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A Necessary and Sufficient Condition That a Set Be Homeomorphic to a Plane Region Bounded by a Finite Number of Nonintersecting Circles.

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A NECESSARY AND SUFFICIENT CONDITION THAT A SET BE HOMEOMORPHIC TO A PLANE REGION BOUNDED BY A FINITE NUMBER OF NONINTERSECTING CIRCLES

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

The Department of Mathematics

by

Robert L. Broussard
B.S., Louisiana State University, 1944
August, 1951
MANUSCRIPT THESSES

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ABSTRACT

A number of papers have been written concerning the characterization of sets homeomorphic with a subset (proper or improper) of the plane or sphere. Among these are R. L. Moore, who gives several characterizations of the plane, Leo Zippin, R. L. Wilder, D. W. Hall, G. S. Young and R. H. Bing.

The main result of this paper is the following theorem: A necessary and sufficient condition that a space $S$ be homeomorphic with a closed, compact, connected subset of the plane bounded by a finite number, $n$, of nonintersecting circles, where $n \geq 2$, is that $S$ be a nondegenerate, compact, continuous curve containing a collection $\mathcal{C}$ of $n$ nonintersecting simple closed curves, such that $S$ is not separated by any pair of points or by any element of $\mathcal{C}$, but $S$ is separated by any simple closed curve which is not an element of $\mathcal{C}$. The necessity of this condition being obvious, it is only necessary to prove the sufficiency of the condition.
The proof of the sufficiency is divided into four parts. In the first part is proved the theorem:

Let $M$ be a locally compact, connected, metric space which can be covered by a finite number of connected open sets of diameter less than $\epsilon$ when $\epsilon$ is positive. Suppose further that $M$ cannot be separated by the omission of any pair of points.

Let $C$ be a compact subset of $M$ and let $x$ and $y$ be common limit points of two components $D_A$ and $D_B$ of $M-C$ such that:

(1) $C$ is locally connected at $x$ and at $y$;

(2) There is a positive $\delta_1$ such that $C - \{C_x, \delta + C_y, \delta\}$ has a finite number of components when $\delta \leq \delta_1$ ($C_x, \delta$ is the component of $C - U(x, \delta)$ which contains $x$).

Then given a positive number $\epsilon_0$, there are:

(1) a $\delta_0$ less than or equal to $\delta_1$;

(2) connected open supersets $U_{\epsilon_0}(C_x, \delta_0)$ and $U_{\epsilon_0}(C_y, \delta_0)$ of $C_x, \delta_0$ and $C_y, \delta_0$ which do not intersect $C_1$ where $C_1 = C - \{C_x, \delta_0 + C_y, \delta_0\}$; and

(3) a simple closed curve $J$ in $D_A + D_B + U_{\epsilon_0} C_x, \delta_0 + U_{\epsilon_0} C_y, \delta_0$ such that $J$ intersects $C_x, \delta_0$.
Gy, s, D, and D, and J does not separate any point of M-J from C.

The proof of this theorem follows closely Bing's proof of the Kline sphere characterization problem.

In the second section it is shown that if C, C,, ..., C, are the elements of \( \mathcal{W} \), then \( S - \bigcup_{i=1}^{n} C_i \) is homeomorphic to a subset of the plane. This is accomplished by showing that \( S - \bigcup_{i=1}^{n} C_i \) is a Peano space which contains at least one simple closed curve and that every simple closed curve, but no arc separates \( S - \bigcup_{i=1}^{n} C_i \).

In the third section it is shown that S does not contain any primitive skew curve of the first or second types and hence, by a theorem of Claytor, S is homeomorphic to a subset of the plane. In the last section it is shown that S is homeomorphic to a region of the plane bounded by n nonintersecting circles.
INTRODUCTION

A number of papers have been written concerning the characterization of sets homeomorphic with a subset (proper or improper) of the plane or sphere.

In [11], R. L. Moore gives three systems of axioms for plane topology and in [12] he proves that the spaces determined by his systems are really homeomorphic with the plane. In [13], Miss Gauhm proves that certain conditions will define a 2-dimensional manifold without boundary among arbitrary Hausdorff spaces. In [16], Leo Zippin shows that in locally compact, locally connected, connected spaces satisfying the Janiszewski theorem, the nondegenerate cyclic elements are homeomorphic with a 2-sphere (or a region of a 2-sphere).

In [10], E. R. van Kampen shows that a P-space which contains at least one simple closed curve and which is separated by every simple closed curve but by no closed

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1 See appendix for footnotes.
are is homeomorphic with a region on a sphere. In [5] S. Claytor shows that a necessary and sufficient condition that a P-continuum K be homeomorphic with a subset of a spherical surface is that:

1. K does not contain any primitive skew curve of type one or type two;
2. each cut point P of K is a boundary point of the closure of every component of K - P.

In connection with this theorem, Hall in [6] shows that if M is a locally connected continuum which is separated by no pair of its points and contains no primitive skew curve of type one, then M contains no primitive skew curve of type two. Further characterizations of the sphere and regions of the sphere are given by Wilder [13, 14] and Zippin [17, 18].

In [8, 9] Hall gives a partial solution to the problem of J.R. Klein: Is a nondegenerate, locally connected, compact continuum which is separated by each of its simple closed curves but by no pair of its points homeomorphic with the surface of a sphere? In [4] Bing gives a complete solution to this problem. In [15] Young gives a simple characterization of 2-manifolds, with or without boundaries, using Bing's solution of the
Kline problem as a tool. For further bibliographical references to other work of this nature see [4, 10].

The purpose of this paper is to prove the following theorem: A necessary and sufficient condition that a space $S$ be homeomorphic with the closed, connected, compact subset of the plane bounded by a finite number, $n$, of nonintersecting circles, when $n \geq 1$, is that $S$ be a nondegenerate, compact, continuous curve containing a collection $\mathcal{N}$ of $n$ nonintersecting simple closed curves, such that $S$ is not separated by any pair of points or by any element of $\mathcal{N}$, but $S$ is separated by any simple curve which is not an element of $\mathcal{N}$.

It is easily seen that this condition is necessary. The problem has been handled nicely by Bing in [4] when $\mathcal{N}$ is empty. In the case that $\mathcal{N}$ is not empty, it is easily seen that $S$ cannot be homeomorphic to the entire 2-sphere, and hence if it is homeomorphic to a subset of the sphere, it is homeomorphic to a bounded subset of the plane.

The proof of the sufficiency of this condition is given in four sections. In the first section is given a generalization of an argument used by Bing [4]. This generalization provides a valuable tool for dealing with certain sets which are separated by all but a finite num-
ber of simple closed curves, but which are not separated by any pair of points. In the second section it is shown that if $C_1, C_2, \ldots, C_n$ are the elements of $\mathcal{N}$, then $S = \sum_{i=1}^{n} C_i$ is homeomorphic to a subset of the plane. In the third section, it is shown that $S$ is homeomorphic to a subset of the plane and in the fourth section, it is shown that $S$ is homeomorphic to a region, i.e. a closed, connected, compact subset, of the plane bounded by $n$ nonintersecting circles.

In the arguments of the first section, the word disrupt is used several times. It shall be said that the point set $M$ disrupts $x$ from $y$ in $D$ if there is an arc from $x$ to $y$ in $D$ but each such arc contains a point of $M$. Also, extensive reference is made to Bing's lemma which is given below.

**Bing's lemma:** Suppose that space is locally connected and cannot be separated by the omission of any pair of its points, that the boundary of the connected domain $D$ is equal to the sum of the mutually exclusive sets $M$, $N$ and $E$, each of which is accessible form $D$, and that $D'$ is a connected subdomain of $D$ such that no point of $D$ either disrupts $D'$ from $E+M$ in $D+E+M$ or disrupts $D'$ from $E+N$ in $D+E+N$. Then there is an open arc from $M$ to $N$ in $D$ that does not
disrupt $D'$ from $E$ in $D+E$.

The following notations are used throughout the paper:

- $S$ is a set satisfying the hypothesis of the problem;
- $\mathcal{T}$ is a collection of $n$ nonintersecting simple closed curves such that no element of $\mathcal{T}$ separates $S$;
- $C_i$ ($1 \leq i \leq n$) is an element of $\mathcal{T}$;
- $x, y$ and $z$ are always points of $S$;
- $U(x)$ is any open subset of $S$ which contains $x$;
- $U(x, \delta)$ is the set of all points of $S$ whose distance from $x$ is less than $\delta$;
- $U_{x, \delta}$ is the component of $U(x, \delta)$ which contains $x$;
- $C_{x, \delta}$ (where $C$ is a subset of $S$) is the component of $C-U(x, \delta)$ which contains $x$;
- $C_1'$ is the set $C-(C_{x, \delta}+C_{y, \delta})$ where $x$ and $y$ are two points of $C$;
- $U_{x, \delta}$ is an open set containing $C_{x, \delta}$ which is of diameter less than $\varepsilon$ but which does not intersect $C_1'$;
- $xy$ is an arc from $x$ to $y$ including the endpoints $x$ and $y$;
- $\langle xy \rangle$ is the open arc $xy-(x+y)$.

All other notations and theorems used are conventional.
§1. Lemma: Let $M$ be a locally connected metric space, and let $L$ be a closed subset of $M$. Let $D$ be a connected open subset of $M$ containing a single component $L_i$ of $L$, such that $D$ is compact. If $M_1$ is a component of $M-L$ then $M_1 \cdot D$ has a finite number of components.

Proof: If $M_1 \subseteq D$ then the conclusion is obvious.

Suppose then, that $M_1$ is not a subset of $D$.

The proof will consist of four steps.

(1) To show that every component of $M_1 \cdot D = M_1 \cdot D \cdot L_i$ is open.

(2) To show that every component of $M_1 \cdot D$ has a limit point on $L_i$.

(3) To show that every component of $M_1 \cdot D$ has a limit point on $M_1 \cdot L_i$.

(4) To show that, in the light of (2) and (3), the assumption that $M_1 \cdot D$ has an infinite number of components contradicts (1).
Let $D^* = D \cdot M_1$.

(1) Let $x$ be any point of a component $D_{b_1}$ of $D^*-L$. Now $D^*-L$ is an open subset of a locally connected metric space and thus it is locally connected. Therefore there is a neighborhood $U$ of $x$ which is a subset of $D_{b_1}$, and $D_{b_1}$ is open.

(2) Suppose $D_{b_1} \cdot L_1 = \phi$. Set $D = D_{b_1} + [(D-D^*) + (D^* - D_{b_1} - L_1)] + L_1$.

Consider $K = D_{b_1} \cdot [(D-D^*) + (D^* - D_{b_1} - L_1)] + L_1$

Also, $D_{b_1} \subseteq M_1$, $D - D^* \subseteq M - M_1$, $D - D^* \subseteq D - D^* + L_1 + D - M - D$.

Therefore, $D_{b_1} \subseteq M_1$, and $D_{b_1} \cdot [D - D^*] + D_{b_1} \cdot [D - D^*] = \phi$.

Now $D_{b_1}$ is a component of $D^* - L_1$ and hence $D_{b_1} \cdot (D^* - L_1 - D_{b_1}) = \phi$.

Also, from part 1, no point of $D_{b_1}$ is a limit point of $M - D_{b_1}$. Therefore, $D_{b_1} \cdot (D^* - L_1 - D_{b_1}) = \phi$. Thus $K = \phi$ and $D$ has a partition. This is a contradiction.

(3) Suppose $D_{b_1} \cdot (M_1 - D) = \phi$. Then $M_1 = (M_1 - D^*)$.

Consider $K = D_{b_1} \cdot [(M_1 - D^*) + (D^* - L - D_{b_1})] + L_1$

As above $D_{b_1} \cdot (D^* - L - D_{b_1}) = \phi$. By supposition $D_{b_1} \cdot (M_1 - D^*) = D_{b_1} \cdot (M_1 - D) = \phi$. Now $D_{b_1} \subseteq D^*$, and hence, $M_1 - D^* \subseteq M - D_{b_1}$. But no point of $D_{b_1}$ is a limit point of $M - D_{b_1}$.

Therefore $D_{b_1} \cdot (M_1 - D^*) \subseteq D_{b_1} \cdot (M - D_{b_1}) = \phi$. Therefore,
K = φ and $M_1$ has a partition. But $M_1$ is a component.
This is a contradiction.

Now $L_1$ and $M_1 - D$ are closed disjoint sets and
hence, $\rho(L_1, M_1 - D) = k > 0$. Since each component of
$D^* - L_1$ has a limit point on $L_1$ and a limit point on $M_1 - D$,
then each component has a diameter greater than or equal
to $k$. Then each component has a point at a distance $k/2$
from $L_1$. Now suppose $D^* - L_1$ has an infinite number of
components. Then there is an infinite set of points $\{x_n\}$
such that, (1) each $x_n$ belongs to a different component
of $D^* - L_1$ and (2) $\rho(L_1, x_n) = k/2$. Now each $x_n \in D$ and
hence there exists an $x$ such that $x$ is a limit point of $\{x_n\}$.
Now each $x_n$ belongs to $\overline{M_1}$ so that $x$ belongs to $\overline{M_1}$. Also,
$\rho(L_1, x_n) = k/2$ for every $x_n$ and hence $x$ does not belong
to $L_1$ and $x$ does not belong to $M_1 - D$. Therefore, $x$ belongs
to $D^*$ and hence to some component $D_b$ of $D^*$. But by (1)
there is a neighborhood $U$ of $x$ which is a subset of $D_b$.
But $U$ contains an infinite number of points of $\{x_n\}$ and
hence $D_b$ contains points of an infinite number of com-
ponents. Therefore, $M_1 \cdot D = \overline{M_1} \cdot (D - L_1)$ has only a finite num-
ber of components.

§2. Lemma: Let $M$ be a locally connected, locally
compact metric space which can be covered by a finite
number of connected domains all of diameter less than $e$
for every $e > 0$. If $z$ is a point of $M$ and $M - z$ is connect-
ed, then $M - z$ can be covered by a finite number of con-
ected domains of diameter less than $e$, none of which
contain $z$.

Proof: By theorem 3.9, page 106 of Wilder [3],
$M$ is locally connected. Since $M$ is locally compact,
for each $x$ belonging to $M$, there is a positive $\delta_x$ such that,
for $\delta$ less than $\delta_x$, $\overline{U(x, \delta)} \cdot M$ is compact.

Let $z$ be any point of $M$ for which $M - z$ is connect-
ed, and let $e$ be any positive number. Let $e_z$ be a positive
number less than $\min(e/2, \delta_x/2)$. Then $M$ can be covered
by a finite number of connected domains of diameter less
than $e_z$. Let $D_1, D_2, \ldots, D_j$ be those which do not contain
$z$ and let $D_{j+1}, D_{j+2}, \ldots, D_n$ be those which do. Set
$D = D_{j+1} + D_{j+2} + \ldots + D_n$. Then $D$ is a connected open set
containing $z$ such that (1) the diameter of $D$ is less
than $e$ and (2) $\overline{D}$ is compact. Then by §1, $D \cdot (M - z)$ has a
finite number of components. These with $D_1, D_2, \ldots, D_j$
give the required covering.

Definition: If $x \in C$ then $C_{x, \delta}$ shall be the
component of $U(x, \delta) \cdot C$ which contains $x$. 
§ 3. Theorem: Let $M$ be a locally compact, connected metric space which can be covered by a finite number of connected domains of diameter less than $e$ when $e > 0$. Suppose further that $M$ cannot be separated by the omission of any pair of points.

Let $C$ be a compact subset of $M$ and let $x$ and $y$ be common limit points of two components $D_A$ and $D_B$ of $M-C$ such that:

1. $C$ is locally connected at $x$ and at $y$;
2. There is a positive $\delta_1$ such that $C-(C_x,\delta_1+C_y,\delta_1)$ has a finite number of components when $\delta \leq \delta_1$.

Then given a positive number $e_0$, there are:

1. a $\delta_0$ less than or equal to $\delta_1$;
2. connected open supersets $U_{e_0}(C_x,\delta_0)$ and $U_{e_0}(C_y,\delta_0)$ of $C_x,\delta_0$ and $C_y,\delta_0$ which do not intersect $C_1$ where $C_1 = C-(C_x,\delta_0+C_y,\delta_0)$; and
3. a simple closed curve $J$ in $D_A+D_B+U_{e_0}C_x,\delta_0+U_{e_0}C_y,\delta_0$ such that (a) $J$ intersects $C_x,\delta_0$, $C_y,\delta_0$, $D_A$ and $D_B$; and (b) $J$ does not separate any point of $M-J$ from $C_1$.

Proof;

Before giving the details, a brief outline of the
proof will be given.

A finite collection \( H_1 \) of connected domains will be obtained such that their sum, \( H^*_1 \), does not separate any point of \( M-H^*_1 \) from \( C'_1 \) in \( M-H^*_1 \), and such that the sum of any pair of nonintersecting elements separates \( H^*_1 \). Collections \( H_2, H_3, \ldots \) will be defined which satisfy corresponding conditions and which are such that the closure of an element of \( H_{n+1} \) is a subset of \( H^*_n \). The collections \( H_1, H_2, \ldots \) will be described in such a way that the common part of their sums is a simple closed curve \( J \) which is a subset of \( D_A+D_B+U_{e_0}C_X, e_0+U_{e_0}C_Y, e_0 \) that does not separate any point of \( M-J \) from \( C'_1 \) in \( M \).

Now consider the details of the proof:

**Description of collection \( H_1 \).** Let \( e_1 = e_0 \) be a positive number less than one one-hundredth of the distance from \( x \) to \( y \). A collection \( H_1 \) of connected domains will be described. The sum of the elements of \( H_1 \) will be denoted by \( H^*_1 \). The collection \( H_1 \) of connected domains \( h_{1,1}, h_{1,2}, \ldots, h_{1,t} \) \( (t > 100) \) will satisfy the following conditions:

1. \( h_{1,i} \) intersects \( h_{1,j} \) \( (1 \leq i \leq t, 1 \leq j \leq t) \) if and only if \( i = j - 1 \) or \( i = j \) or \( i = j + 1 \) \( (h_{1,t+1} = h_{1,1}, h_{1,1-1} = h_{1,t}) \).
(2) If \( z \in M-H_1^* \) then \( h_1^* \) does not separate \( z \) from \( G_1' \) in \( M-H_1^* \);
(3) some point of \( M-H_1^* \) is accessible from \( h_{l,1} \);
(4) the diameter of \( h_{l,1} \) is less than \( e_1 \);
(5) no connected subset of \( R_1^* \) that intersects \( h_{l,1} \) and \( h_{l,1+2} \) \( (1 \leq i \leq t) \) is of diameter less than \( e_1/4 \).

Denote by \( D_1, D_2, \ldots, D_n \) the elements of a finite collection of connected domains covering \( M-(x+y) \) such that the diameter of each is less than \( e_1/300 \). Suppose that each of the domains \( D_1, D_2, \ldots, D_j \) intersects the complement of \( D_{A^*} + D_{B^*} \), each of the domains \( D_{j+1}, D_{j+2}, \ldots, D_k \) is a subset of \( D_{A^*} \) and each of the domains \( D_{k+1}, D_{k+2}, \ldots, D_n \) is a subset of \( D_{B^*} \).

Let \( \alpha_1, \alpha_2, \ldots, \alpha_j \) be a collection of arcs in the complement of \( D_{A^*} + D_{B^*} + x+y \) such that \( \alpha_1 \) \( (i=1, \ldots, j) \) intersects \( D_1 \) and \( C-(x+y) \). Also there are collections of arcs \( \beta_1, \gamma \) \( (i=j+1, \ldots, n) ; \gamma =1,2,3 \) such that:

1. \( \beta_1, \gamma \) intersects \( D_1 \) and \( C \);
2. \( \beta_{i,1} \leq D_{A^*}-(x+y) \) \( (i=j+1, \ldots, k) \); \( \beta_{i,1} \leq D_{B^*}-(x+y) \) \( (i=k+1, \ldots, n) \);
3. \( \beta_{i,2} \leq D_{A^*}-x \) \( (i=j+1, \ldots, k) \); \( \beta_{i,2} \leq D_{B^*}-x \) \( (i=k+1, \ldots, n) \);
(4) \( \beta_{i,3} \leq \overline{D}_A - y \) \((i=j+1, \ldots, k)\)

\( \beta_{i,3} \leq \overline{D}_B - y \) \((i=k+1, \ldots, n)\)

(5) \( \beta_{1,1} \cdot \beta_{1,2} = \beta_{1,2} \cdot \beta_{1,3} = \beta_{1,3} \cdot \beta_{1,4} = x_1 + y_1 \)

\((x_1 \text{ and } y_1 \text{ are endpoints of } \beta_{1,4})\).

Since \( \sum_{j=1}^{j+1} \beta_{j,1} \cdot \sum_{j=1}^{n} \beta_{j,1} \), \( \sum_{j=1}^{n} \beta_{j,2} \) and \( \sum_{j=1}^{n} \beta_{j,3} \) are all closed compact sets there are three positive numbers \( \delta_2, \delta_3 \) and \( \delta_4 \) such that:

(1) \( (U(x, \delta_2) + U(y, \delta_2)) \ast \left( \sum_{j=1}^{j+1} \beta_{j,1} \right) = \phi \);

(2) \( U(x, \delta_3) \ast \sum_{j=1}^{n} \beta_{j,2} = \phi \);

(3) \( U(y, \delta_4) \ast \sum_{j=1}^{n} \beta_{j,3} = \phi \).

Now suppose \( 0 < \delta_0 < \min(\delta_1, \delta_2, \delta_3, \delta_4, \epsilon_1/600) \). Then

(1) \( (C_x, \delta_0) + (C_y, \delta_0) \ast \left( \sum_{j=1}^{j+1} \beta_{j,1} \right) = \phi \);

(2) \( C_x, \delta_0 \ast \sum_{j=1}^{n} \beta_{j,2} = \phi \);

(3) \( C_y, \delta_0 \ast \sum_{j=1}^{n} \beta_{j,3} = \phi \);

(4) \text{diameter of } C_x, \delta_0 \text{ and of } C_y, \delta_0 \text{ is less than } \epsilon_1/300;

(5) \( C' = C - (C_x, \delta_0 + C_y, \delta_0) \) has a finite number of components.

It will be noted that since C is locally connected at \( x \) and at \( y \), there is a neighborhood \( U(x, \lambda) \) of \( x \) and a neighborhood \( U(y, \lambda) \) of \( y \) such that \( U(x, \lambda) \cdot C \) is a subset
of $C_x, \delta_o^*$ and $U(y, \lambda) \cdot C$ is a subset of $C_y, \delta_o^*$. But since, by theorem 2 page 89 in Moore [2], the accessible limit points of $D_A$ and $D_B$ are dense in $C \cdot \overline{D_A}$ and $C \cdot \overline{D_B}$ respectively, then $U(x, \lambda) \cdot C$ and $U(y, \lambda) \cdot C$ both contain accessible limit points of $D_A$ and of $D_B$. Hence, $C_x, \delta_o^*$ contains accessible limit points of $D_A$ and of $D_B$ and $C_y, \delta_o^*$ contains accessible limit points of $D_A$ and of $D_B$.

Owing to the existence of the aros $\beta_{j+1,1}, \beta_{j+1,2}$ and $\beta_{j+1,3}$, no point of $D_A$ disrupts $D_{j+1}$ either from $D_A(C_x, \delta_o^* + C_1')$ or from $D_A(C_y, \delta_o^* + C_1')$. Considering $D_A$, $D_{j+1}$, $D_A \cdot C_x, \delta_o^*$, $D_A \cdot C_y, \delta_o^*$ and $D_A \cdot C_1'$ as $D_*, D^*, M, N$, and $E$ of Bing's lemma, it is seen that there is an arc $\alpha_{j+1}$ from $C_1'$ to $D_{j+1}$ in $D_A \cdot C_1'$ that does not disrupt $C_x, \delta_o^*$ from $C_y, \delta_o^*$ in $D_A + C_x, \delta_o^* + C_y, \delta_o^*$.

Let $D'$ be a component of $D_A - D_{j+1}$ that contains an open arc from $C_x, \delta_o^*$ to $C_y, \delta_o^*$. If $D_{j+2}$ is not a subset of $D'$, let $\alpha_{j+2}$ be an arc in $D_A - D^* + C_1'$ from a point of $D_{j+2}$ to a point of $C_1'$. If $D_{j+2}$ is a subset of $D'$, Bing's lemma can be applied to get an arc $\alpha_{j+2}$ from $D_{j+2}$ to $C_1'$ in $D_A + C_1'$ such that $\alpha_{j+1} + \alpha_{j+2}$ does not disrupt $C_x, \delta_o^*$ from $C_y, \delta_o^*$ in $D_A + C_x, \delta_o^* + C_y, \delta_o^*$. The procedure is described in the following paragraph.

Let $R$ be a point of $D'$. Since no point of $D_A$ disrupts $D_{j+2}$ from $C_1' + C_x, \delta_o^*$ in $D_A + C_1' + C_x, \delta_o^*$, there is an arc $\beta$ from $D_{j+2}$ to $C_1' + C_x, \delta_o^*$ in $D_A + C_1' + C_x, \delta_o^* - R$. A subarc of
\( \beta \) in \( D' + C_1 + \alpha_{j+1} + C_x, \sigma_0 \rightarrow R \) intersects \( D_{j+2} \) and \( C_1 + C_x, \sigma_0 + \alpha_{j+1} \).

Hence, \( R \) does not disrupt \( D_{j+2} \) from \( C_1 + \alpha_{j+1} + C_x, \sigma_0 \) in \( D' + \overline{D'}(C_1 + \alpha_{j+1} + C_x, \sigma_0) \). Similarly, \( R \) does not disrupt \( D_{j+2} \) from \( C_1 + \alpha_{j+1} + C_x, \sigma_0 \) in \( D' + \overline{D'}(C_1 + \alpha_{j+1} + C_y, \sigma_0) \). Applying the lemma, it is found that there is an arc from \( C_x, \sigma_0 \) to \( C_y, \sigma_0 \) in \( D' + \overline{D'}(C_x, \sigma_0 + C_y, \sigma_0) \) which does not disrupt \( D_{j+2} \) from \( C_1 + \alpha_{j+1} \) in \( D' + \overline{D'}(C_1 + \alpha_{j+1}) \). It follows that there is an arc \( \alpha_{j+2} \) from \( D_{j+2} \) to \( C_1 + \alpha_{j+1} \) in \( D' + \overline{D'}(C_1 + \alpha_{j+1}) \) which does not disrupt \( C_x, \sigma_0 \) from \( C_y, \sigma_0 \) in \( D' + \overline{D'}(C_x, \sigma_0 + C_y, \sigma_0) \).

If \( \alpha_{j+2} \) intersects \( C_1 \), set \( \alpha_{j+2} = \alpha_{j+2} \). If \( \alpha_{j+2} \cap C_1 = \emptyset \), let \( \alpha_{j+2} \) be the arc in \( \alpha_{j+2} + \alpha_{j+3} \) from \( D' \) to \( C_1 \). In either case \( \alpha_{j+1} + \alpha_{j+2} \) does not disrupt \( C_x, \sigma_0 \) from \( C_y, \sigma_0 \) in \( D_A + C_x, \sigma_0 + C_y, \sigma_0 \).

Likewise there is an arc \( \alpha_{j+3} \) from \( D_{j+3} \) to \( C_1 \) in \( D_A + C_1 \) such that \( \alpha_{j+1} + \alpha_{j+2} + \alpha_{j+3} \) does not disrupt \( C_x, \sigma_0 \) from \( C_y, \sigma_0 \) in \( D_A + C_x, \sigma_0 + C_y, \sigma_0 \). A continuation of this process provides arcs \( \alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_n \) in \( D_A + D_B + C_1 \) whose sum does not disrupt \( C_x, \sigma_0 \) from \( C_y, \sigma_0 \) in \( D_A + C_x, \sigma_0 + C_y, \sigma_0 \) and does not disrupt \( C_x, \sigma_0 \) from \( C_y, \sigma_0 \) in \( D_B + C_x, \sigma_0 + C_y, \sigma_0 \), and such that \( \alpha_i \) \((i + j+1, \ldots, n)\) intersects \( D_i \) and \( C_1 \).

Let \( G' \) be the collection of all domains \( g \) such that \( g \) is a component of the common part of some domain of \( D_1, D_2, \ldots, D_n \) and the complement of \( C + \alpha_1 + \alpha_2 + \cdots + \alpha_n \).
If P is a point of $D_1$, there is an arc in $D_1$ from P to $\xi_1$. Hence, if $g$ is an element of $G'$, some point of $C_1^+\xi_1^+\xi_2^+\ldots+\xi_n^+$ is accessible from $g$.

Let $g_x^i$ and $g_y^j$ be connected domains which cover $C_x, \delta_0$ and $C_y, \delta_0$, respectively, such that $\rho(x', C_x, \delta_0) < e_1/300$ and $\rho(y', C_y, \delta_0) < e_1/300$ for every $x' \in g_x^i$ and for every $y' \in g_y^j$, but which contain no points of $C_1^+\xi_1^+\ldots+\xi_n^+$. Now let $g_x$ be the union of $g_x^i$ and all members of $G'$ which intersect $g_x^i$ and which do not have accessible limit points on $C_1^+\xi_1^+\ldots+\xi_n^+$. Since every $g'$ has an accessible limit point on $C_1^+\xi_1^+\xi_2^+\ldots+\xi_n^+$ every $g'$ added to $g_x^i$ and every $g'$ added to $g_y^j$ must have accessible limit points on $C_x, \delta_0$ and on $C_y, \delta_0$, respectively. Since every element of $G'$ is of diameter less than $e_1/300$ and since $C_x, \delta_0$ and $C_y, \delta_0$ are of diameter less than $e_1/300$ and since $\rho(x', C_x, \delta_0)$ and $\rho(y', C_y, \delta_0)$ are less than $e_1/300$ for every $x' \in g_x^i$ and for every $y' \in g_y^j$, then the diameters of $g_x$ and $g_y$ are both less than $e_1/100$. Let G be the collection of all elements of $G'$ which have accessible limit points on $C_1^+\xi_1^+\ldots+\xi_n^+$. If $g$ is an element of $G$
then some point of $\mathcal{G}^1 + \mathcal{G}^2 + \ldots + \mathcal{G}^n$ is accessible from $g$ and
either $g \subseteq \mathcal{D}_A$, $g \subseteq \mathcal{D}_B$ or $g \subseteq M-(\mathcal{D}_A + \mathcal{D}_B)$. There exists a
finite collection $\mathcal{G}^A$ of domains of $G$ such that this col-
lection but no collection of fewer elements of $G$ satis-
ifies the condition that the sum of the elements of $\mathcal{G}^A$ is
a connected subset of $\mathcal{D}_A$ and intersects both $\mathcal{E}_X$ and $\mathcal{E}_Y$. Denote the elements of $\mathcal{G}^A$ by $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{q-1}$ where $\mathcal{G}_X$ in-
tersects $\mathcal{G}_1$; $\mathcal{G}_1$ (i=2, ..., q-2) intersects $\mathcal{G}_{i+1}$ and $\mathcal{G}_{q-1}$
intersects $\mathcal{G}_Y$ but $\mathcal{G}_i$ does not intersect $\mathcal{G}_j$ for $j \geq i+2$. Similarly, there is a collection $\mathcal{G}^B$ of elements of $G$ such
that this collection, but no collection of fewer elements
of $G$, satisfies the condition that the sum of the elements
of $\mathcal{G}^B$ is a connected subset of $\mathcal{D}_B$ and intersects both $\mathcal{E}_X$ and $\mathcal{E}_Y$. Denote the elements of $\mathcal{G}^B$ by $\mathcal{G}_{q+1}, \mathcal{G}_{q+2}, \ldots, \mathcal{G}_R$
where $\mathcal{G}_Y$ intersects $\mathcal{G}_{q+1}$; $\mathcal{G}_1$ (i=q+1, ..., r=1) intersects
$\mathcal{G}_{i+1}$, and $\mathcal{G}_R$ intersects $\mathcal{G}_X$ but $\mathcal{G}_i$ does not intersect $\mathcal{G}_j$
for $j \geq i+2$. Denote $\mathcal{E}_X$ by $\mathcal{G}_1$ and $\mathcal{E}_Y$ by $\mathcal{G}_q$.

Let $\mathcal{E}$ denote the set $\mathcal{G}_1 + \mathcal{G}_2 + \ldots + \mathcal{G}_R$ plus all points
of $M-(\mathcal{G}_1 + \mathcal{G}_2 + \ldots + \mathcal{G}_R)$ that it separates from $\mathcal{G}_1$ in $M-(\mathcal{G}_X, \mathcal{G}_Y + \mathcal{G}_R)$. Each component of the common part of $\mathcal{E}$ and an
element of $G$ intersects an element of $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_R$. However, it is to be noted that no such components inter-
sects two $\mathcal{G}_i$'s that do not belong to a consecutive set of
three domains of $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_R$. Denote by $\mathcal{G}_i$ the sum of
$g_i$ and all such components that intersect $g_i$. It will be noted that $g_i$ is of diameter less than $e_1/33$.

If three is a factor of $r$, denote the sum of the first three elements of $g_1', g_2', \ldots, g_r'$ by $h_1$, the sum of the next three elements by $h_2$, \ldots, and the sum of the last three elements by $h_8$. If three is a factor of $r-1$, then $h_1, h_2, \ldots, h_8$ are defined as before except that $h_8$ is the sum of the last four elements of $g_1', g_2', \ldots, g_r'$ instead of the last three. If three is a factor of $r-2$, each of $h_{r-3}$ and $h_8$ is the sum of four elements of $g_1', g_2', \ldots, g_r'$.

Since each element of $h_1$ contains an element $g$ of $G$, then a point of $C_1 + h_1 + h_2 + \ldots + h_8$ is accessible from $h_1$.

Now $h_1$ is of diameter less than $e_1/8$ and the collection $h_1, h_2, \ldots, h_8$ satisfies conditions analogous to conditions (1), (2) and (3) to be satisfied by $h_1, 1, h_1, 2, \ldots, h_1, t$.

Let $h_{1,1}$ be the sum of $h_1, h_2, \ldots, h_n$ where some connected subset of $h_1 + h_2 + \ldots + h_n$ of diameter less than $e_1/4$ intersects $h_1$ and $h_n$ but no such subset intersects both $h_1$ and $h_{n+1}$; let $h_{1,2}$ be the sum of $h_{n+1}, h_{n+2}, \ldots, h_m$ where some connected subset of diameter less than $e_1/4$ intersects $h_{n+1}$ and $h_m$, but no such subset intersects both $h_{n+1}$ and $h_{m+1}$, \ldots, and let $h_{1,t}$ be the sum of $h_{p+1}, h_{p+2}, \ldots, h_s$ where some connected subset of diameter less
than $e_1/4$ intersects $h_{p+1}, h_{p+2}, \ldots, h_s$. Then the collection $h_{1,1}, h_{1,2}, \ldots, h_{1,t}$ satisfies conditions (1), (2), (3), (4) and (5).

**Description of collection $H_2$.** Choose a positive number $e_2$ less than one one-hundredth of the diameter of any connected set in $H^*$ that intersects $h_{1,1}$ and $h_{1,1+2}$ ($i=1, \ldots, t$; $h_{1,t+1}=h_{1,1}$; $h_{1,t+2}=h_{1,2}$). A collection $H_2$ of connected domains $h_{2,1}, h_{2,2}, h_{2,3}, \ldots, h_{2,s}$ will be obtained such that:

1. $h_{2,1}$ intersects $h_{2,j}$ if, and only if, $i=j-1$ or $i=j$ or $i=j+1$ ($h_{2,1-1}=h_{2,s}$; $h_{2,s+1}=h_{2,1}$);
2. if $z \in M-H_2$ then $H_2$ does not separate $z$ from $G_i$ in $M$;
3. some point of $M-H_2^*$ is accessible from $h_{2,1}$;
4. the diameter of $h_{2,1}$ is less than $e_2$;
5. no connected subset of $H_2^*$ that intersects $h_{2,1}$ and $h_{2,1+2}$ is of diameter less than $e_2/4$;
6. if $H(n;i,j)$ denotes $h_{n,1-100^+} \cdots h_{n,i-100^+} \cdots h_{n,j+100}$ where $(h_n,t+r=h_{n,f})$ and if $h_{1,io}$ intersects $h_{2,m_0}$ and if $h_{1,j_0}$ intersects $h_{2,n_0}$ then either $H(2;m_0,n_0) \leq H(1;i_0,j_0)$ and $H(2;n_0,m_0) \leq H(1;i_0,j_0)$ or $H(2;m_0,n_0) \leq H(1;i_0,j_0)$ and $H(2;m_0,n_0) \leq H(1;i_0,j_0)$.
Denote by $L$ the component of the common part of $h_1,2+\ldots+h_1,9$ and the complement of the closure of $h_1,1+\ldots+h_1,10$ which contains $h_1,5+h_1,8$. It will be shown that if $P$ is a point of $h_1,5+h_1,8$ and if $R$ is a point of $L-P$ then $R$ does not disrupt $P$ from $M-H_1^*$ in $M-H_1^*+L$. If $R$ is not a point of $h_1,5+h_1,8$ the result is evident.

Let $PQ$ be an arc in $M-R$ from $P$ to a point of $M-H_1^*$. Let $Q'$ be the first point of $PQ$ in order from $P$ to $Q$ on $M-L$. If $PQ'$ intersects $h_1,3$ then there is an arc from $PQ'-Q'$ to $M-H_1^*$ in $M-H_1^*+h_1,3$ because a point of $M-H_1^*$ is accessible from $h_1,3$. Also, if $PQ'$ intersects $h_1,8$, $R$ does not disrupt $P$ from $M-H_1^*$ in $M-H_1^*+L$. If $PQ'$ intersects neither $h_1,3$ nor $h_1,8$ then $Q'$ is a point of $M-H_1^*$. This demonstrates that $R$ does not disrupt $P$ from $M-H_1^*$ in $M-H_1^*+L$.

Let $G_1$ be a finite collection of connected domains of diameter less than $e_2/1200$ which cover $M$. Let $G$ be the collection of all elements of $G_1$ which intersects $h_1,5+h_1,6$. No point of $L$ disrupts an element of $G$ from $M-H_1^*$ in $M-H_1^*+L$. Repeated applications of Bing's lemma give that there is a collection of arcs $K$ in $M-H_1^*+L$ such that for each element $g \in G$ there is an element $\alpha \in K$ such
that \( \alpha \) intersects \( g \) and \( M-H^* \), but such that 
\[
\sum_{\alpha \in \mathcal{K}} \alpha
\]
do not disrupt \( h_{1,1} \) from \( h_{1,10} \) in \( h_{1,1} + \cdots + h_{1,10} \).

Let \( G' \) be the set of all domains \( g' \) such that \( g' \) is a component of (1) the common part of \( h_{1,1} + \cdots + h_{1,9} \), the complement of \( K^* \), and an element of \( G \), or (2) the common part of the complement of \( K^*+G \) and \( h_{1,1} \) (\( i=2,3,4,7,8,9 \)).

There exists a finite collection \( G'' \) of elements of \( G' \) such that the sum of the elements of \( G'' \) is a connected domain intersecting \( h_{1,1} \) and \( h_{1,10} \) but the sum of no subcollection of \( G' \) having fewer elements than \( G'' \) is a connected domain intersecting \( h_{1,1} \) (\( i=1,\ldots,10 \)).

Assume that \( g_1 \) of \( G'' \) intersects \( h_{1,1} \), \( g_1 \) (\( =1,\ldots,r-1 \)) intersects \( g_{1+1} \) and \( g_r \) intersects \( h_{1,10} \).

There exists a collection \( g'_1, g'_2, \ldots, g'_r \) of connected domains such that \( g'_1 \) intersects \( h_{1,1} \), \( g'_i \) intersects \( g'_{i+1} \) (\( i=1,\ldots,r-1 \)), \( g'_r \) intersects \( h_{1,10} \), and the closure of \( g'_i \) is a subset of \( g_i \).

Let \( E \) denote \( h_{1,1} + g'_1 + \cdots + g'_r + h_{1,10} + \cdots + h_{1,t} \) plus all points of \( H^* \) which it separates from \( M-H^* \) in \( M \). Each component of the common part of \( E \) and an element of \( G' \) intersects one of the domains \( h_{1,1} + g'_1, \ldots, g'_r + h_{1,10} \), but no such component intersects two of these domains that do not belong to a consecutive set of three of these.
Add such components to the ones of $h_1, l, g_1$, $h_1, l_0$ that they intersect to form the sets $h_1, g_1, h_{10}$. It is noted that the diameters of each $g_1$ not intersecting $h_1, g_1, h_1, l_3, h_1, l_4, h_1, l_7, g + h_1, g + h_1, g$ is less than $e_2/400$.

Consecutive elements of $g_1, g_2, \ldots, g_t$ may be combined by threes and fours in a manner previously described so as to get a collection $g_1, l, g_2, \ldots, g_1, u$ such that the collection $h_1, g_1, l, g_1, 2, \ldots, g_1, u, h_{10}, h_1, 11, \ldots, h_1, l_t$ satisfies conditions analogous to conditions (1), (2) and (3) to be satisfied by $h_2, l, h_2, 2, \ldots, h_2, s$. It is noted that the closure of each $g_1, l$ is a subset of $h_1, l, \ldots, h_1, 10$ when $i_0 < i < i_1$ (where $g_1, i_0$ is the last element of $g_1, l, \ldots, g_1, u$ which intersects $h_1, 2$ and $g_1, i_1$ is the first element of $g_1, l, \ldots, g_1, u$ which follows $g_1, i_0$ and intersects $h_1, g$).

In a manner similar to that in which $h_1, l + \ldots + h_1, l_0$ was replaced by $h_1, g_1, l + \ldots + g_1, u + h_{10}$, replace $h_1, l_1 + \ldots + h_1, 10$ by $h_1, l_1 + \ldots + g_1, l + h_{10}$, and $h_1, l_2 + \ldots + h_1, l_20$ by $h_1, l_2 + \ldots + g_1, l_2 + h_{20}$, and $h_1, t - m + \ldots + h_1, t$ (where $m \leq 18$) by $h_t - m + g_t - m, l + \ldots + g_t - m, w + h_t$.

Let $g_1, i_0$ be the fourth element of $g_1, l, g_1, 2, \ldots, g_1, n$ which follows all of those elements that intersect $h_1, i + 3$. Note that $g_1, i_0$, the three domains immediately preceding $g_1, i_0$, and the three domains immediately following $g_1, i_0$, are each a subset of $h_1, l_4$ of diameter less
than \( e_2/100 \).

In the manner described above, replace \( g_1,0^+\ldots+g_1,0+\ldots+g_{11,0} \)
by \( g_1,0+h_2,2^+\ldots+h_2,2^+g_{11,0}^+ \);
replace \( g_{11,0}^+\ldots+g_{21,0}^+ \)
by \( h_2,1^+\ldots+h_2,1^+g_{21,0}^+ \);
and replace \( g_{t-m,0}^+\ldots+g_{1,0}^+ \)
by \( h_2,n^+\ldots+h_2,n^+h_2,1^++h_2,1^+ \). The closure of \( h_2,1^+\ldots+h_2,1^+ \)
is a subset of \( h_2,1^+\ldots+h_2,1^+1^+ \)
and the closure of \( h_2,1^+\ldots+h_2,1^+1^+ \)
is a subset of \( h_1,t-m^+\ldots+h_1,1^+ \). Consecutive elements of \( h_2,1^+\), \( h_2,2^+\),
\( h_2,3^+\), etc., may be combined in a manner previously described so
as to form a collection \( H_2 \) of connected domains \( h_2,1\), \( h_2,2\),
\( h_2,3\), \( h_2,4\), satisfying conditions (1), (2), (3), (4), (5) and
(6). \( e_2 \)

**Description of simple closed curve \( J \).** For each
positive integer \( i \) greater than one, a collection \( H_i \) of
connected domains \( h_1,1^+\), \( h_1,2^+\), \( h_1,3^+\), \( h_1,4^+\) can be described
satisfying conditions analogous to those satisfied by \( H_2 \),
where \( e_1 \) is a positive number less than one one-hundredth
of the diameter of any connected set in \( H_i^+, \) intersecting
\( h_1,1^+ \) and \( h_1,1^+ \). It will be shown that the common
part \( J \) of \( H_1^+, H_2^+, \ldots \) is a simple closed curve in \( D_A + D_B + \bigcup_{e_1} C_x, \bigcup_{e_1} C_y, \bigcup_{e_1} C_z \)
that does not separate any point of
\( H-J \) from \( C_1 \).

As the closure of \( H_i^+ \) is a connected subset of
$H_i^*$ (condition 6) and as each $h_{1,j}$ contains an element of $H_{1,1}, \text{23}$ then $J$ is a nondegenerate continuum. This continuum does not separate any point of $M - J$ from $C_i$ because no $H_i^*$ separates any point of $M - H_i^*$ from $C_i^*$. Also, since $J \subseteq H_i^*$, and since $H_i^* \subseteq D_A + D_B + \epsilon_x \epsilon_y$ (where $\epsilon_x$ and $\epsilon_y$ are open sets of diameter less than $\epsilon_1/33$, containing $C_x, \sigma_0$ and $C_y, \sigma_0$, respectively, but not intersecting $C_i^*$), then $J \subseteq D_A + D_B + \epsilon_0 C_x \sigma_0 + \epsilon_0 C_y \sigma_0$. Let $P$ and $Q$ be any pair of points of $J$. Suppose that $h_{1,P_1}$ and $h_{1,Q_1}$ are elements of $h_{1,1,\ldots,h_{1,n_1}}$ that contain $P$ and $Q$, respectively. For convenience in notation, it will be assumed that it is $H(1; P_1, Q_1)$ that covers the closure of $H(1; P_{1+1}, Q_{1+1})$ and that it is $H(1; Q_1, P_1)$ that covers the closure of $H(1; Q_{1+1}, P_{1+1})$. If $J_{PQ}$ is the common part of $H(1; P_1, Q_1), H(2; P_2, Q_2), \ldots$ and $J_{QP}$ is the common part of $H(1; Q_1, P_1), H(2; Q_2, P_2), \ldots$, it is found that $J = J_{PQ} + J_{QP}$ where $J_{PQ}$ and $J_{QP}$ have only the points $P$ and $Q$ in common. \text{24}

Hence, $J$ is a simple closed curve in $D_A + D_B + \epsilon_0 C_x \sigma_0 + \epsilon_0 C_y \sigma_0$ which does not separate any point of $M - J$ from $C_i^*$. Also, since $J$ contains points of $D_A$ and points of $D_B$ it must intersect $C_x, \sigma_0$ and $C_y, \sigma_0$. 
CHAPTER II

§4. Lemma: Let \( \xi \) be an arc in \( S \) which separates the points \( A \) and \( B \). Let \( xy \) be a minimal subarc of \( \xi \), its endpoints being \( x \) and \( y \), which separates \( A \) and \( B \). Then there is a member \( C_{10} \) of \( \mathcal{U} \) such that \( xy \cdot C_{10} = x + y \).

Proof: \( S - xy \) has two components \( S_1 \) and \( S_2 \) which contain \( A \) and \( B \), respectively. Now there exists a \( \delta_0 \) such that \( U(x, \delta) \) intersects at most one element of \( \mathcal{U} \) and \( U(x, \delta_0) \cap U(y, \delta_0) = \emptyset \).

Since no subarc of \( xy \) separates \( A \) and \( B \), then \( x \) and \( y \) are common limit points of \( S_1 \) and \( S_2 \). Then by §3, there is a simple closed curve \( J \) in \( S \) which intersects \( C_x, \delta \) and \( C_y, \delta \) \( (C_x, \delta \) and \( C_y, \delta \) are components of \( U(x, \delta) \cdot xy \) and \( U(y, \delta) \cdot xy \), and which contain \( x \) and \( y \), respectively) when \( \delta < \delta_0 \) which does not separate any point of \( S - J \) from \( xy - (C_x, \delta + C_y, \delta) \). But the latter is a connected set, and thus \( S - J \) is connected. Therefore, \( J \) must be an element of \( \mathcal{U} \). Let it be \( C_{10} \). Since \( U(x, \delta_0) \) intersects only one element of \( \mathcal{U} \), then there is no other element of \( \mathcal{U} \) which intersects \( C_x, \delta \) and \( C_y, \delta \). But since \( C_{10} \) intersects \( C_x, \delta \)
and $C_y, f$ for every $f < f_0$, then $C_{10}$ must contain $x$ and $y$.

Now let $z$ be any point of $xy-(x+y)$. Then there is a $f_1$ less than $f_0$ such that $z \in xy-(C_x, f_1 C_y, f_1)$. But by §3, $C_{10}$ does not intersect $xy-(C_x, f_1 C_y, f_1)$. Hence, $z \notin C_{10}$. Therefore, $x$ and $y$ are the only points of $xy$ which belong to $C_{10}$ and $xy$ has endpoints only on $C_{10}$.

§5. Lemma: Let $ab$ be any arc in $S$ with endpoints only on an element $C_1$ of $\mathcal{F}$ such that no proper subarc of $ab$ has endpoints only on any element $C_{10}$ of $\mathcal{F}$. Then $ab$ separates $S$.

Proof: Suppose $ab$ does not separate $S$. Let $ac_1b$ and $ac_2b$ be the two different arcs of $C_1$ with ends $a$ and $b$. Now consider $S-(ab+ac_1b)$, a set which has a partition. Let $x$ be any point of $ab+ac_1b$ and suppose that $x$ is not a limit point of some component $S_1$ of $S-(ab+ac_1b)$. Then there is a neighborhood $U$ of $x$ which contains no point of $S_1$. But then a minimal subarc of $(ab+ac_1b)$ separates $S$. This subarc must have endpoints only on some element of $\mathcal{F}$ ($§4$). But $ab$ is the only such subarc. This is a contradiction. Therefore every point of $ab+ac_1b$ is a limit point of every component of $S-(ab+ac_1b)$.
Lemma 5.1: Let $S_1$ be the component of $S-(ab+ac_1b)$ which contains $\langle ac_2b \rangle$. Then one component of $S_1-\langle ac_2b \rangle$ has limit points on $\langle ac_1b \rangle$ and on $\langle ab \rangle$.

Proof of lemma 5.1: Let $Z$ be a point of $\langle ac_1b \rangle$. Now there exists a $\delta_1$ such that $U(Z,\delta_1)^* (ac_2b+ab) = \phi$. In $U(Z,\delta_1)$ there is a point which is an accessible limit point of $S_1$. Let such a point be $Z_1'$. Let $Z_2$ be a point of $S_1$. Then there is in $S_1+Z_1'$ an arc $Z_1'Z_2$ from $Z_1'$ to $Z_2$. Since $Z_1'$ does not belong to $ab+ac_2b$, there is a subarc $Z_1'Z_2'$ of $Z_1'Z_2$ which does not intersect $ab+ac_2b$. But then $Z_1'Z_2'-Z_1' \subseteq S_1-(ac_2b)$ which has $Z_1'$ as a limit point. But $Z_1'Z_2'-Z_1'$ is connected and belongs to some component of $S_1-\langle ac_2b \rangle$. Therefore, some component of $S_1-\langle ac_2b \rangle$ has a limit point on $\langle ac_1b \rangle$. Call this component $S^*$. Suppose $S^*$ has no limit points on $\langle ab \rangle$. Let $S-(ab+ac_1b) = S_1+S_a$. ($S_1+S_a = \phi$). Consider $S-C_1 = S^*+[S_1-(S^*+C_1)] +S_a+\langle ab \rangle$. Let $K=S^*+[S_1-(S^*+C_1)]+S_a+\langle ab \rangle$. Since $S^*$ has no limit points on $\langle ab \rangle$ then $S^* \subseteq S^*+C_1$. But then $S^*+[S_1-(S^*+C_1)]+S_a+\langle ab \rangle = \phi$. Now $S^*+[S_1-(S^*+C_1)]+S_a+\langle ab \rangle = \phi$. Now $S^*+[S_1-(S^*+C_1)]+S_a+\langle ab \rangle \subseteq S^*+[S_1-C_1]+S_a+\langle ab \rangle$. Now $S_1-C_1$ is locally connected and thus $S^*+[S_1-C_1]-S^* = \phi$. 
Also, \( S_(ab+ac_1b) \) is locally connected and thus 
\( S_1(S_a) = \emptyset \), and since \( S_1 \subseteq S_(ab) \), then \( S_1 \cdot ab = \emptyset \).
Therefore, \( K = \emptyset \).

But then \( S-C_1 \) has a partition. This is a contradiction and thus \( S^* \) has limit points on \( <ab> \).
This ends the proof of lemma 5.1.

Now let \( S_2 \) be a second component of \( S_(ab+ab_1c) \).
Since \( S_2 \cdot <ac_2b> = \emptyset \), then \( S_2 \) is also a component of 
\( S_(C_1+ab) \). Therefore \( S_(C_1+ab) \) has as two components 
\( S^* \) and \( S_2 \). Now let \( x \in ab-(a+b) \) and let \( y \in ac_1b-(a+b) \) 
be such that \( x \) and \( y \) are limit points of \( S^* \). Since every 
point of \( ab+ac_1b \) is a limit point of \( S_2 \), \( x \) and \( y \) are 
common limit points of \( S^* \) and \( S_2 \). Now let \( d_0 \) be such that 
\( \left[ U(x,d_0) + U(y,d_0) \right] \cdot (a+b) = \emptyset \). If \( C=C_1+ab \), then it is 
easily seen that \( C-(C_x, d_0 + C_y, d_0) \) is a connected set. Now 
by \( \delta 3 \), there is a simple closed curve \( J \) in \( S \) which inter-
sects \( C_x, d_0 \) and \( C_y, d_0 \) such that \( J \) does not separate any 
point of \( S-J \) from \( C-(C_x, d_0 + C_y, d_0) \). Thus \( S-J \) is connected.
But \( J \) intersects \( C_x, d_0 \) and hence cannot be \( C_1 \); and \( J \) in-
tersects \( C_y, d_0 \) and hence cannot be any element of \( T \)
different from \( C_1 \). This is a contradiction. Therefore 
\( ab \) separates \( S \).

\( \delta 6 \). Lemma: If \( ab \) is an arc in \( S \), then \( ab \) separates
S if and only if there is a subarc of ab with endpoints only on some element $C_{10}$ of $\mathcal{P}$.

Proof: Suppose ab separates S. Let $x$ and $y$ be two points of $S-ab$ which belong to different components of $S-ab$. Let $a'b'$ be a minimal subarc of ab which separates $x$ and $y$. Then by §4, the arc $a'b'$ has endpoints only on some element $C_{10}$ of $\mathcal{P}$.

Suppose ab has a subarc $a'b'$ with endpoints only on an element $C_{10}$ of $\mathcal{P}$. It can be assumed that $a'b'$ does not contain a proper subarc with endpoints only on some other element $C_1$ of $\mathcal{P}$. Now $a'b'$ separates $S$ by §5, (i.e. $S-a'b'=S_1\setminus S_2$). Now $S_1-(ab)$ cannot be empty, else it would be possible to show that a pair of points would separate $S$. Also, $S_2-(ab)$ cannot be empty. Then $(S_1-ab)\cup(S_2-ab)$ is a partition of $S-ab$.

§7. Lemma: Suppose $S^*$ is a domain of $S$ not separated by any pair of points and such that $S-S^*$ has a finite number of components.

Suppose $C_1$ is a member of $\mathcal{P}$ which is a subset of $S^*$. Suppose every component of $S^*-C_1$ has limit points on every component of $S-S^*$.

Then $S^*-C_1$ is connected.
Proof: Let $S_1$ be a component of $S^*-C_1$ and suppose that $S^*-C_1$ has another component $S_2$. There are two possibilities. In case I will be considered the possibility that every point of $C_1$ is a common limit point of $S_1$ and $S_2$. In case II will be considered the possibility that some point of $C_1$ is not a common limit point of $S_1$ and $S_2$.

Case I. Let $x_1$ and $x_2$ be two points of $C_1$. Let $a$ and $b$ be two points of $C_1$ which are separated (on $C_1$) by $x_1+x_2$. Let $\delta = \frac{1}{2}(x_1+x_2, a+b)$. If $C=C_1+S-S^*$, then $C_{x_1,\delta}$ and $C_{x_2,\delta}$ are both subsets of $C_1$ and $C=(C_{x_1,\delta}+C_{x_2,\delta}) = (S-S^*)+C_1-(C_{x_1,\delta}+C_{x_2,\delta})$. Since $C_1-(C_{x_1,\delta}+C_{x_2,\delta})$ consists of two arcs, and since $S-S^*$ has a finite number of components, then by $\delta 3$, there is a simple closed curve $J$ in $S_1+S_2+U_{x_1,\delta}C_{x_1,\delta}+U_{x_2,\delta}C_{x_2,\delta}$ such that no point of $S-J$ is separated from $C=(C_{x_1,\delta}+C_{x_2,\delta})$ by $J$. Now in $S_2+C_1$ there is a subarc $J_1$ of $J$ which has endpoints only on $C_1$. Since $J-J_1$ contains limit points of $S-J$, then no point of $S-J_1$ is separated from $C=(C_{x_1,\delta}+C_{x_2,\delta})$ by $J_1$. But $S_1*J_1=\phi$, and $S_1$ contains limit points on every component of $C=(C_{x_1,\delta}+C_{x_2,\delta})$. Therefore $S-J_1$ is connected. But $J_1$ is an arc with endpoints only on an element of $\mathcal{F}$. Hence, by $\delta 6$, $S-J_1$ has a partition. This is a contradiction, and hence some point of $C_1$ is not a common limit point of $S_1$ and $S_2$. 
Case II. Some point of $C_1$ is not a common limit point of $S_1$ and $S_2$.

In this case some subarc of $C_1$ separates $S^*$. Let $A$ be a point of $S_1$ and let $B$ be a point of $S_2$. Let $x_1ax_2$ be a minimal subarc of $C_1$ which separates $A$ and $B$ in $S^*$. Let $S_1'$ and $S_2'$ be the components of $S^*-x_1ax_2$ which contain $A$ and $B$, respectively. Now $S_1'$ and $S_2'$ must have $x_1$ and $x_2$ as common limit points. Also, $S_1'$ and $S_2'$ have additional limit points on $x_1ax_2$ for otherwise $S^*$ could be separated by the omission of a pair of points. Let $x_3$ be such a limit point of $S_1'$ and $x_4$ be such a limit point of $S_2'$.

Let $\delta = 1/2^{\tau}(x_1+x_2+x_3+x_4)$. If $C=x_1ax_2+S-S^*$ then $C_{x_1,\delta}+C_{x_2,\delta} \subseteq x_1ax_2$, and $x_1ax_2=(C_{x_1,\delta}+C_{x_2,\delta})$ is a single arc which contains a limit point of $S_1'$ and a limit point of $S_2'$. Therefore $C=(C_{x_1,\delta}+C_{x_2,\delta})$ has a finite number of components, each of which contains a limit point of $S_1'$ and each of which contains a limit point of $S_2'$. By $\delta$, there is a simple closed curve $J$ in $S_1'+S_2' + U \cdot C_{x_1,\delta} + U \cdot C_{x_2,\delta}$ such that no point of $S-J$ is separated from $C-(C_{x_1,\delta}+C_{x_2,\delta})$. Now some point of $J$ does not belong to $C_1$. Suppose that such a point lies in the complement of $S_1'$. Then as in case I there is a subarc $J_1$ of $J$ in the complement of $S_1'$ such that: (1) $J_1$ has endpoints only on $C_1$; and (2) no point of $S-J_1$ is separated from $C-C_{x_1,\delta}$.
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(Note that may not have endpoints on $C_{x_1,\delta}$ or on $C_{x_2,\delta}$). But $C-(C_{x_1,\delta}+C_{x_2,\delta})+S_1'$ is a connected set in $S-J_1$. Therefore, $S-J_1$ is connected. But by $\delta\delta$, $S-J_1$ has a partition. This is a contradiction and hence $S^*-C_1$ is connected.

§8. Lemma: If $S$ cannot be separated by any collection consisting of $k$ elements of $\mathcal{T}$ and any finite number of points, then $S$ cannot be separated by any collection consisting of $k+1$ elements of $\mathcal{T}$.

Proof: Consider a collection of $k+1$ elements of $\mathcal{T}$. Let these elements be $C_1, C_2, \ldots, C_{k+1}$. Now suppose $S^* = S - \sum_{2}^{k+1} C_i$. Then by the hypotheses $S^*$ is a connected open subset of $S$ which cannot be separated by the omission of any pair of points. Also, $S - \sum_{1}^{k+1} C_i \cdot 1$ is connected for $1 < i_0 \leq k+1$.

Now if $M$ is any connected open subset of $S$ and $K$ is any closed subset of $M$, then every component of $M-K$ will have limit points on $K$. Therefore, since $S - \sum_{1}^{k+1} C_i \cdot 1$ is connected for $1 < i_0 \leq k+1$, then every component of $S^*-C_1 = S - \sum_{1}^{k+1} C_i$ has limit points on $C_1$ ($1 < i_0 \leq k+1$).

(This property will be used several times in subsequent theorems.) Since $S-S^* = \sum_{2}^{k+1} C_i$, then $S-S^*$ has a finite number of components, and every component of $S^*-C_1$ has
limit points on every component of $S-S^*$. Also, $C_1$ is a subset of $S^*$. Therefore, by §7, $S^*-C_1$ is connected, and thus no collection of $k+1$ elements of $\Gamma$ separates $S$.

§9. Lemma: If $S$ cannot be separated by any collection consisting of $k$ elements of $\Gamma$ and any finite number of points, then $S$ cannot be separated by any collection consisting of $k+1$ members of $\Gamma$ and any finite number of points.

Proof: By induction.

Let $C_1$, $C_2$, $C_3$, ..., $C_{k+1}$ be any $k+1$ elements of $\Gamma$ and let $x_1$ be any point. Suppose $S^*=S-(\sum_{2}^{k+1} C_i+x_1)$. Then $S^*$ is a connected, open subset of $S$ which cannot be separated by any pair of points. Also, $S-S^* = \sum_{2}^{k+1} C_i+x_1$, and has a finite number of components. Since $S-(\sum_{1}^{k+1} C_i+x_1)+x_1$ is connected (§8) and since $S-(\sum_{1}^{k+1} C_i+x_1)+C_{i_0}$ is connected ($1 < i_0 \leq k+1$), then every component of $S^*-C_1=S-(\sum_{1}^{k+1} C_i+x_1)$ has $x_1$ as a limit point and has a limit point on $C_{i_0}$ ($1 < i_0 \leq k+1$). Also, it can be assumed that $C_1$ is a subset of $S^*$. But then by §7, $S^*-C_1$ is connected. Therefore $S$ is not separated by any collection consisting of $k+1$ elements of $\Gamma$ and a single point.
Now suppose that \( S \) cannot be separated by any collection consisting of \( k+1 \) elements of \( \mathcal{P}' \) and by any \( g \) points. Let \( C_1, C_2, \ldots, C_{k+1}, x_1, x_2, \ldots, x_g, x_{g+1} \) be any collection consisting of \( k+1 \) elements of \( \mathcal{P}' \) and any \( g+1 \) points. Suppose \( S^* = S - \left( \sum_{i=1}^{k+1} C_i + \sum_{i=1}^{g+1} x_i \right) \). Then \( S^* \) is a connected, open subset of \( S \) which cannot be separated by any pair of points. Also, \( S - S^* \) has a finite number of components.

Now \( S - \left( \sum_{i=1}^{k+1} C_i + \sum_{i=1}^{g+1} x_i \right) + C_{i_0} \) is connected when \( 1 < i_0 \leq k+1 \) and \( S - \left( \sum_{i=1}^{k+1} C_i + \sum_{i=1}^{g+1} x_i \right) + x_{i_0} \) is connected when \( 1 \leq i_0 \leq g+1 \). Therefore, every component of \( S^* - C_1 \) has \( x_{i_0} \) as a limit point when \( 1 \leq i_0 \leq g+1 \), and has a limit point on \( C_{i_0} \) when \( 1 < i_0 \leq k+1 \). Also, it can be assumed that \( C_1 \) is a subset of \( S^* \). Then by \( \ddagger 7 \), \( S^* - C_1 \) is connected, and thus \( S \) cannot be separated by any collection consisting of \( k+1 \) elements of \( \mathcal{P}' \) and any \( g+1 \) points.

Therefore, by induction, \( S \) cannot be separated by any collection consisting of \( k+1 \) elements of \( \mathcal{P}' \) and any finite number of points.

**\( \ddagger 10 \). Lemma:** \( S \) cannot be separated by the omission of any finite number of points.

**Proof:** Suppose that \( S \) cannot be separated by the
omission of any set of \( k \) points when \( k \geq 2 \). Let \( x_1, x_2, \ldots, x_{k+1} \) be any set of \( k+1 \) distinct points. Suppose that \( x_1, x_2, \ldots, x_{k+1} \) all belong to the same element \( C_{i_0} \) of \( \Pi \) for some \( i_0 \), when \( 1 \leq i_0 \leq n \). Then \( S-C_{i_0} \subseteq S-(x_1+x_2+\cdots+x_{k+1}) \subseteq S-C_{i_0} \). Since \( S-C_{i_0} \) is connected (definition of \( C_{i_0} \)) then so is \( S-(x_1+x_2+\cdots+x_{k+1}) \).

Suppose that two points, say \( x_1 \) and \( x_2 \), do not belong to the same element \( C_{i_0} \) of \( \Pi \). Then any simple closed curve containing \( x_1 \) and \( x_2 \) is not a member of \( \Pi \).

Suppose \( S-\sum_{i=1}^{k+1} x_i \) has a partition. Let \( S_1 \) and \( S_2 \) be two components. Since no set of \( k \) points separates \( S \) then \( x_1 \) and \( x_2 \) are common limit points of \( S_1 \) and \( S_2 \).

Now consider \( S-(x_4+x_5+\cdots+x_{k+1}) \) (note that \( x_4+x_5+\cdots+x_{k+1} \) may be empty). By a repeated application of \( f_2 \) to \( S \), it can be shown that \( S-(x_4+x_5+\cdots+x_{k+1}) \) is a locally compact, connected metric space which can be covered by a finite number of connected domains of diameter less than \( \epsilon \) for every positive \( \epsilon \). Also, \( S-(x_4+x_5+\cdots+x_{k+1}) \) cannot be separated by the omission of any pair of points. Therefore \( S-(x_4+x_5+\cdots+x_{k+1}) \) satisfies the conditions of \( \kappa \) in \( \S \). Also, \( x_1+x_2+x_3 \) satisfies the conditions for \( C \) in \( \S \). Now \( C_{x_1,\sigma} \) and \( C_{x_2,\sigma} \) are just the points \( x_1 \) and \( x_2 \) for and \( \delta \). Also, \( C-(C_{x_1,\sigma}+C_{x_2,\delta}) \) is the point \( x_3 \). Now by \( \S \), there is in \( S_1+S_2+\epsilon C_{x_1,\sigma}+\epsilon C_{x_2,\delta} \).
a simple closed curve \( J \) which intersects \( x_1 \) and \( x_2 \) but which does not separate any point of \( S-(x_4+x_5+\ldots+x_{k+1}) \) from \( x_3 \). Hence, \( S-(x_4+x_5+\ldots+x_{k+1})^J \) is connected. But \( S-J \subseteq S-(x_4+\ldots+x_{k+1})^J \). Therefore, \( S-J \) is connected. This is a contradiction. Therefore no set of \( k+1 \) points separates \( S \). Since by hypothesis, the lemma is true for \( k \) when \( k=2 \), it is true for any finite number of points.

§11. Theorem: \( S \) cannot be separated by any collection consisting of \( k \) elements of \( \mathcal{F} \) and a finite number of points.

Proof: \( S \) cannot be separated by any element of \( \mathcal{F} \) by hypothesis. By §10, \( S \) cannot be separated by any finite collection of points. Then by §9, \( S \) cannot be separated by any element of \( \mathcal{F} \) and a finite collection of points. By §8, \( S \) cannot be separated by any two elements of \( \mathcal{F} \).

Therefore by mathematical induction using §8 and §9, \( S \) cannot be separated by any collection consisting of \( k \) elements of \( \mathcal{F} \) \((k \leq n) \) and a finite number of points.

§12. Lemma: Let \( \mathcal{A}' \) be an arc in \( S \) with endpoints
a' and b'. If $S^*$ is a component of $S-x$, then $S^*$ cannot be separated by the omission of any finite number of points.

Proof: Let $S^*$ be a component of $S-a'b_1$. Let $x$ be a point of $S^*$ and suppose that $S^*-x$ has a partition. Then there are points $A$ and $B$ of $S^*-x$ such that $A$ and $B$ are in separate components of $S-(a'b_1+x)$. Suppose that $A$ belongs to $C_{10} \ (1 \leq i_0 \leq n)$. There is an $e$ such that $U_{A,e}(a'b_1+x) = \emptyset$. But then $U_{A,e}$ is a subset of the component $S_1$ of $S-(a'b_1+x)$ which contains $A$. However, $U_{A,e}$ contains points of $\bigcup_{i=1}^{n} C_i$. Hence, an $A'$ can be picked in $S_1$ such that $A' \notin C_i \ (i=1,\ldots,n)$. Therefore, it can be assumed that $A+B \notin S-\bigcup_{i=1}^{n} C_i$.

Since $S-x$ is connected, then $a'b_1$ separates $A$ and $B$ in $S-x$. Let $\alpha=a'b_1$ be a minimal subarc of $a'b_1$ which separates $A$ and $B$ in $S-x$. Let $S_A$ and $S_B$ be the components of $S-(a'b_1+x)$ which contain $A$ and $B$, respectively. It will be noticed that, since no pair of points separates $S$, $\alpha$ is not a degenerate arc. Since $\alpha$ is a minimal arc separating $A$ and $B$ then $a_1$ and $b_1$ are both common limit points of $S_A$ and $S_B$.

Now $S-x$ satisfies the conditions of §3, and there is a $s_0$ such that when $\alpha_1=\alpha-\alpha_{a_1}, \alpha_{b_1} \alpha, \alpha_{b_1}$, then $\alpha_1$ is
connected when \( \delta < \delta_0 \). Therefore by \( \delta_3 \), there is a simple closed curve \( J \) in \( S_A + S_B + \cup e \alpha_a \delta + U e \alpha_b \delta \) that does not separate any point of \( S - x - J \) from \( \alpha_1 \). Since \( \alpha_1 \) is connected then \( S - x - J \) is connected and \( S - J \) is connected \( (S - J \subseteq S - x - J) \). Therefore, \( J \) must be one of the simple closed curves that does not separate \( S \). Since there is such a \( J \) intersecting \( \alpha_a \delta \) and \( \alpha_b \delta \) for every \( \delta \) less than \( \delta_0 \), and since there is a \( \delta_1 \) such that \( \alpha_a \delta \) intersects at most one element of \( \Pi \) when \( \delta < \delta_1 \), then one element \( C_1 \) of \( \Pi \) must contain \( a_1 \) and \( b_1 \).

Let \( z \) be any point of \( a_1 b_1 \setminus (a_1 + b_1) \). Then there is a \( \delta_2 \) such that \( z \) does not belong to \( \alpha_a \delta_2 + \alpha_b \delta_2 \).

But \( C_1 \) belongs to \( S_A + S_B + \cup e \alpha_a \delta_2 + U e \alpha_b \delta_2 \) and hence \( z \) does not belong to \( C_1 \). Therefore \( a_1 b_1 \) has endpoints only on \( C_1 \).

Also, \( e \) and \( \delta \) can be made sufficiently small so that \( U e \alpha_a \delta + U e \alpha_b \delta \) does not contain \( x \). Hence \( C_1 \subseteq S - x \) and \( x \) does not belong to \( C_1 \).

Now \( a_1 \) and \( x \) are both common limit points of \( S_A \) and \( S_B \) and the component of \( U(x, e) \cdot (x + x) \) which contains \( x \) is just \( x \) itself. Also when \( \delta < \delta_0 \), \( \alpha_a \delta \) is an arc.

Then by \( \delta_3 \), for any \( e \) there is in \( S_A + S_B + \cup e \alpha_a \delta + U(x, e) \) a simple closed curve \( J \) which does not separate any point of \( S - J \) from \( \alpha_a \delta \). But \( \alpha_a \delta \) is connected
and hence \( S = J \) is connected. Now \( \delta \) and \( e \) can be chosen sufficiently small that: (1) \( \bigcup_{\varepsilon \in (0, \delta)} C_\varepsilon = \emptyset \) \( (i=2, \ldots, n) \); and (2) \( U(x, \varepsilon) \cdot C_\varepsilon = \emptyset \). Hence \( J \) is not an element of \( \mathcal{F} \). This is a contradiction. Hence, no point of \( S^* \) separates \( S^* \).

Now suppose that the lemma is true for any set of \( k \) points. Let \( x_1, x_2, \ldots, x_{k+1} \) be \( k+1 \) points of \( S^* \), such that \( S^* = \sum_{i=1}^{k+1} x_i \) has a partition. Then there are points \( A \) and \( B \) in \( S^* = \sum_{i=1}^{k+1} x_i \) such that \( A \) and \( B \) are in separate components of \( S = (a_1 b^1 + \sum_{i=1}^{k+1} x_i) \). As before, it can be assumed that \( A \) and \( B \) do not belong to any element of \( \mathcal{F} \).

Since \( S = \sum_{i=1}^{k+1} x_i \) is connected, by \( \delta 10 \), \( a_1 b^1 \) separates \( A \) and \( B \) in \( S = \sum_{i=1}^{k+1} x_i \). Let \( \alpha = a_1 b_1 \) be a minimal subarc of \( a_1 b_1 \) which separates \( A \) and \( B \) in \( S = \sum_{i=1}^{k+1} x_i \). Let \( S_A \) and \( S_B \) be the components of \( S = (a_1 b_1 + \sum_{i=1}^{k+1} x_i) \) which contain \( A \) and \( B \), respectively. It will be noted that since no finite number of points separate \( S \), by \( \delta 10 \), \( \alpha \) is not a degenerate arc. Since \( \alpha \) is the minimal arc separating \( A \) and \( B \), and since \( S^* \) is not separated by any \( k \) points, then \( a_1, b_1, x_i \) \( (i=1, \ldots, k+1) \) are all common limit points of \( S_A \) and \( S_B \).

It can be shown, by repeated application of \( \delta 2 \), that \( S = \sum_{i=1}^{k+1} x_i \) satisfies the conditions of \( \delta 3 \), and since there is a \( \delta_0 \) such that when \( \alpha = (\alpha = (a_1, \delta + \varepsilon b_1, \varepsilon) \) then \( \alpha \) is connected for every \( \delta < \delta_0 \), then by \( \delta 3 \), there is a
simple closed curve $J$ in $S_A + S_B + U_{e\alpha a_1, e} + U_{e\alpha b_1, e}$ that does not separate any point of $S - \sum_{i=1}^{k+1} x_i - J$ from $\alpha_1$. Since $\alpha_1$ is connected then $S - \sum_{i=1}^{k+1} x_i - J$ is connected and $S - J$ is connected ($S - J \subseteq S - \sum_{i=1}^{k+1} x_i - J$). Therefore, $J$ must be an element of $\omega$. Since there is such a $J$ intersecting $\alpha_{a_1, e}$ and $\alpha_{b_1, e}$ for every $\delta$ less than $\delta_0$, and since there is a $\delta_1$ such that $\alpha_{a_1, e}$ intersects at most one element of $\omega$ when $\delta < \delta_1$, then an element $C_1$ of $\omega$ must contain $a_1$ and $b_1$.

Let $z$ be any point of $a_1 b_1 = (a_1 + b_1)$. Then there are numbers $\delta_2$ and $\epsilon_2$ such that $x_1, x_2, \ldots, x_{k+1}$ and $z$ do not belong to $U_{e_2, \alpha a_1}, \epsilon_2 + U_{e_2, \alpha b_1}, \epsilon_2$. But $C_1$ belongs to $S_A + S_B + U_{e_2, \alpha a_1}, \epsilon_2 + U_{e_2, \alpha b_1}, \epsilon_2$ and hence $x_1, x_2, \ldots, x_{k+1}$ and $z$ do not belong to $C_1$. Therefore, $a_1 b_1$ has endpoints only on $C_1$ and the points $x_1, x_2, \ldots, x_{k+1}$ do not belong to $C_1$.

Now $a_1$ and $x_1$ are both common limit points of $S_A$ and $S_B$ and the component of $U(x, e)(\alpha + x_1)$ which contains $x_1$ is just $x_1$ itself. Also, when $\delta < \delta_0$, $\alpha - \alpha_{a_1, e}$ is an arc. Since $S - \sum_{i=2}^{k+1} x_i$ satisfies the conditions of $\delta 3$, then for any $e$ there is in $S_A + S_B + U_{e\alpha a_1, e} + U(x_1, e)$ a simple closed curve $J$ which does not separate any point of $S - \sum_{i=2}^{k+1} x_i - J$ from $\alpha - \alpha_{a_1, e}$. But $\alpha - \alpha_{a_1, e}$ is connected and hence $S - \sum_{i=2}^{k+1} x_i - J$ is connected. Since $S - J \subseteq S - \sum_{i=2}^{k+1} x_i - J$. 
then $S - J$ is connected. Now $\delta$ and $\epsilon$ can be chosen sufficiently small that: (1) $\bigcup_{\alpha \in C_1, \delta} C_1 = \emptyset$ ($i=2, \ldots, n$); and (2) $U(x_1, \epsilon) \cdot C_1 = \emptyset$. Hence, $J$ is not an element of $\Pi$. This is a contradiction and hence, no finite set of points of $S^*$ separates $S^*$.

§13. Lemma: Let $\alpha$ be an arc in $B = \sum_{1}^{n} C_1$ and let $S^*$ be a component of $S - \alpha$. Then if $S^*$ cannot be separated by the omission of any collection consisting of $k=1$ elements of $\Pi$ and any finite number of points, then $S^*$ is not separated by any collection consisting of $k$ elements of $\Pi$.

Proof: Notice first of all that $S - S^*$ is connected. Now let $C_1, C_2, \ldots, C_k$ be any set of $k$ elements of $\Pi$. Set $S' = S^* - \sum_{1}^{k} C_1$. Now consider $S' - C_1$.

Since $S - \sum_{1}^{k} C_1$ is connected, by §11, then every component of $S' - C_1$ has a limit point on $S - S^*$. Since $S^* - \sum_{1}^{k} C_1 + C_{10}$ is connected when $1 < 10 \leq k$, then every component of $S' - C_1$ has a limit point on $C_{10}$. It may be assumed that $C_1$ is a subset of $S'$. Also, $S'$ is a connected open subset of $S$ which cannot be separated by the omission of any pair of points. Then by §7, $S - C_1$ is connected and hence $S^* - \sum_{1}^{k} C_1$ is connected.
§14. Lemma: Let $\lambda$ be an arc of $S-\sum_{i}^{n} C_{i}$ and let $S^{*}$ be a component of $S-\lambda$. If $S^{*}$ cannot be separated by the omission of any collection consisting of $k-1$ elements of $\eta$ and any finite number of points, then $S^{*}$ cannot be separated by the omission of any collection consisting of $k$ elements of $\eta$ and any finite number of points.

Proof: (by induction). Note that $S-S^{*}$ is just $\lambda$ itself and since $S-\sum_{i}^{n} C_{i}$ is connected, then every component of $S^{*}-\sum_{i}^{n} C_{i}$ has limit points on every component of $S-S^{*}$.

Let $C_{1}, C_{2}, \ldots, C_{k}$ be any collection of $k$ elements of $\eta$ and let $x$ be any point. Then $S^{*}-\sum_{i}^{k} C_{i}$ is connected, by §13, and $S^{*}-\left(\sum_{i=1}^{k} C_{i}+x\right)+C_{i_{0}}$ is connected when $1 \leq i_{0} \leq k$. Set $S^{*}-S^{*}-\left(\sum_{i=1}^{k} C_{i}+x\right)$. This is a connected open subset of $S$ which cannot be separated by the omission of any pair of points. Since $S^{*}-\sum_{i}^{k} C_{i}$ is connected, then every component of $S^{*}-C_{1}$ has $x$ as a limit point, and since $S^{*}-\left(\sum_{i=1}^{k} C_{i}+x\right)+C_{i_{0}}$ is connected, then every component of $S^{*}-C_{1}$ has a limit point on $C_{i_{0}}$ when $1 < i_{0} \leq k$. It may be assumed that $C_{1}$ is a subset of $S^{*}$ and hence by §7, $S^{*}-C_{1}$ is connected. Therefore, $S^{*}-\left(\sum_{i=1}^{k} C_{i}+x\right)$ is connected.

Suppose that $S^{*}-\left(\sum_{i=1}^{k} C_{i}\right)$ is not separated by any collection of $m$ points. Let $x_{1}, x_{2}, \ldots, x_{m+1}$ be any set of
m+1 points. Since \( S = \sum_{i=1}^{n} C_i \) is connected, then every component of \( S^* - \sum_{i=1}^{k} C_i - \sum_{i=1}^{m+1} x_i \) has a limit point of \( \mathcal{A} \), and hence on every component of \( S - \mathcal{S}^* \). Set \( S' \) equal to \( S^* - \left( \sum_{i=1}^{k} C_i + \sum_{i=1}^{m+1} x_i \right) + C_{i_0} \). Since \( S^* - \left( \sum_{i=1}^{k} C_i + \sum_{i=1}^{m+1} x_i \right) + C_{i_0} \) is connected when \( 1 \leq i_0 \leq k \), then every component of \( S' - C_1 \) has a limit point of \( C_{i_0} \). Since \( S^* - \left( \sum_{i=1}^{k} C_i + \sum_{i=1}^{m+1} x_i \right) + x_{i_0} \) is connected when \( 1 \leq i_0 \leq m+1 \) then every component of \( S' - C_1 \) has \( x_{i_0} \) as a limit point. Also, \( S' \) is a connected open subset of \( S \) which cannot be separated by any pair of points, and it can be assumed that \( C_1 \) is a subset of \( S' \). Then by §7, \( S' - C_1 \) is connected and hence \( S^* - \left( \sum_{i=1}^{k} C_i + \sum_{i=1}^{m+1} x_i \right) \) is connected.

Then, by induction, \( S^* - \sum_{i=1}^{k} C_i \) cannot be separated by the omission of any finite set of points.

§15. Theorem: Let \( \mathcal{A} \) be any arc in \( S = \sum_{i=1}^{n} C_i \) and let \( S^* \) be a component of \( S - \mathcal{A} \). Then \( S^* \) cannot be separated by any collection consisting of \( k \) elements of \( \mathcal{F} \) and any finite set of points.

Proof: By §12, \( S^* \) cannot be separated by the omission of any finite number of points. Hence, the lemma is true for \( k=0 \). Now suppose that the lemma is true for \( k=m \). Then by §13 and §14, the lemma is true for \( k=m+1 \). Hence, the lemma is true for any finite \( k \).
\( \tilde{f}16. \) Lemma: No subarc of \( S - \sum_{1}^{n} C_1 \) separates \( S - \sum_{1}^{n} C_1 \).

Proof: By \( \tilde{f}6 \), if \( \alpha \) is an arc of \( S - \sum_{1}^{n} C_1 \) then \( S - \alpha \) is connected. But by \( \tilde{f}15 \), as \( S^* = S - \alpha \), \( S - \alpha - \sum_{1}^{n} C_1 \) \( S - \sum_{1}^{n} C_1 - \alpha \) is connected. Hence, \( \alpha \) does not separate \( S - \sum_{1}^{n} C_1 \).

\( \tilde{f}17. \) Theorem: The set \( S - \sum_{1}^{n} C_1 \) is homeomorphic to a region on a sphere.

Proof: (1) The set \( S - \sum_{1}^{n} C_1 \) is connected, by \( \tilde{f}11 \), locally compact and locally connected. Then \( S - \sum_{1}^{n} C_1 \) is a P-space in the sense used by E. R. van Kampen. (2) The set \( S - \sum_{1}^{n} C_1 \) is a locally compact, locally connected set which cannot be separated by the omission of any point, by \( \tilde{f}11 \). If \( x \) is any point of \( C_1 \), then every \( U(x, \varepsilon) \) contains points of \( S - C_1 \). Thus \( S - \sum_{1}^{n} C_1 \) is nondegenerate. Since \( S \) is a continuous curve, \( S \) is separable. Then by 1.25 chapter III of Wilder, \( [3] \), \( S \) is perfectly separable, and hence any subset of \( S \) is perfectly separable. Also, \( S \) is normal. Then \( S - \sum_{1}^{n} C_1 \) is a nondegenerate, perfectly separable and normal, locally compact, locally connected and connected set. Hence, by 3.32 chapter III of
Wilder [3], $S-\sum_{1}^{n}C_{i}$ is cyclicly connected. Therefore $S-\sum_{1}^{n}C_{i}$ contains at least one simple closed curve.

(3) Let $\beta$ be a simple closed curve in $S-\sum_{1}^{n}C_{i}$. Then $\beta \subseteq S$ and $S-\beta = S_{1}\cup S_{2}$. Now $\beta$ contains a limit point of $S_{1}$. Since $S-\sum_{1}^{n}C_{i}$ is open, there is an $e_{1}$ such that $U(x^{1}, e_{1}) \subseteq S-\sum_{1}^{n}C_{i}$. But $U(x^{1}, e_{1}) \cdot S_{1} \neq \phi$. Therefore $(S-\sum_{1}^{n}C_{i}) \cdot S_{1} \neq \phi$. Similarly $(S-\sum_{1}^{n}C_{i}) \cdot S_{2} \neq \phi$. But then $(S-\sum_{1}^{n}C_{i})-\beta = (S-\sum_{1}^{n}C_{i}) \cdot S_{1}\cup (S-\sum_{1}^{n}C_{i}) \cdot S_{2}$. Therefore $\beta$ separates $S-\sum_{1}^{n}C_{i}$.

(4) Let $ab$ be a closed arc of a simple closed curve $\beta$ of $S-\sum_{1}^{n}C_{i}$. Then by §16, $ab$ does not separate $S-\sum_{1}^{n}C_{i}$.

Then by a theorem of E. R. van Kampen [10], $S-\sum_{1}^{n}C_{i}$ is homeomorphic with a region on a sphere, and hence, to a region of the Euclidean plane.
§18. Lemma: Let $C$ be a finite collection of arcs $\alpha_i$ ($i=1,\ldots,n$) such that: (1) $\alpha_i \cdot \alpha_j \subseteq a_i + b_i$ where $a_i$ and $b_i$ are endpoints of $\alpha_i$ (i.e., the only points of intersection are endpoints); and (2) $\sum_{i=1}^{n} \alpha_i$ is connected. Then the arcs can be rearranged into the order $\alpha_{p_1}, \alpha_{p_2}, \ldots, \alpha_{p_n}$ so that $\sum_{i=1}^{k} \alpha_{p_i}$ is connected when $1 \leq k \leq n$.

Proof: Pick any arc and label it $\alpha_{p_1}$. Suppose that, for some $k_0$ less than $n$, $k_0$ arcs have been labeled $\alpha_{p_1}, \alpha_{p_2}, \ldots, \alpha_{p_{k_0}}$ such that $\sum_{i=1}^{k} \alpha_{p_i}$ is connected when $1 \leq k \leq k_0$. Suppose that no other arc intersects $\sum_{i=1}^{k_0} \alpha_{p_i}$. Then $\sum_{i=1}^{n} \alpha_i$ is not connected, contrary to hypothesis. Therefore some arc must intersect $\sum_{i=1}^{k_0} \alpha_{p_i}$. Label this arc $\alpha_{p_{k_0+1}}$. Then $\sum_{i=1}^{k} \alpha_{p_i}$ is connected when $1 \leq k \leq k_0+1$. Since the conditions of the lemma are obviously satisfied when $k_0=1$, they are satisfied, by mathematical induction, when $k_0=n$.

§19. Lemma: Let $C$ be a collection of a finite number of arcs $\alpha_i$ ($i=1,\ldots,m$) such that: (1) $\sum_{i=1}^{m} \alpha_i$ is
connected; and (2) \( \chi_1 \cdot \chi_j \leq a_1 + b_1 \) where \( a_1 \) and \( b_1 \) are the endpoints of \( \chi_1 \). If \( S^* \) is a component of \( S - C^* \) then no pair of points separates \( S^* \).

Proof: By §12, the lemma is true for \( m=1 \). Suppose that the lemma is true when \( m=k-1 \). Let \( C \) be any collection of \( k \) arcs satisfying the hypothesis. Let \( x \) be any point of \( S^* \) and suppose that \( S^* - x \) has a partition. There are three cases.

Case I. The collection \( C \) contains two arcs \( \chi_1 \) and \( \chi_2 \) where \( \chi_1 = a_1b_1 \) and \( \chi_2 = a_2b_2 \) such that \( a_1 \notin \chi_1 \) (\( i=2, \ldots, k \)) and \( a_2 \notin \chi_1 \) (\( i=1, 3, \ldots, k \)). Since \( \chi_1 \) and \( \chi_2 \) each have only one point in common with \( \sum_{j=3}^{k} \chi_i \), it is obvious that \( \sum_{j=3}^{k} \chi_1 + \chi_1 \) and \( \sum_{j=2}^{k} \chi_1 \) are both connected.

If \( S^* - x \) has a partition, then there are points \( A \) and \( B \) in \( S^* - x \) such that \( A \) and \( B \) lie in different components of \( S - (C^* + x) \). It may be assumed that \( A \) and \( B \) belong to \( S - \sum_{i=1}^{n} C_i \). Since \( \{ \chi_i \} (i=2, \ldots, k) \) is a collection of \( k-1 \) arcs which satisfies the conditions for \( C \), and since \( S^* \) is a subset of component of \( S - \sum_{i=1}^{k} \chi_i \), then \( A \) and \( B \) are in the same component of \( S - \sum_{i=2}^{k} \chi_1 - x \). Therefore, \( \chi_1 \) separates \( A \) and \( B \) in \( S - \sum_{i=2}^{k} \chi_1 - x \).

Now let \( a_1b_1 \) be the minimal subarc of \( \chi_1 \) with
endpoint \(b_1\) which separates \(A\) and \(B\) in \(S - \sum_{3}^{k} \alpha_1 - x\). Since \(\{a_1 b_1, \alpha_1\} \ (i=3, \ldots, k)\) is a collection of \(k-1\) arcs which satisfies the conditions for \(C_i\), and since \(S^*\) is a subset of a component of \(S - \sum_{3}^{k} \alpha_1 - a_1 b_1\), then \(A\) and \(B\) are in the same component of \(S - \sum_{3}^{k} \alpha_1 - a_1 b_1 - x\). Therefore, \(\alpha_2\) separates \(A\) and \(B\) in \(S - \sum_{3}^{k} \alpha_1 - a_1 b_1 - x\).

Let \(a_2 b_2\) be the minimal subarc of \(\alpha_2\), with endpoint \(b_2\) which separates \(A\) and \(B\) in \(S - \sum_{3}^{k} \alpha_1 - a_1 b_1 - x\). Then \(A\) and \(B\) belong to separate components of \(S - \sum_{3}^{k} \alpha_1 - a_1 b_1 - a_2 b_2 - x\). Call these components \(S_A\) and \(S_B\), respectively. Now \(a_1, a_2\) and \(x\) are all common limit points of \(S_A\) and \(S_B\).

Suppose that \(C' = \sum_{3}^{k} \alpha_1 + a_1 b_1 + a_2 b_2 + x\) and set \(\sigma_0\) equal to \(1/2 \min[\rho(a_1, b_1), \rho(a_2, b_2)]\). Then \(C' - (C_{a_1, x, \sigma} + C_{a_2, x, \sigma})\) is connected and \(C' - (C_{a_1, x, \sigma} + C_{a_2, x, \sigma})\) is connected when \(\sigma < \sigma_0\). Also, there is an \(\varepsilon\) such that \((U_{x} C_{a_1, x, \sigma}) - (U_{x} C_{a_2, x, \sigma}) = \emptyset\), when \(\sigma < \sigma_0\). Then by \(\mathcal{S}\), there are two simple closed curves \(J_1\) and \(J_2\) such that: (1) \(J_1 \subseteq S_A + S_B + U_{x} C_{a_1, x, \sigma} + U_{x} C_{a_2, x, \sigma}\) and \(J_2 \subseteq S_A + S_B + U_{x} C_{a_2, x, \sigma} + U_{x} C_{a_1, x, \sigma}\); and (2) \(S - J_1\) is connected and \(S - J_2\) is connected. Now \(J_1\) contains points of \(U_{x} C_{a_1, x, \sigma}\) and \(J_2\) contains points of \(U_{x} C_{a_2, x, \sigma}\). Hence, \(J_1 \neq J_2\). But \(J_1 \cdot J_2\) contains \(x\). Therefore, one of \(J_1\) and \(J_2\), say \(J_1\), is not a member of \(\mathcal{W}\). This is a contradiction.

Case II. The collection \(C\) contains one arc \(\alpha_1\)
where \(\alpha_1 = a_1 b_1\) such that \(a_1 \notin \alpha_1\) \((i=2, \ldots, k)\); and for
every other arc \( \alpha_{1_0} \) of \( C \), where \( \alpha_{1_0} = a_{1_0}b_{1_0} \) (\( i_0 \neq 1 \)), \( a_{1_0} \in \alpha_{j_1} \) and \( b_{1_0} \in \alpha_{j_2} \) for some \( j_1 \) different from \( i_0 \) and \( j_2 \) different from \( i_0 \). Again, let \( A \) and \( B \) be points of \( S^*-x \) such that \( A \) and \( B \) belong to different components of \( S-C^*-x \). Now \( \{ \alpha_i \} \) (\( i=2, \ldots, k \)) is a collection of \( k-1 \) arcs satisfying the conditions for \( C \). Since \( S^* \) is a subset of a component of \( S-\sum_{i=2}^{k} \alpha_i \), \( A \) and \( B \) belong to the same component of \( S-\sum_{i=2}^{k} \alpha_i-x \). Therefore, \( \alpha_1 \) separates \( A \) and \( B \) in \( S-\sum_{i=2}^{k} \alpha_i-x \). Now let \( a_1b_1 \) be the minimal subarc of \( \alpha_1 \), with \( b_1 \) as one endpoint, which separates \( A \) and \( B \) in \( S-\sum_{i=2}^{k} \alpha_i-x \). If \( S_A \) and \( S_B \) are the components of \( S-\sum_{i=2}^{k} \alpha_i-a_1b_1-x \) which contain \( A \) and \( B \), respectively, then \( a_1 \) and \( x \) are both common limit points of \( S_A \) and \( S_B \).

Suppose that \( \delta_0 < \rho(a_1,b_1) \). If \( C' = \sum_{i=2}^{k} \alpha_i+a_1b_1+x \), then \( C'-C_{a_1,b_1} \) is connected when \( \delta<\delta_0 \). But then, by \( \delta_3 \), there is a simple closed curve \( J \) in \( S_A+S_B+U_{a_1,b_1}+U_{e,C_1} \) such that \( S-J \) is connected. Then as in the proof of \( \delta 12 \), it can be shown that \( J \) is one of the elements of \( \mathcal{W} \), call it \( C_1 \), and that \( C_1 \) contains \( a_1 \) and \( x \) but does not contain any other point of \( C' \).

Due to \( \delta 13 \), it can be assumed that \( \sum_{i=1}^{k-1} \alpha_i \) is connected. Since \( \{ \alpha_i \} \) (\( i=1, \ldots, k-1 \)) is a collection of \( k-1 \) arcs satisfying the conditions for \( C \), and since \( S^* \) is a subset of a component of \( S-\sum_{i=1}^{k-1} \alpha_i \), then \( A \) and \( B \) belong
to the same component of $S-\sum_{i=1}^{k-1} \alpha_i - x$. Therefore, $\alpha_k$ separates $A$ and $B$ in $S-\sum_{i=1}^{k-1} \alpha_i - x$. Let the components of $S-C^* - x$ which contain $A$ and $B$ be $S_A^i$ and $S_B^j$, respectively. Now let $z$ be a point of $\alpha_k = (a_k + b_k)$ where $\alpha_k = a_k b_k$.

Suppose that $z$ is a common limit point of $S_A^i$ and $S_B^j$. Set $\delta_0$ equal to $1/2 \min \left[ \rho(a_k, z), \rho(b_k, z) \right]$. If $C^* = C^* + x$, then $C^* = (C^* + C^* + z, \delta_0, x, \delta_0)$ is connected. Then by $\delta 3$, there is a simple closed curve $J$ in $S_A^i + S_B^j + U^{C^*^i} z, \delta_0 + U^{C^*^j} x, \delta_0$ such that $S-J$ is connected. But $J$ contains $z$ and hence, is not identical to $C^*$, and also, $J$ contains $x$ and, hence, intersects $C^*$. This is a contradiction, hence, $z$ is not a common limit point of $S_A^i$ and $S_B^j$.

Now consider the subarc $a_k z$ of $\alpha_k$. If $A$ and $B$ belong to separate components of $S-\sum_{i=1}^{k-1} \alpha_i - z b_k - x$, then $\alpha_i, z b_k$ (i=1,...,k-1) is a collection of $k$ arcs which fall under case I. Hence, $A$ and $B$ must belong to the same component of $S-\sum_{i=1}^{k-1} \alpha_i - z b_k - x$ and $a_k z$ separates $A$ and $B$ in $S-\sum_{i=1}^{k-1} \alpha_i - z b_k - x$. Let $a_k z'$ be the minimal subarc of $a_k z$ with $a_k$ as an endpoint which does this. Let $S_A$ and $S_B$ be the components of $S-\sum_{i=1}^{k-1} \alpha_i - z b_k - a_k z' - x$ which contain $A$ and $B$, respectively. Then $z'$ and $x$ are common limit points of $S_A$ and $S_B$. As before, it can be shown that there is a $\delta_0$, and a simple closed curve $J$ in $S_A^i + S_B^j + U^{C^*^i} z, \delta + U^{C^*^j} x, \delta$ such that $S-J$ is connected, where $C^* = \sum_{i=1}^{k-1} \alpha_i + a_k z' + z b_k + x$. 
But $J$ contains $z'$ and $x$, and hence, $J$ is neither $C_i$ nor $C_i$ ($i=2,\ldots,n$). This is a contradiction.

Case III. If $\alpha_{i_0}$ is any arc of $C$ with endpoints $a_{i_0}$ and $b_{i_0}$, then there is an $\alpha'_{j_1}$ and an $\alpha'_{j_2}$ belonging to $C$ such that: (1) $j_1 \neq i_0$ and $j_2 \neq i_0$; and (2) $a_{i_0} \in \alpha'_{j_1}$ and $b_{i_0} \in \alpha'_{j_2}$. Because of §18, it can be assumed that $\sum_{1}^{k-1} \alpha'_1$ is connected. Let the components of $S-\sum_{1}^{k-1} \alpha'_1-x$ which contain $A$ and $B$ be $S^i_A$ and $S^i_B$, respectively. As before, it can be shown that $\alpha'_k$ separates $A$ and $B$ in $S-\sum_{1}^{k-1} \alpha'_1-x$, and hence, contains limit points of both $S^i_A$ and $S^i_B$.

Let $\alpha'_k$ be an arc such that $\alpha'_k=a_k b_k$, and suppose that there are two distinct points $z_1$ and $z_2$ of $a_k b_k$ which are common limit points of $S^i_A$ and $S^i_B$. Set $\sigma_0$ equal to $1/2 \min \{\rho(a_k+b_k, z_1+z_2), \rho(z_1, z_2)\}$. If $C'=C^*+x$ then $C'-(C_{z_1}', \sigma_0+C_{z_1}', \sigma_0)$ is connected and $C'-(C_{z_2}', \sigma_0+C_{z_2}', \sigma_0)$ is connected. Also, $(C_{z_1}', \sigma_0)*(C_{z_2}', \sigma_0)=\phi$. Then by §3, there are two simple closed curves $J_1$ and $J_2$ such that:

(1) $J_1 \subseteq S^i_A+S^i_B+U e C_{z_1}', \sigma_0+U e C_{z_1}', \sigma_0$ and $J_2 \subseteq S^i_A+S^i_B+U e C_{z_2}', \sigma_0+U e C_{z_2}', \sigma_0$; and (2) $S-J_1$ is connected and $S-J_2$ is connected. Now $J_1$ intersects $C_{z_1}', \sigma_0$ and $J_2$ intersects $C_{z_2}', \sigma_0$ and, hence, $J_1 \neq J_2$. But $J_1$ and $J_2$ both contain $x$; hence, $J_1 \cdot J_2 \neq \phi$. This is a contradiction. Therefore, there are not two distinct points of $a_k b_k-(a_k+b_k)$ which are common
Let \( z \) be a point of \( a_k b_k - (a_k + b_k) \) which is not a common limit point of \( S_A \) and \( S_B \). Consider the two sub-arcs \( a_k z \) and \( zb_k \) of \( a_k b_k \). If \( a_k z \) or \( zb_k \) separates \( A \) and \( B \) in \( S - \sum_{1}^{k-1} x_i - x \), then this case reduces to case II. Therefore, \( A \) and \( B \) belong to the same component of \( S - \sum_{1}^{k-1} x_i - x - a_k z \). Let \( z_1 b_k \) be the minimal subarc of \( zb_k \) which has \( b_k \) as an endpoint and which separates \( A \) and \( B \) in \( S - \sum_{1}^{k-1} x_i - x - a_k z \). Similarly let \( a_k z_2 \) be the minimal subarc of \( a_k z \) which has \( a_k \) as an endpoint and which separates \( A \) and \( B \) in \( S - \sum_{1}^{k-1} x_i - x - z_1 b_k \).

Let \( S_A \) and \( S_B \) be the usual components of \( S - C' \) where \( C' = \sum_{1}^{k-1} x_i + a_k z_2 + z_1 b_k + x \). Then \( z_1, z_2 \) and \( x \) are common limit points of \( S_A \) and \( S_B \). Let \( \delta_0 \) be less than \( \rho(a_k + b_k, z_1 + z_2) \). Then, as before, there are two simple closed curves \( J_1 \) and \( J_2 \) such that: (1) \( J_1 \neq J_2 \); and (2) \( J_1 \cdot J_2 \) contains \( x \); and (3) \( S - J_1 \) and \( S - J_2 \) are both connected. This is a contradiction. Hence, in all cases, \( S^* - x \) is connected.

Now suppose that \( x \) and \( y \) are two points of \( S^* \) such that \( S^* - (x + y) \) has a partition. Let \( S_A' \) and \( S_B' \) be two components of \( S^* - (x + y) \). Then, since no point separates \( S^* \), \( x \) and \( y \) are common limit points of \( S_A' \) and \( S_B' \). Now if \( C' = C^* + x + y \), then for any \( \delta_0 \), \( C' = (C_x^*, \delta_0 + C_y^*, \delta_0) \) is con-
Hence, by §3, there is a simple closed curve $J_1$ in $S_A^+S_B^+U e^i \gamma_x^i, \delta_0^i + U e^i \gamma_y^i, \delta_0^i$ such that $S-J_1$ is connected. Since this is true for any $\delta_0^i$ and $e$, it can be assumed that $J_1 \cdot C^* = \emptyset$. There are now two cases.

Case I. There is in $C$ an arc $a_1$, where $a_1 = a_1 b_1$, such that $a_1 \notin a_1 (i=2, \ldots, k)$. Now $\sum_{i=1}^{k} a_1$ is connected and $\{a_1 \}$ $(i=2, \ldots, k)$ is a collection of $k-1$ arcs satisfying the conditions for $C$. Since $S^*$ is a subset of a component of $S-\sum_{i=1}^{k} a_1$, then $A$ and $B$ are not separated in $S-\sum_{i=1}^{k} a_1 - x-y$, where $A$ and $B$ are points of $S^*-(x+y)$ which belong to different components of $S-(C^*+x+y)$. Therefore, $a_1$ separates $A$ and $B$ in $S-\sum_{i=1}^{k} a_1 - x-y$.

Let $a_1 b_1$ be the minimal subarc, with $b_1$ as an end-point, of $a_1 b_1$ which separates $A$ and $B$ in $S-\sum_{i=1}^{k} a_1 - x-y$. Let $S_A$ and $S_B$ be the components of $S-y-C'$ which contain $A$ and $B$, respectively, where $C' = \sum_{i=1}^{k} a_1 b_1 + x$. Now set $\delta_0 = 1/2 \rho(a_1 b_1)$. Then $C'-(C_1^1, \delta_0 + C_1^1, \delta_0)$ is connected. Since $S-y$ satisfies the conditions for §3, then by §3, there is a simple closed curve $J_2$ in $S_A^+S_B^+U e^i \gamma_x^i, \delta_0^i + U e^i \gamma_y^i, \delta_0^i$ such that $S-y-J_2$ is connected. But $S-J_2 \neq S-y-J_2$ and hence, $S-J_2$ is connected, and $J_2$ intersects $C^*$ and contains $x$, hence $J_2 \neq J_1$ and $J_2 \cdot J_1 \neq \emptyset$. This is a contradiction.

Case II. If $a_1$ is an arc of $C$ such that $a_1 \notin a_1 (i=2, \ldots, k)$, then by §3, there is a simple closed curve $J_1$ in $S_A^+S_B^+U e^i \gamma_x^i, \delta_0^i + U e^i \gamma_y^i, \delta_0^i$ such that $S-J_1$ is connected. Since this is true for any $\delta_0^i$ and $e$, it can be assumed that $J_1 \cdot C^* = \emptyset$. There are now two cases.
"a_k b_k" then there is an $\alpha_{j_1}$ and an $\alpha_{j_2}$ such that:

(1) $j_1 \neq 1_0$ and $j_2 \neq 1_0$; and

(2) $a_1 \in \alpha_{j_1}$ and $b_1 \in \alpha_{j_2}$.

Because of \( \mathcal{E} \), it can be assumed that

\[
\sum_{i=1}^{k-1} \alpha_i
\]

is connected. Also, if \( A \) and \( B \) are points of \( S^*-(x+y) \) which belong to different components of \( S-(C^*+x+y) \), then it can be shown as before that \( \alpha_k \) separates \( A \) and \( B \) in \( S-\sum_{i=1}^{k-1} \alpha_i-x-y \). Let the components of \( S-(C^*+x+y) \) which contain \( A \) and \( B \) be \( S_A \) and \( S_B \). Suppose that there is a point \( z \) on \( \alpha_k-(a_k+b_k) \) which is a common limit point of \( S_A \) and \( S_B \).

Suppose that \( \delta_0 \) is given such that \( \delta_0 < \rho(a_k+b_k, z) \). If \( C' \equiv C^*+x \) then \( C'-(G', \delta_0+C_z', \delta_0) \) is connected. Then by \( \mathcal{E}_3 \), there is a simple closed curve \( J_2 \) in \( S_A+S_B+U \cdot C_z', \delta_0 \) such that \( S-y-J_2 \) is connected. But then \( S-J_2 \) is connected. However, \( J_2 \cdot C_z', \delta_0 \neq \emptyset \), and \( J_2 \) contains \( x \). Hence \( J_2 \neq J_1 \) and \( J_2 \cdot J_1 \neq \emptyset \), a contradiction. Therefore, no point of \( \alpha_k-(a_k+b_k) \) is a common limit point of \( S_A \) and \( S_B \).

Let \( z \) be any point of \( a_k b_k \) and let \( a_k z \) and \( z b_k \) be the two subarcs of \( a_k b_k \) determined by \( z \). If \( A \) and \( B \) are separated in \( S-\sum_{i=1}^{k-1} \alpha_i-a_k z-x-y \) then the case reduces to case I. Suppose, therefore, that \( A \) and \( B \) are not separated in \( S-\sum_{i=1}^{k-1} \alpha_i-a_k z-x-y \). Let \( z_1 b_k \) be the minimal subarc of \( z b_k \) which has \( b_k \) as an endpoint and which separates
A and B in $S^k=\sum_{1}^{k-1} \alpha_1 a_k z-x-y$. Set $C'=\sum_{1}^{k-1} \alpha_1 + a_k z + z b_k + x$ and let $S_A$ and $S_B$ be the components of $S-y-C'$ which contain A and B, respectively. If $\delta_0$ is less than $\rho(z_1, b_k)$, then $C'(z_1, \delta_0 + C_1, \delta_0)$ is connected.

Then by 33, there is a simple closed curve $J_2$ in $S_A + S_B + C_1^i, \delta_0 + C_1^i, \delta_0$ such that $S-y-J_2$ is connected. However, as before, this can be shown to lead to a contradiction.

Therefore, if the lemma is true for $n=k-1$, it is true when $n=k$. Since the lemma is true when $n=1$, then by induction, it is true for any finite $n$.

20. Lemma: If $C$ is a finite collection of arcs $\alpha_i (i=1, \ldots, n)$ where $\alpha_i = a_i a_i^*$ such that $\alpha_i \cdot \alpha_j$ has a finite number of components, then there is a finite collection $C'$ of arcs $\alpha_i' (i=1, \ldots, n')$ such that: (1) $\alpha_i' \cdot \alpha_j' \subseteq a_i + b_i'$ where $a_i'$ and $b_i'$ are endpoints of $\alpha_i'$; and (2) $C' = C'^*$. 

Proof: Let $K$ be a collection such that every member of $K$ is a component of some $\alpha_i \cdot \alpha_j$. Then $K$ has a finite number of members. Now every component of $\alpha_i \cdot \alpha_j$ is a point or an arc. Let $b_{p_1}, b_{p_2}, \ldots, b_{p_k}$ be those members of $K$ which are points and let $b_{p_{k+1}}, b_{p_{k+2}}, \ldots, b_{p_m}$ be the
first and last points of those members of $K$ which are arcs.

Now let $b_1^1, b_2^1, \ldots, b_{q_1}^1$ be the points arranged in the order of occurrence on $a_1^i$ from $a_1^i$ to $a_1^i$. Consider the arcs $a_1^i b_1^i, b_1^i a_1^i, b_j^i b_{j+1}^i$ for all $1 \leq i \leq n$ and $1 \leq j < q_i$. This is a set of at most $n(m+1)$ distinct arcs (some arcs may be represented by two different representations). Let the arcs be labeled $\alpha_1, \alpha_2, \ldots, \alpha_n$, where each distinct arc is counted only once. If $C' = \{\alpha_i^j\}$ $(i=1, \ldots, n')$ then $C'$ satisfies the necessary requirements.

**Proposition.** Let $M'$ be a connected subset of $S$ consisting of the sum of a finite collection $C$ of arcs $\alpha_i^1 (i=1, \ldots, m)$ such that: (1) $M' \cdot C_i$ has a finite number of components when $i=1, \ldots, n$; and (2) if $\alpha_i^j$ and $\alpha_j^k$ are two members of $C$, then $\alpha_i^j \cdot \alpha_j^k \subseteq a_1^i + b_1^i$ where $a_1^i$ and $b_1^i$ are the endpoints of $\alpha_i^j$. Let $A_1 B_1$, $A_1^i A_2^i$ and $A_2^i B_2^i$ be arcs such that: (1) $A_1 B_1 \cdot M' = B_1$ $(i=1, 2)$; (2) $A_1 B_1 \cdot A_2 B_2 = \phi$; (3) $A_1^i A_2^i \subseteq C_i$; (4) $A_1^i A_2^i \cdot (A_1 B_1 + A_2 B_2 + M') = A_1^i + A_2^i$; (5) $(A_1^i + A_2^i) \cdot (A_1^i + A_2^i) = \phi$. Then in $S-(M' + \sum\limits_1^n C_i)$ there is an arc $P_1 P_2$ joining a point of $A_1 B_1$ and a point of $A_2 B_2$.

**Proof:** If $M' \subseteq \sum\limits_1^n C_i$, then $S-(M' + \sum\limits_1^n C_i) = S-\sum\limits_1^n C_i$.
which is connected by §11. Since \( A_1' A_2' \leq C_1' \); \( A_1' A_2' \cdot (A_1 B_1) = A_1 \) and \( (A_1 + A_2) \cdot (A_1' + A_2') = \emptyset \), then \( A_1 B_1 \) and \( A_2 B_2 \) each contains points of \( S - \sum_{l=1}^{n} C_1 \). Let \( P_1 \) be such a point on \( A_1 B_1 \) and let \( P_2 \) be such a point of \( A_2 B_2 \). Then since \( S - \sum_{l=1}^{n} C_1 \) is connected, there is an arc \( P_1 P_2 \) in \( S - \sum_{l=1}^{n} C_1 \).

Suppose, therefore, that \( M' \notin \sum_{l=1}^{n} C_1 \). Set \( M_0 \) equal to \( M' + (A_1 B_1 - A_1) + (A_2 B_2 - A_2) \). If \( M_0 \cdot (C_1 - A_1 A_2') = \emptyset \), let \( \alpha_{m+1} \) be empty, and if \( M_0 \cdot (C_1 - A_1 A_2') \neq \emptyset \), let \( \alpha_{m+1} \) be an arc in \( S - \sum_{l=1}^{n} C_1 + A_1' A_2' \) from \( C_1 - (A_1' A_2') \) to \( M_0 \) such that \( \alpha_{m+1} = C_1 \) and \( \alpha_{m+1} \cdot M_0 \) each consists of a single point. Set \( M_1 = M_0 + \alpha_{m+1} \).

Now suppose that \( M_1 \) has been defined when \( 1 \leq i < k \). If \( M_{k-1} \cdot C_k \neq \emptyset \), let \( \alpha_{m+k} \) be the null set, and if \( M_{k-1} \cdot C_k = \emptyset \), let \( \alpha_{m+k} \) be an arc in \( S - \sum_{l=1}^{n} C_1 + C_k \) from \( C_k \) to \( M_{k-1} \) such that \( \alpha_{m+k} \cdot C_k \) and \( \alpha_{m+k} \cdot M_{k-1} \) each consists of a single point. Define \( M_k \) to be \( M_{k-1} + \alpha_{m+k} \). Then, by induction, \( M_n \) can be defined such that: (1) \( M_n \) contains \( M' \); (2) \( M_n \cdot (C_1 - A_1 A_2') \neq \emptyset \); and (3) \( M_n \cdot C_i \neq \emptyset \) (i=2,...,n).

Let \( B_1' \) be the first point of intersection from \( A_1 \) to \( B_1 \) of \( A_1 B_1 \) with \( M' + \sum_{l=1}^{n} \alpha_m + C_1 + A_1' A_2' + \sum_{l=2}^{n} C_1 \). Then the set \( M \) consisting of \( M_n + (C_1 - (A_1' A_2')) + \sum_{l=1}^{n} C_1 - (A_1 B_1 - B_1') \) \( -(A_2 B_2 - B_2') \) is a closed connected set consisting of the sum of a collection of a finite number of arcs \( \alpha_i' \) (i=1,...,p') such that \( \alpha_1' \cdot \alpha_j' \) has a finite number of
components. Then by §20, there is a finite collection \( C \) of arcs \( \{ \alpha_i \} \) (i=1,...,p) such that: (1) \( \alpha_i \cdot \alpha_j \subseteq a_1+b_1 \) where \( a_1 \) and \( b_1 \) are endpoints of \( \alpha_i \); and (2) \( C^* = M \).

Also, \( S-M-A_1'A_2' \subseteq S-M'-\sum_1^n C_i \). Let \( P_1 \) be a point of \( A_1'B_1'-(A_1'+B_1') \) and let \( P_2 \) be a point of \( A_2'B_2'-(A_2'+B_2') \) and suppose that \( P_1 \) and \( P_2 \) belong to different components of \( S-M-A_1'A_2' \). Now \( A_1'B_1'+A_1'A_2'+A_2'B_2'-(A_1'+A_2'+B_1'+B_2') \) is a connected subset of \( S-M \) and hence \( P_1 \) and \( P_2 \) belong to the same component \( S^* \) of \( S-M \). Let \( A_n'A_n' \) be a minimal subarc of \( A_1'A_2' \) which separates \( P_1 \) and \( P_2 \) in \( S^* \). Let \( S_1 \) and \( S_2 \) be the components of \( S-M-A_1'A_2' \) which contain \( P_1 \) and \( P_2 \), respectively.

Since by §19, no pair of points separates \( S^* \) then:

1. \( A_1^n \neq A_2^n \) and (2) one of the components, say \( S_1 \), has an accessible limit point on \( A_1^n-A_2^n \). Let \( x \) be such a point. Since \( S-(A_1^n+A_2^n) \) is connected, \( S_1 \) also has an accessible limit point on \( M \). Let \( y \) be such a point. Then there is an arc \( xy \) in \( S_1+x+y \) which joins \( x \) and \( y \).

Since \( \sum_1^n \sum_{m+1}^{n+1} +M' \cdot A_1'A_2' = \phi \), there is a \( \sigma_o \) such that \( [U(A_1^n+\sigma_o)\cup(A_2^n+\sigma_o)] \cdot (x+\sum_1^n \sum_{m+1}^{n+1} +M') = \phi \). If \( C' = M+A_1^n+A_2^n \) then \( C'-(C_1^n+\sigma_o+C_2^n+\sigma_o) = M-(C_1^n+\sigma_o+C_2^n+\sigma_o)+A_1^nA_2^n \rightarrow (C_1^n+\sigma_o+C_2^n+\sigma_o) \) which consists of exactly two components.

Then by §3, there is a simple closed curve \( J_1 \) in \( S_1+S_2 \)

\[ +UeC_1^n+\sigma_o+UeC_2^n, \sigma_o \] such that \( J_1 \) does not separate any point
of \( S-J_1 \) from \( C'=\left( C'^{1}_{A_1}, \sigma'^{1}_{0}, A'^{2}, \sigma'^{2}_{0} \right) \). Since this is true for any \( \varepsilon \), it can be assumed that \( \left( U_{i=1,2} C'^{i}_{A_i}, \sigma'^{i}_{0} \right) \cdot xy = \emptyset \). Now there is an arc \( x_1 y_1 \) of \( J_1 \) in \( S_2+x+y \) such that \( x_1 \) and \( y_1 \) belong to \( C_1 \) and \( x_1 y_1 \cdot xy = \emptyset \). But then \( xy \) is an arc joining \( M-\left( C'^{1}_{A_1}, \sigma'^{1}_{0}, A'^{2}, \sigma'^{2}_{0} \right) \) and \( A'^{2}-\left( C'^{i}_{A_i}, \sigma'^{i}_{0}, A'^{i}_{0} \right) \) in \( S-x_1y_1 \). Since \( x_1y_1 \) does not separate any point of \( S-x_1y_1 \) from \( C'=\left( C'^{1}_{A_1}, \sigma'^{1}_{0}, A'^{2}, \sigma'^{2}_{0} \right) \) then \( S-x_1y_1 \) is connected. But by \( \text{Se} \), this is a contradiction.

Therefore, \( P_1 \) and \( P_2 \) belong to the same component of \( S-M-A_1'B_1 \) and there is an arc \( P_1P_2 \) joining \( P_1 \) and \( P_2 \) in \( S-M-A_1'B_1 \) which is a subset of \( S-M-\bigcup_{i=1}^{n} C_i \).

\[ \text{§ 22. Lemma: Let } a_1b_1 \text{ be a minimal separating arc with endpoints only on } C_1 \text{ and let } <a_1r_1b_1> \text{ and } <a_1r_2b_1> \text{ be the two components of } C_1-(a_1+b_1). \text{ Then } <a_1r_1b_1> \text{ and } <a_1r_2b_1> \text{ belong to different components of } S-a_1b_1. \]

\[ \text{Proof: Since } a_1b_1 \text{ is a minimal separating arc, } a_1 \text{ and } b_1 \text{ are common limit points of all components. Suppose } <a_1r_1b_1> \text{ and } <a_1r_2b_1> \text{ belong to the same component } S_1 \text{ of } S-a_1b_1. \text{ Then } C_1 \equiv S_1+a_1+b_1. \text{ Let } \sigma_0 \text{ be such that:} \]

(1) \( \sigma_0 < 1/2 \rho(a_1,b_1) \); and (2) \( \sigma_0 < \rho(a_1+b_1,C_i) \) (1 \( \neq \) 1).

If \( \alpha=a_1b_1 \), then \( <a_1, \sigma_0+b_1, \sigma_0> \cdot C_1 = \emptyset \) (1 \( \neq \) 1) and \( \alpha-(a_1, \sigma_0+b_1, \sigma_0) \) is connected. But then by \( \text{Se} \), there
is a simple closed curve \( J \) which intersects \( a_1, b_1, a_2, b_2 \), and \( S_2 \) such that \( S-J \) is connected. \( S_2 \) is a second component of \( S-a_1 b_1 \). But this is a contradiction. Therefore \( <a_1 r_1 b_1> \) and \( <a_1 r_2 b_1> \) belong to different components of \( S-a_1 b_1 \).

§ 23. Lemma: Let \( ab \) be an arc with endpoints only on some element \( C_1 \) of \( \mathcal{U} \), and let \( <ar_1 b> \) and \( <ar_2 b> \) be the two components of \( C_1-(a+b) \). Then \( <ar_1 b> \) and \( <ar_2 b> \) belong to different components of \( S-ab \).

Proof: Let \( \alpha \) be an arc with endpoints only on \( C_1 \). Let these endpoints be \( a \) and \( b \). Let \( r_1 \) and \( r_2 \) be points of \( C_1 \) which are separated in \( C_1 \) by \( a+b \). Suppose that \( r_1 \) and \( r_2 \) belong to the same component of \( S-ab \). Suppose that \( C_2 \) is the first element of \( \mathcal{U} \) that is intersected by \( ab \) from \( a \) to \( b \). Let \( a_1 \) be the first point and \( b_1 \) be the last point of \( ab \). Then \( r_1 \) and \( r_2 \) belong to the same component \( S \) of \( S-(a_1+b_1) \).

Let \( a_1 y b_1 \) be a subarc of \( C_2 \). Suppose that \( a_1 y b_1 \) separates \( r_1 \) and \( r_2 \) in \( S \). Obviously \( a_1 y b_1 \notin \alpha \). Let \( S_1 \) and \( S_2 \) be the two components which contain \( r_1 \) and \( r_2 \), respectively. Suppose that every point of \( a_1 y b_1 \) is a common limit point of \( S_1 \) and \( S_2 \).
Let $y_1$ and $y_2$ be two points of $a_1yb_1-(a_1+b_1)$ and set $d_1$ equal to $1/2 \min \left[ \rho(y_1, y_2), \rho(y_1 + y_2, a_1 + b_1) \right]$. If $C = a_1 + a_1yb_1 + b_1b$, then $C_{y_1, \mathcal{S}}$ and $C_{y_2, \mathcal{S}}$ is a subset of $a_1yb_1$ when $\mathcal{S} < \mathcal{S}_1$. Since $a$ and $b$ are both common limit points of $S_1$ and $S_2$, then $C = (C_{y_1, \mathcal{S}} + C_{y_2, \mathcal{S}})$ consists of three components, each of which contains a limit point of $S_1$ and a limit point of $S_2$.

Now by §3, there is a simple closed curve $J$ in $S_1 + S_2 + U_0 \in C_{y_1, \mathcal{S}} + U_0 \in C_{y_2, \mathcal{S}}$ such that no point of $S-J$ is separated from $C = (C_{y_1, \mathcal{S}} + C_{y_2, \mathcal{S}})$. Now $C = (C_{y_1, \mathcal{S}} + C_{y_2, \mathcal{S}})$ contains points of $C_2$ and hence $J$ is not equal to $C_2$. Therefore, $J$ contains a point which does not belong to $C_2$. Let $z$ be such a point. Then $z$ lies in the complement of $S_1$ or in the complement of $S_2$, say the first. Since $C_{y_1, \mathcal{S}} + C_{y_2, \mathcal{S}}$ is a subset of $C_2$, then there is a subarc $J_1$ of $J$ in the complement of $S_1$ which has endpoints only on $C_2$. Then $J_1$ does not separate any point of $S-J_1$ from $S_2 + C = (C_{y_1, \mathcal{S}} + C_{y_2, \mathcal{S}})$. But this set is connected and hence $S-J_1$ is connected. But by §6, $S-J_1$ is not connected. This is a contradiction, and thus some point of $a_1yb_1$ is not a common limit point of $S_1$ and $S_2$.

Let $y$ be such a point. Let $a_1y$ be the minimal arc of $aa_1 + a_1y$ with endpoint $a$ which separates $r_1$ and $r_2$ in $S-(yb_1 + b_1b)$, and let $y_2b$ be the minimal arc of $yb_1 + b_1b$. ...
with endpoint b which separates $r_1$ and $r_2$ in $S-a_1$. Let $S_1'$ and $S_2'$ be the two components of $S-(a_1+y_2 b)$ which contain $r_1$ and $r_2$, respectively. Then $y_1$ and $y_2$ are common limit points of $S_1'$ and $S_2'$. Suppose that $y_1$ and $y_2$ both belong to $<a_1 y_1 b>$. 

Set $\delta_1'$ equal to $1/2 \delta(y_1+y_2, a_1+b_1)$. If $C=a_1+y_2 b$, then $C-(C_{y_1}+C_{y_2}, \delta)$ consists of two components when $\delta<\delta_1'$. One of these components contains $a$ and the other contains $b$. Therefore, both $S_1'$ and $S_2'$ have limit points on every component of $C-(C_{y_1}+C_{y_2}, \delta)$. Now by $\delta_3$, there is a simple closed curve $J$ in $S_1'+S_2'+U_y C_{y_1}+U_y C_{y_2}, \delta$ such that no point of $S-J$ is separated from $C-(C_{y_1}+C_{y_2}, \delta)$. Now $C-(C_{y_1}+C_{y_2}, \delta)$ contains points of $C_2$ and hence $J$ contains at least one point which does not belong to $C_2$. Let $z$ be such a point. It can be assumed that $z$ lies in the complement of $S'$.

Since $C_{y_1}+C_{y_2}, \delta$ is a subset of $C_2$, there is a subarc $J_1$ of $J$ in the complement of $S_1'$ which has endpoints only on $C_2$. Then $J_1$ does not separate any point of $S-J_1$ from $S_2'+C-(C_{y_1}+C_{y_2}, \delta)$. But this set is connected and hence $S-J_1$ is connected. Since this is a contradiction, by $\delta_6$, then at least one of $y_1$ and $y_2$ must lie in the complement of $<a_1 y_1 b>$. However, since $a_1+b_1 b$ does not separate $r_1$ and $r_2$ in $S$, then one of $y_1$ and $y_2$, say $y_1$, still belongs to $<a_1 y_1 b>$. Now suppose that
every point of \( a_1 y_1 - a_1 \) is a common limit point of \( S'_1 \) and \( S'_2 \).

Let \( y_3 \) be a point of \( <a_1 y_1> \) and set \( \delta_2 \) equal to \( 1/2 \min \{ \rho(y_1, y_3), \rho(y_3, a_1) \} \). If \( C= (a y_1 + y_2 b) \), then \( C= (C y_1 \delta + C y_3 \delta) \) consists of three components when \( \delta < \delta_2 \). Two of these components contain \( a \) and \( b \), respectively, and the third contains points of \( y_1 y_3 \). Now \( a, b \) and all points of \( y_1 y_3 \) are limit points of \( S'_1 \) and \( S'_2 \).

By \( \delta_3 \), there is a simple closed curve \( J \) in \( S'_1 + S'_2 \) such that no point of \( S-J \) is separated from \( C= (C y_1 \delta + C y_3 \delta) \). Now \( C= (C y_1 \delta + C y_3 \delta) \) contains points of \( C_2 \) and hence some point \( z \) of \( J \) does not belong to \( C_2 \). Assume that \( z \) belongs to the complement of \( S'_1 \). Then as before, there is a subarc \( J_1 \) of \( J \) which belongs to the complement of \( S'_1 \) and has endpoints only on \( C_2 \). Also, no point of \( S-J_1 \) is separated from \( S'_2 - C= (C y_1 \delta + C y_3 \delta) \). But this set is connected and so \( S-J_1 \) is connected. Since, by \( \delta \), this is a contradiction, then some point of \( a_1 y_1 - a_1 \) is not a common limit point of \( S'_1 \) and \( S'_2 \).

Let \( y' \) be such a point. Let \( y'_1 y_1 \) be the minimal subarc of \( y_1 y' \) which separates \( r_1 \) and \( r_2 \) in \( S-(a y'+ y_2 b) \), and let \( a y'_2 \) be the minimal subarc of \( a y' \) which separates \( r_1 \) and \( r_2 \) in \( S-(y_1 y'+ y_2 b) \). Suppose that \( y'_2 \) belongs to \( <a_1 y b_1> \). Let \( S'_1 \) and \( S'_2 \) be the components of \( S-(a y'_2 + y'_1 y_1 \)
+y_2b) which contain r_1 and r_2, respectively.

Set \( d_3 \) equal to \( \phi(y_2', a_1) \). If \( C=ay_2+y_1'y_1+y_2'b \)
then, when \( \delta<\delta_3 \), \( C=(Cy_1', \delta+Cy_2', \delta) \) will consist of at most
three components, two of which will contain \( a \) and \( b \), re-
respectively, and the third (if it exists) of which will
contain \( y_1 \). Therefore, every component of \( C=(Cy_1', \delta+Cy_2', \delta) \)
contains a limit point of \( S_1' \) and a limit point of \( S_2' \).

By \( \delta_3 \), there is a simple closed curve \( J \) in \( S_1'+S_2' \)
+UeCy_1', \delta+UeCy_2', \delta \) such that no point of \( S-J \) is separated
from \( C=(Cy_1', \delta+Cy_2', \delta) \). Now \( C=(Cy_1', \delta+Cy_2', \delta) \) contains points
of \( C_2' \) and hence some point \( z \) of \( J \) does not belong to \( C_2' \).
Assume that \( z \) belongs to the complement of \( S_1' \). Then as
before, there is a subarc \( J_1 \) of \( J \) which belongs to the
complement of \( S_1' \) and has endpoints only on \( C_2' \). Also, no
point of \( S-J_1 \) is separated from \( S'+C=(Cy_1', \delta+Cy_2', \delta) \). But
this set is connected and so \( S-J_1 \) is connected. Since by
\( \delta_6 \), this is a contradiction, then \( y_2' \) does not belong to
\( \langle a_1'y_1 \rangle \). Then \( ay_2' \) and \( y_2'b \) are both subsets of \( \alpha \).
\[ \langle a_1'y_1 \rangle; \text{ suppose } y_1+y_1' \text{ separates } r_1 \text{ and } r_2 \text{ in } S-(ay_2'
+y_2b). \] Then \( y_1+y_1' \) would separate \( r_1 \) and \( r_2 \) in \( S-\alpha \). But
by \( \delta 19 \), no pair of points separates any component of \( S-\alpha \).
Therefore: (1) \( y_1 \neq y_1' \) and (2) some point of \( y_1'y_1'-(y_1
+y_1') \) is a limit point of \( S_1' \), and some point of \( y_1'y_1'-(y_1+y_1') \)
is a limit point of \( S_2' \). Let \( y_4 \) and \( y_4' \) be such limit
points of \( S_1^n \) and of \( S_2^n \), respectively.

Set \( \delta_4 \) equal to \( 1/2 \, \varphi(y_4+y_4', y_1+y_1') \). If \( C=ay_2' +y_1'y_1'+y_2b \), then, when \( \delta<\delta_4 \), the set \( C=(Cy_1'+Cy_1', \delta) \) consists of three components, two of which contain \( a \) and \( b \), respectively, and the third, of which contains \( y_4 \) and \( y_4' \). Therefore, every component of \( C=(Cy_1'+Cy_1', \delta) \) contains a limit point of \( S_1^n \) and a limit point of \( S_2^n \).

By \( \delta_3 \), there is a simple closed curve \( J \) in \( S_1^n+S_2^n \) such that no point of \( S-J \) is separated from \( C-(Cy_1'+Cy_1', \delta) \). Now \( C=(Cy_1'+Cy_1', \delta) \) contains points of \( S_2^n \) and hence some point \( z \) of \( J \) does not belong to \( C_2^n \). Assume that \( z \) belongs to the complement of \( S_1^n \). Then as before, there is a subarc \( J_1 \) of \( J \) which belongs to the complement of \( S_1^n \) and has endpoints only on \( C_2^n \). Also, no point of \( S-J_1 \) is separated from \( S_2^n+C=(Cy_1'+Cy_1', \delta) \). But this set is connected and so \( S-J_1 \) is connected. But by, \( \delta \delta \), this is a contradiction, and hence \( a_1yb_1 \) does not separate \( r_1 \) and \( r_2 \) in \( S-(aa_1+b_1b) \).

If \( \alpha_2 \) is defined to be \( aa_1+a_1yb_1+b_1b \), then \( \alpha_2 \): (1) has endpoints only on \( C_1 \); (2) does not have any subarc with endpoints only on \( C_2 \), and (3) does not separate \( r_1 \) and \( r_2 \).

Since there are only \( k \) elements in \( \pi_k \), this process need be repeated at most \( k-1 \) times to yield an arc \( \alpha_k \).
such that (1) \( \alpha_k \) has endpoints only on \( C_1 \); (2) \( \alpha_k \) does not have any subarc with endpoints only on \( C_1 \) (i.e., \( \ldots, k \)); and (3) \( \alpha_k \) does not separate \( r_1 \) and \( r_2 \). But as a result of \( \S 2 \), \( \alpha_k \) is a minimal separating arc. Therefore by \( \S 2 \), \( \alpha_k \) must separate \( r_1 \) and \( r_2 \). This is a contradiction and hence \( \alpha_k \) must separate \( r_1 \) and \( r_2 \). Therefore, \( \langle ar_1, b \rangle \) and \( \langle ar_2, b \rangle \) must lie in separate components of \( S - \alpha \).

\( \S 24. \) Lemma: If \( a_1b_1 \) is a minimal separating arc then \( S - a_1b_1 \) has only two components.

Proof: Suppose \( S - a_1b_1 \) has at least three components \( S_1, S_2 \) and \( S_3 \). Then \( a_1 \) and \( b_1 \) are common limit points of \( S_1, S_2 \) and \( S_3 \). If \( \alpha = a_1b_1 \) then there is a \( \delta_0 \) sufficiently small that (1) \( \alpha = (\alpha_{a_1}, \delta_0, \alpha_{b_1}, \delta_0) \) is connected; and (2) \( \alpha_{a_1}, \delta_0 \) intersects only one element of \( \Pi \). But by \( \S 3 \), there are two simple closed curves \( J_1 \) and \( J_2 \) such that: (1) \( J_1 \subseteq \mathcal{S}_1 + \mathcal{S}_2 + U \varepsilon \alpha_{a_1}, \delta_0 + U \varepsilon \alpha_{b_1}, \delta_0 \); (2) \( J_2 \subseteq \mathcal{S}_1 + \mathcal{S}_2 + U \varepsilon \alpha_{a_1}, \delta_0 + U \varepsilon \alpha_{b_1}, \delta_0 \); (3) \( S - J_1 \) is connected; and (4) \( S - J_2 \) is connected.

Now \( J_1 \) and \( J_2 \) intersect different components of \( S - a_1b_1 \) and hence \( J_1 \neq J_2 \). But \( J_1 \) and \( J_2 \) both intersect \( \alpha_{a_1}, \delta_0 \) and hence they cannot both be an element of \( \Pi \).
This is a contradiction, and hence $S - a_1b_1$ has only two components.

\[ \text{25. Lemma: Suppose that the arc xay separates } S. \text{ Suppose that } M \text{ is a connected set containing xay such that } M - \langle xay \rangle \text{ is connected. Let } S_0 \text{ be any component of } S - xay. \text{ Then there is an arc xby in } S - S_0 \text{ with endpoints } x \text{ and } y \text{ such that xby does not separate } S. \]

Proof: Let $a_2b_2$ be a minimal separating subarc of xay where $a_2$ is the first point of $a_2b_2$ from $x$ to $y$. Then by §4, $a_2b_2$ has endpoints only on an element of $\mathcal{T}$ (call it $C_1$). Also, by §5, it is seen that no proper subarc of $a_2b_2$ has endpoints only on any element of $\mathcal{T}$.

Let $a_1b_1$ be the maximum subarc of xay with the properties that: (1) $a_2b_2$ is a subarc of $a_1b_1$, (2) $a_1$ and $b_1$ belong to $C_1$. Let $a_2r_1b_2$ and $a_2r_2b_2$ be the arcs into which $a_2b_2$ divides $C_1$. Now if $a_1 = a_2$ and $b_1 = b_2$, then by §23, and §24, $S - a_1b_1$ contains exactly two components, $S_1$ and $S_2$, one of which, say $S_1$, contains $\langle a_2r_1b_2 \rangle$ and the other of which, $S_2$, contains $\langle a_2r_2b_2 \rangle$.

Any component of $S - xay$ is a subset of either $S_1$ or $S_2$. Let $S_0$ be a component of $S - xay$ and suppose that $S_0 \subseteq S_1$. Now consider the arc $xa_2r_2b_2y$. This lies in
$S_{0}$ has $x$ and $y$ as endpoints, and intersects $C_{1}$ only in a single arc.

Now suppose that $a_{1} \neq a_{2}$ or $b_{1} \neq b_{2}$ or both.

Lemma 25.1. Both of the points $a_{1}$ and $b_{1}$ belong to the same one of $a_{2}r_{1}b_{2}$ or $a_{2}r_{2}b_{2}$, say to the first.

Proof of lemma 25.1: Other cases being obvious, suppose that $a_{1} \in \langle a_{2}r_{1}b_{2} \rangle$ and $b_{1} \in \langle a_{2}r_{2}b_{2} \rangle$.

Now $S_{a_{2}b_{2}}$ has a partition and by §23, there are components $S_{1}$ and $S_{2}$ of $S_{a_{2}b_{2}}$ such that $r_{1} \in S_{1}$ and $r_{2} \in S_{2}$. But $M = \langle xay \rangle + x_{a_{1}b_{1}}y$ is a connected set which does not intersect $a_{2}b_{2}$ and this set joins a point of $S_{1}$ and a point of $S_{2}$. This is a contradiction and hence $a_{1}$ and $b_{1}$ belong to the same arc $a_{2}r_{1}b_{2}$ of $C$. The lemma is proved.

Let $C_{1} = a_{1}s_{1}b_{1} + a_{1}s_{2}b_{1}$ where $a_{1}s_{1}b_{1}$ and $a_{1}s_{2}b_{1}$ are the two arcs into which $a_{1}b_{1}$ divides $C_{1}$. Since $a_{1}$ and $b_{1}$ both belong to $a_{2}r_{1}b_{2}$ then $a_{2}$ and $b_{2}$ both belong to one arc of $C_{1}$, say $a_{1}s_{1}b_{1}$. Since $a_{2}b_{2}$ is a subset of $a_{1}b_{1}$ then every component of $S_{a_{1}b_{1}}$ is a subset of a component of $S_{a_{2}b_{2}}$. 
Suppose that $S' = \{x' \mid x' \in S - a_1b_1\}$ and there is an arc $x'y'$ in $S - a_1b_1 + y'$ which joins $x'$ and a point $y'$ of $a_1s_2b_1 = \{a_1 + b_1\}$ and $S'_1 = \{x' \mid x' \in S - a_1b_1$ and $x' \notin S'_2\}$. It is to be shown that $S'_1/S'_2$ is a partition of $S - a_1b_1$.

First suppose $a_1s_2b_1 = a_1b_1$. Then $a_1s_2b_1 = a_1b_1$. But $a_2b_2$ is a subarc of $a_1b_1$ which has endpoints only on $C_1$. This is a contradiction and thus there is an $x_2$ in $a_1s_2b_1$ such that $x_2 \in S - a_1b_1$. Therefore $S'_2$ is not empty.

Now since $a_2$ and $b_2$ are both points of $a_1s_2b_1$ then either $a_2r_1b_2 \subseteq a_1s_2b_1$ or $a_2r_2b_2 \subseteq a_1s_2b_1$. But since $a_1$ and $b_1$ are points of $a_2r_1b_2$ and since either $a_1 \neq a_2$ or $b_1 \neq b_2$, then $a_2r_2b_2 \subseteq a_1s_2b_1$. For similar reasons to those given above, $a_2r_2b_2$ contains at least one point $x_1$ of $S - a_1b_1$. Let $x_1y'$ be any arc joining $x_1$ and a point $y'$ of $\langle a_1s_2b_1 \rangle$. Since $a_2r_2b_2 \subseteq a_1s_2b_1$, then $\langle a_1s_2b_1 \rangle \subseteq \langle a_2r_1b_2 \rangle$, and $J$ joins a point of $\langle a_2r_2b_2 \rangle$ and a point of $\langle a_2r_1b_2 \rangle$. But by §23, $\langle a_2r_1b_2 \rangle$ and $\langle a_2r_2b_2 \rangle$ are subsets of different components of $S - a_2b_2$. Hence $J$ intersects $a_2b_2$ and hence, $a_1b_1$. Therefore $J \notin S - a_1b_1 + y'$.

Therefore $x_1 \in S'_1$ and hence $S'_1$ and $S'_2$ are not empty. Now $S - a_1b_1$ is locally connected, and hence, no point of one component can be a limit point of any combination of other components. Obviously, if any point of a component of $S - a_1b_1$ belongs to $S'_2$, the whole component does. Therefore,
if any point of a component of $S - a_1 b_1$ belongs to $S_1$ then the whole component does. But then $S_1 \cdot \overline{S_2} S_1 \cdot \overline{S_2} = \phi$. Therefore, $S - a_1 b_1 = S_1 \mid S_2$.

Lemma 25.2: If $P$ is any point of $a_1 b_1 \cdot C_1$ then $P$ belongs to $a_1 a_1 b_1$.

Proof of lemma 25.2: Suppose that $P$ belongs to $\langle a_1 s_2 b_1 \rangle$. Since $a_1 b_1 \cdot \langle a_1 s_2 b_1 \rangle \neq \phi$, there is a subarc $PQ$ of $\langle a_1 b_1 \rangle$ such that $P \in \langle a_1 s_2 b_1 \rangle$ and $Q \in \langle a_1 a_1 b_1 \rangle$. Let $Q'$ be the first point of $PQ$ from $P$ to $Q$ on $a_1 a_1 b_1$. Let $P'$ be the first point of $PQ'$ from $Q'$ to $P$ on $a_1 s_2 b_1$. Since $PQ \subseteq \langle a_1 b_1 \rangle$, then $a_1 \notin P'Q'$ and $b_1 \notin P'Q'$. Then $P' \neq Q'$ and $P'Q'$ is an arc with endpoints only on $C_1$. Then by $\delta$, $P'Q'$ separates $S$. Also as a result of $\delta 23$, $a_1$ and $b_1$ are in separate components of $S - P'Q'$. But $M = \langle xay \rangle + xa_1 + b_1 y$ is a connected set joining $a_1$ and $b_1$ which does not intersect $P'Q'$. This is a contradiction, and hence $P$ belongs to $a_1 a_1 b_1$. Lemma 25.2 is proved.

Lemma 25.3: The set $a_1 s_2 b_1 \cdot S_1^i = \phi$ and the set $a_1 s_1 b_1 \cdot S_2^i = \phi$. 
Proof of Lemma 25.3: Obviously \( a_1 s_2 b_1 \cdot S_{1}^{m} = \emptyset \).

Now suppose that \( P \in a_1 s_1 b_1 \). If \( P \in a_1 b_1 \) then obviously \( P \notin S_{2}' \). Suppose then that \( P \notin a_1 b_1 \). Let \( \langle P_1 P_2 \rangle \) be the component of \( a_1 s_1 b_1 - a_1 b_1 \) which contains \( P \). Then \( P_1 \) and \( P_2 \) belong to \( a_1 b_1 \). Suppose that \( P_1 \) belongs to the subarc \( a_1 P \) and the subarc \( P_1 b_1 \) of \( a_1 b_1 \) intersects \( P_2 \), where \( a_1 P \) and \( P_2 \) are the subarcs of \( a_1 s_1 b_1 \). Let \( P_2' \) be the first point of intersection of \( P_1 b_1 \) with \( P_2 \) from \( P_1 \) to \( b_1 \). Let \( P_1' \) be the last point of intersection of \( P_1 P_2' \) with \( a_1 P \) from \( P_1 \) to \( P_2' \). Since by Lemma 25.2, \( P_1 P_2' \) does not intersect \( a_1 s_2 b_1 \), then \( P_1 P_2' \) has endpoints only on \( C_1 \) and by \( \S 6 \), \( S-P_1 P_2' \) is not connected. Also, it is easily seen that by \( \S 23 \), \( P \) and \( a_1 s_2 b_1 \) belong to separate components of \( S-P_1 P_2' \). Hence any arc joining \( P \) and a point \( y \) of \( \langle a_1 s_2 b_1 \rangle \) must intersect \( P_1 P_2' \) and hence \( a_1 b_1 \). Therefore \( P \notin S_{2}' \). Lemma 25.3 is proved.

Now let \( S_0 \) be a component of \( S-xz \). Then \( S_0 \) is a subset of a component of \( S-a_1 b_1 \) which is a subset of one of \( S_1 \) or \( S_{2}' \), say \( S_1 \). Then by Lemma 25.3, \( a_1 s_2 b_1 \) does not intersect \( S_0 \). Now consider the arc \( x a_1 s_2 b_1 y \). It contains no subarc with endpoints only on \( C_1 \). Also, it does not intersect \( S_0 \).
Hence, in all cases, it is possible to obtain an arc \( xy \) in \( S-S_0 \) which does not have a subarc with end-points only on \( C_i \). If \( xy \) separates \( S \) then the argument can be repeated with \( C_2, \ldots, \) etc. In a finite number of steps an arc \( xby \) will be obtained in \( S-S_0 \) such that \( xby \) does not have a subarc with endpoints only on any \( C_i \) \((i=1, \ldots, n)\). Therefore by §6, \( xby \) does not separate \( S \).

§26. Lemma: If \( \Theta \) is a primitive skew curve 
\[(ax)+(xb)+(ay)+(yb)+(az)+(zb)+(xu)+(yu)+(zu)\] of type one in \( S \), then there is a primitive skew curve \( \Theta_1 \) of type one such that
\[\Theta_1 = (ax)'+(xb)+'+(ay)+'+(yb)+'+(az)+'+(zb)+'+(xu)+'+(yu)+'+(zu)\] and \( \Theta_1 \) has the property that neither \((ax)\)', \((xb)\)', \((ay)\)', \((yb)\)', \((az)\)', \((zb)\)', \((xu)\)', \((yu)\)' nor \((zu)\)' separates \( S \).

Proof: Consider any one of the arcs, say \( ax \), and suppose that \( S-ax \) is not connected. Now \( \Theta-ax+a+x \) is connected and by §25, there is an arc \((ax)'\) in \( S-S_0 \) which does not separate \( S \), where \( S_0 \) is any component of \( S-ax \).

Since \( \Theta-ax \) is connected, then it belongs to one component of \( S-ax \). If this component is chosen as \( S_0 \), then \((ax)'\) does not intersect \( \Theta-ax \). Hence, \( \Theta-(ax)+(ax)' \) is a primitive skew curve of type one such that \((ax)\)'.
does not separate $S$.

This process can be repeated for each of the other eight arcs.

§27. Lemma: If $\Theta$ is a primitive skew curve of type one in $S$, then there is a primitive skew curve $\Theta'$, of type one in $S - \sum_{1}^{n} C_{i}$.

Proof: Suppose that $\Theta = ax + xb + ay + yb + az + zb + ux + uy + uz$ and let $\Theta$ be a primitive skew curve of type one in $S$. Then by §26, it can be assumed that no one of the nine arcs of $\Theta$ separates $S$. Note that each of the points $a$, $b$, $x$, $y$, $z$ and $u$ is the endpoint of three arcs. Therefore in small neighborhoods of these points, there are associated points $a'$, $b'$, $x'$, $y'$, $z'$ and $u'$ belonging to $\Theta$ such that: (1) each primed point belongs to $S - \sum_{1}^{n} C_{i}$; and (2) (using $a$ and $a'$ as examples) either $a' = a$ or else $a'a = a \subseteq S - \sum_{1}^{n} C_{i}$, where $a'a$ is a subarc of either $ax$ or $ay$ or $az$.

The proof of the lemma will be divided into two parts. In part $A$ it will be shown that $\Theta$ can be replaced by a $\Theta'$ whose vertices $a_{1}$, $b_{1}$, $x_{1}$, $y_{1}$, $z_{1}$ and $u_{1}$ all lie in $S - \frac{2}{2} C_{1}$. In part $B$ it will be shown that $\Theta'$ can be replaced by a $\Theta'_{1}$ which lies in $S - \sum_{1}^{n} C_{i}$.
Part A: Now consider the point \( a \). If \( a \in S - \sum_{1}^{n} C_{i} \)
set \( a = a' \). Suppose that \( a \) does not belong to \( S - \sum_{1}^{n} C_{i} \).
Then \( a' \) belongs to one of \( ax, ay, \) or \( az \), say \( az \). There
are now two possibilities.

Case I: The point \( x' \) belongs to \( ax \). Now \( ax \) does not separate \( S \) and hence the intersection of \( ax \) with any
element of \( \mathcal{U} \) is a point or an arc or empty. Since \( x'a \) is
a subset of \( xa \) and since \( aa'-a \) is a subset of \( S - \sum_{1}^{n} C_{i} \),
then the intersection of \( x'a' \) with any element of \( \mathcal{U} \) is a
point or an arc or empty.

Let \( AB \) be such an intersection, say with \( C_{1} \),
where \( A \) precedes \( B \) on \( x'aa' \) from \( x' \) to \( a' \). Consider the
set \( M=a'x'x+xb+yb+z+b+ux+uy+uz \). This is a closed con-
nected set composed of a finite number of arcs which in-
tersect only at the endpoints. Also, there is an arc \( A''B'' \)
in \( C_{1} \) such that: \( A'' \neq A, B'' \neq B \); (2) \( AB \subseteq A''B'' \); and
(3) \( A''B'' \cap M = \emptyset \). Now the arcs \( x'A \) and \( Ba' \) have end-
points only on \( A''B'' \) and each intersects \( M \). Hence, by §21,
there is an arc \( A_1B_1 \) in \( S - (M + \sum_{1}^{n} C_{i}) \) which joins \( x'A \) and \( Ba' \).
Hence, the arc \( x'a' \) can be replaced by an arc \( x'A_1B_1a' \)
which does not intersect \( C_{1} \). Since \( x'a' \) intersects only
a finite number of elements of \( \mathcal{U} \), then by repeating this
process, an arc \((x'a')' \) will be obtained which is a subset
of \( S - \sum_{1}^{n} C_{i} \).
Now consider the arc yaa'. Let \( a_1 \) be the first point of intersection of yaa' and \((x'a')'\) from y to a'. Note that \( a_1 y \) may still contain a, but \( a_1 y \) contains no point of any element of \( \mathcal{Y} \) not previously contained by ay. Then the set \( xx' a_1 + a_1 z + a_1 y + xb + yb + zb + xu + y u + zu \) is a primitive skew curve of type one, no arc of which separates \( S \), such that \( a_1 \in S - \sum_1^n C_1 \).

Case II: The point \( x' \) does not belong to ax. In this case \( x' \in xb \) or \( x' \in xu \), say \( x' \in xb \). Since the intersection of xa and any \( C_1 \) is a point or an arc (if it is not empty) and since \( x'x-x \) and \( aa'-a \) are subsets of \( S - \sum_1^n C_1 \), then the intersection of \( x'a' \), where \( x'a'=x'xa' \), with any element of \( \mathcal{Y} \) is either a point, an arc, or empty. If \( M=x'b+a'a'z+yb+zb+uy+yz \), then as in case I, the arc \( x'a' \) can be replaced by an arc \((x'a')'\) in \( S=(M+\sum_1^n C_1) \). Now let \( x_1 \) be the first point of intersection from u to \( x' \) of the arc \( uxx' \) with \((x'a')' \) and let \( a_1 \) be the first point of intersection from y to \( a' \) of the arc yaa' with \((x'a')' \). Note that while \( a_1 y \) and \( ux_1 \) may still contain a and \( x_1 \), respectively, \( a_1 y \) and \( ux_1 \) do not contain any point of \( \sum_1^n C_1 \) not previously contained in ay and ux. Then the set \( x_1 a_1 + a_1 y + a_1 z + x_1 b + yb + zb + x_1 u + yu + uz \) is a primitive skew curve of type one, no arc of which separates \( S \), such that \( a_1 \) \( S - \sum_1^n C_1 \).
Since this can be repeated for each of the six points \( a, b, x, y, z \) and \( u \) it can be assumed that there is a primitive skew curve \( \Theta' \) of type one in \( S \) such that:

1. no one of the nine arcs of \( \Theta' \) separates \( S \); and
2. each of the points \( a, b, x, y, z \) and \( u \) belongs to \( S - \sum_{l=1}^{n} C_l \).

Part B. Consider any arc of \( \Theta' \), say \( ay \). Now \( ay \) does not separate \( S \) and hence its intersection with any element of \( T \) is either an arc, a point or empty. If \( M = \Theta - ay + a + y \), then as in case I of part A, \( ay \) can be replaced by an arc \(( ay)\)' with endpoints \( a \) and \( y \) such that \(( ay)\)' \( \subseteq S - \sum_{l=1}^{n} C_l \). Since this can be done for each of the nine arcs of \( \Theta' \), then there is a primitive skew curve of type one in \( S - \sum_{l=1}^{n} C_l \).

§ 28. Theorem: The set \( S \) is homeomorphic to a subset of the plane.

Proof: By §17, \( S - \sum_{l=1}^{n} C_l \) is homeomorphic to a subset of the plane. Then according to a theorem by S. Claytor [5], \( S - \sum_{l=1}^{n} C_l \) does not contain a primitive skew curve of type one.

Now suppose \( S \) contains a primitive skew curve of type one. Then by §27, \( S - \sum_{l=1}^{n} C_l \) contains a primitive skew
curve of type one. This is a contradiction and hence $S$ cannot contain a primitive skew curve of type one.

Hence, by a theorem of Hall [9], $S$ does not contain a primitive skew curve of type two. Therefore $S$ is a Peanian continuum which does not have any cut points and which does not contain any primitive skew curves of type one or two. Then by Claytor's theorem [5], $S$ is homeomorphic to a subset of a spherical surface.

If the collection $\mathcal{V}$ is empty, then Bing [4] has shown that the set $S$ is homeomorphic to the entire sphere. Assume that $\mathcal{V}$ is not empty. Let $C_1$ be an element of $\mathcal{V}$ and let $C'_1$ be the homeomorph of $C_1$ in $S_2$, the 2-sphere. Then, by Jordan's curve theorem, $S_2 - C_1'$ contains two components $D_1$ and $D_2$. Then the homeomorph of $S - C_1$ in $S_2$ must be a subset of $D_1$ (or $D_2$) else it is easy to show that $S - C_1$ is not connected. Therefore $S$ is homeomorphic to a subset of $D_1$ and hence to a subset of the plane. Note that $S$ is homeomorphic to a bounded subset of the plane.
CHAPTER IV

§ 39. Two theorems and a lemma by A. Gehman [7].

Definition: Let $M$ and $M'$ be point sets in the planes $R$ and $R'$, respectively. Let $f$ be a homeomorphism which carries $M$ into $M'$. Then it is said that $f$ can be extended in the sense of Antoine (A-extended) to a correspondence between $R$ and $R'$ if there is a homeomorphism $F$ of $R$ into $R'$ such that $F(M) = M'$. Note that $F(x)$ is not necessarily equal to $f(x)$ when $x$ belongs to $M$.

Definition: Two plane continuous curves $M$ and $M'$ are in the same interior class with respect to the planes $S$ and $S'$ in which they lie if there exists: (a) a continuous 1-1 correspondence $T$ such that $T(M) = M'$; and (b) a 1-1 correspondence between the set of all simple closed curves in $M$ and the set of all simple closed curves in $M'$, which is such that if $J$ is a simple closed curve in $M$ and $J'$ is the corresponding simple closed curve in $M'$, and if $N$ is the set of all points of $M$ which are interior to $J$ and if $N'$ is the set of all points of $M'$ which are interior to $J'$, then there exists a continuous 1-1
correspondence $W$ such that $W(N)=N'$.  

Of course if $M$ and $M'$ are two simple closed curves, as they are in all applications in this paper, they are in the same interior class.

In the following theorem and lemma the author lists several sets of conditions involving the number of simple closed curves, endpoints, and cutpoints, any set of which will satisfy the theorem (or lemma). Only the set of conditions applicable to the situation in this paper have been copied.

**Theorem 01:** If $M$ is a continuous curve lying in a plane $S$, and $T$ is a continuous 1-1 correspondence such that $T(M)=M'$, where $M'$ lies in a plane $S'$, then $T$ can be $A$-extended to a correspondence between the planes $S$ and $S'$ provided that: (1) $M$ and $M'$ are in the same interior class with respect to $S$ and $S'$; and (2) $M$ has one simple closed curve, less than four endpoints, and the same number of branch points as endpoints.

**Lemma 02:** The plane continuous curve $M$ is reversible if $M$ has one simple closed curve, less than three endpoints, and the same number of branch points as endpoints.

**Note:** The general notion of reversibility of continuous curves is complicated. All plane simple closed
curves are reversible and are the only type considered below.

Theorem 62: Given (1) a point set M lying in a plane S, and a continuous (1-1) correspondence T such that T(M) = M' where M' lies in a plane S'; (2) each component of M is bounded and except for at most one component, each is reversible; (3) if C denotes a simple closed curve in either S or S', C encloses points of at most a finite number of components of either M or M'; (4) for each component M_i of M, the correspondence T between M_i and the component T(M_i) = M'_i of M' can be extended to a correspondence between the planes S and S'; and (5) if M_i and M_j denote any two components of M, then M_i lies in the domain^{26} of S-M_j bounded by the subset B of M_j, if and only if T(M_i) = M'_i lies in the domain^{26} of S-M'_j bounded by T(B). Under these conditions, the correspondence T between M and M' can be A-extended to a correspondence between the planes S and S'.

§30. Lemma: Let S' be the homeomorph of S in the plane R. Let D be a component of R-S'. Suppose that the boundary of D contains a simple closed curve J. Then J ⊆ S' and S'-J is connected.
Proof: Since \( D \) is a component of \( R-S' \), and since \( S' \) is closed, then \( J \) is a subset of \( S' \). Let \( \pi' \) be the collection of elements \( \{C_i'\} \) such that \( C_i' \) is the homeomorph of \( C_i \) (\( i=1, \ldots, n \)). Suppose that \( S' \supseteq J \) has a partition. Then \( J \) is not identical to any member of \( \pi' \).

Hence, there are two points \( P_1 \) and \( P_2 \) of \( J \) such that \( P_1 + P_2 \subseteq S' - \sum \limits_{i=1}^{n} C_i' \). Now let \( \epsilon_1 > 0 \) be sufficiently small that (1) \( U(P_1, \epsilon_1) \cup U(P_2, \epsilon_2) = \emptyset \); and (2) \( U(P_1 + P_2, \epsilon_1) \cup \sum \limits_{i=1}^{n} C_i' = \emptyset \).

By theorem 2, chapter II of Moore [2], there is a point \( P_i' \) in \( U(P_1, \epsilon_1) \cup B \) such that \( P_i' \) is an accessible limit point of \( D \), where \( B \) is the boundary of \( D \) and \( i=1, 2 \).

By 111, \( S' - \sum \limits_{i=1}^{n} C_i' \) cannot be separated by any finite number of points, and hence there are three arcs \( P'_1 \times P'_2 \), \( P'_1 \times P'_3 \) and \( P'_1 \times P' \) in \( S' \) such that \( (P'_1 \times P'_2) \cup (P'_1 \times P'_3) \cup (P'_1 \times P) = \emptyset \) whenever \( i \neq j \). Also, since \( P'_1 \) and \( P'_2 \) are accessible limit points of \( D \), there is an arc \( P'_1 \times P'_2 \) in \( D + P'_1 + P'_2 \).

Now one of the three arcs \( \{P'_1 \times P'_2\} \) (\( i=1, 2, 3 \)), say \( P'_1 \times P'_2 \), is such that \( \langle P'_1 \times P'_2 \rangle \) and \( \langle P'_1 \times P'_3 \rangle \) lie in different components of \( R-(P'_1 \times P'_2 + P'_1 \times P' \) \). Since \( \langle P'_1 \times P'_2 \rangle \) and \( \langle P'_1 \times P'_3 \rangle \) are subsets of \( S' \), then \( S'-(P'_1 \times P'_2 + P'_1 \times P' \) \) has a partition. Since \( \langle P'_1 \times P'_2 \rangle \) is a subset of \( D \), then \( S' - P'_1 \times P' \) has a partition.

But \( P'_1 \times P' \) is a subset of \( S' - \sum \limits_{i=1}^{n} C_i' \) and no subarc
of $P'_1P'_2$ has endpoints on any element of $\mathcal{P}'$. Therefore, by 36, $S'-P'_1P'_2$ is connected. This is a contradiction and hence $S'-J$ is connected.

§51. Lemma: Let $S'$ be the homeomorph of $S$ in the plane $R$. If $\mathcal{P}$ contains more than one element then:

(1) the boundary of $S'$ is the set $\sum_{1}^{n} C'_i$ where $C'_i$ is the homeomorph of $C_i$; (2) there is an element, say $C'_1$, of $\mathcal{P}$ such that $S'-C'_1$ is in the bounded component of $R-C'_1$; (3) $C'_i$ is interior to $C'_1$ when $i=2,\ldots,n$; and (4) $C'_j$ is exterior to $C'_1$ when $i \neq 1, j \neq 1$.

Proof: (1) Let $\mathcal{P}'$ be the collection of simple closed curves which are homeomorphs of elements of $\mathcal{P}$. Let $C'_1$ be any element of $\mathcal{P}'$. Since $C'_1$ is a simple closed curve, then $R-C'_1$ has exactly two components, $D_1$ and $D_2$, such that $C'_1 \subseteq D_1$ and $C'_1 \subseteq D_2$. Now $S'-C'_1$ must be a subset of one of $D_1$ or $D_2$, say $D_1$, else $S-C'_1$ would have a partition. Hence, $D_2 \subseteq R-S'$. Thus $C'_1 \subseteq S'$ and $C'_1 \subseteq R-S'$. Therefore, $C'_1 \subseteq B(S')$. Since this is true for every $i_0$ ($1 \leq i_0 \leq n$), then $\sum_{1}^{n} C'_i \subseteq B(S')$.

Let $D$ be any component of $R-S'$ and let $y$ be any point of $B(D)$. Now by theorem 41, page 261, of Moore [2], $B(D)$ contains a simple closed curve $J$. By §30, $J \not\subseteq S'$.
and \( S' - J \) is connected. Since \( J \) is a simple closed curve then \( R - J = D_1 \mid D_2 \). Because \( J \subseteq S' \) then \( D \) is a subset of one of \( D_1 \) or \( D_2 \), say \( D_1 \).

It is possible to draw arcs \( x_1x_2 \) and \( a_1a_2 \) such that: (1) \( <x_1x_2> \subseteq D \) and \( <a_1a_2> \subseteq S' - J \); (2) \( a_1 + a_2 \) and \( x_1 + x_2 \subseteq J \); and (3) \( a_1 + a_2 \) separates \( x_1 + x_2 \) on \( J \). If \( S' - J \) were a subset of \( D_1 \) then \( <a_1a_2> \) and \( <x_1x_2> \) would both be subsets of the same component \( D_1 \) of \( R - J \). Since \( <a_1a_2> \times <x_1x_2> = \emptyset \), this is impossible. Thus \( S' - J \) is not a subset of \( D_1 \). Since \( S' - J \) is connected then it must belong to \( D_2 \). Therefore \( S' \subseteq \overline{D} \) and \( D_1 \subseteq R - S' \). Thus \( D_1 \) is a component of \( R - S' \) which contains \( D \). Since \( D_1 \) and \( D \) are components of \( R - S' \) and contain points on common, then \( D_1 = D \). Therefore \( y \in J \). Since \( S' - J \) is connected then \( J \) is an element of \( \tau' \), and therefore \( y \in \bigodot_{1}^{n} C' \).

Now let \( y \in B(S') \) and suppose \( y \notin \bigodot_{1}^{n} C' \). Then there are an infinite number of components \( D_1, D_2, \ldots \) of \( R - S' \) and a sequence \( \{y_i\} \) such that \( y_i \in D_1 \) and \( \{y_i\} \rightarrow y \). Since for every \( i \geq 1 \), \( y_i \) belongs to \( D_1 \) and \( y \) belongs to \( S' \) and hence to \( R - D_1 \), there is a sequence \( \{y'_p\} \) such that \( y'_p \) belongs to \( B(D_p) \) and \( \{y'_p\} \rightarrow y \).

But this is a sequence of points of \( \bigodot_{1}^{n} C' \) which converges to a point of \( R - \bigodot_{1}^{n} C' \). Since \( \bigodot_{1}^{n} C' \) is closed, this is a
contradiction. Hence, \( B(S^{'}) \leq \sum_{i=1}^{n} C_{i}' \). But then \( B(S^{'}) = \sum_{i=1}^{n} C_{i}' \).

(2) Since \( S' \) is bounded, then \( R-S' \) contains one unbounded component \( D' \). Then, as in (1), the boundary of \( D' \) is an element \( C_{i}' \) of \( \pi' \). Then \( S' - C_{i}' \) is interior to \( C_{i}' \).

(3) This follows from (2).

(4) Let \( C_{i}' \) be any element of \( \pi' \) different from \( C_{i}' \).

Let \( C_{j}' \) be any element of \( \pi' \) different from \( C_{i}' \). Since \( S - C_{i}' \) is connected and since \( C_{i}' \) is exterior to \( C_{i}' \) then \( S - C_{i}' \) is exterior to \( C_{i}' \). But \( C_{j}' \not\subseteq S' - C_{i}' \) and thus \( C_{j}' \) is exterior to \( C_{i}' \).

§ 32. Lemma: Let \( M \) be the sum of a collection of \( n \) nonintersecting simple closed curves \( C_{1}', C_{2}', \ldots, C_{n} \) in a plane \( R \).

Let \( M' \) be the sum of a collection of \( n \) nonintersecting simple closed curves \( C_{1}', C_{2}', \ldots, C_{n} \) in a plane \( R' \) such that if \( C_{1}' \) and \( C_{2}' \) are any two curves of \( M' \), then \( C_{1}' \) is in the interior of \( C_{2}' \) if and only if \( C_{2}' \) is in the interior of \( C_{1}' \).

Then there is a homeomorphism \( T \) carrying \( R \) into \( R' \) such that \( T(C_{i}') = C_{i}' \) (\( i = 1, \ldots, n \)), where \( C_{i}' \) is one of \( C_{1}', C_{2}', \ldots, C_{n}' \).
Proof: (1) Let $C_1$ be one of the simple closed curves of $M$. Then there is a homeomorphism $f_1$ carrying $C_1$ into $C'_1$. Now $C_1$ and $C'_1$ are in the same interior class with respect to $R$ and $R'$. Therefore by theorem G (Gehman), there is a homeomorphism $f_1$ carrying $R$ into $R'$ such that $f_1(C_1)=C'_1$. Now define $T': M \to M'$ as follows: $T'(x)=f_1(x)$ $(x \in C_1)$. Now $T'$ is a homeomorphism carrying $M$ into $M'$.

(2) By lemma G a simple closed curve is reversible. Also, every simple closed curve in the plane is bounded. Therefore every component of $M$ is bounded and reversible.

(3) Since $M$ and $M'$ have only a finite number of components, obviously any simple closed curve $C$ can contain points of only a finite number of components of $M$ or of $M'$.

(4) For each component $C_1$ of $M$, $T'(C_1)=f_1(C_1)$, and thus $T'(C_1)$ (the notation of Lefschetz, page 2, is $T' C_1$) can be extended to the plane $R$.

(5) Let $C_i$ and $C_j$ be two components of $M$. Now since $C_j$ is a simple closed curve, there is only one bounded component of $R-C_j$ and that is bounded by $C_j$. Any point which belongs to this bounded component of $R-C_j$ is interior to $C_j$. The same is true for $C'_j$. Since $C_i$ is
interior to $C_j$ if and only if $C_j'$ is interior to $C_j'$. Then $C_i$ belongs to the bounded component of $R-C_j$ if and only if $C_i'$ belongs to the bounded component of $R-C_j'$. Hence by theorem 29, there is a homeomorphism $T: R \rightarrow R'$ which carries $M$ into $M'$. Now each of the simple closed curves of $M$ must go into a simple closed curve of $M'$. Let the curve of $M$ which $C_i$ goes into be labeled $C_i'$. Then $T$ carries $C_i$ into $C_i'$.

§33. Theorem: The set $S$ is homeomorphic to the plane region bounded by $n$ nonintersecting circles.

Proof: The proof will consist of two parts. Part I will consider the case when $\mathcal{C}$ consists of a single element. Part II will consider the case when $\mathcal{C}$ consists of more than one element.

Part I. The collection $\mathcal{C}$ consists of a single element $C_i$.

In §28 it was shown that $S$ was homeomorphic to a proper subset of the sphere and that $S-C_i$ was connected. It is easily shown, by projection, that $S$ is homeomorphic to a subset of the plane $R$ such that $S'-C_i'$ is in the bounded domain $D'$ of $R-C_i'$. Then $S' \subseteq \overline{D'}$. Now suppose that there is a point $x$ of $D'$ which belongs to $R-S'$. Let
Let \( D_x \) be the component of \( R-S' \) which contains \( x \). Since
\( C_1' \subseteq S' \) then \( D_x \subseteq D' \). Also since \( S'-C_1' \) is not empty then
\( D_x \) is a proper subset of \( D' \). Now by theorem 41, page 261 of Moore \( ^2 \) the boundary of \( D_x \) contains a simple closed
curve \( J \). Since \( D_x \) is a proper subset of \( D' \), then the
boundary of \( D' \) cannot be identical with \( J \). But by \$30,
\( S'-J' \) is connected. Since \( \mathcal{W} \) contains a single element,
this is impossible. Therefore \( D' \nsubseteq S' \). But then \( \overline{D'} \subseteq S' \)
and \( S'=\overline{D'} \). Let \( C_1'' \) be a circle in \( R \). Then there is a
homeomorphism \( T \) of \( R \) with itself which carries \( C_1'' \) into \( C_1'' \). If \( D'' \) is the bounded component of \( R-C_1'' \), then \( T \) carries \( D' \)
into \( D'' \). But then \( T \) carries \( \overline{D'} \) into \( \overline{D''} \) and \( S \) is homeo-
morphic to \( \overline{D''} \). Therefore \( S \) is homeomorphic to the plane
region bounded by a circle.

**Part II.** The collection \( \mathcal{W} \) consists of more than
one element.

Let \( S' \) be the homeomorph of \( S \) in the plane \( R \) and
let \( C_1' \) be the homeomorph of \( C_1 \) for every element \( C_1 \) be-
longing to \( \mathcal{W} \). Then by \$31: (1) \( S'-C_1' \) is a subset of
the bounded component of \( R-C_1' \) and (2) \( C_j' \) is exterior to
\( C_1' \) when \( \{i \neq 1, j \neq i\} \). Now let \( C_{P_1}', \ldots, C_{P_n}' \) be nonin-
tersecting circles in \( R \) such that: (1) \( C_{P_i}' \) is interior
to \( C_{P_j}' \) when \( i \) is different from \( 1 \) and (2) \( C_{P_j}' \) is exterior
to \( C_{P_1}' \).
to $C^*_i$ when $i \neq j$, $j \neq i^*_j$. Note that $C^*_i$ is interior to $C^*_j$ if and only if $C^*_j$ is interior to $C^*_i$. Then by §32, there is a homeomorphism $T: \mathbb{R} \to \mathbb{R}$ such that $T(C^*_i) = C^*_1$, $C^*_1$ being one of $C^*_1, \ldots, C^*_n$.

Suppose $S''$ is the set into which $T$ carries $S'$.

Then $S''$ is homeomorphic to $S$. By §31, $B(S'') = \sum_{i=1}^{n} C^*_i$.

Since $S'' = \sum_{i=1}^{n} C^*_i$ is connected, then $S'' \subseteq \overline{D'}$ where $D'$ is the component of $\mathbb{R} - \sum_{i=1}^{n} C^*_i$ whose boundary is $\sum_{i=1}^{n} C^*_i$. Note that there is only one such component and that it is bounded.

Now suppose that $x$ is a point of $D'$ which does not belong to $S''$. Then as in part I, it can be shown that there is in $S''$ a simple closed curve $J$ which is not a subset of $\sum_{i=1}^{n} C^*_i$ such that $S'' - J$ is connected. Since this is a contradiction then every point of $D'$ belongs to $S''$. Therefore $\overline{D'} \subseteq S''$ and $S'' = \overline{D'}$. Therefore $S$ is homeomorphic to the plane region bounded by $n$ nonintersecting circles.
SELECTED BIBLIOGRAPHY

BOOKS


PERIODICALS


APPENDIX

1 Numbers in brackets refer to the bibliography.

2 Since \( M \) can be covered by a finite number of connected domains of diameter less than \( e \) for every \( e > 0 \), \( M \) is locally connected (Wilder [3], page 106, theorem 3.9). Then by a repeated application of \( 3.9 \), \( M-\{x+y\} \) can be covered by a finite number of connected domains of diameter less than \( e \) for \( e > 0 \).

3 Let \( x_i \) be a point of \( D_1 \). Let \( y_1 \) be a point of \( C-(x+y) \). Since no pair of points separates \( M \), there are three arcs \( \alpha_{1,1}, \alpha_{1,2}, \alpha_{1,3} \) from \( x_i \) to \( y_1 \) in \( M \) such that \( \alpha_{1,j} \cap \alpha_{1,k} = x_i+y_1 \ (1 \neq j, k \leq 3, j \neq k) \). Then at least one on \( \alpha_{1,j} \) does not intersect \( x+y \). Then let \( \alpha_{1,1} \) be this arc. Let \( y'_1 \) be the first point of \( \alpha_{1,1} \) from \( x_i \) to \( y_1 \) of \( C \). If \( x_i \in M-(D_A+D_B) \) then \( x_i y'_1-y_1 \) is a subset of \( M-(D_A+D_B) \). If \( x_i \in D_A \), then \( x_i y'_1-y'_1 \subseteq D_A \); if \( x_i \in D_B \), then \( x_i y'_1-y'_1 \subseteq D_B \).
4Let \( \alpha_{1,1}, \alpha_{1,2} \) and \( \alpha_{1,3} \) \((i=j+1, \ldots, n)\) be the same as in 3. Let \( y_{i,k} \) be the first point from \( x_i \) to \( y_i \) of \( \alpha_{1,k} \) on \( C \). Let \( \beta_{i,k} \) be the subarc of \( \alpha_{1,k} \) from \( x_i \) to \( y_{i,k} \). Then at most one of \( \beta_{i,k} \) contains \( x \) and at most one of \( \beta_{i,k} \) contains \( y \). Let \( \beta_{1,1} \) be the arc which contains neither \( x \) nor \( y \). \( \beta_{1,2} \) be the arc which does not contain \( x \) but may contain \( y \) and let \( \beta_{1,3} \) be the arc which does not contain \( y \) but may contain \( x \). Then these are the desired arcs.

5Suppose \( D_{j+2} \) contains a point \( x_1 \) of \( D' \) and a point \( x_2 \) of \( M-D' \). Then there is an arc \( x_1x_2 \) from \( x_1 \) to \( x_2 \) in \( D' \) which does not intersect \( C \). Since \( x_1x_2 \) does not intersect \( C \) it must intersect \( \alpha_{j+1} \) as \( B(D') \subseteq C+\alpha_{j+1} \). Then in \( \alpha_{j+1}+x_1x_2 \) there is an arc \( \alpha_{j+2} \) from \( x_2 \) to \( C_1' \) which lies in \( D_A-D'+C_1' \).

Suppose \( D_{j+2} \subseteq M-D' \). Let \( x_1 \) be the first point of \( \beta_{j+2,1} \) from \( D_{j+2} \) to \( C_1 \) on \( C_1+\alpha_{j+1} \), and let \( \beta_{j+2} \) be the subarc from \( D_{j+2} \) to \( x_1 \). Then in \( \beta_{j+2}+\alpha_{j+1} \) there is an arc \( \alpha_{j+2} \) from \( D_{j+2} \) to \( C_1' \). But \( \beta_{j+2}+\alpha_{j+1} \subseteq D_A-D'+C_1' \). In either case, \( \alpha_{j+2} \) is the desired arc.

6In the following discussion \( i=j+1, \ldots, k \). Let \( R \) be any point of \( D_A \). If \( R \notin \beta_{1,1} \) then \( \beta_{1,1} \) joins \( D_1 \) and
C'. Suppose $R \in \beta_{1,1}$. If $R \neq x_1$ then $R \notin \beta_{1,2} + \beta_{1,3}$. Therefore, $R$ does not disrupt $D_1$ from $G_1' + C_x, \sigma_0$ and $R$ does not disrupt $D_1$ from $G_1' + C_y, \sigma_0$. If $R = x_1$ then there is a subarc of $\beta_{1,1}$ which joins $D_1$ and $G_1'$. In any case $R$ does not disrupt $D_1$ from $G_1' + C_x, \sigma_0$ and $R$ does not disrupt $D_1$ from $G_1' + C_y, \sigma_0$. By Bing's lemma there is an arc $J$ from $C_x, \sigma_0$ to $C_y, \sigma_0$ in $D_A + C_x, \sigma_0 + C_y, \sigma_0$ which does not disrupt $D_1$ from $G_1'$ in $D_A + G_1'$. Let $\alpha_1$ be an arc from $D_1$ to $G_1'$ in $D_A + G_1'$. Then $\alpha_1$ does not intersect $J$ and hence $\alpha_1$ does not disrupt $C_x, \sigma_0$ from $C_y, \sigma_0$ in $D_A + C_x, \sigma_0 + C_y, \sigma_0$.

Consider $C_x, \sigma_0$. Since $G_1' + \alpha_1 + \ldots + \alpha_n$ is a closed compact set, then for every $x' \in C_x, \sigma_0$ there is a $\sigma_x < e_1/300$ such that $U_{x'} \sigma_x \cap (G_1' + \alpha_1 + \ldots + \alpha_n) = \emptyset$, where $U_{x'} \sigma_x$ is the component of $U(x'; \sigma_x)$ which contains $x'$. Let $g_x$ be the union of $U_{x'} \sigma_x$ for all $x' \in C_x, \sigma_0$. Then $g_x$ is a connected domain of diameter less than $e_1/100$ which contains $C_x, \sigma_0$, but does not intersect $G_1' + \alpha_1 + \ldots + \alpha_n$.

A set $g_y$ containing $C_y, \sigma_0$ can be obtained with similar properties.

Suppose $g$ is an element of $G$. Suppose also that $g \cdot D_A \neq \emptyset$ and $g \cdot (M-D_A) \neq \emptyset$. Then $g \cdot B(D_A) \neq \emptyset$. But $B(D_A) \subseteq C$ and $g \cdot C = \emptyset$. Therefore, either $g \subseteq D_A$ or $g \subseteq M-D_A$. 
If \( g \subseteq M-D_A \), the same argument gives that \( g \subseteq D_B \) or \( g \subseteq M-D_B \). Hence, \( g \subseteq D_A \) or \( g \subseteq D_B \) or \( g \subseteq M-(D_A+D_B) \).

Since \( \alpha_1+\alpha_2+\ldots+\alpha_n \) does not disrupt \( g_x, \alpha \) from \( g_y, \alpha \) in \( D_A+\alpha_X+\alpha_Y \), there is an arc \( J \) from \( g_x, \alpha \) to \( g_y, \alpha \) in \( D_A+\alpha_X+\alpha_Y \). Every point of \( J-D_A \) belongs to some element of \( G \). Let \( G' \) be the set of all elements of \( G \) which contain a point of \( J-D_A \). Then \( G'+g_x+g_y \) is an open covering of \( J \). Since \( J \) is compact, there is a finite number of elements of \( G' \) which, together with \( g_x \) and \( g_y \), cover \( J \). These elements must all be subsets of \( D_A \). Therefore, there is a finite collection of elements of \( G \) whose sum is a connected subset of \( D_A \) joining \( g_x \) and \( g_y \). Since there is a finite number of domains with this property, there is a smallest number which does this. Let \( G_A \) be such a collection.

10Notation: Let \( G_A-\{\alpha\} \) be the collection of all elements of \( G_A \) except those which have preassigned subscripts belonging to the set \( \{\alpha\} \). Now suppose \( g_x \) intersects more than one element of \( G_A \). Let these elements be \( g_{k_1}, g_{k_2}, \ldots, g_{k_p} \). Let \( G'_{k_1} \) be the union of all components of \( G_A-\{k_1, k_2, \ldots, k_p\} \) which intersects \( g_{k_1} \). Then one of the \( G'_{k_1} \), say \( G'_{k_1} \), must intersect \( g_y \). But then \( G_A-\{k_1, k_2, \ldots, k_p\} \)
connects $g_x$ and $g_y$ (i.e. the union of all elements of $G_A = \{k_2, \ldots, k_p\}$ contains a connected subset which intersects $g_x$ and $g_y$). But this is a contradiction, for $G_A$ is the smallest possible collection of domains which does this. Therefore, $g_x$ intersects only one element of $G_A$.

Call this element $g_2$. Now suppose $g_k$ intersects $g_{k+1}$ but no other element of $G_A = \{1, \ldots, k\}$ for some $k < q-2$ ($g_1 = g_x$).

Set $K = \sum_{1}^{k+1} g_i$. Suppose $K$ intersects more than one element of $G_A = \{2, \ldots, k+1\}$. Denote these elements by $g_{k_1}, g_{k_2}, \ldots, g_{k_p}$. Suppose one of $g_{k_1}$, say $g_{k_1}$, intersects $g_y$. Then $\{g_2, \ldots, g_{k+1}, g_{k_1}\}$ is a subcollection of $G_A$ which joins $g_x$ and $g_y$. This is a contradiction. Therefore, no $g_{k_1}$ intersects $g_y$. But then if $G'_{k_1}$ is the union of all components of $G_A = \{2, \ldots, k+1, k_1, \ldots, k_p\}$ which intersects $g_{k_1}$, one of the $G'_{k_1}$ intersects $g_y$. Say $G'_{k_1}$ is the one.

Then $G_A = \{k_2, \ldots, k_p\}$ is a subcollection of $G_A$ which joins $g_x$ and $g_y$, a contradiction. Therefore, by induction, the collection $G_A$ may be numbered in the desired manner.

Suppose that $g'_{k_1}$ is a component of $E \cdot g_x$ ($g_x$ an element of $G$) which does not intersect any element of $g_1, g_2, \ldots, g_r$. Let $x$ be a point of $g'$ and let $J$ be an arc in $g_{k+1}^y$ from $x$ to a point $y$ of $G_{k+1}^y + \ldots + g_n^y$. Since $x \in E \cdot \sum_{1}^{r} g_i$ then $J$ must intersect $\sum_{1}^{r} g_i$. Let $y'$ be the first
point of $J$ from $x$ to $y$ on $\mathbb{R}(\sum_{i=1}^{r} g_i)$. Then $xy'$ is a connected subset of $g_k$. Also, no point of $xy'$ can be joined to $C_1$ in $M-\sum_{i=1}^{r} g_i$. Therefore, $xy' \subseteq g_k'$. Since $y' \subseteq g_k'$, there is an $\eta_0$ such that $U_{y'}, \eta \subseteq g_k'$ for $\eta < \eta_0$. If every point of $U_{y'}, \eta_1$ belongs to $E$ for some $\eta_1 < \eta_0$ then $U_{y'}, \eta_1 \subseteq g_k'$. But $U_{y'}, \eta_1$ contains points of $\sum_{i=1}^{r} g_i$. Therefore, $U_{y'}, \eta$ contains points of $M-E$ for every $\eta < \eta_0$. Since $C_1$ has only a finite number of components, and since every point of $M-E$ can be joined to $C_1$, then $M-E+C_1$ has a finite number of components. Let $M_1, M_2, \ldots, M_n$ be the components of $M-E+C_1$. Now there is an infinite sequence of points $\{y_i\}$ in $M-E$ which converges to $y'$. It can be assumed that $\{y_i\}$ belongs to $M_1$. But then $y' \in M_1$ and $xy'+M_1$ is a connected set which joins $x$ and $C_1$ but does not intersect $\sum_{i=1}^{r} g_i$, a contradiction. Therefore, $g_k'$ intersects some $g_i$ ($i=1, \ldots, r$).

12 Suppose $g_k$ is an element of $G$ and $g_k'$ is a component of $E \cdot g_k$. Suppose that $g_k'$ intersects $g_1$ and $g_j$ where $j \geq i+3$. Since one of $g_1$ or $g_j$ is different from $g_i$ or $g_k$ then $g_k'$ (and hence $g_k$) is a subset of $D_A$ or of $D_B$, say $D_A$. But then $g_1, g_2, \ldots, g_i, g_k, g_j, g_{j+1}, \ldots, g_q$ contains a subcollection of at most $r-3$ elements of $G$ which lie in $D_A$ and join $g_x$ and $g_y$. But $D_A$ contains the smallest
number of such elements and it contains \( r-2 \) elements.

This is a contradiction, and thus \( g'_k \) does not intersect two elements of \( g_1, g_2, \ldots, g_r \) that do not lie in a consecutive set of three.

13 The only condition that is not obvious is (3).

Suppose \( x \in M \) and \( x \notin (h_1 + h_2 + \ldots + h_3) \). Then there is a connected set \( \mathcal{K} \) in \( M \) which joins \( x \) and some component of \( C'_1 \) but does not intersect \( g_1 + g_2 + \ldots + g_r \). Suppose \( y \in M - (g_1 + g_2 + \ldots + g_r) \). Then \( g_1 + g_2 + \ldots + g_r \) does not separate \( y \) from \( C'_1 \), a contradiction. Therefore, \( K \cap \mathcal{E} = \emptyset \) and \( h_1 + h_2 + \ldots + h_3 \) does not separate \( x \) from \( C'_1 \).

14 Let \( g_1, g_2, \ldots, g_j \) be the elements of \( G \) which intersect \( M - H^*_1 \) and let \( g_{j+1}, g_{j+2}, \ldots, g_n \) be the elements of \( G \) which are subsets of \( H^*_1 \). For each \( i \) \((i=1, \ldots, j)\) let \( \alpha_j \) be a degenerate arc consisting of a point of \( g_i \cdot (M - H^*_1) \). These arcs do not intersect \( L \) and hence cannot disrupt \( E_{1,1} \) from \( E_{1,10} \) in \( H^*_1 \).

Now let \( L, E_{j+1}, \overline{L \cdot (h_{1,1}^*) \cdot H^*_1}, \overline{L \cdot (h_{1,10}) \cdot H^*_1} \) and \( \overline{L \cdot (M - H^*_1)} \) be the sets \( D, D', M, N, \) and \( E, \) respectively, of Bing's lema. Since no point of \( L \) disrupts \( g_{j+1} \) from \( \overline{L \cdot (M - H^*_1)} \), there is an arc from \( \overline{L \cdot (h_{1,1}^*) \cdot H^*_1} \) to \( \overline{L \cdot (h_{1,10}) \cdot H^*_1} \) which does not disrupt \( g_{j+1} \) from \( \overline{L \cdot (M - H^*_1)} \). Therefore,
there is an arc $\alpha_{j+1}$ from $g_{j+1}$ to $\overline{L \cdot (M-H_1^*)}$ that does not disrupt $\overline{L \cdot (\overline{h_{1,1}}\cdot H_1^*)}$ from $\overline{L \cdot (\overline{h_{1,10}}\cdot H_1^*)}$ in $L^+ h_{1,1}^+ h_{1,10}$.

Let $L'$ be the component of $L - \alpha_{j+1}$ which contains an open arc from $\overline{h_{1,1}^* \cdot H_1}$ to $(\overline{h_{1,10}}^* \cdot H_1^*)$.

Suppose $x \in (g_{j+2}) \cdot (M-L')$. Let $y$ be a point of $M-H_1^*$. Let $xy$ be an arc joining $x$ and $y$. Let $y'$ be the first point of $xy$ from $x$ to $y$ on $M-H_1^* - \alpha_{j+1}$. Then in $xy' + \alpha_{j+1}$ there is an arc from $g_{j+2}$ to $M-H_1^*$ which does not intersect $L'$. Suppose $g_{j+2} \subseteq L'$. Let $R$ be any point of $L'$. Then there is an arc in $M-H_1^* + L - R$ from $g_{j+2}$ to $M-H_1^*$, and hence there is in $L' - R + (M-H_1^*) + \alpha_{j+1}$ an arc from $g_{j+2}$ to $(M-H_1^*) + \alpha_{j+1}$. Therefore no point of $L'$ disrupts $g_{j+2}$ from $(M-H_1^*) + \alpha_{j+1}$.

By an application of Bing's lemma, there is in $L'$ an open arc joining $\overline{h_{1,1}^* \cdot H_1}$ and $\overline{h_{1,10}^* \cdot H_1}$ which does not disrupt $g_{j+2}$ from $M-H_1^* - \alpha_{j+1}$.

Thus there is in $M-H_1^* + L$ an arc $\alpha_{j+2}$ from $g_{j+2}$ to $M-H_1^*$ such that $\alpha_{j+1} + \alpha_{j+2}$ does not disrupt $\overline{h_{1,1}^* \cdot H_1}$ from $\overline{h_{1,10}^* \cdot H_1}$ in $L^+ h_{1,1}^+ h_{1,10}$.

Continuing this process provides arcs $\alpha_{j+1}, \ldots, \alpha_n$ such that $\alpha_1 + \alpha_2 + \ldots + \alpha_j + \alpha_{j+1} + \ldots + \alpha_n$ does not disrupt $\overline{h_{1,1}^* \cdot H_1}$ from $\overline{h_{1,10}^* \cdot H_1}$ in $L^+ h_{1,1}^+ h_{1,10}$. It is obvious that $\sum_{i=1}^{n} \alpha_i \cdot (M-H_1^*)$ does not disrupt $h_{1,1}^* \cdot H_1$ from $\overline{h_{1,10}^* \cdot H_1}$ in $h_{1,1}^+ \ldots + h_{1,10}$.
15 Let \( J \) be an arc from \( [h_{1,1} - 1 : h_{1,10}] \) in \( L + h_{1,1} + h_{1,10} \) which lies in \( L \) except for endpoints and which does not intersect \( K^* \). Since \( J \) is connected, \( J \) intersects \( h_{1,i} \) \( (i=2, \ldots, 9) \). Now every point of \( J \) lies in \( h_{1,2} + \cdots + h_{1,9} = K^* \) and hence belongs to at least one element of \( G' \). Since \( J \) is compact there is a finite covering. Since \( J \) intersects \( h_{1,1} \) and \( h_{1,10} \) at least one element of any open covering of \( J \) intersects \( h_{1,1} \) and at least one element intersects \( h_{1,10} \). Also, any covering of \( J \) by open components has for its sum a connected domain. Hence, there is at least one finite collection of elements of \( G' \) whose sum is a connected domain intersecting \( h_{1,1} \) \( (i=1, \ldots, 10) \). Since there is one finite collection which will do this, there is a collection \( G'' \) such that the sum of the elements of \( G'' \), but the sum of no subcollection of \( G' \) with fewer elements than \( G'' \), is a connected domain intersecting \( h_{1,1} \) \( (i=1, \ldots, 10) \).

16 The proof that such an assumption can be made is identical, except for obvious changes in notation, to that of 10.

17 Let \( x_1 \) be a point of \( h_{1,1} \cdot g_1 \) and let \( y_1 \) be a point of \( g_1 \cdot g_2 \). Since \( g_1 \) is a connected domain, there is
an arc $x_1y_1$ in $g_1$ joining $x_1$ and $y_1$. Since $M=g_1$ is closed and $x_1y_1$ is closed and compact, then there is a $\mathcal{J}_1$ such that $\mathcal{C}(x_1y_1) = 2\mathcal{J}_1$. Now let $U_x, \mathcal{J}_1$ be the component of $U(x, \mathcal{J}_1)$ which contains $x$. Set $g'_1$ equal to $\bigcup_{x \in x_1y_1} U_x, \mathcal{J}_1$. Then $g'_1$ is a connected open set whose closure is a subset of $g_1$. Also, $g'_1$ intersects $h_{1,1}$ and $g_2$.

Now suppose that $g'_1$ has been defined for $i$ less than or equal to $m-1$ when $m \leq r$. Let $x_m$ belong to $g_i, g_m$ and let $y_m$ belong to $g_{m+1}$, where $g_{m+1} = h_{1,10}$ if $m = r$. Then as before, a connected open set $g'_m$ can be constructed such that $g'_m$ is a subset of $g_m$ and $g'_m$ intersects $g_{m-1}$ and $g_{m+1}$.

The proof of this statement is identical, except for notation, to 11 and 12.

19 (1) Evident.

(2) Suppose $Z \in M-E$. Then there is a connected set $N$ joining $Z$ and $C'_1$ in $D_{A_1}+C'_1-(h_1+g_1, h_1+g_1+u+h_{10}+h_{11}+\ldots+h_{1,t})$. Since no point of $N$ is separated from $C'_1$ in $D_{A_1}+C'_1$ by $h_1+g_1, h_1+g_1+u+h_{10}+h_{11}+\ldots+h_{1,t}$, then $N \cap E = \emptyset$. Therefore, $Z$ is not separated from $C'_1$ in $M$ by $E$. 
(3) Let \( g' \) be an element of \( G' \). Then either a point of \( K^* \) is accessible from \( g' \) or else a point of \( M-H^*_1 \) is accessible from \( g' \). Since \( (M-H^*_1)^* \subseteq M-E \) then a point of \( M-E \) is accessible from \( g' \).

Let \( g_{1,1_0} \) be any element of \( g_{1,1}, \ldots, g_{1,u} \). Then \( g_{1,1_0} \) is made up of at least three consecutive elements \( g_{j_0-1}, g_{j_0}, g_{j_0+1} \) of \( g_1^* \) \((i=1, \ldots, r)\). Now by the method of construction, \( g_{j_0-1}^* \) contains a subset of \( g_{j_0-1} \), \( g_{j_0}^* \) contains a subset of \( g_{j_0} \) and \( g_{j_0+1}^* \) contains a subset of \( g_{j_0+1} \). Suppose that \( x \) is a point of \( g_{j_0}^* \) which belongs to \( g_{j_0} \). Since \( g_{j_0} \) is an element of \( G' \) then there is an arc \( xy' \) in \( g_{j_0} + y' \) from \( x \) to a point \( y' \) of \( M-H^*_1 \). Let \( y \) be the first point from \( x \) to \( y' \) of \( xy' \) which belongs to \( M-E \).

Suppose that there is a point \( z \) of \( xy-y \) which belongs to \( E-g_{1,1_0} \). Since \( g_{j_0} \) intersects only \( g_{j_0-1} \) and \( g_{j_0+1} \), then \( xy-y \) does not contain any points of \( h_{1,1}, h_1^*, g_2, \ldots, g_{j_0-2}, g_{j_0+2}, \ldots, g_r, h_{1,1_0}, \ldots, h_{1,t} \). Thus \( z \) must be a point of a component \( g^* \) of the intersection of an element \( g' \) of \( G' \) with \( E \) such that \( g^* \) does not intersect \( g_{j_0-1}, g_{j_0} \) or \( g_{j_0+1} \). Let \( Z \) be the set of all points \( z \) of \( xy-y \) which belong to \( E-g_{1,1_0} \). Let \( z' \) be the first point from \( x \) to \( y \) of \( xy \) which belongs to \( Z \). Suppose \( z' \) belongs to \( g_{1,1_0} \). Since \( g_{1,1_0} \) is open then it contains points of \( Z \). This is a contradiction, and thus \( z' \notin g_{1,1_0} \).
Then $z' \in Z$. But from above, $z'$ belongs to some component $g'$ of the intersection of an element of $g'$ of $G'$ with $E$ such that $g' \cdot e_{1,0} = \emptyset$. Since $g'$ is open then it contains points of $x z'-z'$. But then $z'$ is not the first element of $Z$.

Thus the assumption that some point of $x y-y$ belongs to $E-g_{1,0}$ leads to a contradiction. Therefore, $y$ is a point of $M-E$ which is accessible from $g_{1,0}$. Therefore (3) is true for $g_{1,1,\ldots,1,1,u}$. Obviously, the condition is true for $h_{1,1,1,\ldots,1,1}$ ($i=1,1,\ldots,t$).

Since $g_{1,i} (i_0 < i < i_1)$ and $h_{1,j} (j=1,1,\ldots,t)$ are open sets which do not intersect, then $g_{1,1} \cdot h_{1,j} = \emptyset$. Suppose that $g_{1,i} \cdot (M-H_1^*) \neq \emptyset$. Let $y \in g_{1,i} \cdot (M-H_1^*)$.

Now there exists a connected open set $U(y)$ of diameter less than $e_1/4$ containing $y$ which does not contain any point of $x_1$ since $g_1 \subseteq g_1 \subseteq H_1^*$. Since $U(y)$ is open there is a point $x$ belonging to $U(y) \cdot g_{1,i}$. Since $U(y)$ is connected there is an arc $xy$ in $U(y)$ which joins $x$ and $y$. Let $y'$ be the first point of $xy$ from $x$ to $y$ which belongs to $M-H_1^*$. Now consider $xy'-y'$. Since $g_{1,i} (i_0 < i < i_1)$ is a subset of $h_{1,1,\ldots,h_{1,1}}$, then no connected subset of $H_1^*$ of diameter less than $e_1/4$ joins $g_{1,i}$ and $h_{1,j} (i=1,10,11,\ldots,t)$. Hence, since $xy'-y'$ is a subset of $H_1^*$.
which is of diameter less than \( e_1/4 \), and since \( xy'-y' \) intersects \( g_1, i \), it cannot intersect \( h_{1, j} \) (\( j = 1, 10, 11, \ldots, t \)). Since \( xy'-y' \) is a subset of \( U(y) \), then it does not intersect \( \sum_{i=1}^{p} g_i' \). But then \( xy'-y' \) is a connected set which joins \( x \) and \( y' \) and does not intersect \( h_{1, 1} + g_1' + \ldots + g_p' + h_{1, 10} + \ldots + h_{1, t} \). Therefore, \( x \) does not belong to \( E \). But this is a contradiction. Therefore, \( g_1, i \subset (M-H_1) = \emptyset \). Thus \( g_1, i \subset h_{1, 1} + \ldots + h_{1, 10} \).

21 Note here that all of the \( g \)'s have been replaced by elements of \( h_{2,1} \). Also, note that \( h_{2,1} \) is obtained in the last step of the process.

22 The proofs of (1), (2) and (3) are the same as in footnote 19. Properties (4) and (5) follow from the method of combining the sets to form \( h_{2,1} \).

Now consider property (6). Let \( h_{1, i}, 0 \) intersect \( h_{2, m, 0} \) and let \( h_{1, j}, 0 \) intersect \( h_{2, n, 0} \) and suppose \( n > m \).

Now either \((h_{1, i} + \ldots + h_{1, j}, 0) \subset (h_{2, m, 0} + h_{2, n, 0}) \) contains a connected set joining \( h_{1, i} \) and \( h_{1, j} \), or else \((h_{1, i} + \ldots + h_{1, j}, 0) \subset (h_{2, m, 0} + \ldots + h_{2, n, 0}) \) contains a connected set joining \( h_{1, i} \) and \( h_{1, j} \). Suppose the first is true.

Consider \( H(1; i, 0, j, 0) \) and \( H(2; m, 0, n, 0) \). There are elements \( h_{2, k, 0}, h_{2, k_1}, \ldots, h_{2, k_p} \) of \( H_2 \) where \( h_{2, k, 0} \) con-
contains $h_{2,1}; h_{2,k_1}$ contains $h_{2,r+1}; \ldots$; and $h_{2,k_p}$ contains $h_{2,n}$. Notice that $k_{j+1} < k_{j+1}$. Then
\[ h_{2,k_j} + h_{2,k_j+1} + \ldots + h_{2,k_{j+1}} \leq h_{1,10j+1} + \ldots + h_{1,10j+31}. \]

Let $j_1$ and $j_2$ be such that $h_{2,k_{j_1}}$ and $h_{2,k_{j_2}}$

are not elements of $H(2;m_o,n_o)$, but $h_{2,k_j}$ is an element of $H(2;m_o,n_o)$ when $j_1 < j < j_2$. Then $h_{2,m_0}$ is an element of $\{h_{2,1}\}$ when $k_{j_1} < 1 < k_{j_1} + 2$ and $h_{2,n_0}$ is an element of $\{h_{2,1}\}$ when $k_{j_2} - 2 < 1 < k_{j_2}$. Therefore $h_{1,i_0}$ is an element of $h_{1,10j_1}; h_{1,10j_1+1}; \ldots; h_{1,10j_1+42}$. Also, $h_{1,i_0}$ is an element of $h_{1,10j_2}; h_{1,10j_2+1}; \ldots; h_{1,10j_2+22}$.

Now $h_{1,10j_2-20}; h_{1,10j_2+19}; \ldots; h_{1,10j_2+22}$ and thus $H(2;m_o,n_o) \subseteq H(1;i_0,i_0)$. Similarly $H(2;m_o,n_o) \subseteq H(1;i_0,i_o)$.

Let $h_{1,i}$ be any element of $H_1$. Then there are elements $h_{1,i_0}$ and $h_{1,j_1}$ such that: (1) $h_{1,i}$ is an element of $h_{1,i_0} + h_{1,i_0+1}; h_{1,i_0+2}; \ldots; h_{1,j_1-1}$; and (2) there is a subchain $H_2$ of $H_2$ in $h_{1,i_0} + h_{1,i_0+1}; \ldots; h_{1,j_1-1}$ joining $h_{1,i_0}$ and $h_{1,j_1}$. Since the sum of the elements of $H_2$ is a connected subset of $h_{1,i_0} + h_{1,i_0+1}; \ldots; h_{1,j_1-1}$ joining $h_{1,i_0}$ and $h_{1,j_1}$, then some element of $H_2$ intersects $h_{1,i}$ for every $i$ between $j_0$ and $j_1$. Let $h_{2,n_0}$ be the first element of $H_2$.
from $h_{1,j_0}$ to $h_{1,j_1}$ which intersects $h_{1,j+1}$. Let $h_{2,m_0}$ be the first element of $H$ from $h_{2,m_0}$ to $h_{1,j_0}$ which intersects $h_{1,j-1}$. Then $h_{2,m_0} + h_{2,m_0} + 1 + \cdots + h_{2,m_0}$ is a connected set joining $h_{1,j-1}$ and $h_{1,j+1}$. Since the diameter of any such set is greater than or equal to $100e_2$, then the collection $h_{2,m_0}, h_{2,m_0} + 1, \ldots, h_{2,m_0}$ must contain at least 100 elements. Now consider $h_{2,m_0} + 1$. This element cannot intersect $h_{1,i}$ ($i < j$) and it cannot intersect $h_{1,i}$ ($i > j$). Therefore it is a subset of $h_{1,j}$.

Let $R$ be any point of $J_{p_0}$ different from $P$ or $Q$. Then there exists an $i_0$ such that $\rho(R; P+Q) < e_1/200$. Therefore $R$ does not belong to $h_{i_0,P_0} - 100 + \cdots + h_{i_0,P_0} + 100 + h_{i_0,Q_0} - 100 + \cdots + h_{i_0,Q_0} + 100$. But $H(i_0,P_0, Q_0)$. Therefore, since $R$ belongs to $H(i_0,P_0, Q_0)$, $R$ does not belong to $H(i_0,P_0, Q_0)$. Therefore, $R$ does not belong to $J_{p_0}$.

Let $a_1b_1$ be an arc with endpoints only on $Q_1$. If $a_1b_1$ does not contain any subarc with endpoints only
on $G_2$ set $a_2b_2$ equal to $a_1b_1$. If $a_1b_1$ does contain a subarc with endpoints only on $G_2$ set $a_2b_2$ equal to this subarc. Then $a_2b_2C_1 = \emptyset$. In either case $a_2b_2$ has endpoints on one element of $\{C_1, C_2\}$ and no subarc of $a_2b_2$ has endpoints on the other element of $\{C_1, C_2\}$.

Since $\mathcal{P}$ contains only a finite number of elements, there is obtained in a finite number of steps an arc $a_nb_n$ which is a subarc of $a_1b_1$ with endpoints only on some element of $\mathcal{P}$ such that no proper subarc of $a_nb_n$ has endpoints only on any element of $\mathcal{P}$.

The author evidently means the bounded domain.
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EXAMINATION AND THESIS REPORT

Candidate: Robert Lloyd Broussard
Major Field: Mathematics

Title of Thesis: A Necessary and Sufficient Condition That a Set be Homeomorphic to the Plane Region Bounded by a Finite Number of Non-intersecting Circles.

Approved:

[Signature]
Major Professor and Chairman

[Signature]
Dean of the Graduate School

EXAMINING COMMITTEE:

[Signature]

[Signature]

Date of Examination:
July 30, 1951.