. This is because yields for short maturity bonds are much more sensitive to changes in prices of those bonds than yields for bonds with longer maturities.

[57] Quoted LIBOR and swap rates for British pound are initially based on the actual/365 day count and formula (4.6) must be changed accordingly for calculation of sterling spot rates (360 is replaced by 365).

$$r(t,n) = -[\ln S(n)]/n\,. \tag{4.8}$$

The extraction of discount factors from euro interest rate swaps differs since, unlike for the US dollar interest rate swap, the fixed leg of the euro swap is based on the annual payments. The two-year swap will have two fixed leg payments: at the end of the first year and at the end of the second year. The 18-month swap will also have two fixed leg payments: at the maturity it will equal to the swap rate times the swap face value, whereas six months from the swap value date, the fixed leg payment will be equal to the swap rate divided by two times the face value[58]. Hence, equation (4.7) used for the US dollar denominated swaps that is also applicable to the British pound interest rate swaps must be modified accordingly for the euro denominated swaps. The eighteen-month discount factor is found from

$$S(1.5) = \frac{2 - c(1.5)S(0.5)}{2 + c(1.5)}, \tag{4.9}$$

where $S(0.5) = \exp(-r(0.5)0.5)$, *r(0.5)* is the continuously compounded six-month EURIBOR and *c(1.5)* is the 18-month swap rate.

The two-year discount factor is computed in the following way:

$$S(2) = \frac{1 - c(2)S(1)}{1 + c(2)}, \tag{4.10}$$

where *c(2)* is the two-year swap rate and *S(1)* is the one-year discount factor obtained from

$$S(1) = \frac{1}{1 + c(1)}. \tag{4.11}$$

The swap rate quotes for both euro and sterling interest rate swaps are obtained from Bloomberg. If the swap rates are quoted at a different time of the day compared to the EURIBOR and GBP LIBOR quotes, a concern related to the potential errors arising from the non-synchronous data in the sample arises. A few authors augment swap rate data with short-term LIBOR. Dai and Singleton (2000), for example, use a data set of swap rates for maturities from two to ten years and augment it with the six-month LIBOR rate. The source of their swap data is Datastream and the non-synchronicity of quotes arises immediately since LIBOR are quotes as of 11 am London time, while swap rates are recorded at the end of trading day in London which is 5:30 pm London time. It was brought to attention by Rendleman (2004) that Bloomberg allows historical data on swap rates to be collected as of 6 am, 1 pm and 5:30 pm Eastern time of each

[58] The author is grateful to Mark Garofalo for this clarification.

trading day which corresponds to Tokyo, London and New York "closing" time respectively. There is, in fact, no market closing in any of those places since swaps can be traded over-the-counter 24 hours a day, but Bloomberg created these virtual time stamps for a matter of convenience. If one chooses swap rate quotes corresponding to the Tokyo "closing" time, as this paper does, that would perfectly match the timing of the LIBOR/EURIBOR publication (6 am in New York corresponds to 11 am in London). Hence, the subsequent results and conclusions will be clean from the data non-synchronicity.

To mitigate the influence of the interpolation error in the analysis of identification of the futures premium, the criteria for an interpolated yield curve are introduced that would allow it to be included in the final sample. For the Swiss franc spot rate term structure, a yield curve on a particular business day is considered to satisfy the interpolation criteria if the sum of all thirteen absolute fitted errors is below 20 basis points and each absolute error does not exceed five basis points. That leaves fitted daily yield curves for 1,116 business days. For EURIBOR and British pound spot rate term structures, the criteria are the same as above if the spot rates up to a maturity of twelve months were considered only and swap rate data were not available on that day. When the implied spot rates for maturities longer than twelve months obtained from the observed swap rate quotes were also used to fit the spot yield curve, the criteria are 25 basis points and five basis points for the sum and each individual absolute error respectively. These criteria leave 2,064 daily yield curves for EURIBOR/swap and 2,158 spot rate curves for British LIBOR/swap respectively.

To identify the presence of the futures premium, the contracts with expiration of at least 10 days are considered to compute the empirical estimate of the convexity adjustment and to subsequently calculate its predicted value. The futures prices and respective forward prices are matched by the settlement dates of the forward contracts. The total number of matched forward-futures pairs for the Swiss rates is 3,206, for EURIBOR is 13,841, and for the sterling rates is 9,608.

4.5 Estimation Results

The sensitivity of the term structure modeling with regard to the volatility input is not restricted to the chosen specification of the volatility function but it also depends on the way the historical volatility is measured. For instance, daily volatility of the forward rates can be computed using historical rates for the last business week (five days), last month (21 days), last quarter (62 days), or any other period of time. Not only each of those approaches will produce different standard deviation estimates, but there can be a pattern where volatilities calculated using one range of quotes tend to exceed respective standard deviations that were computed using rate data for some other span of time. Such a tendency is clearly observed in the rate data for all three currencies used in this paper. By checking the sample of LIBOR rates for the Swiss franc, it is observed that the daily standard deviation, if measured using the historical quotes for the preceding three months, exceeds the one computed using the rate quotes for the last month in 88.1 percent of cases, while the latter exceeds the one-day standard deviation measured using the quotes for the last five days in 90.0 percent of cases. For the sample of EURIBOR rates, the respective numbers are 90.1 and 92.1 percent, while for the sample of British pound LIBOR, they are 87.0 and 89.6 percent respectively.

Note that the above percentages are for standard deviations that were calculated using a set of spot rates. However, under the HJM model, it is weekly forward rates and their volatilities that are used to construct the evolution of the term structure of rates. The pattern of producing higher volatilities when a larger sample of quotes is used is also valid for the interpolated forward rates. In the sample for the Swiss franc forward rates, the daily standard deviation measured using historical quotes for the last three months, exceeds the one computed using the rate quotes for the last month in 86.9 percent of cases, while the latter exceeds the one-day standard deviation measured using the quotes for the last five days in 88.8 percent of cases. The respective percentages for the EURIBOR sample are 86.2 and 87.1, while for the volatilities of British pound forward rates they are 87.9 and 88.1 percent respectively. Since the predicted value of the interest rate forward-futures price differential is an increasing function of the volatility inputs, these observations imply that the theoretical convexity adjustment will tend to be larger when the historical volatilities are measured using a larger sample of rate quotes. Hence, all else equal, the longer the period for which the forward rate standard deviations are taken, the higher the interest rate futures premium would be.

In this paper all three mentioned ways to calculate the historical volatilities of the forward rates are used: when the sample consists of interpolated forward rates for the last three months (Model 1), the last month (Model 2) and the last week (Model 3). The results, if they differ significantly, will signal that the computation of the predicted convexity adjustment and the implied futures premium can be biased depending on the way the volatility inputs are calculated.

Recall that the other robustness check that this paper uses is the choice of the volatility evolution specification. By utilizing this two-dimensional approach as for the choice of the inputs, i. e. volatility computation (three models) *and* its evolution specification (three types), this paper performs the most comprehensive robustness check of the obtained results that previous literature devoted to the measurement of the interest rate convexity adjustment has been lacking.

Table 4.4 consists of three panels that contain summary statistics of the futures premium results for the three rate samples: Swiss franc, euro and British pound. The Jarque-Bera statistic is highly significant for each subsample of results (not reported in the table) indicating strong non-normality of the futures premium distribution. Bootstrapped 95 percent confidence intervals for the respective means are provided.

Panel A of Table 4.4 shows results for the premium in the Euroswiss franc futures. The average values of the premium for the first three nearby contracts range from 0.18 to 0.54 basis points depending on the type and the model of the volatility computation which essentially tells that the Euroswiss interest rate futures have been priced off the implied LIBOR forward curve. Such results can also signal that in the low interest rate environment observed in Switzerland, the interest rate futures premium tends to be negligible, at least for maturities below one year. Therefore, similar results must be expected if the analysis is performed for Japanese yen interest rate futures.

Panel B of Table 4.4 shows results for the premium in the EURIBOR futures. In all occasions but one (Model 1, Type 1) for maturities less than a year the premium does not exceed one basis point but the range of values it can most likely take widens with maturity and its mean estimate can reach up to four basis points for the 7th nearby contract (Model 1, Type 1, yet again). The widening of the gap between the results with increasing maturity demonstrates that the computation of the predicted value of the convexity adjustment can be quite sensitive to the choice of the combination of the volatility input and its modeled evolution given the high interest rate volatility environment. Model 1 tends to produce a futures premium of a size of a couple of basis points, Model 2 is likely to yield a moderate premium of less than one basis point, whereas Model 3 may show the absence of the premium altogether.

Panel C of Table 4.4 shows results for the premium in the short sterling futures. The findings here are similar to those for the EURIBOR sample: moderate premium of less than one basis point for the contracts with maturities below one year with results for longer maturities being more sensitive to the model choice. Model 1 yields a premium of 2-3 basis points on average for the contracts with maturities beyond one year, Model 2 produces a premium less than one basis point and Model 3 yields no premium.

The major finding from the table can be summarized as following. The choice of the interval for which the volatility inputs are computed (the model) as well as the choice of the volatility specification (the type) affects the results. Longer intervals taken into consideration for the volatility calculation lead to higher predicted values of the convexity adjustment and result in a larger futures premium. The differences in the estimated futures premium are negligible for

Table 4.4 Futures premium measures as the difference between the observed and the predicted forward-futures price differential, basis points

Panel A: Swiss franc (CHF) sample

1st nearby contract

	Model 1			Model 2			Model 3		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
mean	0.20	0.20	0.20	0.19	0.19	0.19	0.18	0.18	0.18
median	0.18	0.18	0.17	0.16	0.17	0.17	0.16	0.16	0.16
stdev	0.81	0.81	0.81	0.81	0.81	0.81	0.81	0.81	0.81
obs.	1102	1102	1102	1102	1102	1102	1102	1102	1102
bootstrapped 95% confidence interval for the mean:									
l. b.	0.15	0.15	0.15	0.14	0.14	0.14	0.14	0.14	0.14
h. b.	0.25	0.24	0.24	0.24	0.24	0.24	0.24	0.23	0.23

2nd nearby contract

	Model 1			Model 2			Model 3		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
mean	0.54	0.51	0.51	0.47	0.46	0.46	0.45	0.45	0.45
median	0.49	0.46	0.44	0.42	0.42	0.41	0.40	0.40	0.40
stdev	1.07	1.06	1.06	1.06	1.06	1.06	1.06	1.06	1.06
obs.	1102	1102	1102	1102	1102	1102	1102	1102	1102
bootstrapped 95% confidence interval for the mean:									
l. b.	0.47	0.45	0.44	0.41	0.40	0.40	0.39	0.39	0.39
h. b.	0.60	0.57	0.57	0.54	0.53	0.52	0.52	0.51	0.51

3rd nearby contract

	Model 1			Model 2			Model 3		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
mean	0.48	0.37	0.36	0.30	0.26	0.25	0.23	0.22	0.22
median	0.44	0.33	0.31	0.24	0.20	0.20	0.18	0.18	0.17
stdev	0.99	0.98	0.98	0.97	0.97	0.97	0.97	0.97	0.97
obs.	1002	1002	1002	1002	1002	1002	1002	1002	1002
bootstrapped 95% confidence interval for the mean:									
l. b.	0.42	0.32	0.30	0.24	0.19	0.19	0.16	0.16	0.16
h. b.	0.54	0.44	0.42	0.36	0.32	0.31	0.29	0.28	0.27

(Table 4.4 continued)

Panel B: Euro (EUR) sample

1st nearby contract

	Model 1			Model 2			Model 3		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
mean	0.54	0.54	0.54	0.52	0.52	0.52	0.51	0.51	0.51
median	0.45	0.45	0.45	0.43	0.43	0.43	0.43	0.43	0.43
stdev	0.79	0.79	0.79	0.79	0.79	0.80	0.79	0.79	0.79
obs.	1994	1994	1994	2033	2033	2033	2049	2049	2049
bootstrapped 95% confidence interval for the mean:									
l. b.	0.51	0.51	0.51	0.49	0.48	0.49	0.48	0.48	0.48
h. b.	0.58	0.57	0.57	0.55	0.55	0.55	0.55	0.55	0.55

2nd nearby contract

	Model 1			Model 2			Model 3		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
mean	0.77	0.74	0.74	0.65	0.64	0.64	0.63	0.62	0.62
median	0.65	0.62	0.62	0.56	0.55	0.55	0.54	0.54	0.54
stdev	1.07	1.06	1.06	1.06	1.06	1.06	1.07	1.07	1.07
obs.	1994	1994	1994	2033	2033	2033	2049	2049	2049
bootstrapped 95% confidence interval for the mean:									
l. b.	0.73	0.70	0.69	0.61	0.59	0.59	0.58	0.58	0.58
h. b.	0.82	0.79	0.79	0.70	0.69	0.69	0.68	0.67	0.67

3rd nearby contract

	Model 1			Model 2			Model 3		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
mean	0.81	0.68	0.65	0.46	0.41	0.39	0.37	0.35	0.35
median	0.67	0.56	0.54	0.41	0.36	0.35	0.33	0.32	0.32
stdev	1.21	1.16	1.16	1.10	1.09	1.09	1.11	1.11	1.11
obs.	1991	1991	1991	2030	2030	2030	2046	2046	2046
bootstrapped 95% confidence interval for the mean:									
l. b.	0.76	0.64	0.61	0.41	0.36	0.35	0.32	0.30	0.30
h. b.	0.86	0.73	0.70	0.50	0.46	0.44	0.41	0.40	0.39

4th nearby contract

	Model 1			Model 2			Model 3		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
mean	1.11	0.81	0.72	0.41	0.29	0.26	0.15	0.11	0.10
median	1.02	0.71	0.61	0.46	0.35	0.31	0.23	0.19	0.19
stdev	1.22	1.21	1.22	1.17	1.17	1.18	1.22	1.22	1.22
obs.	1942	1942	1942	1965	1965	1965	1976	1976	1976
bootstrapped 95% confidence interval for the mean:									
l. b.	1.06	0.76	0.67	0.36	0.24	0.21	0.10	0.06	0.05
h. b.	1.17	0.87	0.78	0.47	0.34	0.31	0.20	0.16	0.15

(Table 4.4 continued)

5th nearby contract

	Model 1			Model 2			Model 3		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
mean	1.62	1.08	0.91	0.49	0.26	0.19	0.09	0.00	-0.02
median	1.40	0.94	0.79	0.50	0.30	0.25	0.16	0.09	0.07
stdev	1.66	1.41	1.39	1.19	1.15	1.16	1.21	1.20	1.20
obs.	1942	1942	1942	1965	1965	1965	1976	1976	1976
bootstrapped 95% confidence interval for the mean:									
l. b.	1.54	1.02	0.85	0.44	0.21	0.14	0.04	-0.05	-0.08
h. b.	1.69	1.15	0.97	0.54	0.31	0.24	0.15	0.06	0.03

6th nearby contract

	Model 1			Model 2			Model 3		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
mean	2.53	1.68	1.44	0.82	0.42	0.32	0.19	0.00	-0.03
median	1.93	1.33	1.15	0.66	0.37	0.30	0.13	0.02	-0.02
stdev	2.48	1.86	1.75	1.56	1.23	1.21	1.57	1.22	1.22
obs.	1942	1942	1942	1965	1965	1965	1976	1976	1976
bootstrapped 95% confidence interval for the mean:									
l. b.	2.42	1.59	1.36	0.76	0.36	0.27	0.12	-0.05	-0.08
h. b.	2.64	1.76	1.53	0.89	0.47	0.38	0.27	0.05	0.02

7th nearby contract

	Model 1			Model 2			Model 3		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
mean	3.90	2.65	2.43	1.59	0.86	0.77	0.68	0.24	0.20
median	2.64	1.92	1.78	1.05	0.68	0.59	0.35	0.14	0.10
stdev	4.05	2.87	2.72	3.40	1.81	1.74	4.15	1.74	1.70
obs.	1742	1742	1742	1758	1758	1758	1769	1769	1769
bootstrapped 95% confidence interval for the mean:									
l. b.	3.72	2.52	2.31	1.45	0.78	0.70	0.53	0.15	0.13
h. b.	4.10	2.78	2.57	1.77	0.95	0.85	0.99	0.31	0.28

Panel C: British pound (GBP) sample

1st nearby contract

	Model 1			Model 2			Model 3		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
mean	0.62	0.61	0.62	0.57	0.57	0.57	0.56	0.56	0.56
median	0.49	0.48	0.48	0.44	0.44	0.44	0.44	0.44	0.44
stdev	1.12	1.12	1.12	1.09	1.09	1.09	1.09	1.09	1.09
obs.	2086	2086	2086	2125	2125	2125	2139	2139	2139
bootstrapped 95% confidence interval for the mean:									
l. b.	0.58	0.57	0.58	0.53	0.53	0.52	0.51	0.51	0.51
h. b.	0.67	0.66	0.68	0.62	0.61	0.62	0.60	0.61	0.61

(Table 4.4 continued)

2nd nearby contract

	Model 1			Model 2			Model 3		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
mean	0.90	0.81	0.82	0.61	0.59	0.59	0.53	0.52	0.52
median	0.71	0.64	0.66	0.51	0.48	0.49	0.44	0.44	0.44
stdev	1.55	1.51	1.51	1.42	1.42	1.42	1.41	1.41	1.41
obs.	2070	2070	2070	2109	2109	2109	2123	2123	2123
bootstrapped 95% confidence interval for the mean:									
l. b.	0.83	0.75	0.76	0.55	0.52	0.53	0.47	0.46	0.46
h. b.	0.96	0.89	0.89	0.67	0.65	0.65	0.59	0.58	0.58

3rd nearby contract

	Model 1			Model 2			Model 3		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
mean	1.02	0.80	0.76	0.41	0.33	0.32	0.23	0.20	0.20
median	0.77	0.61	0.57	0.38	0.31	0.30	0.22	0.21	0.20
stdev	1.79	1.67	1.64	1.41	1.40	1.39	1.39	1.38	1.38
obs.	1990	1990	1990	2029	2029	2029	2043	2043	2043
bootstrapped 95% confidence interval for the mean:									
l. b.	0.95	0.73	0.68	0.35	0.27	0.26	0.17	0.14	0.14
h. b.	1.10	0.87	0.83	0.47	0.40	0.39	0.29	0.26	0.26

4th nearby contract

	Model 1			Model 2			Model 3		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
mean	1.25	0.86	0.73	0.20	0.04	0.00	-0.13	-0.18	-0.20
median	0.92	0.66	0.58	0.27	0.15	0.12	0.01	-0.04	-0.06
stdev	2.04	1.66	1.57	1.55	1.47	1.47	1.54	1.53	1.52
obs.	1772	1772	1772	1811	1811	1811	1825	1825	1825
bootstrapped 95% confidence interval for the mean:									
l. b.	1.15	0.79	0.66	0.12	-0.03	-0.07	-0.20	-0.26	-0.28
h. b.	1.34	0.94	0.80	0.28	0.11	0.07	-0.06	-0.12	-0.13

5th nearby contract

	Model 1			Model 2			Model 3		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
mean	2.57	2.12	1.80	0.30	0.05	-0.01	-0.32	-0.46	-0.48
median	1.95	1.45	1.20	0.33	0.08	-0.01	-0.22	-0.33	-0.36
stdev	2.92	2.77	2.57	1.79	1.74	1.73	1.73	1.70	1.70
obs.	692	692	692	731	731	731	745	745	745
bootstrapped 95% confidence interval for the mean:									
l. b.	2.35	1.93	1.62	0.18	-0.08	-0.15	-0.46	-0.58	-0.61
h. b.	2.78	2.34	1.99	0.44	0.19	0.12	-0.20	-0.34	-0.36

(Table 4.4 continued)

6th nearby contract

	Model 1			Model 2			Model 3		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
mean	3.79	3.04	2.57	0.92	0.56	0.48	0.12	-0.07	-0.09
median	2.73	2.13	1.88	0.80	0.45	0.40	0.02	-0.12	-0.13
stdev	3.90	3.52	2.97	1.86	1.67	1.65	1.69	1.63	1.62
obs.	469	469	469	508	508	508	522	522	522
bootstrapped 95% confidence interval for the mean:									
l. b.	3.46	2.72	2.29	0.77	0.41	0.33	-0.01	-0.21	-0.23
h. b.	4.18	3.34	2.84	1.08	0.70	0.62	0.27	0.06	0.05

7th nearby contract

	Model 1			Model 2			Model 3		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
mean	3.22	1.84	1.76	1.17	0.18	0.14	-0.26	-0.74	-0.76
median	3.77	2.19	1.97	0.81	0.23	0.23	-0.29	-0.73	-0.72
stdev	4.37	3.79	3.73	3.48	2.82	2.79	2.76	2.52	2.51
obs.	164	164	164	203	203	203	211	211	211
bootstrapped 95% confidence interval for the mean:									
l. b.	2.54	1.24	1.18	0.72	-0.24	-0.23	-0.62	-1.06	-1.09
h. b.	3.89	2.40	2.32	1.65	0.56	0.52	0.12	-0.38	-0.43

shorter maturity durations but increase with time to expiration. The results though are less sensitive to the choice of the volatility specification than to the choice of the sample size for volatility inputs computation. The futures premium values obtained using the exponentially dampened volatility specification do not differ much from those when the volatility is modeled as a function of time to maturity. However, if the volatility evolution is specified as a function of maturity time, the premium values tend to be higher. This can be explained by the fact mentioned earlier, namely, that such volatility specification allows the forward rates with longer maturities to carry their original volatilities, which are normally higher than those for rates at the front end of the spot curve, for the rest of their existence until the implied forward contract matures. Such a setup results in larger predicted values of the convexity adjustment and therefore a higher futures premium.

The obtained conclusion is that the estimation of the predicted convexity adjustment as well as the search for the futures premium can be sensitive to the volatility inputs and the produced results may be biased or inconclusive. The obtained size of the futures premium which ranges from zero to four basis points in this paper depending on the model used is too small and thus economically insignificant in the wake of the possible presence of the asynchronicity and the bid-ask spread. There is no evidence of the consistent mispricing of the major Eurocurrency futures traded on Euronext.LIFFE that was reported for the Eurodollar futures in 1980s.

4.6 Forward Rate Agreement (FRA) and Its Pricing

The caveat of using the implied forward prices for the convexity adjustment and the futures premium analysis is that they have a derived or synthetic nature and, hence, represent non-tradable financial instruments. Therefore, the question of whether the futures are traded at a premium to the respective forwards could be rephrased to whether the futures are priced off the implied forward curve. In order to see whether there exists a futures premium, one has to compare futures to a traded instrument that has all features of the forward contract. Such instrument is the over-the-counter traded forward rate agreement (FRA).

According to Questa (1999), FRAs were first introduced in 1984 and a year later the British Bankers Association (BBA) developed a standard contract for the new market, also known as FRABBA. At the end of 2003, the notional amount of outstanding FRAs on rates for all major currencies totaled the equivalent of $10.8 trillion. Smithson (1998) mentions that London, not New York, dominates the FRA trading, even in the US dollar segment. FRAs are available for a variety of maturities and rates. The non-standard term FRAs are those that are fixed on "broken dates", the so called "IMM dates". These are called the IMM FRAs and they are the most convenient for the analysis of the existence of the interest rate futures premium since such contracts have the very same underlying as the Eurocurrency futures, the three-month LIBOR or EURIBOR interest rate, and they have the same expiration dates as the respective interest rate futures. The IMM FRAs offer the highest liquidity since they have the largest volumes of trading among all other types of FRAs and therefore are regarded as a money-market instrument. The FRA rates are quoted by a number of large banks and these quotations are available electronically via Bloomberg and Reuters.

Apart from the difference in cash flows stemming from daily marking-to-market of the Eurocurrency interest rate futures, when comparing the Eurocurrency futures versus the IMM FRAs for hedging or other purposes, attention must be paid to the contracts pricing and settlement rules. First of all, forward rate agreements are priced and settled differently from the implied forwards considered previously in this paper as well as from the Eurocurrency futures. An FRA is quoted as a rate and it actually does not have a price (which is not to be misled with the contract's value). At the expiration date the FRA is cash settled according to the following formula:

$$\frac{(r - r_{FRA})kN}{1 + kr}, \tag{4.12}$$

where r is the underlying rate at the settlement date (the reference rate, e. g. three-month LIBOR) in decimals,

r_{FRA} is the quoted FRA rate in decimals at the time when the transaction is entered,

k is the adjustment factor which is equal to the number of days in the FRA term (usually comprising three, six or 12 months) divided by the day basis[59],

N is the notional value of the contract.

Note in particular, that the futures settlement implies the maturity of underlying deposit of 90 days while the FRA term is based on the actual number of days in the three months that separate the settlement date and the implied deposit's maturity date, and this number is normally higher than 90. The final difference between the IMM FRA and the Eurocurrency futures is that the former has the two business day lag between the date when the interest rate is quoted (the trade date) and the date when the transaction is entered (the spot date). Similarly, two business days separate the date when the contract expires and the reference rate is set for the settlement purposes (the fixing date) and the date when the cash settlement takes place (the settlement date)[60]. Such conventions of the IMM FRA are in line with Euromarkets standards and similar to those that interest rate swap transactions are based on[61]. On the contrary, the Eurocurrency futures are entered and settled on the same date when the price quote or the settlement quote becomes available. In general, however, even despite certain pricing differences mentioned above, the IMM FRAs and the Eurocurrency interest rate futures are functionally equivalent in terms of hedging interest rate risk.

As can be seen from (4.12), a long position in FRA is a bet on rising interest rates in the future and/or a hedge for a future short position in the underlying interest rate. The principal is not exchanged, it serves as a basis to calculate the settlement payment. Since FRAs comprise no exchange of principal, the credit risk is limited to the amount due at the settlement date which is

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[59] k is usually based on the actual/360 day count rule.

[60] The British pound FRAs are settled at the fixing date, not at the settlement date two business days later, and therefore constitute an exception. The adjustment factor for British pound FRA is based on the actual/365 day count which represents another exception.

[61] As a matter of fact, FRAs are closely related to swaps; the former resembles a one-date LIBOR-in-arrears swap.

a function of the difference between the underlying rate at expiration and the locked-in reference rate.

A number of textbooks devoted to derivatives and fixed income instruments (Smithson [1998], Reverre [2001] and Coyle [2001] among others) argue that the FRA rate must be equal to the respective forward rate. Hull (2006) suggests that the FRA rate is usually set equal to the forward rate when FRA is first initiated. It can be shown using a no-arbitrage argument that the FRA rate is equal to the respective forward rate, i. e.

$$r_{FRA} = \left(\frac{1 + r(t+m)(t+m)}{1 + r(t)t} - 1 \right) \frac{1}{m}, \tag{4.13}$$

where $r(t)$ is the annually compounded spot rate for maturity t which is expressed as a fraction of a year, t is the time to expiration of the FRA and m is the time period the reference rate is for, e. g. three months.

The above pricing of an FRA is based on the no-arbitrage argument. The underlying no-arbitrage argument states that neither of the following two zero-cost transactions, borrow a longer maturity deposit at $r(t+m)$, lend a shorter maturity deposit at $r(t)$ and sell an FRA at r_{FRA}, or lend a longer maturity deposit at $r(t+m)$, borrow a shorter deposit at $r(t)$ and buy an FRA at r_{FRA}, will be able to produce a positive cash flow at time t. The pricing formula (4.13) and its underlying argument are, however, subject to two assumptions. First, the two respective interbank market rates are public information and second, it is possible to freely lend as well as borrow funds at those rates. If one considers the IMM FRAs which are fixed on broken dates, the two respective interbank market rates are normally not available. For example, if the contract expires on the last IMM date of 2008, December 15, and today is October 23, then t is equal to 53 and the sum of t and m is 144. The conventional spot rates for such periods do not exist, thus creating limits to arbitrage. Second, even if the conventional maturities (the ones for Eurodollar deposits) are considered, the actual borrowing and lending will likely be conducted at respective bid and ask rate quotes, where the latter is a mark up over the former. Therefore, in the wake of such limits, the IMM FRA rate may not be obtained using (4.13) but rather must be within the specific range so that that arbitrage is not executable given the quoted bid and ask rates. The size of such range depends on the quoted bid-ask spreads of the respective rates and may in all likelihood reach double digits in basis points. Hence, in the presence of the limits to the standard arbitrage, the quoted FRA rate can deviate from the respectively implied forward rate.

Empirical evidence tends to support the validity of the above proposed argument. Table 4.5 shows how frequently the difference between the quoted FRA rate and the implied forward rate lies in a specific range. The FRA rates are for the contracts with standard constant three-month maturities. The implied forward rates are computed using the respective quoted LIBOR rates as in (4.13). The contract is defined as the NxM type, where N is the number of months until the expiration of the contract and M-N is the maturity of the underlying deposit. Table 4.5 presents results for the subsample of the existing contracts with $M \leq 12$ and $M - N = 3$. The data cover the period from 3/19/2003 to 2/28/2007 for the Swiss franc subsample and the British pound subsample, whereas for the euro subsample the period is from 3/19/2003 to 9/25/2007. The data for FRA quoted rates is obtained from Datastream.

Table 4.5 Percentage of occurrences when the difference between the quoted FRA rate and the implied forward rate lies in a specific range

Panel A: Swiss franc (CHF) sample

Range,	Forward Rate Agreement, type								
b. p.	1x4	2x5	3x6	4x7	5x8	6x9	7x10	8x11	9x12
(-1;1)	36.0	27.5	23.9	-	-	23.3	-	-	17.7
(-2,2)	69.2	53.3	47.4	-	-	43.1	-	-	34.6
(-3;3)	89.3	75.5	67.1	-	-	62.8	-	-	46.6
(-4,4)	95.4	90.3	81.4	-	-	75.2	-	-	59.8
(-5;5)	98.0	95.5	90.7	-	-	85.5	-	-	71.8

Panel B: Euro (EUR) sample

Range,	Forward Rate Agreement, type								
b. p.	1x4	2x5	3x6	4x7	5x8	6x9	7x10	8x11	9x12
(-1;1)	27.4	20.4	18.8	27.4	25.7	25.8	20.1	19.7	20.1
(-2,2)	61.6	48.3	46.6	52.7	48.8	47.3	38.7	39.9	38.5
(-3;3)	87.1	71.5	69.0	70.1	66.8	67.5	57.6	55.3	54.5
(-4,4)	93.5	84.4	82.3	80.2	81.0	78.3	74.4	71.0	69.4
(-5;5)	95.4	92.2	88.8	87.6	87.3	85.1	83.1	82.3	79.9

Panel C: British pound (GBP) sample

Range,	Forward Rate Agreement, type								
b. p.	1x4	2x5	3x6	4x7	5x8	6x9	7x10	8x11	9x12
(-1;1)	41.9	30.9	28.9	25.4	26.6	24.8	19.0	18.8	24.3
(-2,2)	69.8	57.1	54.9	48.5	49.7	49.2	36.8	36.9	43.1
(-3;3)	85.2	75.4	75.0	68.4	66.7	67.1	52.1	52.4	60.8
(-4,4)	92.9	86.7	87.2	81.6	77.9	79.5	68.2	63.8	75.1
(-5;5)	97.2	91.8	92.6	89.5	86.3	86.9	77.4	75.5	83.5

The findings demonstrate that the quoted FRA rate does differ from the implied forward rate obtained from the two respective spot LIBOR rates quoted on the same day[62]. The observed difference can be partially attributed to the presence of the bid-ask spread in the quoted FRA rates which normally constitutes one to three basis points. The difference is more frequently observed and is more pronounced for contracts with longer expiration time, i. e. a higher *N*. Such outcome outlines the presence of the limits to arbitrage and suggests that equation (4.13) can serve only as an approximation of the actual FRA rate. Moreover, the spread between the FRA rate and the respective implied forward rate tends to widen considerably in the times of the credit market distress. Figure 4.1 shows the mean, the maximum and the minimum of the absolute value of the FRA/forward rate spread for nine three-month euro FRA contracts (from 1x4 to 9x12) for the period from 6/26/2007 to 9/26/2007. The spread widened considerably in the middle of August 2007, the time when the financial crisis of 2007-2009 has begun, and stayed at the elevated level for the rest of the sample period.

4.7 Estimation of the Futures/FRA Convexity Adjustment

To estimate the FRA rate in the presence of limits to arbitrage, note that the FRA value must be a martingale, i. e. the expected payoff of entering the FRA transaction under risk neutral expectations must be zero. Given the settlement formula for the FRA contract in (4.12), the zero expected payoff condition is

$$\tilde{E}\left(\frac{r - r_{FRA}}{1 + rk}\right) = 0. \tag{4.14}$$

Since r_{FRA} can be treated as a constant, the above formula implies that the FRA rate must be equal to

$$r_{FRA} = \tilde{E}\left(\frac{r}{1 + rk}\right) / \tilde{E}\left(\frac{1}{1 + rk}\right). \tag{4.15}$$

The convexity adjustment in the IMM FRA market is modeled as following. Since the futures is marked to market daily, its rate is equal to the expected LIBOR rate at the expiration. Therefore, the difference between the Eurocurrency futures rate and the respective IMM FRA quoted rate is equal to

[62] The FRA rates are taken as the closing rates as of 4 pm London time, whereas LIBOR spot rates are published at 11 am London time same day. This asynchronicity in quotes, however, would not prevent from executing the arbitrage have the latter been possible.

$$r_{Fut} - r_{FRA} = \tilde{E}(r) - \tilde{E}\left(\frac{r}{1+rk}\right) / \tilde{E}\left(\frac{1}{1+rk}\right) = -\tilde{\text{cov}}\left(r, \frac{1}{1+rk}\right) / \tilde{E}\left(\frac{1}{1+rk}\right) > 0, \qquad (4.16)$$

since the covariance term is always negative. The expression above is the IMM FRA/futures rate differential, or the convexity adjustment for the IMM FRA market, which is different from the futures-forward rate differential shown in Hull (2006), or that derived by Kirikos and Novak (1997). How much should this differential normally be and what does it depend on? First, the formula says that the differential is independent of the maturity and the current level of interest rate. It is expected though that the differential's size will be an increasing function of the underlying rate's expected level and its volatility.

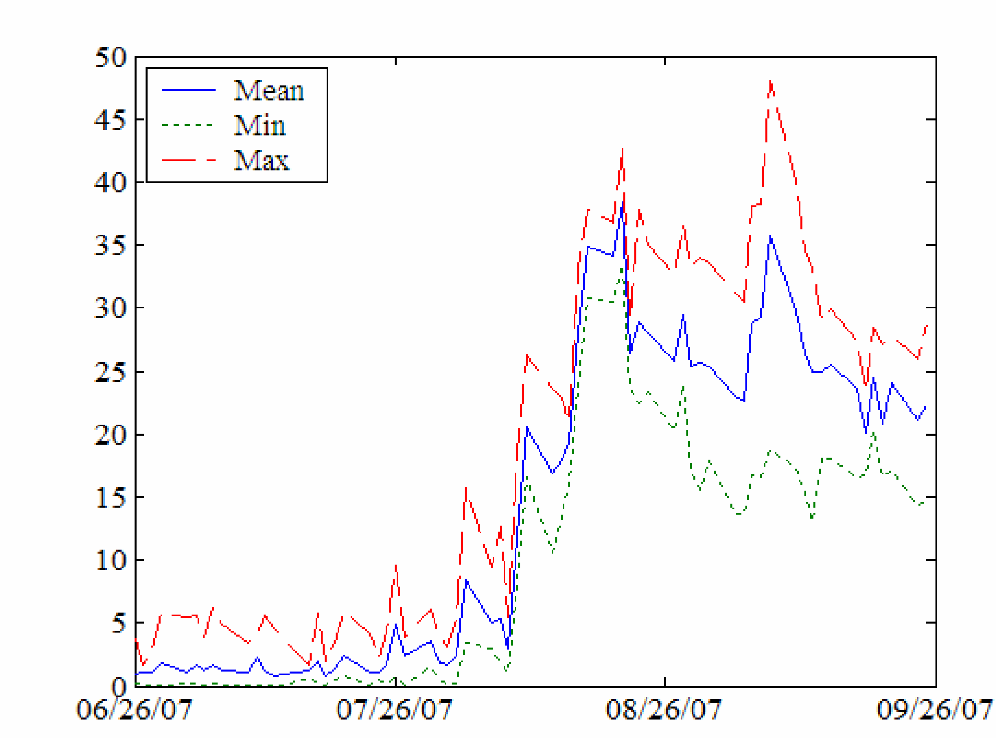

Figure 4.1 Absolute value of the FRA/implied forward rate differential across nine pairs of euro rates for the period from 6/26/2007 to 9/26/2007, basis points

Table 4.6 presents results of the predicted value of the futures/FRA convexity adjustment for all three samples of rates used for the futures premium analysis when arbitrage is not possible to execute. The results are shown for all three volatility specifications described in section 4.3. The initial estimates of the forward rate volatilities are obtained by computing daily standard deviations using a historical sample of rates over the last month. This method is supposed to yield more reliable results than the one where the standard deviations are computed using the

Table 4.6 Predicted values of the futures/FRA convexity adjustment and the futures/forward convexity adjustment, basis points

Type 1 is based on the volatility specification where volatility is a function of maturity time only. Type 2 is based on the volatility specification where volatility is a function of time to maturity. Type 3 is based on the exponentially dampened volatility specification. Initial volatility estimates are computed as the weekly standard deviations using Model 2 (range of rates for the last 21 business days including the current day). Means and standard deviations are presented.

Panel A: Swiss franc (CHF) sample

Futures/FRA convexity adjustment given limits of arbitrage

	Type 1		Type 2		Type 3	
maturity	mean	stdev	mean	stdev	mean	stdev
13 weeks	0.03	0.06	0.03	0.05	0.03	0.05
26 weeks	0.13	0.18	0.08	0.12	0.07	0.11
39 weeks	0.34	0.42	0.17	0.22	0.16	0.22

Futures/forward convexity adjustment if arbitrage is executable:

	Type 1		Type 2		Type 3	
maturity	mean	stdev	mean	stdev	mean	stdev
13 weeks	0.05	0.07	0.04	0.07	0.04	0.07
26 weeks	0.22	0.29	0.14	0.21	0.12	0.21
39 weeks	0.65	0.83	0.35	0.48	0.34	0.48

Panel B: Euro sample

Futures/FRA convexity adjustment given limits of arbitrage:

	Type 1		Type 2		Type 3	
maturity	mean	stdev	mean	stdev	mean	stdev
13 weeks	0.13	0.56	0.13	0.55	0.11	0.49
26 weeks	0.33	1.13	0.27	1.08	0.25	1.07
39 weeks	0.61	1.71	0.46	1.67	0.42	1.69
52 weeks	1.03	2.48	0.69	2.18	0.63	2.32
65 weeks	1.67	3.88	0.98	2.85	0.91	3.10
78 weeks	2.83	6.92	1.38	3.74	1.31	4.05
91 weeks	5.12	13.93	1.96	5.02	1.91	5.19

Futures/forward convexity adjustment if arbitrage is executable:

	Type 1		Type 2		Type 3	
maturity	mean	stdev	mean	stdev	mean	stdev
13 weeks	0.19	0.82	0.18	0.81	0.17	0.71
26 weeks	0.60	2.16	0.52	2.15	0.48	2.02
39 weeks	1.36	4.09	1.08	4.06	1.00	4.00
52 weeks	2.64	7.03	1.91	6.55	1.78	6.75
65 weeks	4.76	11.90	3.14	9.95	2.95	10.48
78 weeks	8.45	20.78	4.93	14.65	4.69	15.37
91 weeks	15.44	38.92	7.60	21.43	7.34	21.72

(Table 4.6 continued)

Panel C: British pound (GBP) sample

Futures/FRA convexity adjustment given limits of arbitrage

	Type 1		Type 2		Type 3	
maturity	mean	stdev	mean	stdev	mean	stdev
13 weeks	0.11	0.17	0.09	0.15	0.10	0.15
26 weeks	0.32	0.39	0.23	0.33	0.22	0.31
39 weeks	0.63	0.70	0.42	0.53	0.37	0.48
52 weeks	0.97	1.05	0.65	0.75	0.54	0.66
65 weeks	1.30	1.47	0.91	1.01	0.77	0.89
78 weeks	1.71	1.97	1.18	1.31	1.06	1.18
91 weeks	2.51	2.78	1.50	1.64	1.44	1.57

Futures/forward convexity adjustment if arbitrage is executable

	Type 1		Type 2		Type 3	
maturity	mean	stdev	mean	stdev	mean	stdev
13 weeks	0.15	0.24	0.14	0.22	0.15	0.22
26 weeks	0.57	0.72	0.43	0.62	0.43	0.60
39 weeks	1.37	1.53	0.96	1.22	0.87	1.14
52 weeks	2.52	2.73	1.77	2.06	1.54	1.89
65 weeks	4.01	4.38	2.85	3.18	2.50	2.92
78 weeks	6.01	6.66	4.24	4.63	3.86	4.32
91 weeks	9.28	9.97	6.05	6.50	5.79	6.29

rates over the last three months since the latter is more prone to the bias in the estimated volatility numbers caused by the interpolation error. For comparison purposes, Table 4.6 also shows respective statistics for the futures/FRA convexity adjustment when there are no limits to arbitrage, which represents the standard futures/forward rate differential.

Two major results are obtained from the table. First, for all three samples, the conventional futures/forward convexity adjustment exceeds the FRA/futures convexity adjustment when there are limits to arbitrage and the discrepancy between those two becomes more evident with the maturity. Second, the average size of the futures/FRA convexity adjustment is extremely low: it does not exceed one basis point for maturities less than a year and is between one and five basis points, depending on the model used, for maturities below two years. Hence, it would be very hard if not impossible to detect the presence of the convexity adjustment in the IMM FRA quotes empirically given the limits of arbitrage and minor data imperfections such as the asynchronicity and the presence of a bid-ask spread in the quoted rates. It would also be reasonable to assume that with such low values of the convexity adjustment in the IMM FRA quotes, in order to cover for their costs, the market makers' offer rate will be a mark up over, and the IMM FRA bid quote will be a mark down from the corresponding Eurocurrency interest rate futures contract. Hence, the IMM FRAs are more likely to be priced off the respective Eurocurrency futures rather than the implied forward curve. Such scenario would make the empirical estimation of the convexity adjustment on the IMM FRA market look like a search for a needle in a haystack. To summarize, this section provides evidence that despite all shortcomings that arise when one is performing the analysis of the convexity adjustment using the implied forward rates, the replacement of the latter by the IMM FRA quotes for the purposes of the analysis is not expected to yield more credibility to the obtained findings. The results would most likely be inconclusive and subject to an error.

4.8 Summary and Conclusions

This paper contributes to the literature by extending the previous research on the convexity adjustment in interest rate futures in two major ways. First, a matching technique is utilized to determine the presence of the convexity adjustment in the futures quotes on each individual trading day. This method helps to identify whether the observed futures prices tend to be overestimated which would suggest the presence of the robust futures premium. The analysis is performed for the three most traded interest rate futures contracts in Europe: EURIBOR futures, short sterling futures and Euroswiss futures. The results suggest that the futures premium is barely present for the contracts with maturities less than one year as its size is likely to be a fraction of a basis point. The innovative feature of the analysis is that it was extended for the EURIBOR futures and the short sterling futures for maturities beyond nine months by using the quoted swap rates. The size of the futures premium for maturities above twelve months is shown to be a function of the model assumptions regarding the volatility parameters and their evolution.

The second contribution of the paper is in dismissing the recent claims that the use of the rates on the forward rate agreements that have same expiration dates as the respective Eurocurrency futures (IMM FRAs) instead of the implied forward rates is supposed to yield more meaningful findings due to the fact that the IMM FRAs are the actually traded forward contracts. The paper extends the statement that can be met in a number of textbooks on derivatives and fixed income that the rate on a forward rate agreement contract is a function of the current term structure and is equal to the implied forward rate. This is not the case in the presence of the limits to arbitrage as the two rates would differ and their difference increases with the maturity.

The fact that the IMM FRA rate is supposed to exceed the implied forward rate and be close to the respective futures rate given the limits of execution of the standard arbitrage allows to explain why the recent research has not been able to detect the presence of the convexity adjustment in the futures relative to IMM FRA market. The paper shows that the whole idea of detecting the empirical presence of the convexity adjustment in the IMM FRA market or lack of thereof is questionable to implement due to the illusory nature of the subject under consideration whose size is supposed to be within the limits of the bid-ask spread. Therefore, the claims that the convexity adjustment has not been empirically priced into IMM FRA quotes lack substance and are subject to revision.

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Chapter 5 Conclusions

This dissertation explores the issues surrounding the nature of the interest rate forward-futures differential, also known as the convexity adjustment. The existence of the convexity adjustment has been justified in the theoretical literature but its empirical presence has been inconclusive and subject to a number of caveats embedded in the analysis such as the asynchronicity, the derived nature of the implied forwards, the Eurocurrency futures pricing design among others. This research work that consists of three essays focuses on looking at the major factors and imperfections that can be met while investigating the size and the nature of the convexity adjustment by incorporating the former and eliminating the latter from the analysis.

In the first part of my dissertation I investigate the magnitude of the forward-futures differential, also known as the convexity adjustment, for Eurodollar interest rate instruments and attempt to identify factors affecting its size. I show that, opposite to theoretical implication, the magnitude of the forward-futures rate differential is much smaller than what was expected, and that its sign is negative on many occasions. I further check for potential data skews and other imperfections that may be behind the obtained results and find that neither asynchronicity bias, nor the unconventional feature of the Eurodollar futures pricing can explain the observed phenomena. The term structure interpolation error and the two business day lag between the fixing (settlement) date and the transaction (value) date to which the implied forward rates and prices are applied cannot be attributed to the observed abnormality either. I also find that if the regression analysis is conducted for the price differential instead of that for the rate differential, it results in a much better goodness-of-fit. However, the negative nature of the differential is not captured by the default factor proxied by the TED spread and remains largely unexplained.

The second essay takes a closer look at the relationship between Eurocurrency interest rate futures prices and forward prices by focusing on the way the Eurocurrency futures settlement procedure affects the forward-futures differential analysis. Sundaresan (1991) argues that the implied forward price from the spot LIBOR term structure is inappropriate for the purposes of comparison with the Eurodollar futures price due to the differences in settlement procedures and introduces a hypothetical forward contract in order to eliminate the presence of the settlement factor. I show that there is no need to come up with hypothetical contracts in order to compare forward prices with those of futures by demonstrating that the difference between the implied forward price obtained from the spot rate term structure and the original Eurodollar futures price at any point of time before maturity is composed of two parts: the element due to marking-to-market and the element arisen from the unconventional settlement of the Eurocurrency futures. The essay further investigates how changing frequency of the marking-to-market may affect the size of its component within the HJM term structure framework. The finding is that the three-month step in the discrete HJM model is quite appropriate since the results with monthly and weekly steps do not lead to significant improvements. Finally, the marking-to-market effect appears to be negligible for maturities of 12 months or less but its size is expected to grow exponentially with the time to maturity. It is also demonstrated that the discrepancy between the forward price and the futures price arisen from the unconventional settlement of the Eurocurrency futures can be hedged using a specific basket of caplets.

The third essay performs the analysis of the convexity adjustment for the three most traded interest rate futures contracts in Europe: EURIBOR futures, short sterling futures and Euroswiss franc futures. I show that, opposite to the earlier studies that conclude about mispricing of the futures, the futures premium is barely detectible for the three contracts, especially for maturities below one year. The futures premium for maturities above twelve months varies across the models and is a subject to model assumptions regarding the volatility input and its evolution. I also extend the statement that can be met in a number of textbooks on derivatives and fixed income that the rate on a forward rate agreement (FRA) contract is a function of the current term structure and is equal to the implied forward rate. This is not the case in the presence of the limits to arbitrage as the two rates would differ and their difference increases with the maturity. This finding allows to challenge the results in recently published works that argue that the convexity adjustment is not priced in by the IMM FRA market makers. On the contrary, the theoretically predicted FRA/futures rate difference is shown to be indistinguishable and therefore would be hard to detect empirically given the presence of asynchronicity and the bid-ask spread in the reported quote data.

Vita

Andrei Shynkevich was born in Minsk, Belarus. In Belarus he earned Bachelor of Science degree in finance from Belarusian State Economic University in 2001. In 2003 Mr. Shynkevich earned Master of Arts in economics from Economic Education and Research Consortium (now Kyiv School of Economics) in Kyiv, Ukraine. Upon the graduation he joined the doctoral program in economics at State University of New York (SUNY) at Stony Brook where he earned Master of Arts in economics as an intermediate degree in 2005. Mr. Shynkevich subsequently transferred to the doctoral program in finance at Louisiana State University where he earned Master of Science in the major discipline in 2008. He expects to earn his Doctor of Philosophy in business administration with a concentration in finance in August 2009.

During his program study at LSU, Mr. Shynkevich taught two different courses at the undergraduate level and served as a teaching assistant for several MBA courses. His research interests are focused on fixed income markets, derivative pricing, risk management applications and investments. He presented his paper at the national annual meeting of the Financial Management Association in Dallas in 2008.

Recently, Mr. Shynkevich has accepted an offer of a tenure-track faculty position at the Department of Finance at Kent State University.