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STATISTICAL ANALYSIS OF THE NON-ERGODIC FRACTIONAL ORNSTEIN–UHLENBECK PROCESS OF THE SECOND KIND

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Abstract. We study the least squares estimator for the drift parameter of the non-ergodic fractional Ornstein–Uhlebbeck process of the second kind. Via Malliavin calculus, we analyze the consistency and the asymptotic distribution of this estimator.

1. Introduction

The purpose of this paper is to analyze the least squares estimator (LSE in the sequel) for the drift parameter of the fractional Ornstein–Uhlenbeck process of the second kind. Let us first describe this stochastic process. It has been introduced in [12] and its definition is related to the Lamperti transform of the fractional Brownian motion. Actually, there are two ways to define the fractional Ornstein–Uhlenbeck process (fOU). The first natural definition is to define it as the solution to the Langevin equation

$$dX_t = -\alpha X_t dt + dB_t \tag{1.1}$$

with some initial condition $X_0 \in \mathbb{R}$, where $(B_t)_{t\geq 0}$ is a fractional Brownian motion with Hurst parameter $H \in (0,1)$. The process is called *ergodic* if $\alpha > 0$ and *non-ergodic* when $\alpha < 0$. The second way to introduce the fOU is via the Lamperti transform of the fractional Brownian motion, that is,

$$X_t = e^{-\alpha t} B_{a_{\alpha,t}} \tag{1.2}$$

where $a_{\alpha,t} = \frac{H}{\alpha} e^{\frac{\alpha t}{H}}$ for every $t \geq 0$. In the case $H = \frac{1}{2}$ these two definitions lead to the same process, but this is not true in the fractional case. Indeed, it has been proven in [7], that the Gaussian processes given by (1.1) and (1.2) have different behavior when $H \neq \frac{1}{2}$.

Therefore, in [12] the authors called the process (1.1) as the fractional *Ornstein–Uhlenbeck of the first kind* and the process given by (1.2) as the *fractional Ornstein–Uhlenbeck of the second kind*. Our paper will focus on the second definition of the fOU process.

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The fractional Ornstein–Uhlenbeck of the second kind can be also defined as the solution to some Langevin type stochastic equation, that is, the process (1.2) coincides (see [12]) with the solution to

$$dX_t^\alpha = -\alpha X_t^\alpha dt + dY_t^\alpha$$

with initial condition $X_0^\alpha = B_{a_0}$, where the noise Y^α is given by the formula $Y_t^\alpha := \int_0^t e^{-\alpha t} dB_{a_{\alpha,s}}$ for every $t \geq 0$.

The above considerations lead us to the study of the process $(X_t)_{t\geq 0}$ defined by

$$\begin{cases} dX_t = \theta X_t dt + dY_t^{(1)}, & t \geq 0 \\ & X_0 = 0, \end{cases} \tag{1.3}$$

where

$$Y_t^{(1)} := \int_0^t e^{-s} dB_{a_s} \text{ with } a_{0,t} := a_t = He^{\frac{t}{H}}, \tag{1.4}$$

$\{B_t, t \geq 0\}$ being a fractional Brownian motion of Hurst index $H \in (\frac{1}{2}, 1)$.

When $H = \frac{1}{2}$, the process $Y_t^{(1)} = \int_0^t e^{-t} dB_{a_s}$ is a standard Brownian motion, by Lévy's characterization theorem. Therefore, the process X given by (1.3) is a standard Ornstein–Uhlenbeck process.

Our purpose is to estimate the parameter θ from the continuous observations of the process $(X_t)_{t\geq 0}$ given by (1.3). We will restrict to the non-ergodic case $\theta > 0$ since the ergodic case $(\theta > 0)$ has been treated in [2].

While the statistical inference of Itô type diffusions has a long history, the statistical analysis for equations driven by fractional Brownian motion (fBm) is obviously more recent. The development of stochastic calculus with respect to the fBm allowed to study such models. We will recall several approaches to estimate the parameters in fractional models but we mention that the below list is not exhaustive:

- The MLE approach in [13], [26] or [29]. In general the techniques used to construct maximum likelihood estimators (MLE) for the drift parameter are based on Girsanov transforms for fractional Brownian motion and depend on the properties of the deterministic fractional operators (determined by the Hurst parameter) related to the fBm. In general, the MLE is not easily computable.
- A pseudo-MLE approach based on the discretization of the equation (1.3) in [6], [28]. This approach allows to simulate better the estimator obtained. Some numerical results are presented in [6] and [28] as well.
- A least squares approach has been proposed in [11]. The study of the asymptotic properties of the estimator is based on certain criteria formulated in terms of the Malliavin calculus (see [22]). See also [8] for a related least squares estimator.
- Other type of estimators, such as minimum L^1-norm estimator, contrast estimators etc, can be found in the monograph [27].
- The statistical inference for the fOU of the second kind has been recently developed in the papers [2] or [3] in the ergodic case. The case of non-ergodic fOU process of the first kind can be found in [5].

We aim is to bring a new contribution to the statistical inference for fractional diffusions by estimating the drift parameter of a non-ergodic fOU process of the second kind. As in [11] or [2], we propose a least squares estimator. Although the formulation of the problem appears rather similar to the one studied in [2], the proofs and the results are quite different. There are several points that make our approach different: first, the behavior of the solution to (1.3) in the non-ergodic case is not the same as in the ergodic case, the covariance and the memory properties of these processes being significantly different; second, in contrast to [2], we use a LSE based on a pathwise integral with respect to the noise and this makes in principle our estimator easier to be simulated; a third significant difference is the behavior or the estimator. While in [2], the estimator is asymptotically normal, in our case we prove that the limit distribution of the LSE is a standard Cauchy distribution.

We structured our paper as follows. In Section 2 we analyze some properties of the fOU process of the second kind. In Section 3 we construct the least squares estimator for the parameter of this process. We also give the asymptotic properties of the estimator, consistency and asymptotic distribution, by using Malliavin calculus. Section 4 (the Appendix) contains the basic elements on fractional Brownian motion and Malliavin calculus.

2. Properties of the Orstein–Uhlenbeck Process of the Second Kind

In this paragraph we extend the results in [12], [2] by giving new properties of the non-ergodic fOU process of the second kind. These properties will be needed in the next section in order to analyze the behavior of the LSE. Let us first note that the unique solution to (1.3) can be written as

$$X_t = e^{\theta t} \int_0^t e^{-\theta s} dY_s^{(1)} \tag{2.1}$$

for every $t \geq 0$, $\theta > 0$ and $H > \frac{1}{2}$ (see [12] or [2]), where $Y^{(1)}$ is given by (1.4). In order to make the analysis of this process easier, we will express the Wiener integral with respect to the process $Y^{(1)}$ as a Wiener integral with respect the fractional Brownian motion B.

Proposition 2.1. *Consider the process* $(\zeta_t)_{t\geq 0}$ *given by*

$$\zeta_t = \int_0^t e^{-\theta s} dY_s^{(1)}, \quad t \geq 0. \tag{2.2}$$

Then for every $t \geq 0$ *we have*

$$\zeta_t = H^{(\theta+1)H} \int_{a_0}^{a_t} s^{-(\theta+1)H} dB_s \tag{2.3}$$

where the integral $\int_{a_0}^{a_t} s^{-(\theta+1)H} dB_s$, *is understood as a Young integral.*

Proof. Using the change of variables formula (4.4), we can write for every $t \geq 0$

$$\begin{aligned}\zeta_t &= \int_0^t e^{-\theta s} e^{-s} dB_{a_s} \\ &= B_{a_t} e^{-(\theta+1)s} - B_{a_0} + \int_0^t (\theta+1) B_{a_s} e^{-(\theta+1)s} ds \\ &= B_{a_t} e^{-(\theta+1)s} - B_{a_0} + (\theta+1) H^{1+(\theta+1)H} \int_{a_0}^{a_t} B_x x^{-(\theta+1)H-1} dx\end{aligned}$$

where we used the change of variables $a_s = x$ in the last integral above. By integrating by parts, we obtain

$$\begin{aligned}\zeta_t &= B_{a_t} e^{-(\theta+1)s} - B_{a_0} + (\theta+1) H^{1+(\theta+1)H} \\ &\quad \times \left[B_{a_t} \frac{a_t^{-(\theta+1)H}}{-(\theta+1)H} + B_{a_0} \frac{H^{-(\theta+1)H}}{(\theta+1)H} + \int_{a_0}^{a_t} \frac{x^{-(\theta+1)H}}{(\theta+1)H} dB_x \right] \\ &= H^{(\theta+1)H} \int_{a_0}^{a_t} x^{-(\theta+1)H} dB_x.\end{aligned}$$

□

Recall that the covariance of the increments of the noise $Y^{(1)}$ satisfies (see [12, Proposition 3.5])

$$\mathbf{E}[(Y_t^{(1)} - Y_s^{(1)})(Y_u^{(1)} - Y_v^{(1)})] = \int_s^t \int_v^u r_H(r,z) dr dz, \quad t > s,\ u > v, \tag{2.4}$$

where

$$r_H(r,z) = H^{2H-1}(2H-1) e^{-(\frac{1}{H}-1)(r-z)} |1 - e^{-(r-z)/H}|^{2H-2}.$$

Note that the kernel r_H is symmetric.

The following lemma will be needed to prove the consistency of the least square estimator.

Lemma 2.2. *Suppose $H \in (\frac{1}{2}, 1)$ and let ζ be given by (2.2). Then*

(i) *For all $\varepsilon \in (0, H)$ the process ζ admits a modification with $(H-\varepsilon)$-Hölder continuous paths, still denoted ζ in the sequel.*

(ii) *As $t \to \infty$*

$$\zeta_t \to \zeta_\infty := H^{(\theta+1)H} \int_{a_0}^{\infty} t^{-(\theta+1)H} dB_t \text{ almost surely and in } L^2(\Omega).$$

Proof. We first prove the point i.. In order to apply the Kolmogorov continuity criterium, we need to evaluate the mean square of the increment

$$\zeta_t - \zeta_s = \int_s^t e^{-\theta r} dY_r^{(1)}$$

with $0 \leq s \leq t$. This is a Gaussian random variable and we will use the formula (4.8) in order to compute its L^2 norm. The covariance of the process $Y^{(1)}$ can be obtained from thecformula (2.4).

We have for every $0 \le s \le t$,

$$\begin{aligned}
&\mathbf{E}[(\zeta_t - \zeta_s)^2] \\
&= \mathbf{E}\left(\int_s^t e^{-\theta r} dY_r^{(1)}\right)^2 \\
&= H^{2H-1}(2H-1)\int_s^t \int_s^t e^{-\theta v} e^{-\theta u} e^{-(\frac{1}{H}-1)(u-v)} |1 - e^{-(u-v)/H}|^{2H-2} du dv \\
&= 2H^{2H-1}(2H-1)\int_s^t \int_s^u e^{-(\theta+1)(u+v)} e^{\frac{u}{H}} e^{\frac{v}{H}} |e^{\frac{u}{H}} - e^{\frac{v}{H}}|^{2H-2} dv du \\
&\le 2H^{2H-1}(2H-1)\int_s^t e^{\frac{u}{H}} du \int_s^u e^{\frac{v}{H}} (e^{\frac{u}{H}} - e^{\frac{v}{H}})^{2H-2} dv \\
&= H^{2H} |e^{\frac{t}{H}} - e^{\frac{s}{H}}|^{2H}.
\end{aligned}$$

Then by using the mean value theorem, we can easily see that for every $0 \le s \le t \le T$

$$E[(\zeta_t - \zeta_s)^2] \le e^{2T} |t-s|^{2H}.$$

By applying the Kolmogorov–Centsov theorem to the centered Gaussian process ζ we deduce item i. of the conclusion.

Concerning the second point ii., we first notice that the Wiener integral

$$\zeta_\infty = H^{(\theta+1)H} \int_{a_0}^\infty s^{-(\theta+1)H} dB_s$$

is well defined as a random variable in $L^2(\Omega)$. In fact, by (4.7)

$$\begin{aligned}
\mathbf{E}\zeta_\infty^2 &= H^{2(\theta+1)H+1}(2H-1)\int_{a_0}^\infty \int_{a_0}^\infty t^{-(\theta+1)H} s^{-(\theta+1)H} |s-t|^{2H-2} ds dt \\
&= H^{2H+1}(2H-1)\int_0^1 \int_0^1 x^{(\theta+1)H-2} y^{(\theta+1)H-2} \left|\frac{1}{x} - \frac{1}{y}\right|^{2H-2} dx dy \\
&= H^{2H+1}(2H-1)\int_0^1 \int_0^1 x^{(\theta-1)H} y^{(\theta-1)H} |x-y|^{2H-2} dx dy \\
&= \frac{(2H-1)H^{2H}}{\theta} \beta(1 + (\theta-1)H, 2H-1) < \infty
\end{aligned}$$

with β denotes the classical Beta function. Note that the parameters of the berta function are strictly positive for $H \in (\frac{1}{2}, 1)$. Moreover, ζ_t converges to ζ_∞ in

$L^2(\Omega)$. Indeed,

$$\begin{aligned}
&\mathbf{E}[(\zeta_t-\zeta_\infty)^2]\\
&= H(2H-1)H^{2(\theta+1)H}\int_{a_t}^{\infty}\int_{a_t}^{\infty} r^{-(\theta+1)H}s^{-(\theta+1)H}|r-s|^{2H-2}dsdr\\
&= H(2H-1)H^{2H}e^{-2\theta t}\int_0^1\int_0^1 x^{(\theta-1)H}y^{(\theta-1)H}|x-y|^{2H-2}dxdy\\
&= \frac{(2H-1)H^{2H}}{\theta}\beta(1+(\theta-1)H,2H-1)e^{-2\theta t}\\
&\to 0 \quad as \quad t\to\infty.
\end{aligned}$$

Now, let us show that $\zeta_t \to \zeta_\infty$ almost surely as $t\to\infty$. By using Borel–Cantelli lemma, it is sufficient to prove that, for any $\varepsilon > 0$

$$\sum_{n\geq 0} P\left(\sup_{n\leq t\leq n+1}\left|\int_t^\infty e^{-\theta s}dY_s^{(1)}\right| > \varepsilon\right) < \infty.$$

For this purpose, let $\frac{1}{2} < \alpha < 1$. As in the proof of Theorem 4 in [1], we can write for every $t > 0$

$$\int_t^\infty e^{-\theta s}dY_s^{(1)} = \beta_\alpha^{-1}\int_t^\infty e^{-\theta s}dY_s^{(1)}\left(\int_t^s (s-r)^{-\alpha}(r-t)^{\alpha-1}dr\right)$$

with $\beta_\alpha^{-1} = \int_t^s (s-r)^{-\alpha}(r-t)^{\alpha-1}dr = \beta(\alpha, 1-\alpha)$.

By Fubini's theorem, we have (see e.g. [18])

$$\int_t^\infty e^{-\theta s}dY_s^{(1)} = \beta_\alpha^{-1}\int_t^\infty (r-t)^{\alpha-1}dr\left(\int_r^\infty (s-r)^{-\alpha}e^{-\theta s}dY_s^{(1)}\right).$$

Cauchy–Schwartz's inequality implies that,

$$\begin{aligned}
&\left|\int_t^\infty e^{-\theta s}dY_s^{(1)}\right|^2\\
&\leq \beta_\alpha^{-2}\left(\int_t^\infty (r-t)^{2(\alpha-1)}e^{-\theta(r-t)}dr\right)\\
&\quad\times\left(\int_t^\infty e^{-\theta(r-t)}dr\left|\int_r^\infty (s-r)^{-\alpha}e^{-\theta s}e^{\theta(r-t)}dY_s^{(1)}\right|^2\right)\\
&= \frac{\beta_\alpha^{-2}\Gamma(2\alpha-1)}{\theta^{2\alpha-1}}e^{-2\theta t}\int_t^\infty e^{-\theta(r-t)}dr\left|\int_r^\infty (s-r)^{-\alpha}e^{-\theta(s-r)}dY_s^{(1)}\right|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sup_{n\leq t\leq n+1}\left|\int_t^\infty e^{-\theta s}dY_s^{(1)}\right|^2\\
&\leq \frac{\beta_\alpha^{-2}\Gamma(2\alpha-1)}{\theta^{2\alpha-1}}e^{-2\theta n}e^{\theta}\int_n^\infty e^{-\theta(r-n)}dr\left|\int_r^\infty (s-r)^{-\alpha}e^{-\theta(s-r)}dY_s^{(1)}\right|^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathbf{E}\left(\left|\int_r^\infty (s-r)^{-\alpha}e^{-\theta(s-r)}dY_s^{(1)}\right|^2\right)\\
&= H^{2H-1}(2H-1)\int_r^\infty\int_r^\infty (u-r)^{-\alpha}e^{-\theta(u-r)}(v-r)^{-\alpha}\\
&\qquad e^{-\theta(v-r)}e^{-(\frac{1}{H}-1)(u-v)}|1-e^{-(u-v)/H}|^{2H-2}dudv\\
&= H^{2H-1}(2H-1)\int_0^\infty\int_0^\infty u^{-\alpha}e^{-\theta u}v^{-\alpha}e^{-\theta v}\\
&\qquad e^{-(\frac{1}{H}-1)(u-v)}|1-e^{-(u-v)/H}|^{2H-2}dudv\\
&= 2H^{2H-1}(2H-1)\int_0^\infty u^{-\alpha}e^{-\theta u}du\\
&\qquad \int_0^u v^{-\alpha}e^{-\theta v}e^{-(\frac{1}{H}-1)(u-v)}(1-e^{-(u-v)/H})^{2H-2}dv\\
&\le 2H^{2H-1}(2H-1)\int_0^\infty u^{-\alpha}e^{-\theta u}du\int_0^u v^{-\alpha}dv\\
&= \frac{2H^{2H-1}(2H-1)\Gamma(2-2\alpha)}{(1-\alpha)\theta^{2-2\alpha}} < \infty.
\end{aligned}$$

Combining this with the fact that $\int_n^\infty e^{-\theta(r-n)}dr = \frac{1}{\theta}$, we obtain

$$\mathbf{E}\left(\sup_{n\le t\le n+1}\left|\int_t^\infty e^{-\theta s}dY_s^{(1)}\right|^2\right) \le \frac{2\beta_\alpha^{-2}\Gamma(2\alpha-1)\Gamma(2-2\alpha)e^\theta}{1-\alpha}e^{-2\theta n}.$$

Consequently

$$\begin{aligned}
&\sum_{n\ge 0} P\left(\sup_{n\le t\le n+1}\left|\int_t^\infty e^{-\theta s}dY_s^{(1)}\right| > \varepsilon\right)\\
&\le \varepsilon^{-2}\mathbf{E}\left(\sup_{n\le t\le n+1}\left|\int_t^\infty e^{-\theta s}dY_s^{(1)}\right|^2\right)\\
&\le \varepsilon^{-2}C^{''}(H,\theta,\alpha)\sum_{n\ge 0}e^{-2\theta n} < \infty
\end{aligned}$$

and the conclusion follows. □

3. Asymptotic Behavior of the Least Squares Estimator

We will construct and analyze the behavior of the LSE for the drift parameter θ in (1.3). The least squares estimator is usually obtained by minimizing the function $\theta \to \int_0^t |\dot{X}_s - \theta X_s|^2 ds$. We will obtain the following form of the LSE

$$\widehat{\theta}_t = \frac{\int_0^t X_s\delta X_s}{\int_0^t X_s^2 ds} \tag{3.1}$$

where the stochastic integral is interpreted as a pathwise (Young) integral.

Notice that in [2] or [11] the authors interpreted the stochastic integral in the nominated of (3.1) as a Skorohod integral with respect to the fractional Brownian motion (as defined in our appendix). We consider a pathwise integral in (3.1), whose simulation is easier since it can be defined as a limit of Riemann sums. We prove that the least squares estimator $\widehat{\theta}_t$ given by (3.1) is strongly consistent and we find its limit distribution as $t \to \infty$.

By replacing in (3.1) X given by (1.3), we can write the LSE $\widehat{\theta}_t$ as follows

$$\widehat{\theta}_t = \theta + \frac{\int_0^t X_s dY_s^{(1)}}{\int_0^t X_s^2 ds}$$

and by (2.1),

$$\widehat{\theta}_t - \theta = \frac{\int_0^t e^{\theta s} \zeta_s dY_s^{(1)}}{\int_0^t e^{2\theta s} \zeta_s^2 ds} \tag{3.2}$$

where ζ_t is defined by (2.2).

3.1. Strong consistency of the least squares estimator. We will analyze separately the nominator and the denominator in the right hand side of (3.2). The following result gives the almost sure convergence of the denominator of (3.2).

Lemma 3.1. *Let $H > \frac{1}{2}$, then, almost surely*

$$e^{-2\theta t} \int_0^t X_s^2 ds = e^{-2\theta t} \int_0^t e^{2\theta s} \zeta_s^2 ds \underset{t \to \infty}{\longrightarrow} \frac{\zeta_\infty^2}{2\theta}$$

with ζ_∞ defined in Lemma 2.2 and ζ_t given by (2.2).

Proof. Recall from proof of Lemma 2.2 that ξ_∞ is a well-defined Gaussian random variable with

$$\mathbf{E}[\zeta_\infty]^2 = \frac{(2H-1)H^{2H}}{\theta} \beta(1 + (\theta - 1)H, 2H - 1) < \infty.$$

Hence $\zeta_\infty \sim \mathcal{N}(0, \frac{(2H-1)H^{2H}}{\theta} \beta(1 + (\theta - 1)H, 2H - 1))$, and this implies that

$$P(\zeta_\infty = 0) = 0. \tag{3.3}$$

The continuity of ζ and point ii. in Lemma 2.2 imply that, for every $t > 0$

$$\int_0^t e^{2\theta s} \zeta_s^2 ds \geq \int_{\frac{t}{2}}^t e^{2\theta s} \zeta_s^2 ds \geq \frac{t}{2} e^{\theta t} (\inf_{\frac{t}{2} \leq s \leq t} \zeta_s^2) \text{ almost surely}$$

and

$$\lim_{t \to \infty} (\inf_{\frac{t}{2} \leq s \leq t} \zeta_s^2) = \zeta_\infty^2 \text{ almost surely.}$$

We deduce that

$$\lim_{t \to \infty} \int_0^t e^{2\theta s} \zeta_s^2 ds = \infty \text{ almost surely.}$$

Hence, we can use l'Hôpital's rule to conclude that

$$\lim_{t \to \infty} \frac{\int_0^t e^{2\theta s} \zeta_s^2 ds}{e^{2\theta t}} = \lim_{t \to \infty} \frac{\zeta_t^2}{2\theta} = \frac{\zeta_\infty^2}{2\theta}$$

almost surely. □

The following theorem gives the strong consistency of the LSE $\widehat{\theta}_t$ (3.1).

Theorem 3.2. *Assume $H \in (\frac{1}{2}, 1)$ and let $\widehat{\theta}_t$ be given by (3.1) for every $t \geq 0$. Then*

$$\widehat{\theta}_t \to \theta \text{ almost surely as} t \to \infty.$$

Proof. Using the chain rule (4.4), we can express the integral with respect to $Y^{(1)}$ from (3.2) as

$$\int_0^t e^{\theta s} \zeta_s dY_s^{(1)} = \frac{1}{2} e^{2\theta t} \zeta_t^2 - \theta \int_0^t e^{2\theta s} \zeta_s^2 ds.$$

Hence

$$\widehat{\theta}_t - \theta = -\theta + \frac{\zeta_t^2}{2e^{-2\theta t} \int_0^t e^{2\theta s} \zeta_s^2 ds}.$$

Now it suffices to apply Lemmas 2.2 and 3.1 to obtain the conclusion. □

3.2. Asymptotic distribution of the estimator LSE. This paragraph is devoted to the investigation of asymptotic distribution of the LSE $\widehat{\theta}_t$ of θ. We start with the following lemma where we express an Young type integral that appears in the expression of the LSE as a Skorohod integral whose square mean can be easier handled.

Lemma 3.3. *Suppose that $H > \frac{1}{2}$. For every $t \geq 0$, let ζ_t be given by (2.2) and denote by*

$$\zeta_t' = \int_0^t e^{(\theta-1)s} dB_{a_s} = H^{-(\theta-1)H} \int_{a_0}^{a_t} s^{(\theta-1)H} dB_s. \tag{3.4}$$

Then, for every $t \geq 0$, we have

$$\begin{aligned}
&\int_0^t \zeta_s e^{(\theta-1)s} dB_{a_s} \\
&= \zeta_t \zeta_t' - H^2 \int_{a_0}^{a_t} s^{-(\theta+1)H} \left(\int_{a_0}^{a_s} r^{(\theta+1)H} \delta B_r \right) \delta B_s \\
&\quad - H^3 (2H-1) \int_{a_0}^{a_t} s^{-(\theta+1)H} \left(\int_{a_0}^{s} r^{(\theta-1)H} |s-r|^{2H-2} dr \right) ds.
\end{aligned}$$

Proof. Recall that δB denotes the Skorohod integral with respect to the fractional Brownian motion B (see the appendix). Using the alternative expression of ζ_t obtained in Proposition 2.1 and using again the change of variables formula (4.4), we will have for every $t \geq 0$,

$$\begin{aligned}
\zeta_t' \zeta_t = \zeta_0' \zeta_0 &+ H^{(\theta+1)H} \int_0^t e^{(\theta-1)s} \left(\int_{a_0}^{a_s} r^{-(\theta+1)H} dB_r \right) dB_{a_s} \\
&+ H^{-(\theta-1)H} \int_0^t e^{-(\theta+1)s} \left(\int_{a_0}^{a_s} r^{(\theta-1)H} dB_r \right) dB_{a_s}
\end{aligned}$$

and then

$$\begin{aligned}
&H^{(\theta+1)H}\int_0^t e^{(\theta-1)s}\left(\int_{a_0}^{a_s} r^{-(\theta+1)H}dB_r\right)dB_{a_s}\\
&=\zeta_t^{'}\zeta_t - H^{-(\theta-1)H}\int_0^t e^{-(\theta+1)s}\left(\int_{a_0}^{a_s} r^{(\theta-1)H}dB_r\right)dB_{a_s}\\
&=\int_0^t e^{-(\theta+1)s}dB_{a_s}\int_0^t e^{(\theta-1)r}dB_{a_r}\\
&\quad - H^{-(\theta-1)H}\int_0^t e^{-(\theta+1)s}\left(\int_{a_0}^{a_s} r^{(\theta-1)H}dB_r\right)dB_{a_s}. \qquad (3.5)
\end{aligned}$$

We want now to change the differentials dB by δB in the last line above. Concerning the integral $\int_{a_0}^{a_s} r^{(\theta-1)H}dB_r$, this change can be done without problems, since the integrand is non-random, due to the remark (4.6). On the other hand, to replace dB_s by δB_s, we need to use the relation (4.5). We will obtain

$$\begin{aligned}
&\int_0^t e^{-(\theta+1)s}\left(\int_{a_0}^{a_s} r^{(\theta-1)H}dB_r\right)dB_{a_s}\\
&=H^{(\theta+1)H}\int_{a_0}^{a_t} s^{-(\theta+1)H}\left(\int_{a_0}^{s} r^{(\theta-1)H}dB_r\right)dB_s\\
&=H^{(\theta+1)H}\int_{a_0}^{a_t} s^{-(\theta+1)H}\left(\int_{a_0}^{s} r^{(\theta-1)H}\delta B_r\right)\delta B_s\\
&\quad + H(2H-1)H^{(\theta+1)H}\int_{a_0}^{a_t} s^{-(\theta+1)H}ds\int_{a_0}^{s} r^{(\theta-1)H}|s-r|^{2H-2}dr. \qquad (3.6)
\end{aligned}$$

By (3.5) and (3.6), we finish the proof. □

We will denote by $\mathcal{C}(1)$ the Cauchy distribution with parameter 1. We will refer to it as the standard Cauchy distribution. Recall that $X, Y \sim \mathcal{N}(0,1)$ are two independent random variables, then $\frac{X}{Y}$ follows a standard Cauchy distribution.

Theorem 3.4. *Let $H > \frac{1}{2}$ be fixed. Then, as $t \to \infty$*

$$e^{\theta t}(\widehat{\theta}_t - \theta) \overset{Law}{\to} 2\theta H^{2(\theta-1)H}\mathcal{C}(1).$$

In order to prove Theorem 3.4 we need the following two lemmas.

Lemma 3.5. *Fix $H > \frac{1}{2}$. Let F be any $\sigma\{B\}-$mesurable random variable such that*

$$P(F < \infty) = 1.$$

Then, as $t \to \infty$

$$\begin{aligned}
&\left(F, e^{\theta t}\int_{a_0}^{a_t} s^{(\theta-1)H}dB_s\right)\\
&\qquad \overset{Law}{\longrightarrow} \left(F, \frac{(2H-1)H^{2\theta H}}{\theta}\beta(1+(\theta-1)H, 2H-1)N\right)
\end{aligned}$$

where $N \sim \mathcal{N}(0,1)$ is independent of B.

Proof. We will use the approach from the proof of Lemma 7 in [10]. It is enough to prove that for any $d \geq 1$, $s_1.....s_d \in [0,\infty)$, we shall prove that, as $t \to \infty$,

$$\left(B_{s_1}, ..., B_{s_d}, e^{-\theta t}\int_{a_0}^{a_t} s^{(\theta-1)H} dB_s\right) \overset{Law}{\to} \left(B_{a_{s_1}}, ..., B_{a_{s_d}}, \sigma N\right), \tag{3.7}$$

where $\sigma = \frac{(2H-1)H^{2\theta H}}{\theta}\beta(1+(\theta-1)H, 2H-1)$. Because the left-hand side in the previous convergence is a Gaussian vector, to get (3.7), it is sufficient to check the convergence of its covariance matrix. Let us first compute the limiting variance of $e^{-\theta t}\int_{a_0}^{a_t} s^{(\theta-1)H} dB_s$ as $t \to \infty$. By (4.7),

$$\begin{aligned}
&\mathbf{E}\left[\left(e^{-\theta t}\int_{a_0}^{a_t} s^{(\theta-1)H} dB_s\right)^2\right] \\
&= H(2H-1)e^{-2\theta t}\int_{a_0}^{a_t}\int_{a_0}^{a_t} v^{(\theta-1)H}u^{(\theta-1)H}|u-v|^{2H-2} dv du \\
&= H(2H-1)H^{2\theta H}\int_{e^{\frac{-t}{H}}}^{1}\int_{e^{\frac{-t}{H}}}^{1} a^{(\theta-1)H}b^{(\theta-1)H}|a-b|^{2H-2} da db \\
&\to \frac{(2H-1)H^{2\theta H}}{\theta}\beta(1+(\theta-1)H, 2H-1) \text{ as } t \to \infty.
\end{aligned}$$

Hence, to finish the proof it remains to check that, for all fixed $s \geq 0$,

$$\lim_{t\to\infty}\mathbf{E}\left(B_s \times e^{-\theta t}\int_{a_0}^{a_t} v^{(\theta-1)H} dB_v\right) = 0.$$

Indeed, for $0 < s < t$,

$$\begin{aligned}
&\mathbf{E}\left(B_s \times e^{-\theta t}\int_{a_0}^{a_t} v^{(\theta-1)H} dB_v\right) \\
&= H(2H-1)e^{-\theta t}\int_{a_0}^{a_t} v^{(\theta-1)H} dv \int_0^s |u-v|^{2H-2} du \\
&\leq H(2H-1)e^{-\theta t}\int_0^s v^{(\theta-1)H} dv \int_0^s |u-v|^{2H-2} du \\
&\quad + H(2H-1)e^{-\theta t}\int_s^{a_t} v^{(\theta-1)H} dv \int_0^s (v-u)^{2H-2} du \\
&:= I_t^1 + I_t^2.
\end{aligned}$$

We will prove that the two summands above converges to zero as $t \to \infty$. It is easy to see that

$$\lim_{t\to\infty} I_t^1 = 0.$$

Concerning the term I_t^2, we can express it as follows

$$\begin{aligned}
I_t^2 &= H(2H-1)e^{-\theta t}\int_s^{a_t} v^{(\theta-1)H} dv \int_0^s (v-u)^{2H-2} du \\
&= H(2H-1)e^{-\theta t}\int_s^{a_t} v^{(\theta-1)H}(v^{2H-1} - (v-s)^{2H-1}) dv.
\end{aligned}$$

Since $1-(1-\frac{s}{v})^{2H-1}=O(1/v)$ as $v\longrightarrow\infty$, there exist two positive constants $v_0>s, c>0$ such that for every $v\geq v_0$ we have $v^{2H-1}-(v-s)^{2H-1}\leq cv^{2H-2}$. Thus, for large t, we can write

$$\begin{aligned}I_t^2 \leq H(2H-1)&\left(e^{-\theta t}\int_s^{v_0} v^{(\theta-1)H}(v^{2H-1}-(v-s)^{2H-1})dv\right.\\&\left.+ce^{-\theta t}\int_{v_0}^{a_t} v^{\theta H+H-2}dv\right)\\&\to 0 \text{ as } t\to\infty\end{aligned}$$

because $H<1$. Then, the conclusion follows. □

Lemma 3.6. *Let* $H>\frac{1}{2}$. *Then, as* $t\to\infty$

$$e^{-\theta t}\int_{a_0}^{a_t} s^{-(\theta+1)H}\left(\int_{a_0}^{s} r^{(\theta-1)H}\delta B_r\right)\delta B_s\to 0 \quad in \quad L^2(\Omega), \tag{3.8}$$

and

$$e^{-\theta t}\int_{a_0}^{a_t} s^{-(\theta+1)H}ds\int_{a_0}^{s} r^{(\theta-1)H}|s-r|^{2H-2}dr\to 0. \tag{3.9}$$

Proof. Let us prove the convergence (3.8). By setting

$$U_s=s^{-(\theta+1)H}\int_{a_0}^{s} r^{(\theta-1)H}\delta B_r$$

we can write

$$-\theta t\int_{a_0}^{a_t} s^{-(\theta+1)H}\left(\int_{a_0}^{s} r^{(\theta-1)H}\delta B_r\right)\delta B_s=\int_0^t U_s\delta B_s$$

and the Malliavin derivative of the integrand U is

$$D_rU_s=s^{-(\theta+1)H}r^{(\theta-1)H}1_{[a_0,s]}(r).$$

By using (4.3) we can bound the L^2 norm of the Skorohod integral as follows

$$\begin{aligned}\mathbf{E}\left(e^{-\theta t}\int_{a_0}^{a_t}U_s\delta B_s\right)^2&=e^{-2\theta t}\mathbf{E}\left[\left(\int_{a_0}^{a_t}U_s\delta B_s\right)^2\right]\\&\leq c_He^{-2\theta t}\left(\int_{a_0}^{a_t}\int_{a_0}^{s}s^{-(\theta+1)}r^{(\theta-1)}drds\right)^{2H}\\&=\frac{c_H}{\theta}e^{-2\theta t}\left(\int_{a_0}^{a_t}s^{-(\theta+1)}(s^\theta-{a_0}^\theta)ds\right)^{2H}\\&\leq\frac{c_H\ln(H)^{2H}}{\theta H^{2H}}t^{2H}e^{-2\theta t}=c(\theta,H)t^{2H}e^{-2\theta t}\to_{t\to\infty}0.\end{aligned}$$

Then we proved the convergence (17). Regarding the convergence (3.9), we have

$$
\begin{aligned}
&e^{-\theta t}\int_{a_0}^{a_t} s^{-(\theta+1)H}ds\int_{a_0}^{s} r^{(\theta-1)H}|s-r|^{2H-2}dr\\
&\le e^{-\theta t}\int_{a_0}^{a_t} s^{-(\theta+1)H}ds\int_{0}^{s} r^{(\theta-1)H}(s-r)^{2H-2}dr\\
&=\beta(1+(\theta-1)H,2H-1)e^{-\theta t}\int_{a_0}^{a_t}\frac{1}{s}ds\to 0\quad as\quad t\to\infty.
\end{aligned}
$$

This finishes the proof. □

Proof of Theorem 3.4. . Using the expression (3.2) of the estimator $\widehat{\theta}_t$, we can write

$$
e^{\theta t}(\widehat{\theta}_t-\theta)=\frac{e^{\theta t}\int_0^t e^{(\theta-1)s}\zeta_s dB_{a_s}}{\int_0^t e^{2\theta s}\zeta_s^2 ds}
$$

with ζ_s from (2.2). We will use Lemma 3.3 in order to give an alternative form of the denominator. We have by Lemma 3.3 with ζ_t' given by (3.4)

$$
\begin{aligned}
e^{\theta t}(\widehat{\theta}_t-\theta)=&\frac{\zeta_t\zeta_\infty}{e^{-2\theta t}\int_0^t e^{2\theta s}\zeta_s^2 ds}\times\frac{\zeta_T'}{\zeta_\infty}\\
&-H^2\frac{e^{-\theta t}\int_{a_0}^{a_t} s^{-(\theta+1)H}\delta B_s\int_{a_0}^{a_s} r^{(\theta-1)H}\delta B_r}{e^{-2\theta t}\int_0^t e^{2\theta s}\zeta_s^2 ds}\\
&-H^3(2H-1)\frac{e^{-\theta t}\int_{a_0}^{a_t} s^{-(\theta+1)H}ds\int_{a_0}^{s} r^{(\theta-1)H}|s-r|^{2H-2}dr}{e^{-2\theta t}\int_0^t e^{2\theta s}\zeta_s^2 ds}\\
:=&A_t^\theta\times B_t^\theta-C_t^\theta-D_t^\theta.
\end{aligned}
$$

By Lemma 3.1, we obtain that

$$
A_t^\theta\longrightarrow 2\theta\quad almost\quad surely\quad as\quad t\longrightarrow\infty
$$

and according the Lemma 3.5 we deduce,

$$
B_t^\theta\overset{Law}{\to}\frac{(2H-1)H^{2\theta H}}{\theta}\beta(1+(\theta-1)H,2H-1)\frac{N}{\zeta_\infty}\quad as\quad t\longrightarrow\infty.
$$

Moreover,

$$
\frac{(2H-1)H^{2\theta H}}{\theta}\beta(1+(\theta-1)H,2H-1)\frac{N}{\zeta_\infty}\overset{Law}{=}H^{2(\theta-1)H}\mathcal{C}(1),
$$

because

$$
\frac{\theta\zeta_\infty}{(2H-1)H^{2H}\beta(1+(\theta-1)H,2H-1)}\sim\mathcal{N}(0,1)
$$

and $N\sim\mathcal{N}(0,1)$ are independent. Thus by Slutsky's theorem, we conclude that

$$
A_t^\theta\times B_t^\theta\overset{Law}{\to}2\theta H^{2(\theta-1)H}\mathcal{C}(1)\quad as\quad t\longrightarrow\infty.
$$

On the other hand, it follows from Lemma 5 that

$$
C_t^\theta\overset{Prob}{\longrightarrow}0\quad as\quad t\longrightarrow\infty,
$$

and

$$D_t^{\theta} \longrightarrow 0 \quad almost \quad surely \quad as \quad t \longrightarrow \infty.$$

Finally, by combining the previous convergences, the proof of Theorem 3.4 is done. □

Remark 3.7. Note that Theorem 3.4 shows that the convergence in distribution of the estimator (3.1) is very fast, at an exponential rate.

4. Appendix: Fractional Brownian Motion and Malliavin Calculus

In this section we describe some basic facts on the stochastic calculus with respect to a fractional Brownian motion. For more a complete presentation on the subject, see [21] and [1].

The fractional Brownian motion $\{B_t, t \geq 0\}$ with Hurst parameter $H \in (0,1)$, is defined as a centered Gaussian process starting from zero with covariance

$$R_H(t,s) = \mathbf{E}(B_t B_s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

We assume that B is defined on a complete probability space $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}$ is the sigma-field generated by B. By Kolmogorov's continuity criterion and the fact

$$\mathbf{E}\left(B_t - B_s\right)^2 = |s-t|^{2H};\ s,\ t \geq\ 0,$$

we deduce that B admits a version which has Hölder continuous paths of any order $\gamma < H$.

Fix a time interval $[0,T]$. We denote by $\mathcal{H}$ the canonical Hilbert space associated to the fractional Brownian motion B. That is, $\mathcal{H}$ is the closure of the linear span $\mathcal{E}$ generated by the indicator functions $1_{[0,t]},\ t \in [0,T]$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle = R_H(t,s).$$

The application $\varphi \in \mathcal{E} \longrightarrow B(\varphi)$ is an isometry from $\mathcal{E}$ to the Gaussian space generated by B and it can be extended to $\mathcal{H}$. If $H \in (\frac{1}{2}, 1)$ the elements of $\mathcal{H}$ may not be functions but distributions of negative order (see [23]).

Therefore, it is of interest to know significant subspaces of functions contained in it. Let $|\mathcal{H}|$ be the set of measurable functions φ on $[0,T]$ such that

$$\|\varphi\|_{|\mathcal{H}|}^2 := H(2H-1)\int_0^T\int_0^T |\varphi(u)||\varphi(v)||u-v|^{2H-2}dudv < \infty.$$

Note that, if $\varphi,\ \psi \in |\mathcal{H}|$,

$$\mathbf{E}(B(\varphi)B(\psi)) = H(2H-1)\int_0^T\int_0^T \varphi(u)\psi(v)|u-v|^{2H-2}dudv.$$

It follows actually from [23] that the space $|\mathcal{H}|$ is a Banach space for the norm $\|.\|_{|\mathcal{H}|}$ and it is included in $\mathcal{H}$. In fact,

$$L^2([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset |\mathcal{H}| \subset \mathcal{H}. \tag{4.1}$$

Let $C_b^\infty(\mathbb{R}^n,\mathbb{R})$ be the class of infinitely differentiable functions $f:\mathbb{R}^n\longrightarrow\mathbb{R}$ such that f and all its partial derivatives are bounded. We denote by $\mathcal{S}$ the class of smooth cylindrical random variables F of the form

$$F=f(B(\varphi_1),...,B(\varphi_n)), \tag{4.2}$$

where $n\geq 1$, $f\in C_b^\infty(\mathbb{R}^n,\mathbb{R})$ and $\varphi_1,...,\varphi_n\in\mathcal{H}$. The derivative operator D of a smooth cylindrical random variable F of the form (4.2) is defined as the $\mathcal{H}$-valued random variable

$$DF=\sum_{i=1}^n\frac{\partial f}{\partial x_i}(B(\varphi_1),...,B(\varphi_n))\varphi_i.$$

In this way the derivative DF is an element of $L^2(\Omega;\mathcal{H})$. We denote by $D^{1,2}$ the closure of $\mathcal{S}$ with respect to the norm defined by

$$\|F\|_{1,2}^2=\mathbf{E}(F^2)+\mathbf{E}(\|DF\|_{\mathcal{H}}^2).$$

The divergence operator δ is the adjoint of the derivative operator D. Concretely, a random variable $u\in L^2(\Omega;\mathcal{H})$ belongs to the domain of the divergence operator $Dom\delta$ if

$$\mathbf{E}\left|\langle DF,u\rangle_{\mathcal{H}}\right|\leq c_u\|F\|_{L^2(\Omega)}$$

for every $F\in\mathcal{S}$, where c_u is a constant which depends only on u. In this case $\delta(u)$ is given by the duality relationship

$$\mathbf{E}(F\delta(u))=E\left\langle DF,u\right\rangle_{\mathcal{H}}$$

for any $F\in D^{1,2}$. We will make use of the notation

$$\delta(u)=\int_0^T u_s\delta B_s,\quad u\in Dom\delta.$$

In particular, for $h\in\mathcal{H}$, $B(h)=\delta(h)=\int_0^T h_s\delta B_s$.

Assume that $H\in(\frac{1}{2},1)$. If $u\in D^{1,2}(|\mathcal{H}|)$, u belongs to $Dom\delta$ and we have (see [21, page 292])

$$\mathbf{E}(|\delta(u)|^2)\leq c_H\left(\|\mathbf{E}(u)\|_{|\mathcal{H}|}^2+\mathbf{E}\left(\|Du\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^2\right)\right),$$

where the constant c_H depends only on H. As a consequence, applying (4.1) we obtain that

$$\mathbf{E}(|\delta(u)|^2)\leq c_H\left(\|\mathbf{E}(u)\|_{L^{\frac{1}{H}}([0,T])}^2+\mathbf{E}\left(\|Du\|_{L^{\frac{1}{H}}([0,T]^2)}^2\right)\right). \tag{4.3}$$

For every $n\geq 1$, let $\mathcal{H}_n$ be the nth Wiener chaos of B, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(B(h)),h\in\mathcal{H},\|h\|_{\mathcal{H}}=1\}$ where H_n is the nth Hermite polynomial. The mapping $I_n(h^{\otimes n})=n!H_n(B(h))$ provides a linear isometry between the symmetric tensor product $\mathcal{H}^{\odot n}$ (equipped with the modified norm $\|.\|_{\mathcal{H}^{\odot n}}=\frac{1}{\sqrt{n!}}\|.\|_{\mathcal{H}^{\otimes n}}$) and $\mathcal{H}_n$. For every $f,g\in\mathcal{H}^{\odot n}$ the following product formula holds

$$\mathbf{E}\left(I_n(f)I_n(g)\right)=n!\langle f,g\rangle_{\mathcal{H}^{\otimes n}}.$$

Finally, It is well-known that $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_n$. That is, any square integrable random variable $F \in L^2(\Omega)$ admits the following chaotic expansion

$$F = \mathbf{E}(F) + \sum_{n=1}^{\infty} I_n(f_n),$$

where the $f_n \in \mathcal{H}^{\odot n}$ are uniquely determined by F.

Fix $T > 0$. Let $f, g : [0,T] \longrightarrow \mathbb{R}$ be Hölder continuous functions of orders $\alpha \in (0,1)$ and $\beta \in (0,1)$ respectively with $\alpha + \beta > 1$. Young [30] proved that the Riemann–Stieltjes integral (so-called Young integral) $\int_0^T f_s dg_s$ exists. Moreover, if $\alpha = \beta \in (\frac{1}{2}, 1)$ and $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function of class $\mathcal{C}^1$, the integrals $\int_0^{\cdot} \frac{\partial \phi}{\partial f}(f_u, g_u) df_u$ and $\int_0^{\cdot} \frac{\partial \phi}{\partial g}(f_u, g_u) dg_u$ exist in the Young sense and the following formula holds:

$$\phi(f_t, g_t) = \phi(f_0, g_0) + \int_0^t \frac{\partial \phi}{\partial f}(f_u, g_u) df_u + \int_0^t \frac{\partial \phi}{\partial g}(f_u, g_u) dg_u, \quad 0 \le t \le T. \tag{4.4}$$

As a consequence, if $H \in (\frac{1}{2}, 1)$ and $(u_t, \ t \in [0,T])$ is a process with Hölder paths of order $\alpha \in (1-H, 1)$, the integral $\int_0^T u_s dB_s$ is well-defined as a Young integral. Suppose moreover that for any $t \in [0,T]$, $u_t \in D^{1,2}$, and

$$P\left(\int_0^T \int_0^T |D_s u_t| |t-s|^{2H-2} ds dt < \infty \right) = 1.$$

Then, by [1], $u \in Dom\delta$ and for every $t \in [0,T]$,

$$\int_0^t u_s dB_s = \int_0^t u_s \delta B_s + H(2H-1) \int_0^t \int_0^t D_s u_r |s-r|^{2H-2} dr ds. \tag{4.5}$$

In particular, when φ is a non-random Hölder continuous function of order $\alpha \in (1-H, 1)$, we obtain

$$\int_0^T \varphi_s dB_s = \int_0^T \varphi_s \delta B_s = B(\varphi). \tag{4.6}$$

In addition, for all $\varphi, \ \psi \in |\mathcal{H}|$,

$$\mathbf{E}\left(\int_0^T \varphi_s dB_s \int_0^T \psi_s dB_s \right) = H(2H-1) \int_0^T \int_0^T \varphi_u \psi_v |u-v|^{2H-2} du dv. \tag{4.7}$$

Note that the above formula holds to any Gaussian process, i.e., if Y is a centered Gaussian process with covariance R in $L^1([0,T]^2)$, then (see e.g. [15])

$$\mathbf{E}\left(\int_0^T \varphi_s dY_s \int_0^T \psi_s dY_s \right) = \int_0^T \int_0^T \varphi_u \psi_v \frac{\partial^2 R}{\partial u \partial v} du dv \tag{4.8}$$

if φ, ψ are such that $\int_0^T \int_0^T \left| \varphi_u \psi_v \frac{\partial^2 R}{\partial u \partial v} \right| du dv < \infty$.

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