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## On Infinite Stochastic and Related Matrices

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## ON INFINITE STOCHASTIC AND RELATED MATRICES

ANDREAS BOUKAS, PHILIP FEINSILVER, AND ANARGYROS FELLOURIS

**ABSTRACT.** We study the Lie structure of the set of infinite matrices associated with bounded operators on  $\ell_\infty$  with the property that their row sums are constant. That includes, in particular, infinite row stochastic and zero-row-sum matrices. We also consider the compactness of these operators as related to the Krein-Rutman theorem, we discuss their Abel limits and we consider their connection to the convergence of Markov chains as well as to sequence transformations and generalized limits.

### 1. Introduction

Infinite matrices appear in many areas of mathematics and physics such as functional equations [21], special functions [25, 26] and quantum mechanics [14]. In fact, Heisenberg's matrix mechanics was formulated in terms of them. They also appear in the solution of infinite linear systems of equations [23] and in probability theory in the form of infinite transition matrices for countable state Markov chains [19, 27].

One can add, subtract and multiply by a scalar, infinite matrices just like the more familiar finite matrices. The main difference between finite and infinite matrices lies in the fact that in order for the product of two infinite matrices to be defined the entrywise infinite sums must converge. However, even if the products involved are defined, associativity of multiplication may not hold. Similarly, the inverse of an infinite matrix, even if it exists, may not be unique. For a review of the analysis of infinite matrices we refer to [6], [11] and [19].

Most of the problems mentioned above are by-passed when dealing with infinite matrices corresponding to bounded linear operators on a Banach space. In this paper we study the structure of the set of infinite row stochastic and, in general, constant row sum, matrices associated with bounded operators on  $\ell_\infty$ .

We summarize the contents of the paper. After a section introducing our notations, Section 3 presents the basic facts about infinite matrices as bounded operators on  $\ell_\infty$ . We continue in Section 4 to examine the Lie structure of the ring of infinite matrices with constant row sums. In particular, the commutator algebra is identified as the set of matrices with all rows summing to zero. Some of the ideal

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structure is studied as well. Section 5 reviews properties of compact operators, especially with regard to the Krein-Rutman theorem, the infinite-dimensional analog of Perron-Frobenius. In Section 6 we review the main theorems on summability theory and see how they appear in our context. Section 7 concludes the paper with Abel convergence of powers of a stochastic matrix in the infinite-dimensional case. Some examples are given in the context of countable state Markov chains.

This paper is an infinite dimensional extension of [4] which dealt with the Lie structure and other properties of generalized stochastic and zero-sum finite dimensional matrices.

## 2. Notations and Conventions

For quick reference we list here the notation introduced and used in the subsequent sections of this paper. We will work over  $\mathbb{R}$  throughout.

The main spaces we will be working with are:

$\ell_\infty$ : The Banach space of real sequences  $x = (x_i)_{i \in \mathbb{N}}$  with

$$\|x\|_\infty := \sup_{i \in \mathbb{N}} |x_i| < \infty .$$

$\ell_1$ : The Banach space of absolutely summable real sequences  $x = (x_i)_{i \in \mathbb{N}}$  with

$$\|x\|_1 := \sum_{i \in \mathbb{N}} |x_i| < \infty .$$

We will typically think of  $\ell_1$  sequences as row infinite vectors and of  $\ell_\infty$  sequences as column infinite vectors.

Our main subject of interest in this paper are infinite matrices related to the above sequence spaces. Unless otherwise explicitly mentioned **the rows of all matrices considered are assumed to be in  $\ell_1$ , i.e. absolutely summable.**

We now indicate the families of matrices we will be working with:

$B(\ell_\infty)$ : The Banach space of bounded linear operators

$$A : \ell_\infty \rightarrow \ell_\infty .$$

$B(\ell_1)$ : The Banach space of bounded linear operators

$$A : \ell_1 \rightarrow \ell_1 .$$

$B_\infty$ : The elements of  $B(\ell_\infty)$  given by infinite matrices, i.e., rows are in  $\ell_1$ , columns in  $\ell_\infty$ , with the  $\ell_1$  norms of the rows uniformly bounded.

$S_\lambda$ : Infinite matrices in  $B_\infty$  with row sum equal to  $\lambda \in \mathbb{R}$ .

$S = \bigcup_{\lambda \in \mathbb{R}} S_\lambda \subseteq B_\infty$ : Constant row sum infinite matrices.

$S_0$ : The vector space of *zero row sum* infinite matrices.

$S_1$ : The vector space of *generalized row stochastic* infinite matrices.

$S_1^+$ : The set of *row stochastic* infinite matrices, i.e., the elements of  $S_1$  that have nonnegative entries.

$J_0$ : The infinite matrix with all entries of the first column equal to 1 and all other entries equal to 0.

$\mathcal{G}$ : The set of invertible elements of  $S$ .

$\hat{S}_1$ : The set of invertible elements of  $S_1$ .

$\cdot^\top$  will denote the transpose of a matrix or infinite vector.

For  $\epsilon > 0$  and  $x \in \ell_\infty$  define the  $\epsilon$ -ball  $B(x, \epsilon) := \{y \in \ell_\infty : \|y - x\|_\infty < \epsilon\}$ .

Note that  $A \in B_\infty$  is in  $S_\lambda$  if and only if

$$AJ_0 = \lambda J_0 .$$

Moreover, a constant row sum matrix is a scalar multiple of a generalized row stochastic matrix and a zero row sum matrix is a generalized row stochastic matrix minus the identity.

### 3. Infinite Matrices as Bounded Operators on Sequence Spaces

A natural extension of the finite dimensional correspondence between linear operators and matrices to the infinite dimensional case is provided by linear mappings  $A : \ell_\infty \rightarrow \ell_\infty$  of the form [16]

$$Ax = ((Ax)_i)_{i \in \mathbb{N}} , \quad (3.1)$$

where

$$(Ax)_i = \sum_{j=1}^{\infty} a_{ij} x_j ; a_{ij} \in \mathbb{R} .$$

Such mappings are thus represented by infinite matrices

$$\hat{A} = (a_{ij}) .$$

In order for  $(Ax)_i$  to be finite and for  $Ax$  to be in  $\ell_\infty$  we require

$$\|\hat{A}\|_\infty := \sup_{i \in \mathbb{N}} \sum_{j=1}^{\infty} |a_{ij}| < \infty . \quad (3.2)$$

Notice that (3.2) implies

$$\sup_{(i,j) \in \mathbb{N}^2} |a_{ij}| < \infty .$$

We point out that even though an infinite matrix of the form (3.1) satisfying (3.2) defines a bounded linear operator on  $\ell_\infty$  the converse is not true, i.e., not every bounded linear operator on  $\ell_\infty$  is of that form [24].

**Proposition 3.1.** *For each infinite matrix  $\hat{A} = (a_{ij})$  satisfying (3.2) and associated through (3.1) with a bounded linear operator  $A$  on  $\ell_\infty$  we have*

$$\|\hat{A}\|_\infty = \|A\| ,$$

where  $\|\cdot\|$  denotes the usual operator norm.

*Proof.* We assume that  $A$  and  $\hat{A}$  are nonzero, otherwise the result is trivial. For  $x \in \ell_\infty$

$$\begin{aligned} \|Ax\|_\infty &= \sup_{i \in \mathbb{N}} |(Ax)_i| = \sup_{i \in \mathbb{N}} \left| \sum_{j=1}^{\infty} a_{ij} x_j \right| \leq \sup_{i \in \mathbb{N}} \sum_{j=1}^{\infty} |a_{ij} x_j| \\ &\leq \sup_{j \in \mathbb{N}} |x_j| \sup_{i \in \mathbb{N}} \sum_{j=1}^{\infty} |a_{ij}| = \|\hat{A}\| \|x\|_\infty . \end{aligned}$$

Thus

$$\|A\| \leq \|\hat{A}\|_\infty .$$

Now let  $\epsilon > 0$  be given. Pick a row  $(a_{i_01}, a_{i_02}, \dots)$  of  $\hat{A}$  with  $\ell_1$  norm greater than  $\|\hat{A}\|_\infty - \epsilon$  and define a vector

$$x_0 := (x_{01}, x_{02}, \dots) \in \ell_\infty ; \|x_0\|_\infty = 1 ,$$

by

$$x_{0j} := \operatorname{sgn} a_{i_0j} ,$$

noting that  $x_0$  depends on  $\epsilon$ . Then

$$\begin{aligned} \|Ax_0\|_\infty &= \sup_{i \in \mathbb{N}} |(Ax_0)_i| = \sup_{i \in \mathbb{N}} \left| \sum_{j=1}^{\infty} a_{ij} x_{0j} \right| \geq \left| \sum_{j=1}^{\infty} a_{i_0j} x_{0j} \right| \\ &= \left| \sum_{j=1}^{\infty} a_{i_0j} \operatorname{sgn} a_{i_0j} \right| = \sum_{j=1}^{\infty} |a_{i_0j}| = \sum_{j=1}^{\infty} |a_{i_0j}| \cdot 1 = \sum_{j=1}^{\infty} |a_{i_0j}| \cdot \|x_0\|_\infty , \end{aligned}$$

so

$$\|Ax_0\|_\infty \geq \sum_{j=1}^{\infty} |a_{i_0j}| \cdot \|x_0\|_\infty > (\|\hat{A}\|_\infty - \epsilon) \cdot \|x_0\|_\infty .$$

That is,

$$\|\hat{A}\|_\infty - \epsilon < \|Ax_0\|_\infty \leq \|A\| ,$$

by definition of the operator norm. Since  $\epsilon$  was arbitrary, we have

$$\|\hat{A}\|_\infty \leq \|A\| .$$

As shown above,

$$\|A\| \leq \|\hat{A}\|_\infty ,$$

so

$$\|A\| = \|\hat{A}\|_\infty .$$

□

*Remark 3.2.* The all-ones infinite matrix,  $J$ , does not play a useful role here as it does in the finite-dimensional case. It does not satisfy (3.2), also see, e.g., [17], and  $J$  does not correspond to a bounded linear operator on  $\ell_\infty$ . However,  $J_0$  satisfies

$$\|J_0\|_\infty = 1$$

and since, in the notation of (3.2),  $a_{ij} = 1$  for  $j = 1$  and  $a_{ij} = 0$  for  $j \neq 1$ ,  $J_0$  corresponds to the bounded linear operator on  $\ell_\infty$  defined by

$$J_0(x_1, x_2, x_3, \dots)^\top = (x_1, x_1, x_1, \dots)^\top .$$

The operator norm  $\|J_0\| = 1$ , since for  $x = (x_1, x_2, x_3, \dots)^\top \in \ell_\infty$ ,

$$\|J_0x\| = |x_1| \leq 1 \cdot \|x\|_\infty ,$$

with equality achieved for

$$x = (1, 0, 0, \dots)^\top .$$

Our first task is to verify that the operators given by matrices form a closed subalgebra of  $B(\ell_\infty)$ .

**Proposition 3.3.**  $B_\infty$  is a unital Banach subalgebra of  $B(\ell_\infty)$ .

*Proof.* First note that  $B_\infty$  is closed under addition and scalar multiplication. Let  $\hat{A} = (a_{ij})$  and  $\hat{B} = (b_{ij})$  be two elements of  $B_\infty$ . Clearly  $\hat{A} + c\hat{B}$  is in  $B_\infty$  for all  $c \in \mathbb{R}$ . To see that  $\hat{A}\hat{B}$  is also in  $B_\infty$  we notice that, by Fubini's theorem,

$$\begin{aligned} \sup_{i \in \mathbb{N}} \sum_{j=1}^{\infty} \left| \sum_{k=1}^{\infty} a_{ik} b_{kj} \right| &\leq \sup_{i \in \mathbb{N}} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |a_{ik} b_{kj}| = \sup_{i \in \mathbb{N}} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |a_{ik} b_{kj}| \\ &= \sup_{i \in \mathbb{N}} \sum_{k=1}^{\infty} |a_{ik}| \left( \sum_{j=1}^{\infty} |b_{kj}| \right) \leq \sup_{k \in \mathbb{N}} \sum_{j=1}^{\infty} |b_{kj}| \sup_{i \in \mathbb{N}} \sum_{k=1}^{\infty} |a_{ik}| \\ &= \|\hat{A}\|_\infty \|\hat{B}\|_\infty < \infty, \end{aligned}$$

proving also the sub-multiplicativity of the norm. To show that infinite matrix multiplication is associative in this case, it suffices to show that the matrix  $\hat{A}\hat{B}$  corresponding to the operator  $AB$  is  $\hat{A}\hat{B}$ . Associativity will then follow from the fact that operator composition is associative. We have, noting that for  $x \in \ell_\infty$ ,  $Bx \in \ell_\infty$ ,

$$\begin{aligned} ((AB)x)_i &= (A(Bx))_i = \sum_{j=1}^{\infty} a_{ij} (Bx)_j = \sum_{j=1}^{\infty} a_{ij} \sum_{k=1}^{\infty} b_{jk} x_k \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} b_{jk} x_k = \sum_{k=1}^{\infty} (\hat{A}\hat{B})_{ik} x_k, \end{aligned}$$

with the interchange of summation order justified as in the above steps for  $AB$ .

The infinite identity matrix has  $\|\cdot\|_\infty$  norm one so  $B_\infty$  is unital. We will show that it is complete.

If  $(A_n)$  is an operator norm convergent sequence in  $B_\infty$  and  $A$  is its limit, then  $(A_n)$  is Cauchy so, for  $n, m \geq n_0$

$$\|A_n - A_m\| = \|\hat{A}_n - \hat{A}_m\|_\infty = \sup_{i \in \mathbb{N}} \sum_{j=1}^{\infty} |a_{n,ij} - a_{m,ij}| < \epsilon.$$

Then, for each  $(i, j)$ , for  $n, m \geq n_0$

$$|a_{n,ij} - a_{m,ij}| < \epsilon,$$

so  $(a_{n,ij})$  is Cauchy. Let  $a_{ij}$  be its limit. Then, using (3.1) we may define a linear operator  $A'$  on  $\ell_\infty$  with associated matrix  $\hat{A}' = (a_{ij})$ . We will show that  $A'$  satisfies (3.2), so it is in  $B_\infty$ , and that  $(\hat{A}_n)$  converges in  $\|\cdot\|_\infty$  to  $\hat{A}'$ . Then

$$\lim_{n \rightarrow \infty} \|A_n - A'\| = \lim_{n \rightarrow \infty} \|\hat{A}_n - \hat{A}'\|_\infty = 0$$

will imply that

$$A = A'$$

so  $A \in B_\infty$ .

For arbitrary  $N \in \{1, 2, \dots\}$  we have

$$\begin{aligned} \sum_{j=1}^N |a_{ij}| &\leq \sum_{j=1}^N |a_{ij} - a_{n_0,ij}| + \sum_{j=1}^N |a_{n_0,ij}| \\ &= \lim_{n \rightarrow \infty} \sum_{j=1}^N |a_{n,ij} - a_{n_0,ij}| + \sum_{j=1}^N |a_{n_0,ij}| \\ &\leq \lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} |a_{n,ij} - a_{n_0,ij}| + \sum_{j=1}^{\infty} |a_{n_0,ij}| < \epsilon + \sum_{j=1}^{\infty} |a_{n_0,ij}|. \end{aligned}$$

Since  $N$  is arbitrary, we have

$$\sum_{j=1}^{\infty} |a_{ij}| < \epsilon + \sum_{j=1}^{\infty} |a_{n_0,ij}|.$$

Taking supremum over  $i$  of both sides we conclude that

$$\|\hat{A}'\|_{\infty} \leq \epsilon + \|\hat{A}_{n_0}\|_{\infty} < \infty,$$

so  $A'$  is in  $B_{\infty}$ . Finally, for arbitrary  $N \in \{1, 2, \dots\}$  and  $n \geq n_0$  we have

$$\sum_{j=1}^N |a_{n,ij} - a_{ij}| = \lim_{m \rightarrow \infty} \sum_{j=1}^N |a_{n,ij} - a_{m,ij}| \leq \lim_{m \rightarrow \infty} \sum_{j=1}^{\infty} |a_{n,ij} - a_{m,ij}| < \epsilon$$

since  $m \rightarrow \infty$  implies  $m > n_0$  also. Thus, by the arbitrariness of  $N$ ,

$$\sum_{j=1}^{\infty} |a_{n,ij} - a_{ij}| \leq \epsilon$$

for all  $i$ , thus

$$\sup_{i \in \mathbb{N}} \sum_{j=1}^{\infty} |a_{n,ij} - a_{ij}| \leq \epsilon,$$

so

$$\lim_{n \rightarrow \infty} \|\hat{A}_n - \hat{A}'\|_{\infty} = 0.$$

Therefore, as described above,  $B_{\infty}$  is topologically closed.  $\square$

We have as well

**Proposition 3.4.** *Acting on the right, on row vectors in  $\ell_1$ , matrices in  $B_{\infty}$  are bounded linear operators on  $\ell_1$ , i.e., elements of  $B(\ell_1)$ .*

*Proof.* For  $x = (x_1, x_2, \dots) \in \ell_1$ ,  $A \in B_\infty$  with corresponding infinite matrix  $\hat{A} = (a_{ij})$ , for  $xA$  we have

$$\begin{aligned} \|xA\|_1 &= \sum_{j=1}^{\infty} \left| \sum_{i=1}^{\infty} x_i a_{ij} \right| \leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |x_i a_{ij}| = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |x_i a_{ij}| \\ &= \sum_{i=1}^{\infty} \left( |x_i| \sum_{j=1}^{\infty} |a_{ij}| \right) \leq \left( \sup_{i \in \mathbb{N}} \sum_{j=1}^{\infty} |a_{ij}| \right) \left( \sum_{i=1}^{\infty} |x_i| \right) \\ &= \|x\|_1 \|\hat{A}\|_\infty, \end{aligned}$$

where we have used Fubini's theorem to interchange the order of summation.  $\square$

*Remark 3.5.* Allen [1], discusses groups of infinite matrices acting on classical sequence spaces.

#### 4. Lie Structure of Infinite Constant Row Sum Matrices

We now consider those elements of  $B_\infty$  with constant row sum,  $S$ . In addition to the associated Lie structures we make some observations about maximal ideals considering the ring structure of  $S$ .

For the multiplicative group structure, we consider the invertible elements of  $S$ . These are the infinite matrices  $\hat{A} \in S$  for which there exists an infinite matrix  $\hat{B} \in S$  such that  $\hat{A}\hat{B} = \hat{B}\hat{A} = I$  where  $I$  is the infinite identity matrix. Such an inverse matrix  $\hat{B}$  exists if the bounded operator  $A$  corresponding to  $\hat{A}$  is invertible and its operator inverse  $A^{-1}$  is in  $B_\infty$ . The boundedness of the inverse follows from the fact that  $\ell_\infty$  is a Banach space. The infinite matrix associated with  $A^{-1}$  is  $\hat{B}$ .

**Lemma 4.1.** *If  $\hat{A}$  and  $\hat{B}$  are infinite matrices of finite  $\|\cdot\|_\infty$ , then*

$$(\hat{A}\hat{B})J_0 = \hat{A}(\hat{B}J_0),$$

where  $J_0$  is the infinite matrix with all-ones in the first column and zeros everywhere else.

*Proof.* The  $i$ 1-th entry of both  $(\hat{A}\hat{B})J_0$  and  $\hat{A}(\hat{B}J_0)$  is

$$\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} a_{ik} b_{km},$$

where the double sum converges absolutely, as the proof of in Proposition 4.2,

$$\left| \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} a_{ik} b_{km} \right| \leq \|\hat{A}\|_\infty \|\hat{B}\|_\infty < \infty.$$

$\square$

**Proposition 4.2.** *With the usual infinite matrix multiplication,  $S$  is a unital Banach algebra. Moreover,  $\mathcal{G}$  is a Banach Lie group.*



*Proof.* Lemma 4.1 shows that  $S$  is closed under multiplication. For, if  $\hat{A} \in S_\lambda$  and  $\hat{B} \in S_\mu$ , then

$$(AB)J_0 = A(\mu J_0) = \lambda\mu J_0$$

so that  $S_\lambda S_\mu \subseteq S_{\lambda\mu}$ .

We already know that  $B_\infty$  is a Banach algebra. We will show that  $S$  is closed in  $B_\infty$ . Let  $\hat{A}_n = (a_{ij}(n))$ ,  $n \in \mathbb{N}$ , be a sequence in  $S$  norm convergent to an element  $\hat{A} = (a_{ij}) \in B_\infty$ . We will show that  $\hat{A} \in S$ . Let  $\epsilon > 0$ . Since each  $\hat{A}_n$  is in some  $S_{\lambda_n} \subseteq S$ , we have

$$\|A_n J_0 - A_m J_0\| = |\lambda_n - \lambda_m| \|J_0\| \leq \|A_n - A_m\| \|J_0\|$$

or, since  $\|J_0\| = 1$ ,

$$|\lambda_n - \lambda_m| \leq \|A_n - A_m\| ,$$

which is bounded above by any given  $\epsilon > 0$  for all sufficiently large  $n$  and  $m$ . Thus  $(\lambda_n)$  is also Cauchy thus convergent to some  $\lambda \in \mathbb{R}$ . Then

$$\left| \sum_{k=1}^{\infty} a_{ik} - \lambda \right| \leq \left| \sum_{k=1}^{\infty} a_{ik} - \sum_{k=1}^{\infty} a_{ik}(n) \right| + |\lambda_n - \lambda| .$$

Choose  $n_0$  large enough so that  $\max\{\|\hat{A}_n - \hat{A}\|_\infty, |\lambda_n - \lambda|\} < \epsilon/2$  for all  $n \geq n_0$ . Then

$$\begin{aligned} \left| \sum_{k=1}^{\infty} a_{ik} - \lambda \right| &\leq \sup_{i \in \mathbb{N}} \sum_{k=1}^{\infty} |a_{ik}(n) - a_{ik}| + |\lambda_n - \lambda| \\ &= \|\hat{A}_n - \hat{A}\|_\infty + |\lambda_n - \lambda| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon , \end{aligned}$$

for all  $n \geq n_0$ . Thus

$$\sum_{k=1}^{\infty} a_{ik} = \lambda ,$$

so  $\hat{A} \in S_\lambda \subseteq S$ .

Finally, the set consisting of the invertible elements of a Banach algebra, is always a Banach Lie group (see p. 82 [12]).  $\square$

**Corollary 4.3.**

$$[S, S] \subseteq S_0 \cap S .$$

*In particular,  $S_0 \cap S$  is a Banach Lie algebra.*

*Proof.* As seen in the above proof, if  $\hat{A} \in S_\lambda$  and  $\hat{B} \in S_\mu$ ,

$$(\hat{A}\hat{B})J_0 = \mu\lambda J_0 ,$$

i.e.,  $\hat{A}\hat{B} \in S_{\mu\lambda} \subseteq S$  and

$$[\hat{A}\hat{B}, \hat{B}\hat{A}]J_0 = \hat{A}\hat{B}J_0 - \hat{B}\hat{A}J_0 = (\lambda\mu - \mu\lambda)J_0 = \mathbf{0} ,$$

so the commutator of two elements of  $S$  is a zero row sum matrix in  $S$ . By Lemma 4.1 and Proposition 4.2,  $S_0 \cap S$  is a vector space closed under the bilinear Lie bracket operation and its elements satisfy the Jacobi identity. We will show that it is also a real Banach space. So let  $(\hat{A}_n)$  be a Cauchy sequence in  $S_0$ . It is then

Cauchy, thus convergent, in  $S$ . Let  $\hat{A}$  be its limit. We will show that  $\hat{A} \in S_0$ . We have

$$\hat{A}J_0 = (\hat{A} - \hat{A}_n)J_0 + \hat{A}_nJ_0 = (\hat{A} - \hat{A}_n)J_0 .$$

Thus

$$\|\hat{A}J_0\|_\infty = \|(\hat{A} - \hat{A}_n)J_0\|_\infty \leq \|\hat{A} - \hat{A}_n\|_\infty \|J_0\|_\infty = \|\hat{A} - \hat{A}_n\|_\infty < \epsilon ,$$

for sufficiently large  $n$ . Since  $\epsilon > 0$  is arbitrary, it follows that

$$\|\hat{A}J_0\|_\infty = 0 ,$$

so

$$\hat{A}J_0 = \mathbf{0} ,$$

which implies that  $A$  is a zero-row sum matrix. □

**Corollary 4.4.**  $S_1 \cap S$  and  $S_1^+$  are closed sub-semigroups of  $S$ .

*Proof.* As shown in the proof of Proposition 4.2, a norm convergent sequence in  $S_1 \cap S$  converges to an element of  $S_1 \cap S$ , since  $\lambda_n = 1$  for all  $n$ . Thus  $S_1 \cap S$  is topologically closed. As shown in Corollary 4.3 the product of two matrices in  $S_1$  is also in  $S_1$ . Moreover, by Proposition 4.2, if the two matrices are in  $S$ , then so also is their product. Thus  $S_1 \cap S$  is closed under multiplication as well therefore it is a closed sub-semigroup of  $S$ . Since  $S_1^+$  is a closed sub-semigroup of  $S_1 \cap S$ , it is also a closed sub-semigroup of  $S$ . □

**Proposition 4.5.**  $\hat{S}_1$  is a closed subgroup of the Banach Lie group  $\mathcal{G}$ .

*Proof.* Since the elements of  $S$  correspond to operators on  $\ell_\infty$ , multiplication is associative and inverses, if they exist, are unique. We will show that the inverse of an infinite matrix in  $S_1 \cap S$  is also in  $S_1 \cap S$ . So let  $\hat{A} \in S_1 \cap S$  be invertible and let  $\hat{A}^{-1}$  be its inverse. Let  $A : \ell_\infty \rightarrow \ell_\infty$  be the bounded operator of the form (3.1) and (3.2) corresponding to  $\hat{A}$ . Since  $\ell_\infty$  is a Banach space, it follows that  $A^{-1}$  is also a bounded operator. Since  $\mathcal{G} \subseteq S$  is a Banach Lie group, it follows that  $\hat{A}^{-1} \in \mathcal{G}$  so  $\hat{A}^{-1} \in S$ . Moreover, by Lemma 4.1,

$$\hat{A}J_0 = J_0$$

implies

$$\hat{A}^{-1}J_0 = J_0 ,$$

so  $\hat{A}^{-1} \in S \cap S_1$ . Thus  $\hat{S}_1$  is a subgroup of  $\mathcal{G}$ . Finally, as shown in Proposition 4.2,  $S \cap S_1$  is norm-closed, so  $\hat{S}_1$  is closed under non-singular convergence. □

There is no infinite dimensional analogue [22] to the famous von Neumann theorem that a closed subgroup of  $GL(n, \mathbb{R})$  is a Lie group. Thus, Proposition 4.5 cannot be used to conclude that  $\hat{S}_1$  is a Lie group as done in the finite dimensional matrix case studied in [3] and [4]. Nevertheless, we can prove the following:

**Proposition 4.6.**  $\hat{S}_1$  is a Banach Lie group whose Banach Lie algebra is  $S_0 \cap S$ .

*Proof.* Since,  $S_0 \cap S$  is a Banach algebra that is also an ideal of  $S$  and since each element of  $S$  is of the form  $I + \hat{A}$  where  $\hat{A} \in S_0 \cap S$  it follows (see page 2 of [13]) that  $\hat{S}_1$  is a Banach Lie group. To show that the Banach-Lie algebra of  $\hat{S}_1$ , i.e. the tangent space at the identity element of  $\hat{S}_1$ , is  $S_0 \cap S$ , as in the finite dimensional case [4], we notice that each  $X \in S_0 \cap S$  is of the form  $\hat{A}'(0)$  where

$$\hat{A}(t) = I + tX \in \hat{S}_1 ,$$

with  $\hat{A}(0) = I$ , where  $t$  is in a sufficiently small interval containing 0 so that  $\hat{A}(t)$  is invertible, i.e. so that  $\|tX\| < 1$ . Thus  $S_0 \cap S$  is contained in the Lie algebra of  $\hat{S}_1$ . For the opposite inclusion, suppose that for  $t$  in a sufficiently small closed interval containing 0 we have a smooth path

$$\hat{A}(t) = (a_{ij}(t)) \in \hat{S}_1 ,$$

with  $\hat{A}(0) = I$  and  $\hat{A}'(0) = X$ . Then

$$XJ_0 = \lim_{h \rightarrow 0} \frac{A(h) - A(0)}{h} J_0 = \lim_{h \rightarrow 0} \frac{1}{h} (A(h)J_0 - A(0)J_0) = 0 ,$$

since

$$A(h)J_0 = A(0)J_0 = J_0 .$$

Thus  $X \in S_0 \cap S$ . □

**4.1. Some remarks on ideals of subrings of  $S$ .** Let  $J'_0 = I - J_0$ . Then, for any  $\hat{A} \in S$ ,  $\hat{A}J'_0 = \hat{A} - \hat{A}J_0 \in S_0$ . Moreover, if  $\hat{X} \in S_0$ , then  $J_0\hat{X} \in S_0$  and  $J'_0\hat{X} \in S_0$ . More generally, if  $\hat{X} \in S$ , then  $J_0\hat{X} \in S$  and  $J'_0\hat{X} \in S_0$ . We notice that

$$\mathcal{I}_0 = \{J_0X : X \in S_0\}$$

is a maximal ideal of  $S_0$ . Similarly,

$$\mathcal{I}_0 = \{J_0X : X \in S_1\}$$

is an ideal of  $S_1$ .

For all  $\hat{A} \in S$  note that  $J_0\hat{A}$  is a compact operator. Moreover, every  $\hat{A} \in S$  has the decomposition

$$\hat{A} = J_0\hat{A} + J'_0\hat{A} ,$$

where  $J_0\hat{A}$  is compact. That is,

$$S \approx \ell_1 \oplus S^0 ,$$

where  $S^0$  are the matrices in  $S$  with zero first row. On  $S_0$ , products in  $\mathcal{I}_0$  vanish. On  $S_1$ ,  $\mathcal{I}_0$  forms a right-zero semigroup, i.e.  $(J_0A)(J_0B) = J_0B$ , for all  $A, B$ . Similar decompositions hold for each row individually and for arbitrary finite subsets of rows.

### 5. Infinite Constant Row Sum Matrices and Compact Operators

The Perron–Frobenius theorem states that a real square matrix with positive entries has a unique maximum real positive eigenvalue and a corresponding eigenvector with positive coordinates. In particular, if the matrix is row-stochastic, then we know that the column vector with each entry equal to 1 is an eigenvector corresponding to the eigenvalue  $\lambda = 1 = r(A)$ , where  $r(A)$  is the spectral radius of  $A$ . If the multiplicity is one, then the Perron-Frobenius applied to the transpose says that every irreducible stochastic matrix has a stationary vector, invariant measure for the corresponding Markov chain, and that the largest absolute value of its eigenvalues is 1.

The Krein–Rutman [20, 7] theorem is the infinite dimensional generalization of the Perron–Frobenius theorem.

We recall the definition of a cone in a Banach space.

**Definition 5.1.** In a real Banach space a closed subset  $C$  is a *cone* provided:

- (i) for all  $\lambda, \mu \geq 0, u, v \in C, \lambda u + \mu v \in C$ ;
- (ii)  $u \in C$  and  $-u \in C$  only if  $u = 0$ .

The theorem can be stated as follows:

**Theorem 5.2.** (*Krein–Rutman*) Let  $X$  be a Banach space, and let  $C \subset X$  be a convex cone such that  $C - C := \{u - v / u, v \in C\}$  is dense in  $X$ , i.e.  $C$  is a total cone. Let  $T : X \rightarrow X$  be a non-zero compact operator such that  $T(C) \subseteq C$  and  $r(T) > 0$  where  $r(T)$  is the spectral radius of  $T$ . Then  $r(T)$  is an eigenvalue of  $T$  with a positive eigenvector, i.e., there exists  $u \in C \setminus 0$  such that  $T(u) = r(T)u$ .

The role of  $X$  and  $C$  in the Krein-Rutman theorem will be played in our case, respectively, by  $\ell_\infty$  and sequences in  $\ell_\infty$  with nonnegative terms. It is known ([7], Thm. 19.3) that if  $X$  is a Banach space and  $C$  is a cone in  $X$  with nonempty interior  $C^\circ$ , then a compact *strongly positive* operator  $T : X \rightarrow X$  has spectral radius  $r(T) > 0$ . By strongly positive we mean that  $T(x) \gg 0$  whenever  $x > 0$ , i.e that if  $x \in C \setminus \{0\}$ , then  $T(x) \in C^\circ$ .

Since compact operators are bounded, the spectral radius of a compact operator  $T$  can be computed with the use of Gelfand’s formula

$$r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} .$$

The problem of finding sufficient conditions for a linear operator

$$T : L^\infty(S, \Sigma, \mu) \rightarrow L^\infty(S, \Sigma, \mu)$$

to be compact, was considered in [9]:

**Theorem 5.3.** Let  $(S, \Sigma, \mu)$  be a positive measure space and let  $k : S \times S \rightarrow \mathbb{R}$  be a measurable function for which:

- (i) there exists a locally  $\mu$ -null set  $N \subseteq S$  (i.e.  $\mu(A \cap N) = 0$  for all  $A \in \Sigma$  with  $\mu(A) < \infty$ ) and a constant  $M > 0$  such that for all  $s \in S \setminus N: k_s \in L^1(S, \Sigma, \mu)$  and  $\|k_s\|_1 \leq M$ , where  $k_s(t) := k(s, t)$ ;
- (ii) there exists a locally  $\mu$ -null set  $N \subseteq S$  such that the set  $K := \{k_s : s \in S \setminus N\}$  is relatively compact in  $L^1(S, \Sigma, \mu)$ .

Then the integral operators

$$T : L^\infty(S, \Sigma, \mu) \rightarrow L^\infty(S, \Sigma, \mu) ; (T\phi)(s) := \int_S k(s, t)\phi(t) d\mu(t) ,$$

$$T_* : L^1(S, \Sigma, \mu) \rightarrow L^1(S, \Sigma, \mu) ; (T_*x)(t) := \int_S k(s, t)x(s) d\mu(s)$$

are compact.

*Proof.* The proof can be found in [9]. □

**Lemma 5.4.** *Let  $X = \ell_\infty$  and let  $C$  be the cone consisting of sequences in  $\ell_\infty$  with nonnegative terms. Then*

$$C^\circ = \{x = (x_i) \in C : \inf_{i \in \mathbb{N}} x_i > 0\} .$$

*Proof.* To show that

$$C^\circ \subseteq \{x = (x_i) \in C : \inf_{i \in \mathbb{N}} x_i > 0\} ,$$

suppose that there exists an  $x = (x_1, x_2, \dots) \in C^\circ$  with  $\inf_{i \in \mathbb{N}} x_i = 0$ . Then, for arbitrary  $\epsilon > 0$  there exists an index  $i_0$  for which  $0 \leq x_{i_0} < \frac{\epsilon}{2}$ . Therefore

$$y := (x_1, \dots, x_{i_0-1}, -\frac{\epsilon}{2}, x_{i_0+1}, \dots) \in B(x, \epsilon) ,$$

because

$$\|y - x\|_\infty = x_{i_0} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon .$$

But  $y \notin C$  so no such  $\epsilon$ -ball can be contained in  $C$ . Thus  $x \notin C^\circ$ . For the reverse inclusion, let  $x = (x_i) \in C$  with  $\inf_{i \in \mathbb{N}} x_i := a > 0$ . Will show that  $B(x, \frac{a}{2}) \subseteq C$  so  $x \in C^\circ$ . Let  $y = (y_i) \in B(x, \frac{a}{2})$ . Then

$$\|y\|_\infty \leq \|y - x\|_\infty + \|x\|_\infty < \frac{a}{2} + \|x\|_\infty < \infty$$

and for each  $i \in \mathbb{N}$

$$|y_i - x_i| < \frac{a}{2} \implies y_i > x_i - \frac{a}{2} > 0 .$$

Therefore  $y \in C$ . □

**Proposition 5.5.** *Let  $\hat{A} = (a_{ij}) \in S$  be an infinite matrix. If*

$$\lim_{n \rightarrow \infty} \sup_{i \in \mathbb{N}} \sum_{j=n}^{\infty} |a_{ij}| = 0$$

*then the linear operator  $A : \ell_\infty \rightarrow \ell_\infty$  corresponding to  $\hat{A}$  is compact. If  $a_{ij} > 0$  for all  $i, j$  and  $C$  is the set of  $\ell_\infty$  sequences with nonnegative terms, then  $A(C) \subseteq C$ . Moreover, if  $\hat{A}$  is row stochastic, then  $r(A) > 0$ .*

*Proof.* In Theorem 5.3 we let  $S$  be  $\mathbb{N}$ ,  $\mu$  be counting measure [2], and  $k(s, t) = a_{st}$ . Then  $L^1(S, \Sigma, \mu) = \ell_1(\mathbb{N})$ ,  $L^\infty(S, \Sigma, \mu) = \ell_\infty(\mathbb{N})$  and the only  $\mu$ -null set is the empty set. The role of  $K$  is played by the set of the rows of  $\hat{A}$  which is  $\ell_1$ -bounded

since the  $\ell_1$  norm of each row is less or equal to  $M := \|\hat{A}\|_\infty$ . By Theorem 5.3 the operator  $T = A$ , defined on  $\phi = (\phi_i)_{i \in \mathbb{N}} \in \ell_\infty$  by

$$(A\phi)_i = \sum_{j=1}^{\infty} a_{ij} \phi_j ,$$

is compact if the set  $K$  is relatively compact. It is known that a bounded subset  $K$  of  $\ell_1$  is relatively compact if and only

$$\lim_{n \rightarrow \infty} \sum_{i=n}^{\infty} |k_i| = 0 ,$$

uniformly for  $k = (k_i)_{i \in \mathbb{N}} \in K$  (see [8], p.6). In our setting, this is equivalent to

$$\lim_{n \rightarrow \infty} \sup_{i \in \mathbb{N}} \sum_{j=n}^{\infty} |a_{ij}| = 0 .$$

If  $\phi = (\phi_i)_{i \in \mathbb{N}} \in C \subset \ell_\infty$ , then

$$(A\phi)_i = \sum_{j=1}^{\infty} a_{ij} \phi_j \geq 0 ,$$

so  $A\phi \in C$  as well. Finally, if  $\hat{A}$  is row stochastic then  $x = (1, 1, ..)$  is an eigenvector of  $\hat{A}$  corresponding to the eigenvalue  $\lambda = 1$ . Thus  $r(A) \geq 1 > 0$ .  $\square$

*Remark 5.6.* Without compactness, strong positivity of  $A$  alone is not sufficient to imply the positivity of the spectral radius of  $A$ . That is because for  $\phi = (\phi_i)_{i \in \mathbb{N}} \in C \setminus \{0\}$  and  $\hat{A}$  with positive entries, the standard estimate would be

$$\inf_i (A\phi)_i = \inf_i \sum_{j=1}^{\infty} a_{ij} \phi_j \geq \inf_k \phi_k \inf_i \sum_{j=1}^{\infty} a_{ij} \geq 0 ,$$

so we cannot be sure that  $A\phi \in C^\circ$ .

In our context, we have the following simple criterion for compactness.

**Proposition 5.7.** *Let*

$$\mathbf{1} := (1, 1, \dots) ; \mathbf{1}_n := (1, 1, \dots, 1, 0, 0, \dots) ; \|\mathbf{1}\|_\infty = \|\mathbf{1}_n\|_\infty = 1 ,$$

where the last 1 is in the  $n$ -th position,  $n \geq 1$ , and let  $\hat{A} \in S \cap S_1$ . If the rows of  $\hat{A}$  satisfy the  $\ell_1$ -relative compactness condition

$$\lim_{n \rightarrow \infty} \sup_{i \in \mathbb{N}} \sum_{j=n+1}^{\infty} |a_{i,j}|$$

then

$$\lim_{n \rightarrow \infty} \|\hat{A}\mathbf{1}_n - \mathbf{1}\|_\infty = 0 .$$

If  $\hat{A}$  is row stochastic, then the converse holds as well.

*Proof.* Notice that for  $\hat{A} = I$

$$\lim_{n \rightarrow \infty} \|\mathbf{1}_n - \mathbf{1}\|_\infty \neq 0,$$

since the sequence  $(\mathbf{1}_n)$  is not even Cauchy. So the result is not true, in general, without the relative compactness assumption. Now

$$\hat{A}\mathbf{1}_n - \mathbf{1} = \hat{A}\mathbf{1}_n - \hat{A}\mathbf{1} = \hat{A}(\mathbf{1}_n - \mathbf{1}) = \left( \sum_{j=n+1}^\infty a_{1,j}, \sum_{j=n+1}^\infty a_{2,j}, \dots \right),$$

so, by the compactness assumption on the rows of  $\hat{A}$ ,

$$\lim_{n \rightarrow \infty} \|\hat{A}\mathbf{1}_n - \hat{A}\mathbf{1}\|_\infty = \lim_{n \rightarrow \infty} \sup_{i \in \mathbb{N}} \left| \sum_{j=n+1}^\infty a_{i,j} \right| \leq \lim_{n \rightarrow \infty} \sup_{i \in \mathbb{N}} \sum_{j=n+1}^\infty |a_{i,j}| = 0.$$

Therefore

$$\lim_{n \rightarrow \infty} \|\hat{A}\mathbf{1}_n - \hat{A}\mathbf{1}\|_\infty = 0.$$

If  $\hat{A}$  is row stochastic, then its entries are non-negative so

$$\lim_{n \rightarrow \infty} \|\hat{A}\mathbf{1}_n - \hat{A}\mathbf{1}\|_\infty = \lim_{n \rightarrow \infty} \sup_{i \in \mathbb{N}} \sum_{j=n+1}^\infty a_{i,j} = \lim_{n \rightarrow \infty} \sup_{i \in \mathbb{N}} \sum_{j=n+1}^\infty |a_{i,j}|,$$

which proves the equivalence of the two conditions in that case. □

See Section 7 below for connections with countable state Markov chains.

### 6. Infinite Stochastic Matrices as Sequence Transformations

A good exposition of the theory of *generalized limits* of sequences and series of real numbers can be found in [6].

Starting with a sequence  $(s_k)_{k \in \mathbb{N}}$ , where  $s_k \in \mathbb{R}, \forall k$ , and an infinite matrix  $A = (a_{n,k})_{(n,k) \in \mathbb{N}^2}$ , we consider the *A-transformed* sequence  $(\sigma_n)_{n \in \mathbb{N}}$  defined by

$$\sigma_n = \sum_{k=1}^\infty a_{n,k} s_k.$$

We are interested in relating the convergence properties of the original sequence and the *A-transformed* one. In particular, if the original sequence consists of the partial sums of a divergent series, it is possible that the *A-transformed* series will converge thus defining a *generalized* or *A-limit* of the original divergent series.

The basic theorems can be found in [6] and read as follows:

**Theorem 6.1.** (*Kojima-Schur*) *Let  $s_k \rightarrow s \in \mathbb{R}$  as  $k \rightarrow \infty$ . The A-transformed sequence*

$$\sigma_n := \sum_{k=1}^\infty a_{n,k} s_k \quad (n > n_0)$$

*converges to a limit  $\sigma \in \mathbb{R}$  as  $n \rightarrow \infty$ , if and only if*

$$\sum_{k=1}^\infty |a_{n,k}| \leq M \quad (\text{for every } n > n_0), \tag{6.1}$$

$$a_{n,k} \rightarrow \alpha_k \in \mathbb{R} \text{ as } n \rightarrow \infty, \text{ for every fixed } k \tag{6.2}$$

and

$$A_n := \sum_{k=1}^{\infty} a_{n,k} \rightarrow \alpha \in \mathbb{R}, \text{ as } n \rightarrow \infty. \tag{6.3}$$

Moreover

$$\sigma_n \rightarrow \alpha s + \sum_{k=1}^{\infty} \alpha_k (s_k - s) \text{ as } n \rightarrow \infty. \tag{6.4}$$

**Theorem 6.2.** (Silverman-Toeplitz) Let  $s_k \rightarrow s \in \mathbb{R}$  as  $k \rightarrow \infty$ . The  $A$ -transformed sequence

$$\sigma_n := \sum_{k=1}^{\infty} a_{n,k} s_k \quad (n > n_0)$$

converges to **the same limit**  $s \in \mathbb{R}$  as  $n \rightarrow \infty$ , if and only if

$$\sum_{k=1}^{\infty} |a_{n,k}| \leq M \text{ (for every } n > n_0), \tag{6.5}$$

$$a_{n,k} \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ for every fixed } k \tag{6.6}$$

and

$$A_n := \sum_{k=1}^{\infty} a_{n,k} \rightarrow 1 \text{ as } n \rightarrow \infty. \tag{6.7}$$

We may prove the following:

**Theorem 6.3.** In the notation of Theorems 6.1 and 6.2, let  $\hat{A} = (a_{nk}) \in S$ .

(i) If  $\hat{A} \in S_1^+$  and the columns of  $\hat{A}$  satisfy condition (6.2), then

$$\sigma_n \rightarrow s + \sum_{k=1}^{\infty} \alpha_k (s_k - s) \text{ as } n \rightarrow \infty.$$

(ii) If  $\hat{A}$  is stochastic and  $a_{nk} \rightarrow 0$  as  $n \rightarrow \infty$  for each  $k$ , then

$$\sigma_n \rightarrow s \text{ as } n \rightarrow \infty.$$

*Proof.* For (i) notice that an  $S$  matrix  $\hat{A}$  satisfies (3.2) so it also satisfies condition (6.1). Since  $\hat{A} \in S_1^+$  it follows that  $\alpha = 1$  and the result follows by the Kojima-Schur theorem. For (ii) notice that stochasticity implies that (6.7) is satisfied and the result follows by the Silverman-Toeplitz theorem.  $\square$

*Remark 6.4.* For example, if  $\hat{A}$  is doubly stochastic, the columns are summable and hence converge termwise to 0.

A main example is the *arithmetic means* matrix

$$\hat{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1/2 & 1/2 & 0 & 0 & 0 & \dots \\ 1/3 & 1/3 & 1/3 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix},$$

corresponding to Cesàro convergence.



## 7. Abel Limits and Convergence of Markov Chains

First we discuss entrywise convergence and review properties of Markov chains on the nonnegative integers. Then we show that for stochastic matrices, convergence of powers always exists in the Abel sense.

*Remark 7.1.* For the remainder of this paper, indexing of rows and columns begins with 0.

**7.1. Recurrence properties of Markov chains.** If we have a stochastic matrix  $A$ , we may consider it as the transition matrix for a Markov chain on  $\mathbb{N} = \{0, 1, 2, \dots\}$ . Thus,

$$A_{ij} = \text{P}(\text{probability of jumping from } i \text{ to } j \text{ in one step})$$

and for the  $n^{\text{th}}$  power

$$(A^n)_{ij} = \text{P}(\text{probability of jumping from } i \text{ to } j \text{ in } n \text{ steps}) .$$

A fundamental question is to determine the limiting behavior of the entries of the powers  $A^n$  as  $n \rightarrow \infty$ .

We review the basic properties and how they relate to powers of  $A$ . There are three principal cases. It is assumed that the chain does not decompose into disjoint systems.

We refer to [18] for criteria mentioned here. In the discussion below, we require the *matrix reduced at  $i$* . This means that row and column  $i$  are deleted, i.e., we are considering the matrix that would be used in forming the  $ii$  minor in the finite-dimensional case.

(i) **Positive recurrent case.** In this case, the chain visits every state infinitely often and there is an invariant distribution  $\pi = (p_0, p_1, \dots)$  with  $\sum_j p_j = 1$  and

$$(A^n)_{ij} \rightarrow p_j ,$$

for all  $i$ . Note that  $\pi \in \ell_1$  and

$$\pi A = \pi$$

is a left fixed point. The Krein-Rutman theorem provides a criterion for positive recurrence.

**Proposition 7.2.** *If the stochastic matrix  $A$  is compact as an operator and strongly positive, with 1 as a simple eigenvalue, then the associated Markov chain is positive recurrent and the powers of  $A$  converge entrywise*

$$\lim_{n \rightarrow \infty} (A^n)_{ij} = p_j ,$$

where  $p_j$  are positive, summing to 1, providing the invariant distribution of the chain.

*Proof.* Recalling our discussion of Section 5, consider  $A$  acting on  $\ell_1$  on the right. Eveson's Theorem shows that the compactness condition is the same whether considered as an operator on  $\ell_1$  on the right or  $\ell_\infty$  on the left. Now we know there is an eigenvector for eigenvalue 1, which is the spectral radius, since the matrix is stochastic. At this point we can invoke Riesz-Schauder theory, [28], since the operator is compact. Thus, the only non-zero spectrum are eigenvalues and

the multiplicities of eigenvalues of  $A$  and the dual operator, which is stochastic acting on  $\ell_\infty$ , are the same. So we have a unique left fixed point in  $\ell_1$ . From Krein-Rutman we know the eigenvector is non-negative, in fact, positive, since the operator is strongly positive. The entrywise convergence follows from Markov chain theory.  $\square$

*Remark 7.3.* Some versions of the Krein-Rutman theorem will already provide that the eigenvalue 1 is simple. The main common feature in all versions is the existence of the positive eigenvector on the left.

*Remark 7.4.* In the next two cases the probabilities converge to zero, i.e.,

$$\lim_{n \rightarrow \infty} (A^n)_{ij} = 0,$$

for all  $i, j$ . This shows why you can not have a general strong limit theorem, since  $A^n \mathbf{1} = \mathbf{1}$  for all  $n \geq 0$ .

(ii) **Null recurrent case.** In this case, the process is recurrent, i.e., every state is visited infinitely often, but there is no invariant measure. There is a nonnegative left eigenvector, but it is not in  $\ell_1$ . Reduce the matrix at one (any) state  $i$ . Then one of the following must hold: (1) the reduced matrix has an unbounded right eigenvector, fixed point or (2) the only nonnegative bounded right fixed point is the zero vector.

**Example 7.5.** The reflected random walk provides a good example for this case. Here the superdiagonal and subdiagonal consist of all  $1/2$ 's except for the entry in the top row,  $A_{01} = 1$ , reflecting at state 0. That is,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \end{pmatrix}.$$

A left formal eigenvector is  $(1, 2, 2, 2, 2, \dots)$ , not in  $\ell_1$ . Reducing at 0 yields the unbounded right eigenvector  $(1, 2, 3, 4, 5, \dots)$ . So the process is recurrent, but return times to a given state have infinite expectation. The probabilities go to zero as the system is trying to normalize the weights  $(1, 2, 2, 2, \dots)$ , which total infinity.

(iii) **Transient case.** Here the process goes off to infinity. If there is no nonzero left eigenvector, the process is transient. If there is a non-normalizable left eigenvector, then transience requires the reduced matrix (at any state) have a nonzero, nonnegative, bounded solution.

**Example 7.6.** A drifting process illustrates one of the criteria. Take a probability distribution on  $\mathbb{N}$ ,  $(p_0, p_1, \dots)$ , with  $p_0 \neq 1$ , so it is not concentrated at 0. Form the Toeplitz matrix

$$A_{ij} = \begin{cases} p_{j-i}, & \text{if } j \geq i; \\ 0, & \text{otherwise.} \end{cases}$$

It looks like

$$A = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 & \cdots \\ 0 & p_0 & p_1 & p_2 & \ddots \\ 0 & 0 & p_0 & p_1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} .$$

It is readily checked that, with  $0 \leq p_0 < 1$ , the only fixed vector on the left is the zero vector. The process eventually makes it way out to infinity.

**7.2. Abel limits.** Ergodic theorems and Markov convergence theorems typically consider Cesàro limits, cf. [28]. We can alternatively consider Abel limits. In both contexts, problems with reducibility and periodicity are automatically handled, so one gets convergence in the generalized sense.

Recall that for a sequence  $(x_0, x_1, \dots)$ , the Abel limit is defined as

$$\lim_{t \uparrow 1} (1 - t) \sum_{n \geq 0} t^n x_n$$

noting that a constant sequence reproduces the correct value and similarly for a convergent sequence. Note that Abel summation concerns the Abel limit of the partial sums of a series.

For Markov limit theorems, of interest is convergence of the powers of a stochastic matrix  $A$ . With  $A$  of norm (at most) one, the geometric series is well-defined and determines the inverse

$$(I - tA)^{-1} = \sum_{n \geq 0} t^n A^n$$

for  $0 < t < 1$ . Thus we have the *Abel limit*, call it  $\Omega$ , of the powers  $A^n$  defined by

$$\lim_{t \uparrow 1} (1 - t)(I - tA)^{-1}$$

see, e.g. [15], [10]. For numerical sequences it is known that the Abel limit will equal the Cesàro limit if the latter exists.

Following [10], Proposition B.1., for, in general substochastic, finite matrices

$$P = (p_{ij})_{1 \leq i, j \leq n} , \quad 0 \leq p_{ij} \leq 1 , \quad \sum_{j=1}^n p_{ij} \leq 1 ,$$

we have

**Proposition 7.7.** *If  $P$  is a stochastic  $(n \times n)$  matrix, then the Abel limit  $\Omega$  exists and satisfies*

$$\Omega = \Omega^2 = P\Omega = \Omega P .$$

*Moreover,  $P$  has a nontrivial fixed point if and only if  $\Omega \neq 0$ .*

The proof starts by showing the uniform boundedness by 1 of the matrix elements

$$\langle e_i, Q(s)e_j \rangle , \quad 0 < s < 1 ,$$

where

$$Q(s) := (1 - s)(I - sP)^{-1}$$

and  $\{e_n : n = 1, 2, \dots\}$  is the standard basis of  $\mathbb{R}^n$ . The Cantor diagonal process yields a convergent subsequence  $Q(s')$ . Showing that its limit  $\Omega$  is the only limit point of  $Q(s)$  produces the Abel limit.

Replacing  $P$  by  $\hat{A} \in S$  with  $\|\hat{A}\| \leq 1$ , we can form

$$Q(s) := (1-s)(I - s\hat{A})^{-1} = (1-s) \sum_{n \geq 0} s^n \hat{A}^n,$$

which will have uniformly bounded entries  $(Q(s))_{ij}$  if  $\hat{A}$  satisfies

$$\|\hat{A}\| \leq 1.$$

The difficulty here is that we may not have strong convergence, so we cannot guarantee that  $\Omega$  will be non-trivial. As noted above, we will have convergence in the compact case of a positive recurrent process to a nontrivial idempotent operator.

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