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Foundations of Differential Geometry.

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THE FOUNDATIONS OF DIFFERENTIAL GEOMETRY

A DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT

OF THE

REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

LOUISIANA STATE UNIVERSITY

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FRANK ATKINSON RICKEY

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ABSTRACT

In Part One of this dissertation, the properties of a simple arc which possesses the osculating circle at every point are studied. If the equation of the arc in vector notation is \( P - Q = \varphi(t) \) and if the osculating circle exists at a single point \( P_0 \) of this arc, it is shown in Theorem I that \( P_0 \) is interior to a sub-arc QR which is rectifiable. This is accomplished by proving that for every partition of the arc QR, the length \( L \) of the inscribed broken line is such that \( L = r \cdot e_{QR} (1 + \epsilon_{QR}) \), where \( r \) is the radius of the osculating circle, \( e_{QR} \) is the plane angle determined by the normal to the osculating plane at the center of the osculating circle and by the points \( Q \) and \( R \), and where \( \epsilon_{QR} \) is a number which vanishes as \( Q \) and \( R \) approach \( P_0 \). Having the result of Theorem I, Theorem II applies the Heine-Borel-Lebesgue theorem to prove that an arc is rectifiable which possesses the osculating circle at every point. Thus in arcs possessing the osculating circle everywhere, \( s \), the length of arc can be used as the parameter replacing the general parameter \( t \). Theorem II gives the result that \( (\frac{ds}{ds})^2 = 1 \) everywhere on such arcs.

In Part Two, nth ordered total variation of a function \( f(x) \) over an interval \((ab)\) is denoted by the symbol \( \nabla^n f \) and is defined by the equation

\[
\nabla^n f = \delta \sum_{i=0}^{m-n} |f(x_i \ldots x_{i+n})| (x_{i+n} - x_i) \quad (i=0, \ldots, m-n)
\]
where $f(x_1,\ldots,x_n)$ is the Ampere-Cauchy function of order $n$ derived from $f(x)$ and where the bound is over all partitions of $(ab)$ given by $a = x_0 < x_1 < \ldots < x_n = b$. Theorems I-IV give the proof that if $\int_a^b f$ is finite, $f'(x),\ldots,f^{(n)}(x)$ exist almost everywhere on the interval $(ab)$. These theorems are next shown to hold when vector functions are used, and in the theorems that follow, it is proved that if $\int_a^b \phi$ exists, the parameter can be taken as length of arc and $(\frac{d\phi}{ds})^2 = 1$ almost everywhere over the given interval. The simplified formula for curvature, $\rho = \phi^2$ is shown to be valid almost everywhere.
# CONTENTS

<table>
<thead>
<tr>
<th>Paragraph</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td><strong>Part One</strong></td>
<td></td>
</tr>
<tr>
<td>The Osculating Circle and Its Arc</td>
<td></td>
</tr>
<tr>
<td>1. Remarks on Vector Notation and Formulas</td>
<td>3</td>
</tr>
<tr>
<td>2. The Osculating Circle</td>
<td>5</td>
</tr>
<tr>
<td>3. Rectifiability of an Arc in the Neighborhood of a</td>
<td>6</td>
</tr>
<tr>
<td>Point at Which the Osculating Circle Exists</td>
<td></td>
</tr>
<tr>
<td>4. Rectifiability of an Arc at Every Point of Which</td>
<td>16</td>
</tr>
<tr>
<td>the Osculating Circle Exists</td>
<td></td>
</tr>
<tr>
<td>5. Use of Length of Arc as Parameter</td>
<td>17</td>
</tr>
<tr>
<td><strong>Part Two</strong></td>
<td></td>
</tr>
<tr>
<td>The Existence of Derivatives</td>
<td></td>
</tr>
<tr>
<td>1. Functions of Bounded Variation</td>
<td>19</td>
</tr>
<tr>
<td>2. The Ampere-Cauchy Derived Functions</td>
<td>19</td>
</tr>
<tr>
<td>3. Definition of the Nth Derivative</td>
<td>20</td>
</tr>
<tr>
<td>4. Bounded Variation of Higher Order</td>
<td>20</td>
</tr>
<tr>
<td>5. Total Variation of Vector Functions</td>
<td>28</td>
</tr>
<tr>
<td>6. Geometric Applications</td>
<td>30</td>
</tr>
<tr>
<td>Bibliography</td>
<td>33</td>
</tr>
</tbody>
</table>
INTRODUCTION

Differential geometers usually assume that the curves and surfaces with which they deal are analytic, that is, the functions which represent the configurations in question can be expanded in power series, which implies the existence of all derivatives of the functions. Very little of a more fundamental character has been done. A notable exception is the theory of rectifiable curves including the celebrated theorem of Lebesgue asserting the existence almost everywhere of tangents to such curves. The writer proposes to make a study of the properties of a simple arc, using the single assumption that it possesses the osculating circle at every point. He further proposes to formulate a sufficient condition for the existence of the derivatives of a function up to a given order which is expressed in terms of the original function only. This investigation therefore falls naturally into two parts.

In the first part vector methods are used to demonstrate the rectifiability of a simple arc at every point of which the osculating circle exists. As a preliminary theorem, it is proved that an arc which possesses the osculating circle at a single point is rectifiable in the neighborhood of that point. The theorem is important because it justifies the usual practice of employing length of arc as the parameter in the equation of the arc. It is proved that the first derivative of the vector function defining the arc then exists and is equal
in absolute value to unity.

The second part is analytic in character. A definition of total variation of nth order is laid down, and the existence almost everywhere of the derivatives up to and including the nth is shown to follow from the assumption that this total nth variation is bounded. This generalization of the theorem of Lebesgue referred to above leads to an elegant sufficient condition for the validity almost everywhere of the usual formula for curvature in terms of length of arc. A theorem by R. L. Smith giving necessary and sufficient conditions for the existence of the nth derivative is made use of at this point.

The notation and properties of vectors have a considerable part in these developments. Hence a brief summary of the vector facts to be used is first given.
PART ONE

THE OSCULATING CIRCLE AND ITS ARC

1. REMARKS ON VECTOR NOTATION AND PROPERTIES.

In this study, the vector determined by the oriented line segment whose initial point is \( P \) and whose terminal point is \( Q \) will be denoted by the symbol \( Q-P \). This notation gives an algebraic form to the process of adding or subtracting vectors. Vectors will also be represented by single Greek letters if the specification of initial and terminal points is not needed. The lengths of the vectors \( Q-P \) and \( a \) will be denoted by \(|Q-P|\) and \(|a|\), respectively.

Vector functions of scalar variables will be expressed in the usual form. Thus \( \varphi(t) \) denotes the vector function \( \varphi \) of the scalar variable \( t \). The vector equation of a simple arc in space may be written

\[
P-Q = \varphi(t)
\]  

We shall make use of the notations product, determinant of two vectors, and determinant of three vectors as introduced by H. L. Smith. The product of two vectors \( \alpha \) and \( \beta \), denoted by \( \alpha \beta \), is defined by the equation

\[
\alpha \beta = |\alpha| |\beta| \cos \angle \alpha, \beta
\]

where \( \angle \alpha, \beta \) represents the angle between \( \alpha \) and \( \beta \). The determinant of two vectors \( \alpha \) and \( \beta \), symbolized by \( |\alpha, \beta|\),

is defined by the equation
\[ \alpha, \beta = |\alpha| |\beta| \varepsilon \sin \alpha, \beta, \]
\[ \varepsilon \text{ being a unit vector perpendicular to the plane of } \alpha \text{ and } \beta \]
and so directed that \( \alpha, \beta, \) and \( \varepsilon \) form a positively oriented triple. The determinant of three vectors \( \alpha, \beta, \) and \( \gamma, \) represented by \( |\alpha, \beta, \gamma|, \) is defined by the equation
\[ |\alpha, \beta, \gamma| = |\alpha, \gamma| \alpha = |\gamma, \alpha| \beta \]
(4)

We list for reference some of the well known properties of these functions. The justification for the name "determinant" is readily seen.

\[ a^2 = |a|^2 \] (5)
\[ a \beta = 0, \text{ if } a \text{ is perpendicular to } \beta \] (6)
\[ |a, \beta| = 0, \text{ if } a \text{ is parallel to } \beta \] (7)
\[ |a, k\alpha, \beta| = 0 \] (8)
\[ |a, \beta| = |a \pm k\beta, \beta| \] (9)

\[ |a, \beta| |\gamma, \delta| = \begin{vmatrix} \alpha \gamma, \alpha \delta \\ \beta \gamma, \beta \delta \end{vmatrix} \] (Lagrange Identity) (10)

\[ |a, \beta, \gamma|^2 = \begin{vmatrix} a^2, a \beta, a \gamma \\ a \beta, a, a \gamma \\ a \delta, a \gamma \\ \alpha \delta, \beta \gamma, \gamma \end{vmatrix} \] (11)

From the definition of \( |\alpha, \beta, \gamma|, \) we notice that for \( |\alpha, \beta, \gamma| \) to be equal to zero it is necessary that (a) at least one of the three elements be of length zero or that (b) \( \alpha, \beta, \) and \( \gamma \) be co-planar.
2. THE OSCULATING CIRCLE.

A simple arc AB is said to possess an osculating circle at a point $P_0$ of the arc if $P_0$ is interior to a sub-arc such that if $P_1$ and $P_2$ are points of this sub-arc distinct from $P_0$ and from each other, then (a) $P_0$, $P_1$, and $P_2$ determine a circle, and (b) this circle approaches a limiting position as $P_1$ and $P_2$ simultaneously approach $P_0$. The limiting circle is the osculating circle. If the word circle is replaced by the word plane, the above definition becomes the definition of the osculating plane. Obviously the osculating plane exists when the osculating circle does and is the plane of the osculating circle. Hereafter, the plane of the osculating circle will be referred to as the osculating plane.

The existence of the osculating circle at $P_0$ implies the existence of the ordinary tangent, it being evident that the secant of the variable circle through $P_0$ and $P_1$ approaches a limiting position (tangency to the osculating circle at $P_0$) as $P_1$ approaches $P_0$. The same is true for the secant through $P_0$.

We now give a more precise definition of the osculating circle:

An osculating circle exists at a point $P_0$ of a simple arc AB if, for every $\varepsilon > 0$, there exists a sub-arc $P_0^eP_n^e$ to which $P_0$ is interior, a fixed point $C$, and a unit vector $\alpha$, which are such that, if $P_1$ and $P_2$ are points of the arc $P_0^eP_n^e$ distinct from $P_0$ and from each other, then

(a) $P_0$, $P_1$, and $P_2$ determine a circle,
(b) \[ |C' - C| \leq \varepsilon \]
(c) \[ ||a',a|| \leq \varepsilon \]

where \( C' \) is the center of the circle determined by \( P_0, P_1, P_2, \) and \( a' \) is the unit vector defined by the equation

\[ a' = \frac{|P_1-C',P_2-C'|}{\sqrt{|P_1-C',P_2-C'|^2}} \]

3. RECTIFIABILITY OF AN ARC IN THE NEIGHBORHOOD OF A POINT AT WHICH THE OSCULATING CIRCLE EXISTS.

Definitions. The length of an arc is defined to be the least upper bound of the lengths of all possible broken lines inscribed in the arc. If this bound is finite, the arc is said to be rectifiable.

In proving that a point of a given curve at which the osculating circle exists is interior to a rectifiable sub-arc, we shall show that the sub-arc chosen is such that its length is not greater than a specified multiple of a definite arc of the osculating circle at the point. To this end, we introduce the following notations for use hereafter in this section:

\( P_0 \) : the point of the given curve at which the osculating circle exists.

\( C \) and \( r \) denote the center and radius, respectively, of the osculating circle at \( P_0. \)

\( P_e'P_e'' \) : the sub-arc to which \( P_0 \) is interior, which by definition is such that for a given \( e > 0, \) if \( P_1 \) and \( P_2 \) are distinct points of arc \( P_e'P_e'', \) then \( |C_{1,2} - C| \leq \varepsilon, \) and \( ||a_{1,2}|| \leq \varepsilon, \) where \( C_{1,2} \) is the center of the circle determined by \( P_0, P_1, \) and \( P_2. \)
if $P_1$ and $P_2$ are distinct from $P_o$ or by $P_1$, $P_2$ and another arbitrary point $P_3$ of arc $P_eP_e'$ if $P_o$ is equal to $P_1$ or $P_2$, and where $a_1$ is the vector $\frac{P_1-C_{1,2}P_2-C_{1,2}}{\sqrt{|P_1-C_{1,2},P_2-C_{1,2}|^2}}$ unit vector perpendicular to the plane of $P_1$, $P_2$, and $C_{1,2}$.

**Lemma I.** If the osculating circle exists at a point $P_o$ of an arc whose vector equation is $P-e = \varphi(t)$ and if $m_1$ and $m_2$ are two half planes determined by the normal at $C$ to the osculating plane and by $P_1$ and $P_2$, respectively, where $P_1$ and $P_2$ are any distinct points of the arc $P_eP_e'$ given by $t=t_1$ and $t=t_2$, $(t_1 \neq t_2)$ respectively, then the dihedral angle formed by $m_1$ and $m_2$ cannot be zero, provided only that $\varphi$ be chosen less than the smaller of $\pi/3$ and $\sqrt{2}/2$ and that $\angle e_{1,2} < \pi/3$.

**Proof.** Let $\beta_1$ and $\beta_2$ be the components perpendicular to $\alpha$ of $P_1-C$ and $P_2-C$, respectively. Then $\gamma_1$, the angle between $\beta_1$ and $\beta_2$, is the plane angle of dihedral angle $m_1m_2$. It will be sufficient to prove that $\sin \gamma_1 \neq 0$.

Now set

$$P_i - C = k_1 \alpha + \beta_i$$

Multiplying both sides by $\alpha$ and noting that $\alpha^2 = 1$, $\alpha \beta_i = 0$, we are able to obtain the result

$$\beta_i = P_i - C - [\alpha(P_i - C)]\alpha$$ \hspace{1cm} (12)

Similarly,

$$\beta_2 = P_2 - C - [\alpha(P_2 - C)]\alpha$$ \hspace{1cm} (13)

It follows that

$$\beta_i^2 = (P_i - C)^2 - [\alpha(P_i - C)]^2$$ \hspace{1cm} (14)
and
\[ \beta^2 = (P_1 - C)^2 - [a(P_2 - C)]^2 \]  \hspace{1cm} (15)

Now by (3), \( |\beta_1, \beta_2| = \sqrt{\beta_1^2 \beta_2^2} \sin \gamma, \) \( \gamma \) being the unit vector perpendicular to the plane of \( \beta_1, \beta_2 \), and so directed that \( \beta_1, \beta_2, \) and \( \gamma \) form a positively oriented triple. Hence, from the fact that \( \gamma^2 = 1 \), and from (4), we have,
\[ \sin \gamma = \left| \frac{\beta_1 \beta_2 - \gamma}{\sqrt{\beta_1^2 \beta_2^2}} \right| \] \hspace{1cm} (16)

Let \( \alpha_{1,2} \) be defined by the equation obtained by replacing \( \gamma \) by \( \alpha \) in (16), i.e.,
\[ \sin \alpha_{1,2} = \left| \frac{\beta_1 \beta_2 - \alpha}{\sqrt{\beta_1^2 \beta_2^2}} \right| \] \hspace{1cm} (17)

Substituting the values of \( \beta_1, \beta_2, \beta_1^2, \) and \( \beta_2^2 \) from (12), (13), (14), and (15) and making use of (9), we obtain from (17)
\[ \sin \alpha_{1,2} = \frac{|P_1 - C, P_2 - C, \alpha|}{\sqrt{(P_1 - C)^2 - [a(P_2 - C)]^2 \sqrt{(P_1 - C)^2 - [a(P_2 - C)]^2}}} \] \hspace{1cm} (18)

The numerator of the fraction in (18) will be zero only if (a) at least one of the vectors \( P_1 - C, P_2 - C, \) and \( \alpha \) is of length zero, or (b) if \( \alpha \) is co-planar with \( P_1 - C \) and \( P_2 - C \).

From the fact that \( |P_1 - C| = r \) and the hypothesis that \( |C - C_1| \leq e \), it easily follows that
\[ |P_1 - C_1| = |P_1 - C| \leq r - e \] \hspace{1cm} (19)

Applying again the condition that \( |C - C_1| \leq e \), we have from (19) and from the fact that \( e < r/3 \) the result that
\[ |P_1 - C| > r/3 \] \hspace{1cm} (20)

From the symmetry in \( P_1 \) and \( P_2 \), it follows that
The fact that $a$ is a unit vector, together with (20) and (21), make (a) impossible.

To show that (b) is impossible, let $M_{12}$ be the midpoint of the chord $P_1P_2$. Then $|C_{12} - M_{12}|$ is the perpendicular distance from $C_{12}$ to the line determined by $P_1$ and $P_2$. Denoting this distance by $d_{12}$, and using the hypothesis that $e < r/3$ and $\angle P_{12}C_{12} < \pi/3$, we find that

$$d_{12} = |P_{12} - C_{12}| \cos \angle P_{12}C_{12}P_2 > \frac{\sqrt{3}}{2}$$

(22)

But $|C_{12} - C| \leq e < r/3$. Therefore

$$d_{12} > |C_{12} - C|$$

(23)

Hence $C$ cannot be co-linear with $P_1$ and $P_2$.

It remains to show that $a$ cannot be co-planar with $P_1 - C$ and $P_2 - C$. By hypothesis and from the definition of the osculating circle, we have $|\alpha, a_{12}| \leq e < \sqrt{2}/2$. Hence

$$\angle \alpha, a_{12} < \pi/4$$

(24)

where

$$a'_{12} = a_{12} \text{ if } \angle \alpha, a_{12} < \pi/2$$

$$a'_{12} = -a_{12} \text{ if } \angle \alpha, a_{12} \geq \pi/2.$$ 

Now the plane angle of dihedral angle $C_{12}P_1P_2$ can be taken to be $\angle D_{12}C_{12}$, where $M_{12}$ is as above taken and where $D$ is the foot of the perpendicular let fall from $C_{12}$ to the plane of $P_1$, $P_2$, and $C$. We have

$$|D_{12} - C_{12}| \leq |D_{12} - C| \leq e < r/3$$

(25)

and from (22),
\[ \left| X_{1,2} - C_{1,2} \right| = \sqrt{3} \]

(25) and (26) give

\[ \mathcal{IMC}_{1,2} = \sin^{-1}\left(\frac{D - C_{1,2}}{\left| X_{1,2} - C_{1,2} \right|} \right) < \sin^{-1}(\sqrt{3}/3) \]  

This means that the angle between \( a'_{1,2} \) and the normal to the plane of \( P_1, P_2, \) and \( C \) is less than \( \sin^{-1}(\sqrt{3}/3) < \pi/4 \). From this fact and from (14), we have the result that the angle between \( a \) and the normal to the plane of \( P_1, P_2, \) and \( C \) is less than \( \pi/2 \), which proves (b) impossible. This also proves that the denominator of the fraction of (18) cannot vanish, for in that case \( a \) would not only be in the plane of \( P_1 - C \) and \( P_2 - C \) but would have to be equal to a multiple of one of these vectors as well. Therefore

\[ \sin e_{1,2} \neq 0 \]  

But

\[ \sin \gamma_{1,2} = \pm \sin e_{1,2} \]

Hence,

\[ \sin \gamma_{1,2} \neq 0 \]

which proves the lemma.

**Lemma II.** Let the osculating circle exist at a point \( P_0 \) of an arc whose vector equation is \( P - Q = \varphi(t) \). Let \( P_1 - Q = \varphi(t_1), P_2 - Q = \varphi(t_2), P_3 - Q = \varphi(t_3), \) and \( P_4 - Q = \varphi(t_4), \) where \( t_1 < t_2 < t_3 < t_4 \), define the points \( P_1, P_2, P_3, \) and \( P_4 \) of the sub-arc \( P_0P_n \), \( e \) being chosen sufficiently small for the conditions of Lemma I to hold. Then
\[ |P_1 - C_1, P_2 - C_2, \alpha| - |P_3 - C_3, P_4 - C_4, \alpha| > 0 \]

Proof. Consider the function \( F(t) \) defined by the equation

\[ F(t) = \left| \varphi(t_t - t_1, t_2, t_3) - (C - \alpha), \varphi(t_t - t_4, t_2, t_3) - (C - \alpha), \alpha \right| \quad (\text{ôstál}) \]

Obviously

\[ F(0) = |P_1 - C_1, P_2 - C_2, \alpha| \quad \text{and} \quad F(1) = |P_3 - C_3, P_4 - C_4, \alpha| \quad (29) \]

We note that \( t_1 - t_1 [t_1, t_2] \) is between \( t_1 \) and \( t_2 \) and that \( t_2 - t_1 [t_2, t_3] \) is between \( t_2 \) and \( t_3 \) on the given interval. Since \( t_1 < t_2 < t_3 \) is easily shown that \( t_1 - t_1 [t_1, t_3] \) is less than \( t_2 - t_1 [t_2, t_3] \). Hence \( F(t) \) satisfies the hypothesis of Lemma I for òstál. We have, therefore, by means of (18) and (28),

\[ F(t) \neq 0 \quad (\text{ôstál}) \quad (30) \]

Since \( f(t) \) is continuous in \( t \) over the interval òstál, it follows from (30) that \( F(0) \) and \( F(1) \) are of the same sign.

This, with (29), gives the lemma.

**Corollary.** If, as in (18), we define \( e_{1,2} \) and \( e_{3,4} \) by the equations

\[ \sin e_{1,2} = \frac{|P_1 - C_1, P_2 - C_2, \alpha|}{\sqrt{(P_1 - C_2 - C_2)^2 + (P_2 - C_2 - C_2)^2}} \]

\[ \sin e_{3,4} = \frac{|P_3 - C_3, P_4 - C_4, \alpha|}{\sqrt{(P_3 - C_4 - C_4)^2 + (P_4 - C_4 - C_4)^2}} \]

where \( P_1, P_2, P_3, \) and \( P_4 \) are as in Lemma II, then

\[ (e_{1,2})(e_{3,4}) > 0 \]

for values of \( e_{1,2} \) and \( e_{3,4} \) between \(-\pi/2\) and \(+\pi/2\).

**Proof.** From Lemma II, it follows that

\[ (\sin e_{1,2})(\sin e_{3,4}) > 0 \quad (31) \]

From (31) the corollary is obvious.
Note. Comparing (16) and (17), we observe that if \( \gamma \) is less than \( \pi/2 \) it is true that \( |e_{12}| = \gamma \). We conclude from (18) that \( e_{12} \) is positive or negative according as \( P_1 - C, P_2 - C \), and a form a positively or negatively oriented triple. The corollary just proved may then be interpreted to mean that \( P_1 - C, P_2 - C \), and a form a triple of similar orientation for all distinct values of \( P_1 \) and \( P_2 \) belonging to the sub-arc \( P_1'P_2' \), provided \( t_1 < t_2 \), where \( t_1 \) and \( t_2 \) are the values of the parameter \( t \) which determine \( P_1 \) and \( P_2 \) respectively from the equation \( P-O = \varphi(t) \).

Lemma III. Assuming the hypothesis and notation of Lemma I, the following statement is true:

\[
\lim_{(q) \to (a,b)} \left( \frac{\sin e_{12}}{P_1 P_2} \right)^2 = \frac{1}{r^2}
\]

where the limit is as the end points of any sub-arc \( QR \) which contains the points \( P_1 \) and \( P_2 \) simultaneously approach \( P_0 \).

By \( P_1' P_2' \) is meant the length of the chord \( P_1 P_2 \).

Proof. From (18), we have

\[
\sin e_{12} = \frac{|P_1 - C, P_2 - C, \alpha|}{\sqrt{p(P_1)}\sqrt{p(P_2)}}
\]

where \( p(P_i) = (P_1 - C)^2 - [\alpha(P_1 - C)]^2 \) \( (i=1,2) \). Dividing both sides of (32) by \( P_1 P_2 \) and denoting \( P_2 - P_1 \) by \( (P_2 - P_1)_1 \), we obtain by means of (9)

\[
\frac{\sin e_{12}}{P_1 P_2} = \frac{|P_1 - C, (P_2 - P_1)_1, \alpha|}{\sqrt{p(P_1)}\sqrt{p(P_2)}}
\]
If both sides of (33) are squared and (11) applied, there results,

\[
\left(\frac{\sin e_d}{P_1 P_2}\right)^2 = \begin{vmatrix}
(P-C)^2 & (P-C)(P_2-P_1) & (P-C)\alpha \\
(P-C)(P_2-P_1) & 1 & (P_2-P_1)\alpha \\
(P-C)\alpha & (P_2-P_1)\alpha & 1
\end{vmatrix}
\sqrt{p(P_1)^2} \sqrt{p(P_2)^2}
\]  

(34)

We consider now the limits of the various elements which compose the right hand member of equation (34), the limits being taken as \((QR) \rightarrow (P_0 P_0)\). From the definition of the osculating circle at \(P_0\), it is obvious that \((P, -C)^2\) approaches \(r^2\) and that \(P, -C\) approaches perpendicularity to \(e\). Hence by (6), \(e(P, -C)\) approaches zero. Furthermore,

\[
\lim_{(QR) \rightarrow (P_0 P_0)} \frac{(P, -C_1)(P_2-P_1)}{1} = 0
\]

and

\[
\lim_{(QR) \rightarrow (P_0 P_0)} \frac{(C - C_1)(P_2-P_1)}{1} = 0
\]

By subtraction,

\[
\lim_{(QR) \rightarrow (P_0 P_0)} \frac{(P, -C)(P_2-P_1)}{1} = 0
\]  

(35)

Again, \((P, -P_1)\) \(1\alpha_{1,2} = 0\) and \(\lim \alpha_{1,2} = \alpha\). Hence, \(\lim (P_2-P_1)\) \(1\alpha = 0\) and \(\lim (P_2-P_1)\) \(1\alpha = 0\)

From these limits, we see that \(p(P_1)\) and \(p(P_2)\) each approach \(r^2\).

Making use of the limits discussed in the last paragraph, we have from (34)

\[
\lim_{(QR) \rightarrow (P_0 P_0)} \left(\frac{\sin e_d}{P_1 P_2}\right)^2 = \begin{vmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{vmatrix} = \frac{1}{r^2}
\]  

\(r^4\)
Corollary. If the osculating circle exists at a point \( P_0 \) of an arc \( P-0 = \varphi(t) \), then there exists a sub-arc QR to which \( P_0 \) is interior and a number \( \varepsilon_{Q\mathcal{R}} \) which are such that, if \( P_1 \) and \( P_2 \) are any distinct points of the arc QR, it is true that

\[
\overline{P_1P_2} \leq r |a_{12}| (1 + \varepsilon_{Q\mathcal{R}})
\]

and

\[
\lim_{(Q\mathcal{R}) \to (P_0 P_0)} \varepsilon_{Q\mathcal{R}} = 0
\]

Proof. Let QR be any sub-arc of the arc \( P_0P_0' \) such that \( P_0 \) is interior to it, \( P_0P_0' \) being subject to the conditions of Lemma I. Then by Lemma III,

\[
\lim_{(Q\mathcal{R}) \to (P_0 P_0)} \left( \frac{\overline{P_1P_2}}{|a_{12}|} \right) = r.
\]

Therefore from the definition of a limit, there exists an \( \varepsilon_{12} \) such that

\[
\overline{P_1P_2} = r \varepsilon_{12} (1 + \varepsilon_{12}), \text{ where } \lim_{(Q\mathcal{R}) \to (P_0 P_0)} \varepsilon_{Q\mathcal{R}} = 0.
\]

Let \( \varepsilon_{Q\mathcal{R}} \) be defined by the equation

\[
\varepsilon_{Q\mathcal{R}} = \overline{B} \left| \varepsilon_{12} \right|
\]

where the bound is over all pairs of points \( P_1, P_2 \) as above chosen. Obviously,

\[
\overline{P_1P_2} \leq r |a_{12}| (1 + \varepsilon_{Q\mathcal{R}})
\]

and

\[
\lim_{(Q\mathcal{R}) \to (P_0 P_0)} \varepsilon_{Q\mathcal{R}} = 0
\]

Theorem I. If the osculating circle exists at a point \( P_0 \) of an arc \( P-0 = \varphi(t) \), then \( P_0 \) is interior to a rectifiable sub-arc of the given arc.
Proof: \( P_0 \) is interior to the sub-arc QR defined in the preceding corollary. We now prove that QR is rectifiable.

Let a partition of the arc QR be formed by the points \( Q_0, Q_1, \ldots, Q_n \) in the following manner: Using the same subscript for the value of the parameter \( t \) which corresponds to a point as is used to distinguish the point, we assume

\[ t_0 = t_1 < t_2 < \ldots < t_n = t_R \]

Thus a partition of the arc QR is determined by the ordered points \( Q_0, \ldots, Q_n \).

Let \( e_{i-1,i} \) be defined by the equation

\[
sin e_{i-1,i} = \frac{|Q_i - C|}{\sqrt{D(Q_i)}^2 \sqrt{D(Q_i)}} \quad \text{(38)}
\]

where \( p(Q_i) = (Q_i - C)^2 - [a(Q_i - C)]^2 \)

Then by the Corollary to Lemma III, there exists a number \( \epsilon_{QR} \) such that

\[
\overline{Q_{i-1}Q_i} \leq r \left| e_{i-1,i} \right| (1 + \epsilon_{QR}) \quad (i = 1, \ldots, n) \quad \text{(39)}
\]

Therefore, letting \( L \) denote the length of the inscribed broken line whose successive vertices are \( Q_0, Q_1, \ldots, Q_n \), it follows that

\[
L \leq r (1 + \epsilon_{QR}) \sum_{i=1}^{n} |e_{i-1,i}| \quad \text{(39')}
\]

Referring to the Corollary to Lemma II and the Note following it, we are able to conclude that

\[
\sum_{i=1}^{n} |e_{i,i+1}| = |\sum_{i=1}^{n} e_{i-1,i}| = \gamma_{QR}
\]

where \( \gamma_{QR} \) is the plane angle of the dihedral angle determined
by the normal at C to the osculating circle and by the points Q and R. Hence (39') becomes

\[ L = r(l + \epsilon_{QR}) \delta_{QR} \]

This is true for values of L given by all partitions of arc QR, and therefore for the least upper bound of L over all partitions. Thus

\[ \text{length of arc } QR = E(L) = r(l + \epsilon_{QR}) \delta_{QR} \]

Therefore QR is rectifiable.

**Corollary.** If the osculating circle exists at a point \( P_0 \) of an arc \( P-O = \varphi(t) \), \( P_0 \) belongs to an open sub-arc which is rectifiable.

**Proof.** It is obvious that if the points Q and R are removed from the arc QR of the preceding theorem, the open arc remaining is rectifiable and contains the point \( P_0 \), \( P_0 \) being interior to the closed arc QR.

4. **RECTIFIABILITY OF AN ARC AT EVERY POINT OF WHICH THERE EXISTS THE OSCULATING CIRCLE.**

If the osculating circle exists at every point of an arc AB defined by \( P-O = \varphi(t) \), we know that every point of AB can be inclosed in a rectifiable, open sub-arc of AB, according to the corollary above proved. Hence AB can be covered with an infinity of open arcs, each of finite length. By the classic Heine-Borel-Lebesgue theorem, arc AB can be inclosed in a finite number of such arcs. Hence arc AB is of finite length. We have then proved the following theorem:
Theorem II. If the osculating circle exists at every point of an arc $AB$, then $AB$ is rectifiable.

5. USE OF LENGTH OF ARC AS PARAMETER IN EQUATIONS OF ARCS HAVING THE OSCULATING CIRCLE AT EVERY POINT.

If an arc $PQ = \varphi(t)$ possesses the osculating circle at every point, we have shown in §4 that the length of arc $s$, measured from a fixed point to a variable point $P$, is finite. Hence $s$ is a monotonic function of $t$ and the equation of the arc can be written

$$PQ = s(s)$$

(40)

Theorem III. If the osculating circle exists at a point $P_0$ of an arc whose equation is $PQ = \varphi(s)$, then $d\varphi ds$ exists at $P_0$ and

$$\left(\frac{d\varphi}{ds}\right)^2 = 1$$

Proof. Let $QR$ be a sub-arc to which $P_0$ is interior such that the conditions of the corollary to Lemma III hold. Let $|\Delta \varphi|$ be taken as $QR$ and let $L_{QR}$ be the length of the inscribed broken line determined by a partition of $QR$ as taken in the proof of Theorem I. Also let $e_{\varphi_{QR}'}$ be defined by (38). Then by (39')

$$L_{QR} \leq r(1 + e_{\varphi_{QR}}) \sum_{j=1}^{m} |e_{\varphi_{QR}' j}| = r(1 + e_{\varphi_{QR}}) \gamma_{QR}$$

and by (39),

$$|\Delta \varphi| = r(1 + e_{\varphi_{QR}}) \gamma_{QR}$$
Hence

\[ \frac{1 - \varepsilon_{QR}}{1 + \varepsilon_{QR}} \leq \frac{\Delta \Phi}{\Delta s} \leq \frac{1 + \varepsilon_{QR}}{1 - \varepsilon_{QR}} \]  

(41)

But \( L_{QR} \) is monotonic increasing as the partition is made finer and finer. We have shown that \( E(L) \), the length of the arc QR is finite. Hence, if \( \Delta s \) is the length of arc QR, we have

\[ \lim_{\text{partition finer}} L_{QR} = \Delta s \]  

(42)

where the limit is as the partition is made finer and finer.

Now \( \varepsilon_{QR} \) is a function of the arc QR and not of its partition. Therefore (41) and (42) give

\[ \frac{1 - \varepsilon_{QR}}{1 + \varepsilon_{QR}} \leq \frac{\Delta \Phi}{\Delta s} \leq \frac{1 + \varepsilon_{QR}}{1 - \varepsilon_{QR}} \]  

(43)

But

\[ \lim_{(QR) \to (PQ)} \varepsilon_{QR} = 0 \]

Hence

\[ \frac{d\Phi}{ds} = \lim_{(QR) \to (PQ)} \frac{\Delta \Phi}{\Delta s} = 1 \]

or

\[ \left( \frac{d\Phi}{ds} \right)^2 = 1 \]
PART TWO

THE EXISTENCE OF DERIVATIVES

1. FUNCTIONS OF BOUNDED VARIATION.

If a function $f(x)$ is defined over an interval $a\leq x \leq b$, the total variation of $f(x)$ on $(ab)$, denoted in this paper by the symbol $\int_a^b f$, is defined by the equation

$$\int_a^b f = \sup \left( \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \right)$$

where the bound is over all partitions $a = x_0 < x_1 < \cdots < x_n = b$. If $\int_a^b f$ is finite, we say that $\int_a^b f$ exists and call $f(x)$ a function of bounded variation.

It is well known that an arc $y = f(x)$ is rectifiable over an interval on which $f(x)$ is of bounded variation. The classic theorem of Lebesgue gives the fact that a function of bounded variation possesses a finite first derivative over any such interval with the exception of at most a set of points of measure zero.

2. THE AMPERE-CAUCHY DERIVED FUNCTIONS.

Any function $f(x)$ of a single variable gives rise to an associated function $f(x_0, x_i)$ of two variables defined for distinct values of these variables by the equation

$$f(x_0, x_i) = \frac{f(x_i) - f(x_0)}{x_i - x_0}$$

Assuming that $f(x_0, x_i), \ldots, f(x_0, x_3, \ldots, x_{n-1})$ have been inductively defined as functions of 2, 3, $\ldots$, $n$ distinct arguments, respectively, we can define $f(x_0, \ldots, x_n)$ by the equation
\[ f(x_0, \ldots, x_n) = \frac{f(x, \ldots, x_n) - f(x_0, \ldots, x_n)}{x - x_0} \quad (2) \]

The functions \( f(x_0, x_1, \ldots, x_n) \) are called the Ampere-Cauchy functions of orders 1, 2, \ldots, \( n \), respectively. It is well known that \( f(x_0, \ldots, x_n) \) is a symmetric function of its \( n+1 \) arguments.

3. A DEFINITION OF THE Nth DERIVATIVE OF \( f(x) \).

Denoting the \( n \)th derivative of \( f(x) \) by \( f^{(n)}(x) \), we shall use the definition of \( f^{(n)}(x) \) given by the equation

\[ f^{(n)}(x) = \lim_{n \to \infty} f(x_0, x_1, \ldots, x_n) \]

where the limit is as \( x_0, \ldots, x_n \) simultaneously approach \( x \), \( x \) remaining between the largest and smallest of the numbers \( x_0, \ldots, x_n \).

That this definition is at least as general as the ordinary one follows from a theorem by Stieltjes\(^1\).

4. BOUNDED VARIATION OF HIGHER ORDER.

We now extend the concept of the total variation of a function to include total variation of the \( n \)th order. We represent the total \( n \)th variation of \( f(x) \) over the interval \( a \leq x \leq b \) by the symbol

\[ \int_a^b \mathcal{V} f \]

and define it by the equation

\(^1\)Cf. Stieltjes, Oeuvres, tome 1, pp. 67-72.
\[
\mathbf{\mathbf{b}}^n \int_{x_i}^{x_{i+1}} f(x) dx = \sum_{i=0}^{m-1} \left| f(x_{i+1}, \ldots, x_{i+n}) (x_{i+n} - x_i) \right|
\]

where the bound is over all partitions \( a = x_0 < x_1 < \ldots < x_m = b \). If \( \mathbf{\mathbf{b}}^n \int_{x_i}^{x_{i+1}} f(x) dx \) is finite, we say that \( f(x) \) is of bounded \( n \)th variation or that \( \mathbf{\mathbf{b}}^n \int_{x_i}^{x_{i+1}} f(x) dx \) exists.

It is obvious that the special case \( n = 1 \) is that of ordinary total variation.

**Lemma I.** If \( u_0, \ldots, u_m \) are any choices of the numbers \( x_0, \ldots, x_m (m \leq n) \), then \( f(u_0, \ldots, u_m) \) is between the largest and smallest of \( f(x_{i+1}, \ldots, x_{i+n}) \) \( (i = 0, \ldots, n-m) \)

**Proof.** Since \( f(u_0, \ldots, u_m) \) is a symmetric function of the \( u \)'s, we may assume \( u_0 < u_1 < \ldots < u_m \). To make the proof by mathematical induction, we notice that the theorem is evidently true if there are no \( x \)'s between the \( u \)'s distinct from them. Assuming the truth of the theorem when there are at most \( k \) of the \( x \)'s between \( u \) and \( u_m \) distinct from \( u_0, u_1, \ldots, u_m \), we now consider the case in which there are \( k+1 \) such \( x \)'s. From the assumption of the truth of the theorem for the \( k \)th case, we have

\[ f(u_0, \ldots, u_m) \text{ is between } f(u_0, \ldots, u_m, \xi) \text{ and } f(u_0, \ldots, u_m, \xi), \]

where \( \xi \) is any one of the \( k+1 \) \( x \)'s between \( u_0 \) and \( u_m \) distinct from \( u_0, \ldots, u_m \). This is true since by definition (2)

\[
f(u_0, \ldots, u_m) = \frac{f(u_0, \ldots, u_m) - f(u_0, \ldots, u_m)}{u_0 - u_m} \quad (4)
\]

\[
f(u_0, \ldots, u_m, \xi) = \frac{f(u_0, \ldots, u_m, \xi) - f(u_0, \ldots, u_m, \xi)}{u_0 - \xi} \quad (5)
\]

\[
f(u_0, \ldots, u_m, \xi) = \frac{f(u_0, \ldots, u_m, \xi) - f(u_0, \ldots, u_m)}{\xi - u_m} \quad (6)
\]
and since the sum of the numerators of (5) and (6) equals the numerator of (4) and the sum of the denominators of (5) and (6) likewise equals the denominator of (4). But there are at most $k$ of the $x$'s distinct from $u_0, \ldots, u_m, \xi$ and between the largest and smallest of them. Hence $f(u_0, \ldots, u_m, \xi)$ is between the largest and smallest of $f(x_c, \ldots, x_{c+n})$ for $(i=0, \ldots, n-m)$. The same is true for $f(u, \ldots, u_m, \xi)$ and hence for $f(u_0, \ldots, u_m)$ which is between $f(u_0, \ldots, u_m, \xi)$ and $f(u, \ldots, u_m, \xi)$.

**Theorem 1.** If $f(x)$ is such that $f^{(n)}(x)$ exists, then $f^{(n)}(x)$ exists everywhere on $(ab)$ except at a set of points at most denumerable.

**Proof.** The Cauchy condition that $f^{(n)}(x_0)$ exist is that

$$\lim_{\substack{\text{h, \ldots, h} \\ \text{simultaneously approach zero}}} \left| f(x_0, x_0 + \text{h}, \ldots, x_0 + \text{h} + \text{n}) - f(x_0, x_0 + \text{k}, \ldots, x_0 + \text{k} + \text{n}) \right| = 0 \quad (7)$$

the limit being taken as the distinct numbers $\text{h}, \ldots, \text{h}, \text{k}, \ldots, \text{k}$, simultaneously approach zero.

Let $K$ be any positive number and let $c_1, \ldots, c_m$ be points of $(ab)$ at which

$$\lim_{(i = 1, \ldots, m)} \left| f(c_i, c_i + \text{h}_{c_i}, \ldots, c_i + \text{h}_{c_i}) - f(c_i, c_i + \text{k}_{c_i}, \ldots, c_i + \text{k}_{c_i}) \right| > K \quad (8)$$

Then, if $d$ is the smallest of the numbers $|c_i - c_{i-1}|, (i = 1, \ldots, m)$, for every $e > 0$ there exist numbers $h_{c_i}, k_{c_i} (i = 1, \ldots, m; j = 1, \ldots, n-1)$ such that each is less in absolute value than $d/2$ and such that

$$\left| f(c_i, c_i + h_{c_i}, \ldots, c_i + h_{c_i}), f(c_i, c_i + k_{c_i}, \ldots, c_i + k_{c_i}) \right| > K - e \quad (i = 1, \ldots, m) \quad (9)$$
Let \( x_0, x_p, \ldots, x_p \) be any partition of \((ab)\) which contains all of the points \( c_i, c_i + h_{cij}, c_i + k_{cij} \) \((i=1, \ldots, m; j=1, \ldots, n-1)\).

Then from Lemma I, it follows that

\[
\begin{align*}
\frac{b}{a} f &\geq \sum_{i=0}^{b-n} |f(x_i, \ldots, x_{i+n})| (x_{i+n} - x_i) \\
&> \sum_{i=1}^{m} f(c_i, c_i + h_{c_i}, \ldots, c_i + h_{c_{i+n}}) - f(c_i, c_i + k_{c_i}, \ldots, c_i + k_{c_{i+n-1}}) \\
&> m(K-e),
\end{align*}
\]

Since \( e \) is arbitrary, it is true that

\[
\frac{b}{a} f \geq mK
\]

(10)

\( \frac{b}{a} f \) and \( K \) being finite, \( m \) must be finite, which means that the number of points of \((ab)\) at which (8) holds is finite. If \( K \) is given successively the values \( \frac{1}{2}, \frac{1}{4}, \ldots \) it becomes evident that the number of points at which (7) fails to hold is at most denumerable.

**Theorem II.** If there exists a point \( x_0 \) belonging to \((ab)\) which is such that \( \bar{f}^{(m)}(x_0) = +\infty \) or that \( \bar{f}^{(m)}(x_0) = -\infty \), then

\[
\frac{b}{a} f = +\infty
\]

**Proof.** Suppose \( x_0 \) such that \( \bar{f}^{(m)}(x_0) = +\infty \). Then let \( \xi_0, \ldots, \xi_{n-1} \) be points of \((ab)\) which are such that for \( \ell = 0, \ldots, n-1 \)

\[
|\xi_\ell - x_0| > \frac{1}{4} (b-a).
\]

Finally, let \( M \) be an arbitrary number such that \( M > f(\xi_0, \ldots, \xi_{n-1}) \). Then there exist numbers \( h_1, \ldots, h_{n-1} \) such that each is less in absolute value than \( \frac{1}{4} (b-a) \) and such that
Now let a partition of \((ab)\) be made by the points \(x_0,\ldots,x_p\)
which contains the points \(x_0,x_0+h,\ldots,x_0+h_{n-1},\xi_0,\ldots,\xi_{n-1}\), but
which contains no other points which lie between the largest and
smallest of \(x_0,x_0+h,\ldots,x_0+h_{n-1}\) or between the largest
and smallest of \(\xi_0,\ldots,\xi_{n-1}\). Then

\[
\frac{b}{a} \sum_{x} f \geq |f(x_0, x_0+h, \ldots, x_0+h_{n-1}) - f(\xi_0, \ldots, \xi_{n-1})|
\]

\[
= f(x_0, x_0+h, \ldots, x_0+h_{n-1}) - f(\xi_0, \ldots, \xi_{n-1})
\]

\[
\geq M - f(\xi_0, \ldots, \xi_{n-1})
\]

Now \(f(\xi_0, \ldots, \xi_{n-1})\) is a fixed number, but \(M\) can be taken arbitrarily large. Hence

\[
\frac{b}{a} \sum_{x} f = + \infty
\]

The proof from the assumption that \(\lim_{(n)}(x_0) = -\infty\) is
similar except that the absolute value signs are retained,
\(M\) being chosen as an arbitrary negative number.

This theorem can be re-stated in the following form:

**Corollary.** If \(\frac{b}{a} \sum_{x} f\) exists, then

\(F^{(n-1)}(x)\) is finite \((a = x = b)\)

and

\(F^{(n-1)}(x)\) is finite \(\quad (\quad )\)

**Theorem III.** If \(\frac{b}{a} \sum_{x} f\) exists, then \(\frac{b}{a} F^{(n-1)}\) and \(\frac{b}{a} f^{(n-1)}\)
exist.

**Proof.** Let \(x_0,\ldots,x_m\) form an arbitrary partition of
\((ab)\), and let \(d\) be the smallest of the numbers \(x_i-x_{i-1}\) \((i=1,\ldots,m)\).
Then, since by Theorem II, \( F^{(n-1)} \) is finite on \((ab)\), for every \( \epsilon > 0 \) there exist numbers \( h_1, \ldots, h_{n-1} \), each less in absolute value than \( \frac{1}{2d} \), such that from the definition in § 3,

\[
|f(x; x_i + h_i, \ldots x_i + h_{i, n-1}) - \frac{F^{(n-1)}(x_i)}{(n-1)!}| \leq \frac{\epsilon}{2m} \quad (i = 0, \ldots, m) \quad (12)
\]

From (12) and the fact that the absolute value of a sum is not greater than the sum of the absolute values, we have

\[
\left| \frac{F^{(n-1)}(x)}{(n-1)!} - \frac{F^{(n-1)}(x_i)}{(n-1)!} \right| \leq \left| f(x; x_i + h_i, \ldots x_i + h_{i, n-1}) - f(x_i; x_i + h_{i, i}, \ldots x_i + h_{i, n-1}) \right| + \frac{\epsilon}{m}
\]

Therefore

\[
\frac{1}{(n-1)!} \left| \frac{F^{(n-1)}(x)}{(n-1)!} - \frac{F^{(n-1)}(x_i)}{(n-1)!} \right| \leq \left| f(x; x_i + h_i, \ldots x_i + h_{i, n-1}) - f(x_i; x_i + h_{i, i}, \ldots x_i + h_{i, n-1}) \right| + \frac{\epsilon}{m}
\]

so that

\[
\frac{1}{(n-1)!} \sum_{i=0}^{m} \left| \frac{F^{(n-1)}(x)}{(n-1)!} - \frac{F^{(n-1)}(x_i)}{(n-1)!} \right| \leq \sum_{i=0}^{m} \left| f(x; x_i + h_i, \ldots x_i + h_{i, n-1}) - f(x_i; x_i + h_{i, i}, \ldots x_i + h_{i, n-1}) \right| + \frac{\epsilon}{m}
\]

\[
\leq \frac{b}{a} \tilde{f} + \epsilon \quad (13)
\]

(13) holds for all possible partitions of \((ab)\). Hence

\[
\frac{b}{a} \tilde{f}^{(n-1)} \leq (n-1)! \left[ \frac{b}{a} f + \epsilon \right] \quad (14)
\]

Similarly,

\[
\frac{b}{a} \tilde{f}^{(n-1)} \leq (n-1)! \left[ \frac{b}{a} \tilde{f} + \epsilon \right] \quad (15)
\]

(14) and (15) give the theorem.
Theorem IV. If \( \frac{b}{a} f \) exists, then \( f^{(n)}(x) \) exists almost everywhere on \((ab)\).

Proof. In order that \( f^{(n)}(x) \) exist, it is necessary and sufficient that

1. \( f'(x_0), \ldots, f^{(n-1)}(x_0) \) exist
2. \( \lim_{h \to 0} \frac{f^{(n)}(x_0+h) - f^{(n)}(x_0)}{h} \) exist

which implies that

\[
f^{(n)}(x_0) = \lim_{h \to 0} \frac{f^{(n)}(x_0+h) - f^{(n)}(x_0)}{h}
\]

where \( \bar{f} \) denotes the separate cases of \( f \) and \( \bar{f} \).

From Theorem I, we know that \( f^{(n-1)}(x) \) exists on \((ab)\) except at a set of points at most denumerable. Hence, from the necessity of condition (1), \( f'(x), \ldots, f^{(n-1)}(x) \) also exist at all of the points of \((ab)\) at which \( f^{(n-1)}(x) \) exists. Thus condition (1) is satisfied at the points where \( f^{(n-1)}(x) \) exists.

Now from Theorem III and the theorem of Lebesgue, we have the result that

\[
\frac{d}{dx} \bar{f}^{(n)}(x) \text{ exists almost everywhere on } (ab)
\]

and

\[
\frac{d}{dx} \bar{f}^{(n)}(x) = \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad theta
Let $x_0$ be any point of $(ab)$ such that $f^{(n-1)}(x_0)$ exists. Then, since $f^{(n-1)}(x)$ exists almost everywhere on $(ab)$, it exists in every neighborhood of $x_0$. Therefore there exists a sequence \{h_n\} which is such that

(a) $\lim_{n} h_n = 0$

(b) $f^{(n)}(x_0+h_n) = F^{(n)}(x_0+h_n)$

(c) $\lim_{n} f^{(n-1)}(x_0+h_n) = f^{(n-1)}(x_0)$

Hence

$$\lim_{h \to 0} \frac{f^{(n)}(x_0+h) - f^{(n)}(x_0)}{h} = \lim_{h \to 0} \frac{f^{(n-1)}(x_0+h) - f^{(n-1)}(x_0)}{h}$$

Thus condition (2) is satisfied almost everywhere on $(ab)$. We have shown (1) to be satisfied. Hence the theorem.

Theorem V. If $\nabla_{x} f$ exists then $\nabla_{x} f$ exists.

Proof. Let $a = x_0 < x_1 < \ldots < x_m = b$ be any partition of $(ab)$.

Then for a continuous function $f(x)$, there exist points of $(ab) \xi_0, \ldots, \xi_{m-n}$ such that

$$x_i < \xi_i \leq x_{i+n-1} (i = 0, \ldots, m-n)$$

and such that

$$f^{(n)}(\xi_i) \leq f(x_i, x_{i+1}, \ldots, x_{i+n-1}) \leq f^{(n)}(\xi_i) \quad (i = 0, \ldots, m-n)$$

(16)

From Theorem III, we see that $F^{(n)}(x)$ and $f^{(n)}(x)$ are bounded on

Hence from (16),
\[ \int \left| f(x_1, x_2, \ldots, x_{m-n}) \right| (x_{i+m-n} - x_i) \leq \left( \frac{x_{i+m-n} - x_i}{(m-n)!} \right) \]
where \( M \) is the larger of the numbers \( \| f^{(m-n)}(x) \| \) and \( \| f^{(m-n)}(x) \| \),
for \( a \leq x \leq b \). Therefore
\[ \sum_{i=0}^{m-n+1} \left| f(x_1, x_2, \ldots, x_{m-n}) \right| (x_{i+m-n} - x_i) \leq (b-a)M \]
This holds for every partition and hence for the least upper bound of such sums over all such partitions. As a consequence,
\[ \int_a^b f = \frac{b-a}{(n-2)!} M \]
which proves the theorem.

5. TOTAL VARIATION OF VECTOR FUNCTIONS.
Corresponding to the definition of total \( n \)th variation of a scalar function \( f(x) \), we define the total \( n \)th variation of a vector function \( \varphi(t) \), denoted by \( \int_a^b \varphi \), by the equation
\[ \int_a^b \varphi = \int_{t_0}^{t_{m-n}} \left| \varphi(t_0, \ldots, t_{i+m-n}) (t_{i+m-n} - t_i) \right| (i=0, \ldots, m-n) \]
where the bound is over all partitions of the interval \((ab)\) such that \( a = t_0 < t_1 < \ldots < t_{m-n} = b \) (\( m = n \)).
Every vector function \( \varphi(t) \) can be expressed in terms of rectilinear components involving scalar functions by means of the equations
\[ \varphi(t) = f(t)\alpha + g(t)\beta + h(t)\gamma \]
where \( \alpha = \beta = \gamma = 1 \) and \( \alpha = \alpha' = \beta' = 0 \) (18)
since \( \varphi(t_0, \ldots, t_{m-n}) \) is a linear function of \( \varphi(t_0), \ldots, \varphi(t_{m-n}) \),
it is evident that

\[ \varphi(t_0 \ldots t_n) = f(t_0 \ldots t_n) a + g(t_0 \ldots t_n) \beta + h(t_0 \ldots t_n) \gamma \quad (19) \]

**Theorem VI.** If \( \varphi(t) \) is related to \( f(t) \), \( g(t) \), and \( h(t) \) as in (18), then a necessary and sufficient condition that \( \varphi \) exist is that \( \varphi \_\_\_\_a \_\_\_\_f \_\_\_\_g \_\_\_\_h \) exist.

Proof. Assume a partition of \((ab) a = t_0 < \ldots < t_n = b\). Then

\[ |\varphi(t_0 \ldots t_{i+n})| = \sqrt{[f(t_0 \ldots t_{i+n}) a + g(t_0 \ldots t_{i+n}) \beta + h(t_0 \ldots t_{i+n}) \gamma]^2} \quad (20) \]

\[ = \sqrt{f(t_0 \ldots t_{i+n})^2 + g(t_0 \ldots t_{i+n})^2 + h(t_0 \ldots t_{i+n})^2} \quad (21) \]

\[ \leq |f(t_0 \ldots t_{i+n})| + |g(t_0 \ldots t_{i+n})| + |h(t_0 \ldots t_{i+n})| \quad (22) \]

\[ (i = 0, \ldots, m-n) \]

From the inequality (22), it follows that

\[ \varphi \_\_\_\_a \_\_\_\_f \_\_\_\_g \_\_\_\_h \quad (23) \]

Now from the equality (21), we have

\[ \varphi(t_0 \ldots t_{i+n}) = \begin{cases} f(t_0 \ldots t_{i+n}) \\ g(t_0 \ldots t_{i+n}) \\ h(t_0 \ldots t_{i+n}) \end{cases} \]

and hence

\[ \varphi \_\_\_\_a \_\_\_\_f \_\_\_\_g \_\_\_\_h \_\_\_\_\_h \quad (24) \]

(23) and (24) give the sufficient and necessary conditions,
respectively, of the theorem.

**Corollary.** If \( \nabla_\alpha^n \varphi \) exists, then \( \varphi^{(n)}(t) \) exists almost everywhere on the interval \( a \leq t \leq b \).

**Proof.** Theorem VI and Theorem IV prove that \( f^{(n)}(t) \), \( g^{(n)}(t) \), and \( h^{(n)}(t) \) exist simultaneously almost everywhere on \( (ab) \). Since \( \varphi^{(n)}(t) = f^{(n)}(t) + g^{(n)}(t) + h^{(n)}(t) \), the corollary is obvious.

**Theorem VII.** If \( \nabla_\alpha \varphi \) exists, then \( \nabla_\alpha \varphi', \ldots, \nabla_\alpha \varphi^{(n-1)} \) also exist and \( \varphi'(t), \ldots, \varphi^{(n)}(t) \) exist simultaneously almost everywhere on \( (ab) \).

**Proof.** From Theorem V and Theorem VI, it follows that \( \nabla_\alpha \varphi, \ldots, \nabla_\alpha \varphi^{(n-1)} \) exist. Hence from the corollary to Theorem VI \( \varphi'(t), \ldots, \varphi^{(n)}(t) \) exist almost everywhere on \( (ab) \), which implies that \( \varphi'(t), \ldots, \varphi^{(n)}(t) \) exist simultaneously on \( (ab) \) almost everywhere.

6. **GEOMETRIC APPLICATION.**

Let us assume that an arc AB is defined by the vector equation \( P-O = \varphi(t) \) for \( a \leq t \leq b \), \( \varphi \) being such that \( \nabla_\alpha \varphi \) exists. Then by Theorem VII, \( \varphi' \) and \( \varphi'' \) exist simultaneously almost everywhere on \( (ab) \). From the same theorem, \( \nabla_\alpha \varphi \) exists and the arc AB is therefore rectifiable. This means that the given arc can be represented by an equation \( P-O = e(s) \), where \( s \) is length of arc. It is obvious that \( e' \) and \( e'' \) exist where \( \varphi' \) and \( \varphi'' \) do, assuming that the derivatives of \( e \) are with
respect to $s$. Now Tonelli\(^1\) has proved that on a rectifiable arc, the limit of the ratio of a chord to its arc as the length of the arc approaches zero is unity almost everywhere on the arc. Hence \(\frac{\mathrm{d}e}{\mathrm{d}s} = 1\) or \(e'^2 = 1\) almost everywhere on the arc $AB$. We have thus proved the following statement:

**Theorem VIII.** If an arc $AB$ is given by the vector equation $P-O = \varphi(t)$ for $a \leq t \leq b$ and if $\frac{b}{a} \varphi$ exists, then the arc $AB$ can be expressed by a vector equation $P-O = e(s)$ for $s_a \leq s \leq s_b$ which is such that $e'$ and $e''$ exist simultaneously almost everywhere and where $e'^2 = 1$ almost everywhere.

**The formula for curvature.** Applying vector notation to the formula from elementary trigonometry for the radius of a circle determined by three points $P_o$, $P_1$, $P_2$, we have

$$r^2 = \frac{(P_1 - P_o)^2(P_2 - P_o)^2(P_2 - P_1)^2}{4|P_1 - P_o, P_2 - P_o|^2} \tag{25}$$

If $P_o$, $P_1$, and $P_2$ are given by $t = t_o$, $t = t_1$, and $t = t_2$, respectively from the equation $P-O = \varphi(t)$, by using the abbreviation $\varphi(t_o) = \varphi_o$, etc., from (25) we may write

$$\frac{1}{r^2} = \frac{4|\varphi_1 - \varphi_o, \varphi_2 - \varphi_o|^2}{(\varphi_1 - \varphi_o)^2(\varphi_2 - \varphi_o)^2(\varphi_2 - \varphi_1)^2} \tag{26}$$

The curvature of the given arc at a point $P_o$ may be defined to be the square root of the limit of (26) as $t$, and $\varphi_2$ simultaneously approach $t_o$.

Now with the aid of (9) of Part One and the definition (2) of Part Two, (26) can be expressed as

A sufficient condition that \( \varphi' = \lim l/r^2 \) exist is that \( \varphi' \) and \( \varphi'' \) exist and \( \varphi' \neq 0 \). In this case (27) gives the usual formula for curvature, i.e. taking the limit of (27) as \( t \to t_0 \) and \( t \to t_0 \):

\[
\rho = \frac{\sqrt{\left| \varphi', \varphi'' \right|^2}}{(\varphi')^\frac{3}{2}}
\]

(28)

Now if \( P-0 = \varphi(t) \) is such that \( \varphi' \) exists, we have from Theorem VIII, the fact that the curvature can be expressed as

\[
\rho = \frac{\sqrt{\left| e', e'' \right|^2}}{(e')^\frac{3}{2}}
\]

and furthermore that \( e' = 1 \). Since \( e'' \) exists, \( e'e'' = 0 \), and by (10) of Part One, \( |e', e''|^2 = e''^2 \). Therefore our formula reduces to

\[\rho = e''^2\]

(29)

almost everywhere on the arc AB.

It is evident that the hypothesis that \( \varphi' \) or \( \varphi'' \) exist, leads to the existence or simplification of many of the formulas of differential geometry, including those of torsion, osculating plane, center of the osculating circle, etc.
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2. Articles.


F. A. Rickey was born in Crowley, Louisiana, in 1903. He spent a large part of his boyhood in North Alabama where his father, Rev. H. W. Rickey served in various pastorates. Returning to Louisiana in 1917, he graduated from the Southwestern Louisiana Institute High School in 1919, the last year of its existence, and received his A.B. degree from Southwestern Louisiana Institute in 1923. He has since studied mathematics in the graduate schools of Tulane University, the University of Chicago, and, during the sessions of 1929-1930, 1933-1934, 1934-1935, and summer of 1930, at Louisiana State University. A graduate minor in physics has been added at the latter institution. In addition to nine years experience as a high school teacher and principal, he held the rank of instructor in the Department of Mathematics at L. S. U. during the 1929-1930 session and that of graduate assistant in mathematics during the latter two years of those above mentioned. His wife, formerly Miss Mary Ellen Patterson, is a graduate of L. S. U., Class of '25. A daughter, Mary Ellen, was born in 1929.