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3-D STOCHASTIC MICROPOLAR AND MAGNETO-MICROPOLAR FLUID SYSTEMS WITH NON-LIPSCHITZ MULTIPLICATIVE NOISE

KAZUO YAMAZAKI

Abstract. We study the stochastic micropolar and magneto-micropolar fluid systems with multiplicative noise in three-dimensional space. Without Lipschitz continuity condition on the noise, we show the existence of a weak martingale solution by applications of Prokhorov and Skorokhod's theorems, followed by de Rham's theorem generalized to processes.

1. Introduction

We study the following micropolar fluid (MPF) and magneto-micropolar fluid (MMPF) systems in $D\times[0,T]$ where D is a bounded simply-connected, Lipschitz, open domain in $\mathbb{R}^3$ respectively:

$$\partial_t u+(u\cdot\nabla)u+\nabla\pi=\chi(\nabla\times w)+(\mu+\chi)\Delta u+f_1(y,t)+g_1(y,t)\partial_t W_1, \tag{1.1a}$$

$$\begin{aligned} &j\partial_t w+j(u\cdot\nabla)w \\ =&-2\chi w+(\alpha+\beta)\nabla\text{div}w+\chi(\nabla\times u)+\gamma\Delta w+f_2(y,t)+g_2(y,t)\partial_t W_2, \end{aligned} \tag{1.1b}$$

$$\begin{aligned} &\partial_t u+(u\cdot\nabla)u-r(b\cdot\nabla)b+\nabla\pi \\ =&\chi(\nabla\times w)+(\mu+\chi)\Delta u+\tilde{f}_1(\tilde{y},t)+\tilde{g}_1(\tilde{y},t)\partial_t W_1, \end{aligned} \tag{1.2a}$$

$$\begin{aligned} &j\partial_t w+j(u\cdot\nabla)w \\ =&-2\chi w+(\alpha+\beta)\nabla\text{div}w+\chi(\nabla\times u)+\gamma\Delta w+\tilde{f}_2(\tilde{y},t)+\tilde{g}_2(\tilde{y},t)\partial_t W_2, \end{aligned} \tag{1.2b}$$

$$\partial_t b+(u\cdot\nabla)b-(b\cdot\nabla)u=-\nu\nabla\times\nabla\times b+\tilde{f}_3(\tilde{y},t)+\tilde{g}_3(\tilde{y},t)\partial_t W_3, \tag{1.2c}$$

where we denoted $y:=(u,w),\tilde{y}:=(u,w,b)$ with u,w,b,π the velocity, microrotational velocity, the magnetic vector fields and the hydrostatic pressure scalar field respectively. Moreover, $W_i, i=1,2,3$ represent the Wiener processes in m_i-dimension respectively. We also denoted physically meaningful quantities: $r=\frac{M^2}{\text{ReRm}}$ where M, Re, Rm are the Hartmann, the Reynolds and magnetic Reynolds numbers respectively, χ the vortex viscosity, μ the kinematic viscosity, j the microinertia, α,β,γ the spin viscosities, $\nu=\frac{1}{Rm}$ all of which we assume to be positive taking into account of conditions such as Calusius-Duhem inequality.

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Finally, $f_i, g_i, \tilde{f}_i, \tilde{g}_i$ are random forces. We denoted $\frac{d}{dt}$ by ∂_t and hereafter denote $\frac{d}{dx_i}$ by ∂_i and assume $j = r = 1$ as well.

We consider these systems with incompressibility, non-slip boundary and perfectly conducting wall conditions with nonrandom initial data:

$$\begin{cases} \nabla \cdot u = \nabla \cdot b = 0, & \forall\, t \in [0, T], \\ (u, w, b)(x, 0) = (u_0, w_0, b_0)(x), & x \in D, \\ u|_{\partial D}(t) = w|_{\partial D}(t) = 0, & \forall\, t \in [0, T], \\ b \cdot n|_{\partial D} = 0, & \forall\, t \in [0, T], \\ (\nabla \times b) \times n|_{\partial D} = 0, & \forall\, t \in [0, T] \end{cases} \tag{1.3}$$

where n is an outward unit normal vector on ∂D.

Let us discuss the rich history of mathematical and physical study concerning the MPF, MMPF and its related systems. Firstly, the system (1.2a)-(1.2c) at $\chi = 0, w \equiv b \equiv \tilde{g}_1 \equiv \tilde{f}_2 \equiv \tilde{g}_2 \equiv \tilde{f}_3 \equiv \tilde{g}_3 \equiv 0$ reduces to the deterministic Navier-Stokes equations (NSE) which has ample engineering applications in fluid mechanics such as design of aircraft and has been investigated mathematically with much intensity ever since the pioneering work in [21] (cf. [6, 24]). Moreover, the system (1.2a)-(1.2c) at $\chi = 0, w \equiv \tilde{g}_1 \equiv \tilde{f}_2 \equiv \tilde{g}_2 \equiv \tilde{g}_3 \equiv 0$ reduces to the deterministic magnetohydrodynamics (MHD) system which describes the motion of electrically conducting fluids and has broad applications in applied sciences (cf. [31]). The global regularity issue of the MHD system in two-dimensional case has in particular attracted much attention recently ([5] and references found therein).

Finally, the system (1.2a)-(1.2c) at $b \equiv \tilde{g}_1 \equiv \tilde{g}_2 \equiv \tilde{f}_3 \equiv \tilde{g}_3 \equiv 0$, which is the system (1.1a)-(1.1b) at $g_1 \equiv g_2 \equiv 0$, reduces to the deterministic MPF system, of which microfluids and micropolar fluids were introduced in [12, 13] respectively. In particular, the micropolar fluids represent the fluids consisting of bar-like elements, e.g. anisotropic fluids, such as liquid crystals made up of dumbbell molecules and animal blood. The study of this system has been conducted by many (e.g. [16, 23, 36, 40]). The deterministic MMPF system, the system (1.2a)-(1.2c) at $\tilde{g}_1 \equiv \tilde{g}_2 \equiv 0$, was studied in [1] and has also found much attraction (e.g. [17, 26, 28, 37]).

In this manuscript, we consider the most general case, namely the stochastic MPF and the stochastic MMPF systems. The study of the stochastic PDE in fluid mechanics has been investigated by many ([9, 10, 19] and references found in [15] for the stochastic NSE related literature and [2, 8, 30, 35] for the stochastic MHD system). To the best of the author's knowledge despite the significance in engineering applications, the study of the stochastic MPF or the stochastic MMPF system is yet to be undertaken. In case $g_i, \tilde{g}_i$ are only additive noise, the existence of solution may be attained by following the pioneering work of [7, 14]. Rather, we work with multiplicative noise which requires a considerably more delicate probabilistic compactness argument using Galerkin approximation combined with Prokhorov's and Skorokhod's theorems which has been used for the stochastic NSE and MHD systems.

In the next section, we set up notations, state preliminaries and then our main result. Thereafter, we prove the main results on the MMPF system. By the way set up notations, it will be clear that the proof in the case of the MPF system will be identical to the proof of the case of the MMPF system.

2. Notations, Preliminaries and Statement of Main Result

2.1. Notations. Let us denote a constant that depends on a, b by $c(a, b)$ and when the constant is not of significance, let us write $A \lesssim B, A \approx B$ to imply that there exists some constant c such that $A \leq cB, A = cB$ respectively.

2.2. Preliminaries and Statement of Main Result. We denote the Lebesgue spaces by $L^p(D)$, Sobolev spaces by $H^m(D), W^{m,p}(D)$, the space of continuous scalar-valued functions with compact support $C_c(D)$. We also denote by $H_0^m = \text{Ker}(\gamma_0)$ where $\gamma_0 : H^m(D) \mapsto L^2(\partial D)$ is the trace operator that is bounded and agrees with the restriction operator $v \mapsto v|_{\partial D}$ for $v \in C^1(\overline{D})$. We denote for the d-dimensional vector-valued functions in $L^p(D)$ with $d = 3$ in particular, $\mathbb{L}^p := (L^p(D))^d$, $\mathbb{W}^{m,p} := (W^{m,p}(D))^d$, $\mathbb{H}_0^m := (H_0^m(D))^d$ and for clarity, for the Lebesgue spaces in time or Ω we shall always write for example $L^p([0, T])$, scalar or vector-valued.

Given any separable Banach space E, we denote by $\mathcal{B}(E)$ the Borel σ-algebra on E and $L_T^p(E), 1 \leq p \leq \infty$ the space of functions endowed with a norm

$$\|v\|_{L_T^p(E)} := (\int_0^T \|v\|_E^p d\tau)^{\frac{1}{p}}$$

if $1 \leq p < \infty$ with a standard generalization at $p = \infty$. We further denote by

$$\begin{aligned}
&\mathcal{V}_1 := \{v \in (C_c^\infty)^3 : \nabla \cdot v = 0\}, \\
&V_1 := \{v \in \mathbb{H}_0^1 : \nabla \cdot v = 0\}, \\
&\mathcal{V}_2 := \{v \in (C^\infty(\overline{D}))^3 : \nabla \cdot v = 0, v \cdot n|_{\partial D} = 0\}, \\
&V_2 := \{v \in \mathbb{H}^1 : \nabla \cdot v = 0,\ v \cdot n|_{\partial D} = 0\}, \\
&H_1 = H_2 := \{v \in \mathbb{L}^2 : \nabla \cdot v = 0,\ v \cdot n|_{\partial D} = 0\}.
\end{aligned}$$

We endow V_1, V_2 with the inner products of

$$((u, v))_1 := \sum_{i=1}^3 (\partial_i u, \partial_i v) \text{ where } (u, v) := \sum_{i=1}^3 \int_D u_i(x) v_i(x) dx,$$

$$((u, v))_2 := (\nabla \times u, \nabla \times v),$$

respectively. We let $V := V_1 \times \mathbb{H}_0^1 \times V_2$ endowed with its norm for $\Phi^i := (X^i, Y^i, Z^i) \in V, i = 1, 2,$

$$((\Phi^1, \Phi^2)) := ((X^1, X^2))_1 + ((Y^1, Y^2))_1 + ((Z^1, Z^2))_2, \quad ((\Phi^i, \Phi^i)) = \|\Phi^i\|^2.$$

In case $Z^i \equiv 0$ as in the solution for the stochastic MPF system (1.1a)-(1.1b), we have $((\Phi^1, \Phi^2)) := ((X^1, X^2))_1 + ((Y^1, Y^2))_1$. We also let $H := H_1 \times \mathbb{L}^2 \times H_2$ endowed with its norm

$$(\Phi^1, \Phi^2) := (X^1, X^2) + (Y^1, Y^2) + (Z^1, Z^2), \quad (\Phi^i, \Phi^i) = |\Phi^i|^2,$$

for which if $Z^i \equiv 0$ as for the system (1.1a)-(1.1b), we have $(\Phi^1, \Phi^2) = (X^1, X^2) + (Y^1, Y^2)$. We introduce $L^p(\Omega, \mathcal{F}, P; L^r(0, T; \mathbb{L}^s))$ endowed with a norm

$$\|u\|_{L^p(\Omega,\mathcal{F},P;L^r(0,T;\mathbb{L}^s))} := \left(E\|u(\omega,\cdot,\cdot)\|^p_{L^r(0,T;\mathbb{L}^s)}\right)^{\frac{1}{p}}.$$

We define three operators A_1, A_2, A_3 with their domains respectively by

$$\begin{aligned}
&< A_1X^1, X^2 >:= -(\mu+\chi)(\Delta X^1, X^2),\ D(A_1) = \mathbb{H}^2 \cap V_1,\\
&< A_2Y^1, Y^2 >:= < -\gamma\Delta Y^1 - (\alpha+\beta)\nabla \text{div} Y^1 + 2\chi Y^1, Y^2 >,\ D(A_2) = \mathbb{H}^2 \cap \mathbb{H}^1_0,\\
&< A_3Z^1, Z^2 >:= \nu < \nabla\times\nabla\times Z^1, Z^2 >,\\
&\qquad D(A_3) = H_1 \cap \left\{b \in \mathbb{H}^2 : (\nabla\times b)\cdot n|_{\partial D} = 0\right\},
\end{aligned}$$

where we recall that $\nabla\times(\nabla\times f) = \nabla(\nabla\cdot f) - \Delta f$. Finally, we let

$$< A\Phi^1, \Phi^2 >:=< A_1X^1, X^2 > + < A_2Y^1, Y^2 > + < A_3Z^1, Z^2 > .$$

Again, in case $b \equiv 0$ as in the case of the system (1.1a)-(1.1b), we have $< A\Phi^1, \Phi^2 >:=< A_1X^1, X^2 > + < A_2Y^1, Y^2 >$.

Next, we define $B(\tilde{y}) := (B_1(\tilde{y}), B_2(\tilde{y}), B_3(\tilde{y}))$ with $\tilde{y} := (u, w, b)$ where

$$\begin{aligned}
&B_1(\tilde{y}) := (u\cdot\nabla)u - (b\cdot\nabla)b - \chi\nabla\times w,\\
&B_2(\tilde{y}) := (u\cdot\nabla)w - \chi\nabla\times u, \qquad B_3(\tilde{y}) := (u\cdot\nabla)b - (b\cdot\nabla)u,
\end{aligned}$$

and consequently with $y := (u, w, 0)$ for the MPF system (1.1a)-(1.1b), $B(y) := (B_1(y), B_2(y), B_3(y))$ where

$$B_1(y) = (u\cdot\nabla)u - \chi\nabla\times w,\ \ B_2(y) = (u\cdot\nabla)w - \chi\nabla\times u,\ \ B_3(y) = 0.$$

We note the following property for $\tilde{y} = (u, w, b)$:

$$< B(\tilde{y}), \tilde{y} >= -2\chi\int(\nabla\times u)\cdot w \le 2\chi\|\nabla u\|_{\mathbb{L}^2}\|w\|_{\mathbb{L}^2} \le \chi\|\nabla u\|^2_{\mathbb{L}^2} + \chi\|w\|^2_{\mathbb{L}^2}$$

due to (1.3), Hölder's and Young's inequalities. This implies

$$\begin{aligned}
&< A\tilde{y}, \tilde{y} > + < B(\tilde{y}), \tilde{y} > \qquad (2.1)\\
\ge&\mu\|\nabla u\|^2_{\mathbb{L}^2} + \gamma\|\nabla w\|^2_{\mathbb{L}^2} + (\alpha+\beta)\|\text{div} w\|^2_{\mathbb{L}^2} + \chi\|w\|^2_{\mathbb{L}^2} + \nu\|\nabla b\|^2_{\mathbb{L}^2} \ge \tilde{c}\|y\|^2
\end{aligned}$$

for some constant $\tilde{c} := c(\mu, \gamma, \alpha, \beta, \chi, \nu) > 0$ and hereafter we assume $\tilde{c} \in (0, 1)$; if not, we can choose a smaller number and relabel it.

Next, we let

$$f := \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},\quad g := \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix},\quad W := \begin{pmatrix} W_1 \\ W_2 \end{pmatrix},$$

$$\tilde{f} := \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{pmatrix},\quad \tilde{g} := \begin{pmatrix} \tilde{g}_1 & 0 & 0 \\ 0 & \tilde{g}_2 & 0 \\ 0 & 0 & \tilde{g}_3 \end{pmatrix},\quad \tilde{W} := \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix}.$$

We can now state our theorems; below $\times m_i$ denotes direct product m_i times.

Definition 2.1. A *weak martingale solution* to (1.1a)-(1.1b) is $(\Omega, \mathcal{F}, \mathcal{F}_t, P, W, y)$ where

1. $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a filtered probability space,
2. $W(t)$ is $(m_1 + m_2)$-dimensional $\mathcal{F}_t$ measurable standard Wiener processes,

(3) $y \in L^p(\Omega, \mathcal{F}, P; L^2(0,T; V_1 \times \mathbb{H}_0^1)) \cap L^p(\Omega, \mathcal{F}, P; L^\infty(0,T; H_1 \times \mathbb{L}^2))$ $\forall\, p \in [1, \infty)$,
(4) for a.e. t, $y(t)$ is $\mathcal{F}_t$-measurable,
(5) for any $v \in D(A)$, a.s. $\forall\, t \in [0,T]$

$$\begin{aligned}&(y(t), v) + \int_0^t < Ay(\tau) + B(y(\tau)), v > d\tau \\ =&(y_0, v) + \int_0^t < f(y(\tau), \tau), v > d\tau + \int_0^t (g(y(\tau), \tau) dW(\tau), v).\end{aligned} \tag{2.2}$$

Definition 2.2. A *weak martingale solution* to (1.2a)-(1.2c) is $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{P}, \tilde{W}, \tilde{y})$ where

(1) $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{P})$ is a filtered probability space,
(2) $\tilde{W}(t)$ is $(m_1+m_2+m_3)$-dimensional $\tilde{\mathcal{F}}_t$ measurable standard Wiener processes,
(3) $\tilde{y} \in L^p(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}; L^2(0,T; V)) \cap L^p(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}; L^\infty(0,T; H))$ $\forall\, p \in [1, \infty)$,
(4) for a.e. t, $\tilde{y}(t)$ is $\tilde{\mathcal{F}}_t$-measurable,
(5) for any $v \in D(A)$, a.s. $\forall\, t \in [0,T]$

$$\begin{aligned}&(\tilde{y}(t), v) + \int_0^t < A\tilde{y}(\tau) + B(\tilde{y}(\tau)), v > d\tau \\ =&(\tilde{y}_0, v) + \int_0^t < \tilde{f}(\tilde{y}(\tau), \tau), v > d\tau + \int_0^t (\tilde{g}(\tilde{y}(\tau), \tau) d\tilde{W}(\tau), v).\end{aligned} \tag{2.3}$$

Theorem 2.3. *Suppose* $y_0 := y(0) \in H_1 \times \mathbb{L}^2$ *and* $f_i, g_i, i = 1,2$ *are nonlinear mappings that are continuous in* t, f *continuous from* $H_1 \times \mathbb{L}^2$ *to* $\mathbb{H}^{-1} \times \mathbb{H}^{-1}$, g_1, g_2 *continuous from* H_1 *to* $H_1^{\times m_1}$ *and* $\mathbb{L}^2$ *to* $(\mathbb{L}^2)^{\times m_2}$ *respectively and*

$$\begin{aligned}&\|f_1(y,t)\|_{\mathbb{H}^{-1}},\ \|g_1(y,t)\|_{H_1^{\times m_1}} \lesssim 1 + \|u\|_{H_1}, \\ &\|f_2(y,t)\|_{\mathbb{H}^{-1}},\ \|g_2(y,t)\|_{(\mathbb{L}^2)^{\times m_2}} \lesssim 1 + \|w\|_{\mathbb{L}^2}.\end{aligned} \tag{2.4}$$

Then there exists a solution to (1.1a)-(1.1b) as in Definition 2.1. Moreover, there exists a unique $\pi \in L^1(\Omega, \mathcal{F}, P; W^{-1,\infty}(0,T; L^2(D)))$ *such that* $\int_D \pi dx = 0$ *in* $(C_c^\infty([0,T]))'$ *and (1.1a)-(1.1b) holds in* $((C_c^\infty([0,T] \times D))')^3$.

Theorem 2.4. *Suppose* $\tilde{y}_0 := \tilde{y}(0) \in H$ *and* $\tilde{f}_i, \tilde{g}_i, i = 1,2,3$ *are nonlinear mappings that are continuous in* t, $\tilde{f}$ *continuous from* H *to* $\mathbb{H}^{-1} \times \mathbb{H}^{-1} \times \mathbb{H}^{-1}$, $\tilde{g}_1, \tilde{g}_2, \tilde{g}_3$ *continuous from* H_1 *to* $H_1^{\times m_1}$, $\mathbb{L}^2$ *to* $(\mathbb{L}^2)^{\times m_2}$ *and from* H_2 *to* $H_2^{\times m_3}$ *respectively and*

$$\begin{aligned}&\|\tilde{f}_1(\tilde{y},t)\|_{\mathbb{H}^{-1}},\ \|\tilde{g}_1(\tilde{y},t)\|_{H_1^{\times m_1}} \lesssim 1 + \|u\|_{H_1}, \\ &\|\tilde{f}_2(\tilde{y},t)\|_{\mathbb{H}^{-1}},\ \|\tilde{g}_2(\tilde{y},t)\|_{(\mathbb{L}^2)^{\times m_2}} \lesssim 1 + \|w\|_{\mathbb{L}^2}, \\ &\|\tilde{f}_3(\tilde{y},t)\|_{\mathbb{H}^{-1}},\ \|\tilde{g}_3(\tilde{y},t)\|_{H_2^{\times m_3}} \lesssim 1 + \|b\|_{H_2}.\end{aligned} \tag{2.5}$$

Then there exists a solution to (1.2a)-(1.2c) as in Definition 2.2. Moreover, there exists a unique $\pi \in L^1(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}; W^{-1,\infty}(0,T; L^2(D)))$ *such that* $\int_D \pi dx = 0$ *in* $(C_c^\infty([0,T]))'$ *and (1.2a)-(1.2c) holds in* $((C_c^\infty([0,T] \times D))')^3$.

Remark 2.5. (1) While we in particular follow the work of [4, 30] closely, the setup differs due to the addition of the micro-rotational velocity w. Moreover, in contrast to the case of the MHD system in [30], we do not have $< B(y), y >= 0$ but appropriate modifications can overcome this difficulty. On the other hand, in contrast to the Boussinesq system, $\chi(\nabla \times u)$ and $\chi(\nabla \times w)$ are more singular than θe_3 of the three-dimensional Boussinesq system which actually raises significant issues in some results on the deterministic side (cf. [11, 38]); indeed, there remain many global regularity results that are known for the latter system but not for the former.

(2) In case dimension is two, MPF and MMPF systems must be set up differently. Indeed, it is clear that a straight-forward generalization of u, w, b to two-dimensional vector fields leads to decoupling of (1.1a) and (1.2a) from w. In two-dimensional case, applying different estimates in Sobolev spaces and using Yamada-Watanabe uniqueness theorem, we have also shown the existence of unique strong solution to both systems under Lipschitz condition on the noise. We chose to present these results in a separate accompanying paper (cf. [39]).

Let us state key lemmas needed to prove Theorem 2.4. We refer readers to [18] for definitions of uniform integrability of a family of stochastic processes, tightness and relative compactness for a family of measures.

Lemma 2.6. *(cf. [22]) Suppose* $(g_k)_k, g \in L^q(0,T; L^q(D))$, $q \in (1, \infty)$ *satisfy*

$$\|g_k\|_{L^q(0,T;L^q(D))} \leq c \quad \forall\, k$$

and $g_k \to g \ (k \to \infty)$ *for almost all* $(x,t) \in D \times (0,T)$. *Then* g_k *converges weakly to* g *in* $L^q(0,T; L^q(D))$.

Remark 2.7. (cf. [30] Remark 6) The results of this lemma hold for the space $L^q(\Omega, \mathcal{F}, P; L^q(D))$ in $\Omega \times D$.

Lemma 2.8. *([27]) A family of probability measures on* E *is relatively compact if and only if it is tight.*

Lemma 2.9. *([33]) For arbitrary sequence of probability measures* $\{P_n\}_n$ *on* $(E, \mathcal{B}(E))$ *weakly convergent to a probability measure* P*, there exists a probability space* $(\Omega, \mathcal{F}, P)$ *and random variables* $\xi, \xi_1, \dots, \xi_n, \dots$ *valued in* E *such that*

(1) $\mathcal{L}(\xi_n) = P_n$,
(2) *the probability law of* ξ *is* P,
(3) $\lim_{n\to\infty} \xi_n = \xi \quad P-$ *almost surely.*

Let us also state de la Vallée-Poussin theorem and Vitali's convergence theorem:

Lemma 2.10. *(cf. [25]) The family* $\{X_\alpha\}_{\alpha \in A} \subset L^1(\mu)$ *is uniformly integrable if and only if there exists* $G(\tau) \geq 0$*, increasing and convex such that*

$$\lim_{\tau \to \infty} \frac{G(\tau)}{\tau} = \infty \text{ and } \sup_\alpha E[G(|X_\alpha|)] < \infty.$$

Lemma 2.11. *(cf. [29]) Let* $(\Omega, \mathcal{F}, P)$ *be the probability space. If* $\{\xi_n\}$ *is uniformly integrable and* $\xi_n(\omega) \to \xi(\omega)$ *P-a.s.* $(n \to \infty)$*, then* $\xi \in L^1(\Omega)$ *and* $\lim_{n\to\infty} \int_\Omega |\xi_n - \xi| dP = 0$.

The following generalization of de Rham's Theorem to processes is due to [20]:

Lemma 2.12. *([20]) Let $D \subset \mathbb{R}^d, d = 2, 3, 4$ be bounded, connected, Lipschitz and open and $(\Omega, \mathcal{F}, P)$ be a complete probability space. Suppose that for $r_0, r_1 \in [1, \infty], s_1 \in \mathbb{Z}$, $h \in L^{r_0}(\Omega, \mathcal{F}, P; W^{s_1,r_1}(0, T; (H^{-1}(D))^d))$ satisfies for any $v \in (C_c^\infty(D))^d$ such that $\nabla \cdot v = 0, P-a.s.$*

$$< h, v >_{((C_c^\infty(D))')^d \times (C_c^\infty(D))^d} = 0 \quad \text{in } (C_c^\infty)'(0, T).$$

Then there exists unique $\pi \in L^{r_0}(\Omega, \mathcal{F}, P; W^{s_1,r_1}(0, T; L^2(D)))$ such that P-a.s. $\nabla\pi = h$ in $((C_c^\infty([0, T] \times D))')^d, \int_D \pi dx = 0$ in $(C_c^\infty([0, T]))'$.

We will use the following elementary inequality repeatedly:

$$(a + b)^p \leq 2^p(a^p + b^p) \quad 0 \leq p < \infty, \ a, b \geq 0. \tag{2.6}$$

3. Proof of Theorem 2.4

3.1. Approximating sequence and a priori estimates. Let $\{c_j\}, \{d_j\}, \{e_j\}$ be the family of eigenfunctions so that

$$\begin{aligned} ((c_j, v^1))_{D(A_1)} &= \lambda_j^1(c_j, v^1) \quad \forall\, v^1 \in D(A_1), \\ ((d_j, v^2))_{D(A_2)} &= \lambda_j^2(d_j, v^2) \quad \forall\, v^2 \in D(A_2), \\ ((e_j, v^3))_{D(A_3)} &= \lambda_j^3(e_j, v^3) \quad \forall\, v^3 \in D(A_3). \end{aligned} \tag{3.1}$$

We may assume that $\{c_j\}, \{d_j\}, \{e_j\}$ are orthonormal basis in $D(A_i), i = 1, 2, 3$ respectively and we have

$$D(A) \subset V \subset H = H' \subset V' \subset D(A)'. \tag{3.2}$$

The eigenfunctions of $D(A)$ are $q_j = (q_j^1, q_j^2, q_j^3)$ where

$$q_j^1 := (c_j, 0, 0), \quad q_j^2 := (0, d_j, 0), \quad q_j^3 := (0, 0, e_j), \tag{3.3}$$

$$((v, q_j^k))_{D(A)} = \lambda_j^k(v, q_j^k) \quad \forall\, v \in D(A), k = 1, 2, 3. \tag{3.4}$$

We look for approximation to (2.3) of the form

$$\begin{cases} \tilde{y}^m(t) \in \text{span}\{q_j, j = 1, 2, \ldots, m\}, \\ \tilde{y}_0^m = \tilde{y}^m(0) \in \text{ span}\{q_j, j = 1, 2, \ldots, m\}, \end{cases} \tag{3.5}$$

and $\tilde{y}_0^m$ converges strongly to $\tilde{y}_0, (m \to \infty)$ in H. We may write by (3.4)

$$\tilde{y}^m(t) = \sum_{k=1}^{3}\sum_{j=1}^{m}((\tilde{y}^m(t), q_j^k))_{D(A)} q_j^k = \sum_{k=1}^{3}\sum_{j=1}^{m} \lambda_j^k(\tilde{y}^m(t), q_j^k) q_j^k. \tag{3.6}$$

We consider $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}, \overline{W})$ where $\overline{W}$ is $(m_1 + m_2 + m_3)$-dimensional Wiener process and study

$$\begin{aligned} &d(\tilde{y}^m(t), q_j^k) + < A\tilde{y}^m(t), q_j^k > dt + < B(\tilde{y}^m(t)), q_j^k > dt \\ &= < \tilde{f}(\tilde{y}^m(t), t), q_j^k > dt + (\tilde{g}(\tilde{y}^m(t), t), q_j^k) d\overline{W}, \quad t \in [0, T], \end{aligned} \tag{3.7}$$

for which locally on $[0, t_m]$, possibly $t_m < T$, the solution is known to exist (see e.g. [34] pg. 121 Theorem 2). To extend to $[0, T]$, we perform an a priori estimate. We first observe that $|\tilde{y}^m|^2 = (\tilde{y}^m, \tilde{y}^m) = \sum_{k=1}^3 \sum_{j=1}^m \lambda_j^k(\tilde{y}^m(t), q_j^k)^2$ due to (3.6).

Now we use Ito's formula and (3.7), multiply the resulting equation by λ_j^k, sum over $k = 1, 2, 3$ and $j = 1, \ldots, m$ to obtain

$$d|\tilde{y}^m(t)|^2 + 2 < A\tilde{y}^m(t), \tilde{y}^m(t) > dt + 2 < B(\tilde{y}^m(t)), \tilde{y}^m(t) > dt \tag{3.8}$$
$$= 2 < \tilde{f}(\tilde{y}^m(t), t), \tilde{y}^m(t) > dt + \sum_{k=1}^{3}\sum_{j=1}^{m} \lambda_j^k (\tilde{g}(\tilde{y}^m(t), t), q_j^k)^2 dt$$
$$+ 2(\tilde{g}(\tilde{y}^m(t), t), \tilde{y}^m(t)) d\overline{W}$$

due to (3.6). Next, for $p \in [4, \infty)$ by Ito's formula and (3.8) we have

$$d|\tilde{y}^m(t)|^p + p|\tilde{y}^m(t)|^{p-2}[< A\tilde{y}^m(t), \tilde{y}^m(t) > + < B(\tilde{y}^m(t)), \tilde{y}^m(t) >]dt \tag{3.9}$$
$$= \frac{p}{2}|\tilde{y}^m(t)|^{p-2}[2 < \tilde{f}(\tilde{y}^m(t), t), \tilde{y}^m(t) > + \sum_{k=1}^{3}\sum_{j=1}^{m} \lambda_j^k (\tilde{g}(\tilde{y}^m(t), t), q_j^k)^2$$
$$+ (p-2)|\tilde{y}^m(t)|^{-2}(\tilde{g}(\tilde{y}^m(t), t), \tilde{y}^m(t))^2]dt$$
$$+ p|\tilde{y}^m(t)|^{p-2}(\tilde{g}(\tilde{y}^m(t), t), \tilde{y}^m(t)) d\overline{W}.$$

We now prove the following proposition:

Proposition 3.1. *Let $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}, \overline{W})$ be a probabilistic system with $\overline{W}$, an $(m_1 + m_2 + m_3)$-dimensional Wiener process. Then the solution $\tilde{y}^m$ to (3.7) satisfies*

$$\overline{E}[\sup_{t \in [0,T]} |\tilde{y}^m(t)|^2] + \overline{E}[\int_0^T \|\tilde{y}^m(t)\|^2 dt] \lesssim 1.$$

Proof. On (3.8), we use (2.1) to obtain

$$d|\tilde{y}^m(t)|^2 + 2\tilde{c}\|\tilde{y}^m(t)\|^2 dt \leq 2 < \tilde{f}(\tilde{y}^m(t), t), \tilde{y}^m(t) > dt \tag{3.10}$$
$$+ \sum_{k=1}^{3}\sum_{j=1}^{m} \lambda_j^k (\tilde{g}(\tilde{y}^m(t), t), q_j^k)^2 dt + 2(\tilde{g}(\tilde{y}^m(t), t), \tilde{y}^m(t)) d\overline{W}.$$

We integrate in time $[0, t]$, use the following standard stopping time

$$\lambda_N := \begin{cases} \inf\{\tau > 0 : |\tilde{y}^m(\tau)| \geq N\} & \text{if } \{\overline{\omega} \in \overline{\Omega} : |\tilde{y}^m(\tau)| \geq N\} \neq \emptyset, \\ \infty & \text{if } \{\overline{\omega} \in \overline{\Omega} : |\tilde{y}^m(\tau)| \geq N\} = \emptyset, \end{cases}$$

take supremum over $\tau \in [0, t \wedge \lambda_N], t \in [0, t_m]$ and $\overline{E}$, the expected value with respect to $\overline{P}$, to obtain

$$\overline{E}[\sup_{\tau \in [0, t\wedge\lambda_N]} |\tilde{y}^m(\tau)|^2] + 2\tilde{c}\overline{E}[\int_0^{t\wedge\lambda_N} \|\tilde{y}^m(\tau)\|^2 d\tau] \tag{3.11}$$
$$\leq |\tilde{y}_0^m|^2 + 2\overline{E}[\int_0^{t\wedge\lambda_N} |< \tilde{f}(\tilde{y}^m(\tau), \tau), \tilde{y}^m(\tau) >| + \sum_{k=1}^{3}\sum_{j=1}^{m} \lambda_j^k (\tilde{g}(\tilde{y}^m(\tau), \tau), q_j^k)^2 d\tau]$$
$$+ 2\overline{E}[\sup_{\tau \in [0, t\wedge\lambda_N]} \left| \int_0^\tau (\tilde{g}(\tilde{y}^m(\eta), \eta), \tilde{y}^m(\eta)) d\overline{W}(\eta) \right|].$$

By (2.5) and Young's inequality we obtain

$$\int_0^{t\wedge\lambda_N} |< \tilde{f}(\tilde{y}^m(\tau),\tau), \tilde{y}^m(\tau) >| d\tau \lesssim \int_0^{t\wedge\lambda_N} (1+|\tilde{y}^m(\tau)|)\|\tilde{y}^m(\tau)\| d\tau$$
$$\leq \frac{\tilde{c}}{2}\int_0^{t\wedge\lambda_N} \|\tilde{y}^m(\tau)\|^2 d\tau + c\int_0^{t\wedge\lambda_N} (1+|\tilde{y}^m(\tau)|^2) d\tau.$$

Thus,

$$2\overline{E}[\int_0^{t\wedge\lambda_N} |< \tilde{f}(\tilde{y}^m(\tau),\tau), \tilde{y}^m(\tau) >| d\tau] \tag{3.12}$$
$$\leq \tilde{c}\overline{E}[\int_0^{t\wedge\lambda_N} \|\tilde{y}^m(\tau)\|^2 d\tau] + c\overline{E}[\int_0^{t\wedge\lambda_N} (1+|\tilde{y}^m(\tau)|^2) d\tau].$$

Next, using $\sum_{k=1}^3\sum_{j=1}^m \lambda_j^k |q_j^k|^2 \lesssim 1$, we estimate by (2.5),

$$\int_0^{t\wedge\lambda_N} \sum_{k=1}^3\sum_{j=1}^m \lambda_j^k (\tilde{g}(\tilde{y}^m(\tau),\tau), q_j^k)^2 d\tau \leq \int_0^{t\wedge\lambda_N} |\tilde{g}(\tilde{y}^m(\tau),\tau)|^2 \sum_{k=1}^3\sum_{j=1}^m \lambda_j^k |q_j^k|^2 d\tau$$
$$\lesssim \int_0^{t\wedge\lambda_N} (1+|\tilde{y}^m(\tau)|^2) d\tau.$$

Hence,

$$2\overline{E}[\int_0^{t\wedge\lambda_N} \sum_{k=1}^3\sum_{j=1}^m \lambda_j^k (\tilde{g}(\tilde{y}^m(\tau),\tau), q_j^k)^2 d\tau] \lesssim \overline{E}[\int_0^{t\wedge\lambda_N} 1+|\tilde{y}^m(\tau)|^2 d\tau]. \tag{3.13}$$

Finally,

$$2\overline{E}[\sup_{\tau\in[0,t\wedge\lambda_N]} \left|\int_0^{\tau} (\tilde{g}(\tilde{y}^m(\eta),\eta), \tilde{y}^m(\eta)) d\overline{W}(\eta)\right|] \tag{3.14}$$
$$\lesssim \overline{E}[\left(\int_0^{t\wedge\lambda_N} |(\tilde{g}(\tilde{y}^m(\eta),\eta), \tilde{y}^m(\eta))|^2 d\eta\right)^{\frac{1}{2}}]$$
$$\leq (1-\tilde{c})\overline{E}[\sup_{\tau\in[0,t\wedge\lambda_N]} |\tilde{y}^m(\tau)|^2] + c\overline{E}[\int_0^{t\wedge\lambda_N} (1+|\tilde{y}^m(\eta)|^2) d\eta]$$

by Burkholder-Davis-Gundy and Hölder's

inequalities, (2.5) and Young's inequality. We consider (3.12)-(3.14) into (3.11) and obtain after absorbing

$$\overline{E}[\sup_{\tau\in[0,t\wedge\lambda_N]} |\tilde{y}^m(\tau)|^2] + \overline{E}[\int_0^{t\wedge\lambda_N} \|\tilde{y}^m(\tau)\|^2 d\tau] \leq 1 + \overline{E}[\int_0^{t\wedge\lambda_N} 1+|\tilde{y}^m(\tau)|^2 d\tau \tag{3.15}$$

as $\tilde{y}_0 \in H$. Gronwall's inequality type argument along with the independence of the bound with respect to m and N which allows us to take the limit $N \to \infty$ completes the proof of Proposition 3.1. ☐

Proposition 3.2. *Let $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}, \overline{W})$ be a probabilistic system with $\overline{W}$, an $(m_1 + m_2 + m_3)$- dimensional Wiener process. Then the solution $\tilde{y}^m$ to (3.7) satisfies*

$$\overline{E}[\sup_{t\in[0,T]} |\tilde{y}^m(t)|^p + \left(\int_0^T \|\tilde{y}^m(t)\|^2 dt\right)^p] \lesssim 1 \quad \forall\, p \geq 1,$$

$$\overline{E}[\left(\int_0^T |\tilde{y}^m(t)|^{p-2} \|\tilde{y}^m(t)\|^2 dt\right)^2] \lesssim 1 \quad \forall\, p \geq 4.$$

Proof. We integrate (3.9) which holds $\forall\, p \geq 4$ over $[0,\tau]$, apply (2.1), take supremum over $\tau \in [0,t]$, square both sides and use (2.6) to obtain

$$\begin{aligned}
&\sup_{\tau\in[0,t]} |\tilde{y}^m(\tau)|^{2p} + (p\tilde{c})^2 \left(\int_0^t |\tilde{y}^m(\tau)|^{p-2} \|\tilde{y}^m(\tau)\|^2 d\tau\right)^2 \\
\lesssim &|\tilde{y}_0^m|^{2p} + \left(\int_0^t |\tilde{y}^m(\tau)|^{p-2} < \tilde{f}(\tilde{y}^m(\tau),\tau), \tilde{y}^m(\tau) > d\tau\right)^2 \\
&+ \left(\int_0^t |\tilde{y}^m(\tau)|^{p-2} \sum_{k=1}^3 \sum_{j=1}^m \lambda_j^k (\tilde{g}(\tilde{y}^m(\tau),\tau), q_j^k)^2 d\tau\right)^2 \\
&+ \left(\int_0^t |\tilde{y}^m(\tau)|^{p-4} (\tilde{g}(\tilde{y}^m(\tau),\tau), \tilde{y}^m(\tau))^2 d\tau\right)^2 \\
&+ \left(\sup_{\tau\in[0,t]} \left|\int_0^\tau |\tilde{y}^m(\eta)|^{p-2} (\tilde{g}(\tilde{y}^m(\eta),\eta), \tilde{y}^m(\eta)) d\overline{W}(\eta)\right|\right)^2.
\end{aligned} \tag{3.16}$$

We take $\overline{E}$ on both sides and first estimate

$$\begin{aligned}
&c\overline{E}[\left(\int_0^t |\tilde{y}^m(\tau)|^{p-2} < \tilde{f}(\tilde{y}^m(\tau),\tau), \tilde{y}^m(\tau) > d\tau\right)^2] \\
\leq &c\overline{E}[\left(\int_0^t |\tilde{y}^m(\tau)|^{p-2} |\tilde{f}(\tilde{y}^m(\tau),\tau)|_{V'}^2 d\tau\right)^2] \\
&+ \frac{(p\tilde{c})^2}{2} \overline{E}[\left(\int_0^t |\tilde{y}^m(\tau)|^{p-2} \|\tilde{y}^m(\tau)\|^2 d\tau\right)^2]
\end{aligned} \tag{3.17}$$

by Young's inequality and (2.6) where the first term is bounded by

$$c\overline{E}[\left(\int_0^t |\tilde{y}^m(\tau)|^{p-2} |\tilde{f}(\tilde{y}^m(\tau),\tau)|_{\mathbb{H}^{-1}}^2 d\tau\right)^2] \lesssim \overline{E}[1 + \int_0^t |\tilde{y}^m(\tau)|^{2p} d\tau] \tag{3.18}$$

due to (2.5)-(2.6) and Hölder's inequality. Next, similarly to (3.13),

$$\overline{E}[\left(\int_0^t |\tilde{y}^m(\tau)|^{p-2} \sum_{k=1}^3 \sum_{j=1}^m \lambda_j^k (\tilde{g}(\tilde{y}^m(\tau),\tau), q_j^k)^2 d\tau\right)^2] \lesssim \overline{E}[1 + \int_0^t |\tilde{y}^m(\tau)|^{2p} d\tau] \tag{3.19}$$

by (2.5), Young's and Hölder's inequalities. Similarly,

$$c\overline{E}[\left(\int_0^t |\tilde{y}^m(\tau)|^{p-4}(\tilde{g}(\tilde{y}^m(\tau),\tau),\tilde{y}^m(\tau))^2 d\tau\right)^2] \lesssim \overline{E}[1+\int_0^t |\tilde{y}^m(\tau)|^{2p}d\tau] \quad (3.20)$$

by (2.5), Young's and Hölder's inequalities. Finally,

$$c\overline{E}[\left(\sup_{\tau\in[0,t]}\left|\int_0^\tau |\tilde{y}^m(\eta)|^{p-2}(\tilde{g}(\tilde{y}^m(\eta),\eta),\tilde{y}^m(\eta))d\overline{W}(\eta)\right|\right)^2] \quad (3.21)$$

$$\lesssim \overline{E}[\int_0^t |\tilde{y}^m(\eta)|^{2p-4}|(\tilde{g}(\tilde{y}^m(\eta),\eta),\tilde{y}^m(\eta))|^2 d\eta] \lesssim \overline{E}[1+\int_0^t |\tilde{y}^m(\eta)|^{2p}d\eta]$$

by Burkholder-Davis-Gundy inequality, (2.5)-(2.6) and Young's inequality. Considering (3.17)-(3.21) in (3.16), Gronwall's inequality type argument implies

$$\overline{E}[\sup_{\tau\in[0,T]} |\tilde{y}^m(\tau)|^{2p}] \lesssim 1 \quad (3.22)$$

$\forall\, p \geq 4$ and hence $\forall\, p \geq \frac{1}{2}$. Moreover, it follows that

$$\overline{E}[\left(\int_0^T |\tilde{y}^m(\tau)|^{p-2}\|\tilde{y}^m(\tau)\|^2 d\tau\right)^2] \lesssim 1. \quad (3.23)$$

Similarly, we integrate (3.10) over $[0,t]$, take supremum over $\tau \in [0,t]$, raise to the power $p \in [1,\infty)$ and use (2.6) to obtain

$$\begin{aligned}(2\tilde{c})^p\left(\int_0^t \|\tilde{y}^m(\tau)\|^2 d\tau\right)^p \lesssim & |\tilde{y}_0^m|^{2p} + \left(\int_0^t |<\tilde{f}(\tilde{y}^m(\tau),\tau),\tilde{y}^m(\tau)>|d\tau\right)^p \\ & + \left(\int_0^t \sum_{k=1}^3\sum_{j=1}^m \lambda_j^k(\tilde{g}(\tilde{y}^m(\tau),\tau),q_j^k)^2 d\tau\right)^p \\ & + \left(\sup_{\tau\in[0,t]}\left|\int_0^\tau (\tilde{g}(\tilde{y}^m(\eta),\eta),\tilde{y}^m(\eta))d\overline{W}(\eta)\right|\right)^p.\end{aligned}$$

Taking $\overline{E}$, we estimate first

$$c\overline{E}[\left(\int_0^t |<\tilde{f}(\tilde{y}^m(\tau),\tau),\tilde{y}^m(\tau)>|d\tau\right)^p] \lesssim \overline{E}[\left(\int_0^t |\tilde{f}(\tilde{y}^m(\tau),\tau)|_{V'}|\tilde{y}^m(\tau)|_V d\tau\right)^p]$$

$$\leq c\overline{E}[\left(\int_0^t (1+|\tilde{y}^m(\tau)|)^2 d\tau\right)^p] + \frac{(2\tilde{c})^p}{2}\overline{E}[\left(\int_0^t \|\tilde{y}^m(\tau)\|^2 d\tau\right)^p]$$

by (2.5), Young's inequality and (2.6) where

$$c\overline{E}[\left(\int_0^t (1+|\tilde{y}^m(\tau)|)^2 d\tau\right)^p] \lesssim 1 + E[\sup_{\tau\in[0,t]} |\tilde{y}^m(\tau)|^{2p}] \lesssim 1$$

due to (2.6), Hölder's inequality and (3.22). Next,

$$c\overline{E}[\left(\int_0^t \sum_{k=1}^3\sum_{j=1}^m \lambda_j^k(\tilde{g}(\tilde{y}^m(\tau),\tau),q_j^k)^2 d\tau\right)^p] \lesssim \overline{E}[\left(\int_0^t 1+|\tilde{y}^m(\tau)|^2 d\tau\right)^p] \lesssim 1$$

by Hölder's inequality, the previous estimates in (3.13), (2.5) and (3.22). Finally,

$$
\begin{aligned}
&c\overline{E}[\left(\sup_{\tau\in[0,t]}\left|\int_0^\tau (\tilde{g}(\tilde{y}^m(\eta),\eta),\tilde{y}^m(\eta))d\overline{W}(\eta)\right|\right)^p]\\
\lesssim&\overline{E}[\left(\int_0^t |(\tilde{g}(\tilde{y}^m(\eta)),\tilde{y}^m(\eta),\eta)|^2 d\eta\right)^{\frac{p}{2}}] \lesssim \overline{E}[\left(\int_0^t 1+|\tilde{y}^m(\eta)|^4 d\eta\right)^{\frac{p}{2}}] \lesssim 1
\end{aligned}
$$

by Burkholder-Davis-Gundy inequality, (2.5), Hölder's inequality and (3.22). In sum after absorbing, for any $p\geq 1$,

$$
\overline{E}[\left(\int_0^T \|\tilde{y}^m(\tau)\|^2 d\tau\right)^p] \lesssim 1. \tag{3.24}
$$

This completes the proof of Proposition 3.2. □

Proposition 3.3. *Let $\delta\in(0,1)$ and $(\overline{\Omega},\overline{\mathcal{F}},\overline{P},\overline{W})$ be a probabilistic system with $\overline{W}$, an $(m_1+m_2+m_3)$-dimensional Wiener process. Then the solution $\tilde{y}^m$ to (3.7) satisfies*

$$
\overline{E}[\sup_{\theta\in[0,\delta]}\int_0^T \|\tilde{y}^m(t+\theta)-\tilde{y}^m(t)\|^2_{D(A)'}dt] \lesssim \delta^{\frac{1}{3}}.
$$

Proof. We estimate the difference of $\tilde{y}^m(t+\theta)-\tilde{y}^m(t)$. Firstly, we notice that

$$
\{\lambda_j^k q_j^k, k=1,2,3, j=1,\dots,m\}
$$

forms an orthonormal basis in $D(A)'$ and introduce a projection P^m in $D(A)'$ onto the span of

$$
\{\lambda_j q_j^k, k=1,2,3, j=1,\dots m\}
$$

defined by

$$
P^m\xi := \sum_{k=1}^3\sum_{j=1}^m \lambda_j^k < \xi, q_j^k > q_j^k.
$$

We integrate (3.7) from $[0,t]$, multiply by $\lambda_j^k q_j^k$, sum over $k=1,2,3, j=1,\dots,m$, use (3.6) to obtain

$$
\begin{aligned}
&\tilde{y}^m(t)+\int_0^t P^m A\tilde{y}^m(\tau)d\tau + \int_0^t P^m B(\tilde{y}^m(\tau))d\tau \qquad (3.25)\\
&=\tilde{y}_0^m+\int_0^t P^m\tilde{f}(\tilde{y}^m(\tau),\tau)d\tau+\int_0^t P^m\tilde{g}(\tilde{y}^m(\tau),\tau)d\overline{W}(\tau).
\end{aligned}
$$

We subtract the difference between $\tilde{y}^m(t+\theta)$ and $\tilde{y}^m(t)$, take $\|\cdot\|_{D(A)'}$, square both sides, take sup over $\theta \in [0,\delta], \delta \in (0,1)$, integrate over $[0,T]$ to obtain

$$\begin{aligned} &\sup_{\theta\in[0,\delta]} \int_0^T \|\tilde{y}^m(t+\theta) - \tilde{y}^m(t)\|_{D(A)'}^2 dt \qquad (3.26) \\ \lesssim &\sup_{\theta\in[0,\delta]} \int_0^T \| \int_t^{t+\theta} P^m A\tilde{y}^m(\tau) d\tau\|_{D(A)'}^2 dt \\ &+ \sup_{\theta\in[0,\delta]} \int_0^T \| \int_t^{t+\theta} P^m B(\tilde{y}^m(\tau)) d\tau\|_{D(A)'}^2 dt \\ &+ \sup_{\theta\in[0,\delta]} \int_0^T \| \int_t^{t+\theta} P^m \tilde{f}(\tilde{y}^m(\tau),\tau) d\tau\|_{D(A)'}^2 dt \\ &+ \sup_{\theta\in[0,\delta]} \int_0^T \| \int_t^{t+\theta} P^m \tilde{g}(\tilde{y}^m(\tau),\tau) d\overline{W}(\tau)\|_{D(A)'}^2 dt. \end{aligned}$$

We take $\overline{E}$ on (3.26) and estimate

$$\begin{aligned} &c\overline{E}[\sup_{\theta\in[0,\delta]} \int_0^T \| \int_t^{t+\theta} P^m A\tilde{y}^m(\tau) d\tau\|_{D(A)'}^2 dt] \qquad (3.27) \\ \lesssim &\overline{E}[\sup_{\theta\in[0,\delta]} \int_0^T \theta \int_t^{t+\theta} \|A\tilde{y}^m(\tau)\|_{D(A)'}^2 d\tau dt] \lesssim T\delta^2 \overline{E}[\sup_{t\in[0,T]} |\tilde{y}^m(t)|^2] \lesssim \delta \end{aligned}$$

by Hölder's inequality, the fact that $P^m \in \mathcal{L}(D(A)', D(A)')$, where $\mathcal{L}(X,Y)$ is the space of bounded linear maps from X to Y, and Proposition 3.1. We also used

$$\int_0^T \int_t^{t+\theta} |f(s)| ds dt = \int_0^\theta [\int_0^T |f(t+s)| dt] ds \leq \theta \|f\|_{L^1([0,T])}$$

which hereafter we shall continue to use without further mentioning. Next,

$$\begin{aligned} &\overline{E}[\sup_{\theta\in[0,\delta]} \int_0^T \| \int_t^{t+\theta} P^m B(\tilde{y}^m(\tau)) d\tau\|_{D(A)'}^2 dt] \qquad (3.28) \\ \lesssim &\overline{E}[\sup_{\theta\in[0,\delta]} \int_0^T \left| \int_t^{t+\theta} \|P^m B(\tilde{y}^m(\tau))\|_{D(A)'} d\tau \right|^2 dt] \\ \lesssim &\overline{E}[\int_0^T \left| \int_t^{t+\delta} \|B(\tilde{y}^m(\tau))\|_{V'} d\tau \right|^2 dt] \end{aligned}$$

because $P^m \in \mathcal{L}(D(A)', D(A)')$. Now we estimate for $\tilde{y}^m = (u^m, w^m, b^m)$,

$$\begin{aligned} &\|B(\tilde{y}^m)\|_{V'} \\ \leq &\|\mathrm{div}(u^m \otimes u^m)\|_{V'} + \|\mathrm{div}(b^m \otimes b^m)\|_{V'} + \chi\|\nabla \times w^m\|_{V'} \\ &+ \|\mathrm{div}(u^m \otimes w^m)\|_{V'} + \chi\|\nabla \times u^m\|_{V'} + \|\mathrm{div}(u^m \otimes b^m)\|_{V'} + \|\mathrm{div}(b^m \otimes u^m)\|_{V'} \\ \lesssim &\|\tilde{y}^m\|_{\mathbb{L}^4}^2 + \|\tilde{y}^m\|_{\mathbb{L}^2} \lesssim 1 + \|\tilde{y}^m\|_{\mathbb{L}^4}^2 \lesssim 1 + \|\tilde{y}^m\|_{\mathbb{L}^2}^{\frac{1}{2}} \|\tilde{y}^m\|_{\mathbb{L}^6}^{\frac{3}{2}} \lesssim 1 + |\tilde{y}^m|^{\frac{1}{2}} \|\tilde{y}^m\|^{\frac{3}{2}} \end{aligned}$$

where we used Hölder's and interpolation inequalities and a Sobolev embedding. Thus, using this in (3.28), we obtain

$$
\begin{aligned}
&\overline{E}[\sup_{\theta\in[0,\delta]}\int_0^T \|\int_t^{t+\theta} P^m B(\tilde{y}^m(\tau))d\tau\|_{D(A)'}^2 dt] \qquad (3.29)\\
\lesssim&\overline{E}[\int_0^T \left|\int_t^{t+\delta} 1+|\tilde{y}^m(\tau)|^{\frac{1}{2}}\|\tilde{y}^m(\tau)\|^{\frac{3}{2}}d\tau\right|^2 dt]\\
\lesssim&\overline{E}[\int_0^T \left|\delta+\sup_{\tau\in[t,t+\delta]}|\tilde{y}^m(\tau)|^{\frac{1}{2}}\delta^{\frac{1}{4}}\left(\int_t^{t+\delta}\|\tilde{y}^m(\tau)\|^2 d\tau\right)^{\frac{3}{4}}\right|^2 dt]\\
\lesssim&\overline{E}[\int_0^T \delta^2+\sup_{\tau\in[t,t+\delta]}|\tilde{y}^m(\tau)|^4\delta+\delta^{\frac{1}{3}}\left(\int_t^{t+\delta}\|\tilde{y}^m(\tau)\|^2 d\tau\right)^2 dt]\\
\lesssim&T\left(\delta^2+E[\sup_{t\in[0,T]}|\tilde{y}^m(\tau)|^4]\delta+\delta^{\frac{1}{3}}\overline{E}[\left(\int_0^T\|\tilde{y}^m(\tau)\|^2 d\tau\right)^2]\right)\lesssim\delta^{\frac{1}{3}}
\end{aligned}
$$

by Hölder's and Young's inequalities and Proposition 3.2. Next,

$$
\begin{aligned}
&\overline{E}[\sup_{\theta\in[0,\delta]}\int_0^T \|\int_t^{t+\theta} P^m\tilde{f}(\tilde{y}^m(\tau),\tau)d\tau\|_{D(A)'}^2 dt] \qquad (3.30)\\
\lesssim&\overline{E}[\sup_{\theta\in[0,\delta]}\int_0^T \left|\sqrt{\theta}\left(\int_t^{t+\theta}\|\tilde{f}(\tilde{y}^m(\tau),\tau)\|_{D(A)'}^2 d\tau\right)^{\frac{1}{2}}\right|^2 dt]\\
\lesssim&\delta\overline{E}[\int_0^T\int_t^{t+\delta}(1+|\tilde{y}^m(\tau)|)^2 d\tau dt]\lesssim\delta^2
\end{aligned}
$$

by Hölder's inequality, $P^m \in \mathcal{L}(D(A)', D(A)')$, (2.5)-(2.6) and Proposition 3.1. Finally,

$$
\begin{aligned}
&\overline{E}[\sup_{\theta\in[0,\delta]}\int_0^T \|\int_t^{t+\theta} P^m\tilde{g}(\tilde{y}^m(\tau),\tau)d\overline{W}(\tau)\|_{D(A)'}^2 dt] \qquad (3.31)\\
\lesssim&\overline{E}[\sup_{\theta\in[0,\delta]}\int_0^T \|\int_t^{t+\theta}\tilde{g}(\tilde{y}^m(\tau),\tau)d\overline{W}(\tau)\|_{D(A)'}^2 dt]\\
\lesssim&\int_0^T\int_D\overline{E}[\left|\sup_{\theta\in[0,\delta]}|\int_t^{t+\theta}\tilde{g}(\tilde{y}^m(\tau),\tau)d\overline{W}(\tau)|\right|^2]dxdt\\
\lesssim&\int_0^T\int_D\overline{E}[\int_t^{t+\delta}|\tilde{g}(\tilde{y}^m(\tau),\tau)|^2 d\tau]dxdt\lesssim\overline{E}[\int_0^T\int_t^{t+\delta}(1+|\tilde{y}^m(\tau)|)^2 d\tau dt]\lesssim\delta
\end{aligned}
$$

due to $P^m \in \mathcal{L}(D(A)', D(A)')$, Burkholder-Davis-Gundy inequality, (2.5)-(2.6) and Proposition 3.1. Using (3.27), (3.29)-(3.31) in (3.26), we obtain

$$
\overline{E}[\sup_{\theta\in[0,\delta]}\int_0^T\|\tilde{y}^m(t+\theta)-\tilde{y}^m(t)\|_{D(A)'}^2 d\tau]\lesssim\delta^{\frac{1}{3}}. \qquad (3.32)
$$

This completes the proof of Proposition 3.3. □

3.2. Application of Prokhorov and Skorokhod theorems. Having completed our a priori estimates, for the rest of the proof where there is much similarity to the cases of the NSE or MHD system, we only sketch it for completeness. We first recall the following useful fact due to [32] (see also [3, 4]):

Lemma 3.4. *For any* $\{\mu_n\}, \{\nu_n\}, \mu_n, \nu_n \geq 0$ *such that* $\mu_n, \nu_n \to 0$ $(n \to \infty)$, *the space*

$$\left\{ z \in L^2(0,T;V) \cap L^\infty(0,T;H), \sup_{n, |\theta| \leq \mu_n} \frac{\left(\int_0^T \|z(t+\theta) - z(t)\|^2_{D(A)'} dt\right)^{\frac{1}{2}}}{\nu_n} < \infty \right\},$$

which we denote by Y_{μ_n,ν_n}, *is a compact subset of* $L^2(0,T;H)$. *Moreover, it is a Banach space if endowed with the norm*

$$\|z\|_{Y_{\mu_n,\nu_n}} := \sup_{t \in [0,T]} |z(t)| + \left(\int_0^T \|z(t)\|^2 dt\right)^{\frac{1}{2}} + \sup_n \frac{1}{\nu_n} \sup_{|\theta| \leq \mu_n} \left(\int_0^T \|z(t+\theta) - z(t)\|^2_{D(A)'} dt\right)^{\frac{1}{2}}.$$

For any $\{\mu_n\}, \{\nu_n\}$ such that $\mu_n, \nu_n \to 0$ $(n \to \infty)$, we denote another Banach space $B_{\overline{P},p,\mu_n,\nu_n} := \{z : \|z\|_{B_{\overline{P},p,\mu_n,\nu_n}} < \infty\}$ where

$$\|z\|_{B_{\overline{P},p,\mu_n,\nu_n}} := \left(\overline{E}[\sup_{t \in [0,T]} |z(t)|^p]\right)^{\frac{1}{p}} + \left(\overline{E}[\left(\int_0^T \|z(t)\|^2 dt\right)^{\frac{p}{2}}]\right)^{\frac{2}{p}} + \overline{E}[\sup_n \frac{1}{\nu_n} \sup_{|\theta| \leq \mu_n} \left(\int_0^T \|z(t+\theta) - z(t)\|^2_{D(A)'} dt\right)^{\frac{1}{2}}].$$

Due to Propositions 3.1, 3.2 and 3.3, $\forall\, p \in [1,\infty), \{\mu_n\}, \{\nu_n\}$ such that

$$\sum_{n=1}^\infty \frac{\mu_n^{\frac{1}{6}}}{\nu_n} \lesssim 1,$$

$\{\tilde{y}^m : m \in \mathbb{N}\}$ is bounded in $B_{\overline{P},p;\mu_n,\nu_n}$. Moreover, we denote by

$$S := C(0,T;\mathbb{R}^{m_1+m_2+m_3}) \times L^2(0,T;H)$$

and $\mathcal{B}(S)$ the σ-algebra generated by Borel sets of S. For $m = 1, 2, \ldots,$ we define $\Phi : \overline{\Omega} \mapsto S$ and a probability measure Π_m by

$$\Phi(\overline{\omega}) = (\overline{W}(\overline{\omega}, \cdot), \tilde{y}^m(\overline{\omega}, \cdot)), \quad \Pi_m(A) := \overline{P}(\Phi^{-1}(A)) \quad \forall\, A \in \mathcal{B}(S).$$

Proposition 3.5. *The family of probability measures* $\{\Pi_m, m \in \mathbb{N}\}$ *is tight.*

Proof. We fix $\epsilon > 0$ and recall that for any $n = 1, 2, \ldots,$

$$\overline{E}[|\overline{W}(t_2) - \overline{W}(t_1)|^{2n}] \leq c(n)|t_2 - t_1|^n, \quad n = 1, 2, \ldots. \tag{3.33}$$

We also fix

$$M_\epsilon := \left(\frac{2c_0 T^2}{\epsilon} \left(\sum_{j=1}^\infty \frac{1}{j^2} \right) \right)^{\frac{1}{4}}, \tag{3.34}$$

for a constant c_0 to be determined below, and set

$$K_\epsilon^1 := \{ v \in C(0,T;\mathbb{R}^{m_1+m_2+m_3}) : n \sup_{t_1,t_2 \in [0,T], |t_2 - t_1| < n^{-6}} |v(t_2) - v(t_1)| \leq M_\epsilon \}$$

for $n \in \mathbb{N}$ which is a compact subset of $C([0,T];\mathbb{R}^{m_1+m_2+m_3})$. We compute

$$\begin{aligned}
&\overline{P}(\{\overline{\omega} : \overline{W}(\overline{\omega}, \cdot) \notin K_\epsilon^1\}) \\
&\leq \sum_{n=1}^\infty \overline{P}(\{\overline{\omega} : \sup_{t_1,t_2 \in [0,T], |t_2 - t_1| < n^{-6}} |\overline{W}(\overline{\omega}, t_2) - \overline{W}(\overline{\omega}, t_1)| > \frac{M_\epsilon}{n} \}) \\
&\leq \sum_{n=1}^\infty \sum_{i=0}^{n^6 - 1} \overline{P}(\{\overline{\omega} : \sup_{iTn^{-6} \leq t \leq (i+1)Tn^{-6}} |\overline{W}(\overline{\omega}, t) - \overline{W}(\overline{\omega}, iTn^{-6})| > \frac{M_\epsilon}{n} \}) \\
&\leq \sum_{n=1}^\infty \sum_{i=0}^{n^6 - 1} \left(\frac{n}{M_\epsilon} \right)^4 \overline{E}[\sup_{iTn^{-6} \leq t \leq (i+1)Tn^{-6}} |\overline{W}(\overline{\omega}, t) - \overline{W}(\overline{\omega}, iTn^{-6})|^4] \\
&\lesssim \sum_{n=1}^\infty \sum_{i=0}^{n^6 - 1} \left(\frac{n}{M_\epsilon} \right)^4 \overline{E} \left[|\overline{W}(\overline{\omega}, (i+1)Tn^{-6}) - \overline{W}(\overline{\omega}, iTn^{-6})|^4 \right] \leq c_0 \frac{T^2}{M_\epsilon^4} \sum_{n=1}^\infty n^{-2}
\end{aligned}$$

by Markov's and Doob's maximal inequalities and (3.33). Choosing this c_0 in (3.34) gives

$$\overline{P}(\{\overline{\omega} : \overline{W}(\overline{\omega}, \cdot) \notin K_\epsilon^1\}) \leq c_0 \frac{(T^2 \sum_{n=1}^\infty n^{-2})}{(\frac{2c_0 T^2}{\epsilon})(\sum_{n=1}^\infty n^{-2})} = \frac{\epsilon}{2}. \tag{3.35}$$

Next, we choose

$$K_\epsilon^2 := \{ z \in Y_{\mu_n,\nu_n} : \|z\|_{Y_{\mu_n,\nu_n}} \leq M_\epsilon, \sum_n \frac{\mu_n^{\frac{1}{6}}}{\nu_n} < \infty \}, \quad M_\epsilon = 2c_1 \epsilon^{-1} \tag{3.36}$$

where c_1 is to be determined below. By Lemma 3.4, K_ϵ^2 is compact. Using our previous observation that $\forall\, p \in [1,\infty), \{\mu_n\}, \{\nu_n\}$ such that $\sum_{n=1}^\infty \frac{\mu_n^{\frac{1}{6}}}{\nu_n} \lesssim 1$, $\{\tilde{y}^m : m \in \mathbb{N}\}$ is bounded in $B_{\overline{P},p;\mu_n,\nu_n}$, it is immediate that

$$\overline{P}(\{\overline{\omega} : \tilde{y}^m(\overline{\omega}, \cdot) \notin K_\epsilon^2\}) \lesssim \frac{1}{M_\epsilon} \|\tilde{y}^m(t)\|_{B_{\overline{P},p;\mu_n,\nu_n}} \leq \frac{c_1}{M_\epsilon}$$

by Chebyshev's and Hölder's inequalities, for some $c_1 \geq 0$. Choosing this c_1 in (3.36) gives

$$\overline{P}(\{\overline{\omega} : \tilde{y}^m(\overline{\omega}, \cdot) \notin K_\epsilon^2\}) \leq \frac{c_1}{M_\epsilon} = \frac{\epsilon}{2}. \tag{3.37}$$

In sum of (3.35) and (3.37), the proof of Proposition 3.5 is complete. □

By the tightness of $\{\Pi_m\}$ according to Proposition 3.5, due to Prokhorov's theorem we obtain its subsequence $\{\Pi_{m_j}\}_j$ such that $\lim_{j \to \infty} \Pi_{m_j} \to \Pi$ weakly. Hence, by Skorokhod's theorem, there exists $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}), (\tilde{W}_{m_j}, \tilde{y}^{m_j}), (\tilde{W}, \tilde{y})$ valued in S such that the probability law of $(\tilde{W}_{m_j}, \tilde{y}^{m_j})$ is Π_{m_j}; therefore, $\{\tilde{W}_{m_j}\}_j$ is a family of $m_1+m_2+m_3$ dimensional Wiener processes, the probability law of $(\tilde{W}, \tilde{y})$ is Π and

$$\lim_{j\to\infty}(\tilde{W}_{m_j}, \tilde{y}^{m_j}) = (\tilde{W}, \tilde{y}) \text{ in } S = C(0,T;\mathbb{R}^{m_1+m_2+m_3}) \times L^2(0,T,H) \ \tilde{P} - \text{a.s.} \tag{3.38}$$

We set $\tilde{\mathcal{F}}_t = \sigma\{\tilde{W}(\tau), \tilde{y}(\tau)\}_{\tau\in[0,t]}$. Then it can be checked that $\tilde{W}(t)$ is a $\tilde{\mathcal{F}}_t$-standard Wiener process; we refer readers to [30] for details.

Next, we show that $\tilde{y}^{m_j}$ satisfies

$$\begin{aligned}
&\tilde{y}^{m_j}(t) + \int_0^t P^{m_j} A\tilde{y}^{m_j}(\tau)d\tau + \int_0^t P^{m_j} B(\tilde{y}^{m_j}(\tau))d\tau \\
=&\tilde{y}_0^{m_j} + \int_0^t P^{m_j}\tilde{f}(\tilde{y}^{m_j}(\tau),\tau)d\tau + \int_0^t P^{m_j}\tilde{g}(\tilde{y}^{m_j}(\tau),\tau)d\tilde{W}_{m_j}(\tau)
\end{aligned} \tag{3.39}$$

in $D(A)'$. We set

$$\begin{aligned}
\xi_m(t) =&\tilde{y}^m(t) + \int_0^t P^m[A\tilde{y}^m(\tau) + B(\tilde{y}^m(\tau))]d\tau \\
&- \tilde{y}_0 - P^m[\int_0^t \tilde{f}(\tilde{y}^m(\tau),\tau)d\tau + \int_0^t \tilde{g}(\tilde{y}^m(\tau),\tau)d\overline{W}(\tau)]
\end{aligned}$$

and $X_m = \int_0^T \|\xi_m(\tau)\|^2_{D(A)'}d\tau$ and note that $X_m = 0$ by (3.7). Thus,

$$\overline{E}[\frac{X_m}{1+X_m}] = 0 \text{ while } \overline{E}[\frac{X_m}{1+X_m}] \le \overline{E}[X_m].$$

Now we let

$$\begin{aligned}
\xi_{m_j}(t) =&\tilde{y}^{m_j}(t) + \int_0^t P^{m_j}[A\tilde{y}^{m_j}(\tau) + B(\tilde{y}^{m_j}(\tau))]d\tau \\
&- \tilde{y}_0 - P^{m_j}[\int_0^t \tilde{f}(\tilde{y}^{m_j}(\tau),\tau)d\tau + \int_0^t \tilde{g}(\tilde{y}^{m_j}(\tau),\tau)d\tilde{W}_{m_j}(\tau)]
\end{aligned}$$

and $Y_{m_j} := \int_0^T \|\xi_{m_j}(\tau)\|^2_{D(A)'}d\tau$. We claim that

$$\tilde{E}[\frac{Y_{m_j}}{1+Y_{m_j}}] = 0$$

which will imply (3.39) as desired. We introduce the regularization

$$\tilde{g}^\epsilon(\tilde{y})(t) := (\rho_\epsilon(\cdot) * \tilde{g}(\tilde{y}(\cdot),\cdot))(t) = \frac{1}{\epsilon}\int_0^t \rho\left(\frac{\tau-t}{\epsilon}\right)\tilde{g}(\tilde{y}(\tau),\tau)d\tau \tag{3.40}$$

with the mollifier

$$\rho_\epsilon(t) = \frac{1}{\epsilon}\rho(\frac{t}{\epsilon}) \tag{3.41}$$

(cf. [24] for its properties). We compute

$$\int_0^T \|\tilde{g}^\epsilon(\tilde{y}(t)) - \tilde{g}(\tilde{y}(t),t)\|^2_{H_1^{\times m_1}\times(\mathbb{L}^2)^{\times m_2}\times H_2^{\times m_3}} dt$$
$$\leq 8\int_0^T \|\tilde{g}(\tilde{y}(t),t)\|^2_{H_1^{\times m_1}\times(\mathbb{L}^2)^{\times m_2}\times H_2^{\times m_3}} dt \lesssim \int_0^T 1 + |\tilde{y}(t)|^2 dt \lesssim 1 \tag{3.42}$$

$\tilde{P}$-a.s. by (2.6), Young's inequality for convolution, (2.5) and (3.38). Thus,

$$\tilde{g}^\epsilon(\tilde{y})(\cdot) \to \tilde{g}(\tilde{y})(\cdot) \quad \text{ in } L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}; L^2(0,T; H_1^{\times m_1} \times (\mathbb{L}^2)^{\times m_2} \times H_2^{\times m_3})). \tag{3.43}$$

We let $X_{m,\epsilon}, Y_{m_j,\epsilon}$ be the analogues of X_m and Y_{m_j} respectively with $\tilde{g}$ replaced by $\tilde{g}^\epsilon$. We introduce the mapping $\phi_{m,\epsilon}$ defined by

$$\phi_{m,\epsilon}(\overline{W}, \tilde{y}^m) = \frac{X_{m,\epsilon}}{1+X_{m,\epsilon}} \text{ and similarly } \phi_{m_j,\epsilon}(\tilde{W}_{m_j}, \tilde{y}^{m_j}) = \frac{Y_{m_j,\epsilon}}{1+Y_{m_j,\epsilon}}.$$

Both mappings are clearly bounded and continuous. Thus,

$$\tilde{E}[\frac{Y_{m_j,\epsilon}}{1+Y_{m_j,\epsilon}}] = \int_S \phi_{m_j,\epsilon}(\omega,x) d\Pi_{m_j} = \overline{E}[\phi_{m_j,\epsilon}(\overline{W}, \tilde{y}^{m_j})] = \overline{E}[\frac{X_{m_j,\epsilon}}{1+X_{m_j,\epsilon}}].$$

This implies

$$\tilde{E}[\frac{Y_{m_j}}{1+Y_{m_j}}] - \overline{E}[\frac{X_{m_j}}{1+X_{m_j}}] = \tilde{E}[\frac{Y_{m_j}}{1+Y_{m_j}} - \frac{Y_{m_j,\epsilon}}{1+Y_{m_j,\epsilon}}] + \overline{E}[\frac{X_{m_j,\epsilon}}{1+X_{m_j,\epsilon}} - \frac{X_{m_j}}{1+X_{m_j}}]$$

where it can be shown via Hölder's and Burkholder-Davis-Gundy inequalities that both terms on the right hand side converge to 0 ($\epsilon \to 0$). Thus, (3.39) is valid.

3.3. Passing to the limit. Due to (3.39), it is clear that our previous estimates for Propositions 3.1, 3.2 and 3.3 now go through for $\tilde{y}^{m_j}$ so that

$$\tilde{E}[\sup_{t\in[0,T]} |\tilde{y}^{m_j}(t)|^p] \lesssim 1 \quad \forall\, p \in [1,\infty), \tag{3.44}$$

$$\tilde{E}[\left(\int_0^T \|\tilde{y}^{m_j}(t)\|^2 dt\right)^p] \lesssim 1 \quad \forall\, p \in [1,\infty), \tag{3.45}$$

$$\tilde{E}[\sup_{\theta\in[0,\delta]} \int_0^T \|\tilde{y}^{m_j}(t+\theta) - \tilde{y}^{m_j}(t)\|^2_{D(A)'} dt] \lesssim \delta^{\frac{1}{3}}, \tag{3.46}$$

which imply by relabeling subsequence

$$\tilde{y}^{m_j} \to \tilde{y} \text{ weak}^* \text{ in } L^p(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}; L^\infty(0,T;H)) \quad \forall\, p \in [1,\infty), \tag{3.47}$$

$$\tilde{y}^{m_j} \to \tilde{y} \text{ weak in } L^p(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}; L^2(0,T;V)) \quad \forall\, p \in [1,\infty). \tag{3.48}$$

Consequently, using (3.47)-(3.48) we can obtain

$$\tilde{E}[\sup_{t\in[0,T]}|\tilde{y}(t)|^p] \lesssim 1, \quad \forall\, p\in[1,\infty), \tag{3.49}$$

$$\tilde{E}[\left(\int_0^T \|\tilde{y}(t)\|^2 dt\right)^p] \lesssim 1, \quad \forall\, p\in[1,\infty), \tag{3.50}$$

$$\tilde{E}[\sup_{\theta\in[0,\delta]}\int_0^T \|\tilde{y}(t+\theta)-\tilde{y}(t)\|^2_{D(A)'}dt] \lesssim \delta^{\frac{1}{3}}. \tag{3.51}$$

Now consider a function $G(\tau)=\tau^4$, $\tau\in\mathbb{R}^+$. Clearly $G\geq 0$, G is increasing and convex satisfying

$$\lim_{\tau\to+\infty}\frac{G(\tau)}{\tau}=\infty.$$

By (3.38), $\tilde{y}^{m_j}\to\tilde{y}$ in $L^2(0,T;H)$ $\tilde{P}$-a.s. and

$$\tilde{E}[G(\|\tilde{y}^{m_j}-\tilde{y}\|^2_{L^2(0,T;H)})] \lesssim \tilde{E}[\sup_{t\in[0,T]}|\tilde{y}^{m_j}(t)|^8T^4]+\tilde{E}[\sup_{t\in[0,T]}|\tilde{y}(t)|^8T^4] \lesssim 1$$

by (2.6), (3.44) and (3.49). Therefore, by de la Vallée-Poussin theorem, $\{\|\tilde{y}^{m_j}-\tilde{y}\|^2_{L^2(0,T;H)}\}_{j=1}$ is uniformly integrable. Now Vitali's Convergence theorem gives

$$\tilde{y}^{m_j}\to\tilde{y} \qquad \text{strongly in } L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P};L^2(0,T;H)). \tag{3.52}$$

Hence $|\tilde{y}^{m_j}-y|^2_H$ converges in L^1 with respect to $d\tilde{P}\times dt$. Denoting subsequence by $\tilde{y}^{m_j}$ again, we have

$$\tilde{y}^{m_j}\to\tilde{y} \qquad \text{in } H \tag{3.53}$$

for a.e. $(\tilde{\omega},t)$ with respect to the measure $d\tilde{P}\times dt$.

Similarly, with same $G(\tau)=\tau^4,\tau\in\mathbb{R}^+$, using (2.5) and (3.44) we can show that $\{\|\tilde{f}(\tilde{y}^{m_j}(t),t)\|^2_{\mathbb{H}^{-1}}\}_{j=1}$ is uniformly integrable by de la Vallée-Poussin theorem. Moreover, by continuity of $\tilde{f}$ from H to $\mathbb{H}^{-1}\times\mathbb{H}^{-1}\times\mathbb{H}^{-1}$, two facts that $\tilde{y}^{m_j}\to\tilde{y}$ in H a.e. (ω,t) according to (3.53) and that $P_{m_j}\to Id$ in the limit, Vitali's Convergence theorem implies

$$P^{m_j}\tilde{f}(\tilde{y}^{m_j}(\cdot),\cdot)\to\tilde{f}(\tilde{y}(\cdot),\cdot) \tag{3.54}$$

in $L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P};L^2(0,T;\mathbb{H}^{-1}))$.

Similarly, with same $G(\tau)=\tau^4$, using (2.5), (3.44), de la Vallée-Poussin Theorem, continuity of $g_i, i=1,2,3$, the fact that $\tilde{y}^{m_j}\to\tilde{y}$ in H a.e. $(\tilde{\omega},t)$ according to (3.53) and Vitali's convergence theorem, we get

$$P^{m_j}\tilde{g}(\tilde{y}^{m_j}(\cdot),\cdot)\to\tilde{g}(\tilde{y}(\cdot),\cdot) \tag{3.55}$$

in $L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P};L^2(0,T;H_1^{\times m_1}\times(\mathbb{L}^2)^{\times m_2}\times H_2^{\times m_3}))$.

Next, we verify

$$P^{m_j}[A\tilde{y}^{m_j}(\cdot)+B(\tilde{y}^{m_j}(\cdot))]\to A\tilde{y}(\cdot)+B(\tilde{y}(\cdot)) \tag{3.56}$$

weakly in $L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P};L^2(0,T;D(A)'))$. As we remarked, it suffices to check

$$A\tilde{y}^{m_j}(\cdot)\to A\tilde{y}(\cdot) \qquad \text{and} \qquad B(\tilde{y}^{m_j}(\cdot))\to B(\tilde{y}(\cdot))$$

weakly in $L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P};L^2(0,T;D(A)'))$. We fix

$$\phi\in L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P};L^2(0,T;D(A)))$$

and observe that

$$\tilde{E}[\int_0^T(\phi,A\tilde{y}^{m_j}-A\tilde{y})d\tau]=\tilde{E}[\int_0^T(A\phi,\tilde{y}^{m_j}-\tilde{y})d\tau]$$

and

$$A\phi\in L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P};L^2(0,T;\mathbb{L}^2)\subset L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P};L^2(0,T;V').$$

By (3.48) this implies

$$\tilde{E}[\int_0^T(\phi,A\tilde{y}^{m_j}-A\tilde{y})d\tau]\to 0\quad (j\to\infty).$$

Similarly, denoting $\tilde{y}=(u,w,b)$ and $\tilde{y}^{m_j}$ analogously, and $\phi=(\phi_1,\phi_2,\phi_3)$, we have

$$\begin{aligned}
&\tilde{E}[\int_0^T(\phi,B(\tilde{y}^{m_j})-B(\tilde{y}))d\tau]\\
=&-\tilde{E}[\int_0^T((u^{m_j}-u)\cdot\nabla\phi_1,u^{m_j})+((u\cdot\nabla)\phi_1,u^{m_j}-u)-((b^{m_j}-b)\cdot\nabla\phi_1,b^{m_j}\\
&-((b\cdot\nabla)\phi_1,b^{m_j}-b)+(\chi(\nabla\times\phi_1),w^{m_j}-w)+((u^{m_j}-u)\cdot\nabla\phi_2,w^{m_j})\\
&+((u\cdot\nabla)\phi_2,w^{m_j}-w)+(\chi(\nabla\times\phi_2),u^{m_j}-u)+((u^{m_j}-u)\cdot\nabla\phi_3,b^{m_j})\\
&+((u\cdot\nabla)\phi_3,b^{m_j}-b)-((b^{m_j}-b)\cdot\nabla\phi_3,u^{m_j})-((b\cdot\nabla)\phi_3,u^{m_j}-u)d\tau].
\end{aligned}$$

For brevity, we continue our estimates on the following:

$$\tilde{E}[\int_0^T((\tilde{y}^{m_j}-\tilde{y}),(\nabla\phi)(\tilde{y}^{m_j}+\tilde{y}))+((\tilde{y}^{m_j}-\tilde{y}),\nabla\phi)d\tau];$$

it is clear that the identical estimates go through for each term. By (3.48), we know that

$$\tilde{y}^{m_j}\to\tilde{y}$$

weakly in $L^p(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P};L^2(0,T;V)),p\geq 1$. Thus, if picking $p=4$ so that its dual is $p'=\frac{4}{3}$, it suffices to show that

$$(\nabla\phi)(\tilde{y}^{m_j}+\tilde{y}),\nabla\phi\in L^{\frac{4}{3}}(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P};L^2(0,T;V')).$$

We now compute

$$
\begin{aligned}
&\tilde{E}[\left(\int_0^T \|(\nabla\phi)(\tilde{y}^{m_j}+\tilde{y})\|_{V'}^2 d\tau\right)^{\frac{2}{3}}]\\
\leq&\tilde{E}[\left(\int_0^T \sup_{\psi\in V,\|\psi\|_V\leq 1}\|(\nabla\phi)(\tilde{y}^{m_j}+\tilde{y})\|_{\mathbb{L}^{\frac{6}{5}}}^2\|\psi\|_{\mathbb{L}^6}^2 d\tau\right)^{\frac{2}{3}}]\\
\lesssim&\tilde{E}[\left(\int_0^T \|\nabla\phi\|_{\mathbb{L}^3}^2(\|\tilde{y}^{m_j}\|_{\mathbb{L}^2}^2+\|\tilde{y}\|_{\mathbb{L}^2}^2)d\tau\right)^{\frac{2}{3}}]\\
\lesssim&\tilde{E}[\sup_{\tau\in[0,T]}(\|\tilde{y}^{m_j}\|_{\mathbb{L}^2}^{\frac{4}{3}}+\|\tilde{y}\|_{\mathbb{L}^2}^{\frac{4}{3}})\left(\int_0^T\|\nabla\phi\|_{\mathbb{L}^2}\|\nabla\phi\|_{\mathbb{L}^6}d\tau\right)^{\frac{2}{3}}]\\
\lesssim&\tilde{E}[\sup_{\tau\in[0,T]}(\|\tilde{y}^{m_j}\|_{\mathbb{L}^2}^{\frac{4}{3}}+\|\tilde{y}\|_{\mathbb{L}^2}^{\frac{4}{3}})\left(\int_0^T\|\phi\|_{D(A)}^2 d\tau\right)^{\frac{2}{3}}]\\
\lesssim&\left(\tilde{E}[\sup_{\tau\in[0,T]}(|\tilde{y}^{m_j}|^4+|\tilde{y}|^4)]\right)^{\frac{1}{3}}\left(\tilde{E}[(\int_0^T\|\phi\|_{D(A)}^2 d\tau)]\right)^{\frac{2}{3}}\lesssim 1
\end{aligned}
$$

by Hölder's and interpolation inequalities and Sobolev embedding. On the other hand, it is immediate that

$$\nabla\phi\in L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P};L^2(0,T;V))\subset L^{\frac{4}{3}}(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P};L^2(0,T;V')).$$

Finally, we show that $\forall\, t$,

$$\int_0^t P^{m_j}\tilde{g}(\tilde{y}^{m_j}(\tau),\tau)d\tilde{W}_{m_j}(\tau)\to\int_0^t\tilde{g}(\tilde{y}(\tau),\tau)d\tilde{W}(\tau) \tag{3.57}$$

weakly in $L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P};H)$. As remarked already it suffices to show

$$\int_0^t\tilde{g}(\tilde{y}^{m_j}(\tau),\tau)d\tilde{W}_{m_j}(\tau)\to\int_0^t\tilde{g}(\tilde{y}(\tau),\tau)d\tilde{W}(\tau)\text{ weakly in }L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P};H).$$

We consider the mollifier from (3.40)-(3.41), recall (3.43) and compute

$$
\begin{aligned}
&\int_0^t\|\tilde{g}^\epsilon(\tilde{y}^{m_j})(\tau)-\tilde{g}^\epsilon(\tilde{y})(\tau)\|_{H_1^{\times m_1}\times(\mathbb{L}^2)^{\times m_2}\times H_2^{\times m_3}}^2 d\tau\\
\leq&\int_0^t\|\tilde{g}(\tilde{y}^{m_j}(\tau),\tau)-\tilde{g}(\tilde{y}(\tau),\tau)\|_{H_1^{\times m_1}\times(\mathbb{L}^2)^{\times m_2}\times H_2^{\times m_3}}^2 d\tau
\end{aligned}
\tag{3.58}
$$

by Young's inequality for convolution. Now we integrate by parts

$$\int_0^t\tilde{g}^\epsilon(\tilde{y}^{m_j})(\tau)d\tilde{W}_{m_j}(\tau)=\tilde{g}^\epsilon(\tilde{y}^{m_j})(t)\tilde{W}_{m_j}(t)-\int_0^t\partial_\tau\tilde{g}^\epsilon(\tilde{y}^{m_j})(\tau)\tilde{W}_{m_j}(\tau)d\tau.$$

We know $(\tilde{W}_{m_j},\tilde{y}_{m_j})\to(\tilde{W},\tilde{y})$ in S $\tilde{P}$-a.s. from (3.38) and we already saw that $P^{m_j}\tilde{g}(\tilde{y}^{m_j}(\cdot),\cdot)\to\tilde{g}(\tilde{y}(\cdot),\cdot)$ in $L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P};L^2(0,T;H_1^{\times m_1}\times(\mathbb{L}^2)^{\times m_2}\times H_2^{\times m_3}))$

from (3.55). Thus, relabeling subsequence by $P^{m_j}\tilde{g}(\tilde{y}^{m_j}(\cdot),\cdot)$,

$$\int_0^t \tilde{g}^\epsilon(\tilde{y}^{m_j})(\tau)d\tilde{W}_{m_j}(\tau) \to \tilde{g}^\epsilon(\tilde{y})(t)\tilde{W}(t) - \int_0^t \partial_\tau \tilde{g}^\epsilon(\tilde{y})(\tau)\tilde{W}(\tau)d\tau \tag{3.59}$$

point-wise almost all $\tilde{\omega}, x$. We again observe that

$$\tilde{g}^\epsilon(\tilde{y})(t)\tilde{W}(t) - \int_0^t \partial_\tau \tilde{g}^\epsilon(\tilde{y})(\tau)\tilde{W}(\tau)d\tau = \int_0^t \tilde{g}^\epsilon(\tilde{y})(\tau)d\tilde{W}(\tau) \tag{3.60}$$

so that (3.59)-(3.60) imply

$$\int_0^t \tilde{g}^\epsilon(\tilde{y}^{m_j})(\tau)d\tilde{W}_{m_j}(\tau) \to \int_0^t \tilde{g}^\epsilon(\tilde{y})(\tau)d\tilde{W}(\tau)$$

point-wise almost all $\tilde{\omega}, x$. On the other hand, $\forall\, j$, it can be shown that

$$\tilde{E}[\|\int_0^t \tilde{g}^\epsilon(\tilde{y}^{m_j})(\tau)d\tilde{W}_{m_j}(\tau)\|_{\mathbb{L}^2}^2] \lesssim \tilde{E}[\int_0^t 1 + |\tilde{y}^{m_j}(\tau)|^2 d\tau] \lesssim 1$$

by Young's inequality for convolution, (2.5) and (3.44). Thus, by Lemma 2.6 and Remark 2.7, $\forall\, h \in L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}; H)$,

$$\tilde{E}[\left(h, \int_0^t \tilde{g}^\epsilon(\tilde{y}^{m_j})(\tau)d\tilde{W}_{m_j}(\tau)\right)] \to \tilde{E}[\left(h, \int_0^t \tilde{g}^\epsilon(\tilde{y})(\tau)d\tilde{W}(\tau)\right)]. \tag{3.61}$$

Similarly, $\forall\, j$

$$\tilde{E}[\|\int_0^t \tilde{g}(\tilde{y}^{m_j}(\tau),\tau)d\tilde{W}_{m_j}(\tau)\|_{\mathbb{L}^2}^2] \lesssim \tilde{E}[\int_0^t 1 + |\tilde{y}^{m_j}(\tau)|^2 d\tau] \lesssim 1$$

by (2.5) and (3.44). Thus, by Lemma 2.6 and Remark 2.7, $\exists\, \eta \in L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}; H)$ such that $\forall\, h \in L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}; H)$,

$$\tilde{E}[\left(h, \int_0^t \tilde{g}(\tilde{y}^{m_j}(\tau),\tau)d\tilde{W}_{m_j}(\tau)\right)] \to \tilde{E}[(h,\eta)].$$

It can be shown that $\eta = \int_0^t \tilde{g}(\tilde{y}(\tau),\tau)d\tilde{W}(\tau)$ by writing

$$\begin{aligned}
&\tilde{E}[\left(h, \int_0^t g(\tilde{y}^{m_j}(\tau),\tau)d\tilde{W}_{m_j}(\tau)\right)] - \tilde{E}[\left(h, \int_0^t \tilde{g}(\tilde{y}(\tau),\tau)d\tilde{W}(\tau)\right)] \\
=&\tilde{E}[\int_D h \int_0^t \tilde{g}(\tilde{y}^{m_j}(\tau),\tau) - \tilde{g}^\epsilon(\tilde{y}^{m_j})(\tau)d\tilde{W}_{m_j}(\tau)dx] \\
+&\tilde{E}[\int_D h[\int_0^t \tilde{g}^\epsilon(\tilde{y}^{m_j})(\tau)d\tilde{W}_{m_j}(\tau) - \int_0^t \tilde{g}^\epsilon(\tilde{y})(\tau)d\tilde{W}(\tau)]dx] \\
+&\tilde{E}[\int_D h \int_0^t [\tilde{g}^\epsilon(\tilde{y})(\tau) - \tilde{g}(\tilde{y}(\tau),\tau)]d\tilde{W}(\tau)dx] := I_1 + I_2 + I_3
\end{aligned}$$

where $I_1, I_3 \to 0$ by simple estimates using Hölder's inequalities while (3.61) already showed that $I_2 \to 0$. This implies (3.57) as desired.

3.4. Existence of pressure term. We let

$$h := -\partial_t u - A_1 u - (u\cdot\nabla)u + (b\cdot\nabla)b + \chi(\nabla\times w) + \tilde{f}_1(\tilde{y},t) + \tilde{g}_1(\tilde{y},t)\partial_t W_1.$$

By Lemma 2.12 it suffices to show $h \in L^1(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P}; W^{-1,\infty}(0,T;\mathbb{H}^{-1}))$. We fix $\phi \in L^\infty(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P}; W^{1,1}(0,T;\mathbb{H}^1)$ and estimate

$$E[\int_0^T\int_D \partial_t u\phi dxdt] \leq E[\int_0^T \|u\|_{\mathbb{L}^2}\|\partial_t\phi\|_{\mathbb{L}^2}dt] \lesssim 1$$

by Hölder's inequalities and (3.49). Next,

$$\begin{aligned} E[\int_0^T\int_D \nabla u\nabla\phi dxdt] \lesssim & E[\int_D \|\nabla u\|_{L^2(0,T)}\|\nabla\phi\|_{L^2(0,T)}dx] \\ \lesssim & E[\|\nabla u\|_{L^2(0,T;\mathbb{L}^2)}\|\phi\|_{W^{1,1}(0,T;\mathbb{H}^1)}] \lesssim 1 \end{aligned}$$

by Hölder's and Gagliardo-Nirenberg inequalities, Minkowski's inequality for integrals and (3.50). Similarly,

$$\begin{aligned} &\int_0^T\int_D [(u\cdot\nabla)u - (b\cdot\nabla)b - \chi(\nabla\times w)]\phi dxdt \\ \lesssim &\int_D \|\tilde{y}\|^2_{L^{\frac{8}{3}}(0,T)}\|\nabla\phi\|_{L^4(0,T)} + \|\tilde{y}\|_{L^2(0,T)}\|\nabla\phi\|_{L^2(0,T)}dx \\ \lesssim &\left(\int_0^T \|\tilde{y}\|^{\frac{8}{3}}_{\mathbb{L}^4}d\tau\right)^{\frac{3}{4}}\|\phi\|_{W^{1,1}(0,T;\mathbb{H}^1)} + \|\tilde{y}\|_{L^2(0,T;\mathbb{L}^2)}\|\phi\|_{W^{1,1}(0,T;\mathbb{H}^1)} \\ \lesssim &\left(\int_0^T \|\tilde{y}\|^{\frac{2}{3}}_{\mathbb{L}^2}\|\tilde{y}\|^2_{\mathbb{H}^1}d\tau\right)^{\frac{3}{4}}\|\phi\|_{W^{1,1}(0,T;\mathbb{H}^1)} + \|\tilde{y}\|_{L^2(0,T;\mathbb{L}^2)}\|\phi\|_{W^{1,1}(0,T;\mathbb{H}^1)} \\ \lesssim &[\sup_t\|\tilde{y}(t)\|^2_{\mathbb{L}^2} + \sup_t\|\tilde{y}(t)\|^{\frac{1}{2}}_{\mathbb{L}^2}\left(\int_0^T\|\nabla\tilde{y}\|^2_{\mathbb{L}^2}d\tau\right)^{\frac{3}{4}} + \|\tilde{y}\|_{L^2(0,T;\mathbb{L}^2)}]\|\phi\|_{W^{1,1}(0,T;\mathbb{H}^1)} \end{aligned}$$

by Hölder's and Gagliardo-Nirenberg inequalities and Minkoswki's inequality for integrals. This leads to

$$E[\int_0^T\int_D [(u\cdot\nabla)u - (b\cdot\nabla)b - \chi(\nabla\times w)]\phi dxdt] \lesssim 1$$

by Hölder's inequalities, (3.49)-(3.50). Next,

$$E[\int_0^T\int_D \tilde{f}_1(\tilde{y}),t)\phi dxdt] \lesssim E[\int_0^T (1+\|\tilde{y}(t)\|_{\mathbb{L}^2})\|\phi\|_{\mathbb{H}^1}dt] \lesssim 1$$

by Hölder's inequality, (2.5) and (3.49). Finally,

$$\begin{aligned}
&E[\int_0^T \int_D \tilde{g}_1(\tilde{y}(t),t)\frac{dW_1}{dt}\phi dxdt] \\
\leq& \int_D \left(E[\sup_{\tau\in[0,T]} |\int_0^\tau \tilde{g}_1(\tilde{y}(\lambda),\lambda)dW_1(\lambda)|^2] \right)^{\frac{1}{2}} \left(E[\|\partial_\tau \phi(x,\tau)\|^2_{L^1([0,T])}] \right)^{\frac{1}{2}} dx \\
\lesssim& \int_D \left(E[\int_0^T |\tilde{g}_1(\tilde{y}(\lambda),\lambda)|^2 d\lambda] \right)^{\frac{1}{2}} \left(E[\|\partial_\tau \phi\|^2_{L^1([0,T])}] \right)^{\frac{1}{2}} dx \\
\lesssim& \left(E[\int_0^T 1 + |\tilde{y}(\lambda)|^2 d\lambda] \right)^{\frac{1}{2}} \left(E[\|\partial_\tau \phi\|^2_{L^1(0,T;\mathbb{L}^2)}] \right)^{\frac{1}{2}} \lesssim 1
\end{aligned}$$

by Hölder's and Burkholder-Davis-Gundy inequalities, (2.5) and (3.49). Thus, we have shown $h \in L^1(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}; W^{-1,\infty}(0,T;\mathbb{H}^{-1}))$.

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