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UNBOUNDED POSITIVE SOLUTIONS OF NONLINEAR PARABOLIC ITÔ EQUATIONS

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Abstract. The paper is concerned with the problem of non-existence of global solutions for a class of semi-linear stochastic parabolic equations of Itô type. Under some sufficient conditions on the initial data, the nonlinear term and the multiplicative white noise, it is proven that, in a bounded domain $\mathcal{D} \subset \mathbb{R}^d$, there exist positive solutions whose $\mathcal{L}^p-$norms will blow up in finite time, while, if $\mathcal{D} = \mathbb{R}^d$, the previous result holds in any compact subset of $\mathbb{R}^d$. One example is given to illustrate an application of the theorems.

1. Introduction

Consider the initial-boundary problem for a nonlinear parabolic equation in domain $\mathcal{D} \subset \mathbb{R}^d$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla^2 u + f(u), \quad t > 0, \\ u(x,0) = g(x), \quad x \in \mathcal{D}, \\ u(x,t) = 0, \quad x \in \partial\mathcal{D}, \end{cases} \tag{1.1}$$

where ∇^2 is the Laplacian operator, $\partial\mathcal{D}$ denotes the boundary of $\mathcal{D}$, and the functions f and g are given functions such that the problem (1.1) has a unique local solution. In 1963 it was first shown by S. Kaplan [7] that, for a certain class of nonlinear functions $f(u)$, the solution of equation (1.1) becomes infinite or explodes at a finite time, provided that the initial state $g(x)$ satisfies appropriate conditions. His result was later extended by Fujita [4] and many others. Since then it has become known that solutions to more general nonlinear parabolic equations may develop singularities in finite time [5], where an extensive references can be found. Physically this phenomenon is manifested as the explosion in combustion, reaction diffusion and branching diffusion problems. It is therefore of interest to examine the effect of a random perturbation to equation (1.1) on the existence of an explosive solution. This consideration has led us to investigate the question of

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nonexistence of a global solution to the following type of parabolic Itô equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla^2 u + f(u) + \sigma(u)\partial_t W(x,t), \quad t > 0, \\ u(x,0) = g(x), \quad x \in \mathcal{D}, \\ u(x,t) = 0, \quad x \in \partial\mathcal{D}, \end{cases} \tag{1.2}$$

with a multiplicative noise, where σ is a given function and $W(x,t)$ is a Wiener random field. To study this type of problems, it is necessary to employ some analytical and probabilistic tools from the theory of stochastic partial differential equations (SPDEs) [1]. In contrast, for stochastic ordinary differential equations, the general results on the explosion and non-explosion of solutions have been well established, (see, e.g., [6]). However, so far, very little is known about such results for SPDEs due to some difficulty in infinite-dimensional stochastic analysis. Therefore one can only hope to resolve such questions for some special cases. Recently we studied the existence of explosive solutions for a class of nonlinear stochastic wave equations. Based on a stochastic energy method, we were able to obtain some sufficient conditions for the blow-up of the second moments of solutions in the L^2-norm [2]. In this paper we shall consider the positive (nonnegative) solutions of nonlinear parabolic Itô equations such as (1.2). By extending Kaplan's approach to the deterministic case [7], we have obtained a certain set of sufficient conditions for such positive solutions to explode in the sense of $\mathcal{L}^p-$norm defined by (2.7) for any $p \geq 1$.

The paper is organized as follows. We shall first recall some basic results for nonlinear stochastic parabolic equations in Section 2. Then, in Section 3, we shall prove the positivity theorem (Theorem 3.3) which states that, under some sufficient conditions on the initial data and the noise term, the solutions to a class of nonlinear stochastic parabolic equations are positive a.s. (almost surely). This theorem was first proved in the book (Theorem 5.3, [1]) for the case $\mathcal{D} = \mathbb{R}^d$. Here this theorem will be extended to the case where $\mathcal{D}$ is a bounded domain and the Laplacian operator is replaced by a self-adjoint uniformly elliptic operator. Section 4 contains the main results of the paper as presented in Theorems 4.1 and 4.2. Under some sufficient conditions, Theorem 4.1 shows the existence of positive solutions in a bounded domain that will explode at a finite time in $\mathcal{L}^p-$norm, while, in the case $\mathcal{D} = \mathbb{R}^d$, Theorem 4.2 affirms a similar result in any compact subset of $\mathbb{R}^d$. Finally in Section 5, we apply the theorems to a special problem to obtain some explicit conditions for explosive solutions.

2. Preliminaries

Let $\mathcal{D}$ be a domain in $\mathbb{R}^d$, which has a smooth boundary $\partial\mathcal{D}$ if it is bounded. We set $H = L^2(\mathcal{D})$ with the inner product and norm are denoted by $(.,.)$ and $\|.\|$ respectively. Let $H^1 = H^1(\mathcal{D})$ be the L^2-Sobolev space of first order and denote by H_0^1 the closure in H^1 of the space of C^1- functions with compact support in $\mathcal{D}$.

Let $W(x,t)$, for $x \in \mathbb{R}^d$, $t \geq 0$, be a continuous Wiener random field defined in a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathcal{F}_t$ (p.38, [1]). It has mean

$E\,W(x,t)=0$ and covariance function $q(x,y)$ defined by

$$E\,W(x,t)W(y,s)=(t\wedge s)q(x,y),\quad x,y\in\mathbb{R}^d,$$

where $(t\wedge s)=\min(t,s)$ for $0\le t,s\le T$.

Consider the initial-boundary value problem for the parabolic Itô equation

$$\begin{cases} \dfrac{\partial u}{\partial t}=Au+f(u,x,t)+\sigma(u,\nabla u,x,t)\partial_t W(x,t), \\ u(x,0)=\,g(x),\quad x\,\in\mathcal{D}, \\ u(x,t)|_{\partial\mathcal{D}}=\,h(x),\quad t\,\in(0,T), \end{cases} \tag{2.1}$$

where $A=\sum_{i,j=1}^{d}\dfrac{\partial}{\partial x_i}[a_{ij}(x)\dfrac{\partial}{\partial x_j}]$ is a symmetric, uniformly elliptic operator with smooth coefficients (say, in $C^3(\overline{\mathcal{D}})$), that is, there exists a constant $a_0>0$ such that

$$b(x,\xi):=\sum_{i,j=1}^{d}a_{ij}(x)\xi_i\xi_j\ge a_0|\xi|^2, \tag{2.2}$$

for all $x\in\overline{\mathcal{D}}$ and $\xi=(\xi_1,\cdots,\xi_d)\in\mathbb{R}^d$. Certain conditions will be imposed on the functions f, σ, g, h later, and ∇u denotes the gradient of u. It is well known that, for the existence and uniqueness question, we can set $h\equiv 0$ in the above equation and consider the corresponding problem with a homogeneous boundary condition.

Now, to regard the equation (2.1) with a homogeneous boundary condition as an Itô equation in the Hilbert space H, we set $u_t=u(\cdot,t)$, $F_t(u)=f(u,\cdot,t)$, $\Sigma_t(u)=\sigma(u,\nabla u,\cdot,t)$ and so on, and rewrite it as

$$\begin{cases} du_t=[A\,u_t+F_t(u_t)]\,dt+\Sigma_t(u_t)\,dW_t,\quad 0<t<T, \\ u_0=g, \end{cases} \tag{2.3}$$

where A is now regarded as a linear operator from H^1 into H^{-1} with domain $H_0^1\cap H^2$, $F_t:H\to H$ is continuous and, for $v\in H_1$, $\Sigma_t(v):C(\overline{\mathcal{D}})\to H$ can be defined as a multiplication operator. In this paper we assume that the covariance function $q(x,y)$ is bounded, continuous and there is $r_0>0$ such that

$$\sup_{x,y\in\mathcal{D}}|q(x,y)|\le r_0,\quad \text{and}\ \int_{\mathbb{R}^d}q(x,x)\,dx<\infty. \tag{2.4}$$

Then we can rewrite equation (2.3) as

$$u_t=g+\int_0^t[A\,u_s+F_s(u_s)]\,ds+\int_0^t\Sigma_s(u_s)\,dW_s, \tag{2.5}$$

where the stochastic integral is well defined (see Theorem 2.4, [1]).

Under the usual conditions, such as the stochastic coercivity, Lipschitz continuity and monotonicity conditions, the equation (2.5) is known to have a unique global strong solution $u \in C\big([0,T];H\big) \cap L^2\big((0,T);H_0^1\big)$ for any $T > 0$ (Theorem 7.4, [1]). Moreover, for a continuous C^2- functional Φ on H, the Itô formula holds

$$\begin{cases} \Phi(u_t) = \Phi(u_0) + \displaystyle\int_0^t [\langle A\,u_s, \Phi'(u_s)\rangle + \big(F_s(u_s), \Phi'(u_s)\big]\,ds \\ + \displaystyle\int_0^t \big(\Phi'(u_s), \Sigma_s(u_s)\,dW_s\big) + \frac{1}{2}\int_0^t Tr\,[\Phi''(u_s)\Sigma_s^\star(u_s)Q\Sigma_s(u_s)]\,ds, \end{cases} \tag{2.6}$$

where Φ', Φ'' denote the first and second Fréchet derivatives of Φ, Q is the covariance operator with kernel q, the star means the conjugate and Tr is the trace of an operator.

On the other hand, if the nonlinear terms are only locally Lipschitz continuous and the monotonicity condition is dropped, one can only assert the existence of a unique local solution. In this case, by the conventional definition, the solution u_t in H is said to explode or blow up if the probability $P_r\{\mathbf{e} < \infty\} = 1$, where $\mathbf{e}$ is the explosion time defined by $\mathbf{e} = \inf\{t > 0 : \|u_t\| = \infty\}$ [6]. In this paper we shall introduce an alternative definition which is closer to the deterministic case. For any $p \geq 1$, we let $\mathcal{L}^p = \mathcal{L}^p(\mathcal{D})$ denote the space of random functions v on $\mathcal{D}$ with norm $\|\!|v|\!\|_p$ such that

$$\|\!|v|\!\|_p := \{E\int_{\mathcal{D}} |v(x)|^p\,dx\}^{1/p} < \infty. \tag{2.7}$$

Then we say that the solution u_t exploses in $\mathcal{L}^p-$norm if there exists a constant $T_p > 0$ such that the left limit

$$\lim_{t\to T_p^-} \|\!|u_t|\!\|_p = \infty,$$

where T_p is called an explosion time. To proceed we shall first take up the issue of positive solutions in the next section.

3. Positive Solutions

To consider positive (nonnegative) solutions, we suppose that, under appropriate conditions as mentioned in the previous section, the parabolic Itô equation (2.1) has a unique strong solution $u(\cdot,t)$ for $t \leq T$. In addition, assume that the following conditions hold:

(P1) There exists a constant $\delta \geq 0$ such that

$$\frac{1}{2}\,q(x,x)\sigma^2(r,\xi,x,t) - \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \leq \delta\,r^2,$$

for all $r \in \mathbb{R}$, $x \in \overline{\mathcal{D}}$, $\xi \in \mathbb{R}^d$ and $t \in [0,T]$.

(P2) The function $f(r,x,t)$ is continuous on $\mathbb{R}\times\overline{\mathcal{D}}\times[0,T]$ such that $f(r,x,t) \geq 0$ for $r \leq 0$ and $x \in \overline{\mathcal{D}}$, $t \in [0,T]$.

(P3) The initial and boundary data $g(x)$ on $\overline{\mathcal{D}}$ and $h(x)$ on $\partial\mathcal{D}$ are both positive and continuous.

Then it can be shown that the solution of equation (2.1) is positive. Notice that, due to the lack of a maximum principle for parabolic equations as in the deterministic case [8], the proof will be quite different. In fact we shall make use of a regularization technique and the Itô formula (2.6). To facilitate the proof of the positivity theorem, we shall first present two technical lemmas in this regard.

Let $\eta(r) = r^-$ denote the negative part of r for $r \in \mathbb{R}$, or $\eta(r) = 0$, if $r \geq 0$ and $\eta(r) = -r$, if $r < 0$. Set $k(r) = \eta^2(r)$ so that $k(r) = 0$ for $r \geq 0$ and $k(r) = r^2$ for $r < 0$. For $\epsilon > 0$, let $k_\epsilon(r)$ be a $\mathbf{C}^2-$regularization of $k(r)$ defined by

$$k_\epsilon(r) = \begin{cases} r^2 - \epsilon^2/6, & r < -\epsilon, \\ -\frac{r^3}{\epsilon}(\frac{r}{2\epsilon} + \frac{4}{3}), & -\epsilon \leq r < 0, \\ 0, & r \geq 0. \end{cases} \tag{3.1}$$

Then it is easy to check that $k_\epsilon(r)$ has the following properties.

Lemma 3.1. *The first two derivatives $k_\epsilon^{'} k_\epsilon^{''}$ of k_ϵ are continuous and satisfy the conditions: $k_\epsilon'(r) = 0$ for $r \geq 0$; $k_\epsilon'(r) \leq 0$ and $k_\epsilon''(r) \geq 0$ for any $r \in \mathbf{R}$. Moreover, as $\epsilon \to 0$, we have*

$$k_\epsilon(r) \to k(r), \quad k_\epsilon^{'}(r) \to -2\,\eta(r) \quad \text{and} \quad k_\epsilon^{''}(r) \to 2\theta(r), \tag{3.2}$$

where $\theta(r) = 0$ for $r \geq 0$, $\theta(r) = 1$ for $r < 0$, and the convergence is uniform for $r \in \mathbb{R}$. □

The next lemma is a consequence of the Itô formula given by equation (2.6).

Lemma 3.2. *Let $u_t = u(\cdot, t)$ denote the solution of the parabolic Itô equation (2.1). Define*

$$\Phi_\epsilon(u_t) = \big(1, k_\epsilon(u_t)\big) = \int_{\mathcal{D}} k_\epsilon\big(u(x,t)\big)dx. \tag{3.3}$$

Then the following formula holds

$$\begin{cases} \Phi_\epsilon(u_t) = \Phi_\epsilon(g) - \int_0^t \int_{\mathcal{D}} k_\epsilon''\big(u(x,s)\big)\, b\big(x, \nabla u(x,s)\big)\, dx\, ds \\ + \int_0^t \int_{\mathcal{D}} k_\epsilon'\big(u(x,s)\big) f\big(u, x, s\big)\, dx\, ds \\ + \int_0^t \int_{\partial\mathcal{D}} k_\epsilon'\big(h(x)\big)\, \frac{\partial}{\partial\nu} u(x,s)\, dS\, ds \\ + \frac{1}{2}\int_0^t \int_{\mathcal{D}} k_\epsilon''\big(u(x,s)\big) q(x,x)\, \sigma^2\big(u, \nabla u, x, s\big)\, dx\, ds \\ + \int_0^t \int_{\mathcal{D}} k_\epsilon'\big(u(x,s)\big)\sigma\big(u, \nabla u, x, s\big)\, dW(x,s) dx, \end{cases} \tag{3.4}$$

where $b(x,\xi)$ is defined by (2.2), dS is the element of surface area on $\partial\mathcal{D}$, and $\frac{\partial}{\partial\nu}$ denotes the differentiation with respect to the conormal vector field $\nu = (\nu_1, \cdots, \nu_d)$

with

$$\nu_i(x) := \sum_{j=1}^{d} a_{ij}(x) n_j, \tag{3.5}$$

and $\mathbf{n} = (n_1, \cdots, n_d)$ *being the unit outward normal vector to the boundary* $\partial\mathcal{D}$.

Proof. By applying the Itô formula (2.6) to $\Phi_\epsilon(u_t)$ given by (3.3), we get

$$\left\{\begin{array}{l} \Phi_\epsilon(u_t) = \Phi_\epsilon(g) + \int_0^t \int_{\mathcal{D}} k'_\epsilon\big(u(x,s)\big) Au(x,s)\,dx\,ds \\ + \int_0^t \int_{\mathcal{D}} k'_\epsilon\big(u(x,s)\big) f\big(u,x,s\big)\,dx\,ds \\ + \frac{1}{2}\int_0^t \int_{\mathcal{D}} k''_\epsilon\big(u(x,s)\big) q(x,x)\,\sigma^2\big(u,\nabla u,x,s\big)\,dx\,ds \\ + \int_0^t \int_{\mathcal{D}} k'_\epsilon\big(u(x,s)\big)\sigma\big(u,\nabla u,x,s\big)\,dW(x,s)dx. \end{array}\right. \tag{3.6}$$

By means of an integration by parts and the Stokes theorem [8], we can obtain the following

$$\left\{\begin{array}{l} \int_0^t \int_{\mathcal{D}} k'_\epsilon\big(u(x,s)\big) Au(x,s)\,dx\,ds = \int_0^t \int_{\partial\mathcal{D}} k'_\epsilon\big(u(x,s)\big)\,\frac{\partial}{\partial\nu} u(x,s)\,dS\,ds \\ \qquad - \int_0^t \int_{\mathcal{D}} k''_\epsilon\big(u(x,s)\big)\,b\big(x,\nabla u(x,s)\big)\,dx\,ds. \end{array}\right. \tag{3.7}$$

Upon substituting (3.7) into (3.6) and noting the boundary condition $u|_{\partial\mathcal{D}} = h$, the equation (3.4) follows. □

With the aid of Lemmas 3.1 and 3.2, we can prove the following positivity theorem.

Theorem 3.3. *Suppose that the conditions (P1), (P2) and (P3) hold true. Then the solution of the initial-boundary problem for the parabolic Itô equation (2.1) remains positive so that* $u(x,t) \geq 0$, *a.s. for almost every* $x \in \mathcal{D}$, $\forall\, t \in [0,T]$.

Proof. In view of Lemma 3.2, by taking an expectation over equation (3.4), we get

$$\left\{\begin{array}{l} E\,\Phi_\epsilon(u_t) = \Phi_\epsilon(g) + E\int_0^t \int_{\mathcal{D}} \{\, k''_\epsilon\big(u(x,s)\big)\,[\frac{1}{2}\,q(x,x)\,\sigma^2\big(u,\nabla u,x,s\big) \\ \qquad -b\big(x,\nabla u(x,s)\big)] + k'_\epsilon\big(u(x,s)\big) f\big(u,x,s\big)\,\}\,dx\,ds \\ \qquad + E\int_0^t \int_{\partial\mathcal{D}} k'_\epsilon\big(h(x)\big)\,\frac{\partial}{\partial\nu} u(x,s)\,dS\,ds. \end{array}\right. \tag{3.8}$$

By making use of condition (P1) and Lemma 3.1, we can deduce from equation (3.8) that

$$\begin{cases} E\,\Phi_\epsilon(u_t) \leq \Phi_\epsilon(g) + \delta\, E \displaystyle\int_0^t \int_{\mathcal{D}} k_\epsilon''\big(u(x,s)\big)\, |u(x,s)|^2\, dx\, ds \\ + E \displaystyle\int_0^t \int_{\mathcal{D}} k_\epsilon'\big(u(x,s)\big) f\big(u,x,s\big)\, dx\, ds \\ + E \displaystyle\int_0^t \int_{\partial\mathcal{D}} k_\epsilon'\big(h(x)\big)\, \frac{\partial}{\partial\nu} u(x,s)\, dS\, ds. \end{cases} \tag{3.9}$$

Note that $\lim_{\epsilon\to 0} E\,\Phi_\epsilon(u_t) = E\,\|\eta\big(u_t\big)\|^2$. By taking the limits termwise as $\epsilon \to 0$ and making use of (3.2) in Lemma 3.1, the equation (3.9) yields

$$\begin{cases} E \displaystyle\int_{\mathcal{D}} |\eta\big(u(x,t)\big)|^2\, dx \leq \int_{\mathcal{D}} |\eta\big(g(x)\big)|^2\, dx \\ + 2\,\delta\, E \displaystyle\int_0^t \int_{\mathcal{D}} \theta\big(u(x,s)\big)\, |u(x,s)|^2\, dx\, ds \\ - 2\, E \displaystyle\int_0^t \int_{\mathcal{D}} \eta\big(u(x,s)\big) f\big(u,x,s\big)\, dx\, ds \\ - 2\, E \displaystyle\int_0^t \int_{\partial\mathcal{D}} \eta\big(h(x)\big)\, \frac{\partial}{\partial\nu} u(x,s)\, dS\, ds. \end{cases} \tag{3.10}$$

By definition of η and conditions (P2) and (P3), we have $\eta(g) = \eta(h) = 0$, $\theta(u)u^2 = \eta^2(u)$ and $\eta(u)f(u,\xi,s) \geq 0$ so that equation (3.10) can be reduced simply to

$$E\,\|\eta\big(u_t\big)\|^2 \leq 2\delta \int_0^t E\,\|\eta\big(u_s\big)\|^2\, ds,$$

which, by means of the Gronwall inequality, implies that

$$E\,\|\eta\big(u_t\big)\|^2 = E \int_{\mathcal{D}} |\eta\big(u(x,t)\big)|^2\, dx = 0 \quad \forall\, t \in [0,T].$$

It follows that $\eta\big(u(x,t)\big) = u^-(x,t) = 0$ a.s. for a.e. $x \in \mathbb{R}^d$ and $t \in [0,T]$. The theorem is thus proved. □

Remark 3.4. The above theorem shows that, under appropriate conditions, a global solution is positive. This is also true for a local solution u_t before the explosion occurs. The proof can be carried out similarly as before by localization, that is, replacing t in the integrals by $t_e = (t \wedge \mathbf{e})$.

4. Unbounded Positive Solutions

To study unbounded solutions, we consider the parabolic Itô equation (2.1) with a homogeneous boundary condition $h \equiv 0$. Before proceeding to the key theorems, we consider the eigenvalue problem for the elliptic equation:

$$\begin{cases} Av = -\lambda v \quad \text{in} \quad \mathcal{D}, \\ v = 0 \quad \text{on} \quad \partial\mathcal{D}. \end{cases} \tag{4.1}$$

It is well known that all the eigenvalues are strictly positive, increasing, and the eigenfunction ϕ corresponding to the smallest eigenvalue λ_1 does not change sign in the domain $\mathcal{D}$ (see pp.451-455, [3]). Therefore we can normalize it in such a way that

$$\phi(x) \geq 0, \quad \int_{\mathcal{D}} \phi(x)\, dx = 1. \tag{4.2}$$

Theorem 4.1. *Let the condition (P1) for the initial-boundary value problem (2.1) be satisfied. In addition we assume that*

(N1) *The function $f(r,x,t)$ is continuous and positive on $\mathbb{R} \times \overline{\mathcal{D}} \times [0,T]$. Moreover there exists a continuous, positive function $F(r)$ on $\mathbb{R}$ which is convex for $r \geq 0$ and*

$$f(r,x,t) \geq F(r), \quad \text{for} \quad r \geq 0,\, x \in \overline{\mathcal{D}},\, t \in [0,T].$$

(N2) *There exists a constant $M > 0$ such that $F(r) > \lambda_1 r$ for $r > M$ and*

$$\int_M^\infty \frac{dr}{F(r) - \lambda_1 r} < \infty.$$

(N3) *Let $h \equiv 0$ in condition (P3) and suppose $\mu_0 \geq M$, where*

$$\mu_0 := (g, \phi) = \int_{\mathcal{D}} g(x)\phi(x)\, dx.$$

Then, for any $p \geq 1$, there exists a constant $T_p > 0$ such that

$$\lim_{t \to T_p^-} \|u_t\|_p^p = \lim_{t \to T_p^-} E \int_{\mathcal{D}} |u(x,t)|^p\, dx = \infty, \tag{4.3}$$

or the solution explodes in the $\mathcal{L}^p$−norm defined by (2.7).

Proof. Under conditions (P1), (N1)–(N3), by Theorem 3.3, the equation (2.1) has a unique positive solution. Suppose the conclusion (4.3) is false. Then there exists a global positive solution u and a real number $p \geq 1$ such that

$$\sup_{0 \leq t \leq T} E \int_{\mathcal{D}} |u(x,t)|^p\, dx < \infty, \tag{4.4}$$

for any $T > 0$. To reach a contradiction, let ϕ be the eigenfunction as given by (4.2) and define

$$\hat{u}(t) := \int_{\mathcal{D}} u(x,t)\phi(x)\, dx \geq 0. \tag{4.5}$$

Since ϕ is positive and normalized, it can be regarded as the probability density function of a random variable ξ in $\mathcal{D}$, independent of W_t, and the above integral can be interpreted as an expectation $\hat{u}(t) = E_\xi\{u(\xi,t)\}$ with respect to this random variable. Since $\hat{u}$ is a linear functional of u, by using (2.3) in equation (4.5), we can get

$$\left\{\begin{aligned} \hat{u}(t) = (g,\phi) &+ \int_0^t \int_{\mathcal{D}} [Au(x,t)]\phi(x)\,dx\,ds \\ &+ \int_0^t \int_{\mathcal{D}} f(u,x,s)\,\phi(x)\,dx\,ds \\ &+ \int_0^t \int_{\mathcal{D}} \sigma(u,\nabla u,x,s)\,\phi(x)\,dW(x,s)\,dx. \end{aligned}\right. \tag{4.6}$$

Recall that A is self-adjoint. So we have $\langle Au,\phi\rangle = (u,A\phi) = -\lambda_1(u,\phi)$. After taking the expectation $E\{\cdot\}$ over equation (4.6) and changing the order of the expectation and an integration by appealing to Fubini's theorem, we obtain

$$\left\{\begin{aligned} &E\,\hat{u}(t) = (g,\phi) - \lambda_1 \int_0^t E\,\hat{u}(s)\,ds \\ &+ \int_0^t E \int_{\mathcal{D}} f(u,x,s)\,\phi(x)\,dx\,ds, \end{aligned}\right.$$

or, in the differential form,

$$\left\{\begin{aligned} &\frac{d\mu(t)}{dt} = -\lambda_1\,\mu(t) + E\int_{\mathcal{D}} f(u,x,t)\,\phi(x)\,dx, \\ &\mu(0) = \mu_0, \end{aligned}\right. \tag{4.7}$$

where we set $\mu(t) = E\,\hat{u}(t)$ and $\mu_0 = (g,\phi)$. In view of condition (N1), the equation (4.7) yields

$$\left\{\begin{aligned} &\frac{d\mu(t)}{dt} \geq -\lambda_1\,\mu(t) \,+\, E\int_{\mathcal{D}} F\big(u(x,t)\big)\,\phi(x)\,dx, \\ &\mu(0) = \mu_0. \end{aligned}\right. \tag{4.8}$$

By condition (N2), $F(r)$ is convex for $r > 0$ so that Jensen's inequality gives us

$$\left\{\begin{aligned} E\int_{\mathcal{D}} F\big(u(x,t)\big)\,\phi(x)\,dx &= E\,E_\xi\,F\big(u(\xi,t)\big) \\ &\geq F\big(E\,E_\xi\,u(\xi,t)\big) = F(\mu(t)). \end{aligned}\right. \tag{4.9}$$

By taking (4.8), (4.9) and condition (N2) into account, we find

$$\left\{\begin{aligned} &\frac{d\mu(t)}{dt} \geq F\big(\mu(t)\big) - \lambda_1\,\mu(t), \\ &\mu(0) = \mu_0, \end{aligned}\right.$$

which implies, for $\mu_0 > M$,

$$T \leq \int_{\mu_0}^{\mu(T)} \frac{dr}{F(r) - \lambda_1 r} \leq \int_M^\infty \frac{dr}{F(r) - \lambda_1 r}. \tag{4.10}$$

But, by conditions (N2) and (N3), the last integral is bounded. Hence the inequality (4.10) cannot hold for a sufficiently large T. This contradiction shows that $\mu(t) = E\int_{\mathcal{D}} u(x,t)\phi(x)\,dx$ must blow up at a time

$$T_e \leq \int_{\mu_0}^{\infty} \frac{dr}{F(r) - \lambda_1 r}.$$

Since ϕ is bounded and continuous on $\overline{\mathcal{D}}$, we apply Hölder's inequality for each $p \geq 1$ to get

$$\mu(t) \leq C_p\{E\int_{\mathcal{D}} |u(x,t)|^p\,dx\}^{1/p},$$

where $C_p = \Big\{\int_{\mathcal{D}} |\phi(x)|^q\,dx\Big\}^{1/q}$ with $q = p/(p-1)$. Therefore we can conclude that the positive solution explodes at some time $T_p \leq T_e$ in $\mathcal{L}^p$-norm for each $p \geq 1$, as asserted by equation (4.3). □

Now we consider the Cauchy problem for equation (2.1) in an unbounded domain $\mathcal{D} = \mathbb{R}^d$, where the boundary condition is omitted. Let $B(R) = \{x \in \mathbb{R}^d : |x| < R\}$ be a ball of radius R in $\mathbb{R}^d$. In this case Theorem 4.1 still holds in the $\mathcal{L}^p-$norm on $B(R)$ for any $R > 0$ as indicated in the following theorem.

Theorem 4.2. *Suppose the conditions (P1) and (N1)–(N3) hold with $\mathcal{D} = \mathbb{R}^d$. Then the solution u of the Cauchy problem for equation (2.1) explodes in the sense of $\mathcal{L}^p(B(R))-$norm, or, for each $p \geq 1$, there is a constant $T_p(R) > 0$ such that*

$$\lim_{t\to T_p^-(R)} E\int_{B(R)} |u(x,t)|^p\,dx = \infty, \tag{4.11}$$

for any $R > 0$.

Proof. Consider the eigenvalue problem (4.1) with $\mathcal{D} = B(R)$ and let ϕ be the eigenfunction normalized as in (4.2). By restricting the solution u to $\overline{B(R)}$, let $\hat{u}(t) = \int_{B(R)} u(x,t)\phi(x)\,dx$ as defined by (4.5). Then, as in the proof of Theorem 4.1, one can proceed to obtain equation (4.6). However, in this case, $u \geq 0$ on the boundary $\partial B(R)$. By Green's identity, instead of $\langle Au, \phi\rangle = -\lambda_1\hat{u}(t)$, one would get

$$\langle Au_t, \phi\rangle = -\lambda_1\hat{u}(t) + \int_{\partial B(R)} u(x,t)[-\frac{\partial\phi(x)}{\partial\nu}]\,dS. \tag{4.12}$$

Since the matrix $[a_{ij}(x)]$ is uniformly positive definite, in view of equation (3.5), $\nu\cdot\mathbf{n} = \Sigma_{i,j}a_{ij}n_in_j > 0$. Hence the conormal $\nu(\mathbf{x})$ is an exterior direction field. Due to the fact that $\phi > 0$ in $B(R)$ and $\phi = 0$ on $\partial B(R)$, we have $\dfrac{\partial\phi(x)}{\partial\nu} \leq 0$. Therefore the extra term in (4.12) is positive so that the differential inequality (4.8) remains valid and the rest of proof can be completed in a similar manner. □

5. Example

As an example, let us consider the following problem in a spherical domain $\mathcal{D} = B(R)$ in $\mathbb{R}^3$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla^2 u + |u|^{1+\alpha} + \sigma_0 (u^2 + |\nabla u|^2)^{1/2}\, \partial_t W(x,t), \\ u(x,0) = a_0\, e^{-\beta|x|}, \\ u(x,t)|_{|x|=R} = 0, \end{cases} \tag{5.1}$$

where $W(x,t)$ is a continuous Wiener random field with the covariance function

$$q(x,y) = b_0 \exp\{-\rho\ (x\cdot y)\}, \quad \text{for} \quad x, y \in B(R). \tag{5.2}$$

All of the above constants $a_0, b_0, \alpha, \beta, \rho, \sigma_0$ are strictly positive and $x\cdot y = \sum_{i=1}^{3} x_i y_i$.

Here we have $A = \nabla^2$ and $\sigma = \sigma_0(u^2 + |\nabla u|^2)^{1/2}$. Clearly we have

$$\frac{1}{2}\, b_0\, \sigma_0^2 \exp\{-\rho|x|^2\}(s^2 + |\xi|^2) - |\xi|^2 \le (\frac{1}{2}\, b_0\, \sigma_0^2 - 1)|\xi|^2 + \frac{1}{2}\, b_0\, \sigma_0^2 s^2.$$

It follows that condition (P1) is satisfied provided that

$$\frac{1}{2}\, b_0\, \sigma_0^2 < 1. \tag{5.3}$$

Obviously the functions $f = |u|^{1+\alpha}$, $g = a_0\, e^{-\beta|x|}$ and $h \equiv 0$ satisfy conditions (P2) and (P3). By Theorem 3.3, if inequality (5.3) holds, the solution u of equation (5.1) is positive.

To determine sufficient conditions for explosion, consider the associated eigenvalue problem for the Laplace equation in $B(R)$. It is not hard to find the smallest eigenvalue $\lambda_1 = (\frac{\pi}{R})^2$ and the corresponding normalized eigenfunction $\phi(x) = \frac{C}{r} \sin \frac{\pi r}{R}$ with $r = |x| \le R$ and $C = \frac{1}{4R^2}$. Let $F(s) = f(s) = |s|^{1+\alpha}$ with $\alpha > 0$ so that condition (N1) holds. Let M be any number greater than $\lambda_1^{1/\alpha} = (\frac{\pi}{R})^{2/\alpha}$. Then, for $s \ge M$,

$$F(s) - \lambda_1\, s = s^{1+\alpha} - \lambda_1\, s\ >\ 0,$$

and, for any $\alpha > 0$, the integral $\displaystyle\int_M^\infty \frac{ds}{s^{1+\alpha} - \lambda_1 s}$ is convergent so that condition (N2) is met. By some simple calculations, we can show that condition (N3) is satisfied if the initial amplitude a_0 is large enough such that

$$\frac{a_0}{R} \int_0^R r \exp\{-\beta r\} \sin \frac{\pi r}{R}\, dr > (\frac{\pi}{R})^{\frac{2-\alpha}{\alpha}}, \tag{5.4}$$

where the integral can be evaluated exactly but will not be given for brevity. If the inequalities (5.3) and (5.4) hold, then, by Theorem 4.1, the solution of the equation (5.1) will blow up in finite time in $\mathcal{L}^p$−norm for any $p \ge 1$. In view of

Theorem 4.2, this is also true for the corresponding Cauchy problem in $\mathbb{R}^3$. Of course, in this case, the $\mathcal{L}^p-$norm is restricted to any ball $B(R) \subset \mathbb{R}^3$.

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