An Algebraic Treatment of Diophantine Analysis

Anthony Andrew Aucoin

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AN ALGEBRAIC TREATMENT OF DIOPHANTINE ANALYSIS

A Dissertation

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requirements for the degree of
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in

The Department of Mathematics

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Without the help and encouragement of Professor W. V. Parker this dissertation would not have been written.
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ABSTRACT

Diophantine analysis, while an old subject, is a collection of special problems. The methods used are very varied and those used to solve one problem will not usually solve a second one. It is possible that this condition exists because Diophantine analysis is usually considered as number theory. While some problems are problems in number theory, a large body of material may be developed by algebraic methods. In this thesis the methods of algebra are exploited.

The material of this thesis divides itself into chapters naturally, according to the degree of the equations. Considered in order are quadratics, cubics, quartics, and equations of degree n. All equations considered are homogeneous polynomials except a few of degree n. All except two equations are generalized to equations of degree n. The solutions are given in terms of arbitrary parameters and are integral for an integral choice of the parameters.

An important concept of this thesis is the introduction of equivalent solutions (due to Professor W. V. Parker). Consider the Diophantine equation

\[(1) \quad f(x_1, \ldots, x_p) = g(y_1, \ldots, y_q),\]

where \(f\) and \(g\) are homogeneous polynomials with integral coefficients, of degrees \(m\) and \(n\) respectively. If \(x_i = \alpha_i, y_j = \beta_j\) is a solution of (1) and there are no integers \(s > 1, \alpha_i', \beta_j'\) such that \(x_i = \alpha_i' s^\lambda, y_j = \beta_j' s^\mu\) where \(\lambda, \mu\) are relatively prime positive integers such that \(\lambda m = \mu n\), then
\( x_i = \alpha_i, \ y_j = \beta_j \) is defined to be a primitive solution of (1). If \( x_i = \alpha_i, \ y_j = \beta_j \) is a primitive solution of (1), then
\[ x_i = \alpha_i t^2, \ y_j = \beta_j t^3, \] where \( t \) is a non-zero integer, is also a solution which is said to be derived from this primitive solution. Two solutions are defined to be equivalent if they may be derived from the same primitive solution.

Each solution obtained is shown, within certain limits, to be general. For a given solution of one of the equations considered, the parameters are chosen in such a way that a solution, equivalent to the given one, is obtained, provided the given solution is not also a solution of a certain equation. Usually, if the original equation is of degree \( n \), the "certain" equation is of degree \( n - 1 \).
CHAPTER I
INTRODUCTION

Diophantine analysis treats the problem of finding rational numbers or integers which satisfy the equation

\[ f(x_1, \ldots, x_r) = 0. \]

Usually \( f \) is a polynomial with integral coefficients. Few problems have been solved when \( f \) is not a polynomial. The solutions are given in terms of arbitrary parameters.

This thesis will be concerned with finding, in terms of parameters, solutions which will be integral for an integral choice of the parameters. The functions will be polynomials except in several problems considered at the end of Chapter V.

While Diophantine analysis is an old subject and has been studied extensively, it remains for the most part a collection of special problems. It appears that this is due to the subject's being unfortunately linked with number theory rather than with algebra. It is true that some problems in Diophantine analysis are problems in number theory, but a large body of material may be developed by strictly algebraic methods. As an insight into the state of development of the subject we quote from Carmichael:

"When we pass to equations of degree higher than the fourth we find but little effective progress has been made. Often it is a matter of great dif-

\[ \text{[1] Robert D. Carmichael, Diophantine Analysis (New York: John Wiley and Sons, Inc., 1915), p. 85.} \]
difficulty to determine whether a given solution is the general solution of a given equation. Indeed this is true, to a large extent, of equations of the third and fourth degrees. Even here there are but few equations for which a general solution is known or for which it is known that no solution at all exists. As the degree of the equation increases, the generality of the known results decreases in a rapid ratio. Only the most special equations of degree higher than the fourth have been at all treated and for only a few of these is one able to answer the question naturally propounded as to the existence or generality of solutions.

If \( x_1 = \alpha_1 \), \( y_1 = \beta_1 \) is a solution of the equation

\[
(1:1) \quad f(x_1, \ldots, x_p) = g(y_1, \ldots, y_j)
\]

where \( f \) and \( g \) are homogeneous polynomials of degrees \( m \) and \( n \), respectively, and there are no integers \( s > 1 \), \( \alpha_i', \beta_i' \) such that \( \alpha_i = \alpha_i' s^1 \), \( \beta_i = \beta_i' s^n \), where \( \lambda, \mu \) are relatively prime positive integers such that \( \lambda m = \mu n \), then \( x_i = \alpha_i \), \( y_i = \beta_i \) is defined to be a primitive solution of the equation. If \( x_i = \alpha_i \), \( y_i = \beta_i \) is a primitive solution of (1:1), then \( x_i = \alpha_i t^\lambda \), \( y_i = \beta_i t^\mu \), where \( t \) is a non-zero integer, is also a solution, which is said to be derived from this primitive solution. Two solutions are defined to be equivalent if they may be derived from the same primitive solution.

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2/ This definition of equivalence of solution as well as its subsequent application is due to Professor W. V. Parker.

A primitive solution of a Diophantine equation has been defined as a set of relatively prime integers which satisfy
example, $x^n - y^n = z^3$, have solutions but do not have primitive solutions. However, $x = \alpha (\alpha^n - \beta^n)^2$, $y = \beta (\alpha^n - \beta^n)^2$, $z = (\alpha^n - \beta^n)^3$ is a primitive solution of this equation according to the definition of this thesis. From a relative prime (primitive) solution other solutions may be obtained by multiplying each by a power of some constant. In this thesis, a primitive solution is obtained by dividing any solution by a power of some constant. It is believed that one procedure is as general as the other.

CHAPTER II

QUADRATIC EQUATIONS

A. Desboves has shown that, given a solution of the equation 

\[ \sum_{ij} a_{ij} x_i x_j = 0, \]

other solutions may be determined. The method is shown in the first theorem.

**Theorem 2:1.** If \( x_i = \alpha_i \) is a solution of the equation

\[ \sum_{ij} a_{ij} x_i x_j = 0, \]

then every solution, which is not also a solution of

\[ \sum_{ij} a_{ij} \beta_i x_j = 0, \]

is equivalent to one of the solutions given by

\[ x_i = \alpha_i s - \lambda_i t, \]

where

\[ s = \sum_{ij} a_{ij} \lambda_i \lambda_j, \quad t = 2 \sum_{ij} a_{ij} \alpha_i \lambda_i \]

and the \( \lambda_i \)'s are arbitrary integers.

**Proof.** If \( x_i \) have the values given by (2:3), then (2:1) becomes

\[ 2st \sum_{ij} a_{ij} \alpha_i \lambda_i = t^2 \sum_{ij} a_{ij} \lambda_i \lambda_j, \]

which is satisfied identically in the \( \lambda_i \)'s if \( s \) and \( t \) are given by (2:4).

Suppose now that \( x_i = \beta_i \) is a given solution of (2:1). If we choose \( \lambda_i = \beta_i \), then \( s = 0 \) and (2:3) becomes \( x_i = -\beta_i t \) which is equivalent to the given solution provided \( t \neq 0 \); that is, provided \( x_i = \beta_i \) is not also a solution of (2:2).

Desboves states that (2:3) gives all solutions of (2:1) if two equivalent solutions are regarded as the same solution. That this is not the case is seen from the proof of Theorem 2:1.
Various quadratic equations will now be solved without considering a particular solution. Each equation considered is such, however, that a particular solution may be noted by inspection.

Theorem 2:2. Every solution of the equation

\[ \sum_{i=1}^{n} a_i x_i^2 = \sum_{i=1}^{n} b_i z^2, \]

where all the a's are different from zero and \( \sum b_i \neq 0 \), is equivalent to one of the solutions given by

\[ x_j = 2K a_j \sum_{i=1}^{n} a_i b_i \alpha_i - K_j b_j \sum_{i=1}^{n} a_i^2 \alpha_i, \quad (j = 1, \ldots, n), \]
\[ z = K \sum_{i=1}^{n} a_i^2 \alpha_i, \]

where \( K \) is the least common multiple of the a's, \( K_j = K/a_j \) and the \( \alpha_i \)'s are arbitrary integers.

Proof. Write (2:5) in the form

\[ \sum_{i=1}^{n} (b_i z^2 - a_i x_i^2) = 0. \]

If

\[ b_j z + a_j x_j = a_j \alpha_j t, \quad (j = 1, \ldots, n) \]
then (2:7) becomes

\[ 2zt \sum_{i=1}^{n} a_i b_i \alpha_i = t^2 \sum_{i=1}^{n} a_i^2 \alpha_i, \]

which is satisfied identically in the \( \alpha_i \)'s if \( z = K \sum_{i=1}^{n} a_i \alpha_i \), \( t = 2K \sum_{i=1}^{n} a_i b_i \alpha_i \). Hence a solution of (2:5) is given by (2:6).

Suppose now that \( x_j = \lambda_j \), \( z = \rho \) is any solution of (2:5).

Then

\[ \sum_{i=1}^{n} a_i \lambda_i^2 = \sum_{i=1}^{n} b_i \rho^2. \]

If

\[ \alpha_j = K \lambda_j + K_j b_j \rho, \quad (j = 1, \ldots, n) \]
then (2:6) becomes

\[ z = \rho R, \quad x_j = \lambda_j R, \quad (j = 1, \ldots, n) \]

where \( R = 2K \sum_{i=1}^{n} b_i (a_i \lambda_i + b_i \rho) \). However if
\[ a_j = K \lambda_j - K b_j \rho, \quad (j = 1, \ldots, n) \]
then \(2:6\) becomes
\[ x_j = \rho S, \quad x_j = \lambda_j S, \quad (j = 1, \ldots, n) \]
where \( S = 2K^3 \sum_i b_i (a_i \lambda_i - b_i \rho). \)

Now if either \( R \) or \( S \) is non-zero, then either \(2:12\) or \(2:14\) is a solution of \(2:5\), each of which is equivalent to the solution \( z = \rho, \quad x_j = \lambda_j. \)

It will now be shown that \( R \) and \( S \) are not both zero unless the solution is the trivial one \( \rho = \lambda_j = 0. \) If \( R = 0, \) then since \( K \neq 0, \)
\[ \sum_i b_i (a_i \lambda_i + b_i \rho) = 0, \]
and hence
\[ \sum_i b_i \rho (a_i \lambda_i + b_i \rho) = 0. \]
From \(2:10\), \(2:15\) becomes
\[ \sum_i a_i \lambda_i (a_i \lambda_i + b_i \rho) = 0. \]
By adding \(2:15\) and \(2:16\) it follows that \( \sum_i (a_i \lambda_i + b_i \rho)^2 = 0, \)
and hence
\[ a_j \lambda_j + b_j \rho = 0, \quad (j = 1, \ldots, n). \]

Similarly, if \( S = 0, \)
\[ a_j \lambda_j - b_j \rho = 0, \quad (j = 1, \ldots, n). \]
Since all the \( a_i \)'s are different from zero, each \( \lambda_j \) must be zero in order that \(2:17\) and \(2:18\) be true simultaneously, and hence \( \rho = 0. \)

If each \( a_i = 1, \)\( b_i = 1, \)\( b_i = 0 \) \((i = 2, 3, \ldots, n), \)\(2:5\) reduces to the sum of \( n \) squares equal to a square. The solution in this case is given by \(5/ \)
\[ z = \sum_i a_i^2, \quad x_i = a_i^2 = \sum_i a_i^2. \]


\[ x_j = 2 a_i, \quad (j = 2, 3, \ldots, n). \] In particular, for \( n = 3, \) the solutions of

---

(2:19) \[ x^2 + y^2 + z^2 = t^2 \]

are

\[ t = \alpha^2 + \beta^2 + \gamma^2, \quad x = \alpha^2 - \beta^2 - \gamma^2 \]

(2:20) \[ y = 2 \alpha \beta, \quad z = 2 \alpha \gamma. \]

Catalan has pointed out that these formulas do not
give all solutions of (2:19), but that all relatively prime
solutions are given by

(2:21) \[ t = m^2 + n^2 + p^2 + q^2, \quad x = m^2 - n^2 - p^2 + q^2 \]

\[ y = 2mn + 2pq, \quad z = 2mp - 2nq. \]

If in (2:20) the choice \( \alpha = m^2 + q^2, \beta = mn + pq, \gamma = mp - nq \) is made, then

\[ t = (m^2 + q^2)(m^2 + n^2 + p^2 + q^2), \]

\[ x = (m^2 + q^2)(m^2 - n^2 - p^2 + q^2), \]

\[ y = (m^2 + q^2)(2mn + 2pq), \]

\[ z = (m^2 + q^2)(2mp - 2nq), \]

which is equivalent to (2:21) provided \( m^2 + q^2 \neq 0 \). If

\( m = q = 0 \), then (2:21) becomes \( t = n^2 + p^2, x = -n^2 - p^2, y = z = 0 \), which is given by (2:20) with \( \alpha = 0, \beta = n, \gamma = p \). Hence in every case, for any solution given by (2:21), an

equivalent solution is given by (2:20).

The solutions of the equation

(2:22) \[ \sum_i d_i x_i^2 = \sum_i d_i z^2 \]

may be determined by Theorem 2:2 by setting \( a_i^2 = b_i^2 = d_i \). After

removing the common factor \( K \) from each expression in (2:6)

(2:23) \[ z = \sum_i d_i \alpha_i^2, \quad x_j = 2 \alpha_j \sum_i d_i \alpha_i - \sum_i d_i \alpha_i^2, \quad (j = 1, \ldots, n) \]
follows as a solution of (2:22). Since (2:23) satisfies (2:22) identically in the \( \alpha' \)'s, it is not necessary to assume that \( d_i \) be a square or that it be positive. It may be shown as above, that (2:23) gives a solution equivalent to any existing solution of (2:22).

Theorem 2:3. The equation

\[
(2:24) \quad \sum_i \left( a_i x_i + b_i y_i \right) \left( c_i x_i + d_i y_i \right) = 0,
\]

has solutions and every solution which is not also a solution of

\[
(2:25) \quad \sum_i a_i c_i (Kx_i + b_i Ky_i)^n = 0,
\]

where \( K \) is the least common multiple of the \( a_i \)'s, \( K_i = K/a_i \), is equivalent to one of the solutions given by

\[
(2:26) \quad x_i = \alpha_i t - b_i K_i \beta_i s, \quad y_i = \beta_i Ks
\]

where

\[
(2:27) \quad s = \sum_i a_i c_i \alpha_i, \quad t = K \sum_i \alpha_i \beta_i (b_i c_i - a_i d_i),
\]

and the \( \alpha_i \) and \( \beta_i \) are arbitrary integers.

Proof. Let \( a_i x_i + b_i y_i = a \alpha_i t, y_i = \beta_i Ks \). Then

\[
x_i = \alpha_i t - b_i K_i \beta_i s.
\]

Substituting in (2:24) gives

\[
t \sum_i a_i c_i \alpha_i^2 = stK \sum_i \alpha_i \beta_i (b_i c_i - a_i d_i)
\]

which is identically satisfied in the \( \alpha' \)'s and \( \beta' \)'s if \( s \) and \( t \) are given by (2:27). Hence (2:26) is a solution of (2:24).

Suppose \( x_i = \lambda_i, y_i = \mu_i \) is any given solution of (2:24).

If \( \alpha_i = K \lambda_i + b_i K_i \mu_i \), \( \beta_i = \mu_i \), then

\[
t = K \sum_i \mu_i (b_i c_i - a_i d_i) (K \lambda_i + b_i K_i \mu_i),
\]

\[
s = \sum_i a_i c_i (K \lambda_i + b_i K_i \mu_i),
\]

\[
= K^2 \sum_i (a_i c_i \lambda_i^2 + b_i c_i \lambda_i \mu_i) + K^2 \sum_i b_i c_i \lambda_i \mu_i + \sum_i a_i b_i c_i K_i^2 \mu_i^2,
\]

and since \( \sum_i (a_i c_i \lambda_i^2 + b_i c_i \lambda_i \mu_i) = - \sum_i (a_i d_i \lambda_i \mu_i + b_i d_i \mu_i^2) \),
Hence \( x_1 = \lambda_1 x_s, y_1 = \mu_1 y_s \) which is equivalent to the given solution if \( x_1 = \lambda_1, y_1 = \mu_1 \) is not also a solution of (2.25).

It will be noted that this also solves the equation
\[
\sum \left( a_i x_i + b_i y \right) \left( c_i x_i + d_i y \right) = 0,
\]
of which (2.5) is a special case. In this case let \( \rho = 1 \) in (2.26).

Likewise the general homogeneous quadratic equation may be solved if it contains either two or three terms which are quadratic in two unknowns, and which are factorable into the product of two linear factors with integral coefficients.

This is shown in the next theorem.

Theorem 2.4. The equation
\[
(2.28) \quad (ay + bz)(cy + dz) = x \sum a_i x_i + z \sum b_i x_i + \sum c_i x_i x_j,
\]
where \( a \) and \( b \) are not both zero, and \( a \) and \( d \) are not both zero, has solutions, and every solution which is not a solution of
\[
(2.29) \quad (bc - ad)(ay + bz) + \sum (ab_i - ba_i) x_i = 0,
\]
is equivalent to one of the solutions given by
\[
y = -d \alpha + \sum b_i \alpha_i + b \sum c_i \alpha_i \alpha_j, \\
z = c \alpha + \sum a_i \alpha_i - a \sum c_i \alpha_i \alpha_j, \\
x_k = \alpha_k \left[ (bc - ad) \alpha_i + \sum (ab_i - ba_i) \alpha_j \right], \quad (k = 1, \ldots, n),
\]
where \( \alpha, \alpha_i \) are arbitrary integers.

Proof. Let
\[
(2.31) \quad ay + bz = \alpha t, \quad x_k = \alpha_k t,
\]
then (2.28) becomes
The combination of (2:31) and (2:32) gives
\[
[(bc - ad)\alpha + \sum_i (ab_i - ba_i)\alpha_i]t_y = [-d \alpha^2 + \sum_i b_i \alpha_i + b \sum_j c_{ij} \alpha_j \alpha_i]t_y,
\]
\[
[(bc - ad)\alpha + \sum_i (ab_i - ba_i)\alpha_i]t_z = [c \alpha^2 - \sum_i a_i \alpha_i - a \sum_j c_{ij} x_i \alpha_i]t_z.
\]
which are satisfied identically in the \( \alpha_i \)'s if
\[
y = -d\alpha^2 + \sum_i b_i \alpha_i + b \sum_j c_{ij} \alpha_j \alpha_i,\\
z = c\alpha^2 - \sum_i a_i \alpha_i - a \sum_j c_{ij} \alpha_i \alpha_j,\\
t = (bc - ad)\alpha + \sum_i (ab_i - ba_i)\alpha_i.
\]
Hence (2:30) is a solution of (2:28).

Suppose now that \( y = \rho, z = \sigma, x_k = \lambda_k \) is any solution of (2:28). If in (2:30), \( \alpha = a\rho + b\sigma, \lambda_k = \lambda_k \), then
\[
y = \rho P, z = \sigma P, x_k = \lambda_k P
\]
where
\[
P = (bc - ad)(a\rho + b\sigma) + \sum_i (ab_i - ba_i)\lambda_i,
\]
which is equivalent to the given solution provided \( P \neq 0 \); that is, provided \( y = \rho, z = \sigma, x_k = \lambda_k \) is not a solution of (2:29).

Interchanging \( a \) with \( c \), \( b \) with \( d \) in (2:30) gives
\[
y = -b\alpha^2 + \sum_i b_i \alpha_i + d \sum_j c_{ij} \alpha_i \alpha_j,\\
z = a\alpha^2 - \sum_i a_i \alpha_i - c \sum_j c_{ij} \alpha_i \alpha_j,\\
x_k = \lambda_k \[(ad - bc)\alpha + \sum_i (ob_i - da_i)\alpha_i], (k = 1, \ldots, n),
\]
a solution equivalent to any existing solution which does not satisfy
\[
(2:34) \quad (ad - bc)\alpha + \sum_i (ob_i - da_i)\alpha_i = 0.
\]

A solution of the sum of \( m \) squares equal to the sum of \( n \) squares may be obtained by Theorem 2:4 or by Theorem 2:2 after a transformation of the equation. The equation
\[
(2:35) \quad \sum_i x_i^2 = \sum_i y_i^2, \quad m > n,
\]
may be written in the form
\[(x_i - y_i)(x_i + y_i) = \sum_{j=1}^{n} y_j^2 - \sum_{j=1}^{n} x_j^2,\]
which is a special case of Theorem 2.4. However, letting
\[y_j = a_j t \quad (j = 1, \ldots, n),\]
(2.35) becomes
\[(2.36) \quad \sum_{j=1}^{n} x_j^2 = \sum_{j=1}^{n} a_j^2 t^2,\]
where \(a_j = 0\) for \(j > n\). Then (2.36) is a special case of
Theorem 2.2, and may be solved for \(x_1, x_2, \ldots, x_m, t\) in terms
of \(m\) parameters. It will be noted that the first method
gives the solution in terms of \(m + n - 1\), while the second
method gives the solution in terms of \(m + n\) parameters.

A combination of Theorem 2.1 and Theorem 2.4 may be
used to obtain a solution equivalent to any solution of (2.28).
For example, consider the equation
\[(2.37) \quad 2x^2 + 3xy + y^2 - xz - yz + yw - 2zw = 0,\]
which may be written in the form
\[(x + y)(2x + y) = xz + yz - yw + 2zw.\]
From (2.30) and (2.33) the two solutions
\[(2.38) \quad x = -p^r + pq - pr + 2qr, \]
\[y = 2p^r - pq - 2qr, \]
\[z = pq - qr, \]
\[w = pr - r^r, \]
and
\[x = -p^r + pq - pr + 2qr, \]
\[y = p^r - pq - 4qr, \]
\[z = -pq + q^r - 2qr, \]
\[w = -pr + qr - 2r^r, \]
are obtained.

Any solution which does not satisfy
(2:40) \[ x + y - w = 0 \]
is equivalent to one given by (2:38), and any solution which does not satisfy
(2:41) \[ 2x + y - z + 2w = 0 \]
is equivalent to one given by (2:39).

Now (2:37), (2:40), and (2:41) have the common solutions \((1, -1, 1, 0)\) and \((-7, 10, 2, 3)\). \((2, -2, -1, -1)\) is a particular solution of (2:37) obtained from (2:38) by choosing \(p = 0, q = r = 1\). From this particular solution Theorem 2:1 gives the solution
\[
\begin{align*}
x &= \alpha^2 + 4 \alpha \beta + 2 \beta^2 - 4 \gamma \delta - 2 \beta \delta + 2 \beta \delta - 4 \gamma \delta, \\
y &= -4 \alpha^2 - 9 \alpha \beta - 4 \beta^2 + 2 \alpha \delta + 2 \beta \delta + 4 \gamma \delta, \\
z &= -2 \alpha^2 - 3 \alpha \beta - \beta^2 - 2 \alpha \delta - \beta \delta - 2 \gamma \delta - \beta \delta + 2 \gamma \delta, \\
w &= -2 \alpha^2 - 3 \alpha \beta - \beta^2 + \alpha \delta + \beta \delta - 3 \alpha \delta - 3 \beta \delta.
\end{align*}
\]

Now (2:42) gives a solution equivalent to any existing one which is not a solution of
(2:43) \[ 3x + 2y + 2z = 0. \]

But (2:37), (2:40), (2:41), and (2:43) have no solutions in common. Hence given any solution of (2:37) we may find a solution equivalent to it from either (2:38), (2:39), or (2:42).
CHAPTER III
CUBIC EQUATIONS

Several methods have been given for solving the equation \( x^3 + y^3 = u^3 + v^3 \). Here a more general equation is solved, of which this is a special case. This is given in the following theorem.

Theorem 3:1. The equation

\[
(ax + by)(mx^3 + nxy + py^3) = (cu + dv)(mu^3 + nuv + pv^3)
\]

has solutions and every solution which is not also a solution of

\[
(mx^3 + nxy + py^3) \frac{(bc - ad)mx + (ben - acp - bdm)y + (bdm - adn + aop)x + (bo - ad)py}{cu + dv} = 0
\]

is equivalent to one of the solutions given by

\[
x = \frac{b}{c} \gamma - (\alpha^3 + n\alpha + mp\beta)(d\alpha + dn\beta - op\beta),
\]
\[
y = \gamma [(\alpha^3 + n\alpha + mp\beta)(c\alpha + dm\beta) - a\beta],
\]
\[
u = (b\alpha + ap\beta)\gamma - d(\alpha^3 + n\alpha + mp\beta)^3,
\]
\[
v = (mb\beta - a\alpha - an\beta)\gamma + e(\alpha^3 + n\alpha + mp\beta)^3,
\]

where \( \alpha, \beta, \gamma \) are arbitrary integers.

Proof. Write (3:1) in the form

\[
ax + by = \frac{mu^3 + nuv + pv^3}{mx^3 + nxy + py^3}
\]

Since \( \gamma^*(mu^3 + nuv + pv^3) = (mx^3 + nxy + py^3)(\alpha^3 + n\alpha + mp\beta^3) \)
(3:5) \[ u = \alpha x - \beta y, \quad v = mx + (\alpha + n \beta)y, \]

(3:6) \[ \frac{ax + by}{ou + dv} = \frac{mu + nuy + py^2}{mx + nxy + py^2} = \frac{a + n \alpha \beta + m \beta}{y^2} \]

may be written in place of (3:4). If \( ou + dv = \delta \), from (3:5) and (3:6) it follows that

\[ (a \alpha + dm \beta)x + [d(\alpha + n \beta) - cp \beta]y = \delta \delta, \]

(3:7) \[ a y'x + b y'y = (\alpha^2 + n \alpha \beta + m \beta^2)\delta. \]

Let

\[ \delta^2 = \begin{vmatrix} c \alpha + dm \beta & d(\alpha + n \beta) - cp \beta \\ a y' & b y' \end{vmatrix} \]

and solve (3:7) for \( x \) and \( y \). With these values, \( u \) and \( v \) may be determined from (3:5), and it is seen that (3:3) is a solution of (3:1).

If \( (x_1, y_1) \) is any solution of \( ax + by = 0 \) or \( mx + nxy + py^2 = 0 \) and \( (u_1, v_1) \) is any solution of \( ou + dv = 0 \) or \( mu + nuy + py^2 = 0 \), then \( (x_1, y_1, u_1, v_1) \) is a trivial solution of (3:1). Such solutions will not be considered.

Suppose that \( x = \lambda, y = \mu, u = \rho, v = \sigma \) is any non-trivial solution of (3:1). Choose \( \alpha = m \lambda \rho + n \mu \rho + p \mu \sigma, \rho = \lambda \sigma - m \rho, \gamma = m \lambda + n \lambda \mu + p \mu, \) then

\[ \alpha^2 + n \alpha \beta + m \beta^2 = (m \lambda^2 + n \lambda \mu + p \mu^2)(m \rho^2 + n \rho \sigma + p \sigma^2) \]

so that when these values are substituted in (3:3) it follows that \( x = \lambda K, y = \mu K, u = \rho K, v = \sigma K, \) where \( K \) is the left hand member of (3:2) with \( x, y, u, v \) replaced by \( \lambda, \mu, \rho, \sigma \). Hence every solution of (3:1) which does not satisfy (3:2) is equivalent to a solution given by (3:3).
The following observation is made concerning the solution. If a non-trivial solution does satisfy (3:2), then

9/ Since the solution is non-trivial, \( mx^* + nxy + py^* \neq 0 \).

\[
\begin{align*}
u &= S[(adn - aop - bdm)x - (bo - ad)py], \\
v &= S[(bc - ad)mx + (bon - aop - bdm)py].
\end{align*}
\]

If the values from (3:8) are substituted in (3:1), then

\[
(ax + by)(mx^* + nxy + py^*) = M^3(ax + by)(mx^* + nxy + py^*)
\]

where

\[
M = \begin{vmatrix} adn - aop - bdm & (ad - bo)p \\ (bo - ad)m & bon - aop - bdm \end{vmatrix}.
\]

Any values \((x, y)\) will satisfy (3:9) if \( M = 1 \) and no value if \( M \neq 1 \). Therefore, in addition to the values given by (3:3), (3:1) is satisfied by all integral solutions of

\[
\begin{align*}
\sqrt{M} u &= (adn - aop - bdm)x - (bo - ad)py, \\
\sqrt{M} v &= (bc - ad)mx + (bon - aop - bdm)py,
\end{align*}
\]

if (3:10) has any integral solutions.

For the equation \( x^* + y^* = u^* + v^* \), \( a = b = c = d = m = p = n = 1 \) and (3:3) becomes

\[
\begin{align*}
x &= \gamma [\gamma^3 - (a - 2 \beta)(a^* - a \beta + \beta^*)], \\
y &= \gamma [(a + \beta)(a^* - a \beta + \beta^*) - \gamma^3], \\
u &= (a + \beta)\gamma^3 - (a^* - a \beta + \beta^*)\gamma^3, \\
v &= (a^* - a \beta + \beta^*)\gamma^3 - (a - 2 \beta)\gamma^3.
\end{align*}
\]

By the use of a multiplicative domain, Carmichael

9/ Robert D. Carmichael, op. cit., p. 62, formulas (4) and (5).

obtains a solution of the equation

\[
(3:11) \quad x^2 + y^2 + z^2 - 3xyz = u^2 + v^2 + w^2 - 3uvw.
\]
He says, "This solution, so readily obtained, unfortunately lacks generality." By a second method he gets expressions for \( x, y, u, v \) in terms of arbitrary parameters \( a, b, z, w \). These are not shown to give the general solution of (3:11).

A more general problem will now be solved which gives the general solution of (3:11) as a special case. Let \( f(x) = f(x_1, \ldots, x_n) \) be a homogeneous polynomial of degree three with integral coefficients. Suppose \( f(x) \) is such that for a set of integers \( a_1, a_2, \ldots, a_n \), not all zero,

\[
\frac{\partial f}{\partial x_j} = 0, \text{ for } x_j = a_j, \ (i,j = 1, \ldots, n).
\]

If \( x_i = a_i s + a_j t \), then

\[
f(x) = s^3 f(a) + s t \sum_j a_j \frac{\partial f}{\partial x_j} + \frac{t^3}{3} a_j \frac{\partial f}{\partial x_j},
\]

and since \( \frac{\partial f}{\partial x_j} = 0 \), \( f(a) = \frac{t}{2} \sum_j a_j \frac{\partial f}{\partial x_j} = 0 \), and hence

\[
f(x) = st^3 \sum_j a_j \frac{\partial f}{\partial x_j} + \frac{t^3}{3} f(x),
\]

Theorem 3:2. The equation

\[
\frac{\partial f}{\partial x_j} = 0, \quad (3:11)
\]

where \( g(y) = g(y_1, \ldots, y_n) \) is a homogeneous polynomial of degree three with integral coefficients, has solutions and every solution which is not also a solution of

\[
\sum_j a_j \frac{\partial f}{\partial x_j} = 0, \quad (3:12)
\]

is equivalent to one of the solutions given by

\[
x_i = a_i s + a_j t, \quad \frac{\partial f}{\partial x_j} = 0, \quad (3:13)
\]

\[
y_j = a_j t, \quad (3:14)
\]

where \( s \) and \( t \) are given by

(3:15) \[ s = g(\beta) - f(\alpha), \quad t = \sum_j a_j^2 \frac{\partial^2 f}{\partial x^2}, \]

and the \(\alpha's\) and \(\beta's\) are arbitrary integers.

Proof. If \(x\) and \(y\) have the values given by (3:14), then (3:12) becomes \[ st \sum_j a_j^2 \frac{\partial^2 f}{\partial x^2} = t' (g(\beta) - f(\alpha)) \]

which is identically satisfied in the \(\alpha's\) and \(\beta's\) if \(s\) and \(t\) have the values given by (3:15).

Let \(x = \lambda, y = \mu\) be any given solution of (3:12). If the choice \(x = \lambda, \beta = \mu\) is made in (3:15), then since \(s = 0\), the solution becomes \(x = \lambda t, y = \mu t\) which is equivalent to the given solution provided \(t \neq 0\); that is, provided \(x = \lambda, y = \mu\) does not satisfy (3:13).

Consider now equation (3:11). Here \(\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0, \)

for \(x = y = z = 1\). Hence the solution is

\[ x = (\lambda^2 + \mu^2 + \nu^2 - 3\lambda\mu\nu) + (\alpha - \beta)^2 + (\alpha - \gamma)^2, \]
\[ y = (\lambda^2 + \mu^2 + \nu^2 - 3\lambda\mu\nu) + (\beta - \alpha)^2 + (\beta - \gamma)^2, \]
\[ z = (\lambda^2 + \mu^2 + \nu^2 - 3\lambda\mu\nu) + (\gamma - \alpha)^2 + (\gamma - \beta)^2, \]
\[ u = 3\lambda(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \alpha\gamma - \beta\gamma), \]
\[ v = 3\mu(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \alpha\gamma - \beta\gamma), \]
\[ w = 3\nu(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \alpha\gamma - \beta\gamma). \]

Now (3:16) gives a solution equivalent to any solution of (3:11) if it does not satisfy

(3:17) \[ x^2 + y^2 + z^2 - xy - xz - yz = 0. \]

It is now shown that (3:17) has no solution other than \(x = y = z\). Suppose \(x = \alpha, y = \beta, z = \gamma\) is any solution of (3:17). Then \(x = \alpha - 1, y = \beta - 1, z = \gamma - 1\) is also a solution.

Now there is no loss of generality in supposing that one of the numbers, say \(\alpha\), is positive. By continuing to subtract
one from the values of $x$, $y$, $z$ a solution is obtained in which $x = 0$ and $y$ and $z$ must be a solution of $y^2 - yz + z^2 = 0$. But this equation has no solutions other than $y = z = 0$. Hence $x = y = z = 0$ is the only solution of (3:17). In has been shown, incidentally, that all the solutions of the equation $x^3 + y^3 + z^3 - 3xyz = 0$ except the trivial one $x = y = z$ are solutions of $x+y+z=0$. These may be obtained from (3:16) by setting $\lambda = \mu = \nu = 0$. Or it is seen by inspection that $x+y+z=0$ is satisfied by $x = \alpha$, $y = \beta$, $z = -(\alpha + \beta)$.

12/ Leonard E. Dickson, op. cit., p. 590.
CHAPTER IV

QUARTIC EQUATIONS

One function \( f \) which satisfies the conditions of Theorem 3:2 is the product of three linear factors in three unknowns. The next consideration will be an equation in which the left-hand member is the product of four linear factors in three unknowns.

Theorem 4:1. The equation

\[
(a_1 + a_2 x_1)(a_3 + a_4 x_2)(a_5 + a_6 x_3) = g(y),
\]

where \( g(y) = g(y_1, \ldots, y_p) \) is a homogeneous polynomial of degree \( n, n \not\equiv 0 \pmod{4} \), has solutions and every solution for which the members of (4:1) do not vanish is equivalent to either a solution given by

\[
x_j = A^{n-1} s^m t^n [t(a_1 a_2 + a_2 a_3) + s a_3 A_{x_j}],
\]

\[
y_k = A^n s^m t^n \beta_n,
\]

or a solution given by

\[
x_j = A^{n-1} s^m t^n [t(a_1 a_2 + a_2 a_3) + s a_3 A_{x_j}],
\]

\[
y_k = A^n s^m t^n \beta_n,
\]

according as \( n = 2m + 1 \) or \( n = 4m + 2 \), where \( A (\neq 0) \) is the determinant of the forms \( \sum_j a_{i_j} x_j \) (\( i = 1, 2, 3 \)) and \( A_{i_j} \) is the cofactor of \( a_{i_j} \) in \( A \), \( s \) and \( t \) are given by

\[
s = a_1 a_2 a_3 \sum_j a_{i_j} (a_1 A_{i_j} + a_2 A_{i_j}),
\]

\[
t = A g(\beta) - a_1 a_2 a_3 \sum_j a_{i_j} A_{i_j},
\]

the \( a_i \) and \( \beta_j \) being arbitrary integers.

Proof. Let

\[
\sum_j a_{i_j} x_j = A^n a_1 s^m t^n, \quad \sum_j a_{i_j} x_j = A^n a_2 s^m t^n \quad (i = 1, 2).
\]
The solution of this system of equations is
\[ (4:6) \quad x_j = A^{-1} s^{-1} t^{-1} \left[ t(a_i A_{ij} + \alpha_i A_{j}) + \beta_i A_{ij} \right] \]
and hence
\[ (4:7) \quad \sum_j x_j = A^{-1} s^{-1} t^{-1} \left[ t \sum_j a_{ij} (\alpha_i A_{ij} + \alpha_i A_{j}) + \beta_i \sum_j a_{ij} A_{ij} \right] \]

The theorem now divides itself into two cases. First let \( n = 2m + 1 \). If
\[ (4:8) \quad y_n = A^\ast s^\ast t^\ast \beta_n, \]
then from (4:5), (4:7), and (4:8), (4:1) becomes
\[ (4:9) \quad A^{n-1} s^{m+1} t^{m+2} \alpha_i \beta_j \left[ s \sum_j a_{ij} (\alpha_i A_{ij} + \alpha_i A_{j}) + \beta_i \sum_j a_{ij} A_{ij} \right] = A^n s^m t^m g(\beta), \]
which is identically satisfied in the \( \alpha \)'s and \( \beta \)'s if \( s \) and \( t \) are given by (4:4).

Second, let \( n = 4m + 2 \). If
\[ (4:10) \quad y_n = A^\ast s^\ast t^\ast \beta_n, \]
then (4:1) becomes (4:9) and hence as before \( s \) and \( t \) are given by (4:4).

Suppose now that \( x_j = \lambda_j \), \( y_n = \mu_n \) is any solution of (4:1). Choose \( \alpha_i = \sum_k a_{ik} \lambda_k \), \( \beta_k = \mu_k \), and then
\[
\begin{align*}
  s &= \left[ \sum_i \sum_k a_{ik} \lambda_i \right] \sum_k a_{ik} A_{ik} + a_{ik} A_{ik} \lambda_k \\
  t &= A \sum_i \sum_k a_{ik} \lambda_k - \sum_k a_{ik} \lambda_k \sum_k a_{ik} A_{ik} \\
  &= A \sum_i \sum_k \left( A a_{ik} - a_{ik} \sum_k a_{ik} A_{ik} \right) \lambda_k
\end{align*}
\]
and hence
\[
\begin{align*}
  s - t &= \left[ \sum_i \sum_k a_{ik} \lambda_i \right] \left( \sum_k a_{ik} A_{ik} + a_{ik} A_{ik} \lambda_k - A \sum_k a_{ik} A_{ik} \right) \\
  &= \left[ \sum_i \sum_k a_{ik} \lambda_i \right] A \sum_k a_{ik} - A \sum_k a_{ik} \lambda_k \\
  &= 0,
\end{align*}
\]
and \( s = t \). Therefore
\[ x_j = t \sum_i a_{ij} \lambda_i \]
\[ t \sum_{j} \sum_{e} a_{ij} A_{ej} \lambda_k = tA \lambda_k. \]

In the first case \( y_k = A^* t^* \lambda_k \) and in the second case \( y_k = A^* t^* \lambda_k \) and so in either case a solution is obtained that is equivalent to any solution.

It is evident that this could be extended to \( 2n \) factors in \( 2n - 1 \) variables with the proper restrictions on \( g(y) \).

A special case of (4:1) is

\[(x + y + z)(-x + y + z)(x - y + z)(x + y - z) = 4A^2,\]

which is the formula for expressing the area \( A \) of a triangle in terms of its sides \( x, y, z \). It is often reduced by a linear transformation \( \text{to} \)

\[ uvw(u + v + w) = t^2 \]

and then solved.

\[ \text{\textsuperscript{13}} \text{ Robert D. Carnichael, op. cit., p. 9.} \]
CHAPTER V

EQUATIONS OF DEGREE N

The theorems of Chapter II as well as Theorem 3:2 will now be generalized.

Suppose $f(x) = f(x_1, \ldots, x_n)$ is a homogeneous polynomial of degree $n$, with integral coefficients, and such that for $x_i = a_i$, $a_i$ integral and not all zero, all its partial derivatives of all orders less than $n - 1$ vanish. If $x_i = a_i s + a_i t$, then by Taylor's theorem

\[ f(x) = st^{n-1} \sum \frac{\partial^n f}{\partial x_i^n} \left( \frac{x_i}{s} \right)^i \frac{t}{s}. \]

Theorem 5:1. The equation

\[ f(x) = g(y), \]

where $g(y) = g(y_1, \ldots, y_m)$ is a homogeneous polynomial of degree $m$, has solutions, and every solution which is not also a solution of

\[ \sum a_i \frac{\partial f}{\partial x_i} = 0, \]

is equivalent to one of the solutions given by

\[ x_i = \alpha_i s t^{\lambda - 1} + \alpha_i t^{\lambda}, \]

\[ y_k = \beta_k t^{\mu}, \]

where $\lambda, \mu$ are relatively prime positive integers such that

\[ \frac{\lambda}{\mu} < 1. \]
\[ \lambda n = \mu m, \alpha, \beta, \gamma, \delta, \lambda, \mu, \nu, \xi, \eta, \zeta, \theta, \phi, \chi, \psi, \omega \text{ are arbitrary integers, and} \]
\[ s = g(\beta) - f(\alpha), \]
\[ t = \sum_{j=1}^{n} a_j \frac{\partial f}{\partial x_j}. \]

**Proof.** If in (5:1), \( x_i, y_k \) have the values given by (5:3), then
\[ st^{n-1} + t^{n-1} f(\alpha) = t^n g(\beta), \]
which is satisfied identically in the \( \alpha \)'s and \( \beta \)'s when \( s \) and \( t \) are given by (5:4). Hence (5:3) affords a solution of (5:1).

Suppose \( x_i = \alpha_i, y_k = \gamma_k \) is any solution of (5:1). If
\[ \alpha_i = \beta_i, \gamma_k = \gamma_i, s = 0 \] and (5:3) becomes \( x_i = \beta_i t^\lambda, y_k = \gamma_i t^\mu, \)
which is equivalent to the given solution provided \( x_i = \beta_i, \gamma_k = \gamma_i \) is not also a solution of (5:2). It will be noted that \( g \) may be identically zero.

**Corollary.** The equation
\[ f(x) = \sum_{j=1}^{n} x_j g_j(y) + g(y), \]
where \( g_j(y) = g_j(y_1, \ldots, y_n) \) and \( g(y) = g(y_1, \ldots, y_n) \) are homogeneous polynomials with integral coefficients of degrees \( n-1 \) and \( n \) respectively, has solutions, and every solution which is not also a solution of
\[ \sum_{j=1}^{n} a_j \frac{\partial f}{\partial x_j} = 0 \]
is equivalent to one of the solutions given by \( x_i = \alpha_i s + \beta_i t, \gamma_k = \gamma_i t, \)
where \( s = g(\beta) - f(\alpha), t = \sum_{j=1}^{n} a_j \frac{\partial f}{\partial x_j} - g_j(\beta) \).

**Theorem 5:2.** The equation
\[ f(x) \sum_{j=1}^{n} d_j x_j = R(y), \]
where \( f \) satisfies the hypothesis of Theorem 5:1 and \( R(y) = R(y_1, \ldots, y_n) \) is a homogeneous polynomial with integral coefficients, of degree \( n-1 \), has solutions and every solution
which is not also a solution of
\[ f(x) \sum_{j} a_j \frac{d^k}{dx_j^k} \left[ \sum_{j} a_j \frac{d^k}{dx_j^k} \sum_{j} d_j a_j \right] = 0 \]
is equivalent to one of the solutions given by
\[ x_i = a_i s + \alpha_i t, \]
\[ y_k = \beta_k t, \]
where
\[ s = A^{-1} \left[ \lambda (AD - BC) \right]^{-1} \left[ D \lambda^2 - BR(\mu) \right], \]
\[ t = A^{-1} \left[ \lambda (AD - BC) \right]^{-1} \left[ AR(\mu) - C \lambda^2 \right], \]
\[ \beta_k = A^{-1} \left[ \lambda (AD - BC) \right]^{-1} \mu_k, \]
where \( A = \sum a_j \frac{d^k}{dx_j^k} \), \( B = f(x) \), \( C = \sum d_j a_j \), \( D = \sum d_j a_j \), \( a_j, \lambda, \mu_k \) being arbitrary integers.

Proof. Let \( x_i, y_k \) have the values given by (5:7).
Then (5:5) becomes
\[ (As^{-1} + Bt^{-1})(Cs + Dt) = t^{-1} R(\beta), \]
which becomes, after dividing out the factor \( t^{-1} \),

16/ It will be shown later that \( t \neq 0 \).

\[ (As + Bt)(Cs + Dt) = R(\beta). \]
By Theorem 5:1 the solution of (5:9) is given by (5:8).

17/ It will be shown later that (5:9) satisfies the hypothesis of Theorem 5:1.

Suppose \( x_i = \rho_i, y_k = \sigma_k \) is any given solution of (5:5).
Then if \( \xi = \rho_i, \mu_k = \sigma_k, \lambda = f(\rho), s = 0 \) and the solution becomes \( x_i = \rho_i K^{-1}, y_k = \sigma_k K^{-1} \) where \( K = A \lambda (AD - BC) \), which is proportional to the given solution provided \( K \neq 0 \); that is, provided \( x_i = \rho_i, y_k = \sigma_k \) is not a solution of (5:6). It will be noted that if \( K \neq 0 \), then \( t \neq 0 \).
One function of interest which satisfies the hypothesis of Theorem 5:1 is the function $D(x) = \left| a_{ij} x_{ij} \right|$, a determinant of order $n$ with $a_{ij}$ integral such that not all the $a$'s in any row or column are zero. If there is any element, say $x_{k*}$, which is distinct from all others, we may select $x_{k*} = 1$ and $x_{ij} = 0$ otherwise. For this choice all partial derivatives of all orders less than $n - 1$ vanish. It is interesting to note that in the solution, the form of the expression for all the $x$'s is the same except for $x_{k*}$. This is illustrated by

$$D = \begin{vmatrix} x & 3y & z \\ u & 7v & 3w & p \\ q & u & 3x & z \\ 0 & 2w & 5z & 0 \end{vmatrix} = r^n,$$

the solution of which is $x = \alpha t^n$, $y = \beta t^n$, $z = \gamma t^n$, $u = \lambda t^n$, $v = st^{-1} + \mu t^n$, $w = \nu t^n$, $p = \rho t^n$, $q = \sigma t^n$, $r = \tau t^n$, where $s = \Pi^n - D'$, $t = \Delta$, $D'$ being $D$ with $x, y, z, u, v, w, p, q$, replaced by $\alpha, \beta, \gamma, \lambda, \mu, \nu, \rho, \sigma$ and

$$\Delta = \begin{vmatrix} \alpha & 0 & \gamma & 0 \\ \lambda & 7 & 3\nu & \rho \\ \sigma & 0 & 3\mu & \gamma \\ 0 & 0 & 5\nu & 0 \end{vmatrix}.$$

It is not necessary, however, that there be one element which is distinct from all others in order that this method apply to certain problems. For example, $D$ may be the circulant, as it was in Theorem 3:2.

Another function which also satisfies the hypothesis of Theorem 5:1 is the function $\prod \sum_j a_{ij} x_j$, where $a_{ij}$ are integers and the determinant $A = |a_{ij}| \neq 0$. For $x_j$ may be chosen
integral, so that \( n - 1 \) of the above factors vanish and hence for this choice of \( x_j \), all partial derivatives of all orders less than \( n - 1 \) vanish. Hence the equation

\[
\prod_j^n \sum_j^a_{i,j} x_j = g(y),
\]

where \( g(y) \) is a homogeneous polynomial with integral coefficients, of degree \( m \), may be solved by means of Theorem 5:1. This equation may, however, be solved by an entirely different method. The method will be illustrated by solving a more general problem.

Let \( P(x) = \sum_k^n \sum_j^a_{i,j,k} x_j \), where \( a_{i,j,k} \) are integers and the determinant \( A_k \), for each \( k \), of the forms \( \sum_j^a_{i,j,k} x_j \) is not zero. Without loss of generality, the notation may be so chosen that \( A_{n,n,k} \neq 0 \), where \( a_{i,j,k} \) is the cofactor of \( a_{i,j,k} \) in \( A_k \). Let \( A_{n,n,k}^{(j)} \) be the determinant obtained from \( A_{n,n,k} \) by replacing the \( j \)th column by \( a_{i,j,k} \), \( a_{i,k} \), \ldots, \( a_{n,k} \), \( R \) the least common multiple of \( A_{n,n,k}^{(j)} \) for each \( k \), and \( R_k = R/A_{n,n,k} \).

Theorem 5:3. The equation

\[
(5:10) \quad P(x) = g(y),
\]

where \( g(y) \) is a homogeneous polynomial with integral coefficients, of degree \( m \), has solutions and every solution which is not also a solution of

\[
(5:11) \quad \sum_k^n \sum_j^a_{i,j,k} x_j R_k x_{n,k} A_k = 0,
\]

is equivalent to one of the solutions given by

\[
(5:12) \quad x_{j,k} = t^j R^{\lambda_j} R_k A_{n,n,k}^{(j)} + st^\lambda R^{\lambda_j} R_k A_{n,n,k} \quad (j = 1, \ldots, n - 1)
\]

\[
(5:12) \quad x_{n,k} = st^\lambda R^\lambda \beta_k \quad (k = 1, \ldots, p)
\]

\[
(5:12) \quad y_j = \delta_j t^\mu R^\mu,
\]

where
and the \( \alpha' \)'s, \( \beta' \)'s, and \( \gamma' \)'s are arbitrary integers, \( \lambda, \mu \), are relatively prime positive integers such that \( \lambda \mu = \mu \). 

Proof. Let 
\[
(5:14) \quad \sum \alpha_{j,k} x_{j,k} = t^R R^\alpha \quad (1 = 1, \ldots, n - 1) \quad (k = 1, \ldots, p)
\]
and set 
\[
(5:15) \quad x_{n,k} = st^\lambda R^\beta \quad (k = 1, \ldots, p)
\]
Now write (5:14) as 
\[
(5:16) \quad \sum \alpha_{j,k} x_{j,k} = t^R R^\alpha - st^\lambda R^\beta a_{n,k}.
\]

Solving (5:16) gives 
\[
(5:17) \quad x_{j,k} = t^R R^\alpha R^\beta A_{n,k} - st^\lambda R^\beta a_{n,k}.
\]

and hence 
\[
(5:18) \quad \sum \alpha_{j,k} x_{j,k} = t^R R^\alpha R^\beta R_k\left[ t \sum \alpha_{j,k} A_{n,k} + s \beta A_k \right].
\]

If \( y_j = t^R R^\alpha \gamma_j \), then by use of (5:14) and (5:18), (5:10) becomes 
\[
\sum \alpha_{j,k} x_{j,k} = t^R R^\alpha R^\beta R_k \left[ t \sum \alpha_{j,k} A_{n,k} + s \beta A_k \right] = t^R R^\alpha R^\beta g(\gamma)
\]
which is identically satisfied in the \( \alpha' \)'s, \( \beta' \)'s, and \( \gamma' \)'s if \( t \) and \( s \) are given by (5:13). Hence (5:12) is a solution of (5:10).

Suppose now that \( x_{j,k} = \beta_{n,k} \), \( y_j = \gamma \) is a given solution of (5:10). 

Choose \( \alpha_{j,k} = \sum \alpha_{j,k} \beta_{n,k} \), \( \beta_{n,k} = \beta_{n,k} \), \( \gamma_j = \gamma \) then

\[18/\]

\( A_{n,k}^{(i)} \) becomes \( \beta_{n,k} \) when \( \alpha_{j,k} \) is replaced by \( \sum \alpha_{j,k} \beta_{n,k} \). 

\[
\begin{align*}
t &= \sum \alpha_{j,k} R \sum \alpha_{j,k} \beta_{n,k} R_k. \\
s &= Rg(\gamma) - \sum \alpha_{j,k} R \sum \alpha_{j,k} A_{n,k}^{(i)} = \sum \alpha_{j,k} R \sum \alpha_{j,k} (\beta_{n,k} A_{n,k} - \beta_{n,k} A_{n,k}).
\end{align*}
\]
Let \( f(x,y) = f(x_1, \ldots, x_r; y_1, \ldots, y_s) \) be polynomials with integral coefficients, homogeneous in both \( x \) and \( y \). Suppose that \( f(x,y) \) is of degree \( m \) in \( x \), \( q \) in \( y \), and \( g(x,y) \) is of degree \( n \) in \( x \) and \( p \) in \( y \), where \( m, n, p, q \) are positive integers and \( m - n \) is relatively prime to \( p - q > 0 \). Since \( m - n \) and \( p - q \) are relatively prime, there exist non-negative integers \( u, v \), \( u \leq p - q \), \( v \leq m - n \), such that \( (m - n)u = (p - q)v + 1 \) and \( (m - n)(p - q - u) = (p - q)(m - n - v) - 1 \).

**Theorem 5:4.** The equation

\[
19/ \quad f(x,y) = g(x,y)
\]

has solutions, and every solution for which the members do vanish, is equivalent to one of the solutions given by
where \( \alpha_i, \beta_j \) are arbitrary integers.

**Proof.** Let \( x_i = \alpha_i \cdot s_i \cdot x^{n-i}, \ y_j = \beta_j \cdot s_j \cdot x^{m-j}. \) Then (5:20) becomes
\[
s^{m-n} \cdot s^{n-m} \cdot x^{n-m} \cdot \left[ f(\alpha, \beta) = s^{m-n} \cdot s^{n-m} \cdot x^{n-m} \cdot g(\alpha, \beta) \right],
\]
which is satisfied identically in the \( \alpha_i \) and \( \beta_j \) if \( s = g(\alpha, \beta), \)
\( t = f(\alpha, \beta). \) Hence (5:21) is a solution of (5:20).

If \( x_i = \lambda_i, \ y_j = \mu_j \) is a given solution of (5:20) then
\( f(\lambda, \mu) = g(\lambda, \mu). \) If \( \alpha_i = \lambda_i, \ \beta_j = \mu_j \) then \( x_i = \lambda_i \cdot [f(\lambda, \mu)]^{m-n}, \)
\( y_j = \mu_j \cdot [f(\lambda, \mu)]^{n-m}, \) which is equivalent to the given solution provided \( f(\lambda, \mu) \neq 0. \)

The proof of Theorem 5:4 is the same if \( m - n < 0 \) and \( p - q < 0. \) The method does not apply, however, in the cases \( m - n > 0, p - q < 0 \) and \( m - n < 0, p - q > 0. \)

Suppose now that for the functions \( f \) and \( g \) considered above, \( f \) is of degree \( m \) in \( x, \ -p \) in \( y; \) \( g \) is of degree \( n \) in \( x, \ -q \) in \( y; \) \( m, n, p, q, \) are positive integers and \( m - n > 0 \) is relatively prime to \( p - q > 0. \) As before, non-negative integers \( u, v \) exist such that \( (m - n)u = (p - q)v + 1 \) and
\( (m - n)(p - q - u) = (p - q)(m - n - v) - 1. \)

**Theorem 5:5.** The equation
\[
(5:22) \quad f(x, y) = g(x, y)
\]
has solutions and every solution for which the members of (5:22) do not vanish is equivalent to a solution given by
\[
(5:23) \quad x_i = \alpha_i \cdot [A(\beta) \cdot B(\beta)]^{m-n} \cdot [f(\alpha, \beta)]^{m-n}, \quad y_j = \beta_j \cdot [A(\beta) \cdot B(\beta)]^{n-m} \cdot [f(\alpha, \beta)]^{n-m},
\]
where \( A(\beta) = \prod \beta_j, \ B(\beta) = \prod \beta_j, \) and the \( \alpha_i's \) and \( \beta_i's \) are...
arbitrary integers.

Proof. If (5:22) is multiplied by $A(y)B(y)$ it becomes

$$ (5:24) \quad A(y)B(y)f(x, y) = A(y)B(y)g(x, y), $$

each member of which is a polynomial with integral coefficients. If $x_i = \alpha_i s^t t^{m-n} y_j = \beta_j s^t t^{m-n}$, then on substituting in (5:24) it follows that

$$ s^t \alpha_i s^t + m-n \cdot v + \frac{\alpha_i}{\beta_j} \left( m-n \cdot v + \frac{\alpha_i}{\beta_j} \right) A(\beta)B(\alpha)f(\alpha, \beta) = s^t \alpha_i s^t + m-n \cdot v + \frac{\alpha_i}{\beta_j} \left( m-n \cdot v + \frac{\alpha_i}{\beta_j} \right) A(\beta)B(\alpha)g(\alpha, \beta), $$

which is satisfied identically in the $\alpha_i$'s and $\beta_j$'s if

$s = A(\beta)B(\beta)g(\alpha, \beta), \quad t = A(\beta)B(\beta)f(\alpha, \beta)$. Hence (5:23) is a solution of (5:22).

If $x_i = \lambda_i, y_j = \mu_j$ is a given solution of (5:22) then

$f(\lambda, \mu) = g(\lambda, \mu)$. If $\lambda_i = \lambda_i, \beta_j = \mu_j$ the solution becomes

$$ x_i = \lambda_i [A(\mu)B(\mu)f(\lambda, \mu)]^{\gamma-\delta}, \quad y_j = \mu_j [A(\mu)B(\mu)g(\lambda, \mu)]^{\gamma-\delta}, $$

which is equivalent to the given solution provided $f(\lambda, \mu) \neq 0$. It is necessary that $\mu_j \neq 0$.

It will be noted that if in Theorem 5:5, $m - n > 0$ and $p - q < 0$, then $u, v$ may be chosen as positive integers such that $(m - n)u = (q - p)v + 1$ and $(m - n)(q - p - u) =$

$(q - p)(m - n - v) - 1$ and the solution in this case is

$$ x_i = \alpha_i [A(\beta)B(\beta)]^{\gamma-\delta} \left[ f(\alpha, \beta) \right]^{\gamma-\delta} \left[ g(\alpha, \beta) \right]^{\gamma-\delta}, $$

$$ y_j = \beta_j [A(\beta)B(\beta)]^{\gamma-\delta} \left[ f(\alpha, \beta) \right]^{\gamma-\delta} \left[ g(\alpha, \beta) \right]^{\gamma-\delta}. $$

If in Theorem 5:4 $f$ is of degree $-m$ in $x$, $-q$ in $y$ and $g$ is of degree $-n$ in $x$ and $-p$ in $y$ the solution is

$$ x_i = \alpha_i [A(\beta)B(\beta)C(\alpha)D(\alpha)]^{\gamma-\delta} \left[ f(\alpha, \beta) \right]^{\gamma-\delta} \left[ g(\alpha, \beta) \right]^{\gamma-\delta}, $$

$$ y_j = \beta_j [A(\beta)B(\beta)C(\alpha)D(\alpha)]^{\gamma-\delta} \left[ f(\alpha, \beta) \right]^{\gamma-\delta} \left[ g(\alpha, \beta) \right]^{\gamma-\delta}, $$

where $C(\alpha) = \sum a_{ij}^m, \quad D(\alpha) = \sum a_{ij}^p$. 

The above methods may be extended to solve certain types of non-homogeneous equations, as is illustrated in the next theorem.

**Theorem 5:6.** The equation

\[(5:25) \quad f(x, y) = g(x, y),\]

where \(f(x, y) = \sum_{i=1}^{m} a_i \prod_{j=1}^{p} x_j^{\alpha_{ij}} y_j^{\beta_{ij}}, \ g(x, y) = \sum_{i=1}^{m} b_i \prod_{j=1}^{p} x_j^{\delta_{ij}} y_j^{\gamma_{ij}}, \ a_i, \ b_i \) are integers, \(\alpha_{ij}, \beta_{ij}, \delta_{ij}, \gamma_{ij}\) being positive integers, has solutions if there exists positive integers \(u_i, v_i, w_i, z_i, M, N\) such that

\[(5:26) \quad \sum_{i=1}^{m} (\alpha_{ij} u_i + \beta_{ij} w_i) = M,
\]

\[(5:26) \quad \sum_{i=1}^{m} (\delta_{ij} v_i + \gamma_{ij} z_i) = N,
\]

and every solution for which the members of (5:25) do not vanish, is equivalent to one of the solutions given by

\[(5:27) \quad x_i = \alpha_i s^{m_i} t^{n_i}, \quad y_i = \beta_i s^{n_i} t^{m_i},
\]

where

\[(5:28) \quad s = f(\alpha, \beta), \quad t = g(\alpha, \beta),
\]

and the \(\alpha_i's\) and \(\beta_i's\) are arbitrary integers.

**Proof.** If \(x_i, y_i\) have the values given by (5:27) then by (5:26), (5:25) becomes \(s^m t^{n+1} f(\alpha, \beta) = s^{n+1} t^n g(\alpha, \beta)\), which is satisfied identically in the \(\alpha_i's\) and \(\beta_i's\) if \(s, t\) are given by (5:28).

If (5:26) is satisfied, there exist non-negative integers \(K, u_i', w_i'\) such that

\[(5:29) \quad \sum_{i=1}^{m} (\alpha_{ij} u_i' + \beta_{ij} w_i') = K, \quad (i = 1, \ldots, m),
\]

\[(5:29) \quad \sum_{i=1}^{m} (\gamma_{ij} v_i' + \delta_{ij} z_i') = K, \quad (h = 1, \ldots, p).
\]

If \(\lambda_i = \lambda; \ y_i = \mu;\) is any integral solution of (5:25) and
there are no integers $u'_i, w'_i$ satisfying (5:29), $d > 1$, and

$\lambda'_i, \mu'_i$, such that $\lambda_i = \lambda'_i d^{u'_i}$, $\mu_i = \mu'_i d^{w'_i}$, then $x_i = \lambda_i$, $y_i = \mu_i$, is said to be a primitive solution of (5:25).

If $x_i = \lambda_i$, $y_i = \mu_i$ is a primitive solution of (5:25), then

$x_i = \lambda_i d^{u_i}, y_i = \mu_i d^{w_i}$, derived from the primitive solution, where $d \neq 0$ is an integer and $u'_i, w'_i$ are non-negative integers satisfying (5:29), is also a solution. Two solutions are said to be equivalent if they may be derived from the same primitive solution.

With this definition we may show that for any solution of (5:25) for which the members do not vanish, an equivalent solution is given by (5:27).

If $x_i = \lambda_i$, $y_i = \mu_i$ is any given solution of (5:25) and the choice $\lambda_i = \lambda_i, \beta_i = \mu_i$ is made then

\begin{equation}
(5:30) \quad x_i = \lambda_i \left[ f(\lambda_i, \mu_i) \right]^{u_i + w_i}, \quad y_i = \mu_i \left[ f(\lambda_i, \mu_i) \right]^{w_i + z_i}.
\end{equation}

Since

\begin{align*}
\sum_i \left[ \alpha_{ij} (u_j + v_j) + \beta_{ij} (z_j + w_j) \right] \\
= \sum_i \left[ \delta_{ij} (u_j + v_j) + \beta_{ij} (z_j + w_j) \right] \\
= M + N + 1,
\end{align*}

the solution is equivalent to the given solution if $f(\lambda_i, \mu_i) \neq 0$.

Theorem 5:6 may be extended, as was Theorem 5:4, to cover equations in which the members are no longer polynomials. The method will be illustrated by a particular example.

Consider

\begin{equation}
\frac{ax^3 u^6}{v^7} + \frac{by^2 v^5}{u^4} = \frac{cx^2}{w}
\end{equation}

which may be written as

\begin{equation}
ax^3 u^7 w + by^2 v^5 w = cx^2 u^2 v^7.
\end{equation}
The solution of this equation is \( x = \alpha^t, \ y = \beta^t, \ u = \gamma t, \ v = \mu t, \ w = \nu t \) where \( s = a \alpha^t + b \beta^t + \gamma t, \ t = c \alpha^t + \beta^t \). It is necessary that \( u, v, \) and \( w \) be different from zero.

If \( x = x', \ y = y', \ u = u', \ v = v', \ w = w' \) is a solution of the above equation and the choice \( \alpha = x', \ \beta = y', \ \gamma = u', \ \mu = v', \ \nu = w' \) is made then \( s = t \) and the solution becomes \( x = x't, \ y = y't, \ u = u't, \ v = v't, \ w = w't \) which, by a natural extension of the definition of equivalence, is equivalent to the given solution.

Two final theorems will now be proved. Both of these theorems are included in the above theory, but because of their interest they will be solved in detail.

Theorem 5:7. The equation

\[
\sum_{i}^{n} \frac{a_i}{x_i} = \frac{c}{y_k}
\]

where all the letters represent integers and \( x_i, y_k \) are unknown, has integral solutions and every solution is equivalent to one of the solutions given by

\[
x_i' = a_{ij}C(\alpha)D(\beta), \quad y_k' = \beta_k D(\beta) \sum_{j}^{n} \frac{a_{ij}A_j(\alpha)B(\alpha)}{B_j(\alpha)}
\]

where \( A_j(\alpha) = \sum_{i}^{n} A_{ij}(\alpha), \ B_j(\alpha) = \sum_{i}^{n} B_{ij}, \ C(\beta) = \sum_{k}^{p} c_k C(\beta), \ D(\beta) = \sum_{k}^{p} d_k D(\beta), \) and \( a_{ij}, \ \beta_k \) (\( \neq 0 \)) are

20/ This theorem is to be published in the National Mathematics Magazine at a future date.

21/ Values of the coefficients which will make some denominator vanish are excluded.
Proof. Equation (5:31) may be written in the form

\[(5:33) \quad D(y) \sum_{j} a_j A_j(x) B(x) = gC(y)B(x).\]

If \(x_{ij} = \lambda_i t, y_k = \mu_k s,\) (5:33) becomes

\[s^{-1} t^{-n-1} D(\beta) \sum_{j} a_j A_j(\alpha) B(\alpha) = s^{-1} t^{-n-1} gC(\beta)B(\alpha),\]

which is satisfied identically in the \(\alpha^j\) and \(\beta^k\) if

\[s = D(\beta) \sum_{j} a_j A_j(\alpha) B(\alpha),\]

\[t = gC(\beta)B(\alpha).\]

Hence a solution of (5:31) is given by (5:32).

Let \(x_{ij} = \lambda_i t, y_j = \mu_j\) be any solution of (5:31). If \(x_{ij} = \lambda_i t, y_k = \mu_k t\) then \(s = t\) and the solution becomes \(x_{ij} = \lambda_i t, y_k = \mu_k t,\) which is equivalent to the given solution provided \(t \neq 0.\) But if \(t = 0,\) then \(B(\lambda) = \sum \lambda_j B_j(\lambda) = 0\) and hence some \(B_j(\lambda) = 0.\) But if some \(B_j(\lambda) = 0,\) then \(\sum \lambda_j = 0\) since \(A_j(\lambda) \neq 0.\) This is impossible from (5:31).

If \(a_i = c = 1, a_x = a_e = \ldots = 0\) (5:31) reduces to

22/ Leonard E. Dickson, op. cit., pp. 688 - 691.


\[(5:34) \quad f(x) = g(y),\]

where \(f(x) = \sum \frac{b_i}{x_{ij}}, g(y) = \sum \frac{b_j}{y_{ij}},\) all the letters represent integers, and \(x_{ij}, y_j\) are unknown, has integral solutions and every solution for which the members do not
vanish is equivalent to one of the solutions given by

$$x_{ij} = \alpha_{ij} A(\alpha) B(\beta) f(\lambda), \quad y_{ij} = \beta_{ij} [A(\alpha) B(\beta)]^{\mu} g(\beta) [f(\lambda)]^{\mu},$$

where $A_i(\alpha) = \prod_{j} a_{ij}$, $A(\alpha) = \prod_j A_i(\alpha)$, $B(\beta) = \prod_j B_j$, and $\alpha_{ij}, \beta_{ij} (\neq 0)$ are arbitrary integers.

Proof. Equation (5:34) may be written in the equivalent form

$$A(x) B(y) f(x) = A(x) B(y) g(y).$$

If $x_{ij} = \alpha_{ij} t$, $y_{ij} = \beta_{ij} s t^{m-1}$, then (5:36) becomes

$$s r_{t} = \prod_{n} p_{n+1}^{t m} A(\alpha) B(\beta) f(\lambda) = s r_{t} = \prod_{n} p_{n+1}^{t m} A(\alpha) B(\beta) g(\beta),$$

which is satisfied identically in the $\alpha$'s and $\beta$'s if

$s = A(\alpha) B(\beta) g(\beta), \quad t = A(\alpha) B(\beta) f(\lambda).$ Hence a solution of (5:34) is given by (5:35).

If $x_{ij} = \lambda_{ij}, \quad y_{ij} = \mu_{ij}$ is a solution of (5:34) and the choice $\alpha_{ij} = \lambda_{ij}, \beta_{ij} = \mu_{ij}$, is made, $s = t$ and the solution becomes

$x_{ij} = \lambda_{ij} t, \quad y_{ij} = \mu_{ij} t$ which is equivalent to the given solution provided $t \neq 0$. But since $\lambda_{ij}, \mu_{ij} \neq 0, \quad t = A(\lambda) B(\mu) f(\lambda) = 0$ if and only if $f(\lambda) = 0.
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Volume II of Dickson's History of the Theory of Numbers is a comprehensive listing of all the important results and methods in Diophantine analysis up to its time of publication.
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The author was born on May 14, 1910 in Houston, Texas. He received his primary education from the parochial schools of Houston, Texas and was graduated in 1930 from San Jacinto High School of that city. He matriculated at Rice Institute in 1930, graduating, with honors in mathematics, in 1934. In 1936 he received the M. A. degree in mathematics from Rice Institute. In 1936 he entered Louisiana State University, where he is at present enrolled.
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