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A CLARK-OCONE TYPE FORMULA UNDER CHANGE OF MEASURE FOR LÉVY PROCESSES WITH L^2-LÉVY MEASURE

RYOICHI SUZUKI

Abstract. The Clark-Ocone formula is an explicit stochastic integral representation for random variables in terms of Malliavin derivatives. In this paper, we prove a Clark-Ocone type formula under change of measure (COCM) for Lévy processes with L^2-Lévy measure.

To show the COCM for L^2-Lévy processes, we develop Malliavin calculus for Lévy processes, based on [11]. By using σ-finiteness of Lévy measure, we obtain a commutation formula for the Lebesgue integration and the Malliavin derivative and a chain rule for Malliavin derivative. These formulas derive the COCM. Finally, we obtain a log-Sobolev type formula for Lévy functionals.

1. Introduction

The representations of functionals of Brownian motions (or Lévy processes) by stochastic integrals are important theorems in Probability theory. It has been widely studied (see, e.g., survey paper by [7]). In particular, the Clark-Ocone (CO) formula is an explicitly martingale representation of functionals of Brownian motions in terms of Malliavin derivatives. If an L^2-random variable F has some regularity in the Malliavin sense, we have

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_t F | \mathcal{F}_t] dW(t),$$

where W is a Brownian motion, $D_t F$ is the classical Malliavin derivative. This formula was shown by Clark, Ocone and Haussmann (see [5, 6, 12, 19]). White noise generalization of the CO formula was proved by [1]. This formula has various applications. For example, the log-Sobolev and Poincare inequalities are obtained in [4]. In the application to mathematical finance, representation of an optimal portfolio is given by this formula (see e.g., [15]).

The CO formula for Lévy processes has been also studied. Løkka ([16]) proved CO formula for functionals of pure jump Lévy processes. White noise generalization of the CO formula for functionals of pure jump Lévy was derived by [10]. Furthermore, we can also see that one for general L^2-Lévy functionals also holds (see [3]).

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Since many applications in mathematical finance require representation of random variables with respect to risk neutral martingale measure, Girsanov transformations versions of this theorem were studied by many people. First, a Clark-Ocone type formula under change of measure (COCM) for Brownian motions was proved by [15]: $F = \mathbb{E}_{\mathbb{Q}}[F] + \int_0^T \mathbb{E}_{\mathbb{Q}}\big[D_t F - F \int_0^T D_t u(s) dW_{\mathbb{Q}}(s) \big| \mathcal{F}_t\big] dW_{\mathbb{Q}}(t)$. They also derived an optimal portfolio of Brownian market by using it. Okur ([20]) generalized it by using white noise theory and derived an explicit representation of hedging strategy of digital option for Brownian market. Huehne ([13]) derived a COCM for pure jump Lévy processes and gave an optimal portfolio. Note that Di Nunno et al. ([9]) and Okur ([21]) also introduced one for Lévy processes using white noise theory. However, their results are different from our results. Our results have different settings and different representation, for more detail, see Remark 4.7 and Theorem 4.4 in this paper.

In this paper, we derive a COCM for Lévy processes with L^2-Lévy measure in section 4:

$$
\begin{aligned}
F = \mathbb{E}_{\mathbb{Q}}[F] &+ \sigma \int_0^T \mathbb{E}_{\mathbb{Q}} \left[D_{t,0} F - F K(t) \middle| \mathcal{F}_{t-} \right] dW_{\mathbb{Q}}(t) \\
&+ \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{Q}}[F(\tilde{H}(t,z) - 1) + z\tilde{H}(t,z) D_{t,z} F | \mathcal{F}_{t-}] \tilde{N}_{\mathbb{Q}}(dt, dz).
\end{aligned}
$$

We precisely define $K(t)$ and $\tilde{H}(t,z)$ and see sufficient conditions for this formula in section 4. Using this result, we obtain log-Sobolev and Poincare type inequalities for Lévy functionals. For that purpose, we adapted Malliavin calculus for Lévy processes based on [11]. Moreover, we show some formulae to show the main theorem, such as chain rule for Malliavin derivative and commutation formulae for integrals and the Malliavin derivative. By using σ-finiteness of Lévy measure (see e.g., [2]), we prove it.

This paper is organized as follows: In Section 2, we review Malliavin calculus for Lévy processes and we also give a chain rule. In Section 3, we first review commutation formulae like [8]. Second, we give some comments about commutation formulae as a remark. Finally, we show another commutation formula. In Section 4, by using results of Section 2 and Section 3, we show a COCM for Lévy processes with L^2- Lévy measure. Using it, we obtain log-Sobolev and Poincare type inequalities for Lévy functionals.

2. Preliminaries

Throughout this paper, we consider Malliavin calculus for Lévy processes, based on, [24] and [11].

For given an infinitely divisible distribution μ on $\mathbb{R}$, we can construct a Lévy process from Lévy-Ito decomposition. For details, see the book by Sato [22].

Given an infinitely divisible distribution μ on $\mathbb{R}$, we have the Lévy-Khintchine representation: there exist unique $\sigma^2 \geq 0$, $\gamma \in \mathbb{R}$ and Lévy measure ν satisfying

$$
\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}} \min(1, |z|^2) \nu(dz) < \infty,
$$

such that its characteristic function has following form:

$$\int_{\mathbb{R}} e^{iuz}\mu(dz) = \exp\left(-\frac{\sigma^2}{2}u^2 + i\gamma u + \int_{\mathbb{R}_0}(e^{iuz} - 1 - iuz\mathbf{1}_{|z|<1})\nu(dz)\right),$$

where $\mathbb{R}_0$ means $\mathbb{R}\backslash\{0\}$. To construct the centered square integral Lévy process, we assume that $\gamma = 0$ and $\int_{\mathbb{R}_0} z^2\nu(dz) < \infty$. In fact, the second condition is equivalent to the existence of second moment of μ.

Second, we give a Lévy process from an infinitely divisible distribution. Let $\{W_t; t \in [0,T]\}$ be a standard Brownian motion and N be a Poisson random measure independent of W defined by

$$N(A,t) = \sum_{s\le t}\mathbf{1}_A(\Delta X_s),\ A \in \mathcal{B}(\mathbb{R}_0),\ \Delta X_s := X_s - X_{s-}.$$

We denote the compensated Poisson random measure by $\tilde{N}(dt,dz) = N(dt,dz) - dt\nu(dz)$, where $dt\nu(dz) = \lambda(dt)\nu(dz)$ is the compensator of N, $\nu(\cdot)$ the Lévy measure of μ. We give a centered square integrable Lévy process $X = \{X_t; t \in [0,T]\}$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\in[0,T]})$, as follows:

$$X_t = \sigma W_t + \int_0^t\int_{\mathbb{R}_0} z\tilde{N}(ds,dz),$$

where $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ is the augmented filtration generated by X.

We consider the finite measure q defined on $[0,T]\times\mathbb{R}$ by

$$q(E) = \sigma^2\int_{E(0)} dt\delta_0(dz) + \int_{E'} z^2 dt\nu(dz), \qquad E \in \mathcal{B}([0,T]\times\mathbb{R}),$$

where $E(0) = \{(t,0) \in [0,T]\times\mathbb{R}; (t,0) \in E\}$ and $E' = E - E(0)$, and the random measure Q on $[0,T]\times\mathbb{R}$ by

$$Q(E) = \sigma\int_{E(0)} dW_t\delta_0(dz) + \int_{E'} z\tilde{N}(dt,dz), \qquad E \in \mathcal{B}([0,T]\times\mathbb{R}).$$

Then for $n \in \mathbb{N}$, and a simple function $h_n = \mathbf{1}_{E_1\times\cdots\times E_n}$, with pairwise disjoint sets $E_1, \cdots, E_n \in \mathcal{B}([0,T]\times\mathbb{R})$, a multiple two-parameter integral with respect to the random measure Q can be defined as $I_n(h_n) := \prod_{i=1}^n Q(E_i)$. Let $L^2_{T,q,n}(\mathbb{R})$ denote the set of product measurable, deterministic functions $h : ([0,T]\times\mathbb{R})^n \to \mathbb{R}$ satisfying

$$\|h\|^2_{L^2_{T,q,n}} := \int_{([0,T]\times\mathbb{R})^n} |h((t_1,z_1),\cdots,(t_n,z_n))|^2 q(dt_1,dz_1)\cdots q(t_n,z_n) < \infty.$$

For $n \in \mathbb{N}$ and $h_n \in L^2_{T,q,n}(\mathbb{R})$, we denote

$$I_n(h_n) := \int_{([0,T]\times\mathbb{R})^n} h((t_1,z_1),\cdots,(t_n,z_n))Q(dt_1,dz_1)\cdots Q(dt_n,dz_n).$$

It is easy to see that $\mathbb{E}[I_0(h_0)] = h_0$ and $\mathbb{E}[I_n(h_n)] = 0$, for $n \ge 1$. In this setting, we introduce the following chaos expansion (see Theorem 2 in [14], Section 2 of [24]).

Theorem 2.1. *Any $\mathcal{F}$-measurable square integrable random variable F has a unique representation*

$$F = \sum_{n=0}^{\infty} I_n(f_n), \quad \mathbb{P}\text{−a.s.}$$

with functions $f_n \in L^2_{T,q,n}(\mathbb{R})$ that are symmetric in the n pairs $(t_i, z_i), 1 \leq i \leq n$ and we have the isometry

$$\mathbb{E}[F^2] = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2_{T,q,n}}.$$

Definition 2.2. (1) Let $\mathbb{D}^{1,2}(\mathbb{R})$ denote the set of $\mathcal{F}$ -measurable random variables $F \in L^2(\mathbb{P})$ with the representation $F = \sum_{n=0}^{\infty} I_n(f_n)$ satisfying

$$\sum_{n=1}^{\infty} n n! \|f_n\|^2_{L^2_{T,q,n}} < \infty.$$

(2) Let $F \in \mathbb{D}^{1,2}(\mathbb{R})$. Then the *Malliavin derivative* $DF : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ of a random variable $F \in \mathbb{D}^{1,2}(\mathbb{R})$ is a stochastic process defined by

$$D_{t,z}F = \sum_{n=1}^{\infty} n I_{n-1}(f_n((t,z),\cdot)), \quad \text{valid for } q\text{−a.e. } (t,z) \in [0,T] \times \mathbb{R}, \mathbb{P} - a.s.$$

(3) For $\sigma \neq 0$, let $\mathbb{D}^{1,2}_0(\mathbb{R})$ denote the set of $\mathcal{F}$ -measurable random variables $F \in L^2(\mathbb{P})$ with the representation $F = \sum_{n=0}^{\infty} I_n(f_n)$ satisfying

$$\sum_{n=1}^{\infty} n n! \int_0^T \|f_n(\cdot,(t,0))\|^2_{L^2_{T,q,n-1}} \sigma^2 dt < \infty.$$

Then for $F \in \mathbb{D}^{1,2}_0(\mathbb{R})$, we can define

$$D_{t,0}F = \sum_{n=1}^{\infty} n I_{n-1}(f_n((t,0),\cdot)), \quad \text{valid for } q\text{−a.e. } (t,0) \in [0,T] \times \{0\}, \mathbb{P} - a.s.$$

(4) For $\nu \neq 0$, let $\mathbb{D}^{1,2}_1(\mathbb{R})$ denote the set of $\mathcal{F}$ -measurable random variables $F \in L^2(\mathbb{P})$ with the representation $F = \sum_{n=0}^{\infty} I_n(f_n)$ satisfying

$$\sum_{n=1}^{\infty} n n! \int_0^T \int_{\mathbb{R}_0} \|f_n(\cdot,(t,z))\|^2_{L^2_{T,q,n-1}} z^2 \nu(dz) dt < \infty.$$

Then for $F \in \mathbb{D}^{1,2}_1(\mathbb{R})$, we can define

$$D_{t,z}F = \sum_{n=1}^{\infty} n I_{n-1}(f_n((t,z),\cdot)), \quad \text{valid for } q\text{−a.e. } (t,z) \in [0,T] \times \mathbb{R}_0, \mathbb{P} - a.s.$$

Remark 2.3. If both $\sigma \neq 0$ and $\nu \neq 0$, we can see $\mathbb{D}^{1,2}(\mathbb{R}) = \mathbb{D}^{1,2}_0(\mathbb{R}) \cap \mathbb{D}^{1,2}_1(\mathbb{R})$.

We next establish the following fundamental result.

Proposition 2.4 (The closability of operator D)**.** *Let $F \in L^2(\mathbb{P})$ and $F_k \in \mathbb{D}^{1,2}(\mathbb{R}), k \in \mathbb{N}$ such that*
(1) $\lim_{k\to\infty} F_k = F$ in $L^2(\mathbb{P})$,
(2) $\{D_{t,z}F_k\}_{k=1}^{\infty}$ converges in $L^2(q \times \mathbb{P})$.
Then $F \in \mathbb{D}^{1,2}$ and $\lim_{k\to\infty} D_{t,z}F_k = D_{t,z}F$ in $L^2(q \times \mathbb{P})$.

Proof. We can show this proposition by the same sort argument as Theorem 12.6 of [9]. Let $F = \sum_{n=0}^{\infty} I_n(f_n), f_n \in L^2_{T,q,n}(\mathbb{R})$ and $F_k = \sum_{n=0}^{\infty} I_n(f_n^k), f_n^k \in L^2_{T,q,n}(\mathbb{R})$. Then by assumption (1), we have $\lim_{k\to\infty} \sum_{n=0}^{\infty} n!\|f_n^k - f_n\|^2_{L^2_{T,q,n}} = 0$. This implies that $\lim_{k\to\infty} f_n^k = f_n$ in $L^2_{T,q,n}$ for all n. From assumption (2), we deduce that

$$\lim_{k,m\to\infty} \sum_{n=1}^{\infty} nn!\|f_n^k - f_n^m\|^2_{L^2_{T,q,n}} = \lim_{k,m\to\infty} \|D_{t,z}F_k - D_{t,z}F_m\|^2_{L^2(q\times\mathbb{P})} = 0.$$

Hence, we obtain

$$\begin{aligned}\lim_{k\to\infty} \sum_{n=1}^{\infty} nn!\|f_n^k - f_n\|^2_{L^2_{T,q,n}} &\leq 2 \lim_{k\to\infty} \sum_{n=1}^{\infty} \liminf_{m\to\infty} nn!\|f_n^k - f_n^m\|^2_{L^2_{T,q,n}} \\ &\leq 2 \lim_{k\to\infty} \liminf_{m\to\infty} \sum_{n=1}^{\infty} nn!\|f_n^k - f_n^m\|^2_{L^2_{T,q,n}} = 0,\end{aligned}$$

because $nn!\|f_n^k - f_n^m\|^2_{L^2_{T,q,n}} \geq 0$ for all n, m, k.
Therefore, we can see that $F \in \mathbb{D}^{1,2}(\mathbb{R})$ and $\lim_{k\to\infty} D_{t,z}F_k = D_{t,z}F$ in $L^2(q \times \mathbb{P})$. $\square$

Next we introduce a chain rule. First we define the following.

Definition 2.5. (1) Let $C_0^\infty(\mathbb{R}^n)$ denote the space of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$ with compact support.
(2) A random variable of the form $F = f(X_{t_1}, \cdots, X_{t_n})$, where $f \in C_0^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}$, and $t_1, \cdots, t_n \geq 0$, is said to be a *smooth random variable*. The set of all smooth random variables is denoted by $\mathcal{S}$.
(3) For $F \in \mathcal{S}$, we define the *Malliavin derivative* operator $\mathcal{D}$ as a map from $\mathcal{S}$ into $L^2(q \times \mathbb{P})$

$$\begin{aligned}\mathcal{D}_{t,z}F := &\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X_{t_1}, \cdots, X_{t_n})\mathbf{1}_{[0,t_i]\times\{0\}}(t,z) \\ &+ \frac{f(X_{t_1} + z\mathbf{1}_{[0,t_1]}(t), \cdots, X_{t_n} + z\mathbf{1}_{[0,t_n]}(t)) - f(X_{t_1}, \cdots, X_{t_n})}{z}\mathbf{1}_{\mathbb{R}_0}(z)\end{aligned}$$

for $(t, z) \in [0, T] \times \mathbb{R}$.

By Lemma 3.1 and Theorem 4.1 in [11], we can see that the closure of the domain of $\mathcal{D}$ with respect to the norm $\|F\|_{\mathcal{D}} := \{\mathbb{E}[|F|^2] + \mathbb{E}[\|\mathcal{D}F\|^2_{L^2_q}]\}^{1/2}$ is the space $\mathbb{D}^{1,2}(\mathbb{R})$ and $D_{t,z}F = \mathcal{D}_{t,z}F$ for all $F \in S \subset \mathbb{D}^{1,2}(\mathbb{R})$. Moreover, by Corollary 4.1 in [11], the set $\mathcal{S}$ of smooth random variables is dense in $L^2(\mathbb{P})$, $\mathbb{D}^{1,2}(\mathbb{R})$, $\mathbb{D}_0^{1,2}(\mathbb{R})$ and $\mathbb{D}_1^{1,2}(\mathbb{R})$.

Proposition 2.6. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a C^1-function with bounded derivative.*
(1) Let $F = (F_1, \cdots, F_n)$. If $F_1, \cdots, F_n \in \mathbb{D}_0^{1,2}(\mathbb{R})$, then $\varphi(F) \in \mathbb{D}_0^{1,2}(\mathbb{R})$ and

$$D_{t,0}\varphi(F) = \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\varphi(F) D_{t,0}F_k. \tag{2.1}$$

(2) Let $F = (F_1, \cdots, F_n)$. If $F_1, \cdots, F_n \in \mathbb{D}_1^{1,2}(\mathbb{R})$, then $\varphi(F) \in \mathbb{D}_1^{1,2}(\mathbb{R})$ and

$$D_{t,z}\varphi(F) = \frac{\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi(F_1, \cdots, F_n)}{z},\ z \neq 0. \tag{2.2}$$

Proof. Since φ is a Lipschitz continuous function, Lemma 5.1 of [11] implies that $\varphi(F) \in \mathbb{D}_0^{1,2}(\mathbb{R})$ and (2.2) holds. Moreover, we can show equation (2.1) and $\varphi(F) \in \mathbb{D}_0^{1,2}(\mathbb{R})$ by a similar step with Proposition 1.30 in [18]. □

To show a chain rule, we introduce the following lemma (Lemma 4.3.3 in [23]).

Lemma 2.7. *Let $n \in \mathbb{N}$. For any $M > 0$ and $\epsilon \in (0, 1)$, there exists a mapping $f \in C_0^\infty(\mathbb{R}^n; [0, 1])$ with partial derivative bounded by $1 + \epsilon$ such that $f(x) = 1$ for all $x \in [-M, M]^n$ and $\operatorname{supp}(f) \subset (-(M + 1), M + 1)^n$.*

Proposition 2.8. *(1) Let $F = (F_1, \cdots, F_n)$, where $F_1, \cdots, F_n \in \mathbb{D}_0^{1,2}(\mathbb{R})$ and $f \in C^1(\mathbb{R}^n)$ for $n \geq 1$. Moreover, assume that $f(F) \in L^2(\mathbb{P})$ and $\sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(F)D_{t,0}F_k \in L^2(\lambda \times \mathbb{P})$. Then $f(F) \in \mathbb{D}_0^{1,2}(\mathbb{R})$ and the following chain rule holds:*

$$D_{t,0}f(F) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(F) D_{t,0}F_k.$$

(2) Let $\varphi \in C^1(\mathbb{R}^n; \mathbb{R})$ and $F = (F_1, \cdots, F_n)$ with $F_1, \cdots, F_n \in \mathbb{D}_1^{1,2}(\mathbb{R})$. Suppose that $\varphi(F) \in L^2(\mathbb{P})$ and

$$\frac{\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi(F_1, \cdots, F_n)}{z} \in L^2(z^2\nu(dz)dtd\mathbb{P}).$$

Then $\varphi(F) \in \mathbb{D}_1^{1,2}(\mathbb{R})$ and

$$D_{t,z}\varphi(F) = \frac{\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi(F_1, \cdots, F_n)}{z}, z \neq 0.$$

Proof. (1) We can show it by similar steps with Theorem 4.3.5 in [23].
Step 1-1: We assume that $f \in C^1(\mathbb{R}^n)$ is bounded and that

$$\sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(F) D_{t,0}F \in L^2(\lambda \times \mathbb{P}).$$

Let $\epsilon \in (0, 1), m \in \mathbb{N}$ and define the sets K_m and V_m by $K_m = [-m, m]^n$ and $V_m = (-m - 1, m + 1)^n$. By Lemma 2.7, there exists a sequence (c_m) in $C_0^\infty(\mathbb{R}^n; [0, 1])$ with $c_m(x) = 1$ for all $x \in K_m$ and $\operatorname{supp}(c_m) \subset V_m$ such that $\frac{\partial c_m}{\partial x_k}$ are bounded by $1 + \epsilon$. We denote $f_m(x) = c_m(x)f(x)$. We will argue that $f_m \in C^1(\mathbb{R}^n)$ with bounded derivative. It is clear that f_m is continuously differentiable, and $\frac{\partial f_m}{\partial x_k}(x) = \frac{\partial c_m}{\partial x_k}(x)f(x) + c_m(x)\frac{\partial f}{\partial x_k}(x)$. Since $(c_m) \in C_0^\infty(\mathbb{R}^n)$, both c_m and $\frac{\partial c_m}{\partial x_k}(x)$ have compact support. It follows that $\frac{\partial f_m}{\partial x_k}(x)$ has compact support, and since it is also continuous, it is bounded. Hence, by Proposition 2.6, we can see that $f_m(F) \in$

$\mathbb{D}_0^{1,2}(\mathbb{R})$ and $D_{t,0}f_m(F) = \sum_{k=1}^n \frac{\partial f_m}{\partial x_k}(F_k)D_{t,0}F_k$. Next note the following: first note that $\frac{\partial c_m}{\partial x_k}$ is zero on K_m°. Since c_m is one on K_m°, we find that $\frac{\partial f_m}{\partial x_k}(x)$ and $\frac{\partial f}{\partial x_k}(x)$ are equal on K_m°.

Since c_m converges pointwise to 1, f_m converges pointwise to f. Hence, we obtain $\lim_{m\to\infty} f_m(F) = f(F)$ a.s. Because $\|c_m\|_\infty \le 1$, the dominated convergence yields that $\lim f_m(F) = f(F)$ in $L^2(\mathbb{P})$. Therefore, we obtain

$$\begin{aligned}
&\left(\mathbb{E}\left[\int_0^T \sum_{k=1}^n \frac{\partial f_m}{\partial x_k}(F)D_{t,0}F_k - \sum_{k=1}^n \frac{\partial f}{\partial x_k}(F)D_{t,0}F_k|^2 dt\right]\right)^{1/2} \\
&= \left(\mathbb{E}\left[\int_0^T \sum_{k=1}^n \mathbf{1}_{(K_m^\circ)^c}|(\frac{\partial f_m}{\partial x_k}(F) - \frac{\partial f}{\partial x_k}(F))D_{t,0}F_k|^2 dt\right]\right)^{1/2} \\
&= \left(\mathbb{E}\left[\int_0^T \sum_{k=1}^n \mathbf{1}_{(K_m^\circ)^c}|[\frac{\partial c_m}{\partial x_k}(F)f(F) + (c_m(F)-1)\frac{\partial f}{\partial x_k}(F)]D_{t,0}F_k|^2 dt\right]\right)^{1/2} \\
&\le \left(\mathbb{E}\left[\int_0^T \sum_{k=1}^n \mathbf{1}_{(K_m^\circ)^c}|\frac{\partial c_m}{\partial x_k}(F)f(F)D_{t,0}F_k|^2 dt\right]\right)^{1/2} \\
&\quad + \left(\mathbb{E}\left[\int_0^T \sum_{k=1}^n \mathbf{1}_{(K_m^\circ)^c}|(c_m(F)-1)\frac{\partial f}{\partial x_k}(F)D_{t,0}F_k|^2 dt\right]\right)^{1/2}.
\end{aligned}$$

We wish to show that each of these terms tend to zero by the dominated convergence. Considering the first term, by definition of c_m, we obtain

$$\left|\sum_{k=1}^n \mathbf{1}_{(K_m^\circ)^c}\frac{\partial c_m}{\partial x_k}(F)f(F)D_{t,0}F_k\right| \le (1+\epsilon)\|f\|_\infty \sum_{k=1}^n |D_{t,0}F_k|$$

which is $L^2(\lambda\times\mathbb{P})$-integrable. Likewise, for the second term we have the $L^2(\lambda\times\mathbb{P})$-integrable bound

$$\begin{aligned}
&\left|\sum_{k=1}^n \mathbf{1}_{(K_m^\circ)^c}(c_m(F)-1)\frac{\partial f}{\partial x_k}(F)D_{t,0}F_k\right| \\
&= \left|\mathbf{1}_{(K_m^\circ)^c}(c_m(F)-1)\right|\left|\sum_{k=1}^n \frac{\partial f}{\partial x_k}(F)D_{t,0}F_k\right| \\
&\le \left|\sum_{k=1}^n \frac{\partial f}{\partial x_k}(F)D_{t,0}F_k\right|.
\end{aligned}$$

Since K_m° increases to $\mathbb{R}$, by the dominated convergence using the two bounds obtained above, we find that both norms tends to zero and therefore we may finally conclude $\lim_{m\to\infty}\sum_{k=1}^n \frac{\partial f_m}{\partial x_k}(F)D_{t,0}F_k = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(F)D_{t,0}F_k$ in $L^2(\lambda\times\mathbb{P})$. Therefore, Proposition 2.4 implies that $f(F)\in\mathbb{D}^{1,2}(\mathbb{R})$ and $D_{t,0}f(F) = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(F)D_{t,0}F_k$.

Step 1-2: Let $f\in C^1(\mathbb{R}^n)$, $f(F)\in L^2(\mathbb{P})$ and $\sum_{k=1}^n \frac{\partial f}{\partial x_k}(F)D_{t,0}F_k \in L^2(\lambda\times\mathbb{P})$. Let $\epsilon\in(0,1), m\in\mathbb{N}$ and define the sets K_m and V_m by $K_m = [-m,m]$ and

$V_m = (-m-1, m+1)$ and and let c_m be the Lipschitz element of $C_0^\infty(\mathbb{R};[0,1])$ with $c_m(x) = 1$ for all $x \in K_m$ and $\operatorname{supp}(c_m) \subset V_m$ that exists by Lemma 2.7 with Lipschitz constant $1+\epsilon$. We denote $C_m(x) = \int_0^x c_m(y)dy$. Note that since c_m is bounded by one and is zero outside of V_m, $\|C_m\|_\infty \le m+1$. In particular, C_m is bounded. Now, defining $f_m(x) = C_m(f(x))$, it is then clear f_m is bounded. Furthermore, f_m is C^1 function and $\frac{\partial f_m}{\partial x_k}(x) = c_m(f(x))\frac{\partial f}{\partial x_k}(x)$. Therefore, we have $|\sum_{k=1}^n \frac{\partial f_m}{\partial x_k}(F)D_{t,0}F_k| = |c_m(f(F))\sum_{k=1}^n \frac{\partial f}{\partial x_k}(F)D_{t,0}F_k| \le |\sum_{k=1}^n \frac{\partial f}{\partial x_k}(F)D_{t,0}F_k|$, hence, $\sum_{k=1}^n \frac{\partial f_m}{\partial x_k}(F)D_{t,0}F_k \in L^2(\lambda \times \mathbb{P})$. Thus, f_m is covered by the previous step of the proof, and we may conclude that $f_m(F) \in \mathbb{D}^{1,2}(\mathbb{R})$ and $D_{t,0}f_m(F) = \sum_{k=1}^n \frac{\partial f_m}{\partial x_k}(F)D_{t,0}F_k$. As in the previous step, we will extend this result to f by applying Proposition 2.4.

To this end, note that since c_m is bounded by one, $|C_m(x)| \le \int_0^x |c_m(y)|dy \le |x|$ and that $\mathbb{E}[|f_m(F) - f(F)|^2] = \mathbb{E}[|C_m(f(F)) - f(F)|^2]$. Since $|C_m(f(F)) - f(F)| \le |C_m(f(F))| + |f(F)| \le 2|f(F)|$, we conclude by the dominated convergence that the above tends to zero, hence $\lim_{m\to\infty} f_m(F) = f(F)$ in $L^2(\mathbb{P})$. Likewise, for the derivatives, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T |\sum_{k=1}^n \frac{\partial f_m}{\partial x_k}(F)D_{t,0}F_k - \sum_{k=1}^n \frac{\partial f}{\partial x_k}(F)D_{t,0}F_k|^2 dt\right] \\
&= \mathbb{E}\left[\int_0^T |c_m(f(F))\sum_{k=1}^n \frac{\partial f}{\partial x_k}(F)D_{t,0}F_k - \sum_{k=1}^n \frac{\partial f}{\partial x_k}(F)D_{t,0}F_k|^2 dt\right] \\
&= \mathbb{E}\left[\int_0^T |(c_m(f(F)) - 1)\sum_{k=1}^n \frac{\partial f}{\partial x_k}(F)D_{t,0}F_k|^2 dt\right],
\end{aligned}
$$

and since $c_m - 1$ tends to zero, bounded by the constant one, we conclude by dominated convergence that the above tends to zero. Therefore, Proposition 2.4 implies that $f(F) \in \mathbb{D}^{1,2}(\mathbb{R})$ and $D_{t,0}f(F) = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(F)D_{t,0}F_k$.

(2) Step 2-1: We assume that $\varphi \in C^1(\mathbb{R}^n)$ is bounded and that

$$
\frac{\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi(F_1, \cdots, F_n)}{z} \in L^2(z^2\nu(dz)dtd\mathbb{P}).
$$

Let $\epsilon \in (0,1), m \in \mathbb{N}$ and define the sets K_m and V_m by $K_m = [-m,m]^n$ and $V_m = (-m-1, m+1)^n$. By Lemma 2.7, there exists a sequence (c_m) in $C_0^\infty(\mathbb{R}^n;[0,1])$ with $c_m(x) = 1$ for all $x \in K_m$ and $\operatorname{supp}(c_m) \subset V_m$ such that $\frac{\partial c_m}{\partial x_k}(x)$ are bounded by $1+\epsilon$. We denote $\varphi_m(x) = c_m(x)\varphi(x)$. By 1-1, we can see $\varphi_m \in C^1(\mathbb{R}^n)$ with bounded derivative. Hence, by Proposition 2.6, we can conclude that $\varphi_m(F) \in \mathbb{D}_1^{1,2}(\mathbb{R})$ and $D_{t,z}\varphi_m(F) = \frac{\varphi_m(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)-\varphi_m(F_1,\cdots,F_n)}{z}, z \neq 0$.

Since c_m converges pointwise to 1, φ_m converges pointwise to φ. Hence, we obtain $\lim_{m\to\infty}\varphi_m(F) = \varphi(F)$ a.s. Because $\|c_m\|_\infty \le 1$, the dominated convergence yields that $\lim \varphi_m(F) = \varphi(F)$ in $L^2(\mathbb{P})$. Moreover, we have

$$
\begin{aligned}
&\lim_{m\to\infty} \frac{\varphi_m(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi_m(F_1, \cdots, F_n)}{z} \\
&= \frac{\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi(F_1, \cdots, F_n)}{z},
\end{aligned}
$$

(t,z,ω)-a.e. On the other hand,

$$\begin{aligned}
&\left|\frac{\varphi_m(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)-\varphi_m(F_1,\cdots,F_n)}{z}\right.\\
&\left.-\frac{\varphi(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)-\varphi(F_1,\cdots,F_n)}{z}\right|\\
&=\left|\frac{c_m(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)\varphi(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)}{z}\right.\\
&\left.-\frac{c_m(F)\varphi(F_1,\cdots,F_n)}{z}-\frac{\varphi(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)-\varphi(F_1,\cdots,F_n)}{z}\right|\\
&\le |c_m(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)|\\
&\times\left|\frac{(\varphi(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)-\varphi(F_1,\cdots,F_n)}{z}\right|\\
&+\left|\frac{\varphi(F_1,\cdots,F_n)(c_m(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)-c_m(F))}{z}\right|\\
&+\left|\frac{\varphi(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)-\varphi(F_1,\cdots,F_n)}{z}\right|\\
&\le 2\left|\frac{\varphi(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)-\varphi(F_1,\cdots,F_n)}{z}\right|\\
&+(1+\epsilon)\|\varphi\|_\infty\sqrt{\sum_{k=1}^n (D_{t,z}F_k)^2}\in L^2(z^2\nu(dz)dtd\mathbb{P})
\end{aligned}$$

because c_m is a Lipschitz continuous function with Lipschitz constant $1+\epsilon$. Therefore, the dominated convergence theorem yields that

$$\begin{aligned}
&\lim_{m\to\infty}\frac{\varphi_m(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)-\varphi_m(F_1,\cdots,F_n)}{z}\\
&=\frac{\varphi(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)-\varphi(F_1,\cdots,F_n)}{z} \text{ in } L^2(z^2\nu(dz)dtd\mathbb{P}).
\end{aligned}$$

Therefore, Proposition 2.4 implies that $\varphi(F)\in\mathbb{D}_1^{1,2}(\mathbb{R})$ and

$$D_{t,z}\varphi(F)=\frac{\varphi(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)-\varphi(F_1,\cdots,F_n)}{z}, z\neq 0.$$

Step 2-2: Let $\varphi\in C^1(\mathbb{R}^n)$, $\varphi(F)\in L^2(\mathbb{P})$ and assume that $\frac{\varphi(F+zD_{t,z}F)-\varphi(F)}{z}\in L^2(z^2\nu(dz)dtd\mathbb{P})$. Let $\epsilon\in(0,1), m\in\mathbb{N}$ and define the sets K_m and V_m by $K_m=[-m,m]$ and $V_m=(-m-1,m+1)$ and and let c_m be the Lipschitz element of $C_0^\infty(\mathbb{R};[0,1])$ with $c_m(x)=1$ for all $x\in K_m$ and $\operatorname{supp}(c_m)\subset V_m$ that exists by Lemma 2.7 with Lipschitz constant $1+\epsilon$. We denote $C_m(x)=\int_0^x c_m(y)dy$. Note that since c_m is bounded by one and is zero outside of V_m, $\|C_m\|_\infty\le m+1$. In particular, C_m is bounded. Now, defining $\varphi_m(x)=C_m(\varphi(x))$, it is then clear φ_m

is bounded. Furthermore, φ_m is C^1 function and

$$\begin{aligned}
&\left|\frac{\varphi_m(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi_m(F)}{z}\right| \\
&= \left|\frac{C_m(\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n)) - C_m(\varphi(F))}{z}\right| \\
&= |z|^{-1}\left|\int_0^{\varphi(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)} c_m(y)dy - \int_0^{\varphi(F)} c_m(y)dy\right| \\
&= |z|^{-1}\left|\int_{\varphi(F)}^{\varphi(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)} c_m(y)dy\right| \\
&\leq |z|^{-1}|\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi(F)| \qquad (2.3)
\end{aligned}$$

hence, $\frac{\varphi_m F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)-\varphi_m(F)}{z} \in L^2(z^2\nu(dz)dtd\mathbb{P})$. Thus, φ_m is covered by the previous step of the proof, and we may conclude that $\varphi_m(F) \in \mathbb{D}_1^{1,2}(\mathbb{R})$ and $D_{t,z}\varphi_m(F) = \frac{\varphi_m(F_1+zD_{t,z}F_1,\cdots,F_n+zD_{t,z}F_n)-\varphi_m(F)}{z}$. As in the previous step, we will extend this result to φ by applying Proposition 2.4.

To this end, note that since c_m is bounded by one, $|C_m(x)| \leq \int_0^x |c_m(y)|dy \leq |x|$, and that $\mathbb{E}[|\varphi_m(F)-\varphi(F)|^2] = \mathbb{E}[|C_m(\varphi(F))-\varphi(F)|^2]$. Since $|C_m(\varphi(F))-\varphi(F)| \leq |C_m(\varphi(F))| + |\varphi(F)| \leq 2|\varphi(F)|$, we conclude by the dominated convergence theorem that the above tends to zero, hence $\lim_{m\to\infty} \varphi_m(F) = \varphi(F)$ in $L^2(\mathbb{P})$. Moreover, by (2.3)

$$\begin{aligned}
&\left|\frac{C_m(\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n)) - C_m(\varphi(F))}{z}\right. \\
&\left.-\frac{\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi(F)}{z}\right| \\
&\leq 2\left|\frac{\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi(F)}{z}\right| \in L^2(z^2\nu(dz)dtd\mathbb{P}).
\end{aligned}$$

Hence, we conclude by the dominated convergence theorem that

$$\begin{aligned}
&\lim_{m\to\infty} \frac{\varphi_m(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi_m(F)}{z} \\
&= \frac{\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi(F)}{z} \text{ in } L^2(z^2\nu(dz)dtd\mathbb{P}).
\end{aligned}$$

Therefore, Proposition 2.4 implies that $\varphi(F) \in \mathbb{D}_1^{1,2}(\mathbb{R})$ and

$$D_{t,z}\varphi(F) = \frac{\varphi(F_1 + zD_{t,z}F_1, \cdots, F_n + zD_{t,z}F_n) - \varphi(F)}{z}.$$

□

By Proposition 2.8, we can immediately derive the following:

Proposition 2.9 (Chain rule)**.** *Let* $\varphi \in C^1(\mathbb{R}^n;\mathbb{R})$ *and* $F = (F_1, \cdots, F_n)$, *where* $F_1, \cdots, F_n \in \mathbb{D}^{1,2}(\mathbb{R})$. *Suppose* $\varphi(F) \in L^2(\mathbb{P})$, $\sum_{k=1}^n \frac{\partial}{\partial x_k}\varphi(F)D_{t,0}F_k \in L^2(\lambda \times$

$\mathbb{P}$), *and* $\frac{\varphi(F_1+zD_{t,z}F_1,\cdots,F_k+zD_{t,z}F_k)-\varphi(F_1,\cdots,F_k)}{z} \in L^2(z^2\nu(dz)dtd\mathbb{P})$. *Then* $\varphi(F) \in \mathbb{D}^{1,2}(\mathbb{R})$,

$$D_{t,0}\varphi(F) = \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\varphi(F)D_{t,0}F_k$$

and

$$D_{t,z}\varphi(F) = \frac{\varphi(F_1 + zD_{t,z}F_1, \cdots, F_k + zD_{t,z}F_k) - \varphi(F_1, \cdots, F_k)}{z},\ z \neq 0.$$

If we take $\varphi(x, y) = xy$, then we can derive the following product rule.

Corollary 2.10. *Let* $F_1, F_2 \in \mathbb{D}^{1,2}(\mathbb{R})$ *and* $F_1F_2 \in L^2(\mathbb{P})$. *Moreover, assume that* $F_1D_{t,z}F_2 + F_2D_{t,z}F_1 + zD_{t,z}F_1 \cdot D_{t,z}F_2 \in L^2(q \times \mathbb{P})$. *Then* $F_1F_2 \in \mathbb{D}^{1,2}(\mathbb{R})$ *and*

$$D_{t,z}F_1F_2 = F_1D_{t,z}F_2 + F_2D_{t,z}F_1 + zD_{t,z}F_1 \cdot D_{t,z}F_2. \tag{2.4}$$

3. Commutation of Integration and the Malliavin Differentiability

In this section, we consider commutations of integration and the Malliavin differentiability, which has an interest of its own and could be applied for other purposes than the one of this paper. First we introduce the following classes.

Definition 3.1. (1) Let $\mathbb{L}^{1,2}(\mathbb{R})$ denote the space of product measurable and $\mathbb{F}$ -adapted processes $G : \Omega\times[0, T]\times\mathbb{R} \to \mathbb{R}$ satisfying $\mathbb{E}[\int_{[0,T]\times\mathbb{R}} |G(s, x)|^2q(ds, dx)] < \infty$, $G(s, x) \in \mathbb{D}^{1,2}(\mathbb{R})$, $q-$a.e. $(s, x) \in [0, T] \times \mathbb{R}$ and

$$\mathbb{E}\left[\int_{([0,T]\times\mathbb{R})^2} |D_{t,z}G(s, x)|^2q(ds, dx)q(dt, dz)\right] < \infty.$$

(2) Let $\mathbb{L}_0^{1,2}(\mathbb{R})$ denote the space of measurable and $\mathbb{F}$ -adapted processes G : $\Omega \times [0, T] \to \mathbb{R}$ satisfying $\mathbb{E}[\int_{[0,T]} |G(s)|^2ds] < \infty$, $G(s) \in \mathbb{D}^{1,2}(\mathbb{R})$, $s \in [0, T]$, a.e. and $\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} \int_{[0,T]} |D_{t,z}G(s)|^2dsq(dt, dz)\right] < \infty$.
(3) Let $\tilde{\mathbb{L}}_1^{1,2}(\mathbb{R})$ denote the space of product measurable and $\mathbb{F}$ -adapted processes $G : \Omega \times [0, T] \times \mathbb{R}_0 \to \mathbb{R}$ satisfying

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}_0} |G(s, x)|^2\nu(dx)ds\right] < \infty, \mathbb{E}\left[\left(\int_{[0,T]\times\mathbb{R}_0} |G(s, x)|\nu(dx)ds\right)^2\right] < \infty,$$

$G(s, x) \in \mathbb{D}^{1,2}(\mathbb{R}), (s, x) \in [0, T] \times \mathbb{R}_0$, a.e. ,

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} \left(\int_{[0,T]\times\mathbb{R}_0} |D_{t,z}G(s, x)|\nu(dx)ds\right)^2 q(dt, dz)\right] < \infty$$

and

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} \int_{[0,T]\times\mathbb{R}_0} |D_{t,z}G(s, x)|^2\nu(dx)dsq(dt, dz)\right] < \infty.$$

We next discuss the commutation relation of the stochastic integral with the Malliavin derivative.

Proposition 3.2. *Let $G : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be a predictable process with $\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} |G(s,x)|^2 q(ds,dx)\right] < \infty$. Then*

$$G \in \mathbb{L}^{1,2}(\mathbb{R}) \text{ if and only if } \int_{[0,T]\times\mathbb{R}} G(s,x)Q(ds,dx) \in \mathbb{D}^{1,2}(\mathbb{R}). \tag{3.1}$$

Furthermore, if $\int_{[0,T]\times\mathbb{R}} G(s,x)Q(ds,dx) \in \mathbb{D}^{1,2}(\mathbb{R})$, then for q -a.e. $(t,z) \in [0,T] \times \mathbb{R}$, we have

$$D_{t,z} \int_{[0,T]\times\mathbb{R}} G(s,x)Q(ds,dx) = G(t,z) + \int_{[0,T]\times\mathbb{R}} D_{t,z}G(s,x)Q(ds,dx), \quad \mathbb{P}\text{−a.s.}, \tag{3.2}$$

and $\int_{[0,T]\times\mathbb{R}} D_{t,z}G(s,x)Q(ds,dx)$ is a stochastic integral in Itô sense.

Proof. We can show the same step as Lemma 3.3 in [8]. □

Next proposition provides a commutation of the Lebesgue integration and the Malliavin differentiability.

Proposition 3.3. *Assume that $G : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is a product measurable and $\mathbb{F}$ -adapted process, η on $[0,T] \times \mathbb{R}$ a finite measure, so that conditions*

$$\begin{aligned}
&\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} |G(s,x)|^2 \eta(ds,dx)\right] < \infty,\\
&G(s,x) \in \mathbb{D}^{1,2}(\mathbb{R}), \qquad \text{for } \eta\text{−a.e. } (s,x) \in [0,T] \times \mathbb{R},\\
&\mathbb{E}\left[\int_{([0,T]\times\mathbb{R})^2} |D_{t,z}G(s,x)|^2 \eta(ds,dx) q(dt,dz)\right] < \infty
\end{aligned}$$

are satisfied. Then we have $\int_{[0,T]\times\mathbb{R}} G(s,x)\eta(ds,dx) \in \mathbb{D}^{1,2}(\mathbb{R})$ and the differentiation rule

$$D_{t,z} \int_{[0,T]\times\mathbb{R}} G(s,x)\eta(ds,dx) = \int_{[0,T]\times\mathbb{R}} D_{t,z}G(s,x)\eta(ds,dx)$$

holds for q -a.e. $(t,z) \in [0,T] \times \mathbb{R}, \mathbb{P}$ -a.s.

Proof. We can show the same step as Lemma 3.2 of [8]. □

Remark 3.4. We already know the following:
(1) If $G(s,x) \in L^1(\eta)$ is a deterministic function, and $\eta([0,T] \times \mathbb{R}) < \infty$ or $\eta([0,T] \times \mathbb{R}) = \infty$, then we can see $\int_{[0,T]\times\mathbb{R}} G(s,x)\eta(ds,dx) \in \mathbb{D}^{1,2}(\mathbb{R})$, and $D_{t,z}\int_{[0,T]\times\mathbb{R}} G(s,x)\eta(ds,dx) = 0 = \int_{[0,T]\times\mathbb{R}} D_{t,z}G(s,x)\eta(ds,dx)$.
(2) Let $\eta(dx,ds) = \delta_{\mathbb{R}_0}(x)\nu(dx)ds$ with $\nu(\mathbb{R}_0) < \infty$.
Then Proposition 3.3 implies that $\int_{[0,T]\times\mathbb{R}_0} G(s,x)\nu(dx)ds \in \mathbb{D}^{1,2}(\mathbb{R})$ and the differentiation rule holds.
(3) We assume ν satisfies $\nu(\mathbb{R}_0) < \infty$ or $\nu(\mathbb{R}_0) = \infty$. Moreover if $G(s,x) = g_1(x)g_2(s)$, where $g_1(x) \in L^1(\nu)$ is a deterministic function and $g_2(s) \in \mathbb{L}_0^{1,2}(\mathbb{R})$ is a stochastic process. Then $\int_{[0,T]\times\mathbb{R}_0} G(s,x)\nu(dx)ds = \int_{\mathbb{R}_0} g_1(x)\nu(dx)\int_{[0,T]} g_2(s)ds = C\int_{[0,T]} g_2(s)ds$, where $C := \int_{\mathbb{R}_0} g_1(x)\nu(dx)$ is a constant number. Therefore, by

Proposition 3.3, we can see $C\int_{[0,T]} g_2(s)ds \in \mathbb{D}^{1,2}(\mathbb{R})$ and the differentiation rule holds.

By using σ-finiteness of ν and Proposition 3.3, we can show the following proposition.

Proposition 3.5. *Let $G \in \tilde{\mathbb{L}}_1^{1,2}(\mathbb{R})$. Then $\int_{[0,T]\times\mathbb{R}_0} G(s,x)\nu(dx)ds \in \mathbb{D}^{1,2}(\mathbb{R})$ and the differentiation rule*

$$D_{t,z}\int_{[0,T]\times\mathbb{R}_0} G(s,x)\nu(dx)ds = \int_{[0,T]\times\mathbb{R}_0} D_{t,z}G(s,x)\nu(dx)ds$$

holds for q -a.e. $(t,z)\in[0,T]\times\mathbb{R}, \mathbb{P}$ -a.s.

Proof. Since ν is σ-finite measure, we can find a sequence $(A_n, n\in\mathbb{N})$ in $\mathcal{B}(\mathbb{R}_0)$ such that $\mathbb{R}_0 = \cup_{n=1}^{\infty} A_n$ and $\nu(A_n)<\infty$. Hence, Proposition 3.3 implies

$$\int_{[0,T]\times\bigcup_{n=1}^k A_n} G(s,x)\nu(dx)ds \in \mathbb{D}^{1,2}(\mathbb{R}), k\in\mathbb{N}$$

and

$$D_{t,z}\int_{[0,T]\times\bigcup_{n=1}^k A_n} G(s,x)\nu(dx)ds = \int_{[0,T]\times\bigcup_{n=1}^k A_n} D_{t,z}G(s,x)\nu(dx)ds.$$

Next, note the following;

$$\lim_{k\to\infty} G(s,x)\mathbf{1}_{\bigcup_{n=1}^k A_n}(x) = G(s,x), \nu\otimes\lambda\otimes\mathbb{P}\text{−a.e.},$$

hence,

$$\lim_{k\to\infty} G(s,x)\mathbf{1}_{\bigcap_{n=1}^k A_n^C}(x) = 0, \nu\otimes\lambda\otimes\mathbb{P}\text{−a.e.},$$

$$|G(s,x)\mathbf{1}_{\bigcup_{n=1}^k A_n}(x) - G(s,x)| = |G(s,x)\mathbf{1}_{\bigcap_{n=1}^k A_n^C}(x)| \le |G(s,x)| \in L^1(\nu\times\lambda)$$

and

$$\begin{aligned}&\left|\int_{[0,T]\times\mathbb{R}_0} G(s,x)\nu(dx)ds - \int_{[0,T]\times\bigcup_{n=1}^k A_n} G(s,x)\nu(dx)ds\right|^2\\ &\le \left(\int_{[0,T]\times\mathbb{R}_0} |G(s,x)|\nu(dx)ds\right)^2 \in L^1(\mathbb{P}).\end{aligned}$$

Then by Lebesgue's dominated convergence theorem, we can see

$$\lim_{k\to\infty}\mathbb{E}\left[\left|\int_{[0,T]\times\mathbb{R}_0} G(s,x)\nu(dx)ds - \int_{[0,T]\times\bigcup_{n=1}^k A_n} G(s,x)\nu(dx)ds\right|^2\right] = 0.$$

Moreover,

$$\lim_{k\to\infty} D_{t,z}G(s,x)\mathbf{1}_{\bigcup_{n=1}^k A_n}(x) = D_{t,z}G(s,x),\ \nu\otimes\lambda\otimes\mathbb{P}\otimes q\text{−a.e.},$$

hence,

$$\lim_{k\to\infty} D_{t,z}G(s,x)\mathbf{1}_{\bigcap_{n=1}^k A_n^C}(x) = 0,\ \nu\otimes\lambda\otimes\mathbb{P}\otimes q\text{−a.e.},$$

$$\begin{aligned}&|D_{t,z}G(s,x)\mathbf{1}_{\bigcup_{n=1}^k A_n}(x) - D_{t,z}G(s,x)| = |D_{t,z}G(s,x)\mathbf{1}_{\bigcap_{n=1}^k A_n^C}(x)|\\&\leq |D_{t,z}G(s,x)| \in L^1(\nu\times\lambda),\end{aligned}$$

and

$$\begin{aligned}&\left|\int_{[0,T]\times\mathbb{R}_0} D_{t,z}G(s,x)\nu(dx)ds - \int_{[0,T]\times\bigcup_{n=1}^k A_n} D_{t,z}G(s,x)\nu(dx)ds\right|^2\\&\leq \left(\int_{[0,T]\times\mathbb{R}_0}|D_{t,z}G(s,x)|\nu(dx)ds\right)^2 \in L^1(q\times\mathbb{P}).\end{aligned}$$

Then Lebesgue's dominated convergence theorem shows

$$\int_{[0,T]\times\mathbb{R}}\mathbb{E}\left[\left|\int_{[0,T]\times\mathbb{R}_0} D_{t,z}G(s,x)\nu(dx)ds - \int_{[0,T]\times\ \cup_{n=1}^k A_n} D_{t,z}G(s,x)\nu(dx)ds\right|^2\right]$$

$\times q(dt,dz)\to 0$ as $k\to\infty$. Therefore, by Proposition 2.4, we can conclude

$$\int_{[0,T]\times\mathbb{R}_0} G(s,x)\nu(dx)ds \in \mathbb{D}^{1,2}(\mathbb{R})$$

and the differentiation rule

$$D_{t,z}\int_{[0,T]\times\mathbb{R}_0} G(s,x)\nu(dx)ds = \int_{[0,T]\times\mathbb{R}_0} D_{t,z}G(s,x)\nu(dx)ds$$

holds for q -a.e. $(t,z)\in[0,T]\times\mathbb{R}, \mathbb{P}$ -a.s. □

4. A Clark-Ocone Type Formula Under Change of Measure for Lévy Processes

In this section, we introduce a Clark-Ocone type formula under change of measure for Lévy processes. First, to use the Girsanov theorem for Lévy processes (see, e.g., Theorem 12.21 in [9]), we assume the following.

Assumption 1. Let $\theta(s,x)<1, s\in[0,T], x\in\mathbb{R}_0$ and $u(s), s\in[0,T]$, be predictable processes such that $\int_0^T\int_{\mathbb{R}_0}\{|\log(1-\theta(s,x))|+\theta^2(s,x)\}\nu(dx)ds<\infty$, a.s., $\int_0^T u^2(s)ds<\infty$, a.s. Moreover we denote

$$\begin{aligned}Z(t) := \exp\Bigg(&-\int_0^t u(s)dW(s) - \frac{1}{2}\int_0^t u(s)^2ds + \int_0^t\int_{\mathbb{R}_0}\log(1-\theta(s,x))\tilde{N}(ds,dx)\\&+\int_0^t\int_{\mathbb{R}_0}(\log(1-\theta(s,x))+\theta(s,x))\nu(dx)ds\Bigg),\quad t\in[0,T].\end{aligned}$$

Define a measure $\mathbb{Q}$ on $\mathcal{F}_T$ by $d\mathbb{Q}(\omega)=Z(\omega,T)d\mathbb{P}(\omega)$, and we assume that $Z(T)$ satisfies the Novikov condition, that is,

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^T u^2(s)ds+\int_0^T\int_{\mathbb{R}_0}\{(1-\theta(s,x))\log(1-\theta(s,x))+\theta(s,x)\}\nu(dx)ds}\right]<\infty.$$

Furthermore we denote $\tilde{N}_{\mathbb{Q}}(dt,dx) := \theta(t,x)\nu(dx)dt+\tilde{N}(dt,dx)$ and $dW_{\mathbb{Q}}(t) := u(t)dt+dW(t)$.

Second, we assume the following.

Assumption 2. We denote

$$\tilde{H}(t,z) := \exp\Bigg(-\int_0^T zD_{t,z}u(s)dW_{\mathbb{Q}}(s) - \frac{1}{2}\int_0^T (zD_{t,z}u(s))^2 ds + \int_0^T \int_{\mathbb{R}_0} \left[zD_{t,z}\theta(s,x) + \log\left(1 - z\frac{D_{t,z}\theta(s,x)}{1-\theta(s,x)}\right)(1-\theta(s,x)) \right] \nu(dx)ds + \int_0^T \int_{\mathbb{R}_0} \log\left(1 - z\frac{D_{t,z}\theta(s,x)}{1-\theta(s,x)}\right) \tilde{N}_{\mathbb{Q}}(ds,dx) \Bigg),$$

and $K(t) := \int_0^T D_{t,0}u(s)dW_{\mathbb{Q}}(s) + \int_0^T \int_{\mathbb{R}_0} \frac{D_{t,0}\theta(s,x)}{1-\theta(s,x)} \tilde{N}_{\mathbb{Q}}(ds,dx)$ and assume that $\sigma \neq 0$. Furthermore, we assume the following:
(1) $F, Z(T) \in \mathbb{D}^{1,2}(\mathbb{R})$, with $FZ(T) \in L^2(\mathbb{P})$,

$$Z(T)D_{t,z}F + FD_{t,z}Z(T) + zD_{t,z}F \cdot D_{t,z}Z(T) \in L^2(q \times \mathbb{P}),$$

(2) $Z(T)D_{t,0}\log Z(T) \in L^2(\lambda \times \mathbb{P}), Z(T)(e^{zD_{t,z}\log Z(T)} - 1) \in L^2(\nu(dz)dtd\mathbb{P})$,
(3) $u(s)D_{t,0}u(s) \in L^2(\lambda \times \mathbb{P})$, $2u(s)D_{t,z}u(s) + z(D_{t,z}u(s))^2 \in L^2(z^2\nu(dz)dtd\mathbb{P})$, s-a.e.
(4) $\log\left(1 - z\frac{D_{t,z}\theta(s,x)}{1-\theta(s,x)}\right) \in L^2(\nu(dz)dtd\mathbb{P})$, $\frac{D_{t,0}\theta(s,x)}{1-\theta(s,x)} \in L^2(\lambda \times \mathbb{P}), (s,x)$ -a.e.
(5) $\sigma^{-1}u, x^{-1}\log(1-\theta(s,x)) \in \mathbb{L}^{1,2}(\mathbb{R})$,
(6) $u(s)^2 \in \mathbb{L}_0^{1,2}$ and $\theta(s,x), \log(1-\theta(s,x)) \in \tilde{\mathbb{L}}_1^{1,2}(\mathbb{R})$,
(7) and $F\tilde{H}(t,z), \tilde{H}(t,z)D_{t,z}F \in L^1(\mathbb{Q}), (t,z)$ -a.e.

We also introduce a Clark-Ocone type formula for Lévy functionals.

Proposition 4.1 (Clark-Ocone type formula for Lévy functionals)**.** *Let* $F \in \mathbb{D}^{1,2}(\mathbb{R})$. *Then*

$$F = \mathbb{E}[F] + \int_{[0,T]\times\mathbb{R}} \mathbb{E}[D_{t,z}F|\mathcal{F}_{t-}]Q(dt,dz) = \mathbb{E}[F] + \sigma\int_0^T \mathbb{E}[D_{t,0}F|\mathcal{F}_{t-}]dW(t) + \int_0^T\int_{\mathbb{R}_0} \mathbb{E}[D_{t,z}F|\mathcal{F}_{t-}]z\tilde{N}(dt,dz). \quad (4.1)$$

Proof. The proof is equal to the one for the Brownian motion case (see, Theorem 4.1 in [9]) and pure jump Lévy case (see, Theorem 12.16 in [9]). □

We also introduce the following

Lemma 4.2. *Let* $F \in \mathbb{D}^{1,2}(\mathbb{R})$. *Then for* $0 \le t \le T$, $\mathbb{E}[F|\mathcal{F}_t] \in \mathbb{D}^{1,2}(\mathbb{R})$ *and*

$$D_{s,x}\mathbb{E}[F|\mathcal{F}_t] = \mathbb{E}[D_{s,x}F|\mathcal{F}_t]\mathbf{1}_{\{s\le t\}}, \quad \text{for } q\text{−a.e. } (s,x) \in [0,T]\times\mathbb{R},\ \mathbb{P}\text{−a.s.}$$

Proof. We can show the same step as Proposition 1.2.8 in [17] □

To show the main theorem, we need the following.

Lemma 4.3. *Under Assumption 1 and Assumption 2, we have*

$$D_{t,0}Z(T) = Z(T)\left[-\sigma^{-1}u(t) - \int_0^T D_{t,0}u(s)dW_{\mathbb{Q}}(s) - \int_0^T\int_{\mathbb{R}_0} \frac{D_{t,0}\theta(s,x)}{1-\theta(s,x)}\tilde{N}_{\mathbb{Q}}(ds,dx)\right] \tag{4.2}$$

$$D_{t,z}Z(T) = z^{-1}Z(T)[\exp(zD_{t,z}\log Z(T)) - 1],\ z \neq 0, \tag{4.3}$$

where

$$\begin{aligned} D_{t,z}\log Z(T) &= -\int_0^T D_{t,z}u(s)dW_{\mathbb{Q}}(s) - \frac{1}{2}\int_0^T z(D_{t,z}u(s))^2 ds \\ &+ \int_0^T\int_{\mathbb{R}_0} \big(z^{-1}A_{t,z}(s,x)(1-\theta(s,x)) + D_{t,z}\theta(s,x)\big)\nu(dx)ds \\ &+ \int_0^T\int_{\mathbb{R}_0} z^{-1}A_{t,z}(s,x)\tilde{N}_{\mathbb{Q}}(ds,dx) + z^{-1}\log(1-\theta(t,z)), z \neq 0. \end{aligned} \tag{4.4}$$

and $A_{t,z}(s,x) = \log\big(1 - z\frac{D_{t,z}\theta(s,x)}{1-\theta(s,x)}\big)$, $z \neq 0$.

Proof. By conditions (1), (2), (5) and (6) in Assumption 2, Proposition 2.8-1 lead to:

$$\begin{aligned} D_{t,0}Z(T) = Z(T)\Bigg[&-D_{t,0}\int_0^T u(s)dW(s) - \frac{1}{2}D_{t,0}\int_0^T u(s)^2 ds \\ &+ D_{t,0}\int_0^T\int_{\mathbb{R}_0} \log(1-\theta(s,x))\tilde{N}(ds,dx) \\ &+ D_{t,0}\int_0^T\int_{\mathbb{R}_0} (\log(1-\theta(s,x)) + \theta(s,x))\nu(dx)ds\Bigg]. \end{aligned} \tag{4.5}$$

From assumption (6) in Assumption 2, Proposition 3.3 implies

$$D_{t,0}\int_0^T u(s)^2 ds = \int_0^T D_{t,0}u(s)^2 ds \tag{4.6}$$

and by Proposition 3.5

$$\begin{aligned} &D_{t,0}\int_0^T\int_{\mathbb{R}_0} (\log(1-\theta(s,x)) + \theta(s,x))\nu(dx)ds \\ &= \int_0^T\int_{\mathbb{R}_0} (D_{t,0}\log(1-\theta(s,x)) + D_{t,0}\theta(s,x))\nu(dx)ds. \end{aligned} \tag{4.7}$$

Since conditions (3)-(5) in Assumption 2 hold, by Proposition 2.8-1, we have

$$D_{t,0}u(s)^2 = 2u(s)D_{t,0}u(s) \tag{4.8}$$

and

$$D_{t,0}\log(1-\theta(s,x)) = -\frac{D_{t,0}\theta(s,x)}{1-\theta(s,x)}. \tag{4.9}$$

From condition (5) in Assumption 2, Proposition 3.2 implies

$$D_{t,0}\int_0^T u(s)dW(s) = \sigma^{-1}u(t) + \int_0^T D_{t,0}u(s)dW(s) \tag{4.10}$$

and

$$D_{t,0}\int_0^T\int_{\mathbb{R}_0}\log(1-\theta(s,x))\tilde{N}(ds,dx) = \int_0^T\int_{\mathbb{R}_0} D_{t,0}\log(1-\theta(s,x))\tilde{N}(ds,dx). \tag{4.11}$$

Hence, by (4.5) - (4.11), we obtain

$$\begin{aligned}
D_{t,0}Z(T) &= Z(T)\Bigg[-\sigma^{-1}u(t) - \int_0^T D_{t,0}u(s)dW(s) - \int_0^T u(s)D_{t,0}u(s)ds \\
&- \int_0^T\int_{\mathbb{R}_0}\frac{D_{t,0}\theta(s,x)}{1-\theta(s,x)}\tilde{N}(ds,dx) + \int_0^T\int_{\mathbb{R}_0}\left(-\frac{D_{t,0}\theta(s,x)}{1-\theta(s,x)} + D_{t,0}\theta(s,x)\right)\nu(dx)ds\Bigg] \\
&= Z(T)\left[-\sigma^{-1}u(t) - \int_0^T D_{t,0}u(s)dW_{\mathbb{Q}}(s) - \int_0^T\int_{\mathbb{R}_0}\frac{D_{t,0}\theta(s,x)}{1-\theta(s,x)}\tilde{N}_{\mathbb{Q}}(ds,dx)\right].
\end{aligned}$$

By (1) and (2) in Assumption 2, Proposition 2.8-2 implies

$$\begin{aligned}
D_{t,z}Z(T) &= \frac{\exp(\log Z(T) + zD_{t,z}\log Z(T)) - Z(T)}{z} \\
&= z^{-1}Z(T)[\exp(zD_{t,z}\log Z(T)) - 1].
\end{aligned}$$

Furthermore, by conditions (5) and (6) in Assumption 2, Proposition 3.2 and Proposition 3.5 show that

$$\begin{aligned}
D_{t,z}\log Z(T) &= -D_{t,z}\int_0^T u(s)dW(s) - \frac{1}{2}D_{t,z}\int_0^T u(s)^2 ds \\
&+ D_{t,z}\int_0^T\int_{\mathbb{R}_0} x^{-1}\log(1-\theta(s,x))x\tilde{N}(ds,dx) \\
&+ D_{t,z}\int_0^T\int_{\mathbb{R}_0}(\log(1-\theta(s,x)) + \theta(s,x))\nu(dx)ds \\
&= -\int_0^T D_{t,z}u(s)dW(s) - \frac{1}{2}\int_0^T D_{t,z}(u(s))^2 ds \\
&+ \int_0^T\int_{\mathbb{R}_0} D_{t,z}\log(1-\theta(s,x))\tilde{N}(ds,dx) \\
&+ \int_0^T\int_{\mathbb{R}_0}(D_{t,z}\log(1-\theta(s,x)) + D_{t,z}\theta(s,x))\,\nu(dx)ds + \frac{\log(1-\theta(t,z))}{z}.
\end{aligned} \tag{4.12}$$

Now we calculate $D_{t,z}(u(s))^2$ and $D_{t,z}\log(1-\theta(s,x))$. By Proposition 2.8-2,

$$D_{t,z}(u(s))^2 = 2u(s)D_{t,z}u(s) + z(D_{t,z}u(s))^2, \tag{4.13}$$

because, $u \in \mathbb{D}^{1,2}(\mathbb{R})$ and condition (3) in Assumption 2. Furthermore we can calculate the following by Proposition 2.8-2:

$$D_{t,z} \log(1-\theta(s,x)) = z^{-1} \log\left(1 - z\frac{D_{t,z}\theta(s,x)}{1-\theta(s,x)}\right). \tag{4.14}$$

From equations (4.12), (4.13), (4.14) and denoting $A_{t,z}(s,x) = \log(1 - z\frac{D_{t,z}\theta(s,x)}{1-\theta(s,x)})$, we have,

$$\begin{aligned} D_{t,z} \log Z(T) = &- \int_0^T D_{t,z}u(s)dW_{\mathbb{Q}}(s) - \frac{1}{2}\int_0^T z(D_{t,z}u(s))^2 ds \\ &+ \int_0^T \int_{\mathbb{R}_0} \left(z^{-1}A_{t,z}(s,x)(1-\theta(s,x)) + D_{t,z}\theta(s,x)\right)\nu(dx)ds \\ &+ \int_0^T \int_{\mathbb{R}_0} z^{-1}A_{t,z}(s,x)\tilde{N}_{\mathbb{Q}}(ds,dx) + z^{-1}\log(1-\theta(t,z)). \end{aligned}$$

□

We next introduce a Clark-Ocone type formula under change of measure for Lévy processes.

Theorem 4.4. *Under Assumption 1 and Assumption 2, we have*

$$\begin{aligned} F = &\mathbb{E}_{\mathbb{Q}}[F] + \sigma\int_0^T \mathbb{E}_{\mathbb{Q}}\left[D_{t,0}F - FK(t)\middle|\mathcal{F}_{t-}\right]dW_{\mathbb{Q}}(t) \\ &+ \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{Q}}[F(\tilde{H}(t,z)-1) + z\tilde{H}(t,z)D_{t,z}F|\mathcal{F}_{t-}]\tilde{N}_{\mathbb{Q}}(dt,dz). \end{aligned}$$

Proof. First we denote $\Lambda(t) := Z^{-1}(t), t \in [0,T]$. Then by the Itô formula (see, e.g., Theorem 9.4 of [9]), we have

$$\begin{aligned} d\Lambda(t) &= \Lambda(t-)\left(\frac{1}{2}u^2(t) - \int_{\mathbb{R}_0}(\log(1-\theta(t,z)) + \theta(t,z))\nu(dz)\right)dt \\ &+ \Lambda(t-)u(t)dW(t) + \frac{1}{2}\Lambda(t-)u^2(t)dt + \int_{\mathbb{R}_0}\Lambda(t-)\left(\frac{1}{1-\theta(t,z)} - 1\right)\tilde{N}(dt,dz) \\ &+ \int_{\mathbb{R}_0}\left[\Lambda(t-)\cdot\frac{1}{1-\theta(t,z)} - \Lambda(t-) + \Lambda(t-)\log(1-\theta(t,z))\right]\nu(dz)dt \\ &= \Lambda(t-)\left[u^2(t)dt + u(t)dW_t + \int_{\mathbb{R}_0}\frac{\theta(t,z)^2}{1-\theta(t,z)}\nu(dz)dt + \int_{\mathbb{R}_0}\frac{\theta(t,z)}{1-\theta(t,z)}\tilde{N}(dt,dz)\right] \\ &= \Lambda(t-)\left[u(t)dW_{\mathbb{Q}}(t) + \int_{\mathbb{R}_0}\frac{\theta(t,z)}{1-\theta(t,z)}\tilde{N}_{\mathbb{Q}}(dt,dz)\right]. \end{aligned}$$

Denoting $Y(t) := \mathbb{E}_{\mathbb{Q}}[F|\mathcal{F}_t], t \in [0,T]$, by condition (1) in Assumption 2, the Beyes rule (see, e.g., Lemma 4.7 of [9]) shows that $Y(t) = \Lambda(t)\mathbb{E}[Z(T)F|\mathcal{F}_t]$.

From (1) in Assumption 2, Lemma 4.2 implies that $\mathbb{E}[Z(T)F|\mathcal{F}_t] \in \mathbb{D}^{1,2}(\mathbb{R})$ holds. We apply Proposition 4.1 to $\mathbb{E}[Z(T)F|\mathcal{F}_t]$, then by Lemma 4.2, we have

$$\mathbb{E}[Z(T)F|\mathcal{F}_t] = \mathbb{E}[Z(T)F] + \int_0^t \int_{\mathbb{R}} \mathbb{E}[D_{s,z}(Z(T)F)|\mathcal{F}_{s-}]Q(ds,dz).$$

Denoting $V(t) := \mathbb{E}[Z(T)F|\mathcal{F}_t]$, $Y(t) = \Lambda(t)V(t)$ holds. Itô's product rule implies that

$$
\begin{aligned}
&dY(t) = \Lambda(t-)dV(t) + V(t-)d\Lambda(t) + d[\Lambda, V]_t \\
&= \Lambda(t-)[\sigma\mathbb{E}[D_{t,0}(Z(T)F)|\mathcal{F}_{t-}]dW(t) + \int_{\mathbb{R}_0} \mathbb{E}[D_{t,z}(Z(T)F)|\mathcal{F}_{t-}]z\tilde{N}(dt, dz)] \\
&+ V(t-)\Lambda(t-)\left[u(t)dW_{\mathbb{Q}}(t) + \int_{\mathbb{R}_0} \frac{\theta(t,z)}{1-\theta(t,z)}\tilde{N}_{\mathbb{Q}}(dt, dz)\right] \\
&+ \Lambda(t-)[\sigma u(t)\mathbb{E}[D_{t,0}(Z(T)F)|\mathcal{F}_{t-}] + \int_{\mathbb{R}_0} \frac{\theta(t,z)}{1-\theta(t,z)}\mathbb{E}[D_{t,z}(Z(T)F)|\mathcal{F}_{t-}]z\nu(dz)]dt \\
&+ \Lambda(t-)\int_{\mathbb{R}_0} \frac{\theta(t,z)}{1-\theta(t,z)}\mathbb{E}[D_{t,z}(Z(T)F)|\mathcal{F}_{t-}]z\tilde{N}(ds, dz) \\
&= \Lambda(t-)\mathbb{E}[\sigma D_{t,0}(Z(T)F)|\mathcal{F}_{t-}]dW_{\mathbb{Q}}(t) + \Lambda(t-)\mathbb{E}[Z(T)Fu(t)|\mathcal{F}_{t-}]dW_{\mathbb{Q}}(t) \\
&+ \Lambda(t-)\int_{\mathbb{R}_0} \frac{\mathbb{E}[D_{t,z}(Z(T)F)|\mathcal{F}_{t-}]}{1-\theta(t,z)}z\tilde{N}_{\mathbb{Q}}(dt, dz) \\
&+ \Lambda(t-)\int_{\mathbb{R}_0} \mathbb{E}\left[Z(T)F\frac{\theta(t,z)}{1-\theta(t,z)}\middle|\mathcal{F}_{t-}\right]\tilde{N}_{\mathbb{Q}}(dt, dz). \qquad (4.15)
\end{aligned}
$$

Now we shall calculate $D_{t,0}(Z(T)F)$ and $D_{t,z}(Z(T)F)$. As for $D_{t,0}(Z(T)F)$, by (1) in Assumption 2, Corollary 2.10 yields that

$$D_{t,0}(Z(T)F) = FD_{t,0}Z(T) + Z(T)D_{t,0}F. \qquad (4.16)$$

Therefore combining (4.16) with (4.2), we can conclude

$$
\begin{aligned}
&D_{t,0}(Z(T)F) = FD_{t,0}Z(T) + Z(T)D_{t,0}F \\
&= FZ(T)\left[-\sigma^{-1}u(t) - \int_0^T D_{t,0}u(s)dW_{\mathbb{Q}}(s) - \int_0^T\int_{\mathbb{R}_0} \frac{D_{t,0}\theta(s,x)}{1-\theta(s,x)}\tilde{N}_{\mathbb{Q}}(ds, dx)\right] \\
&\quad + Z(T)D_{t,0}F = Z(T)\left[D_{t,0}F - F\left(\sigma^{-1}u(t) + K(t)\right)\right]. \qquad (4.17)
\end{aligned}
$$

Next we calculate $D_{t,z}(Z(T)F)$. From condition (1), Corollary 2.10 implies that

$$D_{t,z}(Z(T)F) = FD_{t,z}Z(T) + Z(T)D_{t,z}F + zD_{t,z}Z(T)\cdot D_{t,z}F. \qquad (4.18)$$

We can describe

$$\tilde{H}(t,z) = \exp(zD_{t,z}\log Z(T) - \log(1-\theta(t,z)))$$

by (4.4). Then from (4.3),

$$D_{t,z}Z(T) = z^{-1}Z(T)[(1-\theta(t,z))\tilde{H}(t,z) - 1]. \qquad (4.19)$$

Therefore, combining (4.18) and (4.19), we obtain

$$
\begin{aligned}
&D_{t,z}(Z(T)F) = z^{-1}Z(T)[(1-\theta(t,z))\tilde{H}(t,z) - 1]F \\
&+ Z(T)D_{t,z}F + Z(T)[(1-\theta(t,z))\tilde{H}(t,z) - 1]D_{t,z}F \\
&= Z(T)\left[z^{-1}((1-\theta(t,z))\tilde{H}(t,z) - 1)F + (1-\theta(t,z))\tilde{H}(t,z)D_{t,z}F\right]. \qquad (4.20)
\end{aligned}
$$

From (4.15), (4.17), (4.20), we arrive at:

$$
\begin{aligned}
&dY(t) = \Lambda(t-)\mathbb{E}\left[Z(T)\left[\sigma D_{t,0}F - F\left(u(t) + \sigma K(t)\right)\right]\middle|\mathcal{F}_{t-}\right] dW_{\mathbb{Q}}(t)\\
&+ \Lambda(t-)\int_{\mathbb{R}_0} \mathbb{E}\left[Z(T)\left[F\left(\tilde{H}(t,z) - \frac{1}{1-\theta(t,z)}\right) + z\tilde{H}(t,z)D_{t,z}F\right]\middle|\mathcal{F}_{t-}\right]\\
&\times \tilde{N}_{\mathbb{Q}}(dt,dz)\\
&+ \Lambda(t-)\mathbb{E}[Z(T)Fu(t)|\mathcal{F}_{t-}]dW_{\mathbb{Q}}(t) + \Lambda(t-)\int_{\mathbb{R}_0} \mathbb{E}\left[Z(T)F\frac{\theta(t,z)}{1-\theta(t,z)}\middle|\mathcal{F}_{t-}\right]\\
&\times \tilde{N}_{\mathbb{Q}}(dt,dz)\\
&= \sigma\Lambda(t-)\mathbb{E}\left[Z(T)\left[D_{t,0}F - FK(t)\right]\middle|\mathcal{F}_{t-}\right] dW_{\mathbb{Q}}(t)\\
&+ \Lambda(t-)\int_{\mathbb{R}_0} \mathbb{E}\left[Z(T)\{F\left(\tilde{H}(t,z) - 1\right) + z\tilde{H}(t,z)D_{t,z}F\}\middle|\mathcal{F}_{t-}\right] \tilde{N}_{\mathbb{Q}}(dt,dz).
\end{aligned}
$$

From (5) and (6) in Assumption 2, we have $K(t) \in L^2(\mathbb{P})$ t-a.e. Hence, by (1) in Assumption 2, $\mathbb{E}_{\mathbb{Q}}[|FK(t)|] = \mathbb{E}[|FK(t)|Z(T)] \leq \mathbb{E}[|K(t)|^2]\mathbb{E}[|FZ(T)|^2] < \infty$. Moreover, from (1) in Assumption 2, we have $D_{t,0}F \in L^2(\mathbb{P})$ t-a.e. and $\mathbb{E}_{\mathbb{Q}}[|D_{t,0}F|] = \mathbb{E}[|D_{t,0}F|Z(T)] \leq \mathbb{E}[|D_{t,0}F|^2]\mathbb{E}[Z(T)^2] < \infty$. Then by (7) in Assumption 2 and $F, D_{t,0}F, FK(t) \in L^1(\mathbb{Q})$ t-a.e., the Beyes rule implies

$$
\begin{aligned}
dY(t) &= \sigma\mathbb{E}_{\mathbb{Q}}\left[D_{t,0}F - FK(t)\middle|\mathcal{F}_{t-}\right] dW_{\mathbb{Q}}(t)\\
&\quad + \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{Q}}[F(\tilde{H}(t,z) - 1) + z\tilde{H}(t,z)D_{t,z}F|\mathcal{F}_{t-}]\tilde{N}_{\mathbb{Q}}(dt,dz). \qquad (4.21)
\end{aligned}
$$

Since $Y(T) = \mathbb{E}_{\mathbb{Q}}[F|\mathcal{F}_T] = F, Y(0) = \mathbb{E}_{\mathbb{Q}}[F|\mathcal{F}_0] = \mathbb{E}_{\mathbb{Q}}[F]$, Integrating equation (4.21) gives

$$
\begin{aligned}
F - \mathbb{E}_{\mathbb{Q}}[F] &= \sigma\int_0^T \mathbb{E}_{\mathbb{Q}}\left[D_{t,0}F - FK(t)\middle|\mathcal{F}_{t-}\right] dW_{\mathbb{Q}}(t)\\
&\quad + \int_0^T\int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{Q}}[F(\tilde{H}(t,z) - 1) + z\tilde{H}(t,z)D_{t,z}F|\mathcal{F}_{t-}]\tilde{N}_{\mathbb{Q}}(dt,dz).
\end{aligned}
$$

The proof is concluded. □

Remark 4.5. (1) If $\sigma \to 0$, $u = 0$ and $\nu \neq 0$, then $zD_{t,z}F = D_{(t,z)}F$, we obtain a COCM for pure jump Lévy processes:

$$
F = \mathbb{E}_{\mathbb{Q}}[F] + \int_0^T\int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{Q}}[F(\tilde{H}(t,z) - 1) + \tilde{H}(t,z)D_{(t,z)}F|\mathcal{F}_{t-}]\tilde{N}_{\mathbb{Q}}(dt,dz),
$$

where

$$\tilde{H}(t,z) = \exp\left(\int_0^T\int_{\mathbb{R}_0}\left[D_{(t,z)}\theta(s,x) + \log(1-\frac{D_{(t,z)}\theta(s,x)}{1-\theta(s,x)})(1-\theta(s,x))\right]\nu(dx)ds\right.$$
$$\left.+\int_0^T\int_{\mathbb{R}_0}\log\left(1-\frac{D_{(t,z)}\theta(s,x)}{1-\theta(s,x)}\right)\tilde{N}_{\mathbb{Q}}(ds,dx)\right)$$

and $D_{(t,z)}F$ is a Malliavin difference operator (see Difinition 4.6).
(2) If $\sigma \neq 0$, $\theta = 0$, and $\nu = 0$, then $D_{t,0}F = \sigma^{-1}D_tF$ and we can derive a COCM for Brownian motions: $F = \mathbb{E}_{\mathbb{Q}}[F] + \int_0^T \mathbb{E}_{\mathbb{Q}}[D_tF - F\int_0^T D_tu(s)dW_{\mathbb{Q}}(s)|\mathcal{F}_t]dW_{\mathbb{Q}}(t)$, where D_tF is a classical Malliavin derivative (see Definition 4.6).

Definition 4.6. The *classical Malliavin derivative* is defined by

$$D_tF = \sum_{n=1}^{\infty} nI_{n-1}^W(h_n^W(t,\cdot)),\ \lambda\text{−a.e. } t\in[0,T], \mathbb{P}\text{ −a.s.}$$

for $F \in \mathbb{D}_W^{1,2} \subset L^2(\mathbb{P}) = \{F = \sum_{n=0}^{\infty} I_n^W(h_n^W) : \sum_{n=1}^{\infty} nn!\|h_n^W\|^2_{L^2(\lambda^n)} < \infty\}$, where $I_n(h_n^W) = \int_{([0,T])^n} h_n^W((t_1),\cdots,(t_n))dW_{t_1}\cdots dW_{t_n}, h_n^W \in L^2(\lambda^n)$.

For $F \in \mathbb{D}_J^{1,2} \subset L^2(\mathbb{P}) = \{F = \sum_{n=0}^{\infty} I_n^J(h_n^J) : \sum_{n=1}^{\infty} nn!\|h_n^J\|^2_{L^2((\lambda\times\nu)^n)} < \infty\}$, where $I_n^J(h_n^J) := \int_{([0,T]\times\mathbb{R}_0)^n} h_n^J((t_1,z_1),\cdots,(t_n,z_n))\tilde{N}(t_1,z_1)\cdots\tilde{N}(t_n,z_n), h_n^J \in L^2((\lambda\times\nu)^n)$, the *Malliavin difference operator* for pure jump Lévy functionals is defined by

$$D_{(t,z)}F = \sum_{n=1}^{\infty} nI_{n-1}^J(h_n^J((t,z),\cdot)),\ \lambda\times\nu\text{−a.e. } t\in[0,T]\times\mathbb{R}_0, \mathbb{P}-\text{a.s.}$$

Remark 4.7. To see different points, we review a result of [21]. Let us denote $\mathbb{P}^W$, P^η the Gaussian white noise probability measure on $(\Omega_W, \mathcal{F}_T^W)$ and the pure jump Lévy white noise probability measure on $(\Omega_\eta, \mathcal{F}_T^\eta)$, respectively, where the sample space is the Schwartz space $\mathcal{S}'(\mathbb{R})$, $\mathcal{F}_t^W$ and $\mathcal{F}_t^\eta$ be the augmented filtration generated by the Wiener process and pure jump Lévy process, respectively. Let $\Omega = \mathcal{S}'(\mathbb{R})\times\mathcal{S}'(\mathbb{R})$, $\mathcal{F}_T^W\otimes\mathcal{F}_T^\eta$ and $\mathbb{P} = \mathbb{P}^W\times\mathbb{P}^\eta$. The orthogonal basis for $L^2(\mathbb{P})$ is the family of $\mathbb{K}_\alpha$ with $\|\mathbb{K}\|_{L^2(\mathbb{P})} = \alpha! := \alpha^{(1)}!\alpha^{(2)}!$ and $\mathbb{K}_\alpha := H_{\alpha^{(1)}}(\omega')\cdot K_{\alpha^{(2)}}(\omega'')$, where $(\omega',\omega'')\in\Omega$, $\alpha = (\alpha^{(1)},\alpha^{(2)})$ and $\{\alpha^{(i)}\}_{i=1,2}\in\mathbb{I}$ are multi-indexes defined in section 2 of [21], H_α and K_α are the orthogonal basis for $L^2(\mathbb{P}^W)$ and $L^2(\mathbb{P}^\eta)$ respectively. Moreover, for all $F\in L^2(\mathbb{P})$, there exist unique constants c_α such that $F(\omega) = \sum_{\alpha\in\mathbb{I}^2} c_\alpha\mathbb{K}(\omega)$. For $F\in L^2(\mathbb{P})$ with some condition, Hida Malliavin derivatives are defined as $D_tF = \sum_{\alpha\in\mathbb{I}^2}\sum_{i\geq1} c_\alpha\alpha_i^{(1)}\mathbb{K}_{\alpha^{(1)}-\epsilon^i}e_i(t)$, and $D_{t,x}F = \sum_{\alpha\in\mathbb{I}^2}\sum_{i\geq1} c_\alpha\alpha_{k(i,j)}^{(2)}e_i(t)p_j(x)$, where $\epsilon^k = (0,\cdots,0,1,0,\cdots,0)$ with 1 in the k th position, $k(i,j) = j + \frac{(i+j-2)(i+j-1)}{2}$, $\{e_i(t)\}_{i\geq0}\subset\mathcal{S}(\mathbb{R})$ are Hermite functions on $\mathbb{R}$ and $p_j(x) = \|l_{j-1}\|^{-1}_{L^2(x^2\nu(dx))}xl_{j-1}(x)$, where $\{l_0,l_1,l_2,\cdots\}$ with $l_0 = 1$ is the orthogonalization of $\{1,x,x^2,\cdots\}$ with respect to inner product of $L^2(x^2\nu(dx))$.

In this setting, Okur derived the following equation:

$$F = \mathbb{E}_{\mathbb{Q}}[F] + \int_0^T \mathbb{E}_{\mathbb{Q}}\left[D_t F - F \int_t^T D_t u(s) dW_{\mathbb{Q}}(s) \middle| \mathcal{F}_t\right] dW_{\mathbb{Q}}(t)$$
$$+ \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{Q}}[F(\tilde{H} - 1) + \tilde{H} D_{t,x} F | \mathcal{F}_t] \tilde{N}_{\mathbb{Q}}(dt, dx),$$

for any $F \in L^2(\mathcal{F}_T; \mathbb{P})$, where

$$\tilde{H} = \exp\left(\int_t^T \int_{\mathbb{R}_0} \left[D_{t,x}\theta(s,z) + \log\left(1 - \frac{D_{t,x}\theta(s,z)}{1-\theta(s,z)}\right)(1-\theta(s,z))\right] \nu(dz) ds\right.$$
$$\left. + \int_t^T \int_{\mathbb{R}_0} \log\left(1 - \frac{D_{t,x}\theta(s,z)}{1-\theta(s,z)}\right) \tilde{N}_{\mathbb{Q}}(ds, dz)\right).$$

Of course, to show this equation, we need more conditions, see [21].

Corollary 4.8. *Assume in addition to all assumptions of Theorem 4.4, that u and θ are deterministic functions, then we have*

$$F = \mathbb{E}_{\mathbb{Q}}[F] + \sigma \int_0^T \mathbb{E}_{\mathbb{Q}}[D_{t,0} F | \mathcal{F}_{t-}] dW_{\mathbb{Q}}(t) + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{Q}}[D_{t,z} F | \mathcal{F}_{t-}] z \tilde{N}_{\mathbb{Q}}(dt, dz).$$

Proof. If u and θ are deterministic functions, then we have $D_{t,z}u(s) = 0 = D_{t,z}\theta(s,x)$ and $\tilde{H}(t,z) = 1$. Therefore, thanks to Theorem 4.4, we can get the claimed equation. □

Remark 4.9. If $F \in \mathbb{D}^{1,2}(\mathbb{R})$, $u \equiv 0$ and $\theta \equiv 0$, then we can see that Assumption 1 and Assumption 2 hold and we obtain equation (4.1).

Corollary 4.10. *We assume that $\theta(t,z) \in [-1,1)$ for $(t,z) \in [0,T] \times \mathbb{R}_0$ is a nonrandom function and denote $\nu_{\mathbb{Q}}(dz,dt) = (1+\theta(t,z))\nu(dz)dt$.*
(1) Under Assumption 1 and Assumption 2, we have

$$\mathbb{E}_{\mathbb{Q}}[(F - \mathbb{E}_{\mathbb{Q}}[F])^2] \leq \sigma^2 \int_0^T \mathbb{E}_{\mathbb{Q}}\left[|D_{t,0}F - FK(t)|^2\right] dt$$
$$+ \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{Q}}[|F(\tilde{H}(t,z) - 1) + z\tilde{H}(t,z) D_{t,z}F|^2] \nu_{\mathbb{Q}}(dz, dt).$$

(2) Let $F \in \mathbb{D}^{1,2}(\mathbb{R})$ with $F > \eta$ for some $\eta > 0$ and we assume that $\mathcal{F}_{t-} = \mathcal{F}_t$ for all $t \geq 0$. Moreover, we denote $M(t) = \mathbb{E}_{\mathbb{Q}}[F|\mathcal{F}_t]$ and we assume that $M(t) > 0$ and $M(t) + \mathbb{E}_{\mathbb{Q}}[F(\tilde{H}(t,z) - 1) + z\tilde{H}(t,z)D_{t,z}F|\mathcal{F}_t] > 0$. Then under Assumption 1 and Assumption 2, we have

$$\mathbb{E}_{\mathbb{Q}}[F \log F] - \mathbb{E}_{\mathbb{Q}}[F] \log \mathbb{E}_{\mathbb{Q}}[F] \leq \frac{1}{2}\sigma^2 \int_0^T \mathbb{E}_{\mathbb{Q}}\left[M(t)^{-1}|D_{t,0}F - FK(t)|^2\right] dt$$
$$+ \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{Q}}[M(t)^{-1}|F(\tilde{H}(t,z) - 1) + z\tilde{H}(t,z)D_{t,z}F|^2] \nu_{\mathbb{Q}}(dz, dt).$$

Proof. (1) Theorem 4.4 implies that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[(F-\mathbb{E}_{\mathbb{Q}}[F])^2] &= \mathbb{E}_{\mathbb{Q}}\left[\left(\sigma\int_0^T \mathbb{E}_{\mathbb{Q}}\left[D_{t,0}F - FK(t)\middle|\mathcal{F}_{t-}\right]dW_{\mathbb{Q}}(t)\right.\right.\\
&\left.\left.+\int_0^T\int_{\mathbb{R}_0}\mathbb{E}_{\mathbb{Q}}[F(\tilde{H}(t,z)-1)+z\tilde{H}(t,z)D_{t,z}F|\mathcal{F}_{t-}]\tilde{N}_{\mathbb{Q}}(dt,dz)\right)^2\right]\\
&=\sigma^2\int_0^T\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[D_{t,0}F-FK(t)\middle|\mathcal{F}_{t-}\right]^2\right]dt\\
&+\int_0^T\int_{\mathbb{R}_0}\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[F(\tilde{H}(t,z)-1)+z\tilde{H}(t,z)D_{t,z}F|\mathcal{F}_{t-}]^2]\nu_{\mathbb{Q}}(dz,dt)\\
&\le\sigma^2\int_0^T\mathbb{E}_{\mathbb{Q}}\left[|D_{t,0}F-FK(t)|^2\right]dt\\
&+\int_0^T\int_{\mathbb{R}_0}\mathbb{E}_{\mathbb{Q}}[|F(\tilde{H}(t,z)-1)+z\tilde{H}(t,z)D_{t,z}F|^2]\nu_{\mathbb{Q}}(dz,dt),
\end{aligned}$$

where we use Jensen's inequality and Itô isometry.
(2) First we denote $\zeta(t)=\mathbb{E}_{\mathbb{Q}}[D_{t,0}F-FK(t)|\mathcal{F}_t]$ and $\xi(t,z)=\mathbb{E}_{\mathbb{Q}}[F(\tilde{H}(t,z)-1)+z\tilde{H}(t,z)D_{t,z}F|\mathcal{F}_t]$. The Itô formula (see, e.g., Theorem 9.4 of [9]) implies that

$$\begin{aligned}
&F\log F-\mathbb{E}_{\mathbb{Q}}[F]\log\mathbb{E}_{\mathbb{Q}}[F]\\
&=\sigma\int_0^T(\log M(t)+1)\zeta(t)dW_{\mathbb{Q}}(t)+\frac{1}{2}\sigma^2\int_0^T M(t)^{-1}\zeta(t)^2dt\\
&+\int_0^T\int_{\mathbb{R}_0}\{(M(t)+\xi(t,z))(\log(M(t)+\xi(t,z))-M(t)\log M(t)\\
&-(\log M(t)+1)\xi(t,z)\}\nu_{\mathbb{Q}}(dz,dt)\\
&+\int_0^T\int_{\mathbb{R}_0}\{(M(t)+\xi(t,z))(\log(M(t)+\xi(t,z))-M(t)\log M(t)\}\tilde{N}_{\mathbb{Q}}(dt,dz).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}[F\log F]-\mathbb{E}_{\mathbb{Q}}[F]\log\mathbb{E}_{\mathbb{Q}}[F]\\
&=\frac{1}{2}\sigma^2\mathbb{E}_{\mathbb{Q}}[\int_0^T M(t)^{-1}\zeta(t)^2dt]+\mathbb{E}_{\mathbb{Q}}[\int_0^T\int_{\mathbb{R}_0}\{(M(t)+\xi(t,z))(\log(M(t)+\xi(t,z))\\
&-M(t)\log M(t)-(\log M(t)+1)\xi(t,z)\}\nu_{\mathbb{Q}}(dz,dt)]\\
&\le\frac{1}{2}\sigma^2\mathbb{E}_{\mathbb{Q}}[\int_0^T M(t)^{-1}\zeta(t)^2dt]+\mathbb{E}_{\mathbb{Q}}[\int_0^T\int_{\mathbb{R}_0}M(t)^{-1}\xi(t,z)^2\nu_{\mathbb{Q}}(dz,dt)]\\
&\le\frac{1}{2}\sigma^2\int_0^T\mathbb{E}_{\mathbb{Q}}\left[M(t)^{-1}|D_{t,0}F-FK(t)|^2\right]dt\\
&+\int_0^T\int_{\mathbb{R}_0}\mathbb{E}_{\mathbb{Q}}[M(t)^{-1}|F(\tilde{H}(t,z)-1)+z\tilde{H}(t,z)D_{t,z}F|^2]\nu_{\mathbb{Q}}(dz,dt),
\end{aligned}$$

where we use Jensen's inequality and the following inequality:

$$(x+y)\log(x+y) - x\log x - y(1+\log x) \le \frac{y^2}{x^2}, x>0, x+y>0.$$

□

Remark 4.11. (1) Assume in addition to all assumptions of Corollary 4.10, that u and θ are deterministic functions, then we obtain a Poincare's inequality for Lévy functionals on $\mathbb{Q}$:

$$\mathbb{E}_{\mathbb{Q}}[(F-\mathbb{E}_{\mathbb{Q}}[F])^2] \le \sigma^2 \int_0^T \mathbb{E}_{\mathbb{Q}}\left[|D_{t,0}F|^2\right] dt + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{Q}}[|zD_{t,z}F|^2]\nu_{\mathbb{Q}}(dz,dt).$$

(2) If $F \in \mathbb{D}^{1,2}(\mathbb{R})$, $u \equiv 0$ and $\theta \equiv 0$, then we can see that Assumption 1 and Assumption 2 hold and we obtain a Poincare's inequality for Lévy functionals: $\mathbb{E}[(F-\mathbb{E}[F])^2] \le \int_0^T \int_{\mathbb{R}} \mathbb{E}[|D_{t,z}F|^2]q(dt,dz)$.
(3) Assume in addition to all assumptions of Corollary 4.10, that u and θ are deterministic functions, then we obtain a logarithmic Sobolev inequality for Lévy functionals on $\mathbb{Q}$

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[F\log F] - \mathbb{E}_{\mathbb{Q}}[F]\log\mathbb{E}_{\mathbb{Q}}[F] \le{}& \frac{1}{2}\sigma^2\int_0^T \mathbb{E}_{\mathbb{Q}}\left[M(t)^{-1}|D_{t,0}F|^2\right] dt \\ &+ \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{Q}}[M(t)^{-1}|zD_{t,z}F|^2]\nu_{\mathbb{Q}}(dz,dt).\end{aligned}$$

(4) Assume in addition to all assumptions of Corollary 4.10, that $u \equiv 0$ and $\theta \equiv 0$, then we obtain a logarithmic Sobolev inequality for Lévy functionals:

$$\begin{aligned}\mathbb{E}[F\log F] - \mathbb{E}[F]\log\mathbb{E}[F] \le{}& \frac{1}{2}\sigma^2\int_0^T \mathbb{E}\left[M(t)^{-1}|D_{t,0}F|^2\right] dt \\ &+ \int_0^T \int_{\mathbb{R}_0} [M(t)^{-1}|zD_{t,z}F|^2]\nu(dz,dt).\end{aligned}$$

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