Sociomathematical Norms of Elementary School Classrooms: Crossnational Perspectives on the *Reform of Mathematics Teaching.

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SOCIOMATHEMATICAL NORMS OF ELEMENTARY SCHOOL CLASSROOMS: CROSSNATIONAL PERSPECTIVES ON THE REFORM OF MATHEMATICS TEACHING

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

The Department of Curriculum and Instruction

by

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May, 2000
DEDICATION

To My Parents

I dedicate this dissertation to my parents, GyunHwan Pang and SunHee Hwang, whose sacrifice and vision were a consistent inspiration. I was blessed by the prayer and pride of my parents who were most eager for me to accomplish this work.
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While I had been engaged in researching and writing this dissertation, I realized how blessed I am owing to many people who have consistently supported me in different ways. I wish to express my sincere thanks to all of them and share the joy of this accomplishment with each of them.

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I close by praising my Lord and Savior, Jesus Christ, who has walked with me all the way. I rely on His strength and guidance.
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ABSTRACT

Mathematics education reform in the United States has marshaled large-scale support for instructional innovation, and enlisted the participation and allegiance of large numbers of mathematics teachers. However, there is concern that many teachers have not grasped the full implications of the reform ideals. This study explored the breakdown that may occur between teachers' adoption of reform objectives and their successful incorporation of reform ideals by comparing and contrasting two reform-oriented classrooms.

This study was an exploratory, qualitative, comparative case study using constant comparative analysis. Seven mathematics lessons were video-taped from each class, and intensive interviews conducted with the two teachers. The study provided a detailed description to explore how the participants in each class established a reform-oriented mathematics microculture. Then the two classes were compared and contrasted in terms of their general social norms and sociomathematical norms (Cobb & Yackel, 1996).

The two classes established similar social participation patterns but very different mathematical microcultures. In both classes open-ended questioning, collaborative group work, and students' own problem solving constituted the primary modes of classroom participation. However, in one class mathematical significance was constituted as using standard algorithm with accuracy, whereas the other class established a focus on providing reasonable and convincing arguments. Given these different mathematical foci, students' learning opportunities were seen as unequal. The
students in the latter class had more opportunities to develop conceptual understanding than their counterparts.

This study was nested within a cross-national, collaborative project involving two reform-oriented Korean classrooms. This research report includes a brief joint analysis. As in the U.S., the two Korean classes were similar in their general social norms, but only one class reified mathematically significant distinctions among students' contributions. However, the more successful Korean classroom was very different in character from its U.S. counterpart, with the former focusing on sophisticated conceptual distinctions whereas the latter focused on the social values of students' active participation. A retheorization of sociomathematical norms is offered so as to highlight the importance of this construct in the analysis of reform-oriented classrooms, and to promote a more diverse conceptualization of the possibilities for viable mathematics teaching.
CHAPTER 1

INTRODUCTION

MATHEMATICS EDUCATION REFORM

The call for reform in mathematics education is a reaction to the ample and increasing evidence that mathematics education is not adequately promoting students’ mathematical development. For instance, despite a renewed emphasis on problem solving since 1980, students’ problem-solving performance remained low (National Assessment of Educational Progress, 1992). Students learned even basic mathematical concepts and procedures only at a superficial level. The problems that have motivated the current reform movement include learning without understanding of mathematical ideas (Hiebert & Carpenter, 1992; Hiebert, Carpenter, Fennema, Fuson, Human, Murray et al., 1997; Rosnick & Clement, 1980), increasingly negative mathematical disposition as students advance through school (Brown, Carpenter, Kouba, Lindquist, Silver, & Swafford, 1988; Renga & Dalla, 1993), lack of creative mathematical thinking (Lappan, & Schram, 1989; Peterson, 1988), lack of self-esteem in terms of mathematical ability (McLeod, 1994), lack of problem-solving ability (Charles & Silver, 1988; Hiebert, Carpenter, Fennema, Fuson, Human, Murray et al., 1996), and conceptions of mathematics as a set of rules (Lindquist, 1989; Swafford & Brown, 1990).

The Third International Mathematics and Science Study (TIMSS) provides multinational perspectives on the problems and issues encountered in the United States (National Center for Education Statistics [NCES], 1996, 1997; Schmidt, McKnight, & Raizen, 1997; Schmidt, McKnight, Valverde, Houang, & Wiley, 1997). These include (a) fragmented curricula covering far more topics in a year than is typical internationally; (b)
low level curricular expectations of what should be the basics in mathematics, in particular for the middle school years; (c) inclusive but unfocused textbooks that cover many topics but do so in a comparatively shallow manner; (d) teaching methods focusing on skills rather than thinking and understanding; (e) a decentralized educational system in which agencies and organizations do not always work towards common goals or combined results; and (f) an American ideology of mass production and mass education.

Many of these factors are rooted in legal and cultural aspects of U.S. society, and hence present no immediate prospects for change. However, teaching methods have been critiqued in the U.S., specifically in comparison with Japanese teaching practices (Stigler, Fernandez, & Yoshida, 1996; TIMSS, 1996), and broad-scale efforts have been launched to influence the ways mathematics is taught.

**TEACHER-CENTERED: TYPICAL TEACHING PRACTICES**

The teaching methods perceived as modal in the U.S. are teacher-centered wherein teachers deliver a pre-given mathematical curriculum mainly through explanation and demonstration, asking students to practice the methods (Mullis, Martin, Beaton, Gonzalez, Kelly, & Smith, 1997; TIMSS, 1996). The term *teacher-centered* refers to a teacher's explanations and ideas constituting the focus of classroom mathematical practice. Despite the call for emphasis on problem-solving in mathematics education since the early 1980s (National Council of Teachers of Mathematics [NCTM], 1980), there has been little indication of instructional changes in traditional teacher-centered practices (Cobb, Wood, Yackel, & McNeal, 1992; NCTM, 1989; Stigler, Fernandez, & Yoshida, 1996; TIMSS, 1996).
In a typical U.S. mathematics class, most of the time is devoted to the teacher’s lecture or demonstration and then to students’ individual seatwork. The teacher begins with a brief review of short-answer questions or homework problems. The teacher then demonstrates or explains how to solve the next category of mathematics problems. The teacher’s foremost concern is to display a standard method in a clear and definite way rather than to encourage students to express their own thinking as a means to gaining a conceptual understanding of mathematical principles and processes. During the remainder of the class, students practice the demonstrated method on similar problems while the teacher moves around the room answering individual questions. Students are rarely actively involved in posing problems, offering alternative solution methods, or debating mathematical ideas. Teaching is predominantly telling and showing. Thus, the teaching practice can be described as the delivery of information from a knowledgeable teacher to uninformed students.

This teacher-centered method has its roots in related views of what is mathematics, how the teacher should teach mathematics, and how students can best learn it (Ball, 1988a; Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992; Romberg & Kaput, 1999; Smith, 1996; Thompson, 1984, 1992). Teachers often regard mathematics as a fixed and static body of facts and procedures, mainly for symbol manipulations. School mathematics is arranged in textbooks as specific curricular contents which students are to master to a set criterion. The teacher’s major role is to provide a direct and clear demonstration of particular solution procedures to given mathematics problems, and then to assist students to acquire and consolidate problem-solving skills by giving them a chance to practice, administering periodic tests to check their
competence, and repeating the demonstration or step-by-step instruction whenever needed. In fact, the teacher focuses more on whether students are able to perform (standard algorithms) than on whether they understand (mathematical principles and processes). Students' central role then is to pay their full attention to the teacher's demonstration and explanation, to memorize facts, and to practice routine procedures with many problems until they master them. This interlocking set of conceptions of mathematics and its teaching and learning has become so entrenched as to be the school mathematics tradition (Cobb, Wood, Yackel, & McNeal, 1992; Richards, 1991).

STUDENT-CENTERED: RECOMMENDED TEACHING PRACTICES

Against the common instructional practice described above, educational leaders in the U.S. are seeking to change teacher-centered pedagogy to a student-centered approach. The National Council of Teachers of Mathematics has initiated and propelled a reform movement by the publication of Standards on curriculum, teaching, and assessment (NCTM, 1989, 1991, 1995, 2000). This reform requires substantial changes in the teaching and learning of mathematics. In particular, five major shifts from the current practice have been recommended: (a) toward classrooms as mathematical communities, away from classrooms as simply a collection of individuals; (b) toward logic and mathematical evidence as verification, away from the teacher as the sole authority for right answers; (c) toward mathematical reasoning, away from merely memorizing procedures; (d) toward conjecturing, inventing and problem solving, away from an emphasis on mechanistic answer finding; and (e) toward connecting mathematics, its ideas and its applications, away from treating mathematics as a body of isolated concepts and procedures (NCTM, 1991, p. 3).
The term student-centered refers to students’ contributions and responses constituting the center of mathematics activity. Instead of listening and following a teacher's instruction, the students in a student-centered classroom are expected to have the opportunity to be enculturated into a mathematical discourse in which they invent, explain, and justify their own mathematical ideas and critique others’ ideas. The teacher in a reform classroom is expected to provide worthwhile mathematical tasks on the basis of knowledge of mathematics and students’ understandings in order to engage students’ interests and intellect. The teacher also manages classroom discourse in ways that probe various mathematical ideas and deepen students’ conceptual understanding. In order to promote such discourse, the teacher should be sensitive to students’ engagement in discussions. The teacher specifically needs to listen carefully to their ideas, to ask for clarification or justification, and to decide which responses need to be deeply explored in discussions. Creating a learning environment with mathematically rich tasks and discourse supports the new curricular emphasis of mathematics as problem solving, reasoning and proof, communication, connections, and representation (NCTM, 1989, 2000). The reform-oriented teacher also is expected to continually analyze the effects of the learning environment on students’ knowledge, skills, and disposition. In these respects, the teacher's role in a reform mathematics classroom is to implement new social norms that foster all students’ mathematical learning.

Much of this mathematics education reform movement reflects new cognitive perspectives through which mathematics educators attempt to explicate the nature of mathematics learning. The current reform recommendations are generally geared toward a combination of learning as students’ construction and their mathematical enculturation.
(Davis, 1992; Silver, 1990; Steffe, Nesher, Cobb, Goldin, & Greer, 1996), a substantive
shift from learning as receiving rules.

CHALLENGES OF REFORM

The reform movement has been successful in marshaling large-scale support for
instructional innovation and in enlisting the participation, cooperation, and allegiance of
large numbers of mathematics teachers (Knapp, 1997). Forty-six states developed their
own standards that were aligned with the Standards (Council of Chief State Shool
Officers, 1997). Many texts and programs advocate implementing a “standards-based”
approach. In the TIMSS report, 95% of the eighth grade teachers in the study said that
they are aware of the current reform ideas on mathematics teaching (NCES, 1996).
Moreover, when asked to evaluate their videotaped lessons in terms of current reform
ideas, 70% of the teachers rated the lessons as reasonably in accord with the reform
(Stigler & Hiebert, 1998).

In contrast to the widespread awareness of the standards and the teachers’ self-
evaluation of their teaching practice, there has been a growing concern that many U.S.
teachers do not quite grasp the vision of the current reform ideas (Hiebert et al., 1996;
NCES, 1996, 1997; Research Advisory Committee [RAC], 1997). Teachers often
interpret standards-based reform as a new list of teaching strategies and materials
(Burrill, 1997; Knapp, 1997; Stigler & Hiebert, 1998), rather than regard the reform as a
way to re-conceptualize their understanding of mathematics and its teaching. When
asked to justify why they think that the videotaped lessons in the TIMSS are consistent
with the reform ideals, the majority of U.S. teachers pointed to “surface features, such as
the use of real-world problems, manipulatives, or cooperative learning, rather than to the
deeper characteristics of instruction such as the depth of understanding developed by their students" (Stigler & Hiebert, 1998, p. 45). Similarly, Burrill (1997) identified widespread misinterpretation about standards-based teaching: (a) the teacher is just a facilitator or “guide on the side”, (b) students should never practice, (c) all work should be done in cooperative groups, (d) manipulatives are the basis for all learning, (e) students should write an explanation for every problem they solve, and (f) mathematics should be graded to make students “feel good” (pp. 337-338). While the reform documents have emphasized problem solving as essentially doing mathematics in ways that problem-solving approaches are used for students to construct, investigate, and learn mathematics, the interpretation of it too often is limited primarily to problem-solving strategies and heuristics to be mastered, omitting the development of students’ understanding of mathematics (Bybee, Ferrini-Mundy, & Loucks-Horsley, 1997).

Other studies have shown how difficult it is to see basic instructional changes in mathematics classrooms, even with teachers who are committed to implementing reform recommendations (Carpenter, Franke, & Levi, 1998; Cohen, 1990; Peterson, 1994; Schifter & Fosnot, 1993; Stein, Grover, & Henningsen, 1996). Teachers tended to wait to be told the “right” way to teach mathematics and were eager to change their old teaching strategies in order to implement new ones that have been advocated in the current reform era. However, they didn’t think that they have to fundamentally rethink their views about mathematics and how students should learn mathematics. In other words, they have not reconceived their new teaching processes with respect to students’ learning processes.
Just focusing on new teaching strategies is not sufficient to implement reform ideas. Providing manipulative materials or organizing a classroom into small groups does not guarantee that students are engaging in creative and reflective mathematical activities (Good, Clark, & Clark, 1997; Steffe & Kieren, 1994). Whereas the current reform supports small-group instruction and cooperative learning, there is still a question as to whether small-group instruction promotes the development of students’ abilities and understandings better than whole-group or individualized format (King, 1993; Mulryan, 1995). The real issue is then to understand not the form but the quality of an instructional method — what kinds of mathematical and social exchanges occur and in what ways such exchanges promote students’ understanding of mathematics?

Changing social norms in reform classes is intended to promote the development of “mathematical power” for all students, articulated as (a) learning to value mathematics, (b) becoming confident in one’s own ability, (c) becoming a mathematical problem solver, (d) learning to communicate mathematically, and (e) learning to reason mathematically (NCTM, 1989, 2000). The challenge for teachers is to provide the opportunity for students to experience those elements of mathematical power by being engaged in the classroom mathematical activities. Teachers are expected to use the social structure of the classrooms to nurture students’ development toward mathematical ways of thinking as well as their understanding of specific mathematical concepts and processes. This coordination requires new ways of thinking about the teaching/learning dynamic. Reconceptualizing teaching and learning can pose great difficulty for teachers whose previous experience has been in implementing traditional teacher-centered instruction — even if the teachers are eager and willing to teach differently (Ball, 1993;
Fennema & Nelson, 1997; Schifter & Fosnot, 1993). But these challenges must be met by teachers and teacher educators if the reform intentions are ever to be realized.

THEORETICAL BACKGROUND

It is usually agreed that students' mathematical understandings are influenced by classroom social practices which structure opportunities for learning. But there is a frequent debate about whether learning is a product of students' reorganization of their knowledge structures or a product of social practices (Cobb, 1994; Cobb & Bowers, 1999; Confrey, 1991; Hatano, 1993; Sfard, 1998; Steffe, Nesher, Cobb, Goldin, & Greer, 1996). Taking a pragmatic approach, Cobb and his colleagues have developed an "emergent" theoretical framework that fits well with the reform agenda for instruction (Cobb & Bauersfeld, 1995; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Cobb & Yackel, 1996). They take the theoretical concepts developed from the two different perspectives on learning (i.e., the constructivist and the sociocultural perspective) as tools by which they can focus on different topics in understanding students' mathematical learning in a classroom.

Within the emergent perspective, mathematics learning cannot be fully understood intrapersonally because of its social aspects. As well, analysis in terms of only interpersonal constructs is seen to be inadequate, since it is the learner who must understand mathematical meanings. Building on symbolic interactionism and ethnomethodology (Blumer, 1969; Leiter, 1980; Mehan & Wood, 1975; Voigt, 1994), Cobb and Bauersfeld (1995) describe a reflexive relationship between individual students' thinking and classroom interactions, discourse, and the classroom culture. In this perspective, mathematical meanings are neither decided by the teacher in advance

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nor discovered by students; rather they 
emerge in a continuous process of negotiation
through social interaction. The emergent theory provides a general guide to the
interpretation of classrooms observations in this study.

SOCIAL AND SOCIOMATHEMATICAL NORMS

In investigating students' mathematical learning within the emergent perspective,
Cobb and Yackel (1996) address sociomathematical norms as “the normative aspects of
whole-class discussions that are specific to students’ mathematical activity” (p. 178).
They differentiate general social norms as applicable to any subject matter area from
sociomathematical norms which are unique to mathematics. Sociomathematical norms
are social norms of mathematical explanation and justification.

General social norms are the characteristics that constitute the classroom
participation structure. They include expectations, obligations, and roles adapted by
classroom participants as well as gross patterns of classroom activity (Cobb &
Bauersfeld, 1995; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997). For
instance, the general social norms in a student-centered classroom include the
expectation that students invent, present, and justify their own solution methods and the
role that the teacher listens carefully to students’ contributions and comments on or
redescribes them for further discussion. More specifically, the general social norms
might include the expectation that students provide some ideas that are different from
those that have been offered before, or that are sophisticated, or efficient.

Sociomathematical norms are the more fine-grained aspects of these general
social norms that relate specifically to mathematical practices (Yackel & Cobb, 1996).
For instance, the sense of what constitutes mathematical difference, mathematical
sophistication, or mathematical efficiency within the classroom microculture are sociomathematical norms. Similarly, the understanding of what makes an explanation mathematically acceptable or justifiable is a sociomathematical norm. The sociomathematical norms in a student-centered classroom may include the expectation that students are to present their solution methods by describing actions on mathematical objects rather than simply accounting for calculational manipulations. The sociomathematical norms of difference, sophistication, and efficiency are concerned with relational mathematical values, requiring comparisons and contrasts among multiple solution methods. The sociomathematical norms of acceptability and justification are related to the actual process of making a mathematical contribution to the classroom conversation.

Within the emergent perspective, social norms, including sociomathematical norms, are interactively constituted by social interaction between the teacher and the students (Cobb & Bauersfeld, 1995; Cobb, Gravemeijer et al., 1997; Yackel & Cobb, 1996). The development of sociomathematical norms is based not only on the teacher’s guidance but also on students’ contributions in terms of their explanations, justifications, and argumentations of solution methods. Thus, sociomathematical norms are neither prescriptions prepared by the teacher nor the students own spontaneous accomplishment. Rather, the norms are seen as continually negotiated and redefined by the participation of the teacher and the students in discussions.

While participating in establishing sociomathematical norms, students can develop the capability to make mathematical judgements and, more generally, acquire mathematical beliefs and values which ultimately lead them to become intellectually
autonomous in mathematics (Rasmussen & Yackel, 1999; Yackel & Cobb, 1996). The evolution of sociomathematical norms also provides the teacher with an opportunity to have more direct influence on students’ mathematical development than is true of the teacher who merely attends to surface aspects of the social structure. The teacher can take a proactive role as a representative of the mathematical community in initiating and guiding the establishment of sociomathematical norms (McClain, 1995; McClain & Cobb, 1997). Considering that the teaching recommendations of the current reform era identify more clearly what the teacher is not supposed to do rather than what to do (Smith, 1996), the teacher’s active role in co-constructing sociomathematical norms needs to be closely examined with regard to how the teacher in a reform class pursues his or her pedagogical goals based on students’ participation and contributions.

**UTILITY OF SOCIOMATHEMATICAL NORMS**

The construct of sociomathematical norms is intended to capture the essence of the mathematical microculture established in a classroom community rather than its general social structure (Yackel & Cobb, 1996). The differentiation of sociomathematical norms from general social norms is of great significance because interest is given to the ways of explicating and acting in mathematical practices that are embedded in classroom social structure.

The construct of sociomathematical norms evolved out of a classroom teaching experiment in which Cobb and his colleagues attempted to account for students’ conceptual learning as it occurred in the social context of an inquiry mathematics classroom wherein the teacher and the students together constituted a mathematical community and negotiated mathematical meanings (Cobb & Bauersfeld, 1995). The
researchers designed instructional devices and sequences of specific mathematical content and extensively supported the classroom teacher to foster students’ mathematical learning using those sequences. Cobb and his colleagues analyzed how sociomathematical norms became constituted and stabilized in those project classrooms (e.g., Bowers, Cobb, & McClain, 1999; Cobb et al., 1997; McClain & Cobb, 1997; Stephan, 1998). The frequent examples of sociomathematical norms included what counts as a different, clear, efficient, easy, or acceptable mathematical explanation.

Cobb and his colleagues used another theoretical construct, classroom mathematical practices, together with social norms and sociomathematical norms in order to account for students’ mathematical development from a sociological perspective (Cobb & Yackel, 1996). Whereas sociomathematical norms focus on the evolving criteria for mathematical discussion, classroom mathematical practices are concerned with “the collective mathematical learning of the classroom community” that involves taken-as-shared ways of interpreting, reasoning, symbolizing, or mathematizing in specific task situations (Bowers et al., 1999, p. 26). Sociomathematical norms encompass general criteria for assessing students’ contributions regardless of mathematical content, while classroom mathematical practices are concerned with particular mathematical ideas (Cobb, 1999).

The analysis of the evolution of classroom mathematical practices was of great importance for Cobb and his colleagues because of their interest in (a) explaining students’ mathematical understanding as embedded in their participation in communal classroom processes, and (b) gaining feedback to develop extensive sequences of instructional activities. As a consequence, previous studies tend to briefly document...
sociomathematical norms (and also social norms) mainly as a precursor to the detailed analysis of students' conceptual learning established in the classroom community (e.g., Bowers et al., 1999; Cobb et al., 1997; Stephan, 1998). Thus, the limited use of the sociomathematical norms construct seems to come out of Cobb and his colleagues' developmental research with the specific purpose of developing instructional devices and explaining students' mathematical learning in such a learning environment.

This dissertation explores the possibility of positioning the sociomathematical norms construct as more centrally reflecting the quality of students' mathematical engagement in collective classroom processes. Given the challenges of implementing reform ideals, the sociomathematical norms construct can be critical in understanding whether or not reform-oriented teachers use classroom social structure effectively to develop students' mathematically significant beliefs and values and to enhance their conceptual understanding of mathematics. This study attempts to promote sociomathematical norms as a key theoretical construct in understanding what has been problems in implementing reform and what should be done to solve the problems. Within this dissertation project, I pursue the possibility that the breakdown between teachers' adoption of reform objectives, and their successful incorporation of reform ideals implicates the sociomathematical norms that become established in their classrooms.

**STUDENTS' LEARNING OPPORTUNITIES**

Students' learning opportunity is a fundamental criterion to measure the success of classroom teaching (Cobb & Whitenack, 1996). The principal concern of reform has to do with the extent to which changes in teaching practice translate into changes in the learning opportunities that students will encounter in their mathematics classes. In this
respect, this dissertation analyzes students’ learning opportunities arising from reform-oriented mathematics classrooms.

In coordinating social with psychological perspectives, Cobb and his colleagues see the psychological correlates of sociomathematical norms as consisting of the teacher's and students' mathematical beliefs and values (Cobb & Yackel, 1996). The relationship between sociomathematical norms and personal beliefs and values is considered reflexive. On one hand, individuals contribute to the evolution of sociomathematical norms as they reorganize specifically mathematical beliefs and values. On the other hand, these evolving norms both constrain and enable the ways in which the individuals reorganize these beliefs and values. Consequently, within the coordinated perspective, the teacher is understood as supporting students' mathematical beliefs and values and more generally mathematical disposition, when she or her initiates and guides the negotiation of sociomathematical norms.

Within the emergent perspective, however, the relationship between sociomathematical norms and students' conceptual learning has not been directly addressed. This may come from the specific emphasis on classroom mathematical practices that are considered to be reflexively related to individual students' conceptions and activities (Cobb & Yackel, 1996). This dissertation positions sociomathematical norms as central not only to explaining the quality of students' mathematical engagement but to anticipating the possibility of students' conceptual learning. This is not to say that the analysis of classroom mathematical practices is not important. Moreover, I do not dispute the reflexive relationship between mathematical practices of the classroom community and students' conceptions or interpretations of specific mathematical content.
What I stress is that the analysis of mathematical practices in one classroom over extensive periods of time, which was pragmatically significant for Cobb and his colleagues as instructional designers, is not the optimal tool for the purpose of exploring the problems and issues of implementing reform ideals across classrooms, regardless of the particular mathematical topics or ideas discussed in each lesson.

DISSERTATION PROJECT

This dissertation examines the general social norms, sociomathematical norms, and students’ learning opportunities of two U.S. classrooms that are attempting to implement student-centered instructional methods in mathematics. The classes were selected because of their unequal success in implementing student-centered teaching methods. This is a significant departure from previous research trends on reform where one reform-oriented classroom is extensively studied (e.g., Ball, 1993; Cobb & Bauersfeld, 1995; Cobb et al., 1997; McClain, 1995). Moreover, such reform-minded classrooms tend to be supported by researchers who assist the classroom teachers to transform reform ideals into actual teaching practices. Comparing and contrasting more successful and less successful reform classes, without outsiders’ collaboration, can provide a unique opportunity to reflect on the subtle but important problems and issues of implementing educational reform in the U.S.

Within each classroom, teachers’ approaches are analyzed with regard to their motivation to establish a student-centered classroom microculture. Students’ approaches for participation in such mathematics classrooms are also analyzed. Rather than reducing the complexity of classroom life, this study attempts to provide “thick description” and
deep analyses of the teaching practices within the target classrooms (Geertz, 1973). The following questions define the purposes of this dissertation project:

1. What are the processes that constitute more successful and less successful student-centered pedagogy in the U.S. elementary mathematics classroom? Specifically, in what ways do the teacher and students create such mathematics classrooms? What learning opportunities arise for the students in these classrooms?

2. What are the differences and similarities between more successful and less successful student-centered classrooms, in particular with regard to social norms and sociomathematical norms? What are the challenges for reformers in changing the culture of primary level mathematics teaching?

This dissertation is nested within a larger, crossnational, collaborative project between Korean and the U.S. researchers, including the author of this study, to understand the classroom dynamics of reform-oriented instruction in both countries, and to explore shared challenges in changing the culture of mathematics teaching at the elementary school level. The Korean portion of study was conducted in 1997 and 1998 by a team of four researchers. Full participation in the Korean study encouraged the author of this dissertation to sharpen the distinction between more successful and less successful student-centered classrooms. Moreover, the Korean study raised the possibility of employing the sociomathematical norms construct as central to understanding the qualitatively different mathematical cultures in the target classrooms. Building on the Korean portion of the study, this dissertation analyzes primarily U.S. elementary mathematics classrooms, and then uses the U.S. and Korean data in order to
articulate the problems and issues of reform more broadly. The sociomathematical norms construct is retheorized with regard to the issues of reform. A future systematic comparison of mathematics education reform in Korea and the U.S. will benefit from a redefined theoretical framework on sociomathematical norms offered in the final chapter.

The next chapter includes Korean mathematics education reform and actual classroom episodes from the Korean portion of the study. Chapter 3 reviews the developmental history and utility of sociomathematical norms. Chapter 4 presents research methodology used for this study and Chapter 5 provides detailed analyses of two unequally successful student-centered mathematics classrooms in the U.S. The analyses highlight similar social participation structure but very different mathematical microcultures, which lead unequal learning opportunities on the part of students. Finally, Chapter 6 presents a general overview of this study with concise descriptions of two Korean and two U.S. mathematics teaching practices. A brief joint analysis highlights fundamental differences in the orientation of the more successful Korean classroom and its U.S. counterpart. A retheorization of sociomathematical norms is offered so as to highlight the importance of this construct, and to promote a more diverse conceptualization of the possibilities for viable mathematics teaching.
CHAPTER 2

MATHEMATICS EDUCATION REFORM IN THE UNITED STATES AND SOUTH KOREA

This chapter first reviews the mathematics education reform movements in the U.S. Given the typical and recommended teaching methods described in the previous chapter, this review focuses on theoretical and historical factors that have motivated the current reform movement and reconsiders the challenges of implementing reform ideals. This review serves to provide background information to understand mathematics teaching practices by the two teachers in this study who aspired to establish a student-centered approach.

The next section describes the current mathematics education reform in South Korea. The description highlights the similar aspects of reform between Korea and the U.S. in terms of prevalent teaching practices and recommended methods. The description also serves to provide background information for understanding reform-minded Korean mathematics teaching practices in two elementary school classrooms, which are presented in the third section. As described before, the introduction of Korean mathematics education reform and classroom examples has a special reason. This dissertation study is nested in a larger, international collaborative project between Korea and the U.S. The prior analysis of similarities and differences between the two Korean reform classrooms shed light on the importance of sociomathematical norms as an analytic construct to understand the quality of mathematics instruction, which is explored in this dissertation project with a broader data set, including the two U.S. classroom examples.
MATHEMATICS EDUCATION REFORM IN THE UNITED STATES

As described in the previous chapter, U.S. mathematics educators have sought to change typical teacher-centered teaching methods into student-centered instruction in which the teacher is envisioned to establish a mathematical community on the basis of students' contributions and mathematical ideas. The reform-oriented teacher is expected to manage classroom discourse in ways to provide students with an opportunity to acquire mathematical values and beliefs, and to deepen their conceptual understanding. This section reviews factors that have motivated this reform and then reviews problems that have emerged in transforming reform ideals into actual classroom teaching practices.

MOTIVATING FACTORS FOR THE REFORM

The current mathematics education reform movement has been motivated by diverse factors including the depressing outcomes of typical mathematical instruction described in the previous chapter, and the increasing research knowledge that has offered different perspectives of mathematics learning and teaching. The call for reform also is situated in the historical contexts in which various attempts for better mathematical instruction and their concomitant results have been interwoven. The following is a review of the latter two factors, that is to say, theoretical influences and historical contexts.

Theoretical Influences

Accumulating research on students' learning of mathematics has provided important foundations for the reform movement, because such studies lead to reflection on the nature of mathematical learning and teaching that has been typically assumed. Particular research paradigms and traditions in cognitive theory have shaped mathematics
education in many different ways. The following review of influences of cognitive theories is an attempt to explore the underpinnings that implicitly motivate the current reform movement.

Much of the current mathematics education reform movement reflects new cognitive perspectives by which mathematics educators attempt to explicate the nature of mathematics learning. The reform recommendations are generally geared at a combination of learning as students' construction and their mathematical enculturation (Cobb, 1994; Davis, 1992; Silver, 1990; Steffe, Nesher, Cobb, Goldin, & Greer, 1996). Whereas the previous reform documents (NCTM, 1989, 1991, 1995) contained little explicit discussion of the theoretical perspectives they reflect, the forthcoming Principles and Standards for School Mathematics (NCTM, 2000) acknowledges the influence of these two perspectives of learning mathematics. On one hand, the reform emphasizes the importance of building on individual students' prior knowledge and making connections for their conceptual organization. On the other hand, the reform stresses the processes by which students become active members of a mathematical community in their classroom. The reformers contends that "These two perspectives are clearly interactive, as the learning community is made up of individuals, and individual knowledge is substantially shaped by the interactions that take place within the mathematics classroom" (NCTM, 1998, p. 34). This view of learning mathematics is a substantive extension from learning as information-process based on production systems (e.g., Resnick & Ford, 1981), which had dramatically changed the view of learning as reinforcement between stimuli and responses in behaviorism (e.g. Gagné, 1962).
Constructivist Perspectives

A constructivist approach evolving from Piaget's genetic epistemology assumes that children construct their own knowledge through reflection on their actions in the world, through assimilation and accommodation (von Glasersfeld, 1984, 1995). The constructivist view of mathematics learning has been influential since the early of 1980s: Mathematical learning consists of students' own construction of mathematical concepts and procedures (e.g., Kamii, 1990; Steffe & Blake, 1983; von Glasersfeld, 1991; Wachsmuth, 1983). From the constructivist point of view, students do not simply add new information to their own cognitive structures that have been established. Instead, they connect or construct new relationships among the interpretive structures. Thus a constructivist teacher is very concerned about the possibility that an individual's knowledge structures may either be isolated from each other, rather than integrated together. This perspective has helped to overturn the view of mathematical learning as passive reception and mathematical teaching as the transmission of teachers' knowledge.

Although there have been several versions of constructivism (Confrey, 1995; Ernest, 1996; O'Connor, 1998; Prawat, 1996), the discussion here attempts to capture its fundamental aspects. The constructivist perspective assumes that learning occurs through cognitive conflicts by which the individual's mental structure evolves into more viable structure (von Glasersfeld, 1995). Thus, the main concern of constructivist teaching in mathematics education is to help students enhance their cognitive structures with respect to specific mathematical content (Cobb & Steffe, 1983). Social interaction contributes to this to the extent it raises cognitive conflict and perturbation leading to cognitive reorganization in the process of individual' sense making (Steffe & Kieren, 1994).
Consequently, the crucial role of a constructivist teacher is to provide a learning environment wherein students can confront the limitations of their current understanding of a specific mathematical concept, which in turn leads to conceptual changes (Confrey, 1990a; von Glasersfeld, 1995). For this reason, it is important for a teacher to conjecture about a student’s previous construction of a mathematical topic and to develop extremely detailed teaching strategies in order to modify the student’s thinking (Cobb & Steffe, 1983; Simon, 1995). The teacher continually re-assesses his or her conceptual portrait of the student and the corresponding teaching model based on the effectiveness of the interactions with the student.

Constructivist perspectives inform the recommended teaching practice, student-centered pedagogy, of the current reform in mathematics education. One way is that constructivist-based research provides models of students’ conceptual understandings that can inform teachers’ attempts to create cognitive conflict for students resulting in the evolution of more mathematically powerful knowledge structures. Another way is that constructivist perspectives strongly portray students as active learners and thus encourage a teacher to probe carefully their meaning-making.

Sociocultural Perspectives

Although some mathematics education researchers espoused Vygotsky’s zone of proximal development as a useful theoretical and pedagogical construct in the early 1980s (e.g. Carpenter, 1980; Fuson, 1980), the influence of sociocultural perspectives on mathematics education is relatively recent (Schmittau & Taylor, 1993). The influence has been propelled by anthropological studies which have explored the relations between cultural activities and cognitive development, specifically the comparisons of children’s
mathematical thinking in and out of school culture (Carraher, Carraher, & Schliemann, 1985; Lave, 1988; Minick, 1993; Saxe, 1991; Schliemann & Carraher, 1996). Such studies have often demonstrated that school mathematical knowledge is noticeably inaccessible in out-of-school settings, suggesting that individuals’ arithmetical activities are profoundly influenced by their participation in encompassing cultural practices. These studies urged mathematics educators to broaden their foci of attention so that they become sensitive to incorporating cultural and social dimensions in instruction.

Sociocultural perspectives, inspired by Vygotsky’s work, claim that individuals’ cognitive processes are subsumed by social and cultural processes, locating learning not in the individual’s mind but in the participation of social, cultural, and historical practices (Cobb, 1994; Forman, 1996). This claim reflects a move away from explaining cognition as an individual mental process to understanding the interpersonal context of cognitive growth (Forman, Minick, & Stone, 1993). Sociocultural perspectives conceptualize learning as a situated process, which arises from engagement in socioculturally shared endeavors through the zone of proximal development (Vygotsky, 1978), activity system (Engeström, 1987), cognitive apprenticeship (Collins, Brown, & Newman, 1989), construction zone (Newman, Griffin, & Cole, 1989), or legitimate peripheral participation (Lave & Wenger, 1991). Learning is characterized as mutual appropriation by which the teacher (or master) and the students (or apprentices) continually coopt each other’s contributions until the students are engaged in expected practices (Cobb, 1994; Leont’ev, 1981). In short, learning in the sociocultural perspectives is seen as a process of enculturation into a community of practice.
For learning, sociocultural perspectives are concerned with access to the authentic practice in a community in contrast with the availability of rich instructional resources promoted by constructivism (Forman, 1996). In this respect, mathematical activities in a classroom should reflect what mathematicians do. The teacher, serving as a representative of a mathematical community, organizes classroom activity settings in such a way that students experience the authentic nature of mathematical activities including mathematical ways of knowing, communicating, valuing, justifying, agreeing, arguing, etc (Collins, Brown, & Newman, 1989; Crawford, 1996; Lampert, 1990; Lampert, Rittenhouse, & Crumbaugh, 1996). The main concern of the teacher is whether or not his or her students' classroom practices progress toward those of a socioculturally established mathematics community.

Like constructivist perspectives, sociocultural perspectives inform the recommended teaching practices in the current reform era, but from different points of view. Whereas constructivist perspectives account for students' conceptual development, sociocultural perspectives illuminate the nature and effects of their participation in socially situated activities. Sociocultural perspectives strongly support for, among others, the establishment of a classroom as a mathematical community in which students are engaged in specifically mathematical ways of thinking and interacting (NCTM, 2000).

**Historical Contexts**

For a century, reform efforts in mathematics education have called for significant changes in the ways of teaching and learning of mathematics but with varying emphases: making mathematics more concrete and connecting it with science in the early 1900's; practicing precisely defined mathematical skills in the 1920's; meaningful instruction in
the 1930's; the *New Math* focus on mathematical structure in the 1950's and 1960's; mastering basic mathematical skills in the 1970's; problem solving in the 1980's; and recently changing classroom social norms including the five shifts as described earlier (NCTM, 1970, 1980, 1989, 1991; Pulliam & Patten, 1999; Resnick & Ford, 1981). This section briefly reviews the major reform ideas of the recent half century, partly because the *New Math* movement in the 1950's and 1960's has been compared and contrasted with the current reform (e.g., Usiskin, 1999). Moreover, a review of historical contexts of mathematics education reform within this time frame is practically informative because the two teachers in this study gained their teaching credentials during this period. Specifically, in looking back over their experience of mathematics as students, the teachers evaluated the positive and negative effects of the *New Math* curriculum on their views of mathematics and its teaching (see Chapter 5).

The *New Math* movement was spurred by the shock to the U.S. of the Soviet Union’s successful launching of the Sputnik satellite. This led to questions about the U.S. educational system in which students had little interest in mathematics and scientific careers. Mathematicians promoted the new mathematics emphasis on a well-structured approach based on the abstract and deductive nature of mathematics, the unification in mathematics, and mathematical symbolism (Usiskin, 1999). Students were expected to learn mathematics better if the curriculum presented it in a logical and clear way. The movement stressed students' conceptual understanding of basic mathematical concepts and advocated discovery learning.

Since the *New Math* curriculum was revolutionary in both content and teaching method, training teachers was recognized as crucial (Sarason, 1982). Summer
workshops served as teacher-training sessions for learning the new curriculum and teaching strategies. Thus teachers were under great pressure and tension to transform their curriculum and methods in just a few weeks. As a result, many teachers became angry and frustrated (National Advisory Committee on Mathematical Education, 1975). Contrary to the intentions of the reform, students didn’t learn mathematical structure, still less basic mathematical concepts and skills. For instance, elementary school students, who were not ready to learn mathematics at a formal symbolic level, repeated the teacher’s deductions without understanding (Sarason, 1982).

Although there is disagreement as to whether the New Math movement was a real failure (Usiskin, 1999), there is general agreement that the top-down imposition of the new curriculum and its related policy without understanding school contexts was a serious problem for the reform (Sarason, 1982). Anyway, the failure of the New Math movement prompted the back-to-basics curriculum in the 1970s. Acquiring accurate arithmetic skills had a priority over understanding mathematical concepts. Specific behavioral terms and knowledge hierarchies formed the basis for much of the school mathematics curriculum (Kroll, 1989). Mathematics teachers taught rules and algorithms for rapid computation. Students, as direct receivers of mathematical knowledge, were mainly involved in drill and practice. Learning mathematics meant solving particular kinds of problems accurately. However, finding any value in the back-to-basics movement is uncertain because students generally did not achieve computational proficiency (Usiskin, 1999). This encouraged substantial changes in the next decades.

Problem solving was promoted as the centerpiece of school mathematics in the 1980s (NCTM, 1980). Practicing routine skills for optimal performance at the expense
of understanding was seen as problematic in the 1980s. Instead of finding ways of achieving skill automaticity, the mathematics education community focused on students’ sense-making processes in solving mathematics problems. Practicing skills was left to problem solving situations that required application of the skills. Considerable attention was focused on the individual’s thought processes involved in learning mathematics. Researchers were interested in analyzing solution strategies used by students, comparing and contrasting expert problem solvers with novices, and studying metacognition based on case studies, interviews, and think-aloud protocol analyses (Charles & Silver, 1988; Schoenfeld, 1987, 1994). Such detailed analyses helped mathematics teachers create prescriptive versions of teaching and identify cognitive obstacles students might encounter while solving problems. Mathematical understanding began to be seen in terms of a spectrum rather than in terms of rightness and wrongness of answers. Problem solving is still a major theme emphasized in the current reform movement (NCTM, 1989, 1991, 1995, 2000). However, more emphasis is given to collective problem solving, rather than individual problem solving, in conjunction with the consideration of contextual factors (Lester, 1994).

PROBLEMS IN IMPLEMENTING REFORM

A central role of a reform-oriented teacher is understood as implementing new social norms to better promote students’ mathematical learning. For instance, the teacher is expected to organize classroom activity structure in ways to encourage students to explore and discuss various solution methods for given mathematics problems. The current reform urges the teacher to move from focusing exclusively on students’ habituation and conceptual development, which have been traditionally emphasized, to
incorporating it into the process of their engagement with mathematics in the classroom community. Indeed, the important aspects of mathematical process, for instance problem solving, representing, reasoning, proving, communicating, and making connections, are viewed as part of the new curricular content of mathematics (NCTM, 1989, 2000). Cuoco, Goldenberg, and Mark (1996) claim *habits of mind* as a central principle for mathematics curricula:

> Much more important than specific mathematical results are the habits of mind used by the people who create those results. ... [The goal of a curriculum] is to help high school students learn and adopt some of the ways that mathematicians *think* about problems. ... This includes learning to recognize when problems or statements that purport to be mathematical are, in truth, still quite ill-posed or fuzzy; becoming comfortable with and skilled at bringing mathematical meaning to problems and statements through definition, systematization, abstraction, or logical connection making; and seeking and developing new ways of describing situations. (pp. 375-376)

This emphasis on mathematical process as well as contents blurs a traditional dichotomy between “knowing that” and “knowing how.” As Bauersfeld (1993) put it, “Participating in the processes of a mathematics classroom is participating in a culture of using mathematics, or better: a culture of mathematizing as a practice” (p. 4, quoted in Yackel & Cobb, 1996, p. 459). In this respect, “The classroom is a community of mathematical inquiry and the students are participants in that community, striving toward mapping and understanding mathematical ideas, norms, and rules” (NCTM, 1998, p. 34). This view has been increasingly accepted by the mathematics education community (e.g., Cobb & Bauersfeld, 1995; Lampert, 1990; Schoenfeld, 1994b; Seeger, Voigt, & Waschescio, 1998).

As introduced in the previous chapter, the current reform recommendations have been widely recognized but the outcomes with regard to the real transformation of
teaching practices were evaluated as weak or ineffective to foster students' mathematical learning (Hiebert et al., 1996; Knapp, 1997; RAC, 1997). Teachers often change only the surface routines of their classroom practices, which makes the actual learning process on the part of students remain unchanged. Whereas students' interaction and collaboration are easily recognized by reform-oriented teachers, a deep consideration of in what ways such a change might contribute to students' learning is not always evident in reform-oriented classrooms.

Whereas teachers' misinterpretation of the visions and goals of current reform is manifest (Burrill, 1997; Bybee, Ferrini-Mundy, & Loucks-Horsley, 1997; Stigler & Hiebert, 1998), their misunderstanding is not trivial. Lindquist, Ferrini-Mundy and Kilpatrick (1997) contend that a main impediment to effective reform is a perceived unitary reform pedagogy, which in fact has been eclectically chosen from different psychological theories with their own strengths and weaknesses. Constructivism and socioculturalism are the central theoretical influences on the development of the current reform agenda, as reviewed earlier in this chapter, but the issue of their integration has been theoretically challenging (Cobb, 1994; Confrey, 1995; Hatano, 1993). For instance, Cobb (1994) calls for a pragmatic approach to counteract the acrimony that often arises between promoters of psychological and social dimensions of learning. In examining two metaphors for learning, acquisition and participation, Sfard (1998) questions the possibility of theoretical unification because of their incommensurability, while agreeing with the necessity of both metaphors. Lerman (1996) also claims that adding the social to the individual leads only to an incoherent theory. In reviewing different versions of social constructivism, O'Connor (1998) cautions:
It is naive to think that it will be easy to graft together a truly critical theory that will simultaneously illuminate the global and collective concerns of society, the social nature of knowledge construction in every content area, the nature of individual learning within a local collective, and the complex relation between an individual and the content itself. (p. 63)

The current theoretical limitations for integration are reflected in the difficulties encountered by dedicated researchers and teachers. In interacting with 51 teachers committed to implement standards-based curriculum programs, Manouchehri (1998) found that all the teachers had a difficulty in placing mastery of basic skills or algorithmic knowledge within their reform-oriented teaching. Ball (1993) also experienced a dilemma over the challenge of dual emphases on students’ learning of mathematical concepts of the curriculum and their thinking or participation in mathematical discourse. As she put it, “With my ears to the ground, listening to my students, my eyes are focused on the mathematical horizon” (p. 376). Lampert (1990) found, while attempting to create discourse of school mathematics as closer to that of discipline, “Like teaching someone to dance, it required some telling, some showing, and some doing it with them along with regular rehearsals” (p. 58).

Teachers confront the complexities associated with the reform. On the one hand, they have to make sure students’ understanding of specific mathematical content. On the other hand, they have to promote students’ enculturation toward characteristically mathematical ways of thinking, reasoning, justifying, proving, and communicating through their classroom participation. In traditional mathematics instruction, teachers directly explain mathematical content. Teachers working within the reform visions also agree with a certain degree of direct teaching or modeling (e.g., Lampert, 1990; Wood, Cobb, & Yackel, 1995). What is unclear is when and for what purposes a certain
pedagogical strategy is appropriate within the teacher's objectives and minute-to-minute practices. How can a reform-minded teacher be sure that students will learn the specific mathematical concept, while he or she facilitates their general experience of mathematical ways of thinking in the classroom community? How can the teacher negotiate two divergent teaching objectives, conceptual and social development on the part of students? These key problematics of reform motivate the current study, and the U.S. and Korean contexts.

MATHEMATICS EDUCATION REFORM IN SOUTH KOREA

In comparison with the U.S., South Korean mathematics education reform is more low-key. The compelling word "reform" has been less used within the Korean mathematics education community; though, indeed, reforms have been called for. Korean reform centers around revision of the national mathematics curriculum and concomitant textbooks and teachers’ guidebooks. Whereas educational leaders in Korea have recently attempted to provide for some degree of autonomy at a local school level, the reform documents are very influential leading to directive, coherent, and rather uniform changes. In particular, the guidebooks for teachers provide detailed exemplary instructional procedures for each lesson in line with background knowledge. There is no specific obligation for teachers to follow the guidelines, but almost all Korean teachers use them as the main instructional resources (Kim, Kim, Lyou, & Im, 1996), which consequently serves as bottom-line teaching.

In the U.S., the professional leadership of mathematics teachers such as NCTM has initiated the current reform movement and has made great efforts to change the culture of instructional practices. In particular, NCTM encourages mathematics teachers
to fully engage in the process of the reform movement as directors of their own teaching practices and as partners with researchers or theorists. Korea does not have such a nationally organized professional group of mathematics teachers. The new curriculum has been implemented in rather a top-down format: Selected mathematics teachers are informed as to the changes in curricular emphases and the subsequent instructional implications, and then the teachers inform their colleagues. Some selected teachers are involved in making mathematics textbooks and workbooks. However, the breadth of real engagement is minimal. Generally speaking, mathematics educators develop a mathematics curriculum, textbooks, and guidebooks for teachers, and then teachers implement these well-developed materials.

National differences in educational reform occur not only in the structure, organization, and implementation of curriculum and instruction but also in teacher education and more broadly in the culture of education (Adams & Gottlieb, 1993; Jeong & Armer, 1994; Smith, 1994; Sorensen, 1994). For instance, Smith (1994) regards the heart of Korean elementary teacher education to be a strong disciplinary focus: “Unlike American teachers colleges, where the major for future elementary teachers invariably is ‘elementary education,’ the colleges in Korea want their students to study in depth a specific field or discipline that is related to elementary education” (p. 34). Sorensen (1994) attributes educational success in Korea to “zeal” for education and parental support to study.

These differences may account for the superior mathematics achievement of Korean students, recently in the International Assessment of Educational Progress (IAEP) (Educational Testing Service, 1992) and in the TIMSS (Beaton, Mullis, Martin, 33
Gonzalez, Kelly, & Smith, 1996; Mullis et al., 1997). Korean 13 year old students did better than any of the other 19 countries' counterparts in the IAEP. Korean students in both fourth and eighth grade rank second in the world in mathematics performance, whereas U.S. students score near the international average of the 41 TIMSS countries.

Despite these differences in mathematics achievement and in educational culture, there are noticeable similarities between Korea and the U.S. with regard to the problems and issues that are related to mathematical instruction. This section describes typical Korean teaching practices, recommended methods, and motivating factors. This description serves as background to understand two Korean reform-oriented mathematics classrooms in the final section.

**TYPICAL TEACHING PRACTICES**

The problems in Korea with regard to mathematics education are perceived to be similar to those in the U.S. Good test scores often conceal mathematics learning without deep understanding (Noh, 1998). Students, even with good achievement, develop increasingly negative mathematical disposition and feel lack of self-esteem with regard to their mathematical ability (Kim et al., 1996; Sorensen, 1994). Interestingly, the TIMSS found that the countries with the highest performance in mathematics including Korea and Japan also were those whose students developed the most negative perceptions of mathematics and success in the subject (Beaton et al., 1996). Korean students' mathematical reasoning ability is often not sufficient for solving non-routine problems for which they do not have specific solution strategies in advance (Pang & Jeon, 1995).

These shared problems come from teacher-centered instruction in Korea (Kim et al, 1996). A typical Korean mathematics class is slightly different from a typical U.S.
mathematics class in that Korean teachers assign much less time for individual students’ seatwork than do their U.S. counterparts (Kim et al, 1996; Mullis et al, 1997). Proportionally more time in Korea is spent in whole class demonstrations and explanations. American teachers report using calculators, manipulative materials, and small group format more often than Korean teachers (Zambo & Hong, 1996).

Whereas typical teaching practices in U.S. mathematics classrooms have been extensively studied through microanalysis of video-recordings of mathematics instructions (TIMSS, 1996), those of Korean mathematics classrooms have been little studied in the international contexts. An exception is a study of Korean instruction conducted by Grow-Maienza, Hahn, and Joo (1997, 1999). They observed twenty typical Korean classrooms in five elementary schools and collected data mainly on the organizational structures of classrooms and the activities of teachers and students. In comparing Korean mathematics instruction with other Asian classrooms, the researchers found some similarities such as the pattern of Instruction/Practice/Evaluation, the placement of problems in real-world contexts, the representation of one problem using several modes, the use of concrete demonstration or manipulative materials, and the coherence and progression of the lesson. Nevertheless, they differentiated Korean primary instruction from Japanese instruction because the former focuses primarily on procedures to solutions of the given problems, whereas the latter focuses on students’ problem solving and explanation per se:

What we found generally seems to be typically Korean, a focus on the teacher’s leading questions guiding the whole class through the appropriate procedures in a most systematic way. ... The salience of a teacher-centered whole class organization in mathematics lessons with highly organized IPE (Instruction/Practice/Evaluation) patterns can be said to be typical of Korean
classrooms observed for this study. Teacher behaviors are dominated by question/answer patterns and demonstration of operations in many modes and patterns which lead students through the procedures and conceptual development of the problem, at the same time facilitating student thinking. Student behavior is characterized by choral responses and choral evaluation of individual responses which keep students on task. (Grow-Maienza, Hahn, & Joo, 1999, p. 6)

This observation of "teacher-centered whole class organization" in Korean primary mathematics classes is strongly supported by the more comprehensive TIMSS study wherein Korean elementary school teachers reported that their most common way of organizing a mathematics class is to teach the whole class in which students work together (Beaton et al., 1996; Kim et al., 1996). As Grow-Maienza, Hahn, and Joo (1997, 1999) imply, however, there are differences between Korean and the U.S. mathematics instruction under the same rubric "teacher-centered." For instance, typical Korean teachers orchestrate their teacher-centered lessons more systematically, coherently, completely, and progressively than U.S. counterparts do. It is very noticeable in comparison with U.S. students that Korean students are deeply engaged in teacher-centered lessons and enthusiastically provide choral responses.

Despite the differences noted above, Korean mathematics instruction is indeed teacher-centered in the same way as to U.S. instruction in that teachers' explanations and directions constitute the mainstream of mathematical practices. The teacher retains sole authority as to what should be covered throughout the class period and what is right or wrong. Zambo and Hong (1996) found similar points from their survey on Korean elementary school teachers' beliefs about mathematics problem solving:

Korean teachers also believed... that students should be told the best way to solve types of problems, that hearing other methods of solution tends to confuse children, and that students should be given the correct answer to all the problems that they solve. These strategies present a structured approach with the emphasis
on one path to the solution and do not promote individual thinking or general problem solving ability. (p. 213)

This accords with the more comprehensive TIMSS in which most of Korean elementary and middle school teachers reported that their most common teaching method is to demonstrate how to solve given mathematical problems and then to provide students with similar problems for practice (Kim et al., 1996). Consequently, students are expected to follow the teacher’s demonstrations, and reproduce the teacher’s methods. Students are rarely actively engaged in developing their own solution methods individually, in small groups, or as a whole class. Learning mathematics is basically to receive and practice the teacher’s or textbooks methods.

There are some institutional factors that maintain Korean instruction as teacher-centered. Korean students prepare for a high-stakes examination to enter a good college at the end of secondary school. This exam-driven educational culture often is seen as contributing to Korea’s teacher-centered pedagogy. As the grade level goes up, teachers become increasingly concerned about skillful performance on the examination. As a result, open ended activities or discussions occur infrequently (Kim et al., 1996). Students are occupied with just following teachers’ explanation and practice. As Sorensen (1994) put it,

There is no doubt that teachers “teach to tests.” South Korean students spend an inordinate amount of time memorizing textbook material. But they also practice problems by other than rote means, and they work hard to overcome inadequacies in their schooling. This is encouraged by their parents. (p. 33)

This exam-oriented educational culture may not influence elementary school instruction directly, which can account for many desirable instances of mathematics practices in the study by Grow-Maienza and her Korean colleagues. In fact, the
researchers report what they heard unanimously at the beginning of their study from their collaborating Korean graduate students who observed the target elementary mathematics classrooms: "Classrooms will be totally teacher centered. Teachers will be giving all the information, the students will be giving short answers recited from memory" (Grow-Maienza, Hahn, & Joo, 1999, p. 6).

**RECOMMENDED TEACHING PRACTICES**

Countering the common teacher-centered pedagogy in mathematics, the Ministry of Education in Korea recently developed revised national curricula and teacher guidebooks wherein many characteristics of student-centered teaching methods are consistently recommended (Kang, 1998; Ministry of Education, 1992, 1993, 1997). In particular, the most recent 7th National *Differentiated Curriculum* stresses giving students opportunities to study mathematics based on their individual learning capacity, aptitude, and interest. A reform teacher is expected to select or develop mathematical tasks that are related to students’ everyday life, and to begin with concrete experience before addressing abstract mathematical knowledge. The curriculum urges teachers designing learning environments to “consider concrete manipulative activities and thinking processes, provide opportunities for students to solve mathematical problems and to discover mathematical principles and rules for themselves, ... [and] use open-ended questions which stimulate students’ thinking ability and creativity” (Ministry of Education, 1997, p. 85). These recommendations for enriching learning environment for students are intended to support the curricular emphasis on the understanding of fundamental mathematical concepts, logical thinking, problem solving, communication, and mathematical attitudes. Like in the U.S., the teacher’s role in a reform mathematics
class can be framed as implementing new social norms which facilitate students’ mathematical learning. As we will see, below, the concordance between U.S. and Korean reforms is not accidental, as U.S. recommendations have had a substantial impact here.

**MOTIVATING FACTORS FOR THE REFORM**

In general, mathematics education reform in Korea is motivated by multiple factors such as changes in society, limitations of current instruction and subsequent outcomes, changes in views on students’ mathematical learning, and more broadly the desire to improve the quality of students’ mathematics experience. These factors can be seen in the three interrelated rationales for the most recent national curriculum in Korea identified by Lew (1999): (a) the mathematics curriculum should meet the expectations of a changing society in terms of information, technology, and globalization; (b) previous curricula were rather skill-oriented and fragmentary in conjunction with the expository method of instruction; and (c) previous curricula did not consider various differences among individual students with regard to mathematical abilities, needs, and interests.

The following is an attempt to understand theoretical influences and historical contexts that have been related to the current reform in Korea. A caution must be stressed. The reform documents in Korea contain little explicit discussion of the theoretical perspectives they reflect. Thus, this review should be understood as the author’s own search for the underpinnings that implicitly motivated the Korean reform as emerging from her ability to see contrasts.

**Theoretical Influences**

Much of the current mathematics education reform movement in Korea reflects substantive shifts from learning as receiving to learning as understanding mathematical
knowledge, and from emphasizing problem solving skills and strategies to developing mathematical thinking ability and problem solving ability (Ministry of Education, 1992, 1997). Many recommended teaching methods in the current reform era are consistent with some characteristics of constructivist and sociocultural perspectives, which have been the two most influential theories to the U.S. mathematics education reform.

Constructivist perspectives have not been extensively explored in Korea with regard to their strengths and weaknesses in comparison with the U.S. context. Nevertheless, there is a growing commitment to teaching recommendations that are responsive to a view of learning as an individual's active construction. For instance, teachers are supposed to provide the opportunities for students to interact with various materials, to develop mathematical ideas for themselves, and to reflect on different solution strategies. Implicit in these ideas is the assumption that individual students have different understandings based on the history of their own experience (Steffe & Kieren, 1994).

Another indication of constructivist perspectives is the emphasis on connections among mathematical content. The curriculum and teachers' guidebooks indicate a strong concern of explicit vertical linkages as well as horizontal connections for integrated thinking on the part of students. For instance, teachers are supposed to start with a diagnosis of students' understandings in order to help them connect the current lesson with their previous knowledge structures. Implicit in this is the concern that students may learn mathematical knowledge as isolated so that they cannot retrieve together their knowledge related to solve problems in more complex or novel situations (cf., Hiebert & Carpenter, 1992).
Sociocultural perspectives have been little addressed by the mathematics education community in Korea. However, to be clear, the reform recommendations do stress social aspects of mathematical learning. For instance, teachers are encouraged to manage the classroom atmosphere so students can discuss different ideas without being embarrassed. Teachers are also expected to provide students with non-routine problems with which they are engaged in mathematical thinking as well as they learn problem solving strategies and skills. Nevertheless, these aspects do not guarantee the possible influence of sociocultural perspectives on Korean pedagogy. While the central concern of socioculturalism is socialization towards professional mathematics community, the recommended social aspects for Korean instruction are for the most part seen as a moderator or catalyst for individual students’ cognitive growth. Students’ conceptual understanding of specific mathematical content is perceived more important than their social development in the classroom community.

A review of philosophy of mathematics in Korea may support the claim that the stress on social dimensions of mathematical learning does not reflect sociocultural perspectives. In Korea mathematics is perceived as a domain of rationality based on abstract, formalistic, systematic, metaphysical, and logical characteristics. Similarly, objectivity, universality, and generalizability constitute the important underpinnings of the discipline of mathematics. Implicit in this is the assumption that mathematics is a well-defined field of inquiry that preserves certainty (Ernest, 1998). This assumption sharply contrasts with the view of mathematics as a human activity by which mathematics is negotiated and institutionalized by members of communities. In the sociocultural point of view, mathematical truth is consensual rather than absolute.
Whereas the processes of activity are an essential nature of mathematical learning in sociocultural perspectives, the products of activity are the more important in the Korean educational context.

**Historical Contexts**

Despite the different national contexts, the history of Korean mathematics education during the past half-century reveals many parallels to the U.S. experience (see Park, 1991, for the history of Korean mathematics curriculum). The first mathematics curriculum in the 1950s was centered in everyday life situations. Objectives, content, and methods of mathematics instruction focused on solving real-life problems and considering students' interest. Specifically, mathematical content was forced to be directly related to real-life problems so that students had to calculate many economically specific computations. Mathematics was used mainly as a tool to solve practical problems. The second mathematics curriculum in the 1960s was a strong reaction against the first curriculum. The main characteristic was the consideration of mathematical hierarchy or system over practical usefulness. This second curriculum was regarded as a starting point of a *New Math* movement in Korea, but the more direct influence of the U.S. *New Math* curriculum occurred to the development of third mathematics curriculum in 1970s. The third curriculum, keeping with the basic ideas of the second curriculum, focused on logical thinking and mathematical structure and used a spiral organization of mathematical content. Many new mathematical concepts and symbols appeared even in elementary school mathematics. The discovery learning method was addressed and consistently emphasized as a recommended teaching approach in subsequent curricula.
The fourth curriculum in the early of 1980s was a reaction to the New Math movement. Many mathematical concepts addressed in early school mathematics were reduced in favor of problem solving. The fifth and sixth mathematics curriculum in the late 1980s and in the early 1990s were minor revisions of the previous curriculum in that the mathematical content was reduced in order to enhance problem solving ability, mathematical thinking ability, basic ability and skills, and positive mathematical attitudes. The name of the subject was changed from “arithmetic” to “mathematics,” suggesting that learning mathematics should go beyond acquiring basic problem solving skills, toward developing logical thinking and application ability. Teachers have been encouraged to provide the opportunities for students to solve problems for themselves through individual exploration, small-group cooperative activity, and discussion.

The most recently developed seventh curriculum is significantly different from previous curricula in that it has a level-based differentiated structure (Kang, 1998; Ministry of Education, 1997). The curriculum consists of two parts: (a) a common core curriculum for all students from first to tenth grade with a total of 20 different levels, and (b) selective curriculum with different topics and difficulties in the last two years of high school. The main motivations to this curriculum include increasing concern for individual differences and the desire to provide maximum growth of individual students on the basis of their abilities and needs. Lew (1999) interprets this curriculum as reflecting a constructivist perspective. Since this new curriculum and concomitant textbooks and instructional materials are operated in schools only from 2000, the outcomes are yet to be measured.
INSIGHTS FROM TWO KOREAN REFORM-ORIENTED CLASSROOMS

This capsule summary of Korean mathematics education and its reform provides a backdrop against which to understand the two classrooms discussed next. The classroom data used in this section come from the Korean portion of study in a larger cross-national project in which four researchers explored in detail two Korean reform-oriented elementary mathematics classrooms (see Kirshner, Jeon, Pang, & Park, 1998, for the full report). The full project departs from past international comparisons in which the common objective has been to compare general social norms of typical mathematics classes across countries, for instance by analyzing social interaction patterns, characterizing the style of mathematics instruction in the classroom, and exploring sociocultural influences on the development of specific mathematical concepts (e.g., Easley & Taylor, 1990; Schmidt, Jorde, Cogan, Barrier, Bonzalo, Moser et al., 1996; Stevenson & Stigler, 1992; Stigler, Fernandez, & Yoshida, 1996; TIMSS, 1996; Yang & Cobb, 1995). Such comparisons have provided a more explicit understanding of each country’s own Characteristic Pedagogical Flow (CPF) — recurrent patterns of teaching practices and learning activities, reflecting typical conceptions of instruction (Schmidt, Jorde et al., 1996).

Although comparison of typical classrooms can make a valuable contribution to understanding the dynamics of teaching in a country, it may not always contribute directly to attempts to implement teaching reform. And even when such contributions are possible, they may not be equitable for all countries involved. For instance, student-centered teaching practices observed in typical Japanese classrooms have been identified as more consistent with the U.S. reform recommendations than is true of
typical U.S. classes (Fuson, Stigler, & Barsch, 1989; Schmidt et al., 1996; Schmidt, McKnight, Valverde, Houang, & Wiley, 1997; Stigler & Perry, 1988; Stigler, Fernandez, & Yoshida, 1996; TIMSS, 1996). These results support reform in the U.S. by showing that such instruction is possible on a broad scale, and by illustrating teaching models that might be adopted by U.S. teachers. As a practical matter, however, viewing relatively unsuccessful U.S. teaching methods may not be as helpful for Japanese mathematics educators.

Given the similarities between Korea and the U.S. with regard to typical and recommended teaching practices, focusing on reform-oriented classrooms can produce mutual benefits toward understanding what constitutes the process of implementing reform ideals into actual classroom contexts. It is expected that each country can learn much from the successes and the failures of the other.

This section provides background information of the project with regard to data and methodology followed by descriptions and comparisons of the two classrooms. This section also contains a brief discussion on the importance of sociomathematical norms arising from the analysis of two Korean classrooms.

**DATA AND METHODOLOGY**

During September of 1997, the team of four researchers observed more than one dozen mathematics classrooms that were recommended by district supervisors, in schools attached to universities, or in schools nominated as research schools. Whenever a classroom seemed promising, the researchers conducted an open-ended interview with the teacher focusing on his or her teaching philosophy, and then observed more lessons to confirm the possibility. This extensive search was needed, given the recency of the
reform recommendations, and the infrequency of teachers' explicitly advocating reform allegiances. Finally, two second-grade classes from different schools were chosen, because of their unequal success in implementing student-centered instruction, under the agreement among the researchers.

After preliminary observations, two mathematics lessons in each class were videotaped using three cameras: one for the teacher, another for the students, and the third for the classroom setting as a whole. Additional data sources include audiotapes to capture students' conversation within small groups in both classrooms, field-notes of general classroom activities, copies of individual students' worksheet. As well, there were two interviews with each teacher, the first for clarification of classroom activities as recorded on the videotapes, the second to probe into how they have constructed their own teaching method. The videotaped and audiotaped lessons were transcribed and translated in a two column format, the second column for notes related to the analysis.

The project as an exploratory case study used *grounded theory approach* based on the *constant comparative analysis* for which the primary data sources are classroom video recordings and transcripts. Since this dissertation uses the same methodology, the details of research methodology can be found in Chapter 4.

**ORIENTATION TO THE TWO CLASSROOMS**

This section provides general background information about the two Korean mathematics classrooms, including setting, curricular topics, the sequence of classroom activities, and patterns of social interaction. This description provides a basis for the comparison and contrast of the two classes in the next section.
The Setting

The two schools are located in a suburban area of ChungJu, South Korea. The majority of the students in both schools are from middle- to lower-middle-class families. The two teachers, Ms. G and Ms. C, were highly enthusiastic teachers with more than 20 years’ teaching experience. In particular, Ms. G had voluntarily participated in a regional mathematics club in which elementary school teachers in the same town meet and share their teaching experience in mathematics class. Both teachers tried to create a classroom environment in which students’ discussions and contributions were valued.

Ms. G’s class (class KG) was one of the two second-grade classes in the school, which was the same to Ms. C’s class (Class KC). Each class consisted of around 40 children. Both classrooms were equipped with a personal computer and an overhead video projector connected to a TV screen. Ms. G projected learning objectives and classroom activities typed on the computer to the TV screen. Students in both classes presented their solution methods to given mathematical problems by putting their materials or worksheets on the overhead projector so that other classmates might see it on the screen.

Usually a boy and a girl were paired as partners facing towards the board in the front. During the two observation days, Ms. C used small-group formats by which four to five students sat together and shared their solution methods to a given problem. Ms. G organized her class into small groups only on the first observation day.

Classroom Activities and Interactions

The curricular topics in both classes were three-digit addition and subtraction without carrying/borrowing. Instead of using the textbook and the workbook which are
commonly used throughout Korean elementary schools, Ms. G made several worksheets for each mathematics class. However, the types of problems in the worksheets were similar to those of the textbook or the workbook. Ms. C used one or two worksheets on which students were supposed to solve one problem but with multiple methods, and/or to pose a similar problem at the end of the lesson. The gross pattern of classroom activities was very similar in the two classes. They both began by reviewing related topic. Then they used the whole class discussion followed by individual or small group activities, and finally had summarizing time. The only difference here was that Ms. C encouraged her students to summarize their mathematics activities, whereas Ms. G herself concluded the lessons.

The usual pattern of social interaction in the two classrooms was also similar: (a) students independently solved given problems; (b) the teacher asked students to present their solution methods as well as answers; (c) individual students explained their methods in front; (d) the teacher often repeated and/or amplified students’ explanations, and provided judgments and questions as needed; (e) other students contributed to the discussion on the basis of agreement or disagreement; and (f) the teacher encouraged other students to provide a different solution method. The sequence of teacher-student-teacher-student turn taking was remarkably similar across the classrooms. Direct student-student interaction was rarely found in the whole group discussion in the two classrooms.

**COMPARISON OF THE TWO CLASSROOMS**

The two classrooms are compared and contrasted in terms of general social norms and sociomathematical norms. The comparison of sociomathematical norms
includes three critical episodes from each class that demonstrate how the teacher and the students established the specific norm of mathematical difference.

**Comparison by General Social Norms**

The two Korean classes in this study shared strikingly similar general social norms. There were many similarities with regard to the expectations, obligations, or roles adopted by the teacher and the students across the classrooms. Both classes displayed a classroom participation structure in which:

- The teacher and the students established permissive and open atmosphere so that students’ ideas and even their mistakes were welcomed.
- The discussion pattern of social interaction described above predominated with a sequence of teacher-student turn taking.
- The teacher utilized small group formats to encourage collaboration and discussion among students.
- Students solved mathematics problems for themselves and presented them to the whole class.
- The teacher encouraged students to find different solution methods for a given problem and to provide critiques of their peer’s presentations.
- The teacher supported students’ contributions to the discussion by providing praise and encouragement.

These social norms are compatible with many characteristics that are recommended by educational leaders in both Korea and the U.S. in efforts to reform mathematics education by making it more student-centered. However, the similarities in the social norms exhibited within each class are not entirely coincidental. The teacher’s
questionnaire in the TIMSS study revealed that most Korean primary school teachers heavily consult the teachers' guidebook provided by Ministry of Education in Korea (Kim et al., 1996; Mullis et al., 1997). Ms. G and Ms. C in this study emphasized solving addition and subtraction problems using different methods. But the focus on using various representation methods per se may not be based on the teachers' own reflections on their lesson strategy. The guidebook supplied by the Ministry of Education recommends exactly such a strategy.

Comparison by Sociomathematical Norms

Despite their similarities in social organization, the two classes established dramatically different sociomathematical norms. As described before, Ms. G and Ms. C both stressed finding many different methods to solve a three-digit addition or subtraction problem. However, the two teachers guided the development of very different norms as to what counts as a mathematically different contribution. For comparison, this section first describes how each class established this norm with relevant episodes.

Mathematical Difference in Ms. G's Class

Of special importance in Ms. G's class was the solving of problems by a variety of methods. On the first of the two observation days, a major objective was to include mental computation in the list of methods already learned. On the second day Ms. G broke the worksheet time into two parts. In the first part students obtained an answer to some of the problems. A longer period of time followed in which the students were directed to use multiple solution methods, as illustrated by the following episode.
<Episode KG-1: Finding different solution methods for a problem>

Teacher [T]: Instead of solving the problem using one method, find as many methods as you can. (She walks around and checks individual students' worksheets). Find as many methods as possible. Jinook, why don't you try to find many methods? (She looks at his worksheet.) Did you solve it by mental computation? Find every method. I expect you [Jinook] to do well. Guerae, why don't you do it with different methods? (She walks around all the way.) The first group is doing very well.

After students' individual work time, Ms. G led a whole class discussion and asked for different solution methods to given mathematical problems. For instance, the methods that students had used to solve 460-320 included horizontal presentation of the problem; the usual vertical format; an expanded vertical algorithm in which each place value was added on its own line and then all of these lines were added up for the final answer (rather like the multiplication algorithm); and a method using pictures of coins to represent the various place values. Episode KG-2 shows a student's presentation of her solution methods and Ms. G's criticism with regard to the student's wording of one of her methods. Ms. G encouraged other students to provide a better alternative word.

<Episode KG-2: Presentation of different solution methods and correction>

T: (To the whole class) Look here at all the methods Sulhae used to solve this problem.

Sulhae: For this equation (460-320=140), we can solve it horizontally. We can also do it by mental computation without writing down the equation. And, after computing vertically from the ones digit to hundreds digit, you can add each digit. Or you can reach the answer in vertical lines. You can also do by making story problems.

T: Sulhae explained various methods, but there is something awkward, isn't there? She said that she used vertical lines. I think, there is something awkward about this. How could we say it in other words? We learned it when we studied addition. Who can explain this? (Several students raise their hands and the teacher picks Hyojung.)

Hyojung: It seems more convenient to say digit by digit rather than what Sulhae said, in a vertical line.

T: What does digit by digit mean?
Hyojung: It seems better to get the answer by subtracting from the ones digit, tens digit, and hundreds digit, respectively. Then, add the results from each digit.

Ms. G criticized Sulhæ's procedural language, vertical alignment instead of naming the place values. However, she never probed the nature of the student's understanding. Ms. G seemed satisfied that students would use the approved vocabulary, without regard to the nature of their conceptual understanding. Despite the whole class discussion time on various solution methods, there was little discussion of why different methods worked, how they were related to each other, or even why different methods were important to study. The main concern was getting the correct presentation and the answer of the various methods.

Towards the end of the class Ms. G consolidated and summarized what could be learned from the various methods that had been presented. After reviewing various methods for solving 460-320, Ms. G summarized two different methods and reinforced the standard algorithm as "convenient" (see Episode KG-3). To this end, she also emphasized column alignment and subtracting leftward from the ones digit. Note that braces, {...}, are used to put a brief description in place of a long transcript.

<Episode KG-3: Looking for a convenient method>

T: Which is a convenient method? Let's think about it ... Why don't you look at what I did? Look over here (pointing to the screen displaying the less favored "expanded" method). When I solved 460-320, I did it digit by digit vertically. How did I do? (Points to ones digit.)

Students [Ss]: Ones digit.

T: Subtract from ones digit. Next?

Ss: Tens digit. {Continuing in this way, the teacher summarized the process whereby numbers in each digit are subtracted respectively and then the results from each digit are added.}
T: You have to keep in mind line alignment when you subtract vertically. Next, there is a convenient method, isn't there? Let's look at the convenient method (pointing to the standard algorithm on the screen). Subtract the ones digit, subtract the tens digit, and subtract the hundreds digit. At once, we can reach 140.

Mathematical Difference in Ms. C's Class

Like Ms. G, Ms. C also emphasized multiple methods to solve mathematical problems. In particular, she allowed her students to focus on and discuss various methods for most of the class time by providing only one subtraction problem (one the first day, 460-320; on the second day 476-152). When Ms. C introduced a problem at the beginning of her lesson, she emphasized students' creativity and independence in developing a variety of methods by highlighting that the given problem should be solved by them, not by her. In the following episode, Ms. C made a special point of giving a rationale for multiple solutions.

<Episode KC-1: Rationale for multiple solution methods>

T: Yeah. You yourself do it. It is not me but you who try to do it. Um, find the methods by yourselves. By the way, in which method?

Ss: Various [methods].

T: Yeah. Not one method but ... 

Ss: Various.

T: In various ways. This is important. If you compute using only one method, you may not be able to understand other methods others use. Don't you think so? Try to do in various ways and discuss with your friends, including in what ways your methods differ from your friends' and which is better. Find out this by yourselves so that you can solve any problem similar to this problem. Insofar as you do well this, you are able to compute any subtraction problem regardless of digits. Didn't I say this in the first semester, too?

After students worked together to find multiple methods in their small groups, Ms. C led a whole class discussion by asking them to present their solution methods.

This class activity was the longest part of the class, and produced many discussions of
mathematical concepts. The taken-as-shared meaning of what makes different one
solution method from another had shifted as the lesson progressed (see Pang, 1998, for a
detailed description of such shifts in Ms. C's class). The meaning of “different” methods
included:

- using different materials (e.g., using chopsticks and number cards were counted
  as two different methods);
- using different procedures either in the order of processing (e.g., the subtraction
  from ones digit and the subtraction from hundreds digit were assumed as
different) or in the form of representation even with the same material (e.g., using
  a convenient vertical format where only the answer is represented and using an
  expanded format where the process of adding each place value as well as the
  answer are represented were agreed as different); and
- using different units (e.g., regarding a bundle of 10 chopsticks as 100, not as 10,
  was lavishly praised as insightfully different method).

Most of the time, the teacher accepted students' presentations as different from
what had been offered by other students. But one time, as illustrated in episode KC-2,
Ms. C rejected a student's response, evaluating it as similar from the previous method
used in the discussion.

<Episode KC-2: Rejecting non-difference>

(After several students' presentation with number cards, toothpicks, and linoleum, the teacher
demonstrated subtracting 152 from 476 with her own materials. The material is similar to base
10 blocks except for texture, color, and decorations.)

T: Now, who would like to present in front with another thing? (Students raise their hands.)
What will fifth group use?
S (in the fifth group): Tiles (referring to paper tiles of base 10 blocks).

T: Tiles? Using tiles is the same. This (pointing to her materials which are attached on the board) is the same as tiles. Who will present with other materials except tiles? No tiles, please. (Sangmee responds she would like to use coins. The teacher accepted using coins as different.)

In principle, using different materials might be rewarded by the teacher as a different solution method. But in practice, as seen in the above episode, the teacher did not reward some contributions on the basis of a sense of what is mathematically different. The teacher’s material and paper tiles are the same in that both of them have the concrete characteristics of base 10 blocks: The tens material actually consists of 10 units of the ones material. In this respect, the materials are the same. It is a departure from the teachers’ strategy of using gradation of praise to signal the greater or lesser mathematical interest of a solution. At this point, the teacher elaborated the meaning of “different” materials by distinguishing the mathematically different from the superficially (physically) different.

Despite Ms. C’s consistent emphasis on different methods, there was no explicit discussion of what constitutes the different until the end of the lesson on last observation day. After students’ presentation and discussion of various methods, Ms. C finally indicated the nature of her interest in different methods. The next episode concerns differences between coins and paper tiles. After establishing that the answers are the same regardless of the materials, Ms. C probed for some sense in which the representational forms are different, rejecting many superficial differences until a student was able to express what she thought of as a significant difference. Note that brackets in the episode, "[...]", are used to report low-inference interpretations of words or actions.
<Episode KC-3: Contrasting coins and tiles>

T: Who used this method [using coins]? (Most children raise their hands.) Ah, it seems that all of you can do it with coins. Put down your hands. By the way now I have something that I want to know ... Did what was done with coins and ... what was done with tiles over here produce the same answer? What's the answer?

Ss: 324.

T: 324 (pointing to the tiles on the board) And here? (Points to the projector.)

Ss: 324.

T: 324. The answers are the same, right? So it's the same whether you use paper tiles or coins, but a little bit, a little bit ... The answers are the same, but still there may be a slightly different aspect. It seems to me that there is. Is there something?

Yongho: There is.

T: What? He said there is. Yongho says there is. What? What's there?

Yongho: Coins and tiles...

T: What is the difference between coins and tiles?

Yongho: Shape.

T: Their shapes! Oh! Anything else? Jeongyoung? (The chime rings, which denotes the end of class.)

Jeongyoung: I'll express my opinion. They differ in color.

T: They differ in color! Um. Another [aspect]? Haejin?

Haejin: I'll express my opinion. They differ in size.

T: Differ in size! Size... What size?

Haejin: The tile is ...

T: Please stand up and then express your opinion.

Haejin: Tiles are larger than coins and... the 100-won [Korean unit] coin is larger and...

T: Anything else? another difference?

Byungho: I'll express my opinion. The number of quantity is different.

T: How?
Byungho: Tiles are ... the same as the number of 10. In the case of coins that does not work.

T: I don't know what you mean. Come out here for a moment. What, how? Byungho said something but I can't understand what he meant. (Byungho comes to the front.) What? What did you mean?

Byungho: [Partially inaudible] The number of quantity. In case of coins the number is....(He explains something to the teacher.)

T: Can you represent it? What do you mean by quantity? He says that the quantity is different. What? What do you mean by quantity?

Byungho: Just Size.

T: What size?

Byungho: [Partially audible] In the case of something like tiles, the 10-unit number ... fits into 100-unit number.

T: Aha! Try to represent that. Aha, now I understand what he means. The 10-unit can be embedded like this in the 100-unit (overlapping her hands), but in the case of coins?

Ss: Aha!

Byungho: Coins can't be embedded...

T: Um. I was not sure if I understood what you meant [before, but now I understand]. (To Byungho) Why don't you do it? (Byungho detaches the tiles for 10 on the board and the teacher moves away materials on the projector.) Please, do it over here [on the projector]. Here. Over there (indicating the board) the magnet didn't work well.

Byungho: (Brings a tile for 100 to the projector, but pauses looking at the magnet attached to the back of the tile.)

T: Try it. You can put it down. It's all right to put down the magnet [on the projector]. Just do it. Here are many [tiles for 10]. Here are many [tiles for 10], too.

Byungho: (Starts putting 10-unit tiles on a 100-unit tile on the projector.)

T: I didn't understand what he means but now I understand it. As we get it, like this. (She detaches the tiles left on the board, and puts them next to the projector.) Ah! Like this!

Byungho: (Tries to put another [the 10th] 10-unit tiles, but pauses [because of the lack of the space].)

T: How many are here? Have a try. Here... (The teacher carefully aligns the 10-unit tiles on the 100-unit tiles, making space for the last tile. Byungho puts the 10th 10-unit tile.) Now I understand, can you understand, too?
Ss: Yes.

T: Well, what was said, what Byungho says is that if we superimpose the 10-unit pieces of tiles on a 100-unit piece, the two can become the same; but coins cannot be made [to fit] like that. He said that was the difference. At first I didn't understand what he meant. Aha, now this is a 100-won coin, isn't it? (To Byungho) O.K. Go back to your seat. If there are 1 coin of 100 won and 2 of 10-won coins, uh, let's see how many 10-won coins are there [on the projector]. How many 10-won coins are there? Look over there. How many 10-won coins are there?

Ss: Five.

T: There are five. By the way what portion is five 10-won coins of 100?

Ss: A half.

T: It's a half but, in the case of coins, is that observable or not?

Ss: It's not observable.

T: Byungho explained it well. He seems to mean that this [one of 100 won coin] and this [five of 10 won coin] don't become a half. (To Byungho) Right? But in this case [tiles], he says that the quantities are the same, if we superimpose these pieces. So, when you look at the case of something like tiles, the fact that ten 10-units makes up a 100-unit can be seen more... (She waits for the students to complete this sentence.)

Ss: Easily.

T: Hmm, we [the class] could find this fact also. He [Byungho] found out a very good thing. So one can become a master-of-discovery. As we said before we'll be masters-of-discovery. [She refers to a song sung by the class just before this lesson, containing words something like "if you discover a very good thing in mathematics, you can be a master-of-discovery."]

In keeping with classroom social practices in which students' ideas were solicited and focused, Ms. C sifted through their responses of superficial differences like shape and color until a student hit on a mathematically powerful one. On the basis of specific insights from Byungho the teacher highlighted the contrast between concrete and iconic representations, which are mathematically a very important connection for students to make in retaining their quantitative sense about symbolic algorithms.
Comparison

Whereas Ms. G’s and Ms. C’s classes were similar with regard to focusing on finding different solution methods to a given mathematics problem, the two classes were very different with regard to what made a solution different from previous ones. In the Ms. G’s class, the teacher welcomed all solutions offered by students as being different. Ms. G rarely probed the nature of students’ ideas but checked whether their “different” solution methods produced the same right answer and their presentations included adequate language. Moreover, students’ various ideas were subsumed under the teacher’s own summation of the “convenient” subtraction algorithm.

In the Ms. C’s class, the teacher was quite selective in what she would accept as a different solution. She at first accepted students’ idea that using different materials made different solution methods. However, as students became sophisticated enough to come up with alternative interpretations beyond physical or superficial differences, Ms. C rejected a student’s method of using a similar kind of material that had been offered. Moreover, Ms. C led her students to exploit crucial differences between coins and paper tiles to highlight the meaning of the “different” as the mathematically significant (iconic representation versus concrete representation). In this way, mathematically significant distinctions became embedded within the social practices of the Ms. C’s classroom, but not Ms. G’s classroom.

DISCUSSION

In both of the Korean classrooms, students had many social opportunities to participate and to experience success in their efforts. However, they had very different mathematical learning opportunities. The students in the class KG had the opportunity to
develop skills in solving routine problems, because the mathematical content of the lessons was mainly procedural. But they had little opportunity in terms of developing the conceptual sense of the mathematics they were studying. In contrast, the students in the class KC were continually exposed to mathematically relevant distinctions in their classroom microculture. They had the learning opportunity to make conceptual underpinnings of the mathematics they were studying.

The students in both classrooms were mostly interested in complying with the teacher’s demands and expectations. At their young age, they did not have previous experiences of mathematics as a basis for their own independent ideas about what it means for mathematics to make sense. As a result, both groups of students were particularly vulnerable to the teacher’s views of mathematical sense making as she enacted them in her classroom. In the class KC there was evidence supporting that students were becoming self-motivating in their pursuit of mathematical meaning. For instance, some of the students continued to work on figuring out solutions, even after the teacher had rung the bell to signal a new activity. At the class KG, students remained much more focused on the teacher without those indications of autonomous motivation that signal mathematical empowerment.

This Korean example shows that the learning opportunities within the two classrooms were very much constrained not by the classroom participation structure per se but by the mathematically significant engagements within the structure. In other words, the dynamic engine of learning opportunities was not located in the general social norms of the classroom. Rather learning opportunities arose from the ways in which mathematically significant distinctions were embedded within classroom social
processes. Thus the analysis of mathematics instruction by sociomathematical norms is proposed as a new and promising way to assess the quality of students' learning opportunities towards mathematically powerful ways of knowing and thinking.
CHAPTER 3

REVIEW OF SOCIOMATHEMATICAL NORMS

The two reform-oriented classroom examples from Korea, described in the previous chapter, suggest that sociomathematical norms can be a crucial construct to assess the quality of mathematics instruction. Comparing and contrasting the two mathematics classrooms by social norms and sociomathematical norms made it possible to analyze the extent to which changing teaching practices translated into changes in mathematical learning opportunities that students would encounter in those classrooms. Building on this insight, this chapter reviews the literature on sociomathematical norms and related topics. The chapter is organized into two sections. The first section reviews the origins and the explanations of sociomathematical norms that have been advocated and used in classrooms. The second section critically examines the current understanding of sociomathematical norms, in particular with relation to the purpose of this study.

ORIGINS OF SOCIOMATHEMATICAL NORMS

This section describes sociomathematical norms as introduced by Cobb and Bauersfeld (1995) and Yackel and Cobb (1996) as part of their theoretical framework. This review includes an analysis of how the construct of sociomathematical norms has been used with regard to reform-oriented mathematics teaching by Cobb and his colleagues, and other researchers.

One of the most interesting aspects of this work is the interdependence between the development of a theoretical perspective and the evolution of the classroom teaching experiment. While engaging in a research and development project in inquiry
mathematics classrooms at the elementary school level, they simultaneously grappled with theorizing their classroom experiences and observations (see Cobb & Bauersfeld, 1995). As O'Connor (1998) noted in his review of social constructivism, Cobb and his colleagues “do not subvert their vision to any subsuming theoretical categorization; rather they stubbornly carve out a path that preserves the uniqueness of each classroom and their understanding of it” (p. 58). As Yackel and Cobb (1996) express it in their own words:

There is a reflexive relationship between developing theoretical perspectives and making sense of particular events and situations. The analysis of the particular constitutes occasions to reconsider what needs to be explained and to revise explanatory constructs. Conversely, the selection of particulars to consider reflects one’s theoretical orientation. (p. 459)

The recent origination of the sociomathematical norms construct, together with its emergence through a kind of bricolage, suggest the possible utility of further elaboration of its theoretical and practical implications. Indeed, the following review highlights the embedment of this construct in the particularities of the instructional dynamics encountered in the classrooms that Cobb and his colleagues happened to study. A retheorization of the construct is introduced in Chapter 6, as a reasonable extension or generalization of the construct to broader educational circumstances. Tracing the origins of sociomathematical norms requires a review both of the theoretical orientation and of the classroom mathematical practices. The theoretical perspectives that Cobb and his colleagues have developed are reviewed first, followed by their relation to specific mathematics classroom practices through which the constructs of social and sociomathematical norms were addressed.
THEORETICAL ORIGINS: AN EMERGENT PERSPECTIVE

Cobb’s primary interest in mathematics education was individual students’ conceptual understanding of specific mathematical content from a constructivist framework (Cobb & Steffe, 1983; Steffe, Cobb, & Glasersfeld, 1988). This initial theoretical perspective turned out to be insufficient when he engaged in the classroom-based project in an attempt to analyze students’ mathematical learning in the social contexts (Cobb, Wood, & Yackel, 1993). Cobb came to expand his exclusively psychological constructivist perspective by referring to symbolic interactionism and ethnomethodology, introduced to him through collaboration with German scholars who already had applied symbolic interactionist theory to mathematics education so as to emphasize the interactive, social nature of mathematics learning in a classroom community (e.g., Bauersfeld, 1995; Bauersfeld, Krummheuer, & Voigt, 1988; Voigt, 1985, 1995, 1998).

Cobb and his colleagues contend that psychological (or constructivist) and sociological (or interactional) aspects of mathematical activity need to be coordinated, because of their complementarity in explaining students’ learning processes (Cobb & Bowers, 1999; Cobb, 1994; Cobb & Bauersfeld, 1995; Cobb & Yackel, 1996). In their view, constructivist perspectives do not fully explain the nature of social aspects of learning. Conversely, interactionist perspectives do not fully describe the individual student’s mathematical understandings. This recognition led them to claim that analyses of an individual student’s conceptualization should go together with analyses of his or her engagement in the classroom microculture through social interactions and discourse.
It should be emphasized that they do not subordinate the individual aspects of mathematical learning to the collective aspects, or vice versa (O'Connor, 1998).

This coordinated constructivist and interactionist approach, called the *emergent* perspective (a variety of social constructivism), regards mathematical learning as a process of both active individual construction and enculturation into the mathematical practices (Cobb, 1994). Proposing an interpretive framework for analyzing individual and collective activity at the classroom level, Cobb and Yackel (1996) elaborated their coordinated approach to the two distinct theoretical perspectives. From the social perspective, they suggest three constructs of the classroom microculture: (a) classroom social norms, (b) sociomathematical norms, and (c) classroom mathematical practices. From the psychological perspective, they identify the individual correlates of these social constructs: (a) beliefs about own role, others’ roles, and the general nature of mathematical activity in school, (b) mathematical beliefs and values, and (c) mathematical conceptions and activity. Within this coordinated perspective, Cobb and his colleagues emphasize that mathematical meanings *emerge* in the process of negotiation of social norms (including sociomathematical norms) through social interaction: “The mathematical meanings and practices institutionalized in the classroom were not immutably decided in advance by the teacher, but, instead, emerged during the course of conversations characterized by ... a genuine commitment to communicate” (Cobb, Wood, & Yackel, 1993, p. 93).

Following is a concise description of symbolic interactionism and ethnomethodology, the socially oriented complements to psychological constructivism, as they are used to develop the emergent perspective. From the symbolic interactionist
perspective, Cobb and his colleagues focus on the interactive constitution of mathematical meanings. From the ethnomethodological perspective, they focus on reflexive relationships between the individual and the collective, including classroom interaction, classroom discourse, and classroom culture.

Symbolic Interactionism

Cobb and his colleagues sought to understand what students learn and the processes by which they learn while participating in classroom mathematical activities. They use the theory of symbolic interactionism in connecting the analyses of individual students’ learning with the processes by which the curriculum and the encompassing classroom culture are constituted.

Symbolic interactionism, as social psychology, focuses primarily on the nature of human social interaction (Herman & Reynolds, 1994; Meltzer, Petras, & Reynolds, 1975). As such, symbolic interactionism is based on three basic assumptions:

The first premise is that human beings act toward things on the basis of the meanings that the things have for them. .... The second premise is that the meaning of such things is derived from, or arises out of, the social interaction that one has with one’s fellows. The third premise is that these meanings are handled in, and modified through, an interpretative process used by the person in dealing with the things he encounters. (Blumer, 1969, p. 2)

Thus, symbolic interactionism concerns individuals’ sense-making processes and social processes. In particular, the process of negotiating meanings is seen to mediate individual cognition and the society or culture wherein the cognition is embedded.

Symbolic interactionism considers meanings as social products in that they are formed in and through people’s interactive activities (Blumer, 1969; Herman & Reynolds, 1994; Meltzer et. al, 1975; Voigt, 1994). These socially-created and socially-
shared meanings direct individuals' behavior. Individuals engaged in a joint activity tend to attend to the ways other participants, in particular more knowledgeable ones, interact with objects in the given context. This jointly produced interaction become the basis for the individuals to learn taken-for-granted knowledge. Individuals refer to these shared meanings when making their own interpretations in the process of interaction. In this perspective, norms and their interpretations become established and fortified through collective use, but require continued confirmation through individuals' activities.

Drawing on the symbolic interactionist tenet that the interaction between the individual and the context is always mediated by meanings that originate in social practices, Cobb and his colleagues focus on the analyses of the processes by which a teacher and students jointly constitute classroom-specific regularities such as social norms and sociomathematical norms, and re-negotiate them through ongoing interaction (Cobb & Bauersfeld, 1995; McClain & Cobb, 1997; Yackel & Cobb, 1996). Such analyses explain more direct influence of classroom interaction on the socialization towards mathematical ways of thinking, communicating, and appreciating than is true of psychological constructivism. Building on symbolic interactionism, Cobb and his colleagues regard learning as the interactive constitution of meaning in a classroom community. In the same vein, they regard social interaction as a process of mutual adaptation in which individuals continually negotiate meanings by modifying their original interpretations.

**Ethnomethodology**

Ethnomethodology focuses specifically on how members of a particular group create and understand their daily lives (Leiter, 1980; Mehan & Wood, 1975; Meltzer et
al., 1975). The focus is not on activity itself but rather on the process or method the individual members use to deal with their sense of social structure. Thus, ethnomethodology stresses that social order, including meaning, exists only with the members' accounting (Meltzer et al., 1975). Ethnomethodologists deal with micro-issues such as specifics of conversation or details of action in order to describe the basic and routine grounds of members' interpretations (Leiter, 1980).

The focus of ethnomethodology on individual members' understanding and interpretation fits well with Cobb and his colleagues' initial interest in how the individual student constructs his or her own mathematical meaning. As mentioned before, the classroom teaching project led the researchers to turn their purely constructivist approach toward sociological perspectives which account for the social nature of learning. While acknowledging this theoretical change, however, they attempt to keep individuals' interdependent roles in establishing classroom mathematical practices: "A practice such as inquiry mathematics is interactively constituted in the classroom and does not exist apart from the activities of the individuals who participated in its constitution" (Cobb, Wood, & Yackel, 1993, p. 100). This insistence seems to make them interested in ethnomethodological approaches that preserve the moment-by-moment construction of interpretations by each individual of community.

Cobb and his colleagues adopt the notion of reflexivity from ethnomethodology in order to explain the relationship between the individual and the collective. Reflexivity is a property of social relationships in which accounts and contexts mutually define or elaborate each other (Leiter, 1980). Two properties are reflexively related if the existence of each depends on the other. As such, reflexivity implies that neither an
individual student's mathematical activity nor classroom microculture can be adequately characterized without considering the other.

Cobb's (1995) case studies of second-grade students describe a reflexive relationship between the students' mathematical learning and the social relationships they established. On the one hand, the students' cognitive abilities enabled or constrained the possible patterns which their interaction can take. On the other hand, the relationships the students established determined the types of learning opportunities; that is, the social structure influenced the students' construction of mathematical ways of knowing. Similarly, Cobb and his colleagues postulate a reflexive relationship between the quality of sociomathematical norms and the social situation wherein the norms are developed (McClain & Cobb, 1997), between mathematical themes and individual contributions (Voigt, 1995), and between the individual's engagement in learning and argumentation (Krummheuer, 1995). These reflexive relations propose interdependence between students' acquisition of mathematical knowledge and the social interactions in which it occurs; and, more generally, between psychological perspectives and social perspectives.

PRACTICAL ORIGINS: INQUIRY MATH CLASSROOMS

As mentioned above, the development of social norms (including sociomathematical norms) is based on classroom teaching experiments extending over several years. In tracing this practical origins of the constructs, it should be emphasized that Cobb and his colleagues did not analyze the social norms as an end itself. Rather, their motivation was to account for students' mathematical development as it occurred in the social contexts of their particular project classrooms.
Social Norms

Collaborating with a second-grade teacher, Cobb and his colleagues started the classroom teaching experiment during the 1986-87 school year (see Cobb & Bauersfeld, 1995 for detailed explanation). They attempted to develop instructional settings that were compatible with the implications of constructivist perspectives. Within the first few days of observation, they found an unanticipated issue. The teacher, supported by the research team, encouraged her students to explain their solution methods. However, the students seemed not to present the actual thinking processes they used. Instead, they tended to infer what the teacher might expect, as carried over from their participation in traditional classrooms during their first-grade year. The conflict in the expectations between the teacher and the students led the teacher to initiate the process of negotiating classroom social norms, which are later articulated as obligations and expectations with regard to classroom participation.

Cobb and his colleagues characterize the project classroom as an “inquiry mathematics classroom,” wherein the teacher and students play a significant role as validators of explanations and justifications. Establishing such an inquiry mathematics classroom required the teacher to move toward new kinds of classroom social norms, including the conventions of how to collaborate with others and of how to react to an incorrect answer. The examples of social norms from the project classroom included “explaining and justifying solutions, attempting to make sense of explanations given by others, indicating agreement and disagreement, and questioning alternatives in situations in which a conflict in interpretations or solutions had become apparent” (Cobb & Yackel, 1996, p. 178). Providing a detailed account of how such norms were
renegotiated in a given classroom context (Cobb, Yackel, & Wood, 1989; Cobb et al., 1993), the researchers emphasize that social norms are jointly established by the teacher and students.

These social norms are from the classroom reality which is grounded in the tension between the individual and the collective. For successful interactive learning, as Cobb, Gravemeijer, Yackel, McClain, and Whitenack (1997) point out, the teacher needs to pay special attention to social norms that specifically deal with expectations and obligations for joint activity. In their emergent perspective, social contexts do more than merely influence students' learning since interactive processes are an inherent aspect of learning. At the same time, the individual's participation and contribution are significant for the development of beliefs about the general nature of mathematical activity as well as beliefs about one's own role and others' roles. In this respect, Cobb and Yackel (1996) take such individuals' beliefs to be the psychological correlates of general classroom social norms: "Classroom social norms are seen to evolve as students reorganize their beliefs, and, conversely, the reorganization of these beliefs is seen to be enabled and constrained by evolving social norms" (p. 178).

**Sociomathematical Norms**

As described above, Cobb and his colleagues initially focused on analyzing the interactive processes by which the teacher and students establish and re-negotiate general classroom social norms of explanation and justification. Given that they started the teaching experiment to understand how students learn mathematics and to develop instructional sequences for specific mathematical topics (e.g., place-value numeration), they became interested in ways by which the teacher and students jointly negotiate
mathematical meanings. This interest led them to extend their previous research on general social norms by sociomathematical norms, that are "normative aspects of mathematics discussions specific to students' mathematical activity" (Yackel & Cobb, 1996, p. 461).

Like social norms, sociomathematical norms emerged from the classroom reality. Initially the classroom teacher equally accepted all the students' contributions as part of her attempt to avoid asserting mathematical authority in the classroom. Though the teacher asked for different solution methods, students often repeated what others had already said. The students' ideas of what counted as a different solution varied and were different from the teacher's understanding. This made the classroom discussions unproductive. After discussing this problem with Cobb and his colleagues, the teacher undertook to guide the negotiation of the sociomathematical norm of difference:

As part of the process of guiding the development of an inquiry approach to mathematics in their classrooms, the teachers with whom we worked regularly asked the students if anyone had solved a task in a different way and then either sanctioned or implicitly delegitimized contributions that they did not consider to be mathematically different from those that had been given by other students. It was while analyzing classroom interactions of this type that sociomathematical norms first emerge as an explicit focus of interest for us. (Cobb & Yackel, 1996, pp. 178-179)

It is important to clarify that the teachers in the teaching experiment did not intend to assess their students' contributions: rather they attempted to encourage each student to participate in the classroom discourse and discuss each other's ideas. As Voigt (1995) point out, however, the teachers' indirect evaluation in terms of different reactions to the students' contributions led to the constitution of sociomathematical norms. In the same vein, McClain and Cobb (1997) describe the teacher's proactive role
in guiding the evolution of sociomathematical norms, including the norms of what
counts as mathematically sophisticated, easy, simple, or clear.

As exemplified above, Cobb and his colleagues emphasize that
sociomathematical norms are established by the process of implicit negotiation between
the teacher and the students in a given classroom. Neither the teacher nor the students
have necessarily decided in advance what would count as mathematically significant
difference. Rather, they clarified their sense of mathematical difference while interacting
with each other. The individuals' sense of mathematical values and beliefs contributed to
the process of establishing sociomathematical norms. Conversely, the participation in
the process enabled the individuals to reorganize personal ways of judging what makes a
mathematical contribution acceptable, justifiable, different, sophisticated, etc. Viewed in
this way, Cobb and Yackel (1996) take an individual's mathematical beliefs and values
as the psychological correlate of sociomathematical norms.

To reiterate, the constructs of general social norms and especially
sociomathematical norms originated from a practical attempt to establish an inquiry
elementary mathematics classroom wherein the teacher and the students negotiated such
norms in constituting a community reflecting the practice of mathematics. As O'Connor
(1998) pointed out, there was no preexisting category for those norms.

CURRENT STATUS OF THE THEORY OF SOCIOMATHEMATICAL NORMS

This section is organized into two parts. The first part examines the current
understanding of sociomathematical norms as it has evolved since its introduction. The
second part addresses the limited application of sociomathematical norms and opens an
inquiry toward the possibility of extending the construct with relation to the purpose of
this dissertation. A theoretical extension is discussed in Chapter 6 building on the
analysis of reform-oriented mathematics classrooms in Korea (Chapter 2) and in the
U.S. (Chapter 5) as an attempt to articulate the problems and issues of reform.

**SOCIOMATHEMATICAL NORMS AND MATHEMATICAL DISPOSITIONS**

As noted above, sociomathematical norms are highlighted as reflexively related
to an individual’s sense of mathematical values and beliefs, and more broadly their
mathematical dispositions. As such, Yackel and Cobb (1996) stressed
sociomathematical norms as a significant construct in accounting for how students
develop intellectual autonomy in mathematics as they participate in classroom activity.
As indicated in their theoretical orientation, the researchers initially regarded autonomy
from a psychological perspective as a context-free characteristic of an individual
student’s activity. However, the analysis of sociomathematical norms led them to
redefine autonomy with respect to “students’ participation in the practices of the
classroom community” (Yackel & Cobb, 1996, p. 473). In their project classrooms,
students became increasingly autonomous as they contributed to the process of
negotiating sociomathematical norms. In this respect, the teacher is expected to foster
students’ development of intellectual autonomy by explicitly attending to classroom
social and sociomathematical norms (Rasmussen & Yackel, 1999).

Given the relationship between sociomathematical norms and students’
development of intellectual autonomy, Rasmussen and King (1998) compared students’
performance and autonomy in an inquiry-based calculus classroom with those in a
traditional classroom. Whereas there was no significant difference with regard to the
students’ performance on a final exam, there was striking difference in their level of
autonomy. Students from the inquiry-based class used multiple mathematical ways to argue for and justify their answers, whereas their counterparts mainly checked their calculations and waited for the interviewer's confirmation. The researchers attributed this difference to the different norms that had been established in each classroom as to what counts as an acceptable mathematical justification and what counts as a different mathematical strategy.

While the construct of sociomathematical norms originated from developmental research at the elementary school level, subsequent studies have extended the application of the construct across content and grade levels (Bowers et al., 1999; Cobb, 1999; Cobb et al., 1997; King, 1999; Rasmussen & King, 1998; Rasmussen & Yackel, 1999; Stephan, 1998). The mathematical topics have included number and operation, algebra, measurement, data analysis, calculus, and differential equations. Concomitantly, the grade levels have varied throughout elementary, secondary, and college classrooms. This extension reveals the usefulness of the sociomathematical norms (and social norms) construct for analyzing mathematics instruction, in particular inquiry-based or reform-oriented classroom communities at all levels.

Given the typical classroom reality grounded in the tradition of school mathematics where the teacher demonstrates and the students copy, Cobb and his colleagues imply that teachers can change their classroom mathematical culture through explicit attention to sociomathematical norms. This implication is practically promising, as evidenced by the studies that showed students' dramatic development of mathematical dispositions, autonomy, and increasingly sophisticated knowledge when
compared with their peers' development in a typical mathematics classroom (Cobb et al., 1997; Rasmussen & King, 1998).

**UTILITY OF SOCIOMATHEMATICAL NORMS**

Sociomathematical norms are of great significance in that they are a collective accomplishment and at the same time bear an interdependent relationship to individual students' mathematical ways of knowing and communicating. Nevertheless, the previous review of sociomathematical norms reveals rather limited usage of the norms. Sociomathematical norms (and also social norms) tend to be documented, relatively briefly, as a way to set the *stage* for the analysis of students' collective mathematical development. This does not mean that social and sociomathematical norms are concerned with social aspects of a classroom community, whereas classroom mathematical practices are concerned with mathematical aspects (Bowers et al., 1999). The three constructs reflect different aspects of the classroom microculture and thus are social through and through. However, the trend in previous studies is that the use of sociomathematical norms remains mainly as background information relative to a more central concern for mathematical practices. The frequent examples of sociomathematical norms include the general classroom standards as to what counts as a different, clear, efficient, easy, or acceptable mathematical explanation. This contrasts with "The mathematical practices established by the classroom community [which] can be seen to constitute the immediate, local situations of the students' development" (Cobb & Yackel, 1996, p. 180).

In tracing the origins of sociomathematical norms both theoretically and practically, the distinction between general social norms and sociomathematical norms
was reviewed. As implied in the terms, social norms are applicable to any subject area, while sociomathematical norms and classroom mathematical practices are concerned specifically with mathematics. In order to better understand the current status and usage of sociomathematical norms construct, it is necessary to review what classroom mathematical practices are and how they differ from sociomathematical norms.

Sociomathematical norms are concerned with general criteria for judging what constitutes mathematically acceptable, justifiable, different, sophisticated, etc, explanations. As Cobb (1999) explains, sociomathematical norms cut across mathematical topics. In his example, the criterion of what counts as a mathematically clear explanation could apply to elementary calculation problems or to discussions about relatively sophisticated mathematical ideas. In contrast, classroom mathematical practices involve the specific mathematical content of the discussion and activity (Bowers et al., 1999; Cobb, 1999; Stephan, 1998). As such, classroom mathematical practices are intended to describe the collective learning of a specific mathematical idea established in a classroom community:

Because sociomathematical norms are concerned with the evolving criteria for mathematical activity and discourse, they are not specific to any particular mathematical idea. ... Classroom mathematical practices, in contrast, focus on the taken-as-shared ways of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas. Consequently, if sociomathematical norms are specific to mathematical activity, then mathematical practices are specific to particular mathematical ideas. (Cobb, 1999, pp. 9-10)

Cobb and his colleagues presented detailed analyses of students' collective mathematical learning in a classroom community over a prolonged period of time. For instance, Bowers et al. (1999) document both the evolution of mathematical practices of
a third-grade classroom and the development of individual students' understandings of place value conceptions as they participate in those practices. The practices evolved from counting arrangements of objects by 100s, 10s, and 1s to adding and subtracting by acting symbolically on numerical quantities rather than by depending on the imagery of specific actions. Similarly, Cobb et al. (1997) documented first-graders' matematizing and symbolizing of number concepts and operations, and Stephan (1998) analyzed evolving conceptions of measurement from another first-grade classroom. Cobb (1999) also documented seventh-grade students' mathematical reasoning on statistical data as they participated in specific classroom mathematical practices.

The notion of classroom mathematical practices was crucial for Cobb and his colleagues, both practically and theoretically. First, their classroom-based research was developmental research in which they designed instructional sequences in order to support students' mathematical learning of particular topics. As Cobb (1999) put it, "Our motivation for teasing out a third aspect of the classroom microculture, classroom mathematical practices, stems directly from our concerns as instructional designers" (p. 9). The analysis of students' collective mathematical development was necessary to compare the conjectured learning trajectories they developed in designing instructional sequences with the actual learning trajectory and, consequently, to inform further design efforts (Bowers et al., 1999; Cobb et al., 1997).

Second, their primary motivation in conducting classroom-based research was to explore ways of interpreting students' mathematical learning in the social context. In their theoretical framework, sociomathematical norms are reflexively related to mathematical disposition, whereas classroom mathematical practices are reflexively
related to the individual students' ways of interpreting and understanding specific mathematical conceptions and activities (Cobb & Yackel, 1996). The greater stress on mathematical practices than on sociomathematical norms seems to be supported by a traditional emphasis on mathematical content over disposition. Moreover, Cobb's longstanding constructivist interest is "content-oriented" rather than "thought-oriented" (Cobb & Steffe, 1983, p. 87). The foremost interest of Cobb and his colleagues as social constructivists may be the particular mathematical concepts or ideas students will learn, rather than to the characteristically mathematical thinking and communicating per se.

This review suggests the rather limited use of the sociomathematical norms construct stems from its particular trajectory of development. This dissertation explores the possibility of understanding sociomathematical norms as explicating the general quality of students' engagement in collective mathematical practices of a classroom community. In the same vein, this dissertation seeks for the relationship between sociomathematical norms and students' development regarding specific mathematical content as well as mathematical disposition, as called for in the reform documents.

It should be emphasized that the sociomathematical norms construct serves a specific purpose in this dissertation — to explore what have been problems in implementing the current reform ideals and what might be solutions for solving the problems. The problems of implementing reform include teachers' frequent concern with changing the surface social structure of their classrooms, without monitoring the quality of students' mathematical incorporation into the social fabric. Sociomathematical norms are intended to capture the social aspects of the classroom that are implicated in fostering mathematical ways of knowing and participating. As such, the notion of
sociomathematical norms seems to provide a promising way to look closely at reform-oriented mathematics instruction and the mutual relation between the instruction and students' collective development with regard to both mathematical dispositions and conceptions. Within this dissertation project I pursue the possibility that the breakdown between teachers' adoption of reform objectives, and their successful incorporation of reform ideals implicates the sociomathematical norms that become established in their classrooms.
CHAPTER 4

METHODOLOGY AND PROCEDURES

This chapter is organized into four sections. The first section summarizes general objectives of this dissertation in line with explicit research questions. The second section outlines case study research employed for this dissertation project with rationales and cautionary remarks. The third section describes how data were collected. The final section explains how the data were analyzed.

RESEARCH QUESTIONS

The purpose of this study was to explore the problems and issues of reform by comparing and contrasting the classroom social norms of two U.S. second grade teachers who aspired to implement student-centered teaching approaches. The two classrooms were selected because of their unequal success in transforming the current reform ideals. It should be emphasized that the comparison and contrast between more successful and less successful reform classes could provide a unique opportunity to reflect on the subtle but important issues of implementing reform. The following questions define the purposes of this dissertation project:

1. What are the processes that constitute more successful and less successful student-centered pedagogy in the U.S. elementary mathematics classroom? Specifically, in what ways do the teacher and students create such mathematics classrooms? What learning opportunities arise for the students in these classrooms?

2. What are the differences and similarities between more successful and less successful student-centered classrooms, in particular with regard to social norms
and sociomathematical norms? What are the challenges for reformers in changing the culture of primary level mathematics teaching?

RESEARCH METHODOLOGY

CASE STUDY RESEARCH

This study was an exploratory, qualitative, comparative case study (Yin, 1994) using the *grounded theory approach* based on the *constant comparative analysis* (Glaser & Strauss, 1967; Strauss & Corbin, 1990, 1998) for which the primary data sources were classroom video recordings and transcripts (Cobb & Whitenack, 1996). The grounded theory methodology emphasizes that the development of theoretical constructs should be rooted in the data. A central feature of this method is to compare and to contrast preliminary inferences with new incidents in subsequent data in order to determine if the initial conjectures are sustained throughout the data set. The process of comparisons and contrasts makes provisional inferences stable or elevates them to become powerful explanatory constructs. This process, as Cobb and Whitenack (1996) put it, could be described as “a zigzag between conjectures and refutations” (p. 224).

Case studies are an especially useful mode of inquiry when “the boundaries between phenomenon and context are not clearly evident” (Yin, 1994, p. 13). This is particularly apt for teaching practices, which can be broadly classified using terms like “student-centered” or “teacher-centered” but which only achieve their full definition within the context of particular classroom microculture.

Case studies are also suitable in presenting fundamental information about areas of education where little research has been conducted (Merriam, 1988). It is apparent that there have been many studies of U.S. mathematics teaching practices in
international contexts and within the country. For instance, Stigler, Fernandez and Yoshida (1996) compared U.S. fifth-grade mathematics classes with Japanese counterparts with regard to in what ways students' thinking played a role during instruction. Yang and Cobb (1995) included an analysis of social interaction patterns of U.S. and Chinese classrooms in order to explore sociocultural influences on the development of place-value concepts. The Survey of Mathematics and Science Opportunities investigated mathematics teaching in six countries including the U.S., based on videotapes of 120 classes (Schmidt, Jorde, Cogan, Barrier, Bonzalo, Moser, et al., 1996). The TIMSS videotape study included a relatively large scale investigation of the social norms of typical U.S. middle school classrooms in comparison with those of Japanese and Germany classrooms (TIMSS, 1996). However, these studies in international contexts set out to capture typical schools, not reform classrooms.

Ball (1993) and Lampert (1990) reported about the general social norms in their own mathematics classroom wherein they tried to implement reform recommendations. However, they are not regular mathematics teachers but researchers. There have been many studies that observed the mathematics instruction of classroom teachers as they participated in various teacher development programs (see Fennema & Nelson, 1997). Given that the programs are generally geared at helping teachers transform their mathematics instruction toward reform recommendations, those studies enabled us to explore an important issue of how teacher change occurs. However, the studies were primarily concerned with identifying the effect of such professional programs on the teacher change and thus the classroom observation was a kind of follow-up study. The
focus was not on how teachers grapple with their own values and priorities relative to
the reform ideals in their ongoing teaching career.

Cobb and his colleagues analyzed the social norms and sociomathematical norms
of reform mathematics classrooms (Bowers et al., 1999; Cobb & Bauersfeld, 1995;
Cobb et al., 1997; McClain & Cobb, 1997; Yackel & Cobb, 1996). However, the
classrooms were special mathematics classrooms in that the teachers extensively
collaborated with the researchers whose purpose was to develop instructional sequences
in mathematics and to explore how students learn mathematics as it occurred in the
social context of the classrooms. Consequently, the researchers played a significant role
in influencing the teachers’ instructional methods and, thus, in constituting such social
norms of the classrooms (Cobb, Stephan, McClain, & Gravemeijer, in press).
Naturalistic studies of social and sociomathematical norms in classrooms that are
unsupported by researchers have not yet been undertaken.

This dissertation sought to explore social norms and students’ learning
opportunities in such regular classroom settings. Moreover, this study compared and
contrasted a more successful reform mathematics classroom with a less successful
classroom. This enabled us to explore where and in what ways reform efforts may break
down in the implementation. In this respect, this case study makes new contributions in
informing the reform efforts.

Case studies are particularly useful if the researcher is interested in process
rather than outcome (Merriam, 1998; Yin, 1994). As indicated in the research questions,
one of the purposes of this dissertation was to describe the processes by which a teacher
and students constitute a reform classroom in mathematics. Detailed descriptive
accounts are provided of the teacher's approaches to establishing a student-centered classroom microculture, and the students' approaches to participating in such a mathematics classroom. The focus on the classroom process was necessary for this dissertation to explore how teachers understand current reform ideals and transform them into specific classroom contexts. Where reform breaks down is likely to be investigated only by this intensive focus on classroom process.

CAUTIONARY NOTES

Distinction from Ethnographic Research

This study involved field work, and shares similarities with ethnographic research. However, Yin (1993) differentiates ethnographic research from case study in that the former does not necessarily start with strong theoretical commitments. Ethnographic research stresses carrying out the field work in a reasonably unstructured way, in order that the regularities of what is being studied naturally can surface. But a case study does not seek to establish its research perspectives from (multiple) field-observations. Rather, critical to case studies is the development of theory prior to the conduct of research (Hamel, Dufour, & Fortin, 1993; Stake, 1995; Yin, 1993). Theory in case studies plays a crucial role in helping to "select the cases to be studied in the first place, whether following a single-case or multiple-case (replication) design, [and to] specify what is being explored when you are doing exploratory case studies" (Yin, 1993, p. 4).

In keeping with these strictures, case selection in this research study was theoretically informed. The unit of analysis in this dissertation was classroom and thus the case was a classroom. Since the main purpose of this study was to inform the
implementation of reform recommendations, classrooms had to be selected wherein the student-centered pedagogy was being attempted.

Constructivism and socioculturalism have been the main theoretical impetuses for the current mathematics education reform movement (NCTM, 1989, 1991, 1995, 2000). Both perspectives support student-centered pedagogy but from different points of view. Constructivist based research is relevant to a student-centered approach because it provides models of students' conceptual understandings that can inform teachers' attempts to create cognitive disequilibrium for students resulting in the evolution of more mathematically powerful knowledge structures (Simon, 1995; Simon & Schifter, 1991; von Glasersfeld, 1984). Sociocultural theory is relevant to student-centered teaching, in that it illuminates the nature and effects of students' participation in socially situated activities (e.g., Forman, 1996; Forman, Minick, & Stone, 1993). The emergent perspective of Cobb and his colleagues is an important theoretical tool to use in conjunction with constructivism and sociocultural theory, in that it helps bridge the dualism suggested by the others (Cobb, 1994). Together these theoretical foci helped me make preliminary judgements to guide the selection of classrooms.

According to Yin (1993), theory also plays an important role in deciding what should be explored, particularly in an exploratory case study. This study started with a theoretical initiative of sociomathematical norms as a guideline to observe mathematics classrooms. The established concern with sociomathematical norms enabled me to clarify what should be the focus in analyzing classroom activity and discourse. Despite interactive constitution of sociomathematical norms by all members of the classroom community, the teacher plays a special, proactive role in initiating and guiding the
development of the norms (McClain, 1995). Thus the teacher’s understanding of mathematical beliefs and values, as well as her conceptions of mathematical teaching, had to be integrated into the analyses of teaching practices. These prior considerations dictated that extensive teacher interviews should be undertaken as part of this study. This kind of theory-based method ultimately produces case studies that can be part of a cumulative body of knowledge rather than just isolated empirical inquiries (Yin, 1994).

Another difference between this study and ethnographic studies, is the comparative aspect. I sought not only to describe social norms and learning opportunities in each classroom setting, but also to explore similarities and differences in the sociomathematical norms, which in turn constituted an empirical basis in extending the current understanding of sociomathematical norms. This purpose departs from an ethnographic case study wherein an intensive, holistic description and analysis of a single case is a main focus, in conjunction with the effects of the cultural context (Merriam, 1988; Stake, 1998; Yin, 1994).

**Description of Sociomathematical Norms**

A second caution is related to how to document sociomathematical norms. Cobb and his colleagues have described the developmental process by which sociomathematical norms evolved (or co-evolved) in a project mathematics classroom throughout a semester or a year (Bowers et al., 1999; Cobb et al., 1997; McClain & Cobb, 1997). My limited resources made it impossible to conduct longitudinal studies dealing with the evolutionary process of sociomathematical norms in each classroom. Given the realistically limited resources, this dissertation rather attempted to generate issues that might be informative in implementing reform ideas by comparing unequally...
successful student-centered classrooms in terms of their sociomathematical norms.

Building on such an empirical basis, this dissertation also intended to extend the theory of sociomathematical norms. Thus, this study didn’t seek to trace the development of sociomathematical norms, but to identify the norms as they become established in each classroom. Cobb and his colleagues have observed that the sociomathematical norms in their project classrooms were relatively stable after the first semester of the school year (Cobb et al., 1997). Based on this observation, the data sources of this dissertation were collected in the second semester: Spring in the U.S. and Fall in Korea.

**Scope of the Study**

As mentioned in the previous chapters, this dissertation was nested in a larger, crossnational research project and used U.S. and Korean data. Thus, the scope of this dissertation as well as my own roles and responsibilities should be clearly understood. My role in the Korean data collection and analysis was as an apprentice. I had input into all decisions, but not the final say. The U.S. data were my major focus of concern. In this portion, I had primary responsibility for selecting the observation classes and analyzing the data. It is for this reason that I have chosen to incorporate the results of the Korean analysis in Chapter 2, the review of the literature, so that these results could be part of the dissertation, but without the implication of authorship that would come with later placement. My experience as an apprentice in completing the Korean portion of study had significant impact on the process of collecting and analyzing U.S. data. Among other things, the Korean study shed light on the possibility of extending sociomathematical norms with regard to their definition and applications, in particular with respect to assessing the quality of a reform-oriented mathematics class. For the
purpose of this theoretical initiative, this dissertation included a joint analysis encompassing U.S. and Korean data in Chapter 6, as well as a main analysis of U.S. data in Chapter 5.

It should be noted that intercultural, systematic comparisons of Korean and the U.S. reform remain part of the larger study that is not undertaken here. As reflected in current scholarship, such international judgements are best shared by national representatives of the participating countries (Easley & Taylor, 1990; Schmidt, Jorde, et al, 1996; Stevenson & Stigler, 1992; Stigler, Fernandez, & Yoshida, 1996; TIMSS, 1996). Therefore, these portions of future study will necessarily reflect a more equal collaboration between members of the full research group. This further analysis will have the advantage of employing the extended theoretical perspective on sociomathematical norms developed here.

Viewed in this way, the whole collaborative project is intended to pave the way for further large crossnational studies involving Korea and the U.S. As Schmidt, Jorde, et al. (1996) observed, large scale crossnational research can benefit greatly from previous exploratory research in which the international team has the opportunity to identify, dispute, and achieve consensus on the key issues of the investigation. The new hypotheses uncovered by this exploratory study also would have the potential to inform the theoretical bases for mathematics learning as currently articulated in the literature.

TRUSTWORTHINESS AND CREDIBILITY

An issue which remains to be addressed concerns the validity and reliability of this research. Some qualitative researchers are reluctant to use such terms which have been overspecified in a positivistic, psychometric, and quantitative paradigm. Doing so,
they fear, denigrates central foci of qualitative studies like the understanding of human experience (e.g., Barone, 1992; Eisner, 1992; Janesick, 1998; Wolcott, 1990). In particular, Wolcott (1990) provocatively discusses the absurdity of validity in qualitative research by discussing his life experience wherein validity cannot serve for understanding. Terms such as trustworthiness, credibility, transferability, dependability, and confirmability replace the common quantitative criteria of validity, internal validity, external validity, reliability, and objectivity (Denzin & Lincoln, 1998; Tashakkori & Teddlie, 1998). Qualitative research is inquiry for meaning and understanding. In this context, validity and reliability must be interpreted as probing whether or not such inquiry is trustworthy and credible.

There are some methodological techniques commonly recommended to increase credibility and trustworthiness. A review of such techniques is given in conjunction with how they were implemented in this study. One of the common ways in reporting qualitative research is to present episodes which support general claims (Atkinson, Delamont, & Hammersley, 1988; Bogdan & Biklen, 1998; Glaser & Strauss, 1967). As Cobb and Whitenack (1996) point out, however, a difficulty arises because the interpretation of such episodes often requires the understanding of the rest of classroom data set. They found that understanding specific episodes and making general claims proved to be reflexive in that each depended intrinsically on the other (Leiter, 1980). This mutual dependence in the process of analysis reminded me of the reflexive relationship between the teachers' teaching practices and the classroom contexts wherein they are implemented.
Prolonged engagement is another technique for increasing credibility in conjunction with persistent observation (Lincoln & Guba, 1985; Merriam, 1998). This increases the likelihood that the phenomena being observed are characteristic rather than ideosyncratic. This exploratory case study is based on a limited period of engagement and seven observations of each of two classrooms. In order to capture the characteristics of the classrooms within the constraints, I informed the two teachers that the purpose of this study was to understand the process of mathematical teaching and learning in the U.S. context, not to evaluate their own teaching practices. Before collecting actual data from each classroom, I had several chances to talk with the two teachers and I strongly emphasized my interest in their normal teaching practices. The followup interviews with the teachers indicated that they taught mathematics in their usual manner, without attempting to demonstrate exemplary teaching methods. Of course, this does not compensate fully for the benefits of prolonged observation. However, as discussed above, this is an exploratory case study which permits a greater role for an a priori theoretical orientation, and consequently allows for more limited engagement and observation (Yin, 1993, 1994).

Another technique for increasing credibility is to triangulate key observations and bases for interpretation (Denzin, 1978; Lincoln & Guba, 1985; Patton, 1990; Stake, 1998; Yin, 1994). Among the various triangulation methods, data triangulation refers to comparing and cross-checking the consistency of findings through multiple sources of evidence. The data sources in this study included classroom observations, teacher interviews, and documentary evidence. These different data sources help corroborate the findings concerning classroom teaching practices. For instance, in-depth interviews
with the teachers served as an important tool for validating my interpretations of their instructional approaches.

Analyzing negative cases in conjunction with alternative explanations is another way to enhance the extent to which reconstructions of investigators reflect the original multiple realities (Bogdan & Biklen, 1998; Eisner, 1991; Lincoln & Guba, 1985; Patton, 1990; Stake, 1998). Constant comparative analysis, a main analytic technique used in this study, helped increase credibility because the method requires careful consideration of the extent to which alternative interpretations could be supported from the whole data set (Glaser & Strauss, 1967; Strauss & Corbin, 1990, 1998). Finding the most supportable interpretations of teaching practices included dealing with deviant cases of such interpretations. As well, the back-and-forth interplay with data made grounded theories traceable to the data that gave rise to them. As Cobb and Whitenack (1996) point out, the assertions resulting from this methodology could be justified by backtracking through various phases of analyses, including initial conjectures and even video recordings and transcripts as needed.

DATA COLLECTION

SELECTION OF CLASSROOMS

Making a proper selection of cases (i.e., classrooms in this study) is one of the most important factors in producing a satisfactory qualitative case study (Stake, 1998; Yin, 1994). This was especially true for my study wherein subsequent levels of analysis were conducted on the basis of understanding the individual cases. Whereas in Korea four researchers observed each classroom together and discussed its suitability before
final decisions, slightly different procedures had to be developed because I was the person who finally decided the classrooms to be observed for this study.

Using a kind of purposeful sampling (Patton, 1990), I tried to find promising primary mathematics teachers for this dissertation. I contacted university professors and instructors, elementary mathematics specialists at East Baton Rouge Parish School Board [EBRPSB], a former teacher who worked in the Louisiana State Department of Education, Louisiana Systematic Initiative Project mathematics coordinator, and a recently retired K-12 math supervisor in EBRPSB. These people recommended a total of 17 second grade classrooms that aspired toward student-centered instruction methods in mathematics. I observed the mathematics lessons of these recommended teachers between mid-February and mid-March in 1999 up to three times for some classes in order to ensure the suitability for this study. My advisor sometimes observed the classes with me, and I solicited his input as to whether the classes were displaying student-centered methods. Whenever I observed a class, I wrote field notes which included the general flow of the class activity and brief descriptions of tasks, discourse, learning environment, and evaluation — the four elements of teaching practices emphasized in Professional Standards for Teaching Mathematics (NCTM, 1991). In each class, I focused on whether the students' ideas constituted the foci of classroom discourse and activity. Building primarily on these preliminary observations in line with the general recommendations of Professional Standards (NCTM, 1991), I developed a rubric (Table 4.1) to compare and contrast the teachers' approaches, and to consider their suitability for this study. Note that I started classroom observation with general ideas without rigidly preset criteria as to what constitutes more successful and less
successful reform teaching. As noted before, teaching practices can be broadly classified using terms like "student-centered" or "teacher-centered" but only achieve their full definition within the context of particular classroom contexts.

Table 4.1 Rubric of Student-centered Instruction in Preliminary Observation

<table>
<thead>
<tr>
<th>Criterion 1: Students' participation and ideas are solicited*</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1 Promote communications about mathematics; Ask students to explain and justify their ideas irrespective of the correctness of students' answers</td>
</tr>
<tr>
<td>1.2 Call for individual/collective problem solving and mathematical reasoning</td>
</tr>
<tr>
<td>1.3 Monitor students' participation and encourage each student to participate</td>
</tr>
<tr>
<td>1.4 Establish an open and permissive atmosphere; respect/value students' ideas</td>
</tr>
<tr>
<td>1.5 Provide mathematical tasks that may engage students' conceptual sensitivity</td>
</tr>
<tr>
<td>1.6 Use materials/tools in ways to facilitate students' thoughtful engagement</td>
</tr>
<tr>
<td>1.7 Organize students into small groups for the purpose of exchanging ideas or reaching consensus</td>
</tr>
<tr>
<td>1.8 Draw on students' various background experiences and dispositions</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Criterion 2: Students' ideas become the center of mathematical discussion and activity (The degree by which the teacher uses students' explorations in the path of classroom discourse)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1 Listen carefully to students' mathematical ideas</td>
</tr>
<tr>
<td>2.2 Pose questions that further challenge and extend students' thinking after eliciting it</td>
</tr>
<tr>
<td>2.3 Filter students' multiple ideas to pursue mathematically significant ones in depth</td>
</tr>
<tr>
<td>2.4 Connect students' ideas with mathematical concept, notation, beliefs, values, etc.</td>
</tr>
</tbody>
</table>

*Note: Anything that gets students to reflect on and then express their ideas supports this criterion. 1.1 is the main idea. Given that different teachers may use different techniques to achieve student participation, 1.5-1.8 are rather examples.

I then looked over my field notes and evaluated the teachers' approaches according to each criterion in the rubric as it applied. It was apparent from this evaluation that many teachers did not meet even the most fundamental criterion of

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student-centered instruction (item 1.1 in Table 4.1). These mis-suggestions seem to reflect the prevalent misunderstanding of current reform recommendations.

The comparison and contrast of those teachers observed led me to identify one teacher, Ms. M from school UM, who stood out as the most successful in implementing student-centered teaching, and three possible candidates for a less successful teacher with their different strengths and weaknesses. I chose one of these teachers, Ms. E from school UE, because of her interest in my observations, among other local considerations. It is worthwhile to emphasize that the two teachers were selected because of their unequal success in the extent to which students’ participation and ideas were solicited and became the foci of mathematical discussion and activity. Whereas Ms. E was concerned with students’ participation more than their specific ideas, Ms. M was very concerned with soliciting students’ ideas and bringing them forward as the main topic of whole class discussion. In other words, whereas Ms. E attempted to institute student-centered instruction in ways that were compatible with current mathematics education reform initiatives, Ms. M actually succeeded in displaying a student-centered approach. When I informed them that I would like to observe more mathematics classrooms with video-recordings, both teachers willingly agreed and even volunteered informal accounts of how they generally taught mathematics, including information of classroom organization or lesson structure.

**VIDEOTAPELED MATHEMATICS CLASSROOMS**

After selecting the two classes, I obtained written permissions to conduct this study from the teachers, the students’ parents, and the principals. It was good fortune that the team of researchers was able to gain a clear and convergent picture of what a
more and a less successful student-centered mathematics class in Korea looks like from just two videotaped lessons of each class. For this study, I observed seven mathematics lessons of each class during April in 1999. I observed consecutive mathematics lessons except for occasional disruptions arising from specific situations in the classrooms (e.g., student-teacher's teaching week). When I started collecting classroom data, I had in mind that it would be safer to obtain more data from each of the two classes than we did from Korea in order to increase the likelihood of obtaining clear pictures of the two U.S. classroom microcultures. When I had videotaped a few mathematics lessons of each class, it was apparent that Ms. M displayed a very consistent lesson structure, whereas Ms. E used various forms of instruction. This led me to observe more lessons. With further videotaping I was able to identify consistent patterns in Ms. E's lessons. I felt confident that my data captured the general characteristics of mathematics instruction in each class.

In each class, two or three video cameras were used to capture the teaching practices from different perspectives. In the case of whole class instruction, one of the cameras captured the teacher and a second was used to survey the whole class. This was sufficient to capture students' questions, responses, and general participation. For small group work one of the cameras was trained on the teacher as she circulated among the groups. The other two cameras were used to capture small group interactions. Audiotapes were also deployed to provide a supplement to the small group video-recordings. It was apparent from my review of audiotapes and videotapes each evening that the third camera provided little new information to the data analysis, in particular as I moved on to the latter phase of data collection. Thus the third camera was not used.
later. From the Korean experience I found that the videotapes capturing the teacher were the most important in making transcripts. Thus I always videotaped the teacher, and my other volunteer camerapersons videotaped students or the whole class. A total of 33 videotapes and 13 audiotapes were collected for analyses of classroom teaching practices. I transcribed all video- and audio- taped lessons in a two-column format, the second column for notes related to the analysis.

Using obtrusive video cameras and audio recorders may cause problems in capturing the classroom teaching practices per se. As well there are ethical questions concerning anonymity. However, I tried to minimize the potential problems by letting the teacher and the students know about and agree with the purposes of the taping (Erickson, 1992).

INTERVIEWS WITH TEACHERS

Following initial reviews of the videotapes and audiotapes, the two teachers were extensively interviewed twice during May of 1999: Once to clarify some points that were unclear from the videotaped lessons; the other time to gain information on how they constructed their own teaching methods. Both interviews were audio-taped and transcribed. A total of 12 hours of interviews was taken with the two teachers. Both teachers were remarkably enthusiastic in sharing their various experience and development throughout teaching career. The more intensive second interview was intended to explore significant influences on the teachers’ conceptions of mathematics and its teaching in order to better understand their current practices. The interview probed 12 topic areas: (a) early influences on becoming a teacher, (b) the decision to become a teacher, (c) the teacher education years, (d) early mathematics interests, (e)
early teaching experiences, (f) career path, (g) influence of peers within the school, (h) influence of administrators, (i) professional development (e.g., further courses and degrees, conferences and workshops), (j) professional self-development, (k) mathematics teaching, and (l) educational policies (see the Appendix for interview questions). These topic areas and detailed questions within a topic had been selected on the basis of discussion among the research team members of possible significant factors influencing teaching practices in Korea. I slightly modified it to be used in U.S. contexts.

The planned question sequence was chronological in order to facilitate the teachers' reflection on the construction of their teaching approaches over time. However, the interview plan was modified by the teachers' responses, in order to pursue issues as they arose. The interview was intended to gain the teachers' perspectives on their teaching methods, rather than just to obtain factual information.

DATA ANALYSIS

Data analyses for this study had two stages: Individual analysis of each classroom setting, and comparative analysis of the two classes. For the purposes of this study, interview data were mainly analyzed in relation to classroom teaching practices. Specifically, details of the interviews were included in the analyses whenever they provided useful background information.

The analyses of the individual classes constituted the empirical portion of the study. Stake (1998) claims that case study should be based on the understanding of the case itself. In particular, he cautions against the possibility that the commitment to address an issue or to develop a theory is so strong as to hinder the researcher's
attention to the case. Thus, teaching practices were very carefully scrutinized in a bottom-up fashion. Next, the data from the individual classes were employed for comparisons between a more and a less successful reform instruction. This was done by focusing on the difficulties and successes of the two teachers, and formulating issues and obstacles that may point toward generic problems of reform in the U.S.

**INDIVIDUAL ANALYSIS OF EACH CLASSROOM**

In keeping with the research questions, individual classes were analyzed using the four categories and questions described in Table 4.2. In particular, the second interview inquired into the teacher’s approaches.

**Table 4.2 Interpretive Framework for Individual Analysis of Each Classroom**

<table>
<thead>
<tr>
<th>Category</th>
<th>Main Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classroom Flow</td>
<td>What is the general flow of classroom teaching practices, including classroom activities, gross patterns of interaction, the roles and expectations adopted by the teacher and students, and sociomathematical norms?</td>
</tr>
<tr>
<td>Teacher’s Approaches</td>
<td>What is the teacher’s curricular intention as reflected in her participation in and organization of the class? How are the teacher’s conceptions of mathematics and its teaching related to implemented teaching practices?</td>
</tr>
<tr>
<td>Students’ Approaches</td>
<td>What are the students’ learning intentions as reflected in their patterns of classroom participation?</td>
</tr>
<tr>
<td>Students’ Learning Opportunities*</td>
<td>Given the approaches of the teacher and the students, what kinds of mathematical learning opportunities are likely to arise for students’ concepts, connections, and tools?</td>
</tr>
</tbody>
</table>

*Note: Students’ Learning Opportunities are different from the more objectively characterized Opportunity To Learn (OTL) that includes percentage of time for arithmetic/math/science spent on method, division of lessons to subject matter components, other subject matter areas, etc. (Scheerens & Bosker, 1997, p. 111).

An initial analysis of Korean data had been based on four categories from Cobb and Whitenack (1996), which are “the children’s social relationship, mathematical
meanings, learning opportunities, and mathematical learning" (pp. 219-220). As emerged in the process of analyses, however, it was apparent that we had to modify such categories, partly because of our interest in teaching practices over small group mathematical activities. Keeping in mind the extensively interrelated data collection and analysis within the context of qualitative research (Patton, 1990; Strauss & Corbin, 1990, 1998), the more focused interpretive framework formed the starting point for this dissertation.

Figure 4.1. Relations among Four Categories in the Interpretive Framework

The four categories are closely interrelated. Classroom flow is specifically descriptive including important episodes. Both the teacher’s approaches and students’ approaches are based on the classroom flow in that each approach is inferred from it, and thus consistently grounded in the classroom activities. In other words, the teacher’s approaches and students’ approaches are taken to be the most supportable inferences from the observations reported in the classroom flow. Retrospectively, classroom flow can be consistently explained by the students’ and teacher’s approaches. Finally, the students’
learning opportunities come out of the teacher's approaches and students' approaches. These relationships are summarized in Figure 4.1.

**COMPARATIVE ANALYSIS BETWEEN CLASSROOMS**

The comparative analysis between the two classes had two parts. In the first part, the two classes were compared and contrasted by the four categories of the interpretive framework for this study, and by social and sociomathematical norms. In the second part, the factors influencing the two teachers' instructional goals were compared and contrasted in order to explore how more successful and less successful teaching practices have been constructed.

**Comparison of Teaching Practices**

The teaching practices were compared and contrasted in two ways. First, the differences and similarities between the two classes were analyzed with regard to each category in the interpretive framework (classroom flow, teacher's approaches, students' approaches, and students' learning opportunities). Second, the classes were compared according to the general social norms and the sociomathematical norms of the two classrooms. The general social norms concern the classroom participation structures, whereas sociomathematical norms concern the collective engagement patterns specific to mathematical activity and discourse. The discussion of sociomathematical norms informed the dynamics of implementing reform recommendations in each classroom by focusing on how the teacher and the students struggled together to make sense of their mathematical activities.

The relationship between sociomathematical norms and students' learning opportunities was examined in this part of the analysis. This was intended to explore the
possibility of promoting sociomathematical norms to be significant analytic tools through which we can understand different qualities of the classroom mathematical microculture.

Comparison of Factors Influencing Teachers’ Instructional Goals

The second part of comparative analysis was intended to explore how unequally successful mathematics practices are constructed in these two U.S. classrooms. I focused on identifying the underlying factors (including teacher characteristics, and sociocultural factors) that could account for the differences and the similarities in the implementation of unequally successful reform teaching practices. The portions of the interview data that characterized the teacher’s own reflections on the influences on her general teaching development were especially important as a source of insight here. This affords the possibility of exploring the challenges of reform for teachers and other personnel who are attempting to move teaching practices towards the student-centered ideals.

A review of the literature, coupled with an analysis of the Korean data collected, led to the development of a model of plausible factors which influence the teacher’s own development of instructional goals (see Figure 4.2). This model provided initial direction for the analysis in this part. Table 4.3 describes the main elements of each factor in the model. This is followed by a brief summary of the literature which informed the selection of factors. The factors in the model and their characterization were used, or modified, whenever they applied to the two teachers’ cases with regard to developing their own teaching methods. These emergent modifications and evolution signal the qualitative nature of this study (Janesick, 1998; Patton, 1990).
Figure 4.2 A Proposed Model of Factors Influencing Teacher's Instructional Goals

Table 4.3 Factors Influencing Teacher's Instructional Goals

<table>
<thead>
<tr>
<th>Locus</th>
<th>Factor</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>teacher learning and teaching experience</td>
<td>- mathematical experience as a student</td>
<td>- enjoyment of mathematics</td>
</tr>
<tr>
<td></td>
<td>- personally meaningful teaching models</td>
<td>- prior teaching experience</td>
</tr>
<tr>
<td>knowledge</td>
<td>- mathematical knowledge</td>
<td>- pedagogical content knowledge</td>
</tr>
<tr>
<td>beliefs</td>
<td>- beliefs of mathematics</td>
<td>- beliefs of mathematical teaching and learning</td>
</tr>
<tr>
<td>personality traits</td>
<td>- autonomy; risk-taking</td>
<td>- sensitivity to students’ experience and understanding</td>
</tr>
<tr>
<td>c cultural and educational norms</td>
<td>- normative teaching practices</td>
<td>- expectations of teaching outcomes</td>
</tr>
<tr>
<td>culture curriculum development and administration</td>
<td>- value of education</td>
<td>- structure of curriculum development</td>
</tr>
<tr>
<td>teacher education</td>
<td>- degree of teacher participation</td>
<td>- administrative directive and teacher’s degree of compliance</td>
</tr>
<tr>
<td>professional models and community</td>
<td>- pre-service teacher preparation program</td>
<td>- in-service education</td>
</tr>
<tr>
<td></td>
<td>- availability of alternative models</td>
<td>- teacher’s identification with community</td>
</tr>
</tbody>
</table>
Teacher Factors

Learning and teaching experience

A teacher’s own mathematical learning experience can significantly influence his or her beliefs about the nature of mathematics, and teaching goals. One’s personal history of past successes and failures in mathematics seems to play a major role in shaping teaching practices (Raymond, 1997; Smith, 1996). For example, Raymond (1997) described a teacher whose own dislike of mathematics as a student prompted a strong desire to make sure that her students enjoyed the subject. Reflection on one’s own teaching experiences can be one of the most influential factors in one’s further development as a teacher (Raymond, 1997).

Knowledge

A teacher’s knowledge of and about mathematics is an important element in shaping his or her teaching practices. Many studies demonstrate that teachers’ knowledge of specific mathematical concepts is often weak or even inadequate, raising a critical issue of how the teachers can facilitate students’ conceptual learning (Adams 1998; Babbitt & van Vactor, 1993; Ball, 1991; Even, 1993; Simon, 1993). In addition to understanding specific mathematical content, the quality of the teacher’s understanding of the general nature of mathematical knowledge also is relevant to the quality of a teacher’s classroom practices (Ball, 1988b).

There is increasing evidence that teacher’s pedagogical content knowledge, understanding of how students learn specific subject matter (Shulman, 1986), leads to fundamental changes in their beliefs and instructions in mathematics and is directly

**Beliefs**

Although there have been different perspectives in mathematics education with regard to the degree of consistency between teachers’ beliefs and instructional practices, the significance of teachers’ beliefs as a factor influencing their teaching practice is well established in the mathematics education literature (Battista, 1994; Ernst, 1998; Peterson, Fennema, Carpenter, & Loef, 1989; Franke, Fennema, & Carpenter, 1997; Thompson, 1984, 1992). In particular, Raymond (1997) reports that the teacher’s beliefs about the nature of mathematics are more powerful in explaining teaching practices than beliefs about teaching and learning mathematics.

**Personality traits**

Teacher’s personality traits are relevant to understanding changes (or not) in their teaching practice. Successful implementation of reform ideas requires willingness and ability to take risks such as turning over some classroom authority to students and more generally accepting the uncertainty of instructional change (Lappan, 1997; Nelson, 1997). Pre-service teachers who even have nontraditional beliefs about mathematical teaching often tend to implement more traditional pedagogy, when faced with the various constraints of actual classroom teaching (Brown & Borko, 1992; Eisenhart, Borko, Underhill, Brown, Jones, & Agard, 1993).

Teacher autonomy is another critical element in implementing reform ideas (Castle & Aichele, 1994; Cooney & Shealy, 1997). When teachers become autonomous, they will evaluate instructional materials and practices with regard to their
own beliefs and knowledge, rather than simply copy the claims from the national or local community without reflection. Clarke (1997) argues that teachers’ tendency and opportunity to reflect on their teaching practices are major influences on the changing role of mathematics teachers.

Cultural Factors

Cultural and educational norms

The fact that teaching practices are embedded in each culture provides a rationale to consider cultural and educational norms in studying teaching practice. Such norms may include normative teaching practices, cultural expectations of successful teaching outcomes, the value and role of education, and parents’ support, historical contexts of the mathematics education reform movement, etc.

Curriculum development and administration

The current mathematics education reform movement in the U.S. has been stimulated by the publication of Standards documents by the National Council of Teachers of Mathematics. Similarly, new curriculum development in Korea indicates changes in the teaching and learning of mathematics. Thus, it seems crucial to understand the process of curriculum development and administration, which includes the issue of administrative structure for curriculum development, the degree of teachers’ engagement in curriculum development, administrative directive, and teachers’ degree of compliance.

Teacher education

Pre-service and in-service teacher education programs provide teachers with various mathematical experiences through mathematics content courses, methods
courses, field experiences, student teaching, etc. Unfortunately, most university mathematics courses for teachers are recognized to reinforce the view of mathematics as a set of procedures to be memorized (Battista, 1994; National Research Council, 1989). Against this background, some reform-based in-service teacher education programs have been successful in helping teachers learn to teach mathematics consistent with reform recommendations (e.g., Raymond & Santos, 1995). Philippou and Christou (1998) report that a mathematics preparatory program designed to develop mathematical understanding significantly improved prospective teachers' attitudes towards mathematics, particularly their satisfaction from mathematics, and their sense of its usefulness.

**Professional models and community**

Considering that current reform ideas reflect diverse theoretical perspectives, advocating a single teaching approach is neither possible nor desirable (Lederman & Niess, 1997). Rather, mathematics education reform should offer various teaching approaches so that the teacher considers the strengths and weaknesses of the approaches with regard to his or her own pedagogical intentions in the specific classroom situations. In the process of implementation, the teacher may experience dilemmas from intrinsically competing learning theories and from his or her multifocal teaching intentions. Alternative professional models allow teachers to examine, to reflect on, and to develop their own teaching philosophy on the basis of their considerable commitment towards reform (Kirshner, in press). In this respect, the following will be examined: availability of alternative professional models, the role of professional community, the teacher's identification with professional community, etc.
CHAPTER 5

RESULTS AND ANALYSIS

This chapter is divided into three sections. The first portion provides an individual analysis of class UE where the teacher, Ms. E, attempted to institute student-centered instruction as recommended in current mathematics education reform initiatives. The second section of the chapter provides an individual analysis of another class UM where the teacher, Ms. M, established student-centered instruction. These two individual analyses describe classroom flow in detail with transcribed episodes provided to support the descriptions. The teachers’ curricular intentions and the students’ learning intentions as reflected in the classroom flow are inferred. Finally, students’ learning opportunities in each class are analyzed.

The final section of this chapter provides a comparative analysis of the two classes. The first part of this section reports the similarities and differences of the two classes with regard to the four categories used in the individual analysis -- classroom flow, teacher’s approaches, students’ approaches, and students’ learning opportunities. These similarities and differences are recapitulated by comparing and contrasting social norms and sociomathematical norms of each class in order to explore the possibility of using those norms (in particular, sociomathematical norms) as crucial analytic tools in assessing mathematics instruction of reform-oriented classrooms. The second part of this final section discusses factors that had influenced the two teachers’ instructional goals, which in turn enables an exploration of the challenges of implementing reform ideals.
INDIVIDUAL ANALYSIS OF CLASS UE

SETTING

The school U1 is located in an urban area of Baton Rouge, Louisiana. It is one of 14 elementary magnet schools in the parish. Magnet schools have special programs, ranging from literature and science to arts and technology. School U1 focuses on visual and performing arts. Recently, this school was recognized as one of the two best elementary schools in the United States with regard to the performing arts. Students take a minimum of one hour weekly for visual art, music, dance, drama, and creative writing. The school also provides special arts programs for talented students in visual arts, music, and drama. Talented students are given priority in entering this school and are expected to further develop demonstrated skills through the variety of experiences the school offers. The main admission criteria for other students are their interest and experience in art. The school maintains a racial balance of a 50% black and 50% white ratio. The majority of students are from middle- to lower-middle-class families. The total number of students enrolled in this magnet program is about 400. This school also has a pre-kindergarten program and an extended day program. The latter provides various learning opportunities by tutorial and enrichment experience.

School U1 is committed to providing a supportive educational environment both academically and aesthetically. The school emphasizes students' participatory learning with a strong academic curriculum enriched by various arts programs. The school publishes a newsletter once per month which includes the principal's letter to parents, lists of special events, news at each grade level, reports of students' activities from arts specialists, and lists of honor students in each class.
School U1 has three classes at each grade level from kindergarten to fifth grade with the exception of second grade, which has two. Ms. E’s class is one of the two second grade classes. Ms. E is an enthusiastic teacher with about 30 years teaching experience. Recently, she had an option to retire but decided to continue teaching because of her love of teaching and interacting with children and their parents. Ms. E said that she has tried to teach well, especially in mathematics, because she wants her students to be better than she was. She did not understand mathematical principles and abstract rules when she was as a student, which led her to feel insecure in mathematics. When she was in junior high school, she asked one of her algebra teachers to explain why she could not add 2A and 3B. The teacher’s explanation did not help her understand and the teacher made her feel embarrassed in the class. This contributed to a negative mathematical disposition.

In her teaching, Ms. E tries to emphasize conceptual understanding before introducing mathematical rules or algorithms. Moreover, she attempts to establish a permissive atmosphere in which students are expected to express their ideas without being embarrassed. Despite her consistent efforts to improve her mathematics instruction, teaching mathematics is more difficult for her than teaching science or language arts. Ms. E does not have many in-service experiences regarding mathematics but her voluntary participation in these few workshops about mathematics instruction has greatly influenced her perception of mathematics and mathematical instruction. The mathematics instructors in workshop classes helped her learn mathematics with understanding as well as enjoyment. This positive experience led her to use those instructors’ methods in her own classroom mathematics teaching. Ms. E refers to
and the teachers' manual to prepare her mathematics lessons, but she emphasizes that she does not follow them exactly. Rather she keeps in mind crucial mathematics topics for second graders such as regrouping, borrowing and carrying. These topics are important to reinforce because she believes her students would falter in mathematics in subsequent grades if they did not learn these concepts well in second grade.

Ms. E's class consisted of 12 boys and 13 girls. Students seats are arranged into several small groups. Ms. E periodically changes their seats on the basis of how the students work together. Ms. E agreed to have a child who was about to be suspended from the other second grade classroom because of his misbehavior, and assigned a seat for him close to her. In one of her interviews, however, she expressed some difficulties in managing her class because of the child's indirect influence on other children's behavior. In three of the seven lessons videotaped for this study, students individually worked and sometimes shared manipulative materials in small groups. For the rest of the lessons, Ms. E encouraged students to work together in solving given problems and posing problems as a group.

**CLASSROOM FLOW**

The research questions for this study explore how a teacher and students establish a reform-oriented mathematics classroom. Toward that end, this section describes classroom processes in detail, including general classroom atmosphere and lesson elements. To enhance this description, episodes from the classroom are presented.
**General Atmosphere**

Class UE had an open and permissive atmosphere in which students' ideas, including their mistakes, were welcomed. Students found mistakes made by their peers and their teacher. When a student presented a wrong answer and Ms. E did not recognize it on the spot, another student pointed out the mistake and corrected it. Ms. E lavishly praised the student and provided a treat. In another case, when a student corrected her own mistake by self-checking in the middle of a presentation, Ms. E asked her to explain how she found and corrected the mistake, praising the student throughout their exchange. Ms. E then emphasized the importance of self-checking in solving mathematics problems to the whole class.

Ms. E created a classroom atmosphere where students felt free to ask any questions by emphasizing that there were no dumb questions. For instance, Alex asked the teacher to clarify the meaning of a sentence in a word problem and Ms. E allowed students to use their own various interpretations. When a presenting student made a mistake or could not finish the presentation, Ms. E asked other students to help out the student. However, students were not allowed to help by giving the answer.

The classroom atmosphere was dynamic in that Ms. E frequently used enjoyable activity formats for students. For instance, Ms. E threw a small bag with a basic addition or subtraction problem on it to a student. The student who received the bag had to answer by throwing it back to the teacher. Missing the bag made the class laugh a lot. When a child threw the pouch back to the teacher with a wrong answer or the child spent too much time coming up with an answer, the teacher again threw the pouch back to the same child until s/he said the right answer in time. For a difficult problem,
the teacher provided some facts related to the problem and often wrote them on the board.

In another case, Ms. E led a group of students to simulate the situation described in a given word problem. She also encouraged students to present their solution methods in the front of the classroom for the whole class. Ms. E tried to give her children an equal chance to present and frequently checked whether there was a new student who volunteered to present.

Ms. E usually set a timer to help students focus on what they are supposed to do. Her words, "Pencils up, get set, go" seemed to excite students. The teacher's tone added to students' excitement. Sometimes, if a student started too soon, Ms. E would get the whole class to start again. This kept students' attention. When she sometimes said how much time remained, students shouted and hurried to finish a given task. Ms. E occasionally gave more time to students, even after the limited time passed. Setting time seems to set the pace of instruction. Ms. E provided examples or detailed explanations when students did not understand a given problem. She walked around the classroom while students worked in their small groups. In some cases, Ms. E encouraged students who tried to pose word problems by offering them individual help. However, in most cases, Ms. E checked whether students were on task and working together.

In general, students expressed their excitement about classroom activities. For instance, students had an estimation activity in almost every mathematics lesson in which they had to figure out the number of items being left in a jar after the teacher took a handful out. When Ms. E initiated the estimation activity, students expressed
their excitement. Those who were closest with their estimation number shouted with pleasure. When the class collectively solved an equation, students also expressed their excitement. Students were actively involved in classroom activities. They faithfully followed Ms. E’s directions and solved the given problems individually and collectively.

Ms. E sometimes tested students’ knowledge of basic addition and subtraction facts using a paper-and-pencil format with a time limit. There was no indication of test anxiety on the part of the students. Instead, students laughed a lot when the teacher set a time and asked them to be ready for a test. The students who got a perfect score on the test were praised and recorded on a class bulletin board. Most students in this classroom got a perfect score in basic addition tests. During the observation period for this study, Ms. E asked her students to study subtraction facts. After a preliminary test on subtraction, she emphasized working hard.

Ms. E provided a treat for students who, for instance, guessed the closest estimation number, developed an original idea, pointed out the mathematical mistakes made by others, or got a perfect score on a test. In order to keep students’ attention, she praised some students who were doing what they were supposed to do, and sometimes showed these students’ worksheets and illustrated what they were doing. In general, Ms. E communicated her positive expectation for every student. She also took care of students’ emotional states, specifically when they could not complete their presentation in front of the class. When the teacher required students’ attention, she clapped rhythmically twice and students copied the rhythms, giving their attention to the teacher.
Lesson Elements

This section is organized using the main classroom activities with representative episodes to explore how Ms. E and her students constituted a reform-oriented mathematics classroom. Each mathematics lesson consisted of estimation, problem solving, and other mathematics activities. The sequence of these activities was not fixed. For instance, on one day Ms. E started with estimation, whereas on another day she began with problem solving. The length of a mathematics lesson varied from 50-90 minutes.

Estimation

Each student had brought a bag of 100 items. Before each mathematics lesson, Ms. E took a handful out of one bag and placed what was left in a large glass jar and displayed to the class. Ms. E walked around each group with the jar and students were to guess how many items were left in it. Students had to write down their estimation number with a red pen, which prevented them from changing the number to get a treat. Students then reported their estimation numbers and the teacher wrote them on the board. Ms. E brought the items she took out of the bag and the student who brought the bag went to the front of the classroom and counted the items aloud. Ms. E then asked students to write an equation to figure out the number of items left in the jar. For instance, when a child counted 17 clear plastic items, students wrote 100-17=? in a vertical format. Ms. E often checked whether students understood where those numbers came from and what they were looking for. She always gave students a chance to solve the equation for themselves on their notebooks, and then led them to solve it on the board. After solving the equation, the class sought the closest estimation and that child
would get a treat. Sometimes more than one estimation was closest because students estimated higher and lower than the exact number.

**Strategies for estimation**

Ms. E provided students with some hints for better estimation. During the observation period, students made T-shirts with fish of different sizes and colors on them for a field trip. She asked students to estimate the number of fish on her T-shirt. While showing the shirt for a short time, Ms. E noticed that some students attempted to count the fish one by one. She suggested counting the fish in a row and in a column. In another case, Ms. E asked students for strategies of estimation (see Episode UE-1).

Note how parentheses, brackets, and braces are employed in episodes: parentheses, 

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"(...)", are used to describe actions or movements viewed in the videotape; brackets,
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"[...]", are used to report low-inference interpretations of words or actions; and braces,
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"{...}", are used to put a brief description in place of a long transcript.
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<Episode UE-1: Students' strategies for estimation>

T: Write only your estimation number with your red pen. (She starts walking around with the jar of Brazil nuts.) These are larger. Remember we had this plastic thing (showing an item students used for estimation one day), whatever they were, they were smaller. Remember another day we had different sizes and another day we had q-tips. These are large. So you might keep that in mind when you estimate. Hum. Look at the jar. Hmm. Raise your hand if you think it’s halfway. Raise your hand if you think it’s more than halfway. Less than halfway? Hmm. Do you have the sentence written out? You should be ready to write what you think your answer is. Remember I took a handful out. I took a handful out. (She has walked around the students once and goes to the front.) Before we start, who knows... Does anyone have a strategy how you estimated? Who has a certain strategy? Who would like to describe? What’s your strategy, Benjamin?

Benjamin: I looked at the size and I think how many could fit in my hand, and then, I just, I did...

T: He looked at the size, and he thought how many could fit in his hand. Let’s see how many fit in your hand. (She goes to Benjamin.) Grab it, dear, and let’s see how many you can get in your hand. (Benjamin grabs some but can’t get his hands out of the jar.)
Oops, you are like a monkey in a jar. (She laughs.) Boys and girls, there was a monkey who can have all the candies he can get. So he reached in the jar and grabbed a whole bunch and he tried to get out, (Benjamin takes out something less this time.) Oh! He did better. (To Benjamin) You got less this time. The monkey could not get the candy out. How many did you get?

Benjamin: Five.

T: Five. So, he estimated five. Now wait a minute. Someone needs to think about something. He estimates that he can probably get five. (She opens and closes her hands several times.) That's a good beginning. Okay, who else has another way of estimating? (To Lara) What is yours?

Lara: What I think is, almost like Benjamin's. But the difference is I used two hands to try to do it (motioning with her two hands).

T: UhUh [Agreeing]. So I might use two hands. I didn't necessarily say one hand, did I? Okay. I did say one hand? (A student says “a.”) ‘A’ usually means one. Good thinking. Okay, now anybody else have a way of estimation? Sam?

Sam: Subtract.

T: We will subtract. Alex?

Alex: I estimated five. I looked at your hand and then saw how many she can take out of the jar with two hands, and then I say to myself, then how much is that, she takes out, and I subtract that number.

T: I will give you a hint. I did not put my hand in it to pull it out like he did. I kind of poured them into my hands (tilting the jar). Okay? (Students shout “Oh!”)

In this episode, Ms. E helped students estimate the number of Brazil nuts in a jar by contrasting the size of nuts with that of other items they had used and by asking them to estimate whether the number of the nuts would be more than 50 or not. She then encouraged students to present their idea of how to estimate. Ben explained his strategy as looking at the size of the objects in the jar and guessing how many could fit in his hands. Ms. E repeated what Ben described to the class. By showing her hands, Ms. E indicated that she was able to grab more objects than a child was. Lara proposed using two hands owing to Ms. E’s indication. Sam then mentioned a computational method of
subtraction. Sam seemed to infer what Ms. E might expect, rather than to articulate the method he actually used to estimate. Ms. E kept asking for another way to estimate. Alex provided a comprehensive strategy summarizing the previous contributions of other students. After listening to students’ diverse strategies for estimation, Ms. E explained how she actually took out the objects.

**Subtraction algorithm connected to estimation**

Ms. E connected the estimation activity to teaching the standard algorithm for subtraction that required carrying/borrowing. The following episode clearly shows how she transformed students’ activity of estimation into practicing the subtraction algorithm.

**<Episode UE-2: Using formal algorithm for estimation>**

T: Okay. Now we originally had 100 objects. And I took out one handful. And so, whatever is here (showing the jar) is what you are guessing. We had 100 objects (writing 100) and I took out one handful of something (writing and pointing to the blank space under 100). And this is the answer you are guessing (pointing to the place for the answer in the equation in the vertical format). Write this equation down (100- blank = blank). Now who can tell me when I go back there and get the objects, who can explain how we are gonna use that to figure out what’s in the jar without counting them. Charles, can you explain?

Charles: You count how many is in the bag and how many you took out.

T: How many in, um, that I took out, correct, and then what do I do? Arterrion?

Arterrion: You take them out from the jar and count how many.

T: But I don’t feel like counting all these things. I don’t wanna do that. I wanna take a short cut. What will I do?

Brittany: You should track how many you took out of the jar and subtract it from the 100.

T: Got it, girl! Okay. Let me get on back here. Now I am a kind of... I have to apologize, I was really tricky. (She brings the items she took out.) It’s one handful but look, [long pause] tricky this time (showing the two big stones she took out).

Ss: Ahaaa!
T: All right. Let's subtract. Here we go. Ready? Who can help me? This would be ones column, tens column, or hundreds column? Suzannah? (Shows the items taken out.)

Suzannah: Ones column.

T: In the ones. (She writes 2 under the 100 in the equation.) Okay. See if you can subtract for yourself. Let me see. (She walks around but provides no comments.) Use your pencil. (She comes back to the chalkboard.) Remember the other day Kelsey was up here and she did it. And we helped her out with what she was doing. Today I am going to do it. I want someone to tell me what to do. Who can tell me what I need first? What's the first thing I do? Morgan, tell me what to do. You will let me know what to do.

Morgan: The first thing you do is to subtract. Take away 2 from 0.

T: Take 2 from 0. Okay. Morgan, come here. Take 2 from here (showing her empty hands). (Morgan simulates taking something out from the hands.) Come on, take 2 away. (Morgan does again.) You are not doing it.

Morgan: I can't

T: Why not?

Morgan: Because there is nothing in there.

T: Nothing in there. So who can solve this problem? Who can solve this problem for us? Who can solve our problem? Can you solve this problem, John? You can't solve it? (John slightly shakes his head.) How about you, Mary?

Mary: Take 0 from 2.

T: That's a common, a common thing children do. And they even put those kinds [of problems] on the test to trick you. They say, Aha, 0 is smaller than 2, so let me take away (pointing to 2) from there (pointing to 0 in the ones column).

Ss: No.

T: Billie Jo, what can we do?

Billie Jo: You borrow a 10.

T: We borrow a 10! Can we get a 10 from here, Billie Jo (pointing to 0 in the tens column of 100).

Billie Jo: (Shakes her head).

T: Boys and girls, can I take a 10 from this?
Ss: No.

T: Mary? Can I take a 10 from this 0? (Points to 0 in the tens column of 100.)

Mary: No.

T: Can I get a 10 from hundreds, from this 1? (Points to the 1 in 100.) (Students agree.)

In this episode, the teacher wrote an equation 100-(blank)=(blank) in a vertical format after students reported their estimation. She asked the students to explain how they would use the equation to figure out how many items were left in the jar. In other words, she expected students to use the equation. Charles first suggested counting the objects taken out and Ms. E asked for the next step. Contrary to the teacher’s expectation of using the equation, Arterrion explained a practical and intuitive strategy, that is, counting all the objects remaining in the jar. Ms. E said that “I wanna take a short cut,” indicating that she wanted to use the equation and that subtracting and counting the remainder yield equivalent results. Brittany’s suggestion to subtract fitted into the teacher’s plans, and provided an opportunity to continue. Satisfied with Brittany’s response, Ms. E showed students the objects she took out and wrote the number of objects in the equation 100-2=(blank). Ms. E then encouraged students to solve the equation for themselves. After checking students’ work, she invited students to help her subtract. Ms. E role-played as a student and students role-played as teacher.

However, as evidenced by the statements, “Who can tell me what I need first? What’s the first thing I do?” Ms. E controlled students’ possible responses. In other words, students were expected to subtract 2 from 100 using the standard algorithm rather than inventing their own methods. Eliciting students’ short answers to solve the equation, Ms. E taught the formal algorithm procedure step by step. Since the original
number of items for the estimation activity was always 100, students had to borrow in any equation they made. Showing empty hands and asking students to take something out of them was Ms. E’s strategy to show the need for borrowing.

Ms. E’s concern for using standard algorithm for estimation was manifested when students came up with unexpected ideas while figuring out how many items were left in the jar. A representative episode occurred when the class was estimating the number of Brazil nuts in a jar (see Episode UE-3). Reham counted the nuts Ms. E had taken out. Ms. E wrote an equation 100-12=(blank) in vertical format on the board and asked students to explain how to subtract 12 from 100.

<Episode UE-3: Alex’s idea against the teacher’s expectation of using algorithm>

T: Who can tell me how to do it? Who can tell me how to do it? Someone who has not had a turn yet, a new person. Alex, since you will be absent, what do I have to do first?

Alex: If I took away ... 100 take away 10, it will equal 90. And so if I take away 2 more, it will equal 88.

T: Good gentleman! Look at here. He looked at this 12 and he said, that 12 is close to 10. He said, 100 minus 10 is 90, I just know that (writing 100-10=90 in a vertical format). And I take 2 more away. Good thinking! Super! How can we do it this way now without mental math? (She points to the equation on the board.) How can we do it if it is a little hard for us? Who can explain what we can do? Arterrion? {Arterrion’s explanation is partly inaudible, and Ms. E asks him to explain again.}

Arterrion: 0 minus 2 is 2 ..... and 10 minus 1 is 9.

T: Okay, what I want you to do is, (going to him) I want you to take 2 from 0 (showing two empty hands). Come on, take 2.

This episode illustrated Ms. E’s consistent curricular intention of using formal algorithm in the estimation activity. When she initiated the discussion regarding how to solve 100-12, she expected students to attempt to take away 2 from the 0 in the ones column, as evidenced by the statement, “What do I have to do first?” Unaware of her
expectation, Alex provided a different idea that 100 minus 10 is equal to 90 and 90 minus 2 is equal to 88. Ms. E explained what Alex said to the whole class and praised him for his novel idea. But she immediately returned to her initial interest of using the standard algorithm by asking, "How can we do it this way now without mental math? How can we do it if it is a little hard for us?" Responding to Ms. E's expectation, the class was then involved in doing subtraction with algorithm. Note that the standard algorithm departed from the holistic meaning of subtracting 12 from 100 for estimation (i.e., there were 100 items and 12 of them were taken out) by focusing on each individual digit (i.e., how to subtract 2 from 0). In this way, estimation was shifted from an intuitive conceptual activity into an opportunity to practice standard algorithms.

**Problem Solving**

Ms. E spent a lot of time with the class problem solving. She provided students with a chance to solve problems for themselves and then to present their solution methods in the subsequent whole group discussion. Ms. E also encouraged students to pose similar problems to ones they solved. Ms. E either displayed a mathematics problem on the board or handed it out to each child. In the former case, she simply chose a problem from the resource material which accompanied the textbook she was using. In the latter, she chose a mathematics problem she had made up herself, or collected through her career. Table 5.1 shows the various problems used in the classroom during the seven classes observed. Ms. E used these various problems for different purposes, from checking students' skills in using formal algorithms to
facilitating students' thinking ability with problems for which they had no specific strategies.

<table>
<thead>
<tr>
<th>Name*</th>
<th>Mathematics problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Making Numbers</td>
<td>How many different 2-digit numbers can be made with these cards? 7, 2, 5, and 9 (Each was shown as a number card.)</td>
</tr>
<tr>
<td>Finding a Card</td>
<td>Which card shows a sum of 79? 38+21, 82+17, 76+12, and 64+15 (Each was written in a vertical format.)</td>
</tr>
<tr>
<td>Showing a Sum of 79</td>
<td></td>
</tr>
<tr>
<td>Making the Greatest Sum</td>
<td>Use each number once in the addition problem to get the greatest sum. 2, 4, 6, and 8 (shown with two empty squares in each of two rows with a plus sign, indicating 2-digit addition in a vertical format).</td>
</tr>
<tr>
<td>Marble Problem</td>
<td>Javier has 8 marbles. He picks up 6 more. He gives 1 marble to each of his 5 friends. How many marbles does Javier have left?</td>
</tr>
<tr>
<td>Shell Problem</td>
<td>Roy and Reba are buying shells to make jewelry. They need sixty shells in all for their projects. Roy buys 12 shells. Reba buys 18 shells. How many more shells do they need?</td>
</tr>
<tr>
<td>Seal Problem</td>
<td>There are 32 seals on a rock. 11 more seals climb onto the rock. Then 15 of the seals slide into the water. Now how many seals are on the rock?</td>
</tr>
<tr>
<td>Zoo Problem</td>
<td>24 children were visiting the elephants at the zoo. 8 of the children left to visit giraffes. How many children were still visiting the elephants?</td>
</tr>
</tbody>
</table>

* Note: The class did not use these names. Names are assigned by the author for convenience to refer to the problem in the following text.

**Learning process and visualization emphasized**

Throughout the problem solving activities, Ms. E emphasized that learning how to solve a problem was more important than reaching a right answer. She communicated the idea that the learning process should involve expressing one's own thinking. In Episode UE-4, specifically, she describes a possible benefit of students presenting their thinking to the whole class. The communication of ones' idea might provide the students with a chance to realize where they had difficulties and how they
might possibly overcome those difficulties on the basis of others' suggestions. In this way, Ms. E encouraged students to present their ideas to the whole class.

<Episode UE-4: Learning process over correct answer>

T: Remember, right or wrong is unimportant. What we wanna try to do is to learn the process of problem solving and how to show it. Sometimes you can show your work, then we can understand and you can explain it. We know where you have difficulties, if you tell us how you think. I would like to have a volunteer.

Ms. E also emphasized that students should visualize the problem situation of the given word problem. An example is provided in Episode UE-5. The class was solving the Marble Problem. Ms. E asked students to read the word problem silently. She then provided simple questions which led students to identify information on the problem and to think of appropriate computational methods. Meanwhile, Ms. E used such terms as "imagine" and "visualize" the problem situation. She asked students to draw pictures representing the problem situation as well as to write equations. Ms. E actually arranged students' presentations in such a way as to connect abstract equations (e.g., 8+6=14) with their corresponding pictures (e.g., 8 circles in one line and 6 circles in the other line).

<Episode UE-5: Visualization emphasized for a word problem>

T: Javier has how many marbles, class?

John: Eight.

T: Imagine eight in your hands. What does he do with these marbles? John.

Ss: (Several students read.) He picks up... (Interrupted.)

T: I said John. John?

John: He picks up six more.
T: He picks up six more. So, think of a strategy. If you get more, is this adding or subtraction, class?

Ss: Adding.

T: Then what happens? Then what happens, Morgan?

Morgan: He gives one marble to each of his five friends.

T: Now, close your eyes and visualize five friends. Visualize taking a marble from your hands and giving them each to five friends. Now, we must make another equation. Think. Is this going to be an adding equation or subtracting equation?

Ss: Sub... (Interrupted.)

T: Uh, uh, think, think, think. Raise your hand when you think of it. What is it?

Ss: Subtraction.

In another case, after reading the Zoo Problem, Ms. E asked how many children there were in the classroom. As each student stood up, they counted and found that the number of children in the classroom, 24, was the same number of children in the problem visiting the elephants. In order to help students visualize the problem situation, Ms. E asked them to imagine that they were at the zoo and pointed to the picture of elephants on the front wall. In order to simulate the problem context, she chose eight students and asked them to come to look at the giraffes. The class enjoyed talking about giraffes for a while, especially Lara brought a few interesting stories since she had studied about them as her diorama project. Leading students' collective attention to the problem, Ms. E asked how many were left and the class counted one by one the number of students sitting in their seats and found 16. As seen in Episode UE-6, Ms. E asked students to write an equation for the problem and to include a word showing what the answer represented. As usual, Ms. E gave students an opportunity to figure it out for themselves.
<Episode UE-6: Morgan’s presentation and Ms. E’s intervention>

T: Think about this, when you make your equations out, what’s your mathematical equation about this problem? When you put your answer down, please put your words with it, indicating what that answer stands for. Otherwise the number is isolated, we don’t really know what it means. (She walks around.) Let’s see here who listens. Aha! Morgan listens. Put your finger on the word you put after the answer. And then I would like to see your abstract picture, that demonstrates what happens. (The teacher comes back to the front of the classroom, takes out the projector, and pulls down the screen.) Okay, I need somebody. Let me see. Morgan, come up here and write your equation. The reason I chose her is, because she’s the first person who followed my instruction, writing the word down by the answer.

{Morgan writes 24-8=16 in a vertical format and writes the word "chidren" next to 16. Students point to the misspelling and Ms. E reads "chil-dren" slowly and clearly. Morgan finally changes the misspelling.}

T: Here we go! Okay, this is what I was looking for, the word. Because if this numeral is isolated, it can be 16 giraffes, or 16 elephants, it can be anything. Explain loudly what you did.

Morgan: Uh, first I put 24 and I put minus sign. And, it says, uh, 8 children were going to see the giraffes. So, I put, take away 8. And the answer was 16 children.

T: Okay. I noticed you just put 1 first and then 6 (pointing to the answer 16 on the OHP), and I am thinking that maybe there is a process you didn’t write before you got that answer. Can you tell me how you did loudly, subtracting in ones column first, what we have to do?

Morgan: (Erases the answer 16.) I have to take away 8 from 4. But I can’t, so I borrow 10 from tens column (crossing out 2), and then there is 1 ten now (putting little 1 over 2). So, I have 14 ones (crossing 4 of 24 and putting a little 14), so I take away 8 ones from 14 ones, it equals ... (long pause)

T: 8 plus what equals 14?

Ss: Four. No. (Some of students laugh.)

T: 8 plus what?

Ss: Four, four, six, six, six, 8 plus 6 is 14. (Morgan puts 6 in ones column.)

T: Guess what? The reason I am trying to keep you thinking is when you don’t know that subtraction fact, think what you add to get that.

Ss: Six, six.

T: Here we go. (Morgan points to the tens column and pauses.) (To Morgan) What’re you doing, baby?
Morgan: And then I take away 0 ones from ... (Interrupted.)

T: Are those ones?

Morgan: Oh! No.

T: What are they?

Morgan: Then I take away, 1 tens (pointing to the little 1) from 0 tens. (She writes 1 as a part of the answer.)

T: Okay, look what you said. You're doing fine. But look! You told me that you took away 1 tens from 0 tens (showing empty hands). (Morgan seems to laugh noticing her mistake but she hangs her head.) Okay, one, two, three, four, five, six, seven, eight, nine, ten (picking up 10 cubes). Now, there is 10 here (showing the cubes to her). What are you gonna take away from this 10?

Morgan: 0.

T: Zero! So, you've got that first. Take the bottom numeral from the top. Okay, do again, you got it, you know that!

Morgan: And then, I take away 0 tens from 1 tens. And it equals 1.

T: (clapping) Ya, very good, honey. And I liked that you put 16 children.

In this episode, Morgan wrote down the equation with a word, 24-8=16 children, in a vertical format. When Ms. E asked for an explanation, Morgan mentioned what she put in the equation and partially connected the numbers in it to the problem situation. Morgan did not use a formal algorithm in reaching the correct answer. However, her explanation that "the answer was 16 children" could have been accepted as appropriate in this context, because the class simulated the problem situation and specifically counted the number of students left in their seats to figure out the answer for the problem. Ms. E’s response to Morgan’s explanation clearly shows her expectation. She observed that Morgan wrote 1 for the tens column first, and then 6 for the ones column. Ms. E then asked for a computational process but limited Morgan to use the standard algorithm when she said “subtracting in ones column first.”
Corresponding to the teacher's expectation, Morgan explained how to subtract 8 from 24 using the standard algorithm. In her verbal explanation, Morgan mistook the tens column for the ones and switched the subtrahend with the minuend. In response to Morgan's mistakes, Ms. E provided questions and repeated what Morgan said, which would help Morgan reflect on what she was doing. In particular, when Morgan made her second mistake, switching the subtrahend and the minuend in the tens column, Ms. E used cubes. However, Ms. E's use of a concrete manipulative did not match to what Morgan was supposed to do. Morgan was subtracting 0 tens from 1 tens, whereas the teacher demonstrated subtracting 0 ones from 10 ones. Ms. E gave Morgan a chance to correct her mistakes and to explain again what she was doing. Considering that Morgan's first mistake was related to place value, Ms. E's intervention did not seem to promote conceptual understanding for Morgan, but instead reinforced the order of computation. Note that Morgan laughed when she realized her mistakes, as did the teacher.

**Manipulative materials used for numerical computation**

Ms. E often used manipulative materials in order to help students solve a given word problem. She attempted to connect numerical representations with concrete representations using various materials. This was consistent with Ms. E's general emphasis on visualization. For instance, when students were supposed to solve the Seal Problem individually, Ms. E distributed cubes with which students might trace the changes in the number of seals. After individual students solved the problem, Ms. E initiated whole class discussion and asked students to represent the number of seals in the problem using cubes. Terrance first put 32 cubes on the OHP by random
arrangement to represent 32 seals on the rock. Students counted together while Terrance was putting the cubes one by one. Ms. E started re-arranging the cubes without comments. She made two rods (10 cubes per rod) but did not finish creating another rod; instead she put together the rest in an irregular shape. Alex then put 11 more cubes on the OHP to represent the number of seals which climbed later onto the rock. At that point, Ms. E led students in making and solving the equation for the situation, 32+11=? Students checked their answer by counting the number of cubes in all on the projector. When they confirmed the answer, they shouted with excitement. Finally, Reham took out 15 cubes from the OHP to represent the number of seals that slid into the water. Again, Ms. E asked students to create and solve the equation for the change (43-15=?), as seen in Episode UE-7. She then initiated whole-class discussion and used cubes on the projector to help students understand the computation process of 43-15. Note that there were only 28 cubes on the projector, because Reham had already taken away 15 cubes.

<Episode UE-7: Solving a word problem with equations and cubes>

T: Okay, so, 15 seals have slid into the water. What operation... We now have 43 seals (writing the number 43 on the board) and 15 seals slid into the water. Who can tell me how I write this as a mathematical equation? We’re not gonna say, 15 seals are in the water. We’re gonna write something down, a mathematical equation. What do I write? Kelsey?

Kelsey: Uh, take away.

T: You got it, girl. What goes here? (Points to the empty space under the 43.)

Kelsey: 15.

T: Good. I am so glad (writing 15 under 43 on the board). Now, write a mathematical equation. So, 43 minus 15, boys and girls, see if you can do it by yourself first and then we’ll do it together. If you think you’ve got it done, raise your hands. I wanna see you working your equation. (She walks around.) (To Benjamin) You’ve already done it?
Good. Raise your hands if you’ve done it. (She keeps walking around more groups. Most students raise their hands.) Okay, boys and girls. Let’s see how well we can do. Who would like to tell me how to do this? Who can explain, so I will know what to do. I don’t know what to do, tell me what to do as if I do not know, Sam.

Sam: Uh, you...

T: Okay. Please, take your hands off of the cubes

Sam: You can’t take 3 tens, I mean, 5 tens, I mean 5 ones from the 3 ones.

T: I can’t! Come here. I’ve got 3 ones (picking up three cubes from the OHP, and showing them in her hands). Come and take five. Take five.

(Sam takes out the 3 cubes and at the same time, the teacher counts, one, two, three.)

T: Can’t you take two more? (Sam mimics taking something from the teacher.) You are right. I can’t do it. So, what must I do?

Sam: You should borrow a ten. (Ms. E gets the 3 cubes from Sam and holds them.)

T: We’ve already got three. Take ten more and put them with those three (putting the three cubes on the one side of the OHP).

(Sam counts 10, and put them together with the three, representing 13 ones at the top of the projector.)

T: Okay, he says, I must borrow a ten. Where do I get my ten, Sam?

Sam: From the 4.

T: From the 4 tens. How many tens then?

Sam: 13.

T: No, right here (pointing out the number 4 of 43 on the board), how many tens, I had four of them and you took ten away. How many tens are left?

Sam: Three. (Ms. E crosses out 4 of 43 and write 3 above the 4.)

T: And I add my 10 to the 3, that gives me what? (She crosses out 3 of 43.)

Sam: 13.

T: 13 (writing 13 above the 3 of 43). You know what you are talking about. Okay, Sam. Thank you. Who can help me do the next thing. What’s our next step? Mary, can you tell me what to do?
Mary: You take, 5, I mean 4, I mean 5 away from 3, I mean, 13,

T: (to Mary) Okay, come here, Mary. Here is our 13 (pointing to the set of 13 on the OHP). Remember he put 10 by the 3 we got 13, take 5 from there. (She hides another set of cubes at the bottom.) One, two, three, four, five (As Mary moves the 5 cubes, Ms. E counts.) 13 minus 5 is, what's left, Mary? What's there? Count, so everybody can see.

(Mary starts counting what's been left.)

Ss: One, two, three, four, five, six, seven, eight.

T: 13 minus 5 is what, class?

Ss: Eight.

T: Eight what?

Ss: 8 ones.

T: (Writes 8 under 5 of 15 on the board.) What's next step? Billie Jo.

Billie Jo: Take away one from the three. It's two.

T: So, (writing the number 2 under 1 of 15 on the board) boys and girls. How many seals are left?

Ss: 28.

T: 28

Ss: Yeah, Yeah.

T: How many did you get that? Give yourself a smile face, you guys.

In calling for whole group discussion, the teacher role-played as a student waiting for explanations. When Sam said that he could not subtract 5 ones from 3 ones, Ms. E showed three cubes from the OHP and asked him to take away five. Thanks to Sam's idea of borrowing a ten, Ms. E led him to count ten more cubes and to put them together with the 3 ones. The teacher attempted to connect Sam's action with the formal algorithm of changing the minuend 43 into 3 tens and 13 ones. As noted above,
there had been only 28 cubes (the result of the computation) on the projector so there was no one-to-one corresponding relation between the concrete representation of cubes and the subsequent procedure used to solve the equation 43-15. Ms. E might have used the cubes separately for the first equation (32+11=?) and the second equation (43-15=?). Even in this case, however, she did not seem to use the cubes in a way to foster students' conceptual understanding of regrouping. When Sam suggested borrowing a ten, Ms. E said, "Take ten more and put them with those three" and Sam simply counted 10 cubes. Given that the 28 cubes on the projector had been placed randomly, Ms. E and Sam seemed to add 10 ones to 3 ones rather than regrouping 1 ten of 4 tens to 10 ones to compute ones column. In this way, students had limited benefit of using concrete materials mainly to understand individual computation process in each column (13-5=8 in this episode).

Students' various interpretations encouraged

Ms. E sometimes gave students a similar but more challenging problem after they solved a given problem. An example was that students were asked to make three-digit numbers with 1, 5, and 6, after making two-digit numbers. In another case, Ms. E simply added one sentence — two children join them — to the Zoo Problem after students solved it (see Episode UE-8). This led students to develop different interpretations of the added sentence relative to the problem context. Ms. E allowed them to use their own interpretation.

<Episode UE-8: Teacher's extension of a problem and students' interpretations>

T: Boys and girls, if you... Okay, we're adding something, write on your little space (pointing to the space under the problem glued in a student's notebook). Put, two
children join them, write that. Write two children join them. Now what you need to do? Write the equation, and make a picture to show that.

Alex: Um, who did the two children join?

T: That's a good question. Did they join the ones that were looking at the giraffes? Or did they join the ones that were left behind? I didn't make that clear, did I?

Ss: No.

T: Um, you decide. So, we should have two different interpretations, shouldn't we? (She starts walking around.) So it's very important that you let your words tell people exactly what you mean, or it could be ambiguous like I was. I didn't make that clear. And it's very important, you know, sometimes you ask questions to get things clear. Okay, who decided that they join the students who were looking at the giraffes? Who decided that they join the students that stay behind, and they were not looking at the giraffes? Okay, Logan, will you come and show us how you did yours, show that they went to join the children that stay behind? How did you do that?

(The previous presentation left different sets of 16 and 8 circles with picture of elephants and giraffes respectively. Logan marks out the first two circles in the set of 8 circles representing children looking at giraffes. The teacher interrupts.)

T: These are the ones who went for the giraffes (pointing to the 8 circles with the giraffes). Okay? They join these children that stay behind (pointing to the 16 circles with the elephants). Is that what you did? Okay Well, explain what you're doing here? Oh! you got them from these people [children looking at giraffes]?

Logan: Yes.

T: Ah, that's a third version. In my mind, I was thinking of children from somewhere else, otherwise. Go ahead! That's a good, good thought. I couldn't figure that.

(Logan puts another mark on the first two circles and draws an arrow toward the circles with the elephants in order to represent that two children come back to the children looking at elephants.)

T: The ones who had left joined the others. They went back. Um, good thinking. Now, so he marked out two of them and put them here (pointing to the marked circles and the line). Did you make an equation for that? What was your equation?

Logan: 8 minus 2 equals 6.

T: So, he told us how many left here. Write that down. 8 minus 2 equals 6. So he did subtraction to show that they went back. (Logan writes the equation, 8-2=6 in a vertical format.) So he might have another equation here. Okay, what happens here? (Points to the circles with elephants.) If two join them, then what would that equation be?

Logan: (Writes 16+2=18 in a vertical format.)
In a subsequent interview, Ms. E said that when she provided the new phrase, she expected simply to add two children to those looking at elephants. However, Alex questioned the teacher as to whom the two children joined, raising a variety of alternative possibilities. Ms. E praised Alex and illustrated his question by differentiating the case of the children joining those looking at giraffes from the case of the children joining those looking at elephants. She gave students the chance to choose any interpretation. However, students interpreted the added condition in more than two ways, as evidenced in their presentations and notebooks. Whereas Ms. E thought of children coming from elsewhere to join the existing children, students thought of children moving back and forth between the groups of 16 and the 8 children who were looking at elephants or giraffes respectively. It was apparent that Ms. E’s simplified mathematical assumption violated the pragmatics of the situation as construed by her students in which the total number of children would be fixed (e.g., as in a class field trip). The students were offering spontaneous interpretations of the situations depicted in the problem based on their real world experience and their current mathematics understandings. However, this conflicted with the simplified “school math” interpretations presumed by the teacher.

Initially, Ms. E expected her students to write one of either two equations: 16+2=18 (in case of the children joining those looking at elephants) or 8+2=10 (in case of the children joining those looking at giraffes). However, as students brought new ideas, such as Logan's, Ms. E helped students write two equations for a third case (e.g., 16+2=18 and 8-2=6 in case of children coming back to look at the elephants after watching the giraffes).
Using different methods emphasized

Ms. E frequently solicited students' ideas by asking for different methods for solving a given problem. For instance, when students used both equations (8+6=14; 14-5=9) and concrete representations (putting together 8 circles and 6 circles; taking away 5 circles from 14 circles) to solve the Marble Problem (see Table 5.1), Ms. E asked whether they had found different ways of figuring out the answer (see Episode UE-9).

<Episode UE-9: Teacher's request of different methods>

T: Is there anybody who did it in a different way? How did you do yours, Alex?

Alex: I didn’t do exactly. I just, on the side, I drew, I just put a marble to each of the five friends.

T: Did you draw a picture to showing him doing that?

Alex: UhUh [Agreeing].

T: Come up here and show how you did that, because sometimes [there is] more than one way to do this, class. If you can explain it and if it makes good sense, then you are just as right as another person.

Alex: (Draws 5 people on the Over Head Projector [OHP].)

T: If you feel that you understood this and you wrote right equations and right pictures, if that works for you, give yourself a smiley face and a pat on your back. You are so smart. (Looking at Alex’s picture) Okay, so, these are your five people. They each receive a marble. Did you draw a line from here (picture of friends) to here (circles for marbles) to show they came from them?

Alex: (Starts drawing lines and is interrupted by the teacher.)

T: Is that what you did? I am asking.

Alex: No.

T: Okay, sometimes people don’t understand where the marbles came from. You might need to draw arrows to make it clear cause some people like me need to see it. Very good. That’s good, it shows you understand.
Responding to the teacher’s request for different methods, Alex drew five friends to represent that each of them got one marble, instead of the common strategy of crossing out 5 from 14 circles. Ms. E indicated that connecting each person with one marble would make clear what Alex meant by the picture. Similarly, when students solved the Zoo Problem using the standard algorithm with 24-8=16, Ms. E asked for different methods. Emphasizing that there is more than one way to solve problems, she walked around to observe students’ various representations. On the basis of her observation, Ms. E asked two children to present their methods. One child took away 8 from 24 circles, while the other child drew distinct sets of 24 and 8 circles and then connected by lines 8 of the 24 circles with the other 8 circles.

As exemplified in Episode UE-9, Ms. E sometimes encouraged students to provide multiple solution methods to a given problem and accepted their ideas. She acknowledged that there is more than one way to solve a given problem and emphasized that the explanation of such methods must make sense. However, in many other cases, Ms. E reinforced specific mathematical equations or standard algorithms over students’ various ideas. At one level, Ms. E accepted students’ different ideas. At another level, she revealed her own ultimate interest in students’ various contributions. The following episode gives an example of this dynamic. Students were collectively solving the Shell Problem in their small groups. Ms. E encouraged students to discuss how they would solve the problem with their group members. She also asked them to write their solution methods after they reached consensus. Soon she called for whole class discussion and asked for each group’s method.
<Episode UE-10: Students' various ideas and teacher's interest in using equations>

T: On the Plan [written in the worksheet as a sequence for problem solving], how can you solve the problem? I would like to have one volunteer from every group to read what you decided by consensus in your group. One volunteer, Um, Mary. Listen.

Mary: Solve the problem by adding.

T: She said, they are gonna solve the problem by adding. Is that all you wrote?

Mary: Yes.

T: Okay, is there anyone who would like to read over here? Okay, Alex? [To the whole class] Are you listening? Stop one moment. Someone over here has to listen because he may have something different. You need to learn from your friends.

Alex: How can you solve the problem? [He reads the question in the worksheet.] Add 12 plus 18, and you can find your answer.

T: (To Kelsey’s group) What would you say?

Kelsey: We say, how can you solve the problem? [She reads the question.] They can go to the store and buy more shells or they can go to the beach.

T: Okay, this group, what would you say? Billie Jo.

Billie Jo: I say, {Billie Jo’s presentation is interrupted by Logan falling down from his chair. The teacher provides advice about students’ behavior and asks for their attention.} Add 12 to 18, and see how many they need to make 60.

T: Okay, who will say at this table?

Kayla: They need to buy 30 more shells.

T: You are giving us an answer now. We wanna know how we’re gonna get to that answer. All right, okay, boys and girls. There are many different ways some of you tried to solve the problem. What you wanna do is... what are they going to do? They’re just going to buy more shells. That’s true. But we want mathematical problem solving. Some of you said, they have to buy, they have to add 12 and the 18. Um.. But we know they’ve done that, and we found out what that was. So you added that. But then what does that tell you? Yes, Terrance.

Terrance: They buy 30 more shells to make it.

T: Where did you get the..., adding 30 more?

Terrance: How can you solve the problem? [Reads the question in the worksheet.]

T: Where does the number 30 come from?

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Terrance: Because, I know that, 15 plus 15 is 30, and then I say, 18 plus 12 is 30, and so, then 30 more is 60.

T: Pretty good thinking. 15 and 15 is 30, you know. 18 and 12 is 30. Um. Now, though, Sam. We need 60 all together. We have 30. There is an equation we must make right there to have that number. Raise your hand if you think what kind of equation we must make? Boys and girls. You have told me that 12 plus 18 equals 30 (writing the equation, $12 + 18 = 30$ on the board in a vertical format). Terrance said to me, 15 plus 15 equals 30 (writing the equation, $15 + 15 = 30$ on the board), which is really a nice thought and very true. But now we wanna have 60 all together. And there is an equation we must make. Tell us what we need. So, in writing you plan, what you’re gonna have to do is, you still have one more equation you didn’t talk about. One more thing you must do in your planning, talk with your neighbor to figure out.

(Students discuss and the teacher walks around. The teacher briefly checks Mary’s group and moves to Kelsey’s group.)

T: (To Kelsey) What are you thinking they have to do?

Kelsey: Go to the store and buy more. 30 more.

T: That’s true. But what kinds of mathematics do you need to do to figure out the number. You do have to go to there to get it. But how many? How are you gonna figure that out? What’s your plan? How many do you need, your equation?

Kelsey: We can add 30 more to get...

T: Discuss with your friends.

Morgan: 30 plus 30 is 60.

(Students discuss for a while, and then calls for whole-group discussion)

T: Now, we need to find out how many do they have? It says, Roy buys 12 shells. Reba buys 18 shells. Who can tell me what you already did, what you already told me? You said, 12 plus 18, you get 30. This is how many they have now. (She writes "12+18=30 have") But they need, how many, class? They have this (pointing to the number 30). We wanna find out how many more they need to buy. Then, many of you keep telling me the answer. You haven’t talked about the equation to get it. You are using intuitive thinking, which is good. What equation do I write to find how many more I need to buy? What is it, Lara?

Lara: You need to write 30 plus, 30 plus...(looking at her group members).

T: We have, how many do we have? Lara, look, how many do we have?

Lara: 60 shells.
T: 60. We have... No, we want to have 60. I made a mistake. We have 30. How many more do we need?

Lara: We need, 30 more.

T: Correct. But how did you come up with the number?

Lara: Because I know that 3+3=6. That's easy. I know that 3+3=6. So, I know that 30+30=60.

T: So, your thinking is, what can I add, 30, to get 60? Is that what you are saying?

Lara: Yes.

T: There are several ways to doing that. You can say that 60 minus 30 will be

S: 30.

T: 30 more, you must buy.

When Ms. E initiated the discussion, most students presented their ideas with some ambiguity. Ms. E accepted their contributions but was interested in the two specific equations: 12+18=30 and 60-30=30. Some of the students came up with the answer 30, but did not use the subtraction to figure it out. When Ms. E asked how they got the answer, Terrance explained, "I know that, 15 plus 15 is 30, and then I say, 18 plus 12 is 30, and so, then 30 more is 60." He seemed to use the equation 30+?=60, after adding 12 to 18. Note that Terrance said, "Then 30 more is 60." Given the problem was "how many more shells do they need?", it seemed natural or intuitive for students to come up with the equation 30+?=60. As Ms. E kept asking where the answer 30 came from, Terrance provided a rather irrelevant equation (15+15=30) as well as a reasonable explanation, "30 more is 60." Ms. E acknowledged his contribution but she recognized the equation, 15+15=30, was irrelevant to getting the second addend. Ms. E seemed not satisfied because the students did not use the second equation (60-30=30) she expected. She kept telling, "You still have one more equation
you didn't talk about." At this point, it was not clear whether Ms. E thought of the possibility that students might have made another equation, 30+30=60.

Ms. E then gave students time to discuss more, hoping that some group would come up with the second equation using subtraction. While walking around to check students' engagement in the discussion, Ms. E interacted with Kelsey's group. Kelsey provided a vague explanation, "We can add 30 more to get..." but Morgan clearly expressed a reasonable explanation "30 plus 30 is 60." However, Ms. E did not provide any comment on the spot. Later, when she led whole class discussion again, Ms. E evaluated students' contributions to get the answer as "intuitive" and praised for their thinking. When she said, "You haven't talked about the equation to get it", Ms. E implied that students had not used an appropriate equation. In other words, she seemed not to regard the equation 30+30=60 as mathematically valid. Responding to the teacher's consistent question of writing an equation, Lara provided a rationale to use the equation 30+30=60 to get the answer: "I know that 3+3=6. So, I know that 30+30=60." Ms. E checked Lara's explanation whether she thought of what to add to get 60, and accepted her idea. Acknowledging that there were several ways to get the answer, Ms. E finally revealed her interest in using the equation 60-30=30.

There are two possible interpretations of Ms. E's insistence on 60-30=30. She might have intended to provide semantic (conceptual) grounding for the missing addend interpretation of subtraction. However, Ms. E did not connect the conceptual relationship between 30+X=60 and X=60-30, when she had the opportunities to do so. For instance, when she asked to Terrance, "Where did you get the ..., adding 30 more?", Ms. E might know that he was using the equation, 30+X=60. Thanks to the
Morgan's explanation, Ms. E must have known that students had been using the equation. In the whole class discussion, Lara again explained the rationale of using the equation 30+X=60. Instead of connecting the two equations (30+X=60 and X=60-30), Ms. E directly introduced the subtraction equation 60-30=30 as another (or alternative) way to get the answer 30. This leads to a second and more plausible interpretation of Ms. E's insistence on the subtraction equation. She was interested in using the prescribed form (i.e., the two equations, 12+18=30 and 60-30=30). She had been waiting for the answers based on a desire to follow the form. Because students' did not come up with the subtraction equation, Ms. E introduced it even after students' reasonable thinking. This interpretation is consistent with Ms. E's further instruction. When students were supposed to review their solution process following the problem solving sequence specified on the worksheets (understand, plan, solve, and review), Ms. E asked them to check whether they wrote the two specific equations explicitly. There was little room for students to reflect on their "intuitive" thinking of using 30+X=60 and to develop conceptual grounding for a connection of 30+X=60 and X=60-30.

Teacher's specific strategy presented

Ms. E initiated interesting questions but often provided specific hints or expressed her intentions. In Episode UE-11, for instance, students were solving the problem of making different two-digit numbers with 7, 2, 5, and 9. Students reported many numbers they made and Ms. E wrote them on the board. As students continued to present their numbers, they often repeated numbers which were already contributed. In
those cases, too, Ms. E checked off the numbers. At the end of the students’ report, she asked for a strategy which would encompass all possible numbers.

<Episode UE-11: Ms. E’s strategy presented>

T: Okay, everybody, look and see, 59, 57, 29, 25, 72, 97, 52, 27, 79, 95, 75, 92 (pointing to each number written on the board). Let’s see if we can develop a strategy so we can be sure we have them all. What if we did everything we can do. Start from 7. We can have seventy what?

Ss: 72.

T: And a seventy-

Ss: 75.

T: And a seventy-

Ss: 79.

T: Now let’s start from 2. We have twenty-

Ss: 27, 25, 29.

T: Let’s start with 5.

Ss: 57, 52, 59.

T: Starting with 9.

Ss: 95, 92, 97, 99.

T: Now by doing it that way, that’s a strategy that you can be sure you don’t miss any. Now, notice that you have this and you have reverse (connecting 75 and 57). You have this and you have reverse (connecting 25 and 52). You have this and reverse (connecting 29 and 92). This and reverse (connecting 79 and 97). This [72] and reverse... Where is it... There it is (connecting 72 and 27).

In this episode, Ms. E immediately provided a strategy of starting with a specific number in order to solve the given problem systematically. Though the initial question was challenging in nature, she did not ask students to find their own methods.

Moreover, there was no summary or discussion of why such a strategy guarantees that
all possible cases were included. Instead, she immediately presented her own
observation that each number had its reverse (e.g., 25 and 52). When students solved
the problem themselves before presenting to the whole class, their math notebooks
showed that several of them used either of the two strategies (starting with the same
number or using reversed numbers). Owing to the teacher's direct presentation of the
strategies, students had a limited chance to present their ideas. A more important issue
was that Ms. E’s presentation about reversing numbers seemed to confuse the search
for a systematic method to solve the given problem. In other words, starting with the
same number guarantees a systematic inclusion of all possible combinations, whereas
looking for reversed numbers does not.

Teacher’s conceptual concerns

As seen in many episodes described until now, Ms. E was concerned primarily
with students’ procedural development such as the correct use of formal algorithms.
However, there were some instances, relatively infrequent, where she mediated
classroom discussion for students’ conceptual understanding. Episode UE-12 is an
example. Students were solving the problem of Finding a Card Showing a Sum of 79
— There were four choices: 38+21, 82+17, 76+12, and 64+15. Ms. E asked students to
guess without formal computation, which one they thought would be the answer and
mark it. She emphasized “educated guess” but did not initiate classroom discussion as
to how to figure out a possible answer. Ms. E set up the clock and students solved the
problems on their own within the limited time. Ms. E then called for whole class
discussion.
Lara solved the first choice 38+21 and explained that “I have to add 8 plus 1, 8 plus 1 is 9, and 3 plus 2 is 5.” Given that Lara explained tens digit numbers as ones digit, Ms. E asked her to point to each digit. Satisfied with Lara’s correct response, Ms. E emphasized to the whole class that each column should be straight, drawing a broken line vertically between the ones and the tens column. The following episode includes the interaction between Ms. E and two students who solved the second and third choices in the problem.

<Episode UE-12: Teacher’s conceptual mediation>

T: Who would like to do the next one [82+17] for me? Okay, Billie Jo, come on. Now I want you to make sure that you have done it at your desk before you volunteer to do one up here. Here we go (giving Billie Jo chalk).

Billie Jo: [Works on the second choice, 82+17.] I have to add 2 plus 7. 2 plus 7 is 9 (writing 9). And I have to add 8 plus 1, that’s 9 (writing 9).

T: What digits are in the ones column, Billie Jo?

Billie Jo: 2 and 7 (pointing to the numbers).

T: And uh, what digits are in the tens column?

Billie Jo: 8 and 1

T: If we did not have the two, 2 ones and 7 ones (pointing to 2 and 7), can you write what the equation would be? If these two were not there? (Points to the 2 and 7 in the ones column.) (Billie Jo writes 99.) Okay, look at this. If we took this 2 away (erasing 2 of 82), what would we have there?

Billie Jo: 0.

T: Okay. Can you write that down? This numeral would be ...? (Points to 8 of 82 and the erased place.) (Billie Jo just looks at the numbers.) This would be the tens column (pointing to 8 of 82), and this would be the ones column (pointing to the erased place of 82).

Billie Jo: (Writes 80.)

T: Okay, plus.

Billie Jo: (Writes +.)
T: We don't have any ones (erasing 7 of 17). What would be in ones column?

Billie Jo: (Writes 10.)

T: And what would be the answer for this?

Billie Jo: (Writes 90.)

T: Thank you. Who would like to do the third one [76+12]? Okay, Elena? I am looking for those who quietly raise their hands. (Elena comes to the front.) Tell us aloud what you are thinking.

Elena: [Works on the third choice, 76+12.] You have to first add 6 and 2 together, and that's 8 (writing 8). You have to add 7 and 1, that's 8 ones. So... Yup! 76 and 12 isn't 79.

T: O.K. If we lost all our ones (erasing the digits in the ones column, 6 and 2), can you write what the equation would be? (In this way, she leads Elena to write 70+10=80 in a vertical format.)

In this episode, Billie Jo solved the second choice, 82+17 in the same way as Lara in that she did not differentiate the ones digit numbers from the tens. It is not clear whether students conceptually confused the tens digit numbers with the ones digit. They simply might not have used the signifiers for the tens column (e.g., eight [tens] rather than eighty). In other words, when Billie Jo explained, “8 plus 1, that’s 9” in the middle of solving the problem 82+17, she might mean, “8 [tens] plus 1 [tens], that’s 9 [tens].” In any case, Ms. E checked whether Billie Jo could point to each column. Moreover, she asked Billie Jo to assume that there were no ones digit numbers and to write only the tens digit numbers. Ms. E’s concern for “0” in the ones column made the different place values clear, that is 80+10=90 instead of 8 [tens] plus 1 [tens] is 9 [tens].

Elena solved the third choice, 76+12. Like the previous students, she did not use the signifiers for the tens column while explaining how she was solving the problem. Furthermore, she made a mistake by saying “You have to add 7 and 1, that’s 8 ones.”

Elena seemed not to reflect on what had been going on in the interaction between Ms. E

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and the two students who presented earlier. Unlike other students, Elena expressed the immediate issue of whether or not the third choice is 79. Given that the original task was to simply find which one shows a sum of 79, Elena's claim that the third choice was not 79 could be enough to solve the problem. Facing the fact that Elena also dealt with the tens digit numbers like the ones digit, however, Ms. E mediated conceptually, leading Elena to write the equation 70+10=80. Again, Ms. E's concern for "0" in the ones column made it clear what the 7 and 1 in the tens column stood for. In this way, Ms. E seemed conceptually intent on illustrating place value meanings. Note that Ms. E consistently asked students to explain what they were doing, while they were solving a given equation.

**Authoring problems and group processes**

After students solved a word problem using addition and/or subtraction, Ms. E often asked them to make up similar story problems about the animals used in their diorama projects. To facilitate their work, she led students to present various verbs describing the animals' behavior. For individual problem posing, Ms. E distributed two number plates used to make addition or subtraction problems. One plate had a one digit number and the other had a two digit number. Ms. E provided a caution about using the two numbers: students who would make subtraction problems should use the bigger number first—a constraint not needed for addition. However, she never interacted with students around the problematic of selecting operations either in whole class discussions or in small group sessions.

When Ms. E asked each group to author one problem, she encouraged individual students in the group to make their own problems, discuss them, and decide
which problem would be representative for their group. Ms. E set time limits for individual thinking and group discussion. To initiate students' discussion in their groups, Ms. E asked students to collaborate with each other by differentiating their roles in representing their group's problem. For instance, once one problem was chosen as representative for the group, other students in the group whose problems were not chosen wrote the problem, represented it in a concrete way, solved it, or presented the problem to the whole class. In order to encourage students' collaboration, Ms. E frequently praised the groups where students worked together. Episode UE-13 shows how Lara's group collaborated with each other and Ms. E facilitated their group activity. The group adopted Lara's problem in which she incorporated the group members' names and the assigned animal name: "John found 18 rabbits, Alex found 10 rabbits, Tyler found 8 rabbits, Sam found 9 rabbits, how many did they found [find] in all?"

<Episode UE-13: Students' collaboration in authoring a problem>

T: (in Lara's group) You need to learn how to work with a group. When you work with a group, everyone is not gonna be happy or satisfied. Get it done. Those of you who are waiting, if you are dictating, if you are contributing to your group, (looking at the clock on the wall), two more minutes to write this. (The teacher sets up her clock on the table.) Then we're gonna share. Okay, different person writes the equations.

Lara: Who will write the equations?

(Students point to Sam at the same time and Lara gives paper to him.)

T: Another person can draw a picture and then someone is gonna report.

Tyler: I will do the picture.

Alex: I'm gonna report.

Sam: (Reads loudly as he writes) 28.
Lara: 18 first. (Sam writes 1 over 2 out of 28.)

Sam: 18 plus 10 plus (writing an equation in a vertical format).

Lara: Plus 8 plus 9.

John: Wait, first 8 is in the wrong place (the addend 8 was written close to under the 1 of 10, the second addend).

Lara: Sam ...

Sam: (Erases the 8 and writes 9+8 in a vertical format.) Okay, 18 plus 10 is 28, 28 and 9, [pause], what is 9 plus 8?

John: 9 plus 8? You don’t know what that is?

Sam: 17.

Tyler: I am an illustrator. (He starts drawing circles to represent rabbits.)

Alex: After you all finish, I will look at it. (The group counts the number of rabbits together.)

T: (The clock on the table rings.) Aha, this group already drew their [rabbits], boys and girls, stop for one more minute. Take a look (showing Lara group’s worksheet - Rabbits). They’ve done their equations and they are doing their illustrations. So, you need to think, where you are. Okay, one more minute. Then we are ready to report.

In this episode, Lara asked who would write equations for her problem. In other words, since her problem was chosen as the group’s, Lara was yielding the right to write equations to other students. As Sam was appointed as the group writer, the other two students identified their roles as illustrator and as reporter, respectively. But the students did not wait passively for their roles. They actively monitored Sam’s solution process. Lara first suggested that Sam write 18, rather than 28 (18+10). John commented about alignment. After Sam finished writing the equation, Tyler started drawing rabbits for concrete representation. The teacher praised this group’s collaboration.
**Other Classroom Activities**

Each lesson had one or two extra mathematics activities in addition to estimation and problem solving activities. These extra activities were not usually related to each other, nor were they related to estimation or problem solving activities. Table 5.2 provides a brief description of the main activities. Generally, students seemed to enjoy these mathematics activities. They often laughed a lot while engaged in them.

**Table 5.2 General Classroom Activities Used in Ms. E's Class**

<table>
<thead>
<tr>
<th>Name*</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Composing and Decomposing a 10</td>
<td>One student breaks a tens stick into two parts and the other student represents it by an equation. (e.g., 1+9=10). With the same strategy, students make equations of subtraction (e.g., 10-1=9).</td>
</tr>
<tr>
<td>Throwing a Bag for Basic Facts</td>
<td>The teacher throws a pouch (small bag) to a student as she states basic addition or subtraction facts (e.g., 2 and 7). Then the student throws the pouch back to the teacher with the answer.</td>
</tr>
<tr>
<td>Using Students-drawn Fish to Learn Place Value</td>
<td>Each student draws 10 fish on a piece of paper. Additionally, ten students draw 1 fish each on another paper. By moving the fish among the columns of ones, tens, and hundreds, students learn place-value concept.</td>
</tr>
<tr>
<td>Using Base Ten Blocks for Computation</td>
<td>The teacher asks students to represent a number (e.g., 111) with base ten blocks and leads them to add or subtract some number (e.g., +16).</td>
</tr>
<tr>
<td>Making Numbers with Fingers</td>
<td>Students stand up and show their fingers. As the teacher says a number, they represent it with their fingers. Since the number sometimes is more than 10 (e.g., 22), they have to collaborate with one another.</td>
</tr>
<tr>
<td>Representing a Number in Different Ways</td>
<td>The teacher writes a number on the board (e.g., 6). Students represent the number in different ways (e.g., six, 10-4, VI, 3+3, 1+1+1+1+1+1, 00000, etc).</td>
</tr>
<tr>
<td>Paper-and-pencil Test for Basic Facts</td>
<td>Students are supposed to solve 100 basic addition or subtraction equations (e.g., 9-1=?) written in a vertical format within 5 minutes. Whoever gets a perfect score is praised and recorded on a class bulletin board.</td>
</tr>
</tbody>
</table>

*Note: Names are assigned by the author to refer to the activity in the main text.*
Issues of mathematical depth

Though Ms. E used various and enjoyable activities, there was often a lack of mathematical depth. For instance, in the activity Making Numbers with Fingers, Ms. E asked groups of four students to represent 22 (see Episode UE-14). Students used various combinations of numbers with their fingers. Ms. E walked around checking whether the groups made 22. When she confirmed their answers, the students in each group expressed their excitement.

<Episode UE-14: Making 22 with fingers among four students>

T: Stop, all hands down. Show me 22. (She keeps walking and checking whether each group made 22. She stops in one group.) Okay, good. (The students cheer.) 5, 10, 15, 20, 25, 30 (counting the fingers in another group), oops (covering her mouth with her hand). We are making 22. Work it out, I'll come back. (She moves toward other groups.) 5, 10, 15, 20, 21, 22 (counting the fingers). (The students in this group made more than 22. But while Ms. E counts, they change by folding some fingers.) I want you to demonstrate in the class in a minute. (They cheer and Ms. E keeps checking in three other groups.) Class, may I have your attention now. You did very interesting configurations. For example, I want you to look over here (going toward a group). Look at what children over there did. Take a look at this. (To the group) Hold your hands. (Two students facing each other put their hands up together, making 20. The other two students facing each other add one finger each in the middle of the 20.) I want you to see how they made it balanced and symmetric. It's kind of artistic. These are like statue (pointing to the fingers making 20), count with me class.

Ss: 10, 20, 21, 22.

T: All right. We said 10, 20 (pointing to the corresponding fingers), and this is where you get confused when you count money. We can also count by 5s. Let's count by 5. Ready?

Ss: 5, 10, 15, 20, 21. (some students say 25), 22.

T: Who said 25? Hooo, got you! What is it? (Holds one finger making 21.)

Ss: 21.

In this episode, Ms. E asked students to count the number of fingers represented by their peers. She also picked out something interesting during her
observation of students' activities. She highlighted to the whole class a combination of numbers in which two students facing each other showed 10 fingers each and another two students showed 1 finger each (or 10+10+1+1). Ms. E emphasized the artistic nature of the combination. Similarly, when groups of four students represented a number using their fingers and the rest of the class figured out the number, Ms. E asked students to count the fingers in different ways such as counting by ones, fives, and tens. Though students were excitedly engaged in this activity probably because of their physical movement, the main thing they had to do was counting, which was already too familiar to them. This activity could have been used to deepen students' mathematical understanding of decomposing a number in different ways. For instance, the number 22 can be decomposed differently using four numbers. Within this context, the "artistic" decomposition of 22 by 10+10+1+1 could have been highlighted as mathematically elegant.

The interview data of Ms. E's reflection on this activity support the claim that she was unmindful of the mathematical value of students' engagement in this activity. She explained, "I was thinking of money, a nickel, a dime, a penny. That's a part of counting money one thing. That's one of things in my mind and another thing is just able to maneuver numbers, have a little control over it." In other words, Ms. E did not have specific mathematical objectives for the activity except thinking of multiple addends and reinforcing counting by ones, fives, and tens.

In the activity Composing and Decomposing a 10 shown in Episode UE-15, Ms. E demonstrated the activity with examples. She separated a tens stick into 1 and 9 and asked students to guess which equation they could write. Ms. E then switched the
cubes and led students to experience a turn-around fact or commutativity of addition (e.g., 1+9=9+1=10). She showed one more example by showing a tens stick in one hand and nothing in another hand, which corresponded to the equation 10+0=10. The teacher expressed her concern about the number of equations students made.

<Episode UE-15: Ms. E's concern about the number of equations>

T:  All right. Here we go. Now eyes up here. "As," raise your hands. [She had divided students into “As” and “Bs” for this activity.] You are the recorder. “Bs”, raise your hands. You are the manipulator, manipulate those blocks. Everything you do is going to equal 10. (She picks up a tens stick and shows it.) You are going to show your partner how many ways you can. For example, watch me. (She separates the stick into 1 and 9 and holds them in each hand.) Look, your partner is gonna write 1 plus ...

Ss: Ten, nine, one plus nine!

T: One plus nine equals what? (Writes 1+9= on the board.)

Ss: Ten.

T: Ten (finishing the equation). Wait a minute! Manipulator, look what you are gonna do also. Look, one plus nine, watch what I am gonna do. (She crosses her hands with the cubes.) See that? Watch again. (She switches her hands again.) The recorder, what do you write then?

Ss: Nine plus one equals ten.

T: Okay. (She writes 9+1=10 on the board.) Watch me. This is a tricky one. (She shows one empty hand and the tens stick in the other hand.) What is this?

Ss: Zero plus ten.

T: Equals?

Ss: Ten.

T: (Writes 0+10=10 on the board). Watch me, tricky. (She crosses her hands.)

Ss: Ten plus zero equals ten.

T: (Writes 10+0=10 on the board). Okay, what I want you to do is see how many ways you can make ten. Wait a minute, how many ways you can make ten with this (showing the stick). Okay? I will give you ... Just start and I'll tell you when time's up. Record going down. Now you wanna not make it so big so you have room, but not so tiny you

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can’t see. Begin! (While students were working together in their seats, the teacher walks around each group.) One more minute (setting up the clock).

Ss: Awww!

T: You may not get them all. That’s okay. Do as many as you can. (She checks the front groups.) Okay. Stop. Look this way. Tell how many equations you made.

Ss: 1, 2, 3... (counts) 12, 8..

T: You did a lot. Who remembers... Look this way, eyes up here, Brittany, thank you. Who remembers what we call it if we have 1 plus 9 and 9 plus 1? (Points to 1+9=10 and 9+1=10 on the board.) What do we call this fact?

S: Turn-around.

T: Wait until I call you, please. Mary?

Mary: Turn-around.

T: Turn-around fact. How many, uh, did you... Say softly, how many equations you got, class?

Ss: 12, 9, 8, 11, 10,

T: So, there are many equations you can make. All of them equals 10. Somebody asked me, could we do some subtraction now. Who asked that? Aha. Lara wants to know about some subtraction. Erase your board and switch. Let the manipulator to be the recorder. Now, let’s see how you can do this. Okay, here we go. Watch me. Watch, if I have 10 (showing the stick), minus 3 (taking out 3 cubes and holding them behind her back), what’s left? (She shows the cubes left on her hand.)

Ss: Seven.

T: Seven. So that’s how you are going to do it. Okay, I am gonna give you two minutes. And let’s see how many equations you can get. Go! (She sets up the clock and walks around to provide individual help.) Stop. Eyes this way. How many equations did you get, class?

Ss: 7, 6, 9...

T: How many of you got 9? Who got more than 9? Is that because you remembered to say minus 0 or minus 10? Okay, erase that. Do not switch right now. We have another one for you. Yes, sir?

Alex: Do we have time today?

T: But we don’t have time today. We will do it another day.
Ss: Awww! [Students want to play more.]

Note that Ms. E checked how many equations the students made, but did not ask for which equations there were. Rather, she reinforced students to remember a turn-around fact or the commutativity of addition. The teacher quickly moved to a similar activity where students were supposed to use the same tens stick for subtraction. Ms. E demonstrated how to play by separating one tens stick into 7 and 3, but showing only 7, which corresponded to 10-3=7. After the students played, she again checked how many equations students made and acknowledged that students could make lots of equations. Note that students expressed their enthusiasm in doing this activity, when the teacher called off the lesson.

Memorization emphasized

Ms. E often stressed memorization of basic addition and subtraction facts, and periodically tested students' skills of them. On the bulletin board in the classroom there was a section called “100 equations in 5 minutes: 100% club.” For each perfect score on a paper-and-pencil test, Ms. E would record a check and she marked up to three times for each individual child. Most students were successful with addition problems. Because subtraction was just introduced, only a few students got a perfect score on subtraction problems. To develop students' skills in solving basic subtraction problems (e.g., 12-4, 7-3, 16-8), Ms. E encouraged students to work on such problems every day at home and provided a practice test to show students what they needed to work on. After the practice test, she reminded students that they just started learning subtraction so they should not feel bad about the test results. Only one student got a perfect score on the test. Ms. E hugged the student and put a check mark on the bulletin board. She
then provided some reasons why students should memorize basic facts (see Episode UE-16). She claimed that memorizing basic facts could make students focus on the essence of a problem to be solved and could make them feel free in doing mathematics. Note that this is an instrumental rationale, rather than number sense or some other intrinsic value. With this rationale Ms. E urged students to do practice subtraction problems on a daily basis.

<Episode UE-16: Rationale of memorization>

T: How many did you get close? Good. Just do a little bit every night, it will be easier and easier. Boys and girls, look this way. Give me your attention. Eyes this way. Let me explain something to you about this. To memorize brings you up to different thinking. If you learn your facts just like this (snapping), you can concentrate on problems. You don't have to think, let's see, 4 and 3 is 4, 5, 6, 7 (slowly counting by her fingers), you will do it three times. You can concentrate on the problem you are trying to solve if you know facts. Study them between 5 to 10 minutes every night, and you can get down like that (snapping). And you will find yourself freer in mathematics.

Understanding base 10 system

Ms. E introduced various base systems and led students to experience how numbers could work with different bases. For instance, in the base 4 system, students learned that they can not have 4 ones in ones place; so, they have to carry them to the next higher place. For those special cases, students often chanted, "Oh, oh, we have a problem." Similarly, Ms. E's main strategy to help students understand base 10 system was to problematize specific cases wherein students had 10 ones, 10 tens, or 10 hundreds. For instance, every student drew 10 fish on one piece of paper and 10 students drew 1 extra fish each on another piece of paper. Ms. E then asked students to post fish for the ones place of the board, and initiated a discussion to help students understand base 10 system. She asked whether putting the 10 fish together in the ones place would be okay. With students’ unanimous negative response to the question
saying "You can’t have 10 ones in base 10", Ms. E stimulated them to recall what they did in the special case. This kind of interaction happened in a similar way when students put ten 10-fish pictures in the tens column.

Episode UE-17 illustrates how Ms. E interacted with an individual child helping him understand the base 10 system and its numerical representation. Ms. E always gave students a chance to solve a given problem for themselves. While students were solving in their seats, Ms. E briefly checked their work before calling for whole class discussion. Episode UE-17 is a rare case where Ms. E extensively interacted with one child who was solving an equation. With Ms. E’s guidance, students put the fish-picture on the board and represented it with a three digit number. Ms. E led students to change the number of fish in the tens or ones place on the board, and to write them with numbers. As students became familiar with representing the number of fish, Ms. E connected the change in the number of fish with addition or subtraction. Before Episode UE-17, there were 139 fish pictures on the board in the form of 1 hundred, 3 tens, and 9 ones. Ms. E asked a student to post one more fish in the ones place and wrote the equation 139+1 in a vertical format. The episode starts with Ms. E’s observation of John who was attempting to solve the equation.

<Episode UE-17: Ms. E’s interaction with John solving 139+1>

T: Oh, add it. 139 plus 1. Where is the ones column? What do you have to add together? Which numerals?

John: Nine and the three? (Points to the numbers.)


John: Nine and one.

T: Exactly. What is that?
John: Ten.


John: (Writes 10 under the addend 1.)

T: Is that okay? What's wrong with this?

John: We have ten in the ones [place].

T: What can you do?

John: Ten, put it on the [tens].

T: Show me. Show me what you do.

John: (Crosses out the 10 and starts writing 1 under the plus sign.)

T: Oh-oh, stay in your column. How many? If you put the ten in the tens column, what's in ones column?

John: Zero.

T: Put it then. Under there.

John: (Writes 0 in the left of the 10 crossed out.)

T: Under, goopy! That's side.

John: (Erases the 0 and puts it under 3 of 139.)

T: That's above. (She laughs.) Look, there is under. Look, under! (Points to the place under the crossed out 10.)

John: (Starts erasing the crossed out 10.)

T: Don't erase it.

John: (Writes 0 under the crossed out 10.)

T: Where do you bring the ten?

John: [Inaudible.]

T: In the tens column. Where is tens column? Which number?

John: (Points to the 3 of 139.)

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T: Yeah, write above it.

John: (Points to the left side of the crossed out 10.)

T: Above it, goopy. (She points to the place above 3.)

John: (Writes 10 above the 3.)

T: Here we go. Wait, do you bring 1 ten or 10 tens?

John: [Inaudible.]

T: You said ten? Put 1 there.

John: (Writes 1 above the 10.)

T: Okay. 1 ten and 3 tens, how many tens?

John: Four?

T: Uh-huh [agreement]. Put it down. (She goes back to the front of the classroom and says to the whole class.) Okay, boys and girls, look up here. Eyes this way, one, two, three (showing her fingers). Three. Eyes this way. Look here. It's really important that we go straight under (drawing a broken line between ones and tens column in the equation of 139+1). Look here. 9 ones plus 1 one is how many ones, class?

Ss: Ten.

T: (Writes 10 under 1.) Now I put my ten there. John, tell me what's wrong here?

John: You can't put a ten in ones column.

T: You cannot put a ten in ones column. Let's do something. Who can explain what I am doing? Who can explain, Charles?

Charles: You have to take 1 away and move.

T: I leave 0 and I take 1 which represents 10 (crossing out the 10 and writing 0 under it). And I bring to where?

Charles: To the ones side.

T: This is ones side (pointing to the ones column). I bring it to which side? (Points to the tens column.)

Charles: Tens side.

T: The tens side. I bring it to the tens side. (She writes a little 1 above the 3 of 139.) Now what do I have? 1 tens plus...
Charles: 3 tens

T: Equals how many tens, class?

Ss: Four. (Ms. E writes 4 on the left of the 0.)

In this episode, Ms. E asked John to add the ones digits and he answered 10. Ms. E then said to him, “Put it [10] down. Under it [the addend 1].” Responding to Ms. E’s direction, John wrote the number 10 in the ones column. When Ms. E asked what’s wrong with writing the 10, John easily recognized the problem (i.e., 10 in the ones place), probably because of his experience with various base systems. Moreover, John knew that he should put the ten in the tens column. While he remembered the rule for the base 10 system and what to do when having a ten in the ones column, John seemed to have a difficulty in writing the numbers in appropriate columns. John knew that if he put the ten in the tens column, then 0 would be left in the ones column. However, it took much time for him to write the 0 under the 10 (the sum of ones digit numbers, 9+1). When Ms. E asked, “where do you bring the ten?” John seemed to answer correctly and pointed out the tens column. Ms. E then asked, “write above it [3 in 139]” and John wrote 10 above the 3. It was not clear whether John represented the 10 as 10 ones or 1 ten. Indeed, we can represent “ten” by “10” or by the “1” in “10”. Ms. E’s question, “Do you bring 1 ten or 10 tens?” revealed an important conceptual distinction. Given the previous interactions with Ms. E, John did not seem to mean “10 tens.” John simply answered, “ten”, but it is very complex how to represent the ten keeping up with the conceptual underpinnings.

A possible way would have been to use an expanded form of a standard procedure: 139+1= 10+130=140 in a vertical format. In this way, John could have
understood the 10 as the sum of the ones digit numbers, the 1 in the number as representing the "10" carried over in the tens column, and the 0 in the number as meaning that there was nothing left in the ones column after carrying the "10". In this episode, Ms. E seemed to help John understand the standard convention of base 10 numerical representation. If she was interested in producing a correct answer, there were other easier ways to figure it out such as counting on (i.e., 139, 140) or informal computation (e.g., 9+1=10, 30 more is 40, and 100 more is 140). Note that Ms. E gave the problem to students in the process of changing the number of fish on the board and there were fish-pictures representing 1 hundred, 3 tens, and 10 ones. Students also had lots of concrete experience of trading ten 1-fish into one 10-fish. It might have been easier for John to figure out the answer from the fish-picture and then to represent it with numbers. Moreover, the concrete representation could foster conceptual understanding regarding the process of regrouping 10 ones into 1 tens. Unfortunately, because of the complex meaning of writing the "10" in the tens column, the concrete representation seemed not to be helpful in this specific episode. In other words, the complexity comes from our symbolic representational system in which "1" represents "ten" in the number 10.

Anyway, Ms. E did not use the concrete representation in helping John. She rather fell back on two kinds of procedural knowledge. As evidenced by the words "under" and "above", Ms. E was reinforcing the standard convention of using the formal algorithm. In particular, when John recognized that he should put the "ten" in the tens column, Ms. E's direction of writing above the 3 in the 139 seemed to hinder John from developing conceptual understanding. As Ms. E were not able to approach
this conceptually, she simply asked, “Put 1 there [in the tens column].” A second procedural rule Ms. E referred to in this episode was the rule in base 10 system: You can’t have 10 ones in a ones column. The rule might be conceptually oriented (i.e., renaming 10 ones into 1 tens), but recalling the rule itself was not sufficient. Note that John in this episode recalled the rule correctly and knew what he was supposed to do. However, the rule underspecified what to do in this episode.

After interacting with John, Ms. E emphasized line alignment to the whole class and also simulated the case of putting 10 (the sum of the ones digits) in the ones column. Ms. E checked whether John still recognized the problematic situation and he easily recalled the rule of base 10 system. This seems to reveal that Ms. E did not fully grasp what happening in the interaction with John, or what might be the conceptual difficulties in John’s problem solving process. Confronting the complexities in this episode, Ms. E reverted to her original interest in using a standard algorithm.

**Dealing with students’ wrong answers**

In the activity Representing a Number in Different Ways, students provided various answers. For instance, students’ representations for 6 included “six” and other corresponding words in Italian and Spanish, 6 tally marks, 12-6, 6-0, 10-4, and 13-7. Episode UE-18 starts with a student’s incorrect contribution in renaming 6.

<Episode UE-18: Students’ wrong answers and corrections>

Terrance: 0 minus 6.

T: Okay, 0-6 is not an exact turn-around fact. You can’t take 6 from 0, but I can see you take a turn-around fact (pointing to 6-0 on the board). There is a turn-around fact you can say using 6 and 0 (pointing to 6 and 0, respectively, in 6-0 on the board).

Terrance: 6 plus 0.
T: There are many different ways. Uh, David?

David: 15 minus 8.

T: (Writes 15-8 on the board.) Let me show you one... This is one. In Roman numerals, this is a five (writing V on the board). Does anybody know what to do to make that 5 into 6 in Roman numerals? Who knows? (Several students raise their hands.) Tierany?

Tierany: Put a line on the side.

T: A line on the side (put I next to V). 5 plus I is 6. (Suzannah raises her hand.) (To Suzannah) Yes, Ma'am.

Suzannah: 15 take away 8 is 7.

T: Huh?

Suzannah: 15 take away 8 is 7.

T: Aha, [I] wondered if anybody would catch that. 15 minus 8 is 7. (Ms. E circles 15-8 on the board.) Go and get a treat! If you try something like that, you can get a treat. Good for you.

In this episode, Terrance said an incorrect answer, 0-6 and Ms. E regarded his contribution as using a turn-around fact of 6-0 that had been offered by other students. There are two possible interpretations of why Terrance produced such an incorrect answer. First, he might have overgeneralized a turn-around fact: You can use a turn-around fact with any operation. This kind of overgeneralization is a frequent occurrence in developing knowledge of mathematical rules (Matz, 1980). Second, he might have developed an incorrect rule by replacing addition with subtraction in the correct turn-around rule. This simpler sort of substitution error almost never occurs, as students tend to mimic the many correct instances they encounter in the classroom. Responding to Terrance’s error, Ms. E pointed out 0-6 is not an exact turn-around fact to 6-0 and provided a conceptual rationale saying, “You can’t take 6 from 0." She also acknowledged Terrance’s attempt to use a turn-around fact for 6-0. Ms. E encouraged
him to come up with a correct turn-around fact using 6 and 0. As he was able to provide a correct answer 6 plus 0, Ms. E immediately asked for other multiple ways of renaming 6.

If Terrance's original difficulty was only from incorrectly replacing operation, Ms. E's reaction of directing a correct turn-around fact with 6 and 0 might be enough. However, if Terrance's difficulty was from the overgeneralization of a turn-around fact, Ms. E's mediation was not sufficient. She could have focused on the conceptual misunderstanding behind the overgeneralization. Ms. E could have checked whether Terrance realized the restriction of using a turn-around fact. Given that Ms. E quickly changed the subject around turn-around fact and immediately encouraged other students to present other methods of renaming 6, she seemed either to deal with Terrance's error as trivial or to ignore the deep nature of his conceptual difficulty.

In the same episode, Ms. E did not notice David's computation error (15 minus 8 for renaming 6) and wrote them on the board. Suzannah soon corrected David's mistake and Ms. E praised her and gave her a treat as a reward. Ms. E's praise of her correction of the wrong answer seemed to lead students to closely examine their peers' contributions and the teacher's records on the board in subsequent classroom activities. For instance, when Suzannah said, "7 plus 3", as one way to say 10, Ms. E accidently wrote "7+2" on the board. Immediately, students pointed out her mistake by saying that "7 plus 2 is 9" or "7 plus 3 {is 10}!"

Episode UE-18 illustrated Ms. E's pedagogical emphasis on procedural knowledge over conceptual underpinning. While she did not probe the conceptual difficulty Terrance might have, Ms. E lavishly praised Suzannah who pointed out her
peer's computational mistake. Reflecting Ms. E's primary concern for procedures or algorithms, students' contributions in either whole class or small group sessions were usually concerned with making sure whether they used correct procedures or produced right answers. Episode UE-19 is an exceptional case where students' small group interaction was conceptually based. Morgan and Bethany compared and contrasted their representations of 9. Because Ms. E asked students to copy something different from their partners, Morgan was copying some of Bethany's representations she did not write. While copying, Morgan soon recognized that 1-10 is not appropriate. Moreover, Morgan helped Bethany understand that the bigger number 10 cannot be taken away from the smaller number 1. Note that Morgan's demonstration was the same as the strategy Ms. E used in the case of subtracting a bigger number from a smaller one.

<Episode UE-19: Student's correction of mistake in small group>

(Morgan and Bethany wrote about 9. Morgan is copying Bethany's representations that she didn't get herself. The representations include 9 circles, 9+0, 9-0, 8+1, 10-1, 1-10.)

Morgan: You can't take away 10 from 1. (She crosses out 1-10 in her notebook.)

Bethany: (After seeing that Morgan crosses out 1-10, Bethany scribbles 10-1 and 1-10 in her notebook.)

Morgan: You have one (picking up the number plate in her hand and showing to Bethany), take away 10.

Bethany: (Takes the number plate first and mimics taking out something from the hand.)

Morgan: No.

In the activity Representing a Number in Different Ways, the main strategies for students were words (e.g., ten for 10) and expressions (e.g., 45-35 for 10).

However, they sometimes provided unexpected, creative answers. For instance, in the
Episode UE-20, Elena provided "VV" for 10. Note that Ms. E introduced the Roman numeral V, which was used to represent 6. Ms. E acknowledged that Elena’s answer is not correct, but lavishly praised for her original thinking. Instead of asking Elena to explain how she came up with the idea, the teacher interpreted her answer to the whole class. Suzannah provided a right answer of Roman numeral for 10. However, only Elena got a treat with the teacher’s repeated praise.

\(<\text{Episode UE-20: Elena’s wrong answer but good thinking praised}\>\)

Elena: Uh, Roman numbers.

T: Roman numerals? Do you know what the Roman numeral for 10 is?

Elena: VV [Vee Vee]

T: That’s good thinking! You know, what she said? She said, “V”. She knew that V is for 5 (writing V on the board); so, V.V. (writing another V). You know what? I think that’s extremely good thinking. I am so proud of you. Go and get your treat. That is not the answer, but that was good thinking. Do you know... (To Elena) Go and get the treat! Who does know what the Roman numeral for 10 is? What is it?

Suzannah: X and put two lines.

T: You can put the two lines (after writing X and add two lines both on the top and on the bottom of that letter). But that was such a good thinking on the part of you. I’m so proud of you.

\(\text{Mathematical difference considered}\)

In the activity Representing a Number in Different Ways, the class almost always accepted students’ answers as different from those which had been contributed. An exception occurred when the class represented 10 in multiple methods. Episode UE-21 starts with the teacher’s request of students’ report to the whole class. After three students’ representations by a word, tally marks, and equation, Kelsey and Bethany provided 10 circles and 10 stars respectively. After drawing circles and stars, Ms. E specified the meaning of difference in representation. In other words, simply
using different pictures did not count as mathematically different. Students were expected to recognize what made different from those that had been contributed by their peers.

<Episode UE-21: The meaning of difference in representing 10>

T: Who can tell one way for 10 (writing the number 10 on the board)? Okay, starting with Arterrion. (Each student reports in turn according to their seats. As the student says their answer, Ms. E writes it on the board.)

Arterrion: T- e-n.

T: You wrote the word “t-e-n.” If you put “t-e-n”, give yourself a check. Billie Jo.

Billie Jo: Ten tally.

T: People can tally. (Drawing tally marks) one, two, three, four, five, (pause) one, two, three, four, five. If you did, give yourself a check. Logan? (He is looking at his note.) Let’s do this as quickly as we can.

Logan: 4 plus 6.

T: (Writing 4+6) 4 plus 6. Now, you could’ve written 4 plus 6 in that way. Or you could’ve written 4 plus 6 in this way (writing 4+6 in a vertical format), vertical or horizontal. Kelsey?

Kelsey: Uh, 10 circles.

T: 10 circles.

Bethany: 10 stars.

T: 10 stars! (She laughs, draws stars, counts them by one, and draws one more star.) Now, we’re not gonna have all different ones. Let’s... It’s sufficient, if you made a 10 picture or abstract picture, that’s (showing her second finger, signaling 1). If you did that at all, no matter what your picture is, that is one thing. Okay?

The teacher’s advice on superficial differences in representing a number was not translated into students’ small group activity two days later. Instead of using one number for the whole class, Ms. E led two students to share one number and to compare their representations. She explicitly explained the objective of the activity,
"We are just gonna see how many you can rename and we are gonna compare." As usual, the teacher set the time to represent a given number in various ways. Episode UE-22 begins with Ms. E’s instruction after the assigned time. Consistent with the professed objective of the activity, she asked students to first count how many ways they represented a given number. She then asked them to compare and contrast their representations and to copy something different from their partners.

**<Episode UE-22: Comparing and contrasting two students’ representations of 7>

T: Pencils up! Oh, someone is writing the last one. Thank you. Put out your red pen. First of all, count how many you got down and write that number. How many? Write that number with your red pen. Secondly, look at your partner and you are gonna see how many you got that are the same. Put a check by every one you have that is the same. And write down the equation of every one that is different from yours. If you have the same thing, put a check. If they have something different, you get what they have.

(The students check each other’s work with their red pens, and Ms. E walks around. Kelsey and Logan are partners. Kelsey’s representations are 14-7=7, 7 stars, 7 circles, 7 short underlines, 7X’s, 7e’s, and 7K’s. Logan’s representations are 7+0, 7 circles, 14-7, 7, 8-1, 6+1, 9-2, and 5+2.)

T: (To Kelsey) Did you have all the same that he [Logan] had?
Kelsey: No, only two.
T: Okay, does he have something you don’t have?
Kelsey: Yeah, he didn’t put the stars.
T: (To Logan) Did you put the stars down then? Put stars! (Logan tries to draw 7 stars but seems not to know how to draw them. He substitutes them with another symbols.) Does he have something you don’t have?
Kelsey: Yeah, 7.
T: Put it down. (Kelsey writes 7 and 8-1)

Whereas Logan represented 7 mainly by numerical expressions such as 7+0 and 9-2, his partner Kelsey represented 7 mainly by objects such as 7 stars and 7e’s (eeeeeee). Indeed, most of Kelsey’s representations were the same in that she drew 7
objects with different pictures or letters. However, according to the Ms. E’s request, Logan was supposed to copy Kelsey’s representations which he did not have.

**TEACHER’S APPROACHES**

From the classroom teaching practices described above, the following characteristics of the teacher’s actions can be observed (see Table 5.3).

<table>
<thead>
<tr>
<th>Table 5.3 Characteristics of Ms. E’s Teaching Practices</th>
</tr>
</thead>
<tbody>
<tr>
<td>• provided students with an opportunity to solve a given problem for themselves.</td>
</tr>
<tr>
<td>• walked around and provided individual help while students worked in their seats.</td>
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<tr>
<td>• frequently used small-group format and encouraged students to work with each other.</td>
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<tr>
<td>• encouraged students to present their solution methods.</td>
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<tr>
<td>• shared her positive expectation with every child.</td>
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<tr>
<td>• acknowledged that students could be wrong and accepted students’ mistakes.</td>
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<tr>
<td>• emphasized the learning process of problem solving.</td>
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<tr>
<td>• expressed excitement about students’ novel ideas and lavishly praised.</td>
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<tr>
<td>• repeated or interpreted students’ ideas to the whole class.</td>
</tr>
<tr>
<td>• used manipulative materials and tried to connect symbolic representation with pictorial representation.</td>
</tr>
<tr>
<td>• sometimes posed more challenging questions after students solved a given problem.</td>
</tr>
<tr>
<td>• asked students to pose a word problem for addition and subtraction.</td>
</tr>
<tr>
<td>• sometimes asked for different solution methods to given problems.</td>
</tr>
<tr>
<td>• used her observation of students’ activities for classroom discussion.</td>
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<tr>
<td>• sometimes changed the role of questioning and answering with students.</td>
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<tr>
<td>• used an enjoyable activity format for students.</td>
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<tr>
<td>• sometimes provided her own solution strategies instead of letting students invent them.</td>
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<tr>
<td>• emphasized line alignment and the order of computation when confronted with conceptual complexities in interacting with students.</td>
</tr>
<tr>
<td>• provided direct explanation, a hint, an example or chose the right answer with praise, when students had different answers.</td>
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<tr>
<td>• directly expressed interest in using algorithm, even after students’ novel ideas.</td>
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<tr>
<td>• emphasized memorization on the basis of instrumental rationale.</td>
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<tr>
<td>• often controlled the whole class discussion in one direction.</td>
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<tr>
<td>• rarely probed students’ different ideas.</td>
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<tr>
<td>• sometimes did not use students’ reasonable argument for their answers.</td>
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</table>

The teaching practices in the upper portion of the table are generally consistent with a student-centered approach in which she was concerned about students’ participation in classroom activities and attempted to solicit students’ ideas in many
ways. She encouraged students to solve problems for themselves and to present their ideas to the whole class. In order to encourage students’ willingness to present their ideas, Ms. E emphasized the learning process over a right answer and welcomed their questions and mistakes. She allowed students to use their own interpretations in solving an expanded word problem. She also provided much praise for students’ novel ideas regardless of the correctness of the answer. Moreover, Ms. E sometimes provided a more challenging question and asked for different representations even after students got a right answer. Facilitating students’ discussion and using manipulative materials for formal algorithm also form part of Ms. E’s student-centered teaching practices.

Against this compelling evidence for a student-centered approach, the teaching practices in the lower portion of the table reveal a teacher-centered approach in which Ms. E takes responsibility for directing classroom discussion and authorizing classroom knowledge. At one level, Ms. E seemed to be interested in students’ own ideas and actually managed classroom social processes in order to emphasize their presentations. At another level, Ms. E kept reinforcing her curricular intentions regardless of students’ ideas.

The key to resolve the apparent disparity in Ms. E’s teaching practices is the quality and content of the mathematical ideas discussed in the classroom. The teacher’s primary focus throughout various classroom activities was on whether students correctly used standard algorithms, rather than on whether they understood the conceptual underpinnings they were studying. There are many examples which support the claim that Ms. E’s instructional focus on standard algorithm supercedes any other
foci. In the estimation activity, she led students to use subtraction to figure out how many items were left in a jar. As Arterrion suggested in Episode UE-2, counting the items left in the jar would be practically and conceptually easier than counting the items taken out and solving a difficult equation that always required regrouping from both the hundreds and tens digit. Whereas Ms. E encouraged students to solve such an equation for themselves, she expected them to use a standard algorithm, rather than invent their own solution methods (e.g., Episode UE-3). She encouraged students to present their solution methods, but she controlled the direction of their presentation, as evidenced by the statement, "What should we do first?" (indicating the ones column for the standard subtraction algorithm).

A similar pattern occurs around problem solving activity. Ms. E specifically encouraged students to discuss in their small groups and to present a group's method based on consensus, as seen in Episode UE-10. She apparently listened to their ideas but actually she had been listening for specific equations to solve a given problem. Her primary concern in listening was checking students' contributions against her preconceived solution method. This prevented her from responding to students' explanation, which sometimes included clear and reasonable methods and argumentations.

Ms. E emphasized memorization of basic facts and administered periodic testing to help her students become confident in computation. She expressed much pride in her students' achievement in those tests. Her stress on memorization and automaticity was rooted in her own learning experience of mathematics. Ms. E felt insecure in mathematics and attributed it to the lack of memorization:
A part of my problem is I didn't fully memorize 3 plus 4 is 7. Because of my
trouble to do any equation or problem solving or anything, I didn't ever trust
my answer. Every time I had to add or subtract, I did it at least 3 times, if I got
to correct. I didn't have confidence in my computational skills. I think that's
where it lays with me. ... I was judging how good I was in math only on my
computational skills and how quick I was. I felt deficient.

This was Ms. E's motivation to start emphasizing memorization of basic facts in her
teaching. She had not given students tests on memorization with limited time until she
attended a school-based workshop in which a speaker from another school shared the
positive effects of memorization of basic facts on students' computation skills. The
speaker's experience-based talk made sense to Ms. E and she decided to try it in her
own teaching. She found once they memorized students were excited and proud of
themselves., Ms. E talked with parents to urge them to help their children memorize
basic addition and subtraction facts. At first, parents tended to be reluctant, but they
became satisfied with their children's success with memorization. Consequently,
stressing memorization and automaticity became the most important element in Ms.
E's teaching approach.

To be fair, there were some instances where Ms. E mediated classroom
discussions for students' conceptual understanding. For instance, Ms. E was
conceptually intent on interacting with the students as they considered the different
choices to find a sum of 79 (see Episode UE-12). In particular, she checked whether
the students were able to identify each digit and to compute only with the tens digits in
order to help them understand place value meanings. However, in many cases, Ms. E
seemed not to diagnose the conceptual bases of students difficulties (e.g., Episode UE-
18). When she were not able to proceed conceptually, Ms. E immediately fell back on
using standard algorithms (e.g., Episode UE-17).
There were also some activities in which Ms. E was not focused on formal procedures or algorithms. However, even in many of these activities, Ms. E did not give full attention to the students’ ideas. For instance, she provided her own strategies in the activity Making Numbers, without giving students an opportunity to develop or present their own methods (see Episode UE-11). In the activities Composing and Decomposing a 10 (see Episode UE-15) and Representing a Number in Different Ways (see Episode UE-22), Ms. E mainly checked how many equations students made, rather than what equations they made or how they made them.

In general, the interview data of Ms. E’s reflections on the lessons in this study revealed a lack of mathematical depth for various classroom activities. For instance, in the activity Making Numbers with Fingers, students could have had rich experiences of understanding the various decompositions of a number and comparing/contrasting such decompositions regarding their mathematical elegance. However, Ms. M was less concerned with the mathematical value the activity would offer than with students’ social engagement in the activity with fun and excitement (e.g., Episode UE-14). Thus, students did not have opportunities which might lead to a rich understanding of mathematics.

Overall, the social process by which Ms. E supported students’ engagement in classroom activities was not used to give full attention to their ideas or to develop their mathematical understandings or to challenge them mathematically. In other words, the teacher’s concern and interest, not students’ ideas, were the main factors in directing classroom discussion and authorizing knowledge. These considerations make Ms. E’s basic curricular intention compatible with a teacher-centered approach.
STUDENTS’ APPROACHES

From the classroom activities described above the following characteristics of students’ participation can be observed:

Table 5.4 Characteristics of the Students’ Practices in Class UE

| • solved given problems independently whenever they were supposed to. |
| • volunteered to present their solution methods to the whole class. |
| • pointed out mistakes made by their peers or the teacher. |
| • usually paid attention to others’ presentations. |
| • faithfully followed instructions for activities. |
| • collaborated with each other in their small groups. |
| • asked questions when there was ambiguity or difficulty in the teacher’s instruction. |
| • expressed excitement whenever they found a right answer. |
| • expressed eagerness to be engaged in various classroom activities. |
| • invented their own solution methods for a given subtraction problem, even when the teacher indicated the use of formal algorithm. |
| • sometimes checked the teacher’s response before finishing their presentation. |

From these characteristics, students’ learning intentions as reflected in their participation can be interpreted in different ways. A first possible intention is that students may be interested in making sense of mathematical concepts and processes as they are engaged in various classroom activities. To some extent, the data support this explanation. For instance, when Ms. E added one sentence to the Zoo Problem, Alex asked her to clarify the meaning in the problem context. Moreover, students presented their own interpretations beyond the teacher’s explanation of possible cases (see Episode UE-7). Elena’s “VV” for 10 would be another example which supports the learning intention that students attempted to make sense of mathematical activity (see Episode UE-20).

However, the data as a whole revealed many counter-examples. For instance, consistent with the teacher’s curricular intentions, students frequently showed their interest and concern in the procedure and a correct answer. They often checked the
teacher's responses or expressions while presenting their methods. While doing collective problem solving activities, some groups waited for the teacher's check or confirmation for their decisions. In Episode UE-23, students were solving the Shell Problem. Ms. E encouraged students to discuss how to solve the problem and to reach consensus before writing up one as the group's method. In Lara's group, John suggested an idea of how to approach the given problem and Alex agreed and attempted to write. Sam immediately suggested that they wait for the teacher's direction on when to write. When Alex claimed that Ms. E already allowed another group to write, Lara expressed her preference for the teacher's direct confirmation to her group. This provides further argument against the proposal that students seek to develop their own understanding of mathematical ideas while being engaged in classroom activities. Rather, they are complying with the teacher's expectation that good students faithfully follow her directions.

<Episode UE-23: Students waiting for the teacher's check>

T: How can you solve the problem? What do you need to know? I want you to talk about this with your groups, and then, we're gonna hear from you and write something down. So, talk with your groups, talk with your partners and see what you think. Give your answer. The whole group, the whole table.

(Ms. E visits Arterrion's, Mary's and Reham's groups in order. She checked whether the students in each group discussed with one another and agreed on one method. While the teacher is interacting with other groups, Lara's group members discuss.)

Alex: How can you solve the problem?
John: You can add 12 plus 18, and then ...
Alex: Right, we can add 12 plus 18, and then find out the answer. (He is about to write.)
Sam: Don't write until she says, "Write."
Alex: She says that to that table. Let's go. Let's write.
Lara: She comes to our table, and she says, “All agree?” Then we write.

Alex: You all agree?

Ss: I do. I do.

Another plausible intention on the part of students is that they may be interested in showing up the weaknesses of their peers and their teacher. In Episode UE-18, Suzannah pointed out David’s mistake of obtaining 6 from 15-8. Since Ms. E did not recognize his mistake and wrote the expression on the board, Suzannah’s correction can be seen as pointing out teacher’s insensitivity to students’ responses. Ms. E allowed Suzannah to get a treat with much praise. This episode is somewhat exceptional in that a student corrected her friend’s mistake without the teacher’s initiation. Remember that receiving a treat in this classroom was very honored on the part of students, because Ms. E allowed it only for special cases in which, for example, they had the closest estimation number or provided unexpected but creative ideas. Anyway, there was a tendency after this episode for students to give too much attention to their peer’s contributions and/or the teacher’s records on the board. This led them to immediately recognize others’ mistakes, specifically in the renaming activity, both in the whole class discussion and in the small group activity. Thus, it seems difficult to interpret that students had the original intention of pointing out others’ mistakes in the process of understanding mathematical principles. Rather, they confirm the teacher’s expectation that good students listen carefully to others’ presentations and point out mistakes, if any. Overall, *compliance* with the teacher seemed to be a unifying objective, which is strongly grounded in the whole data set.
STUDENTS' LEARNING OPPORTUNITIES

As described earlier, Ms. E frequently used a small-group format and encouraged students to work together in groups, specifically with posing and solving problems. However, students' collaboration and discussion within small groups were very limited with regard to the development of mathematical ideas. Thus, students' learning opportunities more than likely came from their participation in whole class discussion and their interaction with the teacher. Ms. E's main curricular intention was to reinforce formal algorithm over students' various ideas and students' principal concern was to comply with the teacher's expectations. Considering that students' learning opportunities come from the teacher's and the students' approaches, the students in Ms. E's class seemed to have a limited chance to develop a conceptual basis of the mathematics they were studying.

The instructional objective of practicing computational skills was supported by the teacher's emphasis on line alignment and the order of computation, and her informal and formal testing of students' memorization of basic facts. Within this learning environment, the students in Ms. E's class had many opportunities to develop skills required in solving routine addition and subtraction problems.

There were some classroom activities in which the teacher attempted to connect abstract representations with concrete representations (e.g., Episodes UE-6 and UE-7). However, at most, loose connections between different representation modes (e.g., using numbers and using manipulative materials) gave limited opportunity for students to understand the transition from the informal to the formal. Engaging in a few teacher-directed, hands-on activities seemed not to be enough for students to give mathematical
meaning to the formal procedure they were practicing. Identifying names of different place values (i.e., ones, tens, or hundreds columns) provided students with a limited chance to understand the concept. Overall, there was a lack of mathematical learning opportunities, except those of the mainly procedural, that students might have, even if they actively participated in various classroom activities with enthusiasm.

Whereas students always had a chance to solve a given problem for themselves, they had relatively limited opportunities to present their novel ideas (e.g., starting with one number in the activity of Making Numbers). Though Ms. E emphasized the learning process over a right answer, she was interested in specific methods and sometimes demonstrated them with examples. In this respect, the students in Ms. E’s class had a chance to learn mathematics as a static or fixed discipline. Moreover, they focused on the teacher’s direction and instruction rather than becoming self-motivated in their pursuit for mathematical sense-making.

INDIVIDUAL ANALYSIS OF CLASS UM

SETTING

The elementary school U2 is located in a suburban area of Baton Rouge in Louisiana. Committed to the belief that every child can learn, the principal establishes the vision of this school as providing appropriate instruction by which all children develop critical thinking as well as basic skills, and creating a learning atmosphere in which children feel the joy of learning with their efforts and successes. The academic program of this school includes an integrated curriculum with thematic units. The school maintains a racial balance of 60% black and 40% non-black. The majority of the students are from lower-middle to lower income families.
School U2 has one pre-kindergarten class and three classes at each grade level from kindergarten to fifth grade, except second grade. Ms. M's class was one of four second grade classes and it consisted of 11 boys (7 of them were African-American) and 10 girls (half of them were African-American). During the last week of observation for this study, one African-American girl, who had been taught in a behavior disorder class, was mainstreamed into this class for math lessons.

Ms. M is a highly professional and enthusiastic teacher with 35 years teaching experience. Once she served as a math specialist, observing how teachers in assigned schools taught mathematics and demonstrating instruction. She also had served as a main speaker in various workshops intended to inspire elementary school teachers to develop better instruction. As she came to be familiar with what her students needed to know, Ms. M stopped using mathematics textbooks and workbooks. In her evaluation, mathematics textbooks hindered reflective teaching and focused mainly on rote practice. Instead she continually observed students to see what they knew and what they needed to know to be successful math students. On the basis of this observation, Ms. M prepared for lessons drawing on a whole shelf of her own math resource books.

Ms. M had been teaching the students in this second-grade class since they were in first grade. She judged that almost half of her children were at-risk for failure. Instead of retaining them at the first grade level, she gave them a second chance to learn. Ms. M noted that boys as a whole were quite a bit stronger in math than girls in this class. This contrasts with her previous classes in which girls tended to be stronger.

Ms. M tries to establish a classroom environment in which every student pays attention to the teacher or to students presenting their ideas. For whole-class discussion,
Ms. M asks her students to sit on a rug in the front of the classroom. Some students had assigned seating. Those few students who specifically needed her attention to be involved in classroom activities sat next to her. Because of lack of space, a few students who were excellent in math and voluntarily participated in classroom activities sat on chairs close to the rug. Normally, the desks in the class were arranged in such a way that several students sit together, but students usually worked on the floor as a whole group or with their partners playing a mathematics game. When Ms. M directed them to work on desks, she arranged the desks so that students work together in pairs.

CLASSROOM FLOW

General Atmosphere

This class had an open and permissive atmosphere. When a presenting student made a trivial mistake, other students corrected it. When there was a misunderstanding between the teacher and the students, they reconciled it by explicitly expressing their positions. For instance, Ms. M asked students to present easy multiplication facts after they played a game called Circles and Stars. Michael was turning over his little book in which he wrote down seven multiplication facts while playing. Probably assuming that he was not focusing on her instruction, Ms. M took away Michael's book. Immediately, Michael justified his action by saying that he was just looking for an easy multiplication fact and Ms. M acknowledged her misunderstanding with an apology.

Ms. M usually asked for different answers to a problem. Students provided several answers and nobody laughed at wrong responses. Ms. M required students to explain how they got their answers. When a student who volunteered to present faltered, other students kept quiet, providing the student with a chance to think.
Similarly, when a student’s presentation did not make sense, other students usually waited for clarification rather than immediately responding. Ms. M praised students for their collective good manners. When students expressed a negative response in the middle of a student’s presentation, she gave the presenting student an opportunity to finish his or her argument. Moreover, when students did not know what to do, Ms. M consistently encouraged them to express their ignorance without hesitation. Indeed, the class shared the idea that smart kids know when they don’t know and listen carefully to other students to learn.

Ms. M was concerned with every child participating in her math lesson. She emphasized that good teachers teach everybody, because every child is special. In every lesson, Ms. M provided both hard questions and easy questions. For the easy questions, she expressed her expectation that every child should volunteer to present, and waited for everyone to think. She chose without preference from among the students who raised their hands to answer. However, sometimes Ms. M intentionally chose particular students. In cases where students were supposed to just say an answer to a problem, Ms. M did not forget to call on students who were weak in mathematics. She also used weak students’ contributions in setting up manipulative materials for classroom discussion. For example, the class was discussing how to solve 42-16 using cubes. After taking out 4 tens sticks to make 42, Ms. M asked Nick, who had a great difficulty in counting, to show 42 and to count all of them. When Nick was successful even though Ms. M helped him, she asked the class to clap for him. In cases when many students were confused with alternative ideas during discussion, Ms. M purposely chose strong students for clarification and expected other students to understand by listening.
carefully to their friends. The pace of instruction was rather slow but clear. Whenever most students did not answer, Ms. M repeated her question and gave them enough time to think.

The classroom atmosphere was dynamic in that students actively responded to the teacher and to one another. Students raised their hands for presentation not only when Ms. M asked them to do so but also when they had something to contribute. They even asked Ms. M to pose more challenging problems such as addition with regrouping and multiplication with big numbers. During whole-class discussion, students were sometimes engaged in debate without the teacher’s intervention. When students argued for or against others, they usually provided their own rationale or examples. Since most of class time was used for collective problem solving and game playing, the classroom as a whole was dynamic. On one of the observation days, students had a short time to write about their math lesson. Most students said that they enjoyed math and had lots of fun. Some others wrote about their experience of being a teacher for their game partners, helping each other, and playing well regardless of who was the winner. In one of her interviews, Ms. M said that she tried to make her math lesson as much fun as possible, emphasizing that learning for little children is play.

In order to keep students’ attention on classroom activities, Ms. M highlighted some students who were doing what they were supposed to do by writing their names at the top of the board and drawing the shape of a heart around the names. In contrast, she wrote some students’ names at the bottom, indicating that they were inattentive. Generally, Ms. M expressed positive expectations by saying, “You guys are the smartest in the whole wide world.” When the class moved on to a new activity and the students
needed to change their seats, she often initiated singing a song which implicitly asked for the correct posture for study.

The teacher consistently stressed helping each other. In any phase of a lesson, she praised individual students or groups of students who worked together. Indeed, the normative practice of helping each other seemed to be well established, as evidenced by the students’ voluntary help and affirmative attitude toward a new student, Raven, who was mainstreamed only during their math lessons.

**Lesson Elements**

Each math lesson observed consisted of two content segments, a problem solving session and an activity (mostly in a game format). The activity was implemented through three consecutive phases: teacher’s instructions of how to play (activity instructions), students’ play (activity implementation), and whole-class discussion (activity discussion). Every day math class started at 1 o’clock after recess time and continued for one and a half hours. The time spent in each phase varied depending on the problems and games, but the lesson structure was identical throughout the seven classes. This section describes in detail the four phases of Ms. M’s lesson and presents episodes to enhance the descriptions.

**Problem Solving**

During the observation period for this study, the class covered addition up to three digit numbers (e.g., 26+7, 37+24, 124+150), subtraction with two digit numbers (e.g., 64-10, 42-26), and multiplication by repeated addition (e.g., 2+2+2+2+2, 5X2). Ms. M wrote some computation problems on the board, and the students solved them in their seats. After solving the problems, students individually came to the front and
seated themselves on the rug in three rows. Ms. M waited for all students or else limited
the time to finish.

Ms. M first asked for different answers to a problem, and wrote them without
evaluation. She then asked students to explain how they solved it. When a presenting
student made a mistake, Ms. M gave that student a chance to correct it. When the
student was not able to complete his or her presentation, other students were expected
to provide help. Ms. M wrote what the presenting student said on the board step-by-
step and checked whether what she wrote corresponded to what the student meant. A
student who originally got an incorrect answer sometimes reached the correct answer
while explaining his or her solution method thanks to Ms. M’s questions and what she
wrote during the student’s presentation. Whenever a student provided a vague
explanation, Ms. M asked for clarification. Sometimes the presenting student came to
the board and explained his or her mathematical idea to Ms. M pointing to the
corresponding parts in the equations written on the board. Ms. M then repeated or
amplified to the whole class. Whenever the class moved on to the next problem, she
gave students time to figure out the answer. Sometimes she waited until all students
raised their hands to present.

Students’ thinking emphasized

There were some characteristics of Ms. M’s teaching practice worth noting. She
consistently expressed her interest on how students solved problems. Regardless of the
correctness of an answer, students had a chance to explain their solution methods
without being embarrassed. As exemplified in Episode UM-1, Ms. M praised students’
good manners at not howling out answers to a given problem, and encouraged children
to express what they think without worrying about reactions.

<Episode UM-1: Expressing one's idea without embarrassment>

T: All right. And you know what I notice about our class? I notice about our class that the
children don’t ever laugh at anyone else. Even when they think that the answer’s
different from theirs, they just don’t do that. That’s what makes you special. Children
are free to say what they think, because nobody will laugh at them. When I was in
school, I used to get embarrassed to say things because I was afraid that people would
laugh at me. But you guys don’t do that. So that’s why nobody gets embarrassed when
they say what they think.

Since an expectation in this classroom community was that students express
their ideas, saying the answer without appropriate explanation was not acceptable. An
example of this occurred when the class was discussing multiple methods to solve 26+7.
David’s method was written as 6+7=13, 10+20=30+3=33. Derrick’s method was
written as 7=(4+3), 26+4=30+3=33. (Note that in this classroom the equal sign was
used as a way to represent the process of computation without considering its
mathematical meaning.) The class accepted both methods as appropriate. Chase then
presented his method, “20 plus 7 is 27, plus 6 makes 33.” Ms. M asked him to explain
how he added 6 to 27. Though the teacher and the students provided an example of an
acceptable explanation, such as counting on, Chase couldn’t clarify and, thus, the
teacher didn’t give him credit.

Owing to the teacher’s persistent articulation of students’ thinking processes,
students sometimes anticipated that Ms. M would negatively respond to an explanation
which seemed poorly argued. For instance, in Episode UM-2 the class was discussing a
numerical solution method to solve 42-26, which was initiated by Chase. He suggested
taking away 20 from 40, but was not be able to finish his presentation. It took a long
time for other students to come up with any idea of what to do next. Moreover, their suggestions were incorrect (e.g., subtracting 2 from 6). In this context, Chase suggested simply taking off 4, which could reach the right answer, but he anticipated the teacher's negative evaluation. (Note that students already knew the answer because they had solved the problem using other methods.)

<Episode UM-2: Teacher's negative reaction to wrong argument anticipated>

T: What are you thinking, Chase?

Chase: All you have to do ... I know you're gonna say it's wrong. But all you have to do is to take off 4 from that 20. 19, 18, 17, 16.

T: So, you think so. Tell me again.

Chase: Just take away 4.

T: Because you already know the answer. So you know if you took off 4 you get the right answer. Honey, what if you didn't know the right answer?

Ms. M very often solicited students' ideas. In particular, when students had competing alternative ideas, Ms. M asked students to express their thinking. For instance, in Episode UM-3, she was using counting backwards to solve 42-26.

Accepting Eryn's suggestion, Ms. M wrote down little numbers on the top of the numbers while counting back to keep track of how many they took away. At that point, students disagreed as to whether they should count 42, the first number. Instead of direct explanation, Ms. M allowed students to present their ideas.

<Episode UM-3: Teacher's interest in students' ideas>

T: Okay, let's go. 41, 40, 39, (Ms. M and her students are reading together), 38, 37, 36, 35, 34, 33. (Ms. M writes these numbers in one line and the students count back together.) How many numbers do you think we've done so far? How many do you think we've gone backwards? How can we keep track of how many we've done backwards? Tommy's got his fingers up. He's keeping track that way. Give me a suggestion. How can I keep track of how many I've done backwards? How can I keep track? Eryn?
Eryn: Write one number on the top.

T: (Writes a little Ion the number 41.) Is that gonna confuse you? Are you gonna remember what that one is for? Okay, let's go. 1, 2, 3, 4, 5, 6, 7, 8, 9. (She writes these numbers over the numbers of 41 through 33. Someone says, "you forgot to count the 42."

Ss: Yeah.

T: What? Did I forget to count the 42?

Ss: No, yeah, no ...

T: Some people say yes; some people say no. Tell me how you're thinking? Derrick?

Derrick: Remember when you said, do not, like when 36 plus 24, you are not supposed to circle away 36, because it's the first number.

T: Okay, tell me more. Tell me more. Lainey?

Lainey: Uh, you can't count that number because you already counted it.

T: Tell me more. Eryn, you've got more to tell me? What?

Eryn: You can't take away that number.

T: Okay, tell me more. Chase?

Chase: You can't, you can't use that as 1 because you started off that number.

T: Okay, let's go. Let's see. All right, here we go, 33, let's keep going, 32, 31, 30, 29, 28, 27 (Writing these numbers; she writes 32 under 42, 31 under 41, etc). We had better check how many we used? 9 (pointing to the 9 on the 33). 10, 11, 12, 13, 14, 15 (writing these numbers over the corresponding numbers). Okay, let's go. 26, 25 ... (Interrupted.)

Nahjha: No, you passed it! 26!

Ss: No. No.

Nahjha: No.

T: Okay, I am taking away 26. Have I taken away 26? See that's why it's gonna be confusing. Nahjha just said you took away 26 numbers but have I taken away 26?

Ss: No.

T: How? How? Explain to me what you are thinking.
Nahjha: Oh! (He comes in front of the classroom and points to the numbers on the top.) Because the numbers on the top have to be 26.

T: Say again, Nahjha?

Nahjha: The numbers on the top have to be 26.

T: Okay. (She makes a circle around the top numbers.) Explain it to me. Why? Keep talking to me.

Nahjha: Because we counted backwards with big numbers and we are going from those little numbers.

T: Tell me more. Tell me more.

Note that all four students chosen by Ms. M claimed that they should not count the starting number 42. In her interviews, she explained that sometimes she purposely chose students who were strong in math to present their ideas, in particular when many students seemed to be confused. In the episode above, Nahjha was confused between the numbers counted back and the numbers being taken away. Ms. M reminded him about what she had been doing yet still allowed him to reflect on his confusion and her explanation.

**Different solution methods emphasized**

Another characteristic of Ms. M's teaching practice was her emphasis on different solution methods. After discussing one way of solving a problem, Ms. M asked for different solution methods. Because she did not teach standard algorithms, students provided several solution methods mainly to compose and decompose the numbers of a problem. They suggested alternative methods for subtraction problems such as using a number line or tally marks, because they had trouble using mental computation for subtraction. In any case, Ms. M made sure that the class covered concrete solution methods which were supposed to make sense for every child regardless of his or her
numerical facility. In particular, Ms. M used cubes as a basic method for students to figure out answers. For instance, as mentioned above, students did not know what to do after the computation of 40-20=20 to solve 42-26. As they did not come up with any reasonable idea, Ms. M simulated the problem with cubes. Students easily recognized that they could take away 6 out of 1 tens stick.

After soliciting different solution methods, the teacher emphasized that the answers were the same regardless of the methods used. Under the name of different methods, students sometimes presented mathematically less efficient solution methods than those that had been contributed. Usually Ms. M did not differentiate her reactions to various ways of thinking, emphasizing whatever method that made sense to the individual child was fine. The only exception for this practice happened when she introduced multiplication. For instance, in solving 2+2+2+2 students presented using doubles (i.e., 2+2=4, 2+2=4, 4+4=8) and counting by 2s. Then, one student suggested dividing the last 2 into 1 and 1. Another student changed the first three 2s into 1 and 1, counted the first five 1s and then added the remainders. The third child suggested counting by 1s. In these cases Ms. M simply accepted each method and moved on to the next problem 3+3+3+3, acknowledging that others might have different solution methods. After accepting a student's suggestion of counting by 3s, the teacher revealed her real expectation by calling Michael, who had been talking about using multiplication for repeated addition problems. As seen in Episode UM-4, the teacher even prevented Nahjha from presenting two solution methods because she probably expected that his method could be inefficient. Her interest in multiplicative solution was coded as the
simplest way. When Nahjha used the multiplication methods she had been waiting for, the teacher accepted it with excitement.

<Episode UM-4: Teacher’s real intention in solving repeated addition problems>

(The class is discussing solution methods for 5+5+5+5+5.)

T: Okay. All right. How can you do it? Chase?

Chase: 5 plus 5 is 10, (The teacher connects the first 5s and writes 10), and other 5, 5 plus 5 is 10, (The teacher connects the next two Ss and writes 10), and 5 more equals 25. 10 plus 10 is 20, and 5 more is 25.

T: 10 plus 10 is 20 (connecting 10 and 10 by a line) plus 5 more equals 25. All right. What’s the another way to do it? Nahjha?

Nahjha: I’ve got two ways. One way to do it ... (Interrupted.)

T: Tell me one.

Nahjha: Take one out of each 5 ... (Interrupted.)

T: Okay, but I wanna a simpler way. I don’t wanna wait for more complication. I wanna the simplest way to do it. Quickly.

Nahjha: 5 times 5.

T: What did you just say? (Smiles at Nahjha.)

Nahjha: 5 times 5! (The teacher writes 5x5=)

T: And how do you explain that?

Nahjha: Because there is 5, there is 1, there is five five!

T: 5, 5 what?

Nahjha: Groups of 5

T: 5 groups of 5!

While discussing various solution methods, students sometimes suggested some idea beyond the teacher’s expectation. For instance, Michael presented his method to solve 6+7 by 6+7=(5+1)+(5+2)=(5+5)+(1+2)=10+3=13. Ms. M called his strategy

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"looking for 5." Michael agreed at first but a little bit later he changed it to "looking for 10." Other students preferred Michael’s label and Ms. M accepted it. In another case, the class had been discussing several ways of solving 37+24. In particular, Chelsea used the expanded form of the algorithm (37+24=50+11=61 vertically written). Soon Chase presented his discovery that switching the ones digit numbers in the problem (i.e., 34 +27) also produces the same answer. Ms. M checked his idea with the whole class but she used the commutativity of addition in the middle of computation by saying “7 plus 4 is”, instead of “4 plus 7 is.” Ms. M did not extend Chase’s idea further but quickly moved on to another solution method. In another case, as seen in Episode UM-5 Michael wrote 10X5=50 as a short way of solving 10+10+10+10+10. Ms. M cautioned him and some impatient students said “5 times 10.” In this context, Michael found the commutativity of multiplication (10X5=5X10=50).

<Episode UM-5: Michael’s idea ignored on purpose>

(Class is discussing 10+10+10+10+10. Kelly presented “5 groups of 10.”)

T: All right. Does anyone know how to write in a short way? Do you know how to write in a short way, Michael? Michael, you started all this mess. You are the one who first started to talk to us about multiplication, and you are the one that has got us all interested ... (Interrupted.)

Michael: (Comes in front.) There is an easy way that you can do it. I know how to do it quickly. You can answer quickly. (He writes 10X5=50.) There is an easy way that you can do it.

T: Wait! Look, Michael.

Ss: 5 times 10.

T: Sh!

Michael: Oops!

T: Check it! Yeah.
Michael: (Erases 10 and 5 and says to the teacher.) It doesn't really matter.

T: Well, I really want you to do it the way that it really is right now. I know you can do the turn-around fact. And you know how to do turn-around facts. And I know you know multiplication is the same as addition. You do turn-around facts. Right? But I really want you to do it the way that it really is right now.

Michael: (Finishes writing the equation as 5X10=50.)

In this episode, Ms. M acknowledged Michael's discovery of commutativity but ignored it on purpose. She differentiated the two multiplication facts in order to focus on the semantic meaning of multiplication as repeated addition (10 groups of 5 versus 5 groups of 10), rather than on the results of computation. Throughout the lessons on multiplication a few students presented the commutative property of multiplication, but Ms. M kept differentiating with regard to meaning.

Computation process connected to concrete experience

The teacher frequently connected computation process presented by students with using cubes. For instance, the class was discussing how to solve 37+24. Three students each presented one part of the computation: 30+20=50 (by Kanita), 7+4=11 (by Chase), and 50+11=50+10=60+1= 61 (by Kelly). As seen in Episode UM-6, Ms. M suggested using cubes to check the answer. Instead of demonstration, she helped several students manipulate the cubes step by step in front of the classroom so that other students could observe the process. Note that Ms. M matched the previous mental computation process to individual students' manipulation of cubes.

<Episode UM-6: Using cubes to understand computation process>

T: Kelly thinks the answer is 61. Let's check it out with a real thing. So we can find out what the real answer is and then we can talk about more, because I know you have other ways to solve it.
(At the teacher's request, Erika, Noble, Tommy, and Robert each hold part of the numbers, namely 30, 7, 20, and 4. Ms. M broke down 37+24 because it's difficult for a student to hold, for instance, 3 tens sticks and 7 cubes.)

T: All right. Now, we've got 37 and we've got 24. Let's do it. Let's do it the way Kelly and Kanita, and who was the other person who helped with this? Somebody else helped with this in that problem. There were three of you that helped me. (Chase raises his hand.) You, Chase? Let's do it the way they did it. Who can come and do it the way they did. What's the first thing Kanita did? Chelsea, what's the first thing Kanita did?

Chelsea: Add the 30 and 20 together.

T: Come and do it. She's gonna get the 30 and the 20. (Chelsea takes the 30 and 20 from Erika and Tommy.) She's got the 10s now. Right? (To Chelsea) Move over there. (To Erika and Tommy) And my little friends that were sitting up here, sit down. Thank you very much. (To Chelsea) How much do you have together, Chelsea?

Chelsea: 50.

T: Okay. Look at the children so they can see you. All right. What's the next thing that happens? That's what Chase did. Simone, what happened next?

Simone: 7 plus 4 equals 11.

T: Come and do it. All right. Back up, Chelsea. They can's see your face. Make sure children can see your face. (Simone takes the cubes and holds them together. Noble and Robert go back to their seat.) Brandon, I need you to stop. Okay, look here (pulling Simone). Now what have we got right, how many we've got in Simone's hands? It's little bit handy to hold all those. How many we've got?

Ss: 11.

T: 11, because 7 and 4 is 11. What do we... Oh, the hands are starting to pop even before I open my mouth. Even before I open my mouth, the hands are starting to go up in the air. What are you going to say, Nahjha?

Nahjha: You have enough to make another 10.

T: Do it. (Nahjha hooks the 10 cubes together.) That's what Kelly did. Kelly said, 50 and 10 more is 60. That's what she did. Okay. (To Nahjha) You'd better give it to Chelsea because there is 50. What's 50 and 10 more?

Ss: 60

T: What's Simone got left?

Ss: 1.
T: 61. So, what’s the real answer?

Ss: 61.

T: We do real things to find out.

In conjunction with the connection between numerical computation and concrete representation, the teacher often reminded students of previous activities they had experienced. For instance, when a student suggested using a number line to solve 64-10, Ms. M reminded them of Take Away 10 game in which they kept subtracting 10 from a previous number with a 100 chart. When a problem included the computation of 6+6, she reminded them of dozens of eggs with which students did lots of activities. When a student used counting by 3s to solve a given problem, Ms. M reminded the students of a previous activity where students made three-leaf clover paper and learned how to count by 3s.

Using standard algorithms cautioned

Ms. E was very concerned about students’ use of algorithms. Whenever students voluntarily used the vertical format to solve a problem, Ms. M paid careful attention to whether they understood the different place values. Some students who used vertical format often dealt with tens digit numbers like ones. For instance, in solving 27+25 Chelsea said 40 (20+20) and 12 (7+5), and then 4 plus 1. When Ms. M pointed to the 4 in 40, Chelsea partially corrected her mistake by saying 40 plus 1. Even after the teacher and other students unanimously said 40 plus 10, Chelsea concluded her answer as 5 instead of 50. Acknowledging that Chelsea got the answer 52, Ms. M soon simulated the computation process Chelsea presented with cubes.
Despite the teacher's reluctance to introduce formal algorithms at a second
grade level, some students like Nahjha in Episode UM-7 brought partial knowledge of
algorithm from home. The class was solving 37+24.

<Episode UM-7: Formal algorithm and a student's wrong answer>

Nahjha: (In front of the class) You know how Chelsea did?

T: Yes.

Nahjha: You put it down in ...

T: 37 plus 24 (writing 37+24 in a vertical format), Okay?

Nahjha: My mom told me this. It's what grown-ups do, 30 plus 20 equals 50 (mimicking writing
5 with his finger). (Ms. M writes 5.) And she said this plus and that (pointing to 7 and
4), she put it down here (pointing to the place next to 5). She put down one half of the
11 right here (pointing to the place next to 5), one put it on the top, here (pointing to the
place over the 3 of 37).

Ss: ..... My mom do[es] it! (Students shout with agreement.)

T: You know what? Someone has this answer, 511 (writing 511).

Ss: Oh! Brandon!

T: Brandon did. Brandon, is it possible that the answer could be 511? Would that be a
reasonable answer?

Brandon: No.

T: Listen to Mrs. M. Would it be a reasonable answer? If you had these many candy bars
(pointing to the tally marks for 37) and these many candy bars (pointing to the tally
marks for 24), could it be equal to 511?

Ss: No. No. You can’t! I know!

T: Let me listen to Brandon.

Brandon: 30 ... 60. I thought it’s more!

T: You thought it was more. You know what, Brandon? You made a mistake that lots of
people make. You made a mistake that lots of people make. Look what Brandon did.
Does anybody know what he did? Smart mathematicians can look at the problem, and
they can look at the wrong answer, and they know why people made them.
Brandon: [Inaudible.]

T: What, Brandon?

Brandon: I didn’t want to put the 50 there.

T: Okay, let’s listen to a child who thinks they know what you did. What did he do, Michael?

Michael: Cause he told me. He thought it was 50 and 11. So he had to put them together, like 511 (5 hundred 11). He was telling me.

T: He thought he had to put 50 and 11 together, like 511. Okay. Tell me more, David.

David: He could start with 5 hundred and then 7 and 4, 11. So it’s 511.

T: Um. You know what? You know what Brandon could’ve done right here (drawing a circle around 7 and 4 in ones digit).

S: What?

T: He just could’ve made another ten. He could’ve made another ten. We will talk about that in other time with real things. Then it will give more sense to you. Right? But you know what, Brandon? That’s not something you need to be ashamed of. That’s the mistake lots of people make. But sometimes... Let me tell you what begins to happen. People make that mistake and then they think to themselves, um, 37 and 24, I know that it couldn’t be 511. I know something has to be wrong with that. Just like yesterday when we played the game, sometimes the answers didn’t match. You had to say to yourself, um, well, something is wrong. I’d better go back and do it again to see what’s wrong.

Nahjha’s presentation showed that he remembered only the procedure without the conceptual basis, which Ms. M was concerned about. He described regrouping as splitting the digits in the sum of ones digit numbers in “half” and he was eager to point out where each half was supposed to be placed. Other students immediately indicated that they had the same experience. However, Ms. M did not provide direct comments but connected his contribution with the wrong answer Brandon gave previously. She gave students an opportunity to think of how he got such an unreasonable answer. Indeed, Ms. M frequently used students’ wrong answers to facilitate classroom
discussion. Note that she took care of Brandon’s emotional state by saying that his mistake was common and that looking back at the solution process was more important.

Students’ debate encouraged

During this problem solving phase, students were sometimes engaged in debate, disagreeing with each other. Ms. M allowed each student to present his or her position, explained the argument to the whole class, and usually accepted both sides. For instance, in Episode UM-8, the class was discussing multiple solution methods for 26+7. As David explained his method, Ms. M wrote the following: 6+7=13, 10+20=30+3=33. Trenea said that the 0 in the computation was nothing. Michael said that the 0 was ten, but Trenea disagreed. Nahjha claimed that 0 was ones, differentiating ones place from tens with the specific number 30. After listening to the children, Ms. M explained both Trenea’s and Nahjha’s idea with specific examples.

<Episode UM-8: Students’ debate and teacher’s mediation>

Trenea: You know, the 0 don’t matter. It’s nothing! It’s plain old air.

Michael: No!

T: (Laughing) 0 is air?

Michael: (Looking at Trenea) 0 is the ten.

Trenea: No, 0 is not a ten.

Nahjha: 0 is one!

T: Is 0 one?

Nahjha: Yeah, because ... (coming to the board) see, you start here with this one (pointing to the 30 on the board) and then 10, 20, 30 and the 3, somebody tells us that the 3 is ten (pointing to the 3 of 30) and the other one, the other number is one (pointing to the 0 of 30).
T: Oh-oh, you know what you are saying, Nahjha? You're saying a different thing. Trenea is saying zero (writing "0" on the board). The number zero means nothing. Give me zero cookies (bending her knees). Why don't you giving me any cookies?

Ss: Because...

T: Give me zero cookies.

S: You said zero.

T: Give me zero pencils. (To individual students in the front of the classroom) Why don't you giving me any pencils? Why don't you giving me any pencils?

S: You said zero.

T: Why? What is zero?

Ss: Nothing.

T: Okay, so, what are you gonna give me?

Ss: Nothing!

T: All right, but Nahjha is saying this. Nahjha is saying something different. Nahjha is saying, when you have 30 (writing the number 30), the 3 stands for how many tens sticks ... (underlining the 3 of 30), how many tens sticks you got?

Ss: 3.

T: That's right. So, you've to find your tens stick, and that's what it stands for (showing tens sticks in a bag). And then 0 stands for how many loose ones. So, how many loose ones do we have? (Points to the 0 of 30.)

Ss: Nothing.

T: Nothing, zero. So, you are both right.

Activity (Game) Instructions

After completing discussion of the problems on the board, Ms. M asked students to move to the edge of the rug, and she seated herself next to them. Then she demonstrated an activity students should do for the next phase. Most of these activities were games between two players. Ms. M consistently emphasized that the students
needed to help each other. In fact, students were supposed to share some materials such as dice or score sheets during their play. Ms. M encouraged students to attend to whether their partners needed some help. She explained that students helping each other doesn't mean doing something for their partner. Rather it involves talking together about what to do or how to do it. During most of this phase Ms. M led the activity, but she also encouraged students to participate in a variety of simulated game situations. Whenever they were engaged in a game situation, students appeared to be very excited. For instance, when a student was about to choose a second card in the Turn Over 10 game (see Table 5.5 for a description of this game), the rest of students excitedly approached the middle of the rug where cards had been turned upside down. They shouted the number which could make a 10 when added to the first number. Then the students expressed great excitement if that number was turned over or great disappointment if it was not. If there was any connection between the current activity and previous activities, Ms. M reminded students of the connection. She usually reviewed her instruction including rules and steps for an activity before students would start.

Table 5.5 shows the activities used in the class during the observation period. The game Circles and Stars was used for two consecutive days because on the first day some students had difficulties in following the teacher's instructions, and most of the students did not write expressions employing multiplication, as requested by the teacher. By the second day, through the provision of more specific instruction, Ms. M asked students to represent the total number of stars on each page by making use of the meanings of multiplication as they had learned. For example, if they drew 3 circles
containing 4 stars each, they were to write 3 groups of 4 = 12, 3 sets of 4 = 12, or 3X4=12.

Table 5.5 Activities Used in Ms. M’s Class

<table>
<thead>
<tr>
<th>Activity</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn Over 10</td>
<td>A player sorts a regular deck of plain cards leaving only number cards and kings. A player then mixes the cards. The cards are placed face down making 4 rows of 5 cards. The players alternate turning over two cards and check whether their sum is 10. The kings are used as wild cards. If the sum is 10, the player keeps those cards in his or her tens stack. If the sum is not 10, the player puts them back in their places. When there are 10 cards left to be played, a player puts more cards to make 20. After finishing the game, the players write all the combinations of 10 they made in a notebook.</td>
</tr>
<tr>
<td>How Close To 20</td>
<td>A player sorts out cards to have only number cards. Each player has 5 cards per turn and chooses three out of them attempting to get as close to 20 as possible. In each turn, a player figures out a score by the absolute difference between 20 and the sum of the three cards. The winner is whoever has the lower sum after five turns.</td>
</tr>
<tr>
<td>Raise To 150</td>
<td>A player rolls two dice, adds the numbers on the dice, represents the sum using cubes on a place value board with tens and ones place, and gives the dice to the partner. Each player cumulatively adds numbers and represents them on his or her own board. Whoever gets a 100 waves both hands to the teacher and then she brings a bag. The player puts the 100 in the bag and lays it to the left of the board to represent hundreds place. When one gets 150, both players stop and figure out the sum of the two totals.</td>
</tr>
<tr>
<td>Circles and Stars</td>
<td>Each student makes a little book with 7 pages by folding and cutting a piece of paper. Each player rolls a dice twice — one for drawing circles and the other for drawing stars in each circle. Each player writes expressions employing multiplication. The winner is the person who has the most stars after finishing the 7 pages.</td>
</tr>
<tr>
<td>Solve 80-18 and Write a Real-life Story</td>
<td>Students made up 80-18 during the teacher’s instructions phase. Four students work together with one big piece of paper divided into 4 sections for each individual student. They first solve the problem 80-18 with the description of a solution method, and then write a real-life story about the problem.</td>
</tr>
<tr>
<td>Doing Multiplication with Calculator</td>
<td>Each student folds a piece of paper twice, writes multiplication problems on it, represents the problems with circles and stars, figures out the answers, and checks them with a calculator.</td>
</tr>
</tbody>
</table>
Challenging questions asked

During her step-by-step instruction of how to play a game, the teacher often asked mathematical questions. In case an activity of the day required paper-folding, Ms. M briefly checked students’ knowledge of fraction (e.g., “tell me what this fractional part of the paper is”) and reminded them of previous activities such as using recipes or calendar pieces with which they initially learned fraction. Referring to the characteristic of a game, Ms. M asked even more challenging questions. For instance, when students had to make and represent their own multiplication problems, Ms. M asked why they should draw the same number of stars in each circle. As another example, in the game How Close To 20, she asked which case would be better, when they had the highest score or when they had the lowest score (see Episode UM-9).

<Episode UM-9: Teacher’s challenging question and students’ responses>

T: I have a very difficult question to ask. This is the thinking man’s question. Do you want to have the highest score or the lowest score? Jonathan, this is important. Brandon, you don’t know? So ... do you know? You know? What do you wanna have, Brandon?

Brandon: The highest.

T: Why do you wanna have the highest?

Brandon: Because the higher is the best.

T: In a lot of games, the higher is better. And in a lot of games, Brandon is right. Is he right in this game?

Ss: No, yes, no.

T: Some people say yes; some people say no. Is he right in this game? If you think yes, raise your hands. If you think no, raise your hands. (More students raise hands for no.) Why are you thinking no, Derrick?

Derrick: Because in this game you have to see who can come close to 20. If you have the highest score, then you are not close to 20.

T: Tell me more. Keep talking, keep talking. Tommy? 200
Tommy: Like, if we get to 20 closer then we have a lower score.

T: Tell me more. Keep going. Who else can tell me more, David?

David: It was like, 23, you have 3 more extra than 20.

T: Okay, 3 is better score than 1, David? What’s the better score?

David: 1.

T: So, what do we wanna have?

T: The what, Lainey?

Lainey: The lowest.

T: The lowest score. All right. What, Chase?

Chase: It’s just like in golf. You try to get the lower score, because if you get the higher score, it’s bad. You are trying to get the 20. But if somebody gets 78, it’s not gonna be good, because that’s not 20.

T: I see. That’s right. That will be too far away from 20. You got 78. You would be really far away from 20, wouldn’t you, Chase. Everybody, look at 20 and look at 78 (pointing to 20 and 78, respectively, on the number line).

Ss: God!!!!

T: You would be really far away. All right.

In this episode, the teacher encouraged students to present rationales for their answer. As students had two different choices (highest or lowest score), Ms. M used the explanations of students perceived as strong in math to support that the lowest score would be the correct choice and why. As the discussion came to closure, Chase provided a real-life situation of playing golf where the lowest score is desirable, and a hypothetical case of having a big number 78. It was not clear whether he was suggesting 78 as a sum of three number cards or as the total score. If he meant the total score, then 78 could be an extreme case so that students easily recognized that a bigger score would be far from 20. In fact, 78 could not be the sum of three number cards.
Moreover, the biggest possible sum 30 (three number cards of 10) was better than the lowest sum 3 (three number cards of 1). Regardless, Chase's example was effective enough to finish the discussion.

**Authoring story problems encouraged**

One day during the observation period, the teacher asked students to make a story for the problem 42-26 which they had solved. Episode UM-10 showed how three students made story problems and how Ms. M reacted.

<Episode UM-10: Students' story problems and the teacher's response>

Derrick: 42 oranges were wasted and 26 oranges are picked up. How many more do they have?

T: Okay, 42 oranges ... Tell me again.

Derrick: 42 oranges were wasted. Somebody picked up 26 and put them back. How many more did they have left?

T: Okay, let me see if I got it straight. 46 oranges are ruined ... (Interrupted.)

Derrick: 42!

T: 42 oranges were ruined (looking at the problem on the board). Somebody picks up 26 of the ruined oranges and puts them back?

Derrick: No one ruins. They just fell.

T: 46 oranges were just ruined.

Ss: 42!

T: 42 oranges were ruined and somebody went to get 26 good ones and put them back on the shelf. What's the question?

Derrick: How many more do they have left?

T: How many more do they have left? Um. How many oranges do they have left?

Michael: How many oranges wasn't picked up?

T: How many oranges weren't picked up?
S: How many were left?

T: Brandon, come here, go to your desk [disciplining him]. Okay, who can try, who can do it? Let me see. Try, David. I am very confused about that problem, Derrick. I am gonna think of that for a minute. Okay, David, let’s try yours.

David: 42 dogs are chasing cats. 26 dogs, uh, take a break.

Ss: Uh?

T: Okay, look what he just said. He said, 42 dogs (drawing several Xs), is this art class?

Ss: No.

T: Am I gonna draw dogs or whatever you want me...?

Ss: No.

T: Let’s pretend I have 42 (pointing to the 8 Xs she drew). 42 dogs were chasing cats. 26 of them took a break. I don’t have right numbers here. But how many dogs are left? Would that make sense?

Ss: Yeah.

T: Would that make sense? Okay. All right. Derrick? Try the same problem you told me before with the oranges, but make it make sense to me. Derrick, try again.

Derrick: 42 oranges have fell. (He puts his fingers on his lips.)

T: Move your fingers.

Derrick: 42 oranges have fell to the floor, and 2 people came in and picked them up. How many oranges left on the store [floor]?

T: Okay, 42 oranges ... Look, pretend this is gonna be 42 (drawing several Xs on the board). 42 oranges are on the floor, 2 people come along and picked them up. How many were left? Would that ... (Interrupted.)

Derrick: I mean 26 oranges.

T: Oh, picked 26 of them up? And you wanna know how many there were left on the floor? That would work! There were 42 oranges on the floor, they fell on the floor, and 2 people came along and picked up 26 of them. How many were there left on the floor? Okay, I am gonna take one more problem. Eryn, are you ready? Okay, I am waiting for Noble to look at the problem. Okay, go Eryn.

Eryn: There were ...
Brandon, go to your desk.

There were 42 dogs and 26 of them had their face painted. How many didn't have their face painted?

Wow! Turn to your neighbor. (She asks students to talk about their stories with neighbor.)

In this episode, when Derrick first presented, his wording was ambiguous. Ms. M did not understand his story, which led her to develop two different interpretation. Someone picks up 26 either from the 46 ruined (fallen, or wasted) oranges or from any good oranges. The latter interpretation prevented her from understanding Derrick’s story problem even after Michael clarified the question by “how many oranges wasn’t [weren’t] picked up?” Note that each of Ms. M’s interpretations could produce a legitimate story: one for a take-away problem (as Michael contributed) and the other for a comparison problem (e.g., how many more ruined oranges are there on the shelf than good oranges?). Instead of trying to understand Derrick’s story, Ms. M listened to David, who presented a story using take-away meaning of subtraction. She then encouraged Derrick to revise his problem in a similar way that made sense to her. Derrick presented his story again but more clearly, because of Michael’s articulation and David’s illustrative story. Eryn posed another story for the problem 42-26, which required comparison within a set. Ms. M moved on to the next activity, without comment.

A child’s leading question

During this instruction phase Ms. M specified exactly what students were supposed to do for an assigned activity. However, when she finished explaining how to
play Raise To 150, Chase asked a critical question regarding a person having more than 150.

<Episode UM-11: Chase’s question leading to specify a game rule>

Chase: What if you have more than 150?

T: No, when you get to 150, you have to stop. Your partner doesn’t have 150, you still have to stop. The first person gets to 150, you have to stop. (Derrick raises his hand.)

Derrick: When you roll the dice, and you have 149, and you roll 1 and 5.

T: Oh, listen to Derrick. Derrick said, what if you are on 149 and you roll a 6? That will make it more than what?

Ss: That’s what Chase was talking about.

T: That’s what you were talking about, Chase? I didn’t understand. I am sorry. I am sorry. Oh, just go as close as you can to 150. If you roll a 6, then you’re right, your number is gonna be a little bit larger. Oh, I know what we can do. Chase, I know what we can do. How about... if we only use the part of the number that we need to get a 150? How about that?

Ss: Yeah!

T: So, if it’s, even if it’s 12, we will use 1 of the number. We break it down into 11 and 1. And we will just use the 1 that we need. Because we will ... I really would like it to be exactly 150. Is that okay with you?

Ss: Yeah.

T: Do you understand what I am saying, Jonathan?

Trenea: Then we all have the same number!

T: Well, no, you will have 150, but your partner won’t. I can’t imagine that both of you get to 150.

S: But I know, unless you all both have the same number, but you all are not gonna do that.

T: Probably not.

S: Unless you have at the same time.
Ms. M initially did not understand Chase's question. Soon Derrick articulated Chase's contribution by suggesting an example where a person has 149 and rolls 1 and 5. As Ms. M accepted Derrick's example and asked others to pay attention to the case, students gave the credit to Chase who originally came up with the idea. Acknowledging Chase's contribution, Ms. M tried to explain the case. Meanwhile, she came up with a new idea of what they would do when they had more than 150. Note that Ms. M asked for students' agreement for the new idea, rather than force them to follow her decision.

**Activity (Game) Implementation**

After explaining how to play a game, the teacher gave students materials for the game. Ms. M always asked students to choose their partners. Choosing a partner on a daily basis was intended to help students learn how to get along with a variety of people. Usually two students played together, and they worked on the floor unless the teacher asked them to work at their seats. Ms. M walked around and checked whether students in each group were following her instructions. She spent a substantial amount of time with each student or group, rather than moving quickly from student to student or group to group. She often asked them to explain what they were doing, but sometimes she included questions starting with how or why. When students did not answer, Ms. M suggested tools, such as the number line or related number facts (e.g., 7+7 for 7+8), to help them in their figuring. She frequently expressed her excitement and surprise, adding fun to students' activities.

Ms. M consistently encouraged students to help one another, praising some of them along the way. Visiting an individual group, she often checked whether they were working together. There were usually some students who finished an assigned activity.
earlier than others. Ms. M asked some of them to help students who had difficulties following her game instructions. Sometimes students voluntarily suggested helping their friends in their group or their neighbor. In those cases, students monitored whether their friends faithfully kept the rules or steps for the activity and tried to finish the activity in a timely fashion.

In general, students were actively involved in playing a game with excitement, which sometimes caused Ms. M to ask them to use soft voices. For instance, in playing Turn Over 10, students frequently remembered the places of cards that previously had been turned over. Whenever they turned over 10, students shouted. In particular, when they picked up two wild cards, they looked extremely excited and talked expressively to their neighbor and the teacher. Sometimes students were faced with circumstances Ms. M had not explained in the instruction phase, for example, when they picked up 10 and another number card, or two king cards. Students asked questions about those special cases. Ms. M often dealt with the cases in subsequent whole-group discussion. There were also some students who had difficulty following instructions, in particular with the new game Circles and Stars. Those students drew small circles so that they couldn’t put stars in or forgot to move on to the next page when they finished every turn. However, most students followed Ms. M’s instructions as expected. Some group even played an assigned game twice.

**Teacher’s interaction with students**

Ms. M’s interaction with individual groups was responsive to these students’ needs. If students playing together were strong in math, she briefly checked their work. If students were weak in math and had difficulties in playing a game, she spent time with
them until they were able to work independently. For instance, Ms. M mediated every step between Robert and Nick, who were weak in math, when they were playing Turn Over 10. She used her fingers to help Nick add the two numbers turned over, and asked Robert to guess the second number which could make a 10. Ms. M expressed excitement when Nick made a 10 and Robert turned over two wild cards.

While interacting with individual students, the teacher frequently asked them to explain what they were doing. This often led students to realize their mistakes. For instance, Derrick answered 78 to the problem 80-18 and wrote his method as follows: “I knew 80-10=70+8=78, because you forgot to use the 8.” Ms. M reminded him that he had taken away 10 when he got 70, and asked what more he had to take away. Derrick responded 8, but had not yet realized his mistake. When Ms. M asked what he did, he finally realized that he added the 8 instead of subtracting it and easily corrected his mistake. Similarly, in the game of Circles and Stars the teacher’s questions led students to correct their errors or to find a mismatch between their drawing and its corresponding expression.

When students’ difficulties came from more than trivial mistakes, Ms. M provided more substantial help. For example, Jonathan drew 6 circles with 6 stars in each and wrote 6X6. When the teacher asked how he could figure out the equation, Jonathan found that 2 groups of 6 is 12 by using the number line. As he did not proceed further, Ms. M suggested using cubes, but acknowledged that the problem was hard.

In another case, in solving the problem 80-18, Eryn had a correct representation of 80-18 using tally marks in that she drew 80 tally marks and crossed out the first group of 10 tallies, 5, and then 3 ones. But she wrote 80-18=70-8=78 in a vertical
format and put 78 next to the representation with tally marks. Her explanation on the paper was as follows: “I counted by tens and I took away one ten and I took away eight and when I counted I got seventy eight.” Eryn’s verbal description was the same when Ms. M asked her to explain what she did. Ms. M first counted together the tally marks left, but Eryn did not notice or explain the mismatch between her drawing and the answer. Ms. M pointed out the equation 70−8=78. She then provided a real-life situation for 70−8 saying, “If you have 70 pencils and you take away 8, you are gonna have 78 left?” Eryn realized that 78 is bigger number than 70. As Ms. M did with students strong in math, she left Eryn to correct the mistake for herself.

In the same activity, Nahjha had written 80−18=62 and put 8 tens sticks in one side of his desk and 1 tens stick and 8 cubes on the other side (see Episode UM-12). It was not clear how he got the right answer. He might simply have copied the answer from Michael in front of him, and represented the problem as it was. When Ms. M asked Nahjha to explain how he solved the problem, he said that he took away 18 cubes.

<Episode UM-12: Taking-away meaning of subtraction emphasized>

T: All right, now, have you found out the right answer? Do you know the right answer?

Nahjha: Yes, Ma’am (pointing to 62 in his equation).

T: Let’s see if you have that. That’s not what I see on your paper. 10, 20, 30, 40, 50, 60, 70, 80 (counting the 8 sticks). I still see 80. (Nahjha counts the sticks, too. But the teacher blocks and asks.) Did you take away 18? What did you do? (Points to 18 cubes on one side.)

Nahjha: I took away (holding the 18 cubes).

T: You took it away? Is this 80 to begin with? (Puts the 18 and 80 cubes together.)

Nahjha: No.
T: Is this 80?

Nahjha: No.

T: What’s your first job?

Nahjha: [Inaudible.]

T: Show me 80 take away 18.

Nahjha: (Counts the sticks and separates the group of 80 and the group of 18.) 80 take away 18 (pointing to each group of the sticks).

T: Is this part of your 80? (She puts the separated groups together.) Show me 80. Show me 80. When I come back, I am gonna see 80 on your desk. Raise your hand when you got 80 cubes on your desk, don’t do anything else until I come back.

(After the teacher leaves, Nahjha makes a group of 80 and a group of 18 but with different colors, and separates them. He calls Ms. M when she works with Robert.)

T: You just did an “and” problem. We are doing ... (Interrupted.)

Nahjha: 80 take away 18.

T: But where is ... When you take away the 18, where are they gonna be?

Nahjha: (Points to the group of 80).

T: You just said, 80 “and” 18 (pointing to the groups of cubes).

Nahjha: 80 take away 18.

T: Where is 80?

Nahjha: There (pointing to the group of 80).

T: Then you need to get rid of all those ones you got in there (pointing to the group of 18).

Michael: (To Nahjha) Oh, yeah, take away 18 (pointing to the group of 80).

Nick: Take away 10 and ... (pointing to the group of 80).

T: Listen to Nick. Listen to what Nick says.

Nahjha: Take away these (picking up 2 sticks from the group of 80, and looks at the teacher). I might be ... (Interrupted.)
T: You need to get rid of this, because you are about to be very confused (putting the sticks he picked up on the desk, and pointing to the group of 18). Get rid of them! You don’t need them. 80.

Nahjha: 10 (putting one stick on the side) and (holding another stick) ... (Interrupted.)

Michael: (Goes to Nahjha’s place) equals 70.

Nahjha: 10, 12 (counts, takes out 2 from the stick, and make 8 hooked cubes into loose ones).

T: You are breaking it up because it’s not a 10 any more.

Nahjha: (Breaks up the 2, and separates the group of 62 from the group of 18.) 62. (He looks at the teacher.)

T: So, what’s the answer? So, 80 take away 18 is 62 (pointing to each group of cubes). Okay, you need to draw that for me. Because you did with this, you need to show me a picture of what you did. You need to draw your tens sticks and show me.

In this episode, the teacher emphasized that there was not 80 at the beginning by combining the two groups of cubes, and asked Nahjha to put only 80. After Ms. M left for another group, Nahjha made a group of 80 and a group of 18 again but with different colors. It was not clear what was the motivation for him to separate the groups by colors, despite the teacher’s simple request of putting only 80. There might be many possible interpretations, such as (a) he wants to show the 18, the number taken out; (b) he is thinking of the comparison method; (c) he is emulating formal algorithm; and (d) he is using the colors as negation markers — 80+(-18). When Ms. M came back, the situation was the same in that Nahjha emphasized “80 take away 18” with different color cubes. She seemed not to focus on any possible deep nature of Nahjha’s understanding. Instead she evaluated that Nahjha represented an “and” problem and directly asked him to remove the 18. Michael and Nick in the group supported the teacher’s request by suggesting taking away 18 from the 80. Under the cautionary
attention of the teacher and his friends, Nahjha took away 18, and separated the answer 62 and the subtrahend 18.

**Teacher's assessment by interview**

On the last day of observation, the teacher interviewed individual children to assess students' understanding of multiplication. Ms. M mainly asked two questions: the meaning of multiplication with one problem on the student's paper, and the counting strategy to figure out the answer. For the first question, most students presented an expected answer because they had learned how to state a multiplication equation (e.g., 4 groups of 5, 4 sets of 5, or 4 times 5). For the second question, many students demonstrated counting by ones, and some used more advanced strategies such as counting by twos or fives. In particular, for the problem of 5X6, one student decomposed each 6 into 5 and 1, and counted by fives first and then added ones back. Ms. M sometimes asked for an easier or quicker way than counting by ones. She briefly recorded in her evaluation sheet how each individual student answered.

**Students' various interactions**

As mentioned before, since students chose their partners on a daily basis, the nature of interaction among groups varied. Moreover, because the teacher walked around during this activity phase, students had to work without the teacher's mediation for a relatively long time. Students usually worked well with their partners. They amicably shared materials, faithfully took their turns, and made sure that they were keeping the rules of playing a game. The following are descriptions of different characteristics of interaction in various groups.
Some students, without the teacher's observation, did not pay attention to what their partners were doing. Instead, they were eagerly taking their turns and maintaining good behavior, as exemplified in the following episode.

Episode UM-13: Students taking their own turns

(Brandon and Noble were playing How Close To 20.)

Brandon: (Has 2, 1, 5, 10, and 4. He chooses 10 and 5, touches 2, and then 4, and pauses.) 5 and 10, 15, (pause) 16, 17, 18, 19. 19! (Counts 4 by his fingers, and then finally chooses 4 and writes the combination 4+5+10=19 on the score sheet.)

Noble: (Has 5 and 5) I need more! (He turns over three more cards, 7, 1, and 2.) 5 and 5 is 10 (picking up 5 and 5). 10 [5+5] and 10 [7+1+2], 2 and 1 is 3 (picking up 2 and 1), so I got 20 again!

Brandon: You can't do two [1 and 2].

Noble: Oh, I got 4. Well, I made a mistake, Brandon (putting 1 back). (He writes his combination 5+5+2=12.) Brandon, your turn.

Brandon: (Had already picked up his cards and immediately writes.)

Noble: (Reads Brandon's scores.) 4 plus 5 plus 10 is 19. And 9 plus 9 plus 1 equals 19. Stack here (pointing to the dump file). (Noble has 7 and 1. He puts down two more cards, 2, 8 and counts how many he has. Brandon turns over one more card, 4.) This [2], and 8 (overlapping 8 over 2), I got 11. Brandon, I got 11. So, I put 11, 2 plus 8 plus 1 equals 11 (writing his combination).

Note that Noble first chose four out of the five cards of 5, 5, 7, 2, and 1.

Brandon reminded Noble of the rule of using only 3 cards. Noble accepted his mistake and said what he chose, but Brandon did not challenge Noble's choice of the three cards of 5, 5 and 2. Instead he picked up his cards as soon as Noble selected the cards. In his next turn, Noble got the cards of 7, 1, 2, 8, 4 and made 11 (8+2+1). Noble reported to Brandon. However, Brandon again was not concerned about his partner's work, as long as Noble followed the instructions of playing the game. Note that in the previous instruction students had the collective experience of choosing the best combination to
get as close as possible to 20. When the presenting students made 15 (9+1+5) among the five cards of 5, 9, 1, 2, and 5, they were enthusiastic to point out the best choice of 9, 5, and 5.

As reflective of Ms. M’s frequent request, many students tried to help others. A few students provided related addition facts to help their partners figure out the sum of numbers (e.g., 7 and 3 for 6 and 4). When a student strong in math and a student weak in math played together, the student strong in math played the role of teacher, providing step-by-step instructions of what to do. For instance, Ms. M helped Raven write expressions for her drawings of circles and stars. When leaving for another group, she asked Haley, Raven’s partner, to help her. As evidenced in Episode UM-14, Haley asked questions which led Raven to write each element of the expressions for multiplication. Haley used the models of expressions written through the interaction between Ms. M and Raven as models. Note that Haley offered further help when needed, though she did not finish her own expressions.

<Episode UM-14: Haley playing the role of teacher for Raven>

(Raven has 2 circles of 3 stars)

Haley:  Okay, how many circles are there? (Points to her circles.)

Raven:  2.

Haley:  Write that. (Raven is about to write at the corner of the page.) Not right there, write down here (pointing to the bottom of the page). (Raven writes 2.) How many stars inside each circle?

Raven:  3. (Raven tries to write, but Haley stops her.)

Haley:  But what is here? (Points to “4 groups of 5=20” in the previous page.) How many, it’s groups, how many groups are there? (Points to the circles again.)

Raven:  2.
Haley: 2. Write “groups of.” (She writes “groups of”.) How many stars are there? (Points to stars in Raven’s drawing.)

Raven: (Writes 3.)

Haley: 3 equals. (Raven writes “=”.) How many would that be? Do you know how to count by 3? What is 3 plus 3? (Points to the 3 stars in each circle.)

Raven: 6.

Haley: Equals 6. (Raven writes 6.) Now, what is that? (Points to “4X5 =20” in the previous page.) Times. So, how many circles are there? (Points to the circles again.)

[In a similar way, Haley helped Raven write 2X3=6 and 2 sets of 3=6]

Haley: Can you do it by yourself now? (Raven nods her head.) But if you have any trouble, ask me.

When students strong in math worked together, they sometimes found their own mistakes without their partner’s help. For instance, Lainey and Nahjha were playing Circles and Stars. Lainey had 4 circles with 2 stars in each in her first page and Nahjha was looking at the drawing. Both of them said, “2 sets of 4 equals 8”, and Lainey wrote on her book as 2 sets of 4=8. But she soon realized that she should have put 4 sets of 2. As time went on, they individually wrote expressions for their drawings. Both of them used advanced counting skills such as counting by 5s and 6s. On his last page, Nahjha had 5 circles with 3 stars in each. He wrote “5X3 =”, started counting, and found 2 stars in the second circle. He corrected his mistake by adding one more stars.

Michael and Derrick were also students strong in math. Unlike other students, Derrick carefully paid attention to Michael who was counting his stars in the game Circles and Stars. Derrick provided useful tips but Michael did not take advantage of his suggestions, as seen in the following episode. Instead, Michael kept his own methods of counting.
<Episode UM-15: Michael keeping his way of counting>

(Michael counts the stars and writes 6 on the first page. On his second page, he drew 6 circles with 4 stars in each.)

Derrick: 10. [He seems to add the 6 from the first page and 4 stars in the first circle on the second page.]

Michael: 1, 2, 3, 4, ... (counting the stars on the second page) ..., 24. (He starts counting the stars on the first page.) 25 ... (Interrupted.)

Derrick: You counted it.

Michael: 1, 2, 3, 4 (counting the stars on the second page again) ... (Interrupted.)

Derrick: No, start from 6.

Michael: Four [and] four is 8, 8.

Derrick: Plus 12, equals 20.

Michael: These are two eights. That makes 16. 17, 18, 19, 20, 21, 22, 23, 24. (He starts counting the stars on the first page.) 1, 2... (Interrupted.)

Derrick: Put 24 (writing 24 on the second page).

Michael: Why?

Derrick: 24 plus [6 equals 30].

Michael: I didn’t add this (pointing to the first page).

Derrick: Yes, you did. You said that’s 6 right there (pointing to the first page) and then added... [Interrupted.]

Michael: No, watch. 1, 2, 3, 4, 5, 6, 7, ... (keeps counting) ..., 19, 20 (pointing to each star on the second page) ... (Interrupted.)

Derrick: So, that’s 30.

Michael: 20, 21, 22, 23, 24, (keeps counting the stars on the first page), 25, 26, 27, 28, 29, 30. (He turns to the next page and keeps counting one by one) 31, 32, 33,... (keeps counting)...., 59, God! 60, 61, ... (keeps counting) ..., 80.

In this episode, Michael first counted the stars in the first page and wrote 6. As Michael counted the stars on the second page starting from 1, Derrick suggested he add
the 6. Michael finished counting the stars on the second page by ones and moved on to
the first page. Derrick again reminded him that he already counted the stars, and
strongly encouraged him to start with the number 6. Without listening to Derrick’s
suggestion, Michael started counting again on the second page where he drew 6 circles
of 4 stars but attempted to use counting by fours. When Michael figured out 4 plus 4,
Derrick said, “plus 12 equals 20” implying that he counted the next three circles of four
stars. Michael again did not listen to Derrick and used his own way of knowing that two
eights make 16. But Michael did not pursue the advanced counting strategy and went
back to counting by ones. When Michael kept counting by ones again and again,
Derrick wrote down 24 which was the number of stars on the second page and pushed
Michael to add it to the number of stars on the first page. Despite Derrick’s helpful tips,
Michael ended up counting by ones and gradually became exhausted. It seemed that
Michael was contesting Derrick’s conclusions and using counting by ones as a way to
check his answer. There are at least three possible interpretations for why Michael did
not listen to Derrick: (a) Michael was very confident to his mathematical ability, (b)
Derrick’s suggestion was complex for Michael, and (c) Derrick was interrupting
Michael’s thinking.

As Michael did, during this game, many students expressed their difficulties in
counting stars all together by sighing in the middle of counting as the number increased.
Even though they wrote down the number of stars on each page, students did not come
up with the idea of adding those numbers, instead counting by ones. The only exception
was Trenea who came up with the idea herself though making many mistakes. Ms. M
found Trenea using the idea, but she did not apply it to the whole-group discussion.
even when some students reported their difficulty in counting. Given that Circles and Stars is used as a game, adding together the number of stars from different pages was necessary. However, this game revealed a lack of mathematical purpose. Pedagogically, Circles and Stars was used to introduce multiplication. Indeed, counting stars on each page gave students an opportunity to understand the meaning of multiplication as repeated addition. Moreover, students often used advanced counting strategies such as skip counting when they counted the number of stars on each page. However, this benefit was subverted by adding the number of stars across pages.

The interaction between Nahjha and Chase reveals an example of the dynamic nature of students' own involvement in this activity phase. The two students were playing the game Raise To 150. Interestingly, they got the same number 26, which was represented as 2 tens sticks and 6 loose ones on their own place board. Chase rolled the dice and got 4 and 2. He brought 6 cubes from his resource bag, hooked them together, picked up 4 loose ones on his board, and then added them to the 6 hooked ones to make a tens stick. Nahjha took his turn and had 4 and 2 again. He took out 4 cubes from his resource bag, hooked them together, and counted the 6 loose ones on his place value board in order to make a tens stick. His intention might have been to add the extra 2 cubes after making a 10, but Chase insisted that Nahjha should pick up the addend 6 first. In response to Chase's suggestion, Nahjha tried to show that 4 and 6 could make 10, but Chase intervened and took out the extra 2 cubes and gave Nahjha them. At the same time, Chase put the 6 loose ones back on the board. While Nahjha was hooking together the addend 6 with cubes, Chase separated the ones on the board into 4 and 2 in order to help his partner make a 10 with the 6 new ones and the 4 on
the board. It did not matter whether they first added 4 to the 6 ones on the board and put 2 extra ones (Nahjha's method) or that they divided the 6 ones on the board into 4 and 2, and then added the new 6 to the 4 ones on the board (Chase's method). But for Chase adding 4 first instead of 6 made little sense, which led him to insist on the alternative method. It was not clear why Nahjha accepted Chase's suggestion. Considering his usual pattern of involvement, Nahjha seemingly decided not to simply follow Chase's request. Rather, he accepted Chase's suggestion because he realized that it could also produce the same result.

In many other cases where students were eager to work together, their collaboration was rather limited to social interaction because of their lack of mathematical knowledge. For instance, Erika and Eryn were playing the game Circles and Stars and they monitored each other. As seen in Episode UM-16, in order to figure out 6X6 they counted the stars by ones, realizing that counting by 6s would make their job a lot easier. When Erika counted her stars all together, she counted 200 as the next number after 109. Immediately, Eryn seemed to know Erika's mistake and interrupted her counting, but led her to use another wrong counting system starting with "10 hundred" in place of 110.

<Episode UM-16: Students working together with limited mathematical knowledge>

(Eryn drew 6 circles and 6 stars in each circle.)

Erika and Eryn: Okay, 6 times 6 equals (Eryn writes 6X6=), 1, 2, 3, 4, 5, 6, ... (keep counting the number of stars by ones)..., 36.

Eryn: It will be a lot easier if we knew how to count by 6s.

Erika: (Rolls the dice and draws 2 circles. She again rolls the dice and draws 3 stars.) Oops, again. (She uses Eryn's eraser.) 2 times 3 equals (writing 2X3=), 1, 2, 3, 4, 5, 6 (counting the stars). 6 (writing 6 in the equation).
{They finish playing the game and writing the equations. They count the stars by ones, instead of using the numbers on each page.}

Erika: (Sometimes expresses being overwhelmed in the middle of counting by ones.) I have over 100! (To Eryn) I have over 100. (Eryn joins counting.) 101, 102, ... (keeps counting) ..., 109, 200.

Eryn: Wait! [10 hundred], then 11 hundred!

Eryn and Erika: 12 hundred, 13 hundred, ... (keep counting)..., 17 hundred!

Erika: Yeah, I won the game.

In a similar way, there were many instances where a student was helping his or her partner, but that student also did not know exactly how to figure out what to do. In those cases, the students usually produced incorrect or non-optimal answers. For instance, Trenea and Kelly were playing How Close To 20. Trenea played a leading role in Kelly choosing three cards and, most of the time, Kelly just accepted her suggestion without question. Kelly had five cards: 4, 2, 7, 10, and 6. She initially chose three cards: 10, 6 and 2. Trenea asked her to replace the 6 and 2 with the 4 and 7. Trenea made a mistake in adding up the final three numbers $10+7+4=17+4=20$. Against her strong intervention during Kelly’s turn, Trenea let Kelly write the combinations. However, neither of them realized their mistakes nor found the best combination by using 6 in place of 7. This best combination could be made just by replacing one card among the ones Kelly chose at first.

Activity (Game) Discussion

As students got finished the activity of the day, the teacher had a whole-class discussion at the front of the classroom. Students joined their friends who had already finished the activity and had been waiting on the rug. Ms. M usually asked whether they had fun and said they worked well together during the activity. Most of them agreed.
She highlighted individual students and groups of students who helped each other and followed her directions well during the previous activity. Ms. M often let students pat themselves on the back as a compliment. She shared with students the idea that working together made their class the best. For specific activities such as solving 80-18, Ms. M asked whether the task was easy or difficult. Most students said “easy,” though many of them answered 78. Ms. M usually closed her math lessons with praise showing her love for students and her conviction about their mathematical ability. In particular, in Episode UM-17, Ms. M compared her students with other older students who were reported as not studying mathematics and gave her students credit because they knew how to think about and talk about mathematics. When Ms. M ended each lesson, she proudly said, “I think you do a super job.” Then, she would slow her words to a deliberate rhythm and continue, “You are ...”, and the students would enthusiastically reply, “The best!”

<Episode UM-17: Teacher’s closing remark with praise>

T: All right. I wanna tell you something. I read a paper today. Eyes on me. Bottoms down. Hands in your lap. Ready? I don’t hear bottoms coming down. So I know, I am not ready to open my eyes yet. When I open my eyes, I need to see everybody looking gorgeous and smart. I read a paper today. I read a paper. That older children are afraid of taking math and they are not studying mathematics because they don’t know what it means. They don’t know how to talk about. They don’t know how to think about. You need two thumbs up for yourselves. (Students raise their thumbs up.) You know how to think about mathematics, you know how to talk about mathematics, you are ...

Ss: The best!

T: I love you!

Mathematical discussion encouraged

In this phase the teacher expected students to discuss their experience of playing a game beyond social interaction. However, students sometimes reported who was the
winner. Ms. M listened for a while, but soon expressed her interest in a mathematical
discussion saying, “Don’t tell me who won, because I am not really interested in who
won, but you know what I am interested in? You know me well enough.” A few
students said something about their partners’ mistakes. Ms. M stopped them from
talking about the mistakes and confirmed with students that she was worrying about
their thinking more than anything else.

Ms. M’s concern of students’ thinking was evident when students provided the
right answer without understanding. Ms. M focused not on whether they got the answer
but on how they got the answer. Thus in Episode UM-18, she initially did not accept
Trenea’s contribution because Trenea knew 6X5=30 from a multiplication table at
home. Ms. M encouraged her to figure out the answer in a way that might make sense
at a second grade level. When Trenea came up with the idea of counting by 5s, Ms. M
excitedly accepted her contribution.

<Episode UM-18: Mathematical explanation over memorization emphasized>

(Ms. M draws 6 circles and 5 stars in each on the board.)

T: All right. Now I am gonna practice writing down our multiplication facts, because
you’ve been saying to me over and over. Mrs. M, we wanna study multiplication. Ever
since Michael started this year, you’ve been saying that to me. Mrs. M, we wanna study
multiplication. That’s what we are ready to do. Who can tell me what I’ve got here? (A
few students raise their hands.) How many groups do I have? How many groups do I
have? (Many students raise their hands.) Or how many sets do I have? I wish
everybody’s hand this time. How many groups do I have? (Points to each circle one by
one.) These are sets or groups (pointing to each circle again). How many do I have?
Trenea?

Trenea: 6 times 5.

T: Okay.

Trenea: Equals 30.
T: Trenea says, I’ve got 6 sets with 5 in each one. So, I’ve got 6 groups of 5 (writing 6X5), 6 groups of (pointing to “X”), that’s what times means, 6 groups of 5, she said that’s 30 (finishing the equation, 6X5=30). How did you figure it out, Trenea?

Trenea: Because, I said like, I just knew because I have a times table ... (Interrupted.)

T: We are just in second grade. We don’t just know that. We have to have ways of saying or finding out.

Trenea: I have a times table chart in my house and ... (Interrupted.)

T: But I need to know a better way to find out.

Trenea: 5, 10, 15, 20, 25, 30 (pointing to the circles with her finger).

T: Aha! 5, 10, 15, 20, 25, 30 (pointing to the circles one by one). So, 6 groups of 5 is 30.

Reviewing significant aspects of a game

How the teacher initiated a discussion about a game depended on the nature of the game played and how well students did. For some games, such as Turn Over 10 and Circles and Stars, Ms. M checked what they could do with some possible game situations. In most cases, Ms. M questioned and students answered. As seen in Episode UM-19, however, Ms. M sometimes encouraged students to pose a problem. Brandon provided her with a special case where she had to decide two numbers to make 10 in the game of Turn Over 10. Note that Ms. M questioned only in cases where students needed to decide only one number, for example, 4+?=10.

<Episode UM-19: 10+0 for two wild cards>

T: You make up one to fool me! You think of one to fool me!

Ss: (Raise their hands.) It’s easy!

T: All right, Brandon says, you can fool me.

Brandon: I know.

T: Go, Brandon.
Brandon: 2 blanks.

T: Okay. Not fair! (Students laugh.) No, because ...

Brandon: [Inaudible.]

T: Okay, let me try.

Brandon: Ready? Blank ...

T: Blank (drawing a square). (A few students laugh.)

Brandon: Plus blank ...

T: (Draws another square for the second blank.) This is like those kids that got two wild cards. Equals 10? ( □ + □ = 10) Okay, what did you make your two wild cards? Derrick?

Derrick: 10 and 0, and 10 and 0

T: Robert thought about it. (She puts the number 10 and 0 in each blank and writes another one 0+10=10.)

Michael: Yeah, but how did you get the 0? 2 tens?

Ss: 10 and 10 is 20... Because she said, get something [you want]...

T: Well, Robert thought about that, when he got 2 ... when he got 2 wild cards, he said, can I make both 10s? Then I said, yeah, if you want to. Do what you want. Remember I told you, you can do what you want to.

In this episode, Ms. M immediately connected Brandon’s question with the game situation where students turned over two wild cards. Derrick said that he made 2 tens by 10+0 and 0+10 for the special case. Ms. M also shared her observation of Robert who made the same decision. At that moment, Michael seemed to agree with the equations 10+0=10 and vice versa, but was soon unsure about using them for this game case. Michael’s first question was how players got the 0. Indeed, there was no number card with 0. His implicit second question was that the cards were supposed to sum to 10 per turn. In other words, if they used both wild cards as 10s, they were making 2
tens per turn. With other students’ contributions, Ms. M reminded Michael that a wild
card can be any number, but did not probe his questions deeply. In fact, students had
another special case where they turned over a 10 and another number card. Some
students did not count the case because the sum of the two cards was over 10.
However, others wisely put back the extra number card but kept the 10, which might
correspond to 10+0=10 or vice versa. Moreover, under the shared assumption of wild
cards, the case of two wild cards exclusively could include any combinations of 10
students might make (9+1, 8+2, 5+5, etc). The proposed example of 10+0 was only one
special case. Note that when students selected 0 and 10 for the case of turning over two
wild cards, they were engaged in a particularly mathematical preoccupation, considering
extreme cases.

Specifically, for the game Circles and Stars Ms. M used teddy bears to represent
multiplication and helped students express its meaning using groups or sets (see
Episode UM-20). Ms. M put 3 rows of 3 teddy bears and David tried to represent the
number of bears with multiplication.

<Episode UM-20: David wanting to figure out 3X3 for himself>

T:  Who can tell me how to say this? You don’t know, David?

David:  I know.

T:  You know. Okay, start off then.

David:  6 uh ...

T:  I don’t wanna know the answer. I wanna know how to say it.

David:  3...

T:  He is thinking. The reason why we are such a good class is we give him a chance to
think. Do you need a little bit of help?
David: I know what’s ...

T: You know the answer, don’t you? Okay, you know the answer, but don’t tell me that. I wanna know how to say it. How would I say it?

David: You can tell times ...

T: You could use times. You could. How could you say it?

David: 3 ... You like it in words or in times?

T: Try times.

David: 3 times ...

T: How many sets do I have? I’ve got ... (Interrupted.)

David: 3.

T: And then how many in each set, David?

David: Times 3.

T: Equals

David: Uh, that’s the hard part.

Trenea: No, it’s not. You can count the bears. (Interrupted.)

T: Sh!

David: Leave me alone!

T: He can do it. Go, David.

David: 3 times 3 ... I need to use my fingers.

T: Oh, sure, sure, any way that makes sense to you.

David: (Counts his fingers.) 9.

T: Okay, so he said 3 times 3 equals 9. Who can say it in a different way?

It took some time for David to come up with 3 times 3 for 3 rows of 3 bears in each. As he admitted that it was hard to figure out the answer, Trenea immediately suggested counting the bears. Stopping her from saying anything further, David clearly
expressed his interest in finding the answer for himself. With the teacher's encouragement, he figured out the answer using his fingers, which appeared to make sense to him at that moment.

**Reviewing students’ solution methods**

For some activities, such as solving 80-18 and doing multiplication, the teacher initiated discussion with questions based on her observation of students’ play. For instance, Ms. M asked students to share the ways they showed their solution methods of 80-18 to her. Students’ methods varied including using tally marks, drawing cookies, tracing cubes, drawing circles, counting backwards, and connecting numbers by lines. While Ms. M accepted each method above as different, the class worked together on only the last two methods.

In assessing students’ understanding of multiplication, the teacher found Chase posing challenging problems such as 3X0. When Ms. M asked for its meaning, Chase explained that there were 3 circles but no stars so that the answer would be 0. She expressed her surprise and praised him. As seen in Episode UM-21, Ms. M represented Chase’s problem by drawing three empty circles and initiated whole-class discussion.

*<Episode UM-21: Chase’s problem 3X0 used for class discussion>*

T: Look what Chase did. Chase said this problem. (She starts drawing on a piece of blue paper. Michael moves toward the paper.)

S: I can’t see, Michael.

T: I am gonna let you see, I promise. (She finishes drawing 3 empty circles. She put the paper in the middle of the carpet on the calculator boxes.) Raise your hand when you know what that problem is. (Several students raise their hands.) Robert, do you think you know?

Robert: 3 groups of 0.
T: Look what Robert said. 3 groups of 0 equals what, Robert? (Writes “3X0=” on the paper.)

Robert: 0, 3, 3 (showing three fingers).

T: Is it 3 or is it 0? That’s a tricky answer, isn’t it? Which one is it?

Ss: 0, 3, 0, 3.

T: Now, I wanna listen to Robert. Some people are thinking that the answer is 3 and some people are thinking that the answer is 0. But I wanna listen to Robert and I wanna listen to his explanation. What, Robert?

Robert: 0.

T: Why do you say 0, Robert? Explain it to us.

Robert: Because there is zero, um, zero stars.

T: Aha! Listen to what he said. I am waiting for Nahjha. He said there are zero stars. Say it, Chase?

Chase: There are 3 circles and 0 stars.

T: There are 3 circles, but there are 0 stars. So the answer is what?

Ss: 0.

T: 0 (writing the answer on the paper).

In this episode, Robert explained Chase’s problem as 3 groups of 0, but was confused whether the answer should be 0 or 3. Students also had the two different answers. As usual, Ms. M gave Robert a chance to explain, since he had not finished his turn. Robert finally came up with the same idea as Chase: Because there are 3 stars but no stars, the answer is 0. Expressing her excitement to Robert, Ms. M gave the students another similar problem 77X0. Students correctly answered with similar reasoning.

Using the results of a game

The game Raising To 150 was used as a basis for whole-class discussion.

Individual groups reported their total number, and the teacher wrote on the board with 228

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their initials. Several groups had trouble adding three-digit numbers. One of them did not add hundreds digits, and another group reported 130 as 300. These trivial mistakes were easily corrected by other students. However, when a group reported 149+170=329, others did not catch the mistake. The teacher provided step-by-step instruction with adding digit by digit to get the correct answer. Another group reported 128+126 but did not know how to figure out the answer. David, a student strong in math, solved the problem in place of the teacher by changing it 130+124 and by computing digit by digit.

If students had carefully followed Ms. M's instruction, they could not have had a number bigger than 300. When Nahjha and Chase reported their number 150+154, Michael reminded Ms. M of the rule that they were supposed to use only part of the last number to make exactly 150. Interestingly, when Michael previously reported his group’s number 154+100, other students talked about the rule and Ms. M accepted it by writing Michael's as 150+100. However, this time, she did not accept what he said because this case (150+154=304) might provide something interesting. Note that she expected less than 300 as the total number for each group and intended students to figure out which number was the largest. The number 304 could be a rich example because students had to compare an equal hundreds digit. Because total numbers were expected to range between 200 and 300, students might otherwise compare only tens and ones digits. By Ms. M accepting the number 304, other groups reported their actual numbers which were larger than 300.

After students' reports, Ms. M asked which number was the largest. Four students called on by the teacher chose different numbers, but most of the other
students agreed with the choice 319 (149+170). Ms. M asked students to explain their reason. Episode UM-22 shows the dynamics of the interaction between the teacher and the students presenting their ideas.

<Episode UM-22: Students’ ideas of why 319 is the largest number>

T: I want you to explain why 319 is the largest.

Chelsea: Because that one [319] has the last number, and no one has the 9.

T: Oh, you are looking at the ones number? And you say, no one’s got a 9. That’s how you think that’s the largest number. Okay, who has another way of telling? How do you tell, Lainey?

Lainey: Because, most of them have the ... most of them begin with 200. And some ... and, the bottom [number] just has plain 6 [The number is 306]. And the middle one has zero, that stands for nothing. And then the 300 is like ...

T: So, what you’re saying, Lainey, is in your mind, you got rid of all the 200 ones?

Lainey: Yeah. (The teacher erases the equations that have the total less than 300.)

T: In your mind, you got rid of all the 200 ones, because you knew that there were [larger] numbers, so you were looking at hundreds. That’s what she is saying. Look what she is saying. She is saying, she looked at the hundreds. She got rid of all 200s in her mind. She was looking at 300. Now, she is explaining which one of the 300s she thought was the largest. Okay, go Lainey.

Lainey: Because ... the middle of the ... the middle of the 306. (The teacher points to 306.)

T: There were no tens? Okay.

Lainey: And then, the loose one is just 6.

T: All right.

Lainey: For the top one [304], there is no ten.

T: There is no ten (pointing to the 0 in tens digit). Okay.

Lainey: And then, just 4 loose ones.

T: All right.
Lainey: And for the second one (the teacher points to 300), it's just... There is no ten and no loose ones. And for the... So, the Derrick and Noble [who reported their number 149+170] are the largest.

T: The highest is 319. Tommy?

Tommy: I started like... put the lowest, 300.

T: The lowest is 300.

Tommy: Then, 304.

T: Tommy is putting them in order. 304...

Tommy: Then 306, and then 319.

T: 319.

In this episode, Chelsea claimed that 319 had the largest ones digit. Despite this mathematically invalid reason, Ms. M accepted Chelsea's explanation acknowledging that she expressed her idea. Lainey focused on the hundreds digit and so rejected numbers starting with 2 hundreds. Ms. M checked what Lainey said and explained it to the whole class after erasing all numbers starting with 2 hundreds. With the four numbers left (306, 304, 300, and 319), Lainey kept explaining why 319 was the largest by comparing the tens and the ones digits. Lainey pointed to the place-value concept here, which was crucial, whereas Ms. M did not foster classroom discussion around it, except re-describing what Lainey said to the whole class.

**Asking for children's experience of playing a game**

In another case, the teacher initiated classroom discussion by asking students to talk about their experience of playing a game, instead of simulating game situations or providing leading questions. The main topic of discussion came directly from students' contributions. For instance, in Episode UM-23 Michael shared an interesting experience
that his partner very often got the number card 10, while he got it only one time when playing How Close To 20. Ms. M immediately converted his contribution to a question. If they could draw any cards, which ones might they want (or don’t want) a lot of. This episode shows how the discussion came from students and proceeded through their emergent ideas.

<Episode UM-23: Classroom discussion based on students’ ideas>

T: Tell me something interesting, Michael.

Michael: Tommy was picking all the 10s. And I only got one 10 one time.

T: Tommy kept getting 10. If you were, put your hands down and listen to this thinking person’s question. If you were choosing the cards that you wanted to draw every time, cards that you really wanted to draw, what cards would you wanna draw? What cards would you wanna have, Kanita?

Kanita: 10.

T: You wanna have 10? Okay, what cards would you not wanna have a lot of? You have a bunch of cards in front of you. What cards would you not wanna have a lot of? Chase?

Chase: A lot of 10s.

T: You don’t wanna a lot of 10s. Why not, Chase?

Chase: Because if you had a lot of 10s, you were even further than 20.

T: If you had like ... What if you had 5 tens, you couldn’t have 5 tens because there are only 4 in the deck, that’s right. What if you had 4 tens, and a 9, and you’re looking at ... and you say, Oh! Yeah, you would be really far away, wouldn’t you? All right ... (Interrupted.)

Treena: But if you ... Because 10 and 9 is 19, so you just have 1 more to go, that would be the score.

T: Treena, but what if your score ... What if you had 10 and 9, and 19, and the only other cards you had were 10s?

Ss: Ummm.

Treena: That will be 29 ... (Interrupted.)
Michael: .... had a 10 and a 9. I wanna the next card to be 1.

T: You are wishing for 1, aren't you? Tommy?

Tommy: Aces, because that's only worth 1.

T: Okay, you don't wanna have lots of aces, Tommy, because you can actually have 4 aces, and so that would only be one, two, three, four. Look at that. You are really in trouble. Lainey?

Lainey: If you have only 2, 3, 1, and 4, you can make a 10.

T: So, Lainey says you don't wanna have all low numbers. Tommy says you don't wanna have a bunch of aces. But Chase says you don't wanna have a bunch of 10s.

Chase: If you get so many 10s, you will get over ... (Interrupted.)

Ss: 40.

Chase: And, but you try, you can't get like 1, 3, and 2 because those are numbers, and then you win.

T: If you have all ones, twos, threes, would you win, Chase? If you have all ones, twos, threes?

Michael: But they are good numbers.

T: They are good numbers, but listen to my question. Listen to my question... (Interrupted.)

Chase: Oh, I understand. What you are trying to see, how far to get to 20. But if you have that many, you are far away from 20.

T: Okay, so, here is my question. Here is a very difficult question. And I will let David answer because he is not wiggling his body. He's got his bottom flat, his hands in his lap, and he is looking at me! So, I know he is ready to think. If I said to you, what would be the best hand to draw? What kinds of cards? I don't want you to tell me numbers, but what kinds of cards would you like to get in your 5 cards? You said you don't want all low numbers; you said you don't wanna all high numbers. What kinds of hands would you like to draw? Nahjha?

Nahjha: Middle, in the middle.

T: Explain to me what you are thinking.

Nahjha: Like instead of getting 1, 2, 3, and 4. You can get 5, 6, 7, or 8.
T: You think that would be a better range? (Nahjha nods.) Who has a different idea? Jonathan, you are so important to me. Show me that you’re so important to me. Show me. Tell me more, Robert?

Robert: Um. Big and little.

T: Robert says you wanna get some big numbers and some little numbers.

Ss: Yeah! Yeah!

T: Explain to me what you think, Robert.

Robert: Like, if you have, if you have, uh, one more step to go to 20, I am talking about second one, you have one more step to make 20, you might have 1, and it will give you a better chance to make 20.

Students actively participated in the discussion by supporting their claims and refuting others with specific examples. The discussion centered on the cases where they had 10s or had lower numbers. Students seemed to have difficulties with regard to a general claim. For example, the claim that multiple lower numbers were not sufficient to make 20 was refuted by the case where students made exactly 20 including a lower number. Similarly, students reminded the class that they could make exactly 20 including one 10 to argue against the claim that lots of 10s were not wanted. Students’ difficulties led Ms. M to elaborate her original question with what kinds of cards they might want to draw. In this episode, interestingly, having one 10 is optimal, as it maximizes the chance of summing 20 with 3 cards. However, having all cards in the 5 to 8 range is optimal if they need to talk about all of the cards. The mathematics in this discussion is quite diffuse, as the basic assumptions are not tied down.

Note that Ms. M facilitated the discussion by asking students to explain reasons for their idea, adding details to their explanation, generalizing the specific cases students brought, summarizing their ideas, and providing specific examples in order to prove or
disprove their claims. Thanks to Ms. M's articulated question, Nahjha came up with the idea that middle numbers, instead of lower or larger ones, were the best. Robert provided another idea that the combination of large and small numbers was another solution. It is worthwhile to notice that Robert, perceived as a students who is mathematically weak, suggested the idea out of his engagement in the long but thought-provoking debate. Two weeks later, this discussion was related to when the class had to figure out a similar question to decide which number was better for having the most stars in the game Circles and Stars. Ms. M specifically reminded the class of Robert's idea.

TEACHER'S APPROACHES

Table 5.6 shows the characteristics of Ms. M's teaching practices which can be observed in the previous description of the classroom flow. The characteristics indicate that the teacher's curricular intention is to promote a student-centered instruction. Ms. M plays an active role in creating classroom social norms for the student-centered instruction. For instance, students are expected to present their ideas on the basis of their reasoning without fear of others' reactions. They were supposed to listen to and respect their classmates' ideas. Establishing such norms was not easy, as the teacher mentioned in an interview. She played soccer with her students every day for 15 minutes and had the same amount of time for discussion on what it was like playing together.

Ms. M specifically paid attention to the emotional states of students who provided incorrect answers or who could not finish their presentation. In particular, she emphasized that such incorrect answers were common and looking back at the solution

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process was more important (see Episode UM-7). She promoted her notion of "smart kids", students who know when they don't know and who listen carefully to others to figure out what to do. Ms. M often expressed to her students that they were good mathematicians, who strove to find out why some given methods did not work and who then reviewed the process to find mistakes. Her consistent emphasis on students helping each other and her consistent praise of students' good manners were other strategies employed to create a receptive classroom atmosphere (see Episode UM-1). Ms. M’s positive expectation for her students is another important factor in initiating such a student-centered learning environment (see Episode UM-17).

Table 5.6 Characteristics of Ms. M's Teaching Practices

- provided students with chances to solve problems for themselves.
- asked for different solution methods to given problems.
- encouraged students to present their thinking as well as the answer.
- after a student's presentation, she usually checked whether her interpretation corresponded to what he or she meant, and repeated or amplified to the whole class.
- frequently used manipulative materials.
- shared her positive expectation with every child.
- encouraged students to help each other as well as to be engaged in classroom activities.
- expressed excitement about students' ideas and offered praise.
- when there was debate among students, she let them discuss for a while and then mediated by summarizing the main points in each position.
- posed more challenging questions on the basis of students' contributions.
- emphasized that students should use any method which made sense to them.
- connected students' contributions with the previous activities.
- took care of students' emotional needs.
- used her observation of students' activities for classroom discussion.
- sometimes led students to initiate topics for whole-group discussion.
- sometimes changed the role of questioning and answering with students.
- introduced multiplication by responding to the students' intrinsic interest and request.
- assessed individual student's understanding by conducting an interview.

- occasionally did not probe a student's idea and ignored other possibilities.
- occasionally did not use students' reasonable contributions.
Within this open, student-centered classroom atmosphere, Ms. M solicits students' ideas and uses them as a main source for mathematical discussion throughout her lesson. In other words, creating a student-centered learning environment is important for her to promote mathematical understanding on the part of students. In one of her interviews, Ms. M supported this analysis by sharing what she wanted her students to experience through her math lessons:

I want them to learn to love mathematics. I want them to know that they can do it. ... I want them to know they're powerful, and I want them to know that it's important that they really need to apply themselves. They really need to learn this. They are gonna need in their life because ... I see children making the decision by third or fourth grade [that] they are not gonna learn math. Second, I want them to learn. I wanna use it as a vehicle for them to learn how to work with others because I know that in life that’s what they are gonna have to do. They are gonna have to work with others. Then, last of all, I want them to learn math concepts. But that’s literally last on my list because I believe, with all my heart, that if other things are in a place, they will learn mathematics. If they know that they can do it and if they know it’s important, if they know that they need to apply themselves, and they know it’s gonna be fun, they will do it.

In order to solicit students' ideas and use them as the center of classroom discussion, Ms. M uses several methods. The most noticeable is her lesson structure starting with problem solving, activity instructions, activity implementation, and activity discussion. Some lessons had a more clear connection among the four instructional phases than others. However, the main motivation for using such a structure was Ms. M's consistent attempt to promote mathematical thinking. In one of her interviews, she shared how she developed this lesson structure. When she served as a math specialist, she observed many math lessons across grades in various elementary schools only to find that children were using low-level strategies to solve a given math problem. This observation made her decide to facilitate discussion of various solution
methods in the problem solving session. Ms. M also watched many teachers reinforcing skills without understanding based on the excuse of helping students finish their workbooks. So, she decided not to use workbooks and instead tried to design something else to help students develop math skills. She collected various instructional resources including games and activities. Another critical observation she made was that students did not learn merely by using manipulative materials. Students simply followed the teacher's instruction without thinking. This led Ms. M to provide some questions before students were engaged in activities, and to give them discussion time after activities.

Using a game format in students’ activity reflects Ms. M’s belief that learning is social and play for young students. She tried to make her math lesson as playful as possible. Though she picked up some games that fit her instructional purpose by reviewing her own resource books, she even made up some of them, such as Raise To 150. While encouraging her students to be engaged in a fun game, Ms. M uses her instruction to challenge them to think beyond just playing the game. For instance, she used the game Circles and Stars in response to students’ eagerness to learn multiplication, though multiplication is usually introduced in third grade. In the beginning of this semester, when Ms. M gave students the same three numbers to add, such as 2 plus 2 plus 2, Michael came up with the idea that there were three twos and represented the problem using “times” as he heard from his older sister. His new idea made other students excited and eager to learn multiplication, too. As it neared the end of the semester, Ms. M thought that her students were ready and introduced the game to teach them the meaning of multiplication. On the first day, students were just busy
drawing circles and stars, and figured out the number of stars by counting by ones. As usual, they might have enjoyed the game, but it was not mathematically challenging enough. This led Ms. M to use the same game the next day asking students to represent their drawings as multiplication. Similarly, every game played in the class had a special purpose and the whole-group discussion right after students' activity was a main device by which Ms. M checked students' thinking with regard to the game.

Another strategy for using students' ideas is to allow different solution methods to a given problem. Ms. M encouraged students to express their own ways of knowing. Students sometimes provided less sophisticated mathematical methods than those contributed earlier. Even in these cases, Ms. M usually accepted the methods. The only exception occurred when she introduced multiplication as repeated addition (see Episode UM 4). In order to address the meaning of multiplication, Ms. M expressed her expectation of solving a given problem using "times," rather than decomposing the numbers. While she solicits multiple solution methods, Ms. M did not usually pursue mathematically different methods. For instance, drawing cookies and drawing circles used in solving 80-18 were counted as different methods. Little differentiation among methods, however, seems to be related to the teacher's concern for the individual child. As long as the child uses a method and explains it to the whole class in a reasonable way, any method can be accepted in her class. This is consistent with Ms. M's concern that mathematics should make sense to the individual child. In one of her interviews, she explained:

[T]he minute you show a child a procedure, they stop to think about what makes sense and start to think about the procedure. So, if you say, always start on the right, you say, give them a procedure. Then they start to focus on the
procedure and forget to think about what makes sense. But what I wanna do is I
wanna have them make sense of it in a way that makes sense to them. What I
constantly say to them is that this has got to make sense to you. If it doesn’t
make sense to you, then you need to ask more questions, or you need to do in a
different way. Do it in a way that makes sense to you. Think about a child that
has stopped making sense of it, they simply started just doing what the teacher
says.

While accepting the multiple solution methods students present, Ms. M plays a
proactive role at some points. She makes sure students experience concrete solution
methods, such as using cubes or tally marks to a given problem (see Episode UM-6).
Specifically, when a student uses the vertical format of computation, Ms. M usually
connects each step of abstract computation with using cubes. She strongly disagrees
with teaching formal algorithms at a second grade level, because few students can
understand it. As seen in Episode UM-7, however, students often bring partial
knowledge of algorithms. In one of her interviews, Ms. M explained that she simply
ignored these contribution at the first grade level, but she came to accept it at the
second grade level with special caution as to the meaning of different digits.

Ms. M also takes a strong position in some specific situations to emphasize one
method over another. For instance, Michael discovered the commutativity of
multiplication in the middle of his presentation (see Episode UM-5). While
acknowledging his discovery, Ms. M did not allow him to use the property, and instead
emphasized the different meanings between $5 \times 10$ and $10 \times 5$. Similarly, she emphasized
the “take-away” meaning of subtraction. In Episode UM-10, when Derrick made his
story but with ambiguity, Ms. M did not understand it. She then asked him to revise his
story based on David’s story which clearly used “take-away”. Eryn then posed an
interesting story problem which required comparison within a set. By simply accepting
Eryn's story and moving to next activity, she did not pursue this alternative model of subtraction any further. Similarly, in Episode UM-12, when Nahjha represented two separate groups of 80 and 18, Ms. M asked him to start with only 80. Given that she consistently listened to her students and accepted their various solution methods, these instances appear contradictory. Perhaps Ms. M simply did not know about the comparison interpretation. Another possible interpretation is that she intended to consolidate the fundamental meaning of subtraction as taking away. In fact, students used the term *take away* for the subtraction sign. Earlier in that lesson, some students had solved a subtraction problem, 42-26, as addition and answered 68. In Nahjha’s case, while he represented the minuend and subtrahend separately, he did not provide a reasonable explanation and kept saying “80 take away 18,” when Ms. M asked what he was doing. So, she interpreted his efforts as incorrect addition.

Another strategy for the teacher to use students’ ideas is “kid-watching” as Ms. M calls it. During students’ activity, Ms. M walked around not only checking whether students were on the assigned task but also seeing how they approached a given problem. In one of her interviews, she explained that she knew her students’ general strategies through the problem solving instructional phase, but she was curious whether students use such strategies in their own activity without her. This careful observation led Ms. M to figure out what she needed to teach and emphasize next. Like Chase’s 3X0 problem (see Episode UM-21), she often picked up something interesting from her observation for the whole group discussion. The interview with the individual children is another form of kid-watching. Ms. M informed me that doing an interview was her main
tool for assessing students’ understanding, in line with looking carefully at how they solved problems in every phase of her regular lesson.

Ms. M also often challenged students’ thinking (see Episode UM-9) and led students’ contributions toward mathematically interesting discussions beyond reporting who was the winner or knowing the right answer. Her emphasis on thinking was initiated partly from her experience as a math student. She made straight As in math because she was really good at memorizing procedure, but she was not as strong in math as she hoped to be. She shared her hope in one of her interviews, “I want you [the student] to be a much stronger thinker mathematically than I am or was.”

Another teaching strategy is to simply give students many chances to explain their thinking. Ms. M did not hurry to correct students’ mistakes before listening to their explanation. She did not interrupt students’ debate until they finished defending their position. Instead, Ms. M summarized the points of each position to facilitate productive debate among students (see Episodes UM-8 and UM-23). Even when there were competing alternative answers, she did not use her authority as teacher to choose the correct answer. Rather, Ms. M used the contributions of students strong in math and expected other students to understand by listening carefully to their friends (see Episode UM-3). In one of her interviews, she expressed her disappointment that students in the current class had not discovered that counting up is an easy way to solve a subtraction problem. In fact, she said that there was no method except providing them with more experience and waiting for them.

With all the above strategies, Ms. M was willing to take a risk in her math lessons by giving students many opportunities to develop their own ways of knowing.
In one of her interviews, when asked whether her lessons generally went as planned, she described her students as a “bomb” in that she was not able to predict what they would present and how they would do. Nevertheless, she expressed love of her lessons and confidence in how to handle unexpected students’ responses on the basis of her long teaching experience.

Ms. M was heavily influenced by traditional interpretations of Piaget’s work. In her interviews, she said that studying Piaget made her begin to understand how young children learn and to realize the crucial importance of observing students and asking them questions in order to probe what they know. She emphasized that learning for little children is play, and that early introduction of standard algorithms was developmentally inappropriate. She tried to make her mathematics lessons as much fun as possible. As well, she eschewed teaching formal algorithms at the second grade level and instead gave students the opportunity to explore their own informal ideas which in turn would help them understand the standard procedures and algorithms. Ms. M’s commitment to allowing her children to develop their own intuitions and ideas reflects a traditional interpretation of Piaget's genetic epistemology (Confrey, 1990b) at the time when Ms. M was introduced to educational theory.

While her pedagogical priority was to promote students’ positive dispositions (socialization) toward mathematics, Ms. M did not completely discard her interest in conceptuality. Influenced by this Piagetian interpretation, Ms. M saw herself as a facilitator rather than initiator of teaching mathematical content. But she discharged her concern for content through careful selection of students to give explanation in group discussion.

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Recall that one of Ms. M's basic strategies was to choose students who were strong in math to present their ideas in cases where many students seemed to be confused. This teaching technique can be analyzed in two ways. From an individual perspective, Ms. M can be seen as a highly manipulative teacher. She used students strong in math to insert mathematical ideas into the discussion that she herself was unwilling to do through the assertion of her authority. This creates the possibility that students would understand their location in the hierarchy of the classroom through Ms. M's attribution of their mathematical talents. In other words, Ms. M’s teaching strategy seemed to deny students the opportunity to establish their own location in the classroom hierarchy through the strengths or weaknesses of their ideas as evaluated by their peers.

Alternatively, from a communal perspective, knowledge is not conceived as the possession of individuals. Instead the class as a whole is taken as the source of knowledge. From this perspective, Ms. M’s teaching technique might be analyzed as addressing the strengths and weaknesses of the class, not an individual. When the focus is on the overall function of the classroom mathematical community as a whole organism, the location of individual students becomes irrelevant. This better perspective seemed to guide Ms. M's pedagogy. In other words, Ms. M herself seemed to have a communitarian perspective in which the well being of the individuals in the class was a function of the successes and failures of the class as a whole.

**STUDENTS' APPROACHES**

Table 5.7 describes the characteristics of students' participation in Ms. M’s class inferred from the classroom flow reported above.
Table 5.7 Characteristics of the Students’ Practices in Class UM

- solved given problems independently whenever it was required.
- volunteered to present their solution methods and ideas.
- pointed out mistakes made by other students or the teacher.
- whenever they did not understand or had a question about the teacher’s instruction, they asked for clarification.
- without the teacher’s initiation, they were often engaged in debate and argued for or against the ideas discussed.
- articulated other students’ explanations, when the teacher did not understand the original contribution.
- faithfully followed instructions in playing a game.
- often complied with rules of interaction without really attending to each other’s ideas.
- collaborated with each other regardless of the teacher’s presence.
- were eager to show their work to the teacher.
- sometimes asked for more challenging problems.
- some students stuck to their original ways of approaching a given task in the students’ activity phase, even when presented with alternatives.
- attempted to support or dispute the ideas of other students or the teacher with specific examples.

In general, the students attempted to comply with the teacher’s instructions.

Ms. M encouraged students to solve given problems in a way that made sense to them and to explain their solution methods to the whole class. In keeping with her expectation, students invented their own ways of knowing and volunteered to present their ideas throughout a lesson. With Ms. M’s proactive role, students contributed to the establishment of classroom social norms by which individual students’ ideas and presentations were carefully examined and valued. Students helped each other and usually gave the presenting student a chance to finish his or her presentation. They also played games keeping in mind the teacher’s directions.

However, compliance with the teacher does not fully explain students’ engagement in their classroom activities. For instance, students pointing out mistakes made by peers or the teacher might be interested in showing off their own ability by revealing others’ weaknesses. To some extent, the data supports this possible
explanation. For example, when the class was discussing the pattern of taking away 10, Michael said that he observed that every time they took away 10, the numbers had the same ones digit number. Michael added, "No one understands except me," followed by his peers' immediate response, "I understand!" Michael's confidence in his mathematical ability might explain why in Episode UM-15 he used the strategy of counting by ones to figure out the total number of stars without listening to his partner's helpful suggestions.

However, the data support another alternative and even stronger interpretation of students' motivation. They tried to make sense of mathematics and their experience, and they were interested in understanding mathematical concepts and processes. To establish this interpretation, the following analyses illustrate the nature of students' concern when they pointed out others' mistakes.

In one case, the class was solving the problem 124+150 and students provided four different answers, including 274 offered by Chase. When Chase explained his method, he made a mistake by saying, "Those two hundreds in the first place tell me how many tens." Michael immediately tried to check what Chase meant by asking, "You mean hundreds?" Chase's mistake must have been trivial because he mentioned the two hundreds in the hundreds digit. When Ms. M gave Chase the opportunity to finish his presentation, he easily corrected his mistake. Michael's interruption seems to reveal that he was faithfully keeping the classroom social norm that students were supposed to listen carefully to a presenting student and to try to make sense of it. This interpretation is supported by Episode UM-10 where Michael tried to understand the essence of Derrick's story problem for 42-26. When Ms. M could not understand the
story because of Derrick’s use of confusing words, Michael clarified Derrick’s story by making an appropriate question using the context Derrick presented.

Similarly, Episode UM-24 illustrates that students tried to make sense of their peer’s presentation. The class was discussing multiple solution methods to the problem 2 + 2 + 2 + 2. Chase explained his method in which he added the first three 2s, then added 1s and ended up with 8. Ms. M at first interpreted his method as adding 2s one by one. As Chase claimed that he added 1 from the last 2, she pointed out that 6 plus 1 is 7 and did not give credit to his presentation. Brandon explained that Chase decomposed the last 2 into 1 and 1, and another student agreed. Brandon’s explanation made Ms. M understand Chase’s original contribution, emphasizing again that mathematics should make sense. Brandon’s presentation appeared to come from his attempt to understand the computation process Chase was explaining.

<Episode UM-24: Brandon articulating Chase’s presentation>

Chase:  Well, I had 2 plus 2, and I know that’s 4, plus like that other 2 makes three 2s, that’s ...

T:  So you said ... (Interrupted.)

Chase:  That makes 6. You have that other 2 left on that side, so you just take 1 from that 2, and then it equals 8.

T:  So you said 2 plus 2 is 4 and 4 plus 2 is 6, 6 plus 2 is 8 (connecting the first 2s and writing 4, and then connecting the 4 and the third 2 and writing 6, and connecting the 6 and the last 2).

Chase:  No, take that 2, take that 2 and put 1 and then you have 8.

T:  That wouldn’t make sense though. 6 and 1 wouldn’t be 8. Do it again. Do it so it makes sense. There is no 1. That’s 2 (underlining the last 2).

Brandon:  I know what he means.

T:  Do you know what he means, Brandon? What does he mean?

Brandon:  It means, take the 2 and turn into 1, and you will have 1 more.
S: Yes.

T: Turn 2 into 1 plus 1 (writing “1+1” over the last 2). Then 6 plus 1 is 7, and then 7 plus 1 is 8. That’s what you meant, Chase? That makes sense. Math has to make sense, doesn’t it?

Many characteristics in Table 5.7 support the interpretation that making sense of their experience is a unifying objective of students’ engagement in mathematics activity. Whenever they did not understand or had a question about the teacher’s instruction, students asked for clarification. An example of this was when Chase asked an unexpected question for the case where they had more than 150 in the game Raise To 150, which led the class to specify a new rule of the game (see Episode UM-11).

In another case, Michael disagreed when Ms. M excitedly accepted the case where a student turned over two wild cards and used both by making 10+0 and 0+10 (see Episode UM-19). For Michael this did not make sense because there was no number card for 0 and the cards were supposed to sum to one 10 per turn. There were many instances in which individual students tried to solve a given problem for themselves (see Episode UM- 20) and were eager to solve more challenging problems (see Episode UM- 21).

Debates between students, specifically without the teacher’s initiation, also indicate their attempt to understand the ideas being discussed. An example of this was during Episode UM 8 where two students debated the different cases for using 0 — one to represent nothing and the other to put 0 as a ones digit. In order to prove or disprove competing claims, students used real-life situations (see Episode UM-9) and specific examples or counterexamples (see Episode UM-23).
STUDENTS' LEARNING OPPORTUNITIES

The teacher in this classroom played an active role in promoting students' mathematical understanding. Along with the teacher's approach, the students attempted to make sense of mathematics and their participation in the classroom activities. This joint approach fostered learning opportunities for students to understand mathematical concepts and processes contributing to number sense.

Because students in this classroom chose their partners on a daily basis, they had different kinds of social and mathematical relationships. In keeping with the social norm that they were expected to work together, students seemed to establish good social relationships with one another. However, intellectual challenge in their small groups was constrained by their mathematical knowledge and disposition (see Episodes UM-13 and UM-16). Thus, students' learning opportunities seemed to come from instructional phases based on the teacher's mediation (e.g., problem solving and activity discussion) rather than from their own activity phase.

As mentioned earlier, the teacher tried to establish critical classroom social norms in order to enhance students' mathematical understanding. One was that mathematics should make sense to the individual child. This seemed to spur students to develop dispositions to gain understanding from their own engagement. In fact, students were eager to invent their own strategies to figure out given problems. Another important norm was that students were expected to present reasonable solution methods as well as answers. Chase in this classroom even knew that his idea would not be accepted by the teacher, though he had the right answer, because he could not provide a rationale for his thinking (see Episode UM-2). The expectation of
students' explanations for their ideas seemed to lead students to learn the importance of thinking about reaching an answer.

The teacher's consistent request for different methods to a given problem and students' contributions made students realize that there were many valid methods to solve a mathematics problem. As evidenced by many problems discussed in this classroom, different levels of mathematical ability among students produced various solution methods ranging from more concrete (using cubes or tally marks) to more abstract representation (using number lines or numerical computation). While discussing those methods, students had the opportunity to connect abstract computation processes to concrete materials. This connection might serve as a critical basis on which students gave intuitive meaning to each computation process, which otherwise might have been meaningless to them.

Moreover, discussing multiple solution strategies might also give a student the chance to compare and contrast his or her own method with others, and to begin to see that certain methods have advantages over others. For instance, students in this class began to see that using multiplication (e.g., 5x2) in place of repeated addition (e.g., 2+2+2+2+2) was mathematically efficient. Similarly, students realized that counting on was mathematically more sophisticated than counting all.

The discussion of various methods of decomposing and composing numbers in given problems gave students the opportunity to experience properties of addition and multiplication, such as commutativity and associativity (e.g., 26+7=26+(3+4)= 26+(4+3)=(26+4)+3=30+3=33). Given that the students were at the second grade level, they might not be expected to understand those mathematical properties involved in
the computation process. In fact, the class used the equal sign to record the process of computation, which was not mathematically correct (e.g., 26+4=30+3=33 for the same problem 26+7). But at least several students could implicitly use such properties to compute effectively. In particular, David applied such properties even to three digit numbers (e.g., 128+126=130+124). Some students like Chase and Michael found the commutativity of addition and multiplication and shared with the whole class. Students had an opportunity to learn more advanced strategies by participating in discussions where their peers who made mathematically strong contributions.

The teacher allowed her students to pose some problems and to solve them. Students themselves attempted to make sense of mathematics as well as comply with the teacher’s instructions. This led Michael to develop his own strategy to solve 5X6, that is, counting by 5s and then adding 1s. Similarly, a few students made interesting multiplication problems using 0, which produced a rich whole-group discussion by which students could reinforce the meaning of multiplication (see Episode UM-21).

The focus on students’ multiple ideas about given problems rather than their answers showed that mathematics is based on logical reasoning. In particular, the teacher’s rejection of students’ contributions which showed little reasonable explanation revealed that mathematics should make sense to others as well as to themselves. The exploration of problems and their solution strategies also presented mathematics as a dynamic discipline rather than a static or fixed one. The teacher’s praise for students behaving like “smart kids” or “good mathematicians” created the opportunity for students to learn the importance of reflection in solution processes while doing mathematics.
The teacher's request for making a story-problem to a subtraction problem provided another learning opportunity by which students could see how subtraction might be used in real-life situations. As seen in Episode UM 10, the teacher stressed the "take-away" interpretation of subtraction using David's and Derrick's story problems. In the same episode, Eryn presented a mathematically significant story problem in which comparison within a set was required, but her presentation was not given specific attention by the teacher. In the subsequent students' activity phase, students made another story problem to a different subtraction problem 80-18. Many students used the same context that David and Derrick used in their stories, which included "separate" action or the "take-away" meaning of subtraction. The students used such verbs as "take away", "fall down", "eat", and "run away", and came up with the same question of "how many left". Outside of this trend, there were a few students who began to use the comparison meaning of subtraction, but did not complete their stories. For instance, Kelly wrote her story as follows: "Once upon a time, in the store, A [a] girl bought 80 eggs. A laddie [lady] bought 18 eggs." Since the students had difficulty in figuring out the answer to the problem, the class spent the subsequent discussion time solving the problem with multiple solution methods various students used, leaving the story problems unexplored. Thus, the learning opportunity of using subtraction for comparison problems was not pursued by the teacher, but left to the individual child who might try to make sense of their experience.

The students in this class could learn mathematics with enjoyment and confidence. The teacher's careful concern for involvement and autonomy in math activities by all students helped them value themselves and their peers as mathematical
problem solvers. The teacher avoided computation algorithms for students to follow. Instead, she provided her students with various activities from which they could learn mathematics. In developing their own solution methods and articulating them to others, students might become confident in their mathematical ability and learn to take responsibility for their own learning through making sense of their experience.

The most important thing was that the students in this class had the opportunity to engage in some critical aspects of mathematical activity including inventing their own solution methods, explaining why such methods worked, arguing for or against the ideas being discussed, moving back and forth between general assertions and specific examples, and raising and judging questions grounded in their mathematical experience. In other words, students had the opportunity to be enculturated in the particular classroom microculture where specifically mathematical ways of knowing, valuing, and arguing were emphasized. In particular, various debates among students throughout any phase of a lesson helped them develop an appreciation for valid arguments and to locate themselves in the debates on the basis of their own judgements.

COMPARATIVE ANALYSIS BETWEEN CLASS UE AND CLASS UM

The purpose of this comparison is to better understand how unequally successful reform-oriented mathematics classes were constructed and to articulate the problems and issues of implementing reform ideals. The comparative analysis had two parts. In the first part, the two classes, Class UE and Class UM, were compared and contrasted in two ways. First, the classes are compared by the four categories of the interpretive framework for this study, and second, by the social and the sociomathematical norms.
In the second part, the factors influencing teachers’ instructional goals are compared and contrasted in order to explore how more successful and less successful teaching practices have been constructed. Note that there were two interviews with each teacher lasting a total of 6 hours. The first was to clarify some points that were unclear from the videotaped lessons. The second was to gain information on how the teacher developed her teaching methods. The data from the second interview were used for this part of the analysis. General reform documents were also employed in discussing the teachers’ professional development processes.

**COMPARISON OF TEACHING PRACTICES**

The first section provides similarities and differences between Class UE and Class UM with regard to each category in the interpretive framework for this study — classroom flow, teacher’s approaches, students’ approaches, and students’ learning opportunities. The second section consolidates the previous comparison and contrast by addressing social and sociomathematical norms.

*Comparison by Four Categories*

**Classroom Flow**

The general atmosphere in the two classrooms was very similar. In both classrooms teachers established open and permissive atmospheres in which students’ ideas and their mistakes were welcomed. Both classes were dynamic in that students actively responded to the teacher and one another. Both teachers tried to give each student an equal chance to present. Moreover, when students expressed a negative response in the middle of their friend’s presentation, the teacher gave the presenting student an opportunity to finish his or her argument. Both teachers emphasized that
working hard was more important than getting a right answer. Students in both classrooms frequently expressed their excitement about various classroom activities. The teacher and the students in each classroom laughed a lot while engaged in mathematics activities.

Gross patterns of classroom activities were similar. Class UE included estimation, problem solving, and collective activities with various manipulative materials. Class UM included problem solving, activity instructions, activity implementations, and activity discussions. The sequence of these classroom activities was fixed in Class UM, but variable in Class UE.

The patterns of social interaction in the two classrooms were similar in several respects: (a) the teacher initiated an activity or gave students a mathematical problem, (b) students independently solved the given problems, (c) the teacher asked students to report their solution methods to the whole class, (d) students presented their solution methods, and (e) the teacher mediated the classroom discussion. The difference was that Ms. E in Class UE tended to control the discussion directing it toward a particular orientation, whereas Ms. M in Class UM tended to facilitate the discussion. For instance, Ms. E often evaluated students' answers or expressed her expectation of what students would present. However, Ms. M provided further questions for clarification after a student's presentation, which often led to other students' participation in the discussion based on their agreement or disagreement. The sequence of teacher-student-teacher-student turn taking was prevalent in Class UE, whereas direct student-student interaction was often found in the whole-group discussion in Class UM.
Teacher’s Approaches

There were many similarities in the two classrooms with regard to the expectations, roles, and obligations adopted by each teacher (see Table 5.8). For instance, both teachers stressed group cooperation and provided encouragement and positive expectations for the students accomplishments. The class often used manipulative materials. This contributed to an enjoyable activity format for students.

Table 5.8 Comparison: Teachers’ Approaches

<table>
<thead>
<tr>
<th>Degree of Similarity*</th>
<th>Ms. E in Class UE</th>
<th>Ms. M. in Class UM</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>- provided students with chances to solve problems for themselves.</td>
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<td></td>
<td>- encouraged students to present their solution methods/strategies.</td>
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<td></td>
<td>- repeated or amplified students’ ideas to the whole class.</td>
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<td></td>
<td>- expressed excitement about students’ novel ideas.</td>
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<td></td>
<td>- sometimes changed the role of questioning and answering with students.</td>
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<td></td>
<td>- shared her positive expectation to every child.</td>
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<td></td>
<td>- took care of students’ emotional states, especially when they made mistakes.</td>
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<td></td>
<td>- emphasized the process of problem solving.</td>
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<td></td>
<td>- often used manipulative materials and tried to connect symbolic/abstract</td>
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<td>representation with pictorial/concrete representation.</td>
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<td></td>
<td>- encouraged students to work each other.</td>
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<td></td>
<td>- frequently used an enjoyable activity format.</td>
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<td></td>
<td>- asked students to pose a story problem for addition or subtraction.</td>
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<td></td>
<td>- circulated and provided some help while students worked in their groups.</td>
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<td></td>
<td>- lavishly provided praise and encouragement to support students’ efforts.</td>
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<td></td>
<td>- when students were confused with directions, she provided detailed</td>
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<td></td>
<td>illustration with examples.</td>
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<td></td>
<td>- sometimes asked for different methods.</td>
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<td></td>
<td>- picked out something interesting from students’ responses, but it was not</td>
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<td></td>
<td>necessarily mathematically significant.</td>
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<td></td>
<td>- posed a more challenging problem based on students’ contributions.</td>
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<td>(Table Continued)</td>
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<tr>
<td>Degree of Similarity*</td>
<td>Ms. E in Class UE</td>
<td>Ms. M. in Class UM</td>
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<tr>
<td>-----------------------</td>
<td>----------------------------------------------------------------------------------</td>
<td>----------------------------------------------------------------------------------</td>
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<tr>
<td>•</td>
<td>- sometimes provided her own solution strategies, instead of letting students invent them.</td>
<td>- did not provide her own methods.</td>
</tr>
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<td></td>
<td>- taught formal algorithm for subtraction with step-by-step instruction using students' contributions.</td>
<td>- frequently checked whether the teacher's interpretation corresponded to what students meant after their presentation.</td>
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<td></td>
<td>- specifically emphasized line alignment and the order of computation.</td>
<td>- eschewed formal algorithm.</td>
</tr>
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<td></td>
<td>- provided direct explanation, a hint, an example or chose the right answer with praise, when students had different answers (or difficulties).</td>
<td>- carefully checked whether students understood place value concept, when they used a vertical format of computation.</td>
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<td></td>
<td>- even after a student's novel idea, the teacher directly expressed interest in using algorithm.</td>
<td>- students were allowed to argue for a while, when there were debates among them. Later the teacher mediated by summarizing the main argument in each position.</td>
</tr>
<tr>
<td></td>
<td>- emphasized memorization of basic addition/subtraction facts, and tested memorization periodically using a paper-and-pencil format.</td>
<td>- emphasized using any method which made sense to students.</td>
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<td></td>
<td>- very often set time in doing classroom activities.</td>
<td>- connected students' contributions with previous activities they completed.</td>
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<tr>
<td>4</td>
<td></td>
<td>- assessed individual student's understanding by conducting an interview.</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>- sometimes led students to initiate the topics for whole-group discussion.</td>
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<tr>
<td>0</td>
<td></td>
<td>- introduced multiplication responding to her students' interest and request.</td>
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</tbody>
</table>

* Note: Degree of similarity: • = a lot; 4 = somewhat; 0 = very little

There were also differences in that Ms. E emphasized the standard algorithm with line alignment and the order of computation, whereas Ms. M did not converge...
towards a standard method to solve problems. Differences were also noticeable when
students had different answers. When there seemed to be misunderstanding about the
solutions, Ms. E usually explained by giving examples. Otherwise, she selected the
right answer with praise. In contrast, Ms. M tended to let the students argue for a while
and then mediated the discussion by summarizing the main argument in each position.

**Students' Approaches**

There were many similarities with regard to the expectation, obligations, or
roles adopted by students across the classrooms. The students solved given problems
independently, presented their solution methods in the whole class, and complied with
the teacher's instruction (see Table 5.9). The main difference lay in the concern about
right answers. The students in classroom UE expressed their excitement when they got
the answer. Some students waited for the teacher's confirmation while doing their
group activities. However, in Class UM, the answer itself was not the main focus of
discussion. Students often argued for or against ideas without the teacher's initiation.
Moreover, some students used their ways of approaching a given task in their small
groups. Given that the expectations adopted by students stem from the teacher, this
difference can be fully understood in conjunction with the different roles adopted by
the teacher, as described above.

**Students' Learning Opportunities**

Students' learning opportunities within the two classrooms were very much
constrained by the mathematically significant distinctions embedded within the
classroom discourse. Table 5.10 shows the dramatic differences with regard to the
learning opportunities in both classrooms.
<table>
<thead>
<tr>
<th>Degree of Similarity*</th>
<th>The Students in Class UE</th>
<th>The Students in Class UM</th>
</tr>
</thead>
<tbody>
<tr>
<td>•</td>
<td>- solved given problems independently whenever they were supposed to.</td>
<td>- focused on finding multiple solution methods to given mathematics problems</td>
</tr>
<tr>
<td></td>
<td>- presented their individual or group solution methods to the whole class.</td>
<td>- were eager to show their work to the teacher.</td>
</tr>
<tr>
<td></td>
<td>- usually listened carefully to their friends' explanations.</td>
<td>- whenever students did not understand or had a question to the teacher's instruction, they asked for clarification.</td>
</tr>
<tr>
<td></td>
<td>- complied with the teacher's instruction.</td>
<td>- were more engaged in their group activities and discussion.</td>
</tr>
<tr>
<td></td>
<td>- collaborated with each other while working together.</td>
<td>- attempted to support or dispute the ideas by other students or the teacher with specific examples.</td>
</tr>
<tr>
<td></td>
<td>- pointed out mistakes made by others or the teacher.</td>
<td>- didn't seem to be concerned about right answer</td>
</tr>
<tr>
<td>€</td>
<td>- sometimes asked the teacher to clarify when her instructions were confusing.</td>
<td>- articulated their peers' explanation, when the teacher did not understand the original contribution.</td>
</tr>
<tr>
<td></td>
<td>- gave more attention to the teacher whenever requested.</td>
<td>- some students kept their ways of approaching a given task.</td>
</tr>
<tr>
<td></td>
<td>- some students checked the teacher's response before finishing their presentation.</td>
<td>- sometimes asked for more challenging problems.</td>
</tr>
<tr>
<td></td>
<td>- when students found the right answer, they expressed excitement.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- while doing collective problem solving activities, some groups waited for the teacher's check or confirmation for their decisions.</td>
<td></td>
</tr>
</tbody>
</table>

* Note: Degree of similarity: • = a lot; € = somewhat; ○ = very little

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<table>
<thead>
<tr>
<th>Degree of Similarity*</th>
<th>Class UE</th>
<th>Class UM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>- the mathematical content was mainly procedural.</td>
<td>- had the opportunity to develop conceptual basis of the arithmetic being studied.</td>
</tr>
<tr>
<td></td>
<td>- learned mathematics as a static or fixed discipline.</td>
<td>- learned mathematics as a dynamic discipline.</td>
</tr>
<tr>
<td></td>
<td>- developed skills in solving only routine mathematical problems.</td>
<td>- enculturated in the particular classroom microculture where specific mathematical ways of knowing, valuing, and arguing were emphasized.</td>
</tr>
<tr>
<td></td>
<td>- had limited opportunities to present their novel strategies or ideas.</td>
<td>- self-motivated in their pursuit for mathematical meaning.</td>
</tr>
<tr>
<td></td>
<td>- focused more on the teacher’s instruction, without autonomous motivation for mathematical sense-making.</td>
<td>- had the chance to compare and contrast their own methods with peers, and to begin to see that certain methods had mathematically significant advantages over others.</td>
</tr>
<tr>
<td></td>
<td>- limited learning opportunities with regard to the transition from informal to formal strategy of doing computation.</td>
<td>- had the chance to value themselves and their peers as mathematical problem solvers.</td>
</tr>
</tbody>
</table>

* Note: Degree of similarity: • = a lot; □ = somewhat; ○ = very little

Whereas socially both groups of students had many opportunities to participate and to experience success in their efforts, the mathematical content in Class UE was mainly procedural. Those students had the opportunity to develop skills in solving problems (e.g., using standard algorithms for computation), but they had very limited opportunities to learn the transition from informal to formal strategy of doing computation. In other words, they had little opportunity to reflect on the conceptual underpinnings of the mathematics they were studying. In contrast, the students in Class UM were continually exposed to mathematically significant ways of knowing, valuing
and arguing. For instance, they had the chance to compare and contrast their own methods with peers, and to begin to see that certain methods had mathematically significant advantages over others. They had the opportunity to make conceptual sense of the mathematics they were studying. Moreover, there was evidence that the students in Class UM were becoming self-motivated in their pursuit for mathematical meaning.

**Comparison by Social and Sociomathematical Norms**

**Social Norms**

The general social norms concern the classroom participation structures. Class UE and Class UM had many similarities with regard to this participation structure including:

- The two classes displayed an open and permissive approach in which students’ ideas were solicited and even their mistakes were welcomed. Both teachers paid special attention to the emotional states of a child who was not able to finish his or her presentation or produced an incorrect answer. The two teachers also put an emphasis on the learning process rather than on getting an answer.

- Small-group formats were used to encourage collaboration.

- Whole-class discussion followed individual or small-group activity.

- Both teachers provided students with an opportunity to discuss alternative solutions.

- The students in both classes were expected to solve a given problem by themselves and to explain their solution methods to the whole class.

- The students were expected to be actively involved in classroom activity, to make sense of others’ explanations, and to ask for clarification as needed.
Both teachers repeated or amplified students' contributions.

Finding others' mistakes was encouraged by both teachers.

The differences between Class UE and Class UM included:

Ms. E directly evaluated or commented on students' contributions, whereas Ms. M encouraged other students to participate in discussion based on agreement or disagreement.

Ms. E somewhat controlled the direction of class discussion, whereas Ms. M encourage students to debate, with minimal interruption.

Sociomathematical Norms

Despite the similarities in the social organization, remarkably different sociomathematical norms had been established in Class UE and Class UM. Recall that sociomathematical norms concern the quality of students' collective engagement in mathematical practices of a classroom community. For instance, although both teachers frequently used an enjoyable activity format, how the teacher handled the activity influenced the content and quality of students' experiences.

The mathematical content of Class UE was primarily procedural. Ms. E acknowledged students' novel ideas or strategies, but ultimately led classroom discussion in such ways so as to emphasize the standard algorithm or a specific procedure (see Episodes UE-3, UE-6, and UE-10). Within this classroom microculture, the students had the opportunity to develop skills in solving routine mathematical problems. Moreover, being accurate or automatic in solving a given mathematical problem was perceived as more important than being creative or insightful. In contrast, Ms. M was concerned about students' engagement in characteristically mathematical
ways of thinking, valuing and communicating. Producing a right answer with a wrong argument was not accepted in this classroom (e.g., Episode UM-2). When students were engaged in debate, sometimes without the teacher’s initiation, they had to communicate in such a way as to convince others on the basis of mathematical thinking or with (counter)examples (see Episodes UM-8, UM-9, and UM-23). In this way, students came to develop a norm of what counts as a mathematically justifiable or acceptable explanation. Moreover, students had a chance to develop important mathematical dispositions, for example, being confident in their mathematical ability, being eager to solve problems using their own methods, and being autonomous in their pursuit of mathematical meaning (see Episodes UM-11, UM-15, and UM-20).

**COMPARISON OF FACTORS INFLUENCING TEACHERS’ INSTRUCTIONAL GOALS**

This section explores how unequally successful mathematics practices were constructed in the U.S. classrooms. The interview data that pertains to the two teachers’ personal reflections on the influences on their professional development were used as a source of insight to identify the underlying factors that might account for the differences and the similarities in implementing reform ideals in teaching mathematics. This, in line with analysis of general reform documents, affords exploration of the challenges of moving teaching practices toward student-centered approaches.

**Ms. E’s Case**

**Early Learning and Teaching Experience**

Ms. E had successful elementary school years. Her teachers thought that Ms. E was smarter in math than she felt she was. She was fast enough in computation due to
quick mental math. Ms. E had a significant mathematical experience in an algebra class and in a geometry class. She explained:

I went to junior high school. I did well there, but I received my first B in my whole life. That made me decide that I was not smart in math... I can still remember asking my teacher the question, “Why can’t you add 2A to 3B?” She said, “You can’t add apples and oranges,” and continued to say, “get the same fruit.” The explanation did not satisfy my confusion. She truly didn’t want the question asked. I was so confused at that point in time. I needed somebody to make sense of what I didn’t know what’s happening... That was my difficult time. That was a turning point of my attitude toward math. The next turning point was in geometry in high school. I always made As in everything, but in my math I couldn’t get my As. I felt bad. At high school, when I took geometry, I read those theorems and proofs, and they made sense. I can see it! They had those pictures, I can see it. It made perfect sense. I did well.

These experiences had a great influence on Ms. E’s teaching practices. She did not want her students to feel deficient in mathematics as she did. Because she felt embarrassed when she asked the algebra question which led her not to ask any more question in the class, Ms. E wanted to make her math lessons open for students to feel free to ask any question. Throughout her own learning experience, Ms. E often did not understand why some mathematical principles worked. She copied her teachers’ methods without understanding. This led her to make sure that she explained to her students how mathematical concepts and processes worked, before introducing a rule or a formula.

Ms. E decided to be a teacher because teaching was second nature to her. As the oldest child in her family, she often had to teach her sisters. She liked to teach young children and, more generally, to do something with them. Ms. E explained that she learned basic yet important principles of teaching in her family where there were a lot of explanation, communication, and love. Ms. E also had some special teachers who influenced her view of what good teaching looks like. Those teachers took care of her
and she enjoyed their classes. Among others, Ms. E regarded her first-grade teacher as the best model for her teaching. Her first grade teacher frequently asked students to go to the chalkboard, to sit down immediately after solving given problems on the board, to see how their friends did, and to explain how they solved the problems.

When she decided to be an elementary school teacher, Ms. E also had an image of what she didn’t want to be. Her third grade teacher copied many equations from her notebook on the board and Ms. E had to copy and solve them all. This teacher often asked students to draw circles and put numbers in them for computation, and to draw squares and put numbers to make a 100 chart, which was boring to Ms. E.

In general, Ms. E did not enjoy her teacher preparation program. University professors usually lectured. Moreover, they did not consider school contexts so their instruction was not practical. They were telling beginning teachers what to do in a classroom, but they were not practicing what they were saying. With regard to mathematics, Ms. E took a class based on *New Math*. In that class she saw circles, squares and triangles in equations, but never understood any purpose behind using shapes with equations. Ms. E said that her university experience did not help her learn how to teach mathematics.

However, Ms. E recalled that she had a good student-teaching experience in that the supervising teacher helped her be strong and strict enough in managing students’ classroom behavior. In her practicum, she could prepare for mathematics lessons more easily because she had a teacher’s manual which included the students’ text and provided useful information for teaching at hand. She felt free to develop her own teaching style rather than being expected to copy her supervisor’s style. In her student-
teaching, Ms. E attempted to teach in such a way that students would enjoy her lessons. She specifically remembered physically moving children to dramatize mathematical operations in her classrooms. When Ms. E later served as a supervising teacher for 10 years, she wanted to be a good model, yet give student-teachers a chance to develop their own instructional approaches.

In 1966, Ms. E started her teaching career in a small elementary school where the whole community, including parents, was close to each other. She taught second grade for a few years. In her first year of teaching, Ms. E was fortunate to be around a few mature teachers who served as good mentors. They mainly helped Ms. E adapt to the school culture by sharing materials for similar lesson units and advising on how to handle difficulties in managing a classroom. Ms. E had a chance to discuss something in regard to her lessons, but these discussions did not help much because the teaching styles of her mentors were basically traditional.

Professional Development and Teaching

Ms. E got her master's degree in educational media in the early 1970s. Since she did not enjoy classes in her teacher education program, Ms. E wanted to try something different for her master's program. She willingly chose educational media because using technology began to be a focus at that time and she was interested in incorporating audiovisuals into her lessons. Ms. E started the master's program because she thought that an advanced degree would help her get employment after a four year maternity leave. But she was easily re-hired at the same school because it started a kindergarten for which Ms. E was certified. She taught kindergarten for 13 years. In teaching mathematics in kindergarten, Ms. E used lots of games and counting activities with
students' physical movement and manipulative materials. This teaching experience had an influence on her teaching of mathematics when she came to teach second grade children again.

Ms. E had worked at the same elementary school for most of her career and recently moved to a new urban magnet school. Ms. E earned a master's plus 30 by taking basic science classes in the late 1980s. She enjoyed the science classes where the instructors used lots of hands-on activities and encouraged teachers to work as a group. Taking those classes influenced Ms. E's mathematics teaching in that she came to use lots of concrete activities within a partners or small group format.

Under a professional development program for in-service teachers, Ms. E had a chance to take either university or workshop classes in the early of 1980's. Until then, Ms. E had taught mathematics on the basis of her personal learning experience and her teaching. She knew what she didn't want to do in her mathematics lesson, but did not know how to teach mathematics differently, except for adapting some methods from her model teachers. She had wished to visit various classrooms where she could observe good teaching. She felt competent in teaching language arts but not in mathematics. Because she wanted to teach mathematics better, Ms. E signed up for workshop classes by the Learning Institute. She took an evening class and summer workshops for about 5 years, which was over and beyond mandated requirements. She regarded the workshops as beneficial and they improved how she taught mathematics. She said that the instructors showed her a way to teach mathematics better. She elaborated further:

They [the instructors, one of whom was Ms. Richardson] didn’t say you can do this, you can do that. We were their students and they were the teachers and they did that kind of teaching. They did the kind of teaching they were trying to
teach us. I was a recipient and I was learning. I didn’t know anything about base 2, base 3, base 4. That was eye-opening for me. I heard it, but I didn’t understand it. And when I did it, I understood. And I thought, huh, that’s the way of doing it.

The workshop classes greatly influenced Ms. E’s mathematics teaching. In fact, she identified taking those classes as one of the most influential factors in shaping her mathematics teaching. She began to teach different base systems using Unifix cubes in a way that she was taught in those classes. She thought that learning different base systems would help students understand the base 10 system and, thus, understand place value concept, the most important mathematical concept for second grade. Ms. E said that the instructors of the workshop classes illustrated the logical steps necessary to understand mathematical principles. The fact that Ms. E was able to understand mathematics in the classes was special to her, because she had been learning it without conceptual underpinnings throughout her schooling. She explained:

I needed to know why. Don’t just tell me multiply. Go through the steps, go through the long way, the way that it may have been developed in the first place. Show me the thinking that people went through before they came to that spot ... And I’ll remember why that short-cut works and how to do it. And I understand you can invert and multiply. With regrouping, I understand why you have to carry this here and you can borrow from this here, because I have seen this 10, because you cannot have a 10 in the ones place, because that was a rule.

Ms. E also remembered that the mathematics instructors were dynamic and excited about mathematics. She remembered how eager they were to help teachers be enthusiastic about teaching mathematics, too. Moreover, they expected teachers to ask any question without being embarrassed. Regardless of the nature of the questions, they were compassionate in sharing their ideas. Within such an open and supportive learning environment, Ms. E enjoyed learning mathematics and learning how to teach it. She

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came to enjoy teaching mathematics and wanted to learn more. She became more concerned about whether her students were enjoying mathematics, because she experienced that people can learn mathematics better when they enjoy it. Ms. E attempted to use a variety of enjoyable formats for mathematics activities in order to keep students’ attention and interest. She intentionally welcomed students’ wrong responses in the process of their making sense of mathematics.

The workshop instructors had also emphasized using manipulative materials as a way to foster students’ intuitive thinking and providing different representational modes by which individual children might learn in the best way appropriate for each of them. Since Ms. E realized the importance of concrete representations through her learning experience in geometry, she definitely used lots of manipulative materials in teaching kindergarten and second grade.

For Ms. E, the workshop classes were “just the first time I really got into the theory of mathematics.” For instance, she came to understand why a teacher needs to present mathematical content using multiple methods — because students have different routes to understanding. Some students, like Ms. E, need to see how a mathematical principle works by picture or illustration whereas others are able to understand abstractly. In the same vein, she tried to teach the idea that there are many ways to approach a given mathematics problem. Recently, Ms. E volunteered to participate in an Exxon teacher-training program in mathematics. Though her principal chose another teacher, Ms. E learned from the program and its manipulative materials through what that teacher shared.
Influence of Her Colleagues

Throughout her career, Ms. E found that teachers are usually generous in sharing materials and ideas. However, she thought that the conversation among teachers was limited to classroom management and not directly related to teaching methods. For Ms. E, "Teachers are really very isolated in their own classroom." She also did not have many opportunities to discuss instruction-related ideas with her school administrators. Though Ms. E thought that she had autonomy in developing her own teaching methods throughout her teaching career, she basically regarded administrators as people who tell teachers about responsibilities and confirm whether or not they comply. Whenever her principals made observations, Ms. E has had good evaluations. Most recently her current principal observed and liked Ms. E's mathematics lesson on base 3 and 4 with Unifix cubes. The principal mainly checked whether students were on tasks and how Ms. E managed the lesson as planned.

Reform Implementation

Ms. E sees reform movements administered in a top-down pattern as problematic. She attributed the failure of the New Math movement in 1960s to a top-down administration of the new curriculum with a lack of teacher participation and understanding. Ms. E agreed with the idea of teaching mathematical structure and processes emphasized in the movement. However, she recalled that the university instructors taught the new mathematics as a product which she was supposed to implement based on what she learned from the university classes. Generally, Ms. E has been familiar with reform recommendations through district workshops wherein teachers were told what should be emphasized in mathematics instruction. Current
reform documents such as *Benchmarks* (AAAS, 1993) were sent to schools and using them in planning a lesson has been mandatory. Ms. E regarded using manipulative materials in mathematics as one of the important reform recommendations. The following shows how Ms. E understood the administration of reform ideals:

I think what they [reformers] just try to do is to catalogue what they think we must do. They try to make an outline of what’s important and they are gonna test us to make sure we meet them. If we don’t meet them, we’re flunked [laughing] ... There are a bunch of checks we have to put.

In particular, Ms. E complained about the impracticality of a required lesson plan. She was supposed to include many items in a lesson plan including multiple intelligence, *Benchmarks* statements, and other government references. Ms. E explained, “You just have to put it down. It’s just work. Every year they add what should be written on a lesson plan, not taking anything away.” After all, she had to write a complicated lesson plan for administrators and, at the same time, prepare a simple plan to use for her actual teaching. Similarly, when teachers were supposed to work on the same page in a workbook across schools, Ms. E simply put down the page number in her lesson plan and taught what she felt needed to be taught. She defended herself saying, “You’ve gotta do what’s right for the children.” Ms. E said that some teachers did nothing in the old days when they were supposed to develop a lesson plan for actual practice, which she thought led administrators to require teachers to write a mandatory and more complicated lesson plan.

Ms. E said that the parish called meetings for teachers when it developed a new curriculum or adapted a new mathematics textbook. Teachers had to listen to an expert from a publishing company or from the parish. For instance, when the parish began to
encourage teachers to use a calculator in their mathematics classrooms, an expert demonstrated how to use it. Ms. E found this to be an ineffective aspect of administration. “The problem is there, there are so many things I want to do in so few hours. ... I don’t have enough reflective time during the day.” Ms. E was overwhelmed by the amount of curriculum to be taught. While acknowledging that a teacher was supposed to choose the content to be focused on in a mathematics lesson, Ms. E thought that current teaching materials required too many things, putting pressure on many teachers to cover it all. In one of her interviews, she supported her claim by referring to an article on international comparison of curriculum which showed that the U.S. curriculum is too broad and too shallow. On the basis of her teaching experience, Ms. E came to narrow down her curriculum and tried to focus on essential topics for her students such as regrouping. She explained:

I rather go deep and get something across, what they really need to know and find out what children in third grade are handicapped, if they don’t get it in second grade. In my mind, they are handicapped, if they can not regroup, they can not borrow and carry, and if they don’t know the basic facts. If they still count on their fingers, they can’t utilize their computational skills and problem solving. Now, that’s the deep down things that they need to have. ... Teachers just follow them [textbooks]. You may notice that I don’t use my textbook too much because I want to be the one to decide, not the book. So, I pick and choose what’s important and skip around.

During interviews, Ms. E expressed pride in how well her students performed on a standardized test, the Iowa Test of Basic Skills. She believes standardized tests are needed for individual teachers to evaluate their teaching practices. However, she does not believe in the early administration of such a test because it has influenced her priority of mathematics content.
Summary Reflections

Though she did not enjoy learning mathematics as a student, as a teacher Ms. E came to realize that learning and teaching mathematics could be fun and joyful. She considered mathematics important because of its practical usefulness in a real world and regarded mathematics as a second language, but did not expand this view. Personal teaching experience was influential in shaping Ms. E's teaching approaches in mathematics. While teaching mathematics to her students, Ms. E improved her own mathematical skills and sometimes her understanding. This experience motivated her to use group activities by which students had a chance to teach one another. Before expecting students to gain similar cognitive benefits by teaching their peers, Ms. E knew that she had to work on developing students' social skills in working with partners or in groups. Her teaching experience often made Ms. E feel confident in the way she taught mathematics. She had former students who came back to her class and expressed how much they enjoyed her mathematics lessons, such as the one on different base systems with Unifix cubes. Ms. E felt successful in her mathematics teaching when students expressed their excitement in doing mathematics, when they were competent in their mathematical ability and proud of themselves, and when they offered mathematical ideas or encouraged their peers to keep thinking.

As described before, throughout her interviews, Ms. E reflected on what were the significant factors in the development of her mathematics teaching approach, one of the most influential of which was her enrollment in workshop classes. More importantly, she commented that her teaching approach was constantly evolving as she tried to teach better. It was a gradual development, rather than a development characterized by
importation of completely new approaches: "I am still trying to do better. I'll probably keep on growing and growing. I'll never be satisfied."

Ms. M's Case

Early Learning and Teaching Experience

Ms. M went to a small Catholic school in which teachers were very strict in terms of school work yet were warm in loving their students. Ms. M had to memorize many procedures in mathematics and was extremely good at memorization. Whereas the teachers in the school emphasized memorization first, they were also intent on students’ understanding and asked students to explain what they were doing. Ms. M specifically remembered her seventh and eighth grade teachers who were always full of enthusiasm for teaching and who later served as her tutor when Ms. M had difficulties in algebra in high school. Similar to her early school years, Ms. M had to memorize many theorems from the beginning of high school yet she managed well.

Being enamored with her teachers, Ms. M wanted to be a teacher, particularly like her seventh and eighth grade teachers. In addition to her love for her teachers, Ms. M generally enjoyed interacting with people and she had lots of teaching experience as the oldest child in her family and as a leader in Bible classes. She also enjoyed watching how children learn. After all, the decision to be a teacher was an easy one for her. When Ms. M was in college, New Math was coming out which its emphasis on understanding. She said that the way she was teaching in her classroom was similar to the way she was taught in college. Ms. M regarded her college education as practical and useful. The instructors in education classes provided students with an opportunity to visit elementary school classrooms and to discuss their observations. Moreover, they focused
on how to teach each subject area. Ms. M’s student-teaching, however, was not helpful in learning how to teach mathematics because the supervising teacher was not interested in mathematics instruction. At that time, Ms. M felt free to try her own teaching approach, one she did not learn from her supervisor.

Ms. M taught her first two years in a Catholic school in Louisiana with young nuns as her mentors who helped her learn practical teaching techniques such as how to choose what to teach out of so many instructional resources. At that time, Ms. M afforded students the opportunity to explain how to solve a problem, but never encouraged them to invent their own methods or to develop alternatives. She explained, “I basically always showed them a way to do it.” After teaching first and second grade two more years in a public school, Ms. M taught kindergarten in Arizona for 10 years while raising her own children.

**Professional Development and Teaching**

While she was in Arizona teaching kindergarten, Ms. M worked on her master’s degree in early childhood and studied Piaget. Ms. M recalled the time when she really began to understand how young children learn and when she realized the crucial importance of observing students and asking them questions in order to probe what they know. Ms. M began to collect instructional resources, study them, and try them out with her children to see what worked. She specifically remembered a book *Math Their Way* in which the author emphasized the ideas that Ms. M began to believe such as providing children with hands-on activities for active learning of mathematics. She also began to read professional books, in particular, those written by Ms. Richardson, a math educator who published mathematics materials for elementary teachers.
When Ms. M's family came back to Louisiana, she taught first and second grade. Her primary instructional concern was to create a classroom atmosphere where kindergarten children learned by doing and with excitement. She explained, "That's when I put children in groups and began to just experiment all different kinds of ways, where the classroom looked different, where they were talking all the time. What I was saying to them was, it's important to talk each other, it's important to explain what you are doing." Ms. M got a master's degree in reading but she thought it was useless because there was no relationship between the course work and her classroom teaching.

As mentioned above, memorization was easy for Ms. M and she did not have trouble with it because she ultimately understood what she was doing. However, Ms. M never encouraged her students to memorize a procedure in mathematics, because she has seen the negative impact of memorization. When he was in fourth grade, Ms. M's son told her he did not want to learn mathematics because he could not understand it except by memorization. Ms. M's daughter, a gifted student throughout her school years, made all As in mathematics, but had a difficulty in taking advanced placement calculus, because the teacher covered a chapter per day and she could not memorize that much that fast. Ms. M was mortified when her daughter told her she had not understood mathematics since eighth grade, but kept good grades through memorization. Ms. M also heard from many elementary school teachers that they did not understand mathematics as a student and had to memorize it rather than learn, which in turn prevented them from teaching mathematics for understanding. When she served as a mathematics specialist, Ms. M found that many students memorized procedures in using standard algorithm without understanding place value.
Ms. M believed that introducing standard algorithms in early grades might be developmentally inappropriate. Ms. M’s strong disagreement with teaching algorithm in early grades has been reinforced by the interplay between her teaching experience and her assiduous study. Personal teaching experience gave Ms. M an opportunity to make sense of her readings and others’ talks at national conferences. Her study in turn helped to confirm her observation of how often students memorized only procedures in mathematics without understanding conceptual underpinnings. Among many studies, Ms. M said that she has been deeply influenced by one professor who has written and spoken about the negative influence of teaching algorithm in early grades. Ms. M could unquestionably agree with the professor, mainly because she had seen it through her own teaching experience.

The elementary school where Ms. M has been teaching for about 20 years is special to her because the school as a whole attempted to change teaching under a previous principal’s aspiration. The principal, who once taught side by side with Ms. M, treated individual teachers like professionals and put them in a situation wherein they discussed their instructional approaches and shared resources. For staff development, most other schools invited experts from the outside and teachers were supposed to listen. However, the principal in the Ms. M’s school said that the teachers did not need outside experts because they were ready to be the best. Instead, the principal encouraged teachers to discuss with one another in faculty meetings, which in turn motivated the teachers to study and share readings voluntarily. The teachers even made a study group after school in an effort to understand reform ideals and to discuss how they might apply them in their classroom situations.

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In the beginning of this school reform, the students did poorly on a standardized test. Nevertheless, rather than discourage the teachers, the principal acknowledged their attempts because what they were doing made sense to her such as using hands-on activities and de-emphasizing memorization. The principal then initiated a discussion in hopes that teachers might find a way to compromise toward higher test results in keeping with their original attempts. In Ms. M’s school, most teachers came to be state presenters and thirteen of them, including Ms. M, were national presenters, until a new superintendent moved many of these teachers into different schools under a desegregation policy.

Ms. M informed her principal about workshops for teachers hosted by her friends at one school in Arizona. The principal accepted the idea of having workshops and, as a result, the teachers in Ms. M’s school had a super Saturday, for about five years. The teachers invited 500 other teachers to come into their school on a Saturday and had workshops on math, science, and language arts. These workshops became popular, not only within the state, but outside of the state as well so that teachers in other states often participated. Ms. M was eager to communicate with colleagues who held similar kinds of workshops in other states. She brought *Math Their Way* to Louisiana and began to teach in workshops. Ms. M invited Ms. Richardson as a workshop speaker and they began working together. Initially, Ms. M started working with Ms. Richardson and her study group in which teachers across the country met every summer to exchange their classroom observation and discuss how their students learned mathematics. Ms. M identified working with Ms. Richardson for about 12 years as the most influential factor in developing her mathematics teaching approach.
Influence of Colleagues

Whenever Ms. M tried something new in her mathematics teaching, she had to check with her principal. However, she has been treated as a professional and she felt comfortable talking with administrators. She took responsibilities for her pedagogical decision. For instance, when she decided not to use mathematics textbooks about 12 years ago, her principal accepted it but asked her to administer a test with her children after teaching each unit to make sure they were learning. Ms. M willingly agreed to this compromise because she believed that her students were able to pass a test once they made sense of the mathematics they were studying. This belief was expressed when she later had to advise many elementary school teachers who attributed standardized tests as the main obstacle to changing their mathematics instructions away from stressing algorithms. Ms. M would explain to them:

My children will probably do just as well as yours. They just use an extra line. They say 30 and 20 is 50, and 9 and 7 is 16, 50 and 16 is whatever, they will say it like that. All they need is just a little bit more space, but it will make sense to them. So, I won’t ever have confused them. I won’t ever tell them anything like, “Who cares if mathematics doesn’t make sense?”

Ms. M characterized her teaching in this way, “They [other teachers] are guided by their math book. I am one of the few people who are guided by children. ... My experience of school has never been except to teach children in a way that makes sense to them.”

From the beginning of her career, Ms. M has considered herself a professional. She has steadily worked hard to be a better teacher. Unlike the previous principals in Ms. M’s school, the current principal began to mandate what teachers should do and made rules such as assigning workbook pages. When the principal talked about those rules being followed in Ms. M’s classroom, she replied, “You can tell me what you
don’t like about my teaching or you can tell me what you don’t like about what my children learn. But I won’t be told how to teach. I’ve been teaching 35 years. ... I feel pretty bossy. But I won’t be told how to teach.” Ms. M kept her teaching approach in which she did not use math textbooks nor emphasized memorization. Instead of forcing Ms. M to comply, the principal compromised by requiring Ms. M to regularly test her children’s mathematics understanding.

**Reform Implementation**

With regard to reform movements, Ms. M was actively involved except when she taught kindergarten. By reading professional materials and trying ideas from those readings in her classroom, Ms. M was familiar with reform ideals before they became popular. In particular, she said that the current reform movement documented in the *Standards* has been very influential in her teaching of mathematics. She commented that the forthcoming revision of the *Standards* should take a stronger position against teaching algorithm in early grades than it does now. Ms. M talked about how the previous principal used to “fuss” at her. But eventually she came to recognize her unique capabilities:

> You [Ms. M] just see something, you understand it, and you just jump a ditch and you are there and you are doing it. She [the principal] said, nobody else does it like that, or not many other people do it like that. ... You never move slowly, you just jump in with both feet, and then just get started, and then start filtering it out what doesn’t make sense and what really work.

**Summary Reflection**

While she loved mathematics as a student, Ms. M said that her love for mathematics as well as her love for watching children learn mathematics has been really developed over the past 15 years. Ms. M considered mathematics an especially
important subject because of its practical usefulness, for instance, in getting a decent job. She found reading to be less difficult for most children so that they learn to read at some level, whereas mathematics is more difficult so that they need to work harder to understand it. Observing how children make sense of mathematics came to be fascinating to Ms. M. She wanted to develop her own teaching approach so as to facilitate the children themselves as the source of understanding mathematics. The main challenge in the process was that many times a new approach did not work. For instance, when she tried to use manipulative materials, she could not figure out how to manage the class. In those cases, Ms. M consulted with her colleagues about classroom management and lesson structure to keep students’ attention. However, when it came to struggling with herself with regard to students’ understanding of mathematics, she explained, “I don’t think I ever ask anybody about understanding because I think that was my own search about understanding.” She recalled:

With math, there were no models. I didn’t know where I had to go to look. I knew Math Their Way works. I knew [what Ms. Richardson said] works. But what I couldn’t figure out was how to make it ... how I need to look in the classroom. In other words, I understood the philosophy for years, but I didn’t know how to translate that into what to do tomorrow in the classroom. So, I just became fascinated with that whole thing and realized that a bunch of people across the United States were making the same kind of transition, making the same step or path toward what this is supposed to look like in the classroom.

Ms. M said that her teaching approach changed in definite ways rather than just evolving as she gained more experience. She explained:

I really think I probably came out of school thinking that I could pour the information into their heads. And I think I had definite experiences along the way that taught me that wasn’t gonna work. And so, then I began the search. Okay, if they have to seek the knowledge for themselves, what are the best methods for talking them into wanting them to search it out for themselves, and then motivating them to actually do it, and then what experience will. And I
think I am still evolving in that way, still searching out really good lessons or really good ways to get the children to think about certain things, and to express it. I think now they are evolving more, but I do believe there was the time when I finally realized that you cannot give that information to someone else. They have to get it for themselves. And I think that was a real eye-opener for me.

Challenges of Reform: Insights from Ms. E's case and Ms. M's Case

The comparison of Ms. E's case and Ms. M's case is not intended to evaluate a more successful teacher against a less successful teacher or vice versa. Both teachers deserve a great deal of credit for their lifelong efforts toward better mathematics instruction. The goal of comparison is to raise some subtle but crucial issues that may be significant for successful implementation of reform ideals in actual classroom teaching practices.

Teaching Practice as Inquiry

The two teachers' cases in this study support previous studies in which teachers reported their classroom teaching experience as the most important factor in shaping their actual teaching practices or beliefs about pedagogy (Raymond, 1997; Smylie, 1989). Ms. E identified her reflection on her teaching experiences as the most influential factor in her further evolution as a teacher. Though some other elements (e.g., taking workshop classes) led Ms. E to change her instructional approach, Ms. E regarded her mathematics pedagogy as constantly evolving through her teaching experience. While teaching students, Ms. E learned and improved her mathematical skills and realized that learning and teaching mathematics could be enjoyable. Students' feedback became an important indicator of her success in teaching mathematics. Personal teaching experience was also important to Ms. M in developing her mathematics instruction. Like Ms. E, she had kindergarten teaching experience which later influenced how she
taught second grade children. Ms. M attempted to establish an open classroom atmosphere, as she had done in kindergarten, in which children could learn mathematics by doing and with excitement.

Ms. E’s and Ms. M’s cases reveal that teachers can learn very differently from their teaching practices. A primary pedagogical concern for Ms. E was whether or not her students enjoyed mathematics and they gained confidence in mathematics. Thus, Ms. E’s learning how to teach mathematics through her teaching practice was mainly related to providing exciting mathematical activities and checking students’ mathematical skills such as the memorization of basic number facts. Ms. E encouraged her students to explain their solution methods to the whole class. However, her listening to students’ explanations was not attentive to the understanding of how they solved problems. Instead, she was listening for something in particular, checking whether students used a particular standard method and evaluating the correctness of that method. In other words, Ms. E had limited chance to learn how students think about mathematics arising from her teaching practice.

In contrast, interacting with students was fundamental for Ms. M to realize how students learn mathematics and, thus, how she should teach it. At first, she did not assume how much young children knew about math until she listened to them. After listening to them, she came to believe that young children know math before they learn math. Her teaching was like watching children and helping their thinking move forward by constantly asking herself, “What do they need to know next to be successful mathematics students?” As Ms. M explained, her mathematics teaching was guided, not by textbooks, but by students. An example of this is that she decided to teach a standard
formal algorithm at the point at which students understood place value and already had experiences in inventing various informal computations. She observed that students too often use algorithms by rote memorization only. Ms. M's teaching was a continuous inquiry into students' mathematical thinking and, in this respect, daily classroom practice was the main medium for her in developing her own mathematics pedagogy.

A more profound difference between Ms. E and Ms. M is found in the different opportunities that classroom teaching practices offered them to investigate ideas they learned from others. Both Ms. E and Ms. M had a chance to learn Math Their Way and to work with Ms. Richardson. Ms. E regarded taking the workshop classes from Ms. Richardson as a turning point in her attitude toward mathematics and her actual classroom teaching. In those classes, Ms. E learned to understand mathematics and that it was fun. She learned how to teach mathematics, not by what the instructors said, but by what they actually did in their teaching. Ms. M was also fascinated with the ideas illustrated in Math Their Way, such as using manipulative materials for young children and pursuing students' understanding before introducing rules. Indeed, Ms. M began to teach these ideas to other teachers through workshops. Ms. M also joined Ms. Richardson's study group.

The question then is how might we explain the different teaching practices in Ms. E's and Ms. M's classrooms coming out of the same influence. A possible explanation is that the two teachers used their classroom teaching differently. For Ms. E, as exemplified in her teaching of different base systems, a classroom was a site of implementing or copying knowledge from others, specifically experts. In other words, Ms. E's approach was to import the methods observed outside into her own classroom.
setting. In contrast, Ms. M's attempt was to reconstruct her own teaching, informed by outside models. Her teaching practice served as a filter by which she was able to differentiate what worked and what did not work for students. This experimentation with her children served as the catalyst whereby she made sense of outside teaching models.

Ms. E's and Ms. M's cases bring to mind two teachers who participated in a professional teacher development program called Cognitively Guided Instruction (e.g., Carpenter, Franke, & Levi, 1998; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996). From the program, the two teachers learned about students' thinking on various mathematics concepts and processes and how to use it in their classroom teaching. However, one of them did not perceive her classroom practice as a context for learning about students' mathematical thinking and, thus, her interaction with students was not generative of such learning. In contrast, the other struggled to understand students' thinking in her mathematics teaching and, thus, she used her classroom interaction to enlarge her knowledge of students' thinking. Overall, teachers' epistemological stances about their own learning experience in a classroom need to be highlighted with regard to their impact on their teaching practices (Carpenter, Franke, & Levi, 1998).

**Personality Traits and Beyond**

The common factors between Ms. E and Ms. M in developing their mathematics teaching include commitment and efforts to improve their teaching throughout their professional career. An important factor was that both teachers had self-motivation to develop or change their teaching approach. Though Ms. E was somewhat unsure of her
mathematics teaching and in doubt about her mathematical ability, she had a strong
desire for her students to feel competent in mathematics and to enjoy learning it. This
promoted her participation in professional teacher development programs. Similarly,
after realizing that her teaching method was sometimes problematic for her children,
Ms. M was eager to search for a new approach which would help them make sense of
mathematics. Both teachers’ dissatisfaction with their previous practice and initial self-
motivation were crucial to their subsequent active learning.

A difference between Ms. E and Ms. M with regard to personality traits is the
extent to which each would take a risk in the midst of ambiguity or uncertainty.
Encouraging children to present their various strategies and reasoning, rather than
directly telling them how to solve problems, may leave teachers feeling uncertain
because they cannot anticipate the direction classroom discourse may take. Using
manipulative materials or exciting activity formats may make teachers feel out of
control. Expressing this kind of uncertainty, Ms. M regarded her children as a “bomb”.
However, when her mathematics classes did not go as well as expected, Ms. M tried to
learn from the experience and identified possible reasons to plan a better activity next
time. Because she believed that students learn mathematics by their own meaning-
making process, Ms. M took risks in her own teaching and allowed her students to
wrestle with different ideas to solve a mathematics problem. In contrast, Ms. E was less
open in taking risks in her own teaching and in allowing her students to be puzzled or
frustrated as they learned mathematics. For her, learning mathematics was basically to
listen to what the teacher said and to faithfully follow it. When students were confused
with directions or ideas, Ms. E provided detailed illustration with examples. She was
not comfortable watching her children struggle with different ideas. Implementing reform ideals means taking risks, specifically to most teachers who were taught in traditional mathematics classrooms. Investigating where teachers feel uncomfortable when they begin to move toward new forms of practice and how they may overcome such discomfort or even increase their willingness to take risks is needed in the current mathematics education reform.

Another difference between Ms. E and Ms. M is the ways they considered their professional autonomy. Both teachers professed that they had autonomy in developing their own teaching methods. The difference was that Ms. M determined the focus and course of her own professional development, whereas Ms. E was rather vulnerable and did not always make decisions about her professional development. Ms. M made instructional decisions against common teaching practices such as the use of textbooks and believed that she was the one who knows her children and, thus, needs to make judgement about them. She also consistently pursued her agenda of understanding students’ mathematical thinking and took advantage of various workshops which emphasized this. In contrast, Ms. E was rather obedient in conceptualizing and carrying out the methods she learned in workshops. It seems critical that for teachers to have professional autonomy they need to evaluate alternatives in teaching mathematics along with what they value and attend to empirical evidence in confirming or disconfirming such alternatives (Cooney & Shealy, 1997).

Another difference between the two teachers in this study is how they dealt with tests for students’ mathematical learning. Whenever Ms. M tried something new in her mathematics teaching, she consulted with principals, who asked her to at least
administer periodic tests to check her children's learning. Since she believed that her students were able to pass tests if they really understood what they were taught, Ms. M gladly accepted the challenge. Tests, specifically standardized ones, were not used as a main indicator by which Ms. M evaluated the success (or failure) of her mathematics teaching. Rather, students' explanations in the class as well as Ms. M's own interviews with them were the main assessment tools for their learning and her teaching practice.

In contrast, Ms. E was very concerned about tests. The focus of her teaching was heavily influenced by tests her students were mandated to take, particularly standardized tests. She was proud of her children's performance when they memorized basic number facts and computed well with standard algorithms. The result of tests, either her self-tests on numerical skills or standardized tests, was a principal index of the effectiveness of her mathematics teaching.

**Teachers' Perceptions of Curriculum Development and Administration**

Ms. E's and Ms. M's cases in this study show teachers' different perspectives on the structure of curriculum development and administration, and their concomitant outcomes. Ms. E regarded administrators as people whose main job was checking whether teachers complied with policies or expectations. She used instructional resources such as teachers' manuals and the district curriculum guide as something she was supposed to cover thoroughly. Even though Ms. E professed that she had instructional authority to choose the content focus of her lessons, she felt burdened by too much content. While Ms. E was in a position where she was informed about the new curricula and teaching skills by policy-makers and experts, Ms. M actively participated in the process of curriculum administration. She was familiar with the new
curriculum and teaching method before it became popular, owing to her professional reading and experimentation with her students. Moreover, Ms. M did not perceive various instructional resources as something mandated. Rather, she decided what to teach and what not to teach.

The current mathematics education reform movement is different from the previous one with regard to curriculum development and administration. For instance, in the new math era in the 1960s, in a top-down fashion, mathematicians developed curriculum that teachers had to learn in a short period of time, mainly in summer workshops. In contrast, the current movement has been initiated mainly by the NCTM, a professional organization of mathematics teachers. The three volumes of *Standards* were developed as a framework or guideline leaving room for differences in states, districts, or schools. Nevertheless, Ms. E’s and Ms. M’s cases reveal the importance of teachers’ personal positioning with regard to policy. For Ms. E, there was no difference between the previous and the current movements; so, her view of administrative directives remained an external and mandated constraint. Partly because of their lack of empowerment regarding their own classrooms, some teachers like Ms. E may make instructional changes superficially despite great commitment and effort toward reform.

**Participation in a Supportive Community**

The degree of participation in a supportive community for teachers is a noticeable difference between Ms. E and Ms. M in developing their own teaching approaches. Ms. E believed that teachers in the U.S. are left alone in their classrooms, struggling to develop their own norms for good teaching. Ms. M commented that the sorts of opportunities she had for professional growth are a rarity in the profession. Ms.
M was at a school which served as a supportive community for teachers as learners. The principal encouraged individual teachers to feel autonomous and professional in developing their own instructional approaches but, at the same time, she encouraged teacher collaboration. The interaction among the teachers at the school determined the context-specific meaning and application of current reform ideals. In other words, the school as a learning community helped teachers activate the reform in their classrooms by allowing them to raise questions on their current instruction, search for alternatives, try on new approaches, share with colleagues (including the principal), and weigh their approaches against others' pedagogical alternatives. The community also supported the teachers in keeping their efforts toward reform, even when the immediate outcomes, such as test results, were not promising. Additionally, Ms. M had another supportive community, the study group with Ms. Richardson. She admitted that simply talking with other teachers, who have been attempting to improve their teaching practice by exploring how their children learn mathematics, was a great help to her.

Ms. M's case channels our attention toward collaborative communities where groups of teachers are committed to improve their practice. In fact, many recent studies of teachers' attempts toward reformed mathematics teaching suggest the importance of teacher-directed local communities that provide shared goals and collaboration (Campbell & White, 1997; Secada & Adajian, 1997; Stein, Silver, & Smith, 1998; Stocks & Schofield, 1997). The benefits of participating in those communities include teachers' sense of a collective responsibility for students' learning, shared resources, increased instructional expertise, and reduced feelings of isolation. The message is that there may be a need to reconceptualize aligning teaching to the reform, not primarily as...
an individual teacher’s isolated accomplishment, but as a community’s collaborative enterprise.

Ms. E’s case allows us to articulate the nature and the functioning of collaboration in local communities such as schools. Throughout her long career, Ms. E developed good relationships with her colleagues and administrators. However, the dialogues with them did not challenge her instructional approach. Teachers shared instructional materials, but were private about their teaching. In general, the norms of school culture were nonjudgmental rather than critical of others’ teaching practices. This culture would have provided psychological and social support for teachers in changing their practice toward the reform ideals. However, if any school wide commitment toward reform would be actualized, an uncritical culture may not provide professional or intellectual support teachers need to transform traditional teaching. In working together in a supportive community for reform, participants need to establish new norms for discourse concerning their instructional changes, obstacles and dilemmas of change, as well as the more general nature of mathematics teaching and learning. While accepting the importance of supportive communities for teachers, there is a need to explore the ways such communities provide teachers with opportunities to challenge their teaching practice, and to discover how participants perceive their collaboration with one another.
CHAPTER 6

CONCLUSIONS AND IMPLICATIONS

This chapter is organized into three sections. The first section begins with a general overview of this study, including its purpose, research questions, methods, and limitations. Then a concise sketch is presented of the mathematics teaching practices in the two U.S. classes studied as part of this dissertation project, and two Korean classes studied within the broader crossnational research project of which this dissertation is a part. This section concludes with an analysis of the importance of effective sociomathematical norms for students' learning opportunities, as illustrated in the U.S. and Korean classes. Given this importance, the current status of theorizing about sociomathematical norms is reviewed in the second section, and an elaboration of this construct is suggested. The third section presents implications for reform as the extension of the sociomathematical norms construct paves the way to embrace diverse implementations of reform ideals.

OVERVIEW, SUMMARY, AND IMPORTANCE

OVERVIEW OF THE STUDY

Educational leaders in both the U.S. and South Korea are seeking to change the prevailing teacher-centered pedagogy of mathematics to a student-centered pedagogy (Ministry of Education, 1992, 1997; NCTM, 1989, 1991, 1995, 2000). The term teacher-centered refers to a teacher's explanations and ideas constituting the focus of classroom mathematical practice, whereas the term student-centered refers to students' contributions and responses constituting the focus of classroom practice. The teacher in a reform mathematics classroom is expected to provide worthwhile mathematical tasks...
and to be sensitive to students' conceptual understanding and their engagement in classroom discourse and activity.

Cobb and his colleagues developed an "emergent" theoretical framework that fits well with the reform agenda for instruction (Cobb & Bauersfeld, 1995). In this perspective, mathematical meanings are seen as emerging in a continuous process of negotiation through social interaction. In investigating students' mathematical learning within the emergent perspective, Cobb and his colleagues addressed sociomathematical norms as "the normative aspects of whole-class discussions that are specific to students' mathematical activity" (Cobb & Yackel, 1996, p. 178). They differentiate general social norms as applicable to any subject matter area from sociomathematical norms which are unique to mathematics. The examples of sociomathematical norms have included the norms of what count as an acceptable, a justifiable, an easy, a clear, a different, an efficient, an elegant, and a sophisticated explanation (Bowers et al., 1999; Cobb et al., 1997; McClain & Cobb, 1997; Yackel & Cobb, 1996).

In the U.S., the reform movement has been successful in marshaling large-scale support for instructional innovation, and in enlisting the participation and allegiance of large numbers of mathematics teachers (Knapp, 1997). However, despite the widespread endorsement of reform, there is concern that many teachers have not grasped the full implications of the reform ideals (Hiebert et al., 1996; RAC, 1997). Teachers too easily adopt a new teaching techniques, but without reconceptualizing how such a change in teaching strategies relates to fostering students' conceptual understanding or mathematical dispositions (Burrill, 1997; Stigler & Hiebert, 1998).
Numerous studies have examined the challenges faced by dedicated and committed teachers as they struggle to understand and come to terms with reform (Ball, 1993; Fennema & Nelson, 1997; Lampert, 1990; Schifter, 1996; Schifter & Fosnot, 1993; Wood, Cobb, & Yackel, 1995). However, past research has been limited mainly to studying general social norms of typical classes or exploring social and sociomathematical norms of specific reform classes wherein researchers are supporting teachers in their attempt to change their instructional methods. Naturalistic studies of social and sociomathematical norms in unsupported reform-oriented classrooms have not yet been undertaken. Moreover, the previous research trend was to provide an extensive analysis of one single reform-oriented classroom. Close contrasts and comparisons of unequally successful reform classes have rarely been conducted in previous research on reform.

Given the challenges in implementing reform, this study intended to explore the breakdown that may occur between teachers' adoption of reform objectives and their successful incorporation of reform ideals. To this end, the study compared and contrasted the classroom social norms of two U.S. second grade teachers who aspired to implement reform. The two classrooms were chosen because of their unequal success in activating the reform recommendations. The comparison and contrast between more successful and less successful reform classes provided a unique opportunity to reflect on possibly subtle but crucial issues with regard to reform implementation. This study provided detailed descriptions of the processes that constituted unequally successful student-centered pedagogy in the U.S. elementary mathematics classrooms. Since the principal concern of reform is to connect changes in teaching practices with changes in
learning opportunities that students will encounter in the class, students’ learning opportunity was analyzed in the target classrooms.

This study was an exploratory, qualitative, comparative case study (Yin, 1994) using constant comparative analysis (Glaser & Strauss, 1967; Strauss & Corbin, 1998) for which the primary data sources were classroom video recordings and transcripts (Cobb & Whitenack, 1996). As a kind of purposeful sampling (Patton, 1990), the classroom teaching practices of 17 second grade teachers recommended as reform-oriented were preliminary observed and analyzed. Two classes were selected that clearly aspired to student-centered classroom social norms, but that appeared to differ in the extent to which students’ ideas became the center of mathematical discourse. Seven mathematics lessons in each of these classes were videotaped, audio-taped, and transcribed. A total of 12 hours of interviews was taken with the two teachers to trace their construction of their teaching approaches. These interviews were audio-taped and transcribed. Additional data included students’ papers and projects. This methodology parallels the study of two Korean classes undertaken by the crossnational research team (Kirshner, Jeon, Pang, & Park, 1998).

The purpose of exploratory study is to articulate new issues and problems, rather than to definitively answer questions (Yin, 1993, 1994). The small number of classes, and small number of observations of each class, do not provide a basis for firm generalization. However, qualitative case study is well established as a methodology for generating theoretical and empirical insights to be pursued in subsequent broader based studies (Yin, 1994). Particularly, large scale crossnational research can benefit greatly from previous exploratory research in which the international team has the opportunity
to identify, dispute, and achieve consensus on the key issues of the investigation (Schmidt et al., 1996).

**SUMMARIES OF FOUR CLASSROOM TEACHING PRACTICES**

Despite the limitations of exploratory research, a clear and convergent picture of each classroom teaching practice emerged from the analysis. As noted above, this study was nested within a larger, crossnational, collaborative project to compare and contrast the classroom dynamics of reform-oriented instruction in Korea and the U.S. The Korean portion of the study was conducted by a team of four researchers including the author of this dissertation. The U.S. portion was conducted independently, though informed by the methods and theoretical focus of the larger project. Thus, the two U.S. classroom examples constitute the main data sources for this dissertation project. However, in order to formulate and articulate the issues of reform more broadly, this chapter uses both Korean and the U.S. data sources as well as other published reports of classroom research. To enable the subsequent discussion, portraits of the teaching practices observed in the Korean classrooms are presented along with portraits of the two U.S. classrooms. The two Korean teachers are identified as Ms. G (in Class KG) and Ms. C. (in Class KC). The two U.S. teachers are identified as Ms. E (in Class UE) and Ms. M. (in Class UM). A more complete account of the Korean classes is available in Kirshner, Jeon, Pang, and Park (1998).

**Mathematics Teaching and Learning in Ms. G’s Classroom**

In many ways, Ms. G in Korea encouraged students’ participation in order to make their ideas and judgments the focus of classroom attention. For instance, she lavished praise and positive expectations on the students, organized a variety of
classroom participation structures, asked for explanations, different methods, and critiques of previous methods, stressed the importance of sharing ideas and collaborating in small groups, and frequently repeated or rephrased students' explanations for the whole class.

Despite these compelling social norms that are generally consistent with student-centered pedagogy, Ms. G's consistent focus remained the procedural methods for accomplishing various tasks, and the correctness of the answers. For instance, when Ms. G criticized one student's presentation in which she used the word "vertical lines" instead of naming the place values, Ms. G never probed the nature of the student's conceptual understanding (see Episode KG-2). Ms. G seemed satisfied that another student was able to provide a better vocabulary, "digit by digit", and quickly changed the subject of discussion. Indeed, the two students' explanations were the same in terms of their conceptual understanding. Throughout her lesson, Ms. G emphasized using different methods to solve a given problem and welcomed all students' solutions as being different. There was little discussion of what counts as mathematically different, or even of why using different methods is important. Toward the end of the class, Ms. G consolidated and summarized standard subtraction algorithm as the "convenient" method, after listening to various methods presented by students (see Episode KG-3). What was mathematically significant in this classroom was getting the correct answer and giving presentations using the recommended standard algorithm.

The students' main approach in Ms. G's class was to comply with the teacher's expectations rather than to pursue their own understandings. Owing to the teacher's general emphasis on their participation, the students sometimes provided criticism and
question but those were limited to procedural aspects of the mathematical content being considered. As a result of Ms. G's and their own approaches, students learning opportunities tended to be limited to procedures for correctly adopting standard algorithms.

Mathematics Teaching and Learning in Ms. C's Classroom

Ms. C in Korea established similar social norms similar to Ms. G's, in that students' ideas constituted the central concern of lessons. Ms. C especially emphasized that students should solve a given subtraction problem for themselves in many different ways. She carefully observed students' individual or collective work and picked out mathematically insightful contributions for presentation in the subsequent whole class discussion. An example was that she became very excited when a group of four students developed the use of arbitrary units in representing numbers. They decided to regard a bundle of 10 chopsticks not as 10 but as 100. Another crucial episode revealing Ms. C's teaching approach was when she guided a mathematics discussion through which students had a chance to understand what was the mathematical difference between using coins and using tiles — iconic representations only symbolize the relative quantities through unequal sizes of the coins, whereas concrete representations actually embed the quantitative relations in the physical structure of the tiles (see Episode KC-3). Note that Ms. C, like Ms G, asked for different solution methods to solve a problem and encouraged students to present their invented methods. But rather than applauding all contributions equally, Ms. C carefully selected mathematically significant contributions as described above. In this way, Ms. C initiated and guided development of a sociomathematical norm for what counts as a mathematically different explanation.
The students in Ms. C's classroom not only complied with her demands for participation, but even tried to invent their own mathematical ideas and to participate fully in their group activities. In these approaches, the students had the learning opportunity to construct significant understandings of mathematical concepts: differences between iconic and concrete representational modes, the take-away interpretation of subtraction, and the arbitrariness of the unit in arithmetic representations. Moreover, there was evidence of students becoming self-motivating in their pursuit of mathematical meaning. For instance, some of the students continued to work on figuring out solutions even after the teacher had rung the bell to signal a new activity.

Mathematics Teaching and Learning in Ms. E's Classroom

Like the two Korean teachers, Ms. E in the U.S. was successful in establishing classroom social norms compatible with a student-centered approach. Among other things, Ms. E actively facilitated students' participation in the classroom mathematics activities and discussions by employing enjoyable formats, emphasizing visualization of the given problem situations, giving students many opportunities to solve problems individually or collectively and to present their methods to the class, expressing excitement about students' novel ideas, and asking students to author story problems within a group.

Despite this exemplary form of student-centered instruction, the content and qualities of Ms. E's teaching focussed primarily on procedural knowledge. To be clear, in some cases, Ms. E expressed her interest in conceptuality but those cases were somewhat infrequent (e.g., Episode UE-12). Ms. E listened to students' various
contributions but usually turned out to control the classroom discourse toward one direction — using standard algorithm or a specific equation for a given mathematics problem. This concern occurred across different classroom activities. For instance, in an estimation activity, when Alex presented his own mathematical reasoning to solve 100-88, Ms. E praised him but immediately guided students to use formal algorithm (see Episode UE-3). She frequently initiated classroom discussions by saying, for example “what’s the first thing I do [to solve this problem]?” which signaled that students were expected to attempt to add or subtract from the ones column (see Episode UE-2). In a problem solving activity, Ms. E solicited students to solve a problem but reinforced their use of specific equations, even after they provided mathematically reasonable alternative methods (e.g., Episode UE-10). As well, Ms. E often provided her own solution strategies or ideas (e.g., Episode UE-11).

Reflecting Ms. E’s practices, students often expressed keen excitement when they got right answers. But they sometimes waited for their teacher’s confirmation rather than develop their own rationales or arguments while engaged in group activities (e.g., Episode UE-23). As a result of Ms. E’s and of their own approaches, students’ learning opportunities were somewhat limited to acquire procedural skills to solve routine problems with accuracy and confidence. Whereas the students were actively involved in classroom mathematical activities, they had little chance to develop the mathematical understandings that could inform their activities. In these respects, the important sociomathematical norms of this class included mathematical accuracy and automaticity.
Ms. M in the U.S. also established classroom social norms by which students' contributions and ideas were focused. Like Ms. E, she was concerned about students' participation in classroom mathematical activities and discussions. Unlike Ms. E, Ms. M focused on students' own sense-making processes while they were participated in the classroom community. Her primary interest was to create an effective classroom community in which students invent, explain, and justify their own solution methods or ideas. Ms. M encouraged her students to argue or debate for extended periods of time — especially when there were competing solution methods or ideas — rather than providing her judgement (e.g., Episode UM-3). Only after students' full contributions to the discussion did she summarize the main argument in each position (e.g., Episode UM-8).

While focusing on students' mathematical thinking, Ms. M urged them to specifically mathematical ways of valuing and communicating. Producing only a correct answer without a mathematically justifiable process was rejected. For instance, Chase knew the answer for 42-26 because the class solved it using unifix cubes. When he presented his numerical solution method, Chase claimed to take away 4 more after subtracting 20 from 40. As he was not able to provide mathematical evidence of taking away 4 more, Ms. M did not accept his contribution as valid (see Episode UM-2). Similarly, merely recalling a multiplication fact from a times table seen at home, without a mathematical explanation, was not valued (see Episode UM-18).

In conjunction with Ms. M's approach, her students tended to use their own ways of approaching a given task both in whole group and in small group settings.
Moreover, they were often engaged in mathematical debates without the teacher's initiation or mediation. In these approaches, the students had the learning opportunities to construct conceptual underpinnings of the mathematics they were studying, even as they were continually exposed to mathematically significant ways of knowing, valuing, and arguing.

**IMPORTANCE OF SOCIOMATHEMATICAL NORMS**

The four classrooms presented above established very similar social norms including an open and permissive learning environment, stressing group cooperation, connecting concrete representation by manipulative materials to numerical computation process, employing enjoyable activity formats for students, orchestrating individual or small group session followed by whole group discussion, emphasizing multiple solution methods, expecting students' active participation, and providing the teacher’s amplification of students’ contributions. These are general social norms that are compatible with current reform recommendations (NCTM 1989, 1991, 2000).

Despite these similar social participation structures, the two classes within each country were remarkably different in terms of sociomathematical norms. In one class (KG from Korea, UE from the U.S.), students experienced mathematics on the basis of rather fixed procedure the teacher consistently emphasized. The students' mathematical ways of thinking and valuing were limited to find out the pre-determinded rules. Similarly, their mathematical ways of arguing and justifying were concerned mainly with following the rules, rather than with their own sense-making. In these respects, being accurate or automatic was evaluated as a more important contribution to the classroom community than being insightful or creative. In contrast, the students in the other class
(KC from Korea, UM from the U.S.) learned mathematics on the basis of their own sense-making processes. A specific solution method or idea was little emphasized over students’ various ways of approaching to a given mathematics problem. The students were continually engaged in significant mathematical processes by which they could develop an appreciation of characteristically mathematical ways of thinking, communicating, arguing, proving, and valuing.

The similarities and differences between the two teaching practices within a country clearly show that students’ learning opportunities do not arise from general social norms of a classroom community. Instead, they are closely related to its sociomathematical norms. Thus, this study suggests that reform efforts highlight the importance of sociomathematical norms that become established in the classroom microculture. It was apparent from this study that sociomathematical norms are an important construct reflecting the quality of students’ mathematical engagement and predicting their conceptual learning opportunities. However, the theoretical elaboration of this important construct has thus far been limited.

RETHEORIZING SOCIOMATHEMATICAL NORMS

As reviewed in Chapter 3, the sociomathematical norms construct was developed out of a classroom teaching experiment in which Cobb and his colleagues designed instructional devices for specific mathematical content and supported the classroom teacher to foster students’ mathematical learning using those devices (Bauersfeld & Cobb, 1995; Cobb & Yackel, 1996). The researchers attempted to account for students’ conceptual understanding embedded in the social context of an inquiry mathematics classroom. As instructional designers, the researchers were...
interested in analyzing the collective mathematical learning of the classroom community (i.e., *classroom mathematical practices*). In their analyses, the evolving criteria for mathematical discussion (i.e., *sociomathematical norms*) were described mainly as a precursor to the detailed analysis of communal learning process. In contrast, my interest in assessing the potential of various reform-oriented classes for developing students’ conceptual understanding and positive dispositions toward mathematics led me to position the sociomathematical norms construct as more central. This different interest now leads me to reconsider current theorizations of sociomathematical norms.

One result of studying classrooms in which the teacher is supported by the research team is the likelihood of viewing relatively effective reform classrooms. Indeed, the existing literature on sociomathematical norms is based almost exclusively on successful classrooms in which students have ample opportunity to learn and to develop mathematically. As a consequence, the sociomathematical norms construct has been developed around positive instances related to students’ development of mathematical dispositions, autonomy, and increasingly sophisticated knowledge (Cobb et al., 1997; Rasmussen & King, 1998; Yackel & Cobb, 1996). The current study provides an opportunity to extend the sociomathematical norms construct to a wider range of classrooms. For instance, Class KG in Korea and Class UE in the U.S. established general classroom social norms as recommended in the reform literature. However, their procedure-based instruction tended to consolidate mathematical values of correctness, algorithm procedures, and automaticity prior to, and often in opposition to, sense-making and creativity. The two classes had developed a reasonable discourse structure, but one that does not reflect the culture of mathematical inquiry. It is unclear
from the existing theorization how such classes might be analyzed as promoting sociomathematical norms.

Another aspect of the developmental trajectory of the sociomathematical norms construct concerns the vision of reform operative in the classroom. In the existing literature, the classroom teacher has been supported by the researchers toward the development of a particular discursive structure. In these inquiry classrooms, the teacher and students together constituted a mathematical community and negotiated mathematical meanings by asking for justifications and by challenging others’ explanations. The role of the teacher in this community is to mediate such conversations as they tend toward increasingly sophisticated mathematical forms.

The result of this developmental history is clearly evident in the theorization of sociomathematical norms as related to the mathematical character of the classroom discursive practices. The sociomathematical norms thus far identified include the extent to which classroom participants have mastered the distinctly mathematical notions of what counts as a different, sophisticated, efficient, or elegant contributions to the discussion (Stephan, 1998; Voigt, 1995; Yackel & Cobb, 1996). This notion of sociomathematical norms has been very adequate in analyzing students’ mathematical involvement in the sorts of inquiry-based mathematics classrooms studied by Cobb and his colleagues. However, it is not clear the same notion would extend to broader classroom circumstances, as might evolve without the oversight of a research team. The practice in the current study of examining already existing reform-oriented classrooms invites such speculations.
An account of a reform oriented classroom that does not function according the model of the inquiry classroom as studied by Cobb and his colleagues can be found in regard to Christopher Healy's *Build-A-Book* geometry course (Healy, 1993a, 1993b). Although the course does produce a community of geometers, Healy is not positioned in the classroom as mediating the discourse. At the beginning of the course, Healy gives his students a few geometric statements as a starting point for discussion. But from then on students, in small group and whole class formats, develop definitions, argue for or against conjectures proposed by their peers, and finally produce a geometry book with their agreed upon definitions and results. Healy's sole concern is with the character, quality, and intensity of students' engagement in the evolving mathematical community of the classroom.

There is some evidence in Healy's accounts of his classes that students do develop discursive practices and even mathematical understandings that are mathematically valuable, and these could be used within the current framework to analyze the sociomathematical norms of the class. But Healy himself does not mediate the classroom discussions. Rather, the most salient outcome of the course that he does mediate is the characteristically mathematical sense commitment to mathematical invention. Thus this major accomplishment of Healy's classroom would go unanalyzed according to the current notions of sociomathematical norms. A broader definition of sociomathematical norms is needed to account for this aspect of mathematical enculturation.

There are signs of an emerging recognition of the current limitations of the sociomathematical norms construct. Cobb (1999) himself somewhat broadened the
sociomathematical norms construct relating to "the evolving criteria for mathematical activity and discourse" (p. 9). However, despite this change in definition, the illustrations of the construct have retained their original character (e.g., Bowers et al., 1999; Cobb, 1999).

In summary, the current interpretation of sociomathematical norms lack both theoretical breadth and practical utility. There seems to be a natural definition of sociomathematical norms that eliminates these limitations. The construct can be related to mathematical enculturation in general, rather than just to mathematical enculturation accomplished as a result of a particular discourse strategy of the teacher. In other words, an extension of the sociomathematical norms construct is to include all aspects of classroom microculture that reflect and support particularly mathematical modes of engagement. Enculturation here needs to be broadly defined to include engagement with people, problems, tools, or oneself in a classroom community (Kirshner, in press). This extension of sociomathematical norms includes, but in no ways needs to be limited to, teacher's mediation of mathematics discussions.

The extended notion allows us to see a teacher as promoting sociomathematical norms to the extent that she or he attends to concordance between the social processes of the classroom, and the characteristically mathematical ways of engaging. The extension of sociomathematical norms in this study also broadens the range of domains which can be addressed. Within earlier theorization, sociomathematical norms are reflexively related to individual's mathematical beliefs and values and, consequently, their intellectual autonomy (Cobb & Yackel, 1996; Yackel & Cobb, 1996). The extended definition of sociomathematical norms can include their psychological
correlates broadly not only mathematical beliefs and values but also feelings, patterns of thinking and arguing, metacognition, logical reasoning, and use of problem solving heuristic (Kirshner, in press).

IMPLICATIONS FOR REFORM: EMBRACING DIVERSITY

This study supports the growing realization of the reform community that reforming mathematics teaching is a matter neither of changing the social structure of instruction nor of adding a few new techniques to an existing repertoire. Rather it involves reconceptualizing how students' engagement in the social fabric of the classroom may enable them to develop increasingly sophisticated ways of mathematical knowing and valuing. This reconceptualization never comes easily, even for teachers who are dedicated and committed to aligning their teaching practices to reform. Ms. E's case in the U.S. and Ms. G's case in Korea warn of the possibility that simply changing classroom social norms promotes neither students' conceptual learning opportunities nor their social engagement toward characteristically mathematical ways of thinking and communicating. Although the students in both of these classrooms had positive and enjoyable experiences in their mathematics classes, opportunities for enhancing their specifically mathematical development were somewhat limited. Ms. M's case in the U.S. and Ms. C's case in Korea show the possibility that students may acquire conceptual underpinnings of mathematics they are studying as they actively participate in the social processes which include explanation, justification, argumentation that are specific to mathematical activity and discourse. This is a case where reform efforts turn out to be successful. In this respect, the construct of sociomathematical norms, not general social
norms, should be focused for initiating and evaluating mathematics education reform efforts as they occur at the classroom level.

The four classroom teaching practices examined in this study also reveal that the simple dichotomy between student-centered and teacher-centered pedagogy obscures the variety of mathematics education reform possibilities. Class KG and Class UE displayed student-centered instruction at one level. The general social norms established in both classes, which were compatible with reform recommendations, were very different from those norms in typical teacher-centered mathematics classes. However, the detailed analyses of both Class KG and Class UE (see Chapter 5) illustrated that they displayed teacher-centered instruction at another level, because the ultimate focus of mathematical activity and discourse was on the teachers’ methods, rather than on the students’ contributions. Similarly, the discussion below will highlight the very significant differences in pedagogical intention and learning opportunities between the more successful student-centered classrooms of Ms. C in Korea and Ms. M in the U.S.

EMBRACING DIVERSE VISIONS OF REFORM

As summarized at the beginning of this chapter, current reform emphasizes students’ development with regard both to specific mathematical content and to mathematical dispositions (NCTM, 1989, 1991, 2000). Stemming from Piaget’s genetic epistemology, psychological constructivism provides valuable insight into the process of students’ conceptual development (von Glasersfeld, 1984, 1991, 1995). In order to understand students’ mathematical enculturation, there has been increasing interest in theorizing learning mathematics as a social process (e.g., Forman, Larreamendy-Joerns,
Stein, & Brown, 1998; Lampert, 1990; Seeger, Voigt, & Waschescio, 1998). Such perspectives are needed to better understand how participating in a classroom community in which students' mathematical thinking and ideas are valued and discussed, can lead to students' appreciation of the role of mathematics in a society, confidence in solving difficult mathematical problems, eagerness to make sense of mathematics on their own ways, perseverance in confronting challenging problems, flexibility in approaching multiple solution methods, and autonomy in thinking about and using mathematics (NCTM, 1989, 2000). In these respects, substantive consideration of students' development of mathematical dispositions has been compelling in the current reform era.

The transition from students' conceptual development to its incorporation with social development has remained challenging both theoretically and practically (Anderson, Reder, & Simon, 1996, 1997; Cobb, 1994; Confrey, 1995; Greeno, 1997; Hatano, 1993; Kirshner & Whitson, 1998; Lerman, 1996, 1997, 2000; O'Connor, 1998; Sfard, 1998; Steffe & Thompson, 2000). The current theory of sociomathematical norms has provided theoretical support for one sort of coordination of social and psychological objectives, but only in terms of a unitary conception of reform as teachers' explicit mediation of classroom discourse. However, a unitary conception of reform has been cited as a potential problem of the reform movement (Lindquist, Ferrini-Mundy, Kilpatrick, 1997). The retheorization, here, of sociomathematical norms as aspects of reflecting students' mathematical engagement broadens the possible range of what may count as effective reform-oriented mathematics teaching.
The classroom analyses of the teaching practices Ms. M in the U.S. and Ms. C in Korea reveal very different models of successful reform implement. As summarized above, Ms. M's pedagogical priority was to create a learning environment in which students were able to engage in significant mathematical discussions. She rarely considered the implications or conceptual subtleties of what her students came up with. Rather, she focused on whether her students had the opportunity to build their own mathematical understanding as they participated in the classroom activity and discourse. When compared with Ms. C's class (see below), Ms. M seemed to be more successful in establishing a more empowering classroom mathematical community owing to her more focused interest in students' mathematical socialization. There was much evidence in Class UM that students were becoming self-motivating in their pursuit of mathematical meaning. The examples included students' request for clarification of the problems/game situations given by Ms. M, their frequent debates from different interpretations without Ms. M's initiation or guidance, and their persistence in developing and using their own solution methods both in whole group and in small group settings.

As discussed in Chapter 5, Ms. M was heavily influenced by traditional interpretations of Piaget's work. She emphasized that learning for little children is play, and that early introduction of standard algorithms was developmentally inappropriate. While her pedagogical priority was to promote students' positive dispositions (socialization) toward mathematics, Ms. M did not completely discard her interest in conceptuality. Influenced by this Piagetian interpretation, Ms. M saw herself as a facilitator rather than initiator of teaching mathematical content. But she discharged her
concern for content by carefully soliciting the contributions of stronger students in cases where the risk of conceptual confusion was high. From an individualist perspective, this teaching technique might be seen as creating a hierarchy among the students according to the various roles instituted by the teacher. But Ms. M herself seemed to have a communitarian perspective in which the well being of the individuals in the class was a function of the successes and failures of the class as a whole.

Whereas Ms. M was constrained by Piagetian assumption in teaching specific mathematical content, Ms. C from Korea was much more willing to mediate classroom discourse for students' conceptual development. Unlike Ms. M, in the course of supporting students' mathematical engagement and discussion, Ms. C developed a specific conceptual agenda for use of different materials. When compared with Ms. M's class, the quality of mathematical content dealt with in Ms. C's class was much more sophisticated. Ms. C masterfully attended to concordance between the social processes of the classroom and students' engagement toward development of specific mathematical concepts. She began with a subtraction problem and emphasized solving the problem with different methods. Her students contributed to the classroom discourse through their invention, explanation, justification, and argumentation of various solution methods. Meanwhile, the students had the opportunity to experience characteristically mathematical ways of thinking and communicating that had been established in the classroom community. Ms. C paid careful attention to her students' mathematical understandings embedded in solving and discussing the given subtraction problem. The students became more sophisticated with regard to what constitutes mathematically different. Based on their individual and collective activity of solving one
problem using various manipulative materials, Ms. C initiated and orchestrated discussion in ways that allowed the students to develop a mathematically significant concept, that is the difference between concrete and iconic representational systems. The understanding of what makes a solution mathematically different was embedded within the activity structure of the lessons. Indeed, the concept was addressed not by the teacher’s authoritative knowledge but by the students’ participation in the classroom practices. Students’ engagement in the task was central to compare and contrast different representational modes.

Ms. M’s and Ms. C’s teaching practices reveal the successful aspects of implementing reform ideals but in different ways — focusing primarily on students’ mathematical engagement and coordinating classroom social processes with students’ conceptual development. The retheorization of sociomathematical norms helps to broaden the perspective of reform, and to progress past notions of a unitary reform model.

Recall that there were similar aspects in Korea and the U.S. with regard to comparison and contrast between a more successful and a less successful mathematics classroom. This observation provides a more caution for the Korean reform movement than for its U.S. counterpart. As reviewed in Chapter 2, the characteristics of typical Korean mathematics teaching practices included teacher-centered whole class instruction in a systematic, coherent, and progressive way (Grow-Maienza et al., 1999). In general, whole-class teaching by well-prepared, skilled teachers in East Asia are appreciated in comparison to western individualized instruction (Stevenson & Lee, 1995). The prevalent Korean teacher-centered teaching method has contributed at least

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to students’ superior mathematical achievement in international contexts. As reviewed above, it was compelling for Korean mathematics educators to advocate instructional changes from teacher-centered toward student-centered pedagogy. However, any kind of rash implementation of such changes may lead to lose the current well-structured Korean teaching practices and, to make it worse, the changes may not promote students’ conceptual understanding of mathematics as intended. Given this, changing teaching practices should be conducted with great delicacy and consideration in Korea.

Reform is fundamentally about significant change, and the teacher remains the key to change. The extent to which significant change occurs depends a great deal on how the teacher comes to make sense of reform and respond to it. It is not an overstatement that real instructional change occurs only at the classroom level, as teachers grapple with their own values and priorities relative to the ideals promoted for the profession. Teachers need to be empowered in developing alternatives or integrating different aspects of reform agenda with regard to their own diverse pedagogical motivations (Kirshner, in press). This study with a broadened definition of sociomathematical norms paves a way by which teachers and reformers open towards diverse but viable mathematics teaching approaches that are compatible with the current reform recommendations.

AFTERWORD

Reflecting on my work following a rigorous dissertation defense meeting, I want to address two methodological concerns that stem from unique aspects of my study. One is the apparent circularity between the selection of classes of a certain type, and the subsequent analyses of them. The other is related to ethical concerns that arise from the

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comparison and contrast of two unequally successful teaching practices.

Using a kind of purposeful sampling (Patton, 1990), I needed to locate a more successful and a less successful reform-oriented mathematics class for intensive study. One of the most important factors in a satisfactory qualitative case study is making a proper selection of cases (Stake, 1998; Yin, 1994). This was especially true for my study for two reasons. First, teachers report their familiarity with and allegiances to reform but their actual classroom teaching practices often do not reflect full understanding of its implications (Cohen, 1990; Peterson, 1994). Thus, searching for a more successful reform class was challenging and needed an extensive search, in conjunction with careful consideration of the distinction between more successful and less successful student-centered classrooms.

Second, subsequent levels of analysis were conducted on the basis of understanding of individual classes. The analysis of the individual classes constituted the empirical portion of the study. Following Stake (1998)'s claim that case study should be based on a keen understanding of the case itself, teaching practices in each class were very carefully scrutinized in a bottom-up fashion. These detailed individual analyses were then employed for comparisons and contrasts of the teaching practices and the factors influencing the teachers' instructional goals.

As outlined in Chapter 4, seventeen classes recommended as reform-oriented were observed to get an initial sense of the extent to which students' contributions and ideas were solicited and became the foci of classroom discourse and activity. On the basis of these preliminary observations, including selected revisits, two classes were selected because of their unequal successes in implementing reform.
The detailed analyses of classroom teaching practices with representative episodes (see Chapter 5) found that the two classes constituted qualitatively different mathematical engagements, though building on similar social participation structures. This analysis seemed to prove the proper selection of each class. More importantly, however, the similarities and differences of the two classes might have been predicted because of the selection of such classes.

While acknowledging a degree of circularity in this design, it's important to note that the individual and comparative analyses of the two classes did more than merely confirm the suitability of class selection. At the outset, similarities and differences of the two classes (data collection phase) could be broadly described. But they only achieved their full definition through detailed analysis of each classroom microculture (data analysis phase). In other words, the selection of classes was based on the product of teaching practices (i.e., what more or less successful student-centered instruction looked like overall). In contrast, the extensive analyses of classes detailed the processes of how unequally successful teaching practices were actually constituted. Detailed descriptive accounts were provided of teachers’ approaches to creating student-centered classroom microcultures, and students’ approaches to participating in such classrooms. The focus on the classroom process is necessary to explore how teachers understand reform ideals and transform them into their specific classroom contexts (Carpenter, Franke, & Levi, 1998; Cobb & Bauersfeld, 1995; Fennema & Nelson, 1997).

Another argument against a simple circularity in the research design was the retheorization of sociomathematical norms. The analyses of classroom teaching

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processes illustrated the importance of sociomathematical norms as reflecting the quality of students' mathematical engagement and anticipating their learning opportunities. This re-newed importance urged me to broaden the scope of the theoretical construct to capture the insight emerging from this study. This indicates that the process of identification of unequally successful teaching practices was also a part of definition, not confirmation.

The second issue to be addressed is the ethics of identifying unequally successful teaching practices without reference to "objective" criteria like students' test scores, cognitive analyses of their abilities, and the like. First of all, it should be emphasized that the focus of the comparison in this study was on the classroom microculture as a whole, not the individual teacher or students. Indeed, the analysis was extensively grounded in the patterns of classroom discussion and participation. Now, students' learning opportunity within such a learning environment was highlighted as a fundamental criterion to assess the quality of classroom mathematics teaching. However, learning opportunity was analyzed as a function of the students' as well as the teacher's participation patterns. This is a direct implication of the emergent perspective employed in this study, which sees classroom dynamics as reflexively related to all of the participants (Cobb & Yackel, 1996; Voigt, 1995).

That said, it is widely recognized that the teacher does have a special role in developing the elementary mathematics classroom microculture (Ball, 1993; McClain, 1995; Yackel, 1995). At their young age, the students may not have much experience to use as a basis for their own independent ideas about what it means for mathematics to make sense. Thus, the teacher's views of mathematical sense making, as she enacted
them in her classroom, would have a great influence on the nature of students’ mathematical engagement. So this research does make indirect judgments as to the relative effectiveness of the teachers’ approaches. Indeed, the major application of this research project is taken to reside in its implications for teaching.

How can the decision to render judgments as to teaching efficacy be defended? The reform movement has reached a critical juncture at which increasingly large numbers of mathematics teachers are identifying themselves as reform educators (Knapp, 1997; NCES, 1996, 1997; Stigler & Hiebert, 1998). As mentioned above, however, there has been a growing concern that many teachers do not quite grasp the vision of the current reform ideals (Burrill, 1997; Hiebert et al., 1996; RAC, 1997). Given this challenge of reform, I felt that it is necessary to make indirect evaluations of effectiveness of reform-oriented teaching practices. If mathematics educators can say in general terms that not all teachers who subscribe to reform movement fully understand its implications, then it is necessary to study specific teachers and specific classrooms to better understand the breakdown between teachers’ adoption of reform objectives, and their successful incorporation of reform ideals. The comparison of unequally successful reform-oriented classes in this study provides a unique contribution in articulating subtle but crucial issues and obstacles that may be indicative of generic problems of reform.
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APPENDIX

INTERVIEW QUESTIONS

INTRODUCTION

Thank you for giving me your valuable time. I would love to listen your life as a teacher. You have taught elementary school students for a long time. You have much expertise in teaching. Today, I would like to listen your stories with regard to teaching, such as what you have done, how you have taught, what difficulties you have encountered, etc. in order to better understand how you have become the excellent teacher that you are continuing to evolve into.

Please, feel free to say whatever you would like to say. I will be very happy if you regard me as an apprentice, practicing teacher, rather than an interviewer. Indeed, I graduated from the same kind of university to be an elementary school teacher, and am still deeply interested in how teachers construct their teaching practice. I know that this interview will take up lots of time. But I hope it will be rewarding for you by giving us the opportunity to think through the foundation of your teaching approach together. The prepared questions are based on my understanding of schools and teaching. So, there might be some inappropriate questions in the United States contexts. Please, correct me or paraphrase as needed.

PART 1. EARLY INFLUENCES ON BECOMING A TEACHER

To begin can you tell me about your elementary school experience? Where did you grow up? At what age did you begin school? Do you remember your first day of school? [e.g., Did you enjoy elementary school?, Do you remember teachers from the elementary school days?, Were there any experience, good or bad, which makes you remember to this day?, Did you ever think of becoming a teacher yourself at the time?]

Next, please tell me about your family. Are members of your family professionals like yourself? Did they encourage you to work hard in school? (Did they have expectations that you would become a professional?) Would you say that they encouraged you to become a teacher, or did they discourage you?

PART 2. THE DECISION TO BECOME A TEACHER

Tell me a little bit about how you finally decided to go into teaching. Can you remember the actual moment you reached your decision? Was it a difficult decision? Was teaching your first choice of career? What things appealed to you about it, and what reservations did you have at the time? Did you have an image of what kind of teacher you wanted (or expected) to be?
POSSIBLE INFLUENCES TO EXPLORE

Intrinsic interest (e.g., love working with kids)
Extrinsic interest (e.g., status, wages)
Commitment (e.g., importance of role)
Opportunities (e.g., accepted to the university, not skilled in other areas)
Strategic concerns (e.g., likelihood of employment, desire for a family)
Advice from influential others (e.g., parents, teachers, friends)

PART 3: THE TEACHER EDUCATION YEARS

EXPECTATIONS

Tell me a bit about your university experiences in becoming a teacher. What was it like to be an education student? Where did you go to school? How did you decide on this school? What were your expectations for your teacher training experiences? Did you expect to enjoy the program? Did you feel that you already had a good idea about what elementary school teaching is like? Did you feel that you were going to learn some skills to be more effectively? Or did you feel that you were starting at the ground level, and were going to rebuild your conceptions about teaching anew?

SOCIALIZATION

How do students relate together in a teacher training program? Did you have friends with whom you studied? (If so, what was the nature of your discussions?) Did you usually work on the specific assignments given by yourself, or did you also discuss personal interpretations of what becoming a teacher means? Do you think your experience was typical of other students? How did you feel about your program of studies, your professors, etc. Did you think that they really had all the answers, or did you compare what they were doing with your own ideals/expectations?

INTELLECTUAL INFLUENCES

Let’s think substantively about your courses and experiences in your student teaching. What was the nature of the messages you were getting from your professors? Would you say they focused on practical matters, or on philosophical matters? Which of these aspects interested you most? How did you decide to become a second grade teacher?

PRACTICUM INFLUENCES

The practicum or student teaching experiences are an important part of the teacher training program. What was your learning experience as a practicing teacher? What sort of exposure at the classroom did you get during your program? Do you remember your supervising teacher(s)? Tell me about their styles of teaching. How did
this compare to your own ideas at the time? How did you prepare your lessons at that
time? Did you feel free to develop your own style, or were expected to do things the way
your supervising teacher did them? How did the messages you were getting about
teaching at the school site compare with the messages you were getting at the university?
If they were inconsistent, did you experience this as a conflict, or did you just do the
academic things at the university and the practical things at the school?

PART 4: EARLY MATHEMATICS INTERESTS

As you know, my special interest is mathematics. Was teaching mathematics one
of the things you looked forward to in your career; or was it neutral, or a bit of a
negative? Why?

When you were students, what was your favorite subject? What sort of
mathematics student were you? Did you enjoy mathematics, any way? Did you want to
teach in a way similar to how you were taught? Or did you think you would teach it
differently to make it more enjoyable for your students than it was for you?

PART 5: EARLY TEACHING EXPERIENCES

Tell me about your first year(s) teaching. When and where did you start? Tell me
what you remember about the teaching approaches you used to start off with? Were they
similar to what you had expected when you first decided to go into teaching? Were they
similar to what you had expected when you were in your teacher education program?
Did you feel that finally had the opportunity to teach the way you wanted to? or did you
mostly focus on fitting into the way things were done at that school? Did your initial
successes/failures cause you to reconsider the approaches you were using?

PART 6: CAREER PATH

In Korea, teachers rotate school to school every several years. What schools have
you taught at? What schools have been the most enjoyable/successful for you? Why?
Have you been in any special elementary school? Could you describe the characteristics
of schools you taught at? Is the current school somehow different from other elementary
schools? What were pros and cons to being a teacher in a special elementary schools?

PART 7: INFLUENCE OF PEERS WITHIN THE SCHOOL

As you began your teaching, what sorts of relationships did you have with your
fellow teachers? Would you say you are close friends with some of the people you taught
with; casual acquaintances? Did you discuss your problems or thoughts on a casual
basis? Or are you pretty much independent? Are you typical of the United States
teachers in this regard? If you wanted to try something new, would you feel that you
should consult your colleagues, or would you just go ahead and do your own thing? Do
you think colleagues might disapprove of certain ways of teaching, if you decided to
implement them? As you became an experienced excellent teacher, was there some difference in relationships with your colleagues?

PART 8: INFLUENCE OF ADMINISTRATORS

As you began your teaching, and even as it has continued, what sorts of relationships did you have with your schools' or districts' administrators? Do you discuss your problems or thoughts with them? Do they observe you periodically and make suggestions? If so, is their advice often useful, or do you feel they're out of step with what you're trying to do in the classroom? If you wanted to try something new, would you consult your administrators? Are there pressures, overt or covert, to teach in a particular way? From your experience, do you think that teachers have autonomy in developing their own teaching methods?

In Korea, schools have collective supervision either at the request of the supervisor or as part of the school's policy. Is there similar concept of collective supervision here? (One teacher teaches publicly and observers discuss pros and cons of the demonstrative teaching with the teacher.) Did you teach mathematics publicly before? What was the experience like? Did you use your typical methods, or did you perform more in the way you think they might have expected? How did you evaluate your teaching at that time? What kinds of comments have you gotten from others (including colleagues, supervisors, and principals?)

PART 9: PROFESSIONAL DEVELOPMENT

In Korea, elementary school teachers are supposed to get a retraining course in a regular basis. Is there something like this in the United States? Could you describe how the United States elementary school teachers are supposed to develop professionally?

Have you taken other courses/degrees since you first started teaching? What motivated you? (e.g., increased status/pay/opportunity, desire for self-improvement, to prevent burnout, academic interests) Have these experiences been positive experiences for you? Did they meet your expectations? Have they caused significant changes in the ways you teach? Why or why not?

Have you participated in any math conferences/workshops/seminars? What was experience like? What motivated you? Did they fulfill your objectives? Was such experience helpful in your math teaching? How?

PART 10: PROFESSIONAL SELF-DEVELOPMENT

Have your teaching approaches changed in some definite ways, or would you say they've just evolved as you've gained more experience? Could you tell me about these changes. Much of the academic and government structures for teachers are designed to
tell teachers how they should teach. Are there some respects in which you have taken responsibility for deciding your own directions for change?

In Korea, some schools have their own local math circles, and teachers discuss their methods. Teachers who teach at the same grade level have a meeting once per week. The purpose of the meeting is to share various teaching experience among teachers, as well as cross-checking what they cover in math classes. Is there something similar here? If so, how does it work? Are the directions set by teachers, or are there outside influences? Are (voluntary) contacts with other teachers influential in developing your teaching?

In what directions have you tried to develop your math teaching methods? What has influenced you to make such changes? In what ways have you been successful or unsuccessful? What challenges do you face in these changes?

PART 11: MATHEMATICS TEACHING

Is mathematics one of your most (or least) favorite subjects to teach? Why? Do you consider it an especially important subject? Why? How has mathematics figured into your general changes in teaching? Has it just been one subject among many that have changed or has it led the way? Why?

PART 12: REFORM MOVEMENT; EDUCATIONAL POLICIES

Throughout your teaching career, you must have been through various reform movement in the United States. I read about reform movement in the United States mainly by reform documents and papers. I would like to listen to the teacher's voice. Could you describe anything about reform movement as you have experienced? In what ways have you been familiar to reform recommendations? (If any) How have the reform ideals influenced in developing your teaching methods? Specifically, I am interested in the current mathematics education reform movement in the United States. How do you feel about the NCTM standards, or more recently the proposal? Are they sensible? Are they realistic?

Now, if you can identify, what was the most influential factors that significantly changed your math teaching approaches?
JeongSuk Pang, the youngest daughter of GyunHwan Pang and SunHee Hwang, was born in Puyo, South Korea, on December 23, 1971. She later moved to Kong-Ju and graduated from the Attached High School of Kong-Ju College of Education in 1990. She then majored in elementary education with a special focus on mathematics at Seoul National University of Education. She received her bachelor of arts degree in 1994 graduating with the highest academic honor.

JeongSuk continued her study in elementary mathematics education in graduate school at the Korea National University of Education. She received her master of education degree in 1996. During the 1995-1996 academic year, her major advisor, Dr. Pyung Kook Jeon, was a visiting professor at Louisiana State University. He facilitated JeongSuk's decision to come to the United States.

JeongSuk started her doctoral studies in elementary mathematics education in the Department of Curriculum and Instruction at Louisiana State University in the spring of 1996. She held a research assistantship in the department for four years. JeongSuk will be awarded the degree of Doctor of Philosophy in May, 2000.
DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: JeongSuk Pang

Major Field: Curriculum and Instruction

Title of Dissertation: Sociomathematical Norms of Elementary School Classrooms: Crossnational Perspectives on the Reform of Mathematics Teaching

Approved:

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Major Professor and Chairman

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EXAMINING COMMITTEE:

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Date of Examination:

March 13, 2000