1998

Stress and Strength of Laminated Composite Containing a Circular Hole.

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STRESS AND STRENGTH OF LAMINATED COMPOSITE CONTAINING A CIRCULAR HOLE

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

The Department of Mechanical Engineering

by

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December 1998
ACKNOWLEDGMENTS

The author wishes to express his sincere gratitude to his major professor, Dr. Su-Seng Pang, for his invaluable guidance and continued support throughout this research. It is with his encouragement that this study is completed. The author would also like to thank Dr. George Z. Voyiadjis, his minor advisor and Boyd Professor of Civil Engineering, Dr. Medhy Sabbaghian, Chevron Professor and Chairman of Mechanical Engineering Department, Dr. John R. Collier, Ike East Professor of Chemical Engineering and the president of Faculty Senate of LSU, Dr. Wanjun Wang of Mechanical Engineering, and Dr. Michael Tom of Mathematics Department, for their valuable advices and comments.

Mr. Richard H. Lea, the president of Specialty Plastics, Inc., has continuously given the author inspiration and support which are essential for the author to complete his Ph.D. degree. The author would also like to extend his thanks to Dr. Sam Ibekwe of Mechanical Engineering Department at Southern University and Dr. Guoqiang Li, a Research Associate in Mechanical Engineering at LSU, for their assistance in conducting experimental work. A special thanks is owed to Dr. Xiaohua Lu of Mechanical Engineering Department at University of Houston for the instrumental demonstration during the author’s two visits to his laboratory.

This research was made possible through funding by the following agencies:

- National Institute of Standards and Technology (NIST) through Advanced Technology Program (ATP)
- NASA/Marshal Space Flight Center through Partnership Program
- Louisiana Board of Regents and Specialty Plastics, Inc.
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ABSTRACT

This dissertation is to study the stress and strength of a finite laminated plate containing a circular hole subjected to tension. Efforts were made in three aspects: theoretical modeling, computer simulation using finite element method, and experimental evaluation using unidirectional and crossply Scotchply laminates. The stress concentration factor, normal stress distribution and tensile strength of laminates with various hole sizes and gage lengths were investigated.

Theoretically, two models were proposed to describe the normal stress distribution of a laminate with finite width and finite length. By defining a width correction function and a size correction function, the normal stress in a laminate with finite width and finite size was predicted in terms of the normal stress in an infinite laminate. The introduction of width correction function and size correction function was based upon the elastic solutions of finite and infinite isotropic plates containing a circular hole. Experimental work was performed on laminated specimens made of unidirectional and crossply Scotchply laminates. Width effect, or hole size effect, was studied using 2" wide specimens with hole size varying from 0.1" to 1.5" (the corresponding d/W ratio from 0.05 to 0.75). Specimens with different lengths were tested. Both initial tensile strength and ultimate (rupture) strength were obtained for specimens with various hole sizes and gage lengths. Strain gages were also used in several specimens to check normal stress distribution. Finite element method was extensively used in the stress analysis of the unidirectional and crossply laminates in this study. COSMOS software was utilized which has the capability to model laminated composites. Composite quadrilateral plate elements
were used for the laminates. A total of 29 cases were analyzed to obtain the stress concentration factor, normal stress distribution, and initial tensile strength for laminates with various hole sizes and lengths.

The results of this study showed that the proposed models provide better stress and strength predictions than existing models. The predictions from the proposed models agree well with the experimental data and results from the finite element analysis.
CHAPTER 1. INTRODUCTION

The studies of laminated theory have advanced to the point where the failure criteria have been fully established. Criteria like Tsai-Hill and Tsai-Wu are well accepted in the failure analysis of laminated composites. On the other hand, however, the applications of these failure criteria are still limited to academic and research purposes but rarely to the engineering design and analysis practices. The two major barriers are: it is inconvenient to determine the material failure properties required in applying these failure criteria, and most importantly, there are simply no analytical expressions of either stress or strain components available for this purpose. By realizing this limitation, extensive researches have been performed during the past two decades on the notched strength of composite laminates containing intentionally induced circular holes or cracks. The purpose is to develop strength prediction models or criteria which can be used to evaluate the failure of some simple but commonly used structural components, such as a laminate containing holes or cracks subjected to tension or bending.

As a result, several analytical models have been developed so far including the linear elastic fracture mechanics model (LEFM), equivalent stress models (such as point stress model and average stress model), damage zone model, and progressive failure model. Of these models, the LEFM model and equivalent stress models drew more attention than other models. The early studies using LEFM approach were conducted by many researchers including Waddoups, et al. (1971), Whiteside, et al. (1973), and Cruse (1973). But there is a difference of opinion on whether LEFM is applicable to composite containing discontinuities and results so far indicate that LEFM can only be applied in limited case.
The equivalent stress approach was first proposed by Whitney and Nuismer (1974) for predicting the notched strength of laminate composites under tensile loading and was then extended for different applications by many other researchers (such as Karlak, 1977; Nuismer and Labor, 1979; Pipes, et. al., 1979 & 1980; Tan, 1987, 1988, 1989 & 1994; El-Zein and Reifsnider, 1990, Bradshaw and Pang, 1991; Yao, 1992; Zhao et al., 1995 and 1996; and Keith and Kedward, 1997).

For the case of orthotropic laminates containing a circular opening or a straight crack under normal loads, Whitney and Nuismer (1974) proposed two failure models based on the maximum stress failure criterion: Point Stress Criterion (PSC) and Average Stress Criterion (ASC). These models assumed that the strength of a notched orthotropic plate was controlled by the normal stress adjacent to the circular hole. To represent the point stress and average stress status, the characteristic parameters $a_o$ and $d_o$ were introduced, both denoting the distance from the hole edge. Konish and Whitney (1975) developed an approximate solution in the form of a polynomial to describe the normal stress distribution adjacent to a circular hole in an infinite orthotropic plate. This approximate solution is based upon the overall laminate properties and is useful in assessing the notched strength of an orthotropic plate containing a circular hole. Pipes et al. (1979 & 1980) proposed a three parameter model for predicting the strength of the composite laminate containing a circular hole under tension. By introducing radius shift parameters and notch sensitivity factors, the influence of the stacking sequence was considered in this strength model.

First ply failure criterion (FPFC) was proposed by Tan (1987) for predicting the notched strength of a laminated plate containing an elliptical opening under in-plane...
loadings. Two models were developed for two types of in-plane loadings: uniaxial tension and combined loadings. Both models were based on point stress and average stress failure criteria. Stresses were evaluated on the basis of each ply rather than on overall laminate, and a quadratic failure criterion was applied for better prediction. The first ply failure model assumed that when the stress at a distance $d_0$ from the hole edge in any ply reached the unnotched first ply failure strength, the notched plate was considered to be failed. It should be noted that in Tan's models the failure criterion can be applied along an elliptical path which is at a distance $d_a$ from the edge of the elliptical hole. An approximate stress analysis was then performed by Tan (1988) to obtain the normal stress distribution of an orthotropic laminate containing an elliptical hole under normal tension. This stress distribution was assumed to be a polynomial function in addition to the isotropic solution. El-Zein and Reifsnider (1990) studied the notched strength of unidirectional off-axis, angle ply, and $[0/\pm\theta]$, laminates containing a circular hole under tensile loading. Both point stress and average stress criteria were employed, and average stress criterion was also applied on the ply level. Their model provided good predictions for off-axis and $[0/\pm\theta]$, laminates but not for angle-ply laminates because of the diversity of the failure modes.

Despite some critics on point stress criterion and average stress criterion, these two models along with the modified models proposed by Tan have been generally accepted in engineering design applications, especially in aerospace industry. In order to better understand the background of this research topic, a description of the equivalent stress models, along with some basic theories and/or models which are directly related to this study is discussed in more detail in the next chapter.
CHAPTER 2. EXISTING THEORIES AND MODELS

2.1 Equivalent Stress Models

2.1.1 Point Stress Criterion

The point stress failure criterion (PSC) was first proposed by Whitney and Nuismer in 1974 for the case of an orthotropic plate with a circular hole under normal loads, as shown in Fig. 1. This criterion states that when the normal stress at some distance $d_o$ from the hole edge, $\sigma_x(0, R+d_o)$, reaches the failure strength of an unnotched plate, $\sigma_f$, failure will occur in the notched plate. That is

$$\sigma_x(0, R+d_o) = \sigma_f \quad (1)$$

Whitney and Nuismer claimed that the parameter $d_o$, which is called the characteristic length, is a material constant which can be determined experimentally. And they also claimed that this criterion can predict the hole size effect on the notched strength of the laminate since the stress concentration factors at the point $(0, R+d_o)$ are different for different hole sizes. In the original point stress and average criteria proposed by Whitney and Nuismer, the stress distribution for an isotropic plate was actually used to simulate the orthotropic plate. To see how to determine the "characteristic length" $d_o$, consider an infinite isotropic plate containing a central circular hole. The normal stress along the $y$ axis passing through the hole center is (Timoshenko and Goodier, 1979):

$$\sigma_x = \sigma_0 \left[ 1 + \frac{1}{2} \left( \frac{R}{y} \right)^2 + \frac{3}{2} \left( \frac{R}{y} \right)^4 \right] \quad (2)$$
Applying point stress criterion, we have

\[ \frac{\sigma_f^N}{2} \left( 2 + \left( \frac{R}{R + d_0} \right)^2 + 3 \left( \frac{R}{R + d_0} \right)^4 \right) = \sigma_f^U \]  

or

\[ \frac{\sigma_f^N}{\sigma_f^U} = \frac{2}{2 + \xi_1^2 + 3\xi_1^4} \]  

where

\[ \xi_1 = \frac{R}{R + d_0} \]

The "characteristic length" \( d_0 \) can be solved from the above equations once the failure stress for both notched and unnotched laminates are determined.
2.1.2 Average Stress Criterion

When the average stress of the region from the hole edge to some distance \( a_0 \) from the hole edge reaches the strength of the unnotched plate, \( \sigma_f \), the failure occurs in the notched plate. This is the so-called average stress failure criterion (ASC) which was proposed by Whitney and Nuismer (1975). This criterion can be expressed as

\[
\frac{1}{a_0} \int_0^{a_0} \sigma_x(0,y) dy = \sigma_f
\]

Again for an infinite isotropic plate containing a circular hole of radius \( R \),

\[
\frac{\sigma_f^N}{2a_0} \int_0^a \left[ 2 + \left( \frac{R}{y} \right)^2 + 3 \left( \frac{R}{y} \right)^4 \right] dy = \sigma_f^U
\]

or

\[
\frac{\sigma_f^N}{\sigma_f^U} = \frac{2(1-\xi_2)}{2-\xi_2^2-\xi_2^4}
\]

where

\[
\xi_2 = \frac{R}{R+a_0}
\]

As in the case of point stress criterion, the parameter \( a_0 \) was treated as a material constant which is determined experimentally.
2.1.3 Modified Point Stress Model by Karlak

In Whitney and Nuismer's point stress criterion it is assumed that the characteristic distances are material parameters and are independent of the hole size. Karlak (1977) conducted tensile tests on graphite/epoxy laminates containing circular holes of various sizes. Based on his test data, Karlak concluded that the characteristic length is not a material constant but related to the square-root of the hole radius. Upon this observation, Karlak proposed a Modified Point Stress Criterion which incorporates the relationship between the hole radius, \( R \), and the characteristic length, \( d_0 \), into the notched strength formula. Using the expression derived from the original Point Stress Criterion for the notched strength of laminates containing a circular hole:

\[
\frac{(\sigma_f^N)^e}{\sigma_f^U} = \frac{2}{2 + \xi_1^2 + 3\xi_1^4} \tag{8}
\]

Solving for \( d_0 \):

\[
d_0 = \frac{\sqrt{6R}}{\sqrt{1 + \left[1 - 24 \left(1 - \frac{\sigma_f^U}{(\sigma_f^N)^e}\right)^{1/3}\right]} - R} \tag{9}
\]

the value of \( d_0 \) can be calculated from the test results of notched and unnotched strength values for given values of \( R \). Karlak plotted his data and the data by Whitney and Kim (1977) and Nuismer and Whitney (1975) for graphite/epoxy laminates containing holes of different sizes on a \((\sigma_f^N)/\sigma_f^U\) vs. \( d_0 \) format where \( d_0 \) is calculated from Eq. (9). Curve fitting these data with

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\[ d_0 = k_0 \sqrt{R} \]  

(10)

Karlak showed that the best fit was obtained by the following relationships:

\[
\begin{align*}
  d_0 &= 0.15\sqrt{R} \quad \text{for} [±45/0/90]_s \\
  d_0 &= 0.11\sqrt{R} \quad \text{for} [90/0/±45]_s
\end{align*}
\]

where the units for \( d_0 \) and \( R \) are both inches. Rewriting Eq. (8) using \( d_0 = k_0 R^{1/2} \) yields the strength formula based on Karlak's model:

\[
\frac{(\sigma^N_f)^\infty}{\sigma_f^U} = \frac{2}{2 + \left[ 1 + \frac{k_0}{\sqrt{R}} \right]^2 + 3 \left[ 1 + \frac{k_0}{\sqrt{R}} \right]^4}
\]

(11)

The underlying assumption in the Karlak's model is that the characteristic length \( d_0 \) is related to the hole radius according to Eq. (10). The parameter \( k_0 \) depends on laminate configuration and material properties and therefore must be experimentally determined.

2.1.4 Three-Parameter Model

The modification of Whitney and Nuismer's point stress criterion proposed by Karlak (1977), which incorporates the dependence between the characteristic length, \( d_0 \), and the hole radius, \( R \), was further extended by Pipes, Wetherhold and Gillespie (1979 and 1980). Two separate models were proposed to predict the notched strength of a composite laminate containing circular holes (1979) and straight cracks (1980) which can be applied to any multi-directional orthotropic laminate. Instead of using stress expression for infinite isotropic plate as Whitney and Nuismer did, the approximate stress distribution of an
infinite orthotropic laminate along the \( x = 0 \) section (Konish & Whitney, 1975), which is perpendicular to the loading direction, was used:

\[
\sigma_x(0, y) = \frac{\sigma}{2} \left\{ 2 + \left( \frac{R}{y} \right)^2 + \left( \frac{R}{y} \right)^4 - (K_T^\infty - 3) \left[ 5 \left( \frac{R}{y} \right)^6 - 7 \left( \frac{R}{y} \right)^8 \right] \right\}
\]  

(12)

where \( K_T^\infty \) denotes the stress concentration factor at the edge of the hole:

\[
K_T^\infty = 1 + \sqrt{\frac{2}{A_{22}}} \left[ \sqrt{A_{11}A_{22} - A_{12}^2} + \frac{A_{11}A_{22} - A_{12}^2}{2A_{66}} \right]
\]

(13)

and \( A_{ij}, i,j = 1,2,6, \) are the components of extensional stiffness matrix:

\[
A_{ij} = \sum_{k=1}^{N} \left( \overline{Q}_v \right)_k \left[ h_k - h_{k-1} \right]
\]

where

\[
\overline{Q}_{11} = \Omega_{11}m^4 + 2(\Omega_{12} + 2\Omega_{66})m^2n^2 + \Omega_{22}n^4
\]

\[
\overline{Q}_{12} = (\Omega_{11} + \Omega_{22} - 4\Omega_{66})m^2n^2 + \Omega_{12}(m^4 + n^4)
\]

\[
\overline{Q}_{16} = -mn\Omega_{22} + m^3n\Omega_{11} - mn(m^2 - n^2)(\Omega_{12} + 2\Omega_{66})
\]

\[
\overline{Q}_{22} = \Omega_{11}n^4 + 2(\Omega_{12} + 2\Omega_{66})m^2n^2 + \Omega_{22}m^4
\]

\[
\overline{Q}_{26} = -mn\Omega_{22} + mn\Omega_{11} + mn(m^2 - n^2)(\Omega_{12} + 2\Omega_{66})
\]

\[
\overline{Q}_{66} = (\Omega_{11} + \Omega_{22} - 2\Omega_{12})m^2n^2 + \Omega_{66}(m^2 - n^2)^2
\]

in which \( m = \cos \theta, n = \sin \theta, \) and \( \theta \) is the angle between the \( x \)-axis and the fiber direction, and \( \Omega_{ij} \) can be related to the engineering constants of the laminate by.
Based upon Karlak, the characteristic dimension $d_0$ is taken to be a function of the hole radius, i.e., Eq. (10). It was assumed that

$$d_0 \propto R^m$$

(14)

and by introducing a new parameter, $c$, defined as notch sensitivity factor, the following form was given:

$$d_0 = (R / R_0) / c$$

(15)

where $R_0$ is a reference notch radius that is introduced so that the quotient in parentheses becomes non-dimensional. Choosing $R_0 = 1$" for algebraic simplicity and applying point stress criterion together with assumption (15), the notched strength was obtained as

$$\frac{(\sigma_f^N)^\infty}{\sigma_f^U} = \frac{2}{2 + \lambda^2 + 3\lambda^4 - (K^\infty_T - 3)(5\lambda^6 - 7\lambda^8)}$$

(16)

where

$$\lambda = \frac{1}{1 + \frac{R^{m-1}}{R_0^m c^{-1}}}$$

(17)
Once the parameters $m$ and $c$ as well as the unnotched strength $\sigma_i^u$ are determined experimentally, the strength of notched laminate can be calculated from the expression. This model is called "modified Whitney" model and it is a three-parameter model, i.e., the unnotched strength $\sigma_i^u$, the notch sensitivity factor $c$, and the exponential parameter $m$.

The exponential parameter $m$ is bounded between $0 < m < 1$. When $m = 0$, the Whitney and Nuismer's point stress criterion recovered; while at $m = 1$, the notched strength is independent of notch radius. It should be pointed out that to experimentally determine the mentioned three parameters in this model, three tensile specimens have to be used: an unnotched specimen, and two notched specimens with two different hole sizes. In comparison, only two tensile specimens are required in Whitney and Nuismer's original model. It was found that the arbitrary selection of $R_0$ affects the notch sensitivity factor. For example, when $m$ is held constant, decreasing $R_0$ yields a smaller notch sensitivity prediction. In other words, different values of $R_0$ will result in different values of parameters $m$ and $c$.

### 2.1.5 Stress Field Intensity Model

The stress field intensity model (Yao, 1992) defines a normalized effective stress parameter, the field intensity function, $f_{RD}$, over the region near the notch, instead of at a point:

$$f_{RD} = \frac{1}{V} \int_D \phi(\tilde{r}) \psi(\tilde{r}) dV$$

(18)

where $D$ is the region and $V$ is the volume of $D$. $\phi(\tilde{r})$ is a weight function and $f(\tilde{r})$ is the failure surface. For a plane problem, Eq. (18) can be written as
\[ f_{RC} = \frac{1}{S} \int_{D} f(\theta) \varphi(r) \, ds \]  

where \( S \) is the area of region \( D \). Theoretically, the function \( f(\theta) \) can be obtained by combining the failure criteria and laminated plate theory. Some features of this model include: (1) It considers the multi-axial stresses near the notch; (2) It considers the stress gradient near the notch, and when the theoretical stress concentration factor, \( K_T \), is reduced to 1 \( (\varphi(r) = 1) \), \( f_{RD} \) is the normalized effective stress of unnotched laminates; and (3) \( D \) is related to the damage zone to some extent. It can be seen that this mode is a more general form of PSC and ASC. It also makes more physical sense because it considers a finite damage in front of the hole instead of just a single point or a finite linear length. However, except for a few special cases which reduces it to either PSC, ASC, or LEFM, no other problems have been solved due to the difficulties in obtaining function \( f(\theta) \).

2.1.6 Extended Point Stress Criterion and Average Stress Criterion

For an isotropic laminate under in-plane tension, the normal stress is uniformly distributed across the laminate thickness. But when the laminate is subjected to a bending load, the stress varies in the thickness direction. Take a crossply laminate as an example. Under the in-plane tensile load, all of the 0° plies are in the same stress level, and rest all the 90° plies are also in the same stress level. If the stress in one 0° ply (or 90° ply) reaches the failure strength, all of the 0° plies (or 90° plies) will fail simultaneously. But for the cross-ply laminates under bending, each 0° ply or 90° ply is at different stress level. The stress at one point or the average stress over a length within on ply may not represent
the stress level or stress concentration of the laminate. Therefore, the original Point Stress Criterion and Average Stress Criterion, both were developed for the case of tension, may not be directly applied to the case of bending. It is more reasonable to include the stress variation across the laminate thickness in the failure models. By this consideration, some modifications were made by Zhao et al. (1995 and 1996) to the two criteria to evaluate the notched strength of laminates under bending.

The first one is the Extended Point Stress Failure Model. This model extends Whitney and Nuismer's point stress model by defining the "point stress" as the average stress of the plies with same fiber angle across the laminate thickness. For the laminates other than on-axis unidirectional or cross-ply, the maximum stresses usually will not occur along a radial line parallel to the applied bending moment but at some angles. Thus the modified point stress criterion should be applied along a path which encloses the elliptical opening. The extended Point Stress Criterion becomes

\[
\frac{2}{h_a} \int_0^{h/2} (\sigma_{\alpha})_{\alpha} \sqrt{\frac{x^2}{(a+b)^2} + \frac{y^2}{(b-a)^2}} \ dz = \sigma_f
\]  

(20)

where \( \sigma_\alpha \) is the stress of the ply with fiber angle \( \alpha \), \( h_a \) is the total thickness of the plies with fiber angle \( \alpha \), \( h \) is the total thickness of the laminate, \( \sigma_f \) is the failure strength of the material along or perpendicular to the fiber direction, and \( \alpha_i \) is a characteristic parameter to evaluate the failure strength of the laminate.

The other model is the Extended Average Stress Failure Model. In the case of in-plane loadings, the average stress criterion includes the contribution of the stress
distribution along the radial direction in the vicinity of the hole edge. For notched laminates under bending, the contribution of stress distribution along the thickness direction should also be considered. This modified model can be written as

$$\frac{2}{a_2 h} \int_0^{h/2} \int_0^{R-a_2 h/2} \sigma_a(\theta, r, z) dr dz = \sigma_f$$

where $R$ is the radial distance from the edge to the center of the hole, $\theta$ is the location of the section where maximum stress occurs, and $a_2$ is another characteristic parameter to measure the failure strength of the laminates under bending.

The determination of parameters $a_1$ and $a_2$ requires the stress field information as well as the failure load which is normally determined experimentally. Since a theoretical solution of stress distribution to the problem is not available, finite element method can be used to provide the stress information. A detailed description of this work is included in "Appendix A. Bending Strength Prediction – A Preliminary Model."

2.2 Finite Width Effect

In section 2.1, we discussed the two well-known notched strength evaluation criteria, point stress criterion and average stress criterion proposed by Whitney and Nuismer (1974), and several modified criteria and/or models. Despite the different assumptions and treatments involved in these models, there are two things that are in common: (1) A stress expression for infinite plate is used; and (2) Various parameters are determined experimentally using finite width tensile specimens. However, rarely in engineering practice can a plate be assumed to be of unlimited size when investigating the strength
weakened by a circular hole. In most cases the dimensions of such plates are small. Therefore, it is necessary to find out the effect finite width on the stress and strength of the plate containing a circular.

2.2.1 Isotropic Plate Containing a Circular Hole

The case of a circular hole in an infinite isotropic plate subject to uniaxial tension was first solved a hundred years ago by Kirsh (1898) who found the stress concentration factor is 3. The stress distribution along the cross-section passing through the hole center and normal to the loading direction can be written as (Timoshenko and Goodier, 1979):

\[ \sigma_x = \sigma_0 \left[ 1 + \frac{1}{2} \left( \frac{R}{y} \right)^2 + \frac{3}{2} \left( \frac{R}{y} \right)^4 \right] \quad (22) \]

where \( R \) is the radius of the hole and \( \sigma_0 \) is the applied tensile stress. This is actually the stress expression used by Whitney and Nuismer in their point stress criterion and average stress criterion (1974) for evaluating the strength of composite laminates.

For a finite width plate containing a circular hole subjected to uniaxial tension, there are many formulas available for calculation of the maximum stress around the hole. Among the researchers who made contributions to this topic are Howland (1930), Coker and Filon (1931), Wahl and Beeuwkes (1934), Sjostrom (1950), Heywood (1952), Isida (1953), Koiter (1957), Christiansen (1968), Belie (1972), Shilkrut and Ben-Gad (1985), etc. The best known results are Heywood’s empirical formula (1952) and the analytical work by Howland (1930).

Heywood’s formula is for calculating the stress concentration factors of a finite width plate with a circular hole of any sizes:

15
\[ K_g = \frac{2 + (1 - \frac{R}{w})^3}{1 - \frac{R}{w}} \]  

or

\[ K_n = 2 + \left(1 - \frac{R}{w}\right)^3 \]  

where \( K_g \) and \( K_n \) are stress concentration factors based on gross stress and nominal (net) stress, respectively.

Howland (1930) solved the problem of an infinitely long strip containing a central circular hole loaded with a uniform tension. By introducing dimensionless coordinates \( \xi = x/(W/2), \eta = y/(W/2) \), \( \rho = r/(W/2) \), and \( \lambda = R/(W/2) = d/W \), where \( a \) and \( b \) are radius of the hole and half width of the strip, respectively, the problem is reduced to determination of a bi-harmonic function \( U(\xi, \eta) \) which satisfies

\[ \nabla^4 U(\xi, \eta) = \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) U(\xi, \eta) = 0 \]  

and the boundary conditions at the contour \( \eta = \pm 1 \):

\[ \sigma_y = \frac{1}{b^2} \frac{\partial^2 U}{\partial \xi^2} = 0; \quad \tau_{xy} = -\frac{1}{b^2} \frac{\partial^2 U}{\partial \xi \partial \eta} = 0 \]  

at the hole edge \( r = a \), or \( \rho = \lambda \):

\[ \sigma_\rho = \frac{1}{b^2} \left( \frac{1}{\rho^2} \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial U}{\partial \rho} \right) = 0 \]  

\[ \tau_{\rho \theta} = -\frac{1}{b^2} \frac{\partial}{\partial \rho} \left( \frac{1}{\rho} \frac{\partial U}{\partial \theta} \right) = 0 \]  

and at infinity \( x = \xi b \rightarrow \infty \):
Howland developed the stress function $U(\xi, \eta)$ as a series:

$$U(\xi, \eta) = U_0^{(1)}(\xi, \eta) + U_0(\xi, \eta) + U_1(\xi, \eta) + U_2(\xi, \eta) + \ldots, \quad (29)$$

The terms of this series are bi-harmonic functions with the properties that the function $U_0(\xi, \eta)$ could be the solution of the problem for an infinitely large plane with a circular hole loaded with uniaxial tensile stress:

$$U^0(\rho, \theta) = \frac{b^2 \sigma_0}{4} \left[ \rho^2 - 2 \lambda^2 \ln \rho + (2 \lambda^2 - \frac{\lambda^4}{\rho^4} - \rho^2) \cos 2\theta \right] \quad (31)$$

For convenience this function is split into two parts:

$$U_0^{(1)} = \frac{b^2 \sigma_0}{4} \rho^2 (1 - \cos 2\theta)$$

$$U_0 = \frac{b^2 \sigma_0}{4} \left[ (2 \lambda^2 - \frac{\lambda^4}{\rho^4}) \cos 2\theta - 2 \lambda^2 \ln \rho \right]$$
The function \( U_1(\xi, \eta) \) of Eq. (29) nullifies the stresses due to the function \( U_2(\xi, \eta) \) at the strip boundary \( \eta = \pm 1 \) but introduces a stress function at the contour of the hole. Generally, the function \( U_{2n} + U_{2n+1} \) gives a zero stress at the boundary \( \eta = \pm 1 \) if the function \( U_{2n-1} + U_{2n} \) gives zero stress at the contour of the hole. The final solution Howland obtained is:

\[
\sigma_\rho / \sigma_0 = \frac{1}{2} (1 + \cos 2\theta) + 2m_0 + \frac{d_0}{\rho^2} + 2 \sum_{n=1}^{\infty} \left[ \frac{n(2n+1)d_{2n}}{\rho^{2n+2}} \right] \cos 2n\theta
\]

\[
\sigma_\theta / \sigma_0 = \frac{1}{2} (1 - \cos 2\theta) + 2m_0 + \frac{d_0}{\rho^2} - 2 \sum_{n=1}^{\infty} \left[ \frac{n(2n+1)d_{2n}}{\rho^{2n+2}} \right] \cos 2n\theta
\]

\[
\tau_{\rho\theta} / \sigma_0 = -\frac{1}{2} \sin 2\theta + 2 \sum_{n=1}^{\infty} \left[ n(2n-1)\left( \frac{l_{2n}\rho^{2n-2} - \frac{e_{2n}}{\rho^{2n}}} \right) + n(2n+1)\left( \frac{m_{2n}\rho^{2n} - \frac{d_{2n}}{\rho^{2n+2}}} \right) \right] \sin 2n\theta
\]

2.2.2 Orthotropic Laminates Containing a Circular Hole

Great efforts has been made by S.C. Tan to consider the finite width effect of laminated composites containing discontinuities (1994). Assuming that the normal stress profile for a finite plate is identical to that for an infinite plate except for an FWC factor, he used a finite-width correction (FWC) factor which is defined as a scale factor to multiply the notched infinite solution to obtain the notched finite width plate solution. Based on this assumption, the following relation is obtained:
\[
\frac{K_T}{K^\infty_T}\sigma^\infty_y(x,0) = \sigma_y(x,0) \tag{33}
\]

where \(K_T/K^\infty_T\) is the FWC factor, and \(K_T\) and \(K^\infty_T\) are stress concentration factor at the hole edge for a finite width plate and an infinite width plate, respectively. It should be noted that in Tan’s analysis the uniaxial tensile load is applied along the \(y\) direction. The normal stress along the \(x\)-axis for a laminate containing an elliptical hole is (Lekhnitskii, 1968; Tan, 1987):

\[
\sigma^\infty_x(0,y) = \sigma_x + \sigma_x \text{Re} \left\{ \frac{1}{\mu_1 - \mu_2} \left[ -\frac{-\mu_2(1-i\mu_1\lambda)}{\sqrt{\gamma^2 - 1 - \mu_1^2\lambda^2}(\gamma + \sqrt{\gamma^2 - 1 - \mu_1^2\lambda^2})} + \frac{\mu_1(1-i\mu_2\lambda)}{\sqrt{\gamma^2 - 1 - \mu_1^2\lambda^2}(\gamma + \sqrt{\gamma^2 - 1 - \mu_1^2\lambda^2})} \right] \right\} \tag{34}
\]

where \(\gamma = y/a\), \(\lambda = b/a\), and \(\mu_1\) and \(\mu_2\) are the solutions of the characteristic equation:

\[
a_{22}\mu^4 - 2a_{26}\mu^3 + (2\alpha_{12} + a_{66})\mu^2 - 2a_{16}\mu + a_{11} = 0 \tag{35}
\]

where \(a_{ij}, i,j = 1,2,6\), are the compliances of the laminate with 1 and 2 parallel and transverse to the loading direction, respectively. There are four roots from equation (35), but only two roots that have a positive imaginary part should be used. Using the equilibrium condition

\[
\frac{aK_T}{K^\infty_T} \int_{-\frac{W}{2a}}^{\frac{W}{2a}} \sigma^\infty_y(x,0)dy = a \int_{-\frac{W}{2a}}^{\frac{W}{2a}} \sigma_y(x,0)dy = \frac{W}{2}\sigma_0 \tag{36}
\]

yields

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\[
\frac{K_T^p}{K_T} = 1 - \frac{d}{w} + \text{Re}\left\{ \frac{1}{\mu_1 - \mu_2} \left[ \frac{\mu_2}{1 + i\mu_1\lambda} \left( 1 - \frac{2a}{w} - i\mu_1\lambda \left( \frac{2a}{w} \right) \right) \right. \right.
\]
\[
\left. - \sqrt{1 - \left( 1 + \mu_1^2 \lambda^2 \right) \left( \frac{2a}{w} \right)} \right) \left. - \frac{\mu_1}{1 + i\mu_2\lambda} \left( 1 - \frac{d}{w} - i\mu_2\lambda \left( \frac{2a}{w} \right) - \sqrt{1 - \left( 1 + \mu_2^2 \lambda \right) \left( \frac{2a}{w} \right)^2} \right) \right\} \right\} \quad (37)
\]

For an orthotropic laminate containing an elliptical hole, Tan derived the following expression:

\[
\frac{\sigma_y^p(x,0)}{\sigma_0} = \frac{\lambda^2}{(1-\lambda)^2} + \frac{(1-2\lambda)}{(1-\lambda)^2 \sqrt{\gamma^2 - 1 + \lambda^2}} + \frac{\lambda^2 \gamma}{(1-\lambda)(\gamma^2 - 1 + \lambda^2)^{3/2}}
\]
\[
- \frac{\lambda^7}{2} \left( K_T^p - \frac{2}{\lambda} \right) \left[ \frac{5\gamma}{(\gamma^2 - 1 + \lambda^2)^{7/2}} - \frac{7\lambda^2 \gamma}{(\gamma^2 - 1 + \lambda^2)^{9/2}} \right] \quad (38)
\]

Since no other literature but the work by Tan has been found on studying finite width effect of laminated composite, the comparison between the theoretical models proposed in this study and the existing models will be primarily based upon Tan's models.
CHAPTER 3. OBJECTIVES OF THE CURRENT RESEARCH

3.1 Motivation

Since laminated composites containing holes have numerous engineering applications, the strength models on this type of structural components are of particular interest. Currently, the strength models based on Whitney and Nuismer's equivalent stress models and Tan's modified models have not only been accepted by many researchers but also been applied to engineering design practice, especially in aerospace and the Air Force industry. On the other hand, there have been some questions concerning the assumptions and conclusions from these models. There is a need to clarify some of these questions and to develop some improved analysis models.

3.2 Questions Associated with Existing Models

3.2.1 Characteristic Lengths – Not Material Constants

The Point Stress Criterion and Average Stress Criterion use two length parameters $d_0$ and $a_0$ as the measure of failure, and these two parameters were assumed as the material constants. But actually, these two so-called "characteristic lengths" are not material constants. This conclusion has been made by several researchers (Karlak, 1977; Tan, 1991; etc.). The preliminary experimental results of this study also confirmed this point. Despite this conclusion, some researchers still use "characteristic lengths" as a strength measurement for laminated composite containing a hole. On the other hand, since these "characteristic lengths" are related to plate geometry, and for each laminate configuration made of specific material, two tests are required to determine the "characteristic lengths."

The result for one specific plate is not applicable to the plates with different configurations.
or different materials. Apparently the practical importance of these criterions/models is quite limited.

3.2.2 WN's Models – Equivalent Stress Models

For an orthotropic laminate containing a circular hole subjected to tension, the maximum stress is apparently at the hole edge. In my understanding, Whitney used the stress distribution of an infinite isotropic plate as a reference to compare with the actual stress distribution of the laminate at failure. As illustrated in Fig. 2, suppose the actual stress distribution for a finite width laminate at failure is represented by the thick line. There always exists a point along the curve at which the stress is equal to the material failure strength $\sigma_f$. By shifting the isotropic stress curve up or down, one curve can always be found so that it intersects the actual stress curve at the point C. Then the Point Stress Criterion says the laminate fails when the stress of the reference or equivalent curve reaches strength of the unnotched laminate.
3.2.3 Controversy in Using Infinite Plate Solution for Finite Width Plates

Some of the preliminary experimental results in this study showed that the "characteristic lengths" based on Whitney and Nuismer's point stress and average stress criteria are physically unacceptable because they are out of the plate range. This can actually be seen from the formulations for the two criteria. For example, consider Eq. (4) which represents point stress criterion,

\[
\frac{\sigma^N_f}{\sigma^U_f} = \frac{2}{2 + \xi_1^2 + 3\xi_1^4}
\]  

Since both \(\sigma^N_f\) and \(\sigma^U_f\) are known from experiments, the parameter \(\xi_1\), which is actually related to the \(d_0/R\) ratio, can be determined. Apparently, \(d_0\) increases as \(R\) increases so that at certain value of \(R\), the calculated \(d_0\) becomes greater than the value of \((W/2 - R)\). That is to say, the reference point C in Fig. 2 is at some point outside the plate. In this case, the point stress criterion certainly loses its physical meaning. The reason to cause this controversy is the usage of the stress expression for finite width plate.

3.2.4 Limitation of PWG Model

In the three-parameter model proposed by Pipes, Wetherhold, and Gillespie (1980), it is assumed that the exponential parameter, \(m\), is bounded by \(0 < m < 1\). However, for certain materials notched strength data may lead to negative values of \(m\), which is again not a material constant. The issue was addressed by Yost (1981). Moreover, the application of this model requires three separate tensile tests. Combining these two factors, the application of this model is very limited.
3.2.5 About the Assumption Used in Tan’s Models

The most practical strength prediction models are proposed by S.C. Tan who has a large number of publications dealing with finite width laminated plate containing a hole, including a recent book "Stress Concentrations in Laminated Composites" (1994). He proposed several models to incorporate the finite width influence. A brief description of his model has been given in section 2.2.2. The basic assumption in Tan’s models is that the notched finite width solution can be obtained by multiplying the notched infinite-plate solution by a finite width correction factor. This factor can be determined from equilibrium consideration. This assumption seems oversimplified: it does not consider the influence of finite width to the configuration of stress distribution, and it does not reduce to isotropic solution when isotropic plate is considered. In fact, a comparison between the finite element analysis conducted in this study and Tan’s model shows that Tan’s model underestimates the stress concentration factors for almost all cases, and Tan’s model cannot describe appropriately the stress distribution of finite width laminate with a circular hole.

3.3 Objectives of Study

This research investigates the stress and strength of a finite orthotropic laminate containing a circular hole subjected to uniform tensile loading. The specific objectives of the study are stated as follows:

(1) To develop a theoretical model for a laminate with finite width containing a circular hole under tensile load. The consideration of width correction factor should not only be limited to the maximum stress at the hole edge, but also be the stress distribution along the cross section. Therefore, such a model could be used to
predict not only the stress concentration factor, but the normal stress distribution as well.

(2) To predict the initial strength of an orthotropic laminate containing a circular hole subjected to tension, and to establish a relation between the hole size and tensile strength. This strength prediction will be based on the proposed theoretical model and the maximum stress failure criterion.

(3) To conduct an experimental study on unidirectional and crossply laminates with various hole sizes subjected to tensile load. The tensile specimens made of unidirectional and crossply laminates will be used. The tensile strength of specimens with various hole size will be experimentally determined. The normal stress distribution will also be verified.

(4) To perform a computer simulation of the laminates using finite element method. Laminates with various hole sizes will be modeled using composite layer elements. All of the theoretical and experimental results will be verified by finite element method.

(5) To study the effect of plate length on stress concentration factor and tensile strength of an orthotropic laminate with a circular hole. Experimental and finite element analysis will also be carried to compare with theoretical predictions.
4.1 Finite Width Effect

To discuss the effect of hole size on the stress and strength of a finite width plate, consider an isotropic plate first since the associated analytical solutions are readily available. According to Tan, the finite width correction (FWC) factor, which is defined by Tan as $K/K^*$, can be calculated based on Eqs. (33) – (37). For the case of an isotropic plate, we have

$$\frac{K}{K^*} = \frac{w^2}{W^2} \int_{R}^{R} \frac{\sigma_y(0,y)dy}{\sigma_y(0,y)} = \int_{R}^{R} \frac{\sigma_y(0,y)dy}{2\sigma_0} = \frac{W}{2\sigma_0}$$  \hspace{1cm} (39)

i.e.,

$$\frac{K}{K^*} \int_{R}^{R} \frac{\sigma_0}{\sigma_y(0,y)} \left[ 1 + \frac{1}{2} \left( \frac{R}{y} \right)^2 + \frac{3}{2} \left( \frac{R}{y} \right)^4 \right] dy = \frac{W}{2\sigma_0}$$  \hspace{1cm} (40)

From which

$$\frac{K}{K^*} = \frac{1}{1 - 2 \left( \frac{d}{W} \right)^2 - \frac{1}{2} \left( \frac{d}{W} \right)^4}$$  \hspace{1cm} (41)

Table 1 gives the calculated results for a finite width isotropic plate with $d/W$ ratios ranging from 0.1 to 0.5 along with the analytical results from Howland solution. It can be seen that the deviation increases as the $d/W$ ratio increases. Apparently, Tan’s model underestimates the stress concentration at hole edge for finite width plate, especially for the cases of large hole size.
Table 1. Stress Concentration Ratio K/K’ for Finite Width Isotropic Plate

<table>
<thead>
<tr>
<th>d/W</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Howland</td>
<td>1.010</td>
<td>1.047</td>
<td>1.120</td>
<td>1.247</td>
<td>1.440</td>
</tr>
<tr>
<td>Tan</td>
<td>1.020</td>
<td>1.021</td>
<td>1.052</td>
<td>1.102</td>
<td>1.185</td>
</tr>
<tr>
<td>Deviation</td>
<td>0.99%</td>
<td>2.76%</td>
<td>6.11%</td>
<td>11.82%</td>
<td>17.69%</td>
</tr>
</tbody>
</table>

To understand what causes these deviations, the distribution of normal stress $\sigma_x$ along $x = 0$ section is examined. Figure 3 plots the stress distribution for finite width plates five different $d/W$ ratios, i.e., $d/W = 0.1, 0.2, 0.3, 0.4, \text{ and } 0.5$. Three results shown are the infinite plate solution by Timoshenko, as expressed by Eq. (2), the finite width plate solution by Howland, as expressed by Eq. (32), and the results obtained from Tan’s model. What these curves suggest is that the stress distribution for a finite width plate cannot be generated by simply multiplying a factor to that of an infinite plate because the profiles of the two curves are not proportional. A noticeable difference is the change of curvature of the curves for finite width plate at the portion close to the straight edges. Obviously, this change in slope in the finite stress curve is associated with the finite width. To see whether this is also a characteristic for finite width composite plates, a preliminary finite element analysis was first conducted in this study on unidirectional laminate (the details of finite element analysis will be discussed later in a separate section). Figure 4 presents the FEA results for $[0]_{13}$ (unidirectional) Eglass/Epoxy plates with $d/W$ ranging from 0.1 to 0.5. The results are quite convincing: the normal stress distribution curves for unidirectional laminates show clear similarity with those of finite isotropic plates.
Stress Distribution along x = 0 Section
(Isotropic, d/W = 0.1)

Fig. 3 Normal Stress Distribution, Finite Isotropic Plate
(a) d/W = 0.1  (b) d/W = 0.2  (c) d/W = 0.3
(d) d/W = 0.4  (d) d/W = 0.5
(Figure continued)
Stress Distribution along $x = 0$ Section
(Isotropic, $d/W = 0.2$)

\begin{center}
\includegraphics[width=\textwidth]{stress_distribution}
\end{center}

\textbf{Fig. 3 (b)}

(Figure continued)
Stress Distribution along $x = 0$ Section
(Isotropic, $d/W = 0.3$)

Fig. 3 (c)

(Figure continued)
Stress Distribution along $x = 0$ Section
(Isotropic, $d/W = 0.4$)

Figure 3 (d)

(Figure continued)
Stress Distribution along $x = 0$ Section (Isotropic, $d/W = 0.5$)

Fig. 3 (e)
Effect of Finite Width
(FEA Results, Unidirectional Scotchply, L/W = 2)

Fig. 4 Finite Width Effect on Normal Stress Distribution
The profile of the stress distribution in a finite width plate, both isotropic and anisotropic, is determined by two coupled factors. One is the existence of the circular hole, which results in the highest stress at the hole edge (stress concentration). The other is the finite width effect (or hole size effect, or straight edge effect) which changes not only the stress concentration level but also the stress distribution profile. While the first factor can readily be dealt with using the available analytical solution(s), the consideration of finite width effect is not obvious. In the finite width orthotropic laminate model proposed by Tan (1994), an infinite laminate solution is adopted along with an equilibrium equation. The resulting stress distribution curve for finite width laminate is the one simply shifted vertically from the infinite laminate curve where the shifting factor is the Finite Width Correction Factor as Tan defined.

4.2 Width Correction Function

Considering the characteristics of the stress curve for finite width plate, a Width Correction Function, rather than a constant width correction factor used by Tan, is introduced in this study. This Width Correction Function should have the following two basic features:

- The normal stress distribution at $x = 0$ section for a finite width orthotropic laminate containing a circular hole subjected to tension can be approximated by multiplying this function to the corresponding infinite width laminate.
- The finite isotropic plate solution can be retrieved when this function is applied to an isotropic plate.
Mathematically we have the following relation:

$$\sigma_x(0,y) = f(y)\sigma_x^\infty(0,y)$$ \hspace{1cm} (42)$$

where $\sigma_x(0,y)$ and $\sigma_x^\infty(0,y)$ are stress distribution for finite width and infinite laminates, respectively, $f(y)$ is the Width Correction Function which is defined as

$$f(y) = \eta \frac{g_x(y)}{g_x^\infty(y)}$$ \hspace{1cm} (43)$$

here $g_x(y)$ and $g_x^\infty(y)$ are stress distribution profile functions for finite width and infinite isotropic plates, and the coefficient $\eta$ is determined from equilibrium condition:

$$\int_R^{w/2} f(y)\sigma_x^\infty(0,y)dy = \frac{W}{2} \sigma_0$$ \hspace{1cm} (44)$$

The stress distribution profile functions can be obtained based on Howland’s finite width plate solution and Timoshenko’s infinite plate solution, respectively:

$$g_x(y) = 1 + 2m_0 - \frac{d_0}{(2y/W)^2} - 2 \sum_{n=1}^{\infty} (-1)^n \left[ \frac{n(n+1)d_{2n}}{(2y/W)^{2n+2}} + \frac{(n+1)(2n-1)e_{2n}}{(2y/W)^{2n+2}} \right]$$

$$+ n(2n-1)l_{2n}(2y/W)^{2n-2} + (n+1)(2n+1)m_{2n}(2y/W)^{2n}$$ \hspace{1cm} (45)$$

$$g_x^\infty(y) = \frac{\sigma_x^\infty(0,y)}{\sigma_0} = 1 + \frac{1}{2} \left( \frac{R}{y} \right)^2 + \frac{3}{2} \left( \frac{R}{y} \right)^4$$ \hspace{1cm} (46)$$
Fig. 5 Width Correction Function
4.3 Prediction of Stress and Strength

Using the proposed Width Correction Function, the stress concentration factor as well as the normal stress distribution for an orthotropic laminate containing a circular hole can be predicted. Calculations have been carried out for unidirectional Scotchply laminate containing a circular hole with $d/W$ being 0.1, 0.2, 0.3, 0.4, and 0.5. Finite element method has also been used to compare with the model prediction. The detailed results are provided in "Chapter 7. Results and Discussions."

The main objective of stress analysis is strength prediction. While the strength of a material is a material property, the strength of a structure or structural component such as a plate containing a hole is not a material constant. The later is a function of material properties as well as structure configurations. To avoid future confusion, it is necessary to provide a definition for the strength for a notched plate. The tensile strength of a plate, either isotropic or anisotropic, containing a circular is defined as the applied uniaxial tensile stress at which the plate fails. The applied tensile stress corresponding to initial failure is the "initial tensile strength," and the applied tensile stress corresponding to rupture is the "rupture or ultimate tensile strength." For a finite width orthotropic laminate containing a circular hole subjected to uniaxial tension, we have

$$\sigma_x(0, y) = g_x(y)\sigma_0$$

as a plate containing a hole is not a material constant. The later is a function of material properties as well as structure configurations. To avoid future confusion, it is necessary to provide a definition for the strength for a notched plate. The tensile strength of a plate, either isotropic or anisotropic, containing a circular is defined as the applied uniaxial tensile stress at which the plate fails. The applied tensile stress corresponding to initial failure is the "initial tensile strength," and the applied tensile stress corresponding to rupture is the "rupture or ultimate tensile strength." For a finite width orthotropic laminate containing a circular hole subjected to uniaxial tension, we have

$$\sigma_x(0, y) = g_x(y)\sigma_0$$

37
and for an infinite orthotropic laminate,

\[ \sigma_x^\infty(0,y) = \bar{g}_x^\infty(y)\sigma_0^\infty \]  \hspace{1cm} (48)

where \( g_x(y) \) and \( g_x^\infty(y) \) are stress distribution profile functions for finite width and infinite laminated plates, respectively, and \( \sigma_0 \) and \( \sigma_0^\infty \) are the applied tensile load on finite width and infinite laminate, respective. When initial failure occurs at both laminates,

\[ \sigma_x(R) = \sigma_x^\infty(R) = \sigma_f \]  \hspace{1cm} (49)

or

\[ \bar{g}_x(R)\sigma_s = \bar{g}_x^\infty(R)\sigma_0^\infty \]  \hspace{1cm} (50)

where \( \sigma_f \) is the strength of the material, and \( \sigma_s \) and \( \sigma_0^\infty \), by definition, are the tensile strength of finite width and infinite laminate, respectively. Equation (50) can be rewritten as

\[ \sigma_s = \frac{\bar{g}_x^\infty(R)}{\bar{g}_x(R)}\sigma_0^\infty \]  \hspace{1cm} (51)

The ratio \( \frac{\bar{g}_x^\infty(R)}{\bar{g}_x(R)} \) represents the strength reduction due to the finite width. Because of the difficulty in obtaining \( g_x^\infty(R) \) analytically, we assume that \( \frac{\bar{g}_x^\infty(R)}{\bar{g}_x(R)} = \frac{\bar{g}_x^\infty(R)}{g_x(R)} \). That is, the strength reduction due to finite width for an orthotropic laminate is assumed to be the same as that of the isotropic plate. With this assumption, Eq. (51) can now be simplified as

\[ \sigma_s = \frac{g_x^\infty(R)}{g_x(R)}\sigma_0^\infty \]  \hspace{1cm} (52)
where \( g_x(y) \) and \( g_x(\gamma) \) are expressed by Eqs. (45) and (46), respectively. Since the initial failure strength of an infinite laminate containing a circular hole can easily be correlated to the strength of the unnotched laminate, theoretically this relation can be used to predict the initial tensile strength of a laminate containing a circular hole. That is

\[
\sigma_s = \frac{g_x^\infty(R)}{g_x(R)} \frac{\sigma_f}{K^\infty}
\]

where \( \sigma_f \) is the strength of material and \( K^\infty \) the stress concentration factor of infinite laminate.

It should be noted that since this theoretical model is a single-stress, the failure criterion used in the strength analysis is also a single component criterion – the maximum stress criterion. It is known that the maximum stress criterion works only in the applications where the failure is single stress dominated. The experiments conducted in this study (see the corresponding charters for details) show for both unidirectional and crossply laminated specimens subjected to uniaxial tension experience single stress dominated failure. But this criterion does not applicable to general angle-ply laminates, or unidirectional and crossply laminate subjected to off-axis loading.

4.4 Summary

To study the finite width effect, or hole size effect, a Width Correction Function was first introduced which has two features: The normal stress distribution at \( x = 0 \) section for a finite width orthotropic laminate containing a circular hole subjected to tension can be approximated by multiplying this function to the corresponding infinite width laminate;
The finite isotropic plate solution can be retrieved when this function is applied to an isotropic plate. The theoretical basis for this function is the analytical solution for a finite width isotropic plate with a circular hole by Howland (1930) and the solution for an infinite isotropic plate with a circular hole by Timoshenko (1959). It is assumed that the profile of normal stress distribution for a finite width laminate with a circular hole is similar to that of an identical isotropic plate. Then the normal stress distribution of a finite width orthotropic laminate containing a circular hole can be predicted based on plate geometry, material properties, and equilibrium consideration. This proposed mode can be used to predict the stress concentration factor at the hole edge and the normal stress distribution. It can also be used to predict the initial tensile strength of an orthotropic laminate using Maximum Stress Criterion. A comparison between the proposed theoretical model and Tan’s model is provided in Table 2.

Table 2. Comparison of Proposed Model and Tan’s Model

<table>
<thead>
<tr>
<th></th>
<th>Proposed Model</th>
<th>Tan’s Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite width correction</td>
<td>width correction function</td>
<td>width correction factor</td>
</tr>
<tr>
<td>Theoretical basis</td>
<td>solutions of finite width isotropic plate and infinite laminate.</td>
<td>Solutions of infinite isotropic plate and infinite laminate</td>
</tr>
<tr>
<td>Equilibrium condition</td>
<td>Satisfied</td>
<td>Satisfied</td>
</tr>
<tr>
<td>Stress concentration factor</td>
<td>Consistent with FEA results</td>
<td>Underestimated; Inconsistent with FEA results</td>
</tr>
<tr>
<td>Stress distribution profile</td>
<td>Consistent with FEA results</td>
<td>Similar to that of infinite plate; Inconsistent with FEA results</td>
</tr>
</tbody>
</table>
CHAPTER 5. TENSILE STRESS AND STRENGTH PREDICTION MODEL – FINITE LENGTH CORRECTION

5.1 Finite Length Consideration

While many previous work can be found in studying the finite width effect on a plate containing a hole (mainly on isotropic plates), few studies have been found on the effect of finite length of the plate. There may be two reasons. First, the length effect may not be of the same significance as the width effect to the stress and strength of the plate. The existence of a hole in a finite width plate reduces the cross-sectional area and results in the increase of stress level on the cross section. Increasing the hole size with a fixed plate width or decreasing the width with fixed hole size will further raise the stress concentration at the hole edge. In contrast, the effect of changing plate length on stress distribution and stress concentration is not obvious. Second, theoretical difficulties make it almost impossible to solve this problem analytically. According to the results from Shilkrut and Ben-Gad (1985) on an isotropic plate containing a circular hole, the length of the plate does affect the maximum stress at the hole edge, especially when the hole is relative large, say $d/W > 0.5$. To see whether this is the case for orthotropic composite laminate, preliminary studies have been conducted through both finite element method and experiments. Figure 6 provides the normal stress distribution at $x = 0$ section for unidirectional Scotchply laminate containing a circular with $d/W = 0.5$. Four different $L/W$ ratios were analyzed in the FEA: $L/W = 1.0, 1.5, 2.0, \text{ and } 3.0$. The stress concentration factors for the four cases were found to be 8.715 for $L/W = 1.0$, 6.384 for $L/W = 1.5$, 5.385 for $L/W = 2.0$, and 5.198 for $L/W = 3.0$. Here $L$ is the length of a finite plate. For tensile
specimen, $L$ is the gage length which is the total specimen length subtracted by two tab lengths. We used 1.5" tab for all the tensile specimens used in this study; therefore, the actual length of a specimen is $L + 3$ (inches). Table 3 shows the result of tensile test using eight unidirectional Scotchply specimens with four different gage lengths. The width of the specimens is 2" and the diameter of the hole is 1" (thus the $d/W$ ratio is 0.5). The gage lengths of the specimens are 3" ($L/W = 1.5$), 4" ($L/W = 2.0$), 5.0" ($L/W = 2.5$) and 6.0" ($L/W = 3.0$). The results in Table 3 suggest that the gage length does affect the initial strength of the laminates, especially when the $L/W$ ratio is small. However, the length effect on initial strength is not as significant as the finite width effect.

Table 3. Initial Tensile Strength of Specimens with Different $L/W$ Ratios

<table>
<thead>
<tr>
<th>$L/W$</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Tensile Strength (ksi)</td>
<td>Set I</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>29.33</td>
<td>33.52</td>
<td>35.77</td>
<td>35.85</td>
</tr>
<tr>
<td>Set II</td>
<td>28.01</td>
<td>34.00</td>
<td>34.23</td>
<td>35.70</td>
</tr>
</tbody>
</table>

Note: For all of the specimens in this table, $d/W = 0.5$.

To better understand the effect of finite width as well as finite length on the stress distribution and eventually the strength, some analytical work will be conducted in this study. The basic concept is similar to that used in finite width correction as discussed in the previous chapter. But instead of using a Width Correction Function, a Finite Size Correction Function will be used in this case. The theoretical basis for this function is an analytical solution for a finite isotropic plate, i.e., an isotropic plate of finite width and length with a circular hole subjected to tension, as well as the Timoshenko’s solution for infinite isotropic plate containing a circular hole.
Effect of Finite Length
(FEA Results, Unidirectional Scotchply d/W = 0.5)

Fig. 6   Effect of Finite Length on Normal Stress Distribution

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5.2 Stress Formulation

The stress distribution of a finite plate can be solved by applying theory of elasticity. Bailey and Hicks (1960) used this approach to solve two special cases which they called symmetrical ($u_x = u_y$, and $\sigma_x = \sigma_y$) and anti-symmetrical displacement ($u_x = -u_y$, and $\sigma_x = -\sigma_y$).

The stress components in a two dimensional problem can be expressed in polar coordinate system as

\[
\begin{align*}
\sigma_r & = \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial r^2} \\
\sigma_\theta & = \frac{\partial^2 \phi}{\partial r^2} \\
\tau_{r\theta} & = -\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \phi}{\partial r} \right)
\end{align*}
\]

(54)
where $\phi$ is the Airy's stress function which satisfies the biharmonic equation

$$\nabla^4 \phi = 0$$

(55)

and

$$\nabla^4 = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

in 2-D Cartesian coordinates and

$$\nabla^4 = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)$$

in polar coordinates. The numerical solution to the problem is based on the following solution in polar coordinates (Timoshenko and Goodier, 1979):

$$\phi = a_0 \ln r + b_0 r^2 + c_0 r^2 \ln r + d_0 + \sum_{n=2}^{\infty} \left[ a'_n r^n + b'_n r^{n+2} + \frac{a_n}{r^n} + \frac{b_n}{r^{n-2}} \right] \cos n\theta$$

$$+ \sum_{n=2}^{\infty} \left[ c'_n r^n + d'_n r^{n+2} + \frac{c_n}{r^n} + \frac{d_n}{r^{n-2}} \right] \cos n\theta$$

(56)

where $n$ is even, $n = 2, 4, 6, \ldots$, and $a_0, a_n, a'_n$, etc., are unknown constants to be determined from boundary conditions. According to Timoshenko and Goodier (1979), $c_0$ can be taken as zero here. From Eq. (54), the stress components in polar coordinates, $\sigma_r, \sigma_\theta$, and $\tau$, can be calculated, as shown in Eq. (57). It should be noted the constants in Eq. (56) carries the dimension of length in various order. To make it easy for the numerical determination of these constants, a new set of non-dimensional constants are used in Eq. (57) while still keeping the original notations.
\[
\sigma_r = a_0 \left( \frac{R}{r} \right)^2 + 2b_0 + \sum_{n=2}^{\infty} \left[ n(n-1)a_n \left( \frac{r}{R} \right)^{n-2} + (n+1)(n-2)b_n \left( \frac{r}{R} \right)^n \right] + \sum_{n=2}^{\infty} \left[ n(n-1)c_n \left( \frac{r}{R} \right)^{n+2} + (n+2)(n-1)d_n \left( \frac{r}{R} \right)^n \right] \cos n\theta
\]

\[
\sigma_\theta = -a_0 \left( \frac{R}{r} \right)^2 + 2b_0 + \sum_{n=2}^{\infty} \left[ n(n-1)a_n \left( \frac{r}{R} \right)^{n-2} + (n+2)(n+1)b_n \left( \frac{r}{R} \right)^n \right] + \sum_{n=2}^{\infty} \left[ n(n-1)c_n \left( \frac{r}{R} \right)^{n+2} + (n+2)(n+1)d_n \left( \frac{r}{R} \right)^n \right] \sin n\theta
\]

\[
\tau_{r\theta} = \sum_{n=2}^{\infty} \left[ (n-1)a_n \left( \frac{r}{R} \right)^{n-2} + (n+1)b_n \left( \frac{r}{R} \right)^n \right] - \frac{R}{r} \left[ (n-1)b_n \left( \frac{r}{R} \right)^n \right] \sin n\theta - \sum_{n=2}^{\infty} \left[ (n-1)c_n \left( \frac{r}{R} \right)^{n-2} + (n+1)d_n \left( \frac{r}{R} \right)^n \right] \cos n\theta
\]
The boundary conditions for the problem are:

(i) The internal boundary, or the hole edge, should be free of stress:

\[ \sigma_r \bigg|_{r=R} = \tau_{r\theta} \bigg|_{r=R} = 0 \]  \hspace{1cm} (58)

(ii) The stress at external boundary should be consistent with the applied load:

\[ \sigma_x \bigg|_{x=\pm L/2} = \sigma_y \bigg|_{y=\pm W/2} = 0 \]
\[ \sigma_y \bigg|_{y=\pm W/2} = \tau_{xy} \bigg|_{x=\pm L/2} = 0 \]  \hspace{1cm} (59)

In order to apply the boundary conditions at outer border, it is preferred to use the stress expressions in Cartesian coordinates which can be obtained from the relation:

\[ \sigma_x = \sigma_r \cos^2 \theta + \sigma_\theta \sin^2 \theta - \tau_{r\theta} \sin 2\theta \]
\[ \sigma_y = \sigma_r \sin^2 \theta + \sigma_\theta \cos^2 \theta + \tau_{r\theta} \sin 2\theta \]
\[ \tau_{r\theta} = \frac{1}{2} (\sigma_r - \sigma_\theta) \sin 2\theta + \tau_{r\theta} \cos 2\theta \]  \hspace{1cm} (60)

where the angle \( \theta \) is defined in Fig. 8.
The stress components in the Cartesian coordinate system are

\[ \sigma_x = 2b_0 \left[ 1 - \left( \frac{R}{r} \right)^2 \cos 2\theta \right] + \sum_{n=2}^{\infty} (n+1)b_n \left[ \left( \frac{r}{R} \right)^n \left[ 2\cos n\theta - n\cos(n-2)\theta \right] \right. \]

\[ + (n-1) \left( \frac{r}{R} \right)^{n-2} \cos(n-2)\theta - \left( \frac{R}{r} \right)^{n+2} \cos(n+2)\theta \]

\[ + \sum_{n=2}^{\infty} (n-1)b_n \left[ \left( \frac{R}{r} \right)^n \left[ 2\cos n\theta + n\cos(n+2)\theta \right] \right. \]

\[ + (n+1) \left( \frac{R}{r} \right)^{n+2} \cos(n+2)\theta + \left( \frac{r}{R} \right)^{n-2} \cos(n-2)\theta \]

\[ + \sum_{n=2}^{\infty} (n+1)d_n \left[ \left( \frac{r}{R} \right)^n \left[ 2\sin n\theta - n\sin(n-2)\theta \right] \right. \]

\[ + (n-1) \left( \frac{r}{R} \right)^{n-2} \sin(n-2)\theta - \left( \frac{R}{r} \right)^{n+2} \sin(n+2)\theta \]

\[ + \sum_{n=2}^{\infty} (n-1)d_n \left[ \left( \frac{R}{r} \right)^n \left[ 2\sin n\theta + n\sin(n+2)\theta \right] \right. \]

\[ + (n+1) \left( \frac{R}{r} \right)^{n+2} \sin(n+2)\theta + \left( \frac{r}{R} \right)^{n-2} \sin(n-2)\theta \]

\[ \sigma_y = 2b_0 \left[ 1 + \left( \frac{R}{r} \right)^2 \cos 2\theta \right] + \sum_{n=2}^{\infty} (n+1)b_n \left[ \left( \frac{r}{R} \right)^n \left[ 2\cos n\theta + n\cos(n-2)\theta \right] \right. \]

\[ - (n-1) \left( \frac{r}{R} \right)^{n-2} \cos(n-2)\theta + \left( \frac{R}{r} \right)^{n+2} \cos(n+2)\theta \]

\[ + \sum_{n=2}^{\infty} (n-1)b_n \left[ \left( \frac{R}{r} \right)^n \left[ 2\cos n\theta - n\cos(n+2)\theta \right] \right. \]

\[ - (n+1) \left( \frac{R}{r} \right)^{n+2} \cos(n+2)\theta - \left( \frac{r}{R} \right)^{n-2} \cos(n-2)\theta \]

\[ + \sum_{n=2}^{\infty} (n+1)d_n \left[ \left( \frac{r}{R} \right)^n \left[ 2\sin n\theta + n\sin(n+2)\theta \right] \right. \]

\[ + (n-1) \left( \frac{r}{R} \right)^{n-2} \sin(n-2)\theta - \left( \frac{R}{r} \right)^{n+2} \sin(n+2)\theta \]

\[ + \sum_{n=2}^{\infty} (n-1)d_n \left[ \left( \frac{R}{r} \right)^n \left[ 2\sin n\theta - n\sin(n+2)\theta \right] \right. \]

\[ + (n+1) \left( \frac{R}{r} \right)^{n+2} \sin(n+2)\theta + \left( \frac{r}{R} \right)^{n-2} \sin(n-2)\theta \]
\[-(n+1) \left( \frac{R}{r} \right)^{n+2} \cos(n+2)\theta - \left( \frac{r}{R} \right)^{n-2} \cos(n-2)\theta \]

\[-\sum_{n=2}^{\infty} (n+1)d_n \left[ \left( \frac{r}{R} \right)^{n} \right] [2 \sin n\theta + n \sin(n-2)\theta] \]

\[-(n-1) \left( \frac{r}{R} \right)^{n-2} \sin(n-2)\theta + \left( \frac{R}{r} \right)^{n+2} \sin(n+2)\theta \]

\[-\sum_{n=2}^{\infty} (n-1)d_n \left[ -\left( \frac{R}{r} \right)^{n} \right] [2 \sin \theta - n \sin(n+2)\theta] \]

\[-(n+1) \left( \frac{R}{r} \right)^{n+2} \sin(n+2)\theta - \left( \frac{r}{R} \right)^{n-2} \sin(n-2)\theta \]

\[
\tau_{xy} = -2b_0 \left( \frac{R}{r} \right)^2 \sin 2\theta + \sum_{n=2}^{\infty} (n+1)b'_n \left[ n \left( \frac{r}{R} \right)^{n} - (n-1) \left( \frac{r}{R} \right)^{n-2} \right] \sin(n-2)\theta
\]

\[-\left( \frac{R}{r} \right)^{n+2} \sin(n+2)\theta \] - \sum_{n=2}^{\infty} (n-1)b_n \left[ n \left( \frac{R}{r} \right)^{n} - (n+1) \left( \frac{r}{R} \right)^{n+2} \right] \sin(n+2)\theta

\[+ \left( \frac{r}{R} \right)^{n-2} \sin(n-2)\theta \] + \sum_{n=2}^{\infty} (n+1)d'_n \left[ n \left( \frac{r}{R} \right)^{n} - (n+1) \left( \frac{r}{R} \right)^{n+2} \right] \cos(n-2)\theta \]

\[-\left( \frac{R}{r} \right)^{n+2} \cos(n+2)\theta \] - \sum_{n=2}^{\infty} (n-1)d_n \left[ n \left( \frac{r}{R} \right)^{n} - (n+1) \left( \frac{r}{R} \right)^{n+2} \right] \cos(n+2)\theta

\[+ \left( \frac{r}{R} \right)^{n-2} \cos(n-2)\theta \]

Applying the boundary conditions at the hole edge gives the following relations between the unknown constants:

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\[ a_0 = -2b_0 \]
\[ a_n = \frac{1}{n}[b'_n - (n-1)b_n] \]
\[ a'_n = -\frac{1}{n}[b'_n + (n+1)b_n'] \]
\[ c_n = \frac{1}{n}[d'_n - (n-1)d_n] \]
\[ c'_n = -\frac{1}{n}[d'_n + (n+1)d_n'] \] (62)

Taking into consideration the symmetry of the problem, i.e., \( \sigma_t(\theta) = \sigma_t(-\theta) \), \( \sigma_\theta(\theta) = \sigma_\theta(-\theta) \), and \( \tau_{\theta\theta}(\theta) = -\tau_{\theta\theta}(-\theta) \). It requires that all the constants \( d_n \) and \( d'_n \) be zero. Then by Eq. (62), the constants \( c_n \) and \( c'_n \) are also zero. Thus the stress expressions can be greatly simplified.

In polar coordinate system:

\[ \sigma_r = a_0 \left( \frac{R}{r} \right)^2 + 2b_0 - \sum_{n=2}^{\infty} \left\{ n(n-1)a'_n \left( \frac{r}{R} \right)^{n-2} + (n+1)(n-2)b'_n \left( \frac{r}{R} \right)^n \right\} \cos n\theta \]
\[ +n(n+1)a_n \left( \frac{R}{r} \right)^{n+2} + (n+2)(n-1)b_n \left( \frac{R}{r} \right)^n \] (63)
\[ \sigma_\theta = -a_0 \left( \frac{R}{r} \right)^2 + 2b_0 + \sum_{n=2}^{\infty} \left\{ n(n-1)a'_n \left( \frac{r}{R} \right)^{n-2} + (n+2)(n+1)b'_n \left( \frac{r}{R} \right)^n \right\} \cos n\theta \]
\[ +n(n+1)a_n \left( \frac{r}{R} \right)^{n+2} + (n-2)(n-1)b_n \left( \frac{r}{R} \right)^n \] \[ \tau_{\theta\theta} = \sum_{n=2}^{\infty} \left\{ (n-1)a'_n \left( \frac{r}{R} \right)^{n-2} + (n+1)b'_n \left( \frac{r}{R} \right)^n - (n+1)a_n \left( \frac{R}{r} \right)^{n+2} \right\} \sin n\theta \]

The expression of radial stress, \( \sigma_r \), is exactly the same as that obtained by Shilkrut and Ben-Gad (1985) except for different notation of constants. Considering the correlation
between the constants, as given in Eq. (62), the corresponding stresses in Cartesian coordinates become

\[
\sigma_x = 2b_0 \left[ 1 - \frac{R}{r} \cos 2\theta \right] + \sum_{n=2}^{\infty} (n+1)b_n \left[ \left( \frac{r}{R} \right)^n \left[ 2 \cos n\theta - n \cos (n-2)\theta \right] 
\right.
\]
\[\left. + (n-1) \left( \frac{r}{R} \right)^{n-2} \cos (n-2)\theta - \left( \frac{r}{R} \right)^{n+2} \cos (n+2)\theta \right] \]
\[+ \sum_{n=2}^{\infty} (n-1)b_n \left[ \left( \frac{r}{R} \right)^n \left[ 2 \cos n\theta + n \cos (n+2)\theta \right] \right. \]
\[\left. + (n+1) \left( \frac{r}{R} \right)^{n+2} \cos (n+2)\theta + \left( \frac{r}{R} \right)^{n-2} \cos (n-2)\theta \right] \]
\[\sigma_y = 2b_0 \left[ 1 + \frac{R}{r} \cos 2\theta \right] + \sum_{n=2}^{\infty} (n+1)b_n \left[ \left( \frac{r}{R} \right)^n \left[ 2 \cos n\theta + n \cos (n-2)\theta \right] \right. \]
\[\left. - (n-1) \left( \frac{r}{R} \right)^{n-2} \cos (n-2)\theta + \left( \frac{r}{R} \right)^{n+2} \cos (n+2)\theta \right] \]
\[\tau_{xy} = -2b_0 \left( \frac{R}{r} \right)^2 \sin 2\theta + \sum_{n=2}^{\infty} (n+1)b_n \left[ \left( \frac{r}{R} \right)^n - (n-1) \left( \frac{r}{R} \right)^{n-2} \right] \sin (n-2)\theta \]
\[\left. - \left( \frac{R}{r} \right)^{n+2} \sin (n+2)\theta \right] - \sum_{n=2}^{\infty} (n-1)b_n \left[ \left( \frac{r}{R} \right)^n - (n+1) \left( \frac{r}{R} \right)^{n+2} \right] \sin (n+2)\theta \]
\[+ \left( \frac{r}{R} \right)^{n-2} \sin (n-2)\theta \]
In order to determine the unknown constants, not only sufficient equations should be derived, but also it is necessary that the series of these equations be truncated. Suppose the upper bound of the summation is $N$, then the total number of unknowns would be $2(N/2) + 1 = N+1$: $b_n, b'_n, n = 2, 4, 6, ..., N$, and $b_0$. Therefore, a total of $N+1$ equations are needed to solve the problem. The approach used in this study is the "point matching" method: the boundary conditions are satisfied at certain number of chosen discrete boundary points. Recalling that the relations (62) are derived from the boundary conditions at the inner hole edge, Eq. (58), thus the stress expressions, Eqs. (63) and (64), satisfy the boundary conditions at the inner boundary. It is also noted that the symmetric condition has already been used in eliminating constants $c_n, c'_n, d_n$, and $d'_n$. Therefore, only the outer boundary condition at first quadrant should be satisfied, i.e., at $x = L/2$ and $0 \leq y \leq W/2$: $\sigma_x = \sigma_0, \tau_{xy} = 0$; at $0 \leq x \leq L/2$ and $y = W/2$: $\sigma_x = 0, \tau_{xy} = 0$.

Before solving the constants numerically, the stresses at the outer boundary are further expressed as functions of $\theta$ rather than $r$ and $\theta$. Considering the external boundary in the first quadrant. Along the vertical boundary, we have

$$r = \frac{L/2}{\cos \theta} = \frac{L}{2 \cos \theta}$$

and along the horizontal boundary,

$$r = \frac{W/2}{\sin \theta} = \frac{W}{2 \sin \theta}$$

thus the stresses at the external boundary are functions of $\theta$ only. The numerical results of stresses are presented in "Chapter 7. Results and Discussions."
5.3 Size Correction Function

In Chapter 4, the effect of finite width on normal stress distribution is represented by a "Width Correction Function." Similarly, we introduce a "Size Correction Function" for the present case in which both finite width and finite length are considered. This Size Correction Function should have two basic features:

- The normal stress distribution at $x=0$ section for a finite width orthotropic laminate containing a circular hole subjected to tension can be approximated by multiplying this function to the corresponding infinite width laminate.

- The finite isotropic plate solution can be retrieved when this function is applied to an isotropic plate.

Mathematically we have the following relation:

$$\sigma_x(0, y) = h(y)\sigma_x^\infty(0, y)$$

(65)

where $\sigma_x(0, y)$ and $\sigma_x^\infty(0, y)$ are stress distribution for finite (width and length) and infinite laminates, respectively, $h(y)$ is the Size Correction Function which is defined as

$$h(y) = \eta \frac{p_x(y)}{p_x^\infty(y)}$$

(66)

here $p_x(y)$ and $p_x^\infty(y)$ are stress distribution profile functions for finite and infinite isotropic plates, respectively, and the coefficient $\eta$ is determined from equilibrium condition:

$$\int_{-W/2}^{W/2} h(y)\sigma_x^\infty(0, y)dy = \frac{W}{2} \sigma_0$$

(67)
The stress distribution profile functions $p_x(y)$ and $p_x^e(y)$ can be obtained based on the stress formulation in section 5.2 and Timoshenko’s infinite plate solution, respectively:

$$p_x(y) = \frac{\sigma_x(0, y)}{\sigma_0}$$ 

(68)

where $\sigma_x(0, y)$ can be obtained from Eq. (64a) by letting $\theta = \pi/2$ and $r = y$:

$$\sigma_x(0, y) = 2b_0\left[1 + \left(\frac{R}{y}\right)^2\right] + \sum_{m=1}^{\infty} (2m + 1)b_{2m}'\left[2(1 + m)(-1)^m\left(\frac{y}{R}\right)^{2m+1}ight.$$  

$$+ (2m - 1)(-1)^{m-1}\left(\frac{y}{R}\right)^{2m-2} - (-1)^m\left(\frac{y}{R}\right)^{2m+1}\right]\right]$$

$$+ \sum_{m=1}^{\infty} (2m - 1)b_{2m}\left[2(1 + m)(-1)^{m+1}\left(\frac{R}{y}\right)^{2m+2} + (-1)^m\left(\frac{R}{y}\right)^{2m-2}\right]$$  

(69)

and

$$p_x^e(y) = \frac{\sigma_x^e(0, y)}{\sigma_0} = 1 + \frac{1}{2}\left(\frac{R}{y}\right)^2 + \frac{3}{2}\left(\frac{R}{y}\right)^4$$

(70)

Using the theoretical model developed in this chapter, the stress concentration factor at the hole edge, normal stress distribution, $\sigma_x(0, y)$, and initial tensile strength, can be predicted following the same procedure described in Chapter 4, by Eqs. (47) through (53). It is worthy noting that although three stress components, $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ or $\sigma_r$, $\sigma_\theta$, and $\tau_{r\theta}$ are involved in the derivation, the only stress component used in the Size Correction Function is the normal stress $\sigma_x$. The inclusion of other two stress components is to
determine the unknown constants by applying boundary conditions. For this reason, this proposed model can be used to predict the normal stress along $x = 0$ section, $\sigma_x(0, y)$, for a finite laminate containing a circular hole, rather than the complete stress field. Also for this reason, when this model is used to predict the initial tensile strength of a laminate containing a circular hole, the applications are limited to the cases of single stress dominated failure modes.
CHAPTER 6. MODEL VERIFICATION

6.1 Verification by Finite Element Method

A linear elastic finite element analysis has been performed to verify the proposed theoretical models using software COSMOS. Composite quadrilateral plate elements were used to model the notched laminate.

A composite quadrilateral plate element is a 4-node multi-layer element with membrane and bending capabilities for the analysis of structural models. Each node has six degrees of freedom (three translations and three rotations). The material properties associated with each layer are specified by defining the material group and fiber angle for the layer. Three sets of coordinate systems are used to accomplish the anisotropic feature: the material coordinates, the element (or local) coordinates, and the global coordinates. The material coordinates are associated with the material properties of each layer and are characterized by rotation with respect to the element coordinate system. The element (local) coordinates are determined by the relative position of nodal points within an element. The global coordinate system is the reference coordinate system to establish the geometry of the structure. In all of the models developed in this study, Cartesian coordinates were used as the global coordinates. A total of 9 material properties need to be used for each material group: $E_1$, $E_2$, $E_3$, $G_{12}$, $G_{23}$, $G_{13}$, $v_{12}$, $v_{23}$, and $v_{13}$, where $E_1$, $E_2$, $E_3$ are the modulus of elasticity in the 1st, 2nd, and 3rd material direction, respectively, $G_{12}$, $G_{23}$, $G_{13}$ are the shear modulus in the 1-2 plane, 2-3 plane, and 1-3 plane, respectively, and $v_{12}$, $v_{23}$, and $v_{13}$ are the Poisson's ratios relating the 1st and 2nd, 2nd and 3rd, and 1st and 3rd directions, respectively.
Since the material coordinates are defined using local (element) x-coordinate (i.e., from the first node to the second node) as reference, it has to make sure the material directions are correctly represented in the FEA mesh. In all the FEA meshes for laminated composites in this study, the local (elemental) x-axis for all elements are chosen to be identical to the global x-axis. This can be found in the FEA meshes that each element has at least one side parallel to the global x-axis. Figure 9 shows two of the meshes used in the finite element analysis. Due to symmetry, only one fourth of the plate was modeled. The symmetrical boundary conditions are applied at the two sides of symmetry. An evenly distributed uniaxial stress $\sigma_0 = 1$ is applied at the end; therefore, the calculated results are actually $\sigma_x/\sigma_0$. A total of 29 cases have been studied in the finite element analysis to verify the proposed tension models as well as to demonstrate the hole size effect (or finite width effect). Table 4 shows the cases performed in the finite element analysis.

Table 4. Cases Studied in Finite Element Analysis

<table>
<thead>
<tr>
<th>Material</th>
<th>d/W</th>
<th>L/W</th>
<th>Purpose</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isotropic</td>
<td>0.1, 0.2, 0.3, 0.4, 0.5</td>
<td>2.5</td>
<td>Hole size effect</td>
</tr>
<tr>
<td>Unidirectional</td>
<td>0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.75</td>
<td>2.5</td>
<td>Width effect</td>
</tr>
<tr>
<td>Crossply</td>
<td>0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.75</td>
<td>2.5</td>
<td>Width effect</td>
</tr>
<tr>
<td>Unidirectional</td>
<td>0.5</td>
<td>1.0, 1.5, 2.0, 3.0, 4.0</td>
<td>Length effect</td>
</tr>
<tr>
<td>Crossply</td>
<td>0.5</td>
<td>1.0, 1.5, 2.0, 3.0, 4.0</td>
<td>Length effect</td>
</tr>
</tbody>
</table>

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\[ d/W = 0.5, \ L/W = 2.0 \]

\[ d/W = 0.5, \ L/W = 3.0 \]

Fig. 9 FEA Meshes
6.2 Verification by Experiments

6.2.1 Specimens

To study the hole size effect or edge effect experimentally, specimens with various hole sizes were made. These specimens were cut from 1/8” Scotchply laminates. A 2” width was used for all specimens since the widest grip available from Instron or MTS company is the 2” hydraulic flat specimen grip. And because the Instron machine at LSU is not equipped with the 2” grips, all tensile tests were performed on the MTS machine in the Mechanical Engineering Department at Southern University which is equipped with a pair of 2” hydraulic grips. Four groups of specimens were used in the experimental study:

- Unidirectional specimens (2”×8”×1/8”): 9 hole sizes; 2 sets; 18 total specimens.
- Crossply specimens (2”×8”×1/8”): 9 hole sizes; 2 sets; 18 total specimens.
- Unidirectional specimens: 4 different lengths; 8 total specimens.
- Crossply specimens: 4 different lengths; 8 total specimens.

The hole sizes and/or lengths, along with the numbering of the specimen, are listed in Tables 5 and 6.

<table>
<thead>
<tr>
<th>Group 1: Unidirectional</th>
<th>Group 2: Crossply</th>
</tr>
</thead>
<tbody>
<tr>
<td>U00-1</td>
<td>C00-1</td>
</tr>
<tr>
<td>U00-2</td>
<td>C00-2</td>
</tr>
<tr>
<td>U01-1</td>
<td>C01-1</td>
</tr>
<tr>
<td>U01-2</td>
<td>C01-2</td>
</tr>
<tr>
<td>U02-1</td>
<td>C02-1</td>
</tr>
<tr>
<td>U02-2</td>
<td>C02-2</td>
</tr>
<tr>
<td>U04-1*</td>
<td>C04-1*</td>
</tr>
<tr>
<td>U04-2</td>
<td>C04-2</td>
</tr>
<tr>
<td>U06-1</td>
<td>C06-1</td>
</tr>
<tr>
<td>U06-2</td>
<td>C06-2</td>
</tr>
<tr>
<td>U08-1*</td>
<td>C08-1*</td>
</tr>
<tr>
<td>U08-2</td>
<td>C08-2</td>
</tr>
<tr>
<td>U10-1</td>
<td>C10-1</td>
</tr>
<tr>
<td>U10-2</td>
<td>C10-2</td>
</tr>
<tr>
<td>U12-1</td>
<td>C12-1</td>
</tr>
<tr>
<td>U12-2</td>
<td>C12-2</td>
</tr>
<tr>
<td>U15-1</td>
<td>C15-1</td>
</tr>
<tr>
<td>U15-2</td>
<td>C15-2</td>
</tr>
</tbody>
</table>

* These are the specimens in which strain gages were attached.
Table 6. Length and Numbering of Specimen

<table>
<thead>
<tr>
<th>Description</th>
<th>Length (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6</td>
</tr>
<tr>
<td>Group 3: Unidirectional</td>
<td>UL06-1</td>
</tr>
<tr>
<td></td>
<td>UL06-2</td>
</tr>
<tr>
<td>Group 4: Cross ply</td>
<td>CL06-1</td>
</tr>
<tr>
<td></td>
<td>CL06-2</td>
</tr>
</tbody>
</table>

Drilling can cause damage in composites. Matrix cracking may occur from the hole boundary; small chips may come off at the hole surface, and fiber may break out at the drill-exit side of the specimen. Studies showed that these problems could be eliminated with proper precaution and drilling techniques such as using a carbide-tip drill bit or a diamond-coated drill, and/or using a thin plate of glass/epoxy or aluminum on the top and at the bottom of the specimen before drilling (Tan, 1994). The machining of the holes for all of the specimens were done by Dynomach, Inc., Port Allen, Louisiana. The holes were machined using a 1-mm diameter diamond coated drill in a CNC machine. The Scotchply laminated plates were manufactured by 3M company.

For laminates containing a cutout, the failure is normally initiated from the hole boundary because of the stress concentration. The stress concentration due to machine grips does not generally cause any problem (Adams and Odom, 1991). Therefore, rectangular tabs with a 90° cut-off angle are commonly used because of the simplicity in preparation. In this study, two types of tabs were used in specimens. The 1/16" thick aluminum tabs were first used for Group 1 specimens (unidirectional specimens with various hole sizes). Later, 1/8" Scotchply tabs were used for Groups 2, 3, and 4 specimens.
The Scotch-Weld 2216 B/A epoxy adhesive by 3M was used. It has been found that using Scotchply tabs results in higher bond strength between the specimen and tabs. The properties of the unidirectional Scotchply laminates are given in Table. 7, which were provided by 3M company. The properties of crossply Scotchply laminates can easily be obtained from the table.

Table 7. Properties of Scotchply Laminates

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Longitudinal Modulus $E_{11}$</td>
<td>60.70 GPa (8.8 Mpsi)</td>
</tr>
<tr>
<td>Transverse Modulus $E_{22}$</td>
<td>24.80 GPa (3.6 Mpsi)</td>
</tr>
<tr>
<td>Shear Modulus $G_{12}$</td>
<td>12.00 GPa (1.74 Mpsi)</td>
</tr>
<tr>
<td>In-Plane Poisson's Ratio $v_{12}$</td>
<td>0.23</td>
</tr>
<tr>
<td>Longitudinal Tensile Strength $X_T$</td>
<td>0.965 GPa (140 ksi)</td>
</tr>
<tr>
<td>Transverse Tensile Strength $Y_T$</td>
<td>0.031 GPa (4.5 ksi)</td>
</tr>
<tr>
<td>Shear Strength $S$</td>
<td>0.072 GPa (10.44 ksi)</td>
</tr>
<tr>
<td>Resin Content (by weight)</td>
<td>32 %</td>
</tr>
<tr>
<td>Fiber Content (by weight)</td>
<td>68 %</td>
</tr>
<tr>
<td>Number of Layers</td>
<td>13</td>
</tr>
<tr>
<td>Layer Thickness</td>
<td>0.254 mm (0.01 in.)</td>
</tr>
<tr>
<td>Total Thickness of Laminate</td>
<td>3.302 mm (0.13 in.)</td>
</tr>
</tbody>
</table>

6.2.2 Testing Equipment

All the tensile tests were conducted at an MTS 810 TestStar system in the Mechanical Engineering Department at Southern University. This testing machine is equipped with a pair of hydraulic grips with 2" grip width (the grips at LSU’s Instron Machine is 1" wide) which makes it possible to conduct the “wide” plate tensile tests.
Fig. 10 Schematic of Hydraulic Material Testing System
The MTS 810 TestStar IIs system is an integrated testing packages which includes a Model 318 load unit with integrally mounted actuator and servovalves, a hydraulic power supply, and the MTS TestStar IIs control system. A schematic diagram of the system is demonstrated in Fig. 10. The TestStar IIs control system has three major parts: the TestStar system software running on a personal computer, the digital controller, and a remote station control panel. These function together provides a fully automated test control. The TestStar IIs digital controller Software is the heart of the TestStar IIs system. Through a series of menus, the main TestStar IIs window provides quick access to all the controls needed for test setup. These controls include windows for assigning transducers, setting limits, auto-zeroing sensors, and—when necessary—setting up parameters such as error limits and tuning. The various windows are easily accessible but conveniently hidden when not needed. The TestStar IIs control system runs on the Windows NT operating system from Microsoft Corporation. This software environment simplifies data sharing and makes it easy to integrate the TestStar IIs control system into computer networks.

6.2.3 Test Procedure

There are two purposes of the tensile tests: The main purpose is to obtain both initial and rupture strengths of the laminate specimens; while the other is to verify the stress distribution.

While the ultimate (rupture) strength can easily be obtained from the recorded “load vs. displacement” curves, the initial failure strength for each specimen was manually recorded since no noticeable drop associated with initial failure was identified from the “load vs. displacement” curves. During the tests, the tensile specimen needs to be
monitored by two persons with one on each side. Once a fiber breakage was noticed, or a cracking sound was heard (which is an indication of fiber breakage), the corresponding load was recorded. In almost all cases, the visible fiber breakage and audible cracking sound occur at the same time. All 48 specimens listed in Tables 5 and 6 were tested to obtain the initial and rupture strength for laminates with various hole size and gage length.

![Diagram of Strain Gages](image)

**Fig. 11 Location of Strain Gages**

Four specimens were used for strain measurement purpose. These specimens are listed in Table 5 with an "*". Three strain gages were attached to each of these specimen: a CEA-06-125UW-350 single gage, with an active area of 0.125" × 0.10" (3.18mm × 2.54mm), attached at the hole edge and two 0/90 rosette (CEA-06-125WT-350), with an active area of 0.125" × 0.18" (3.18mm × 4.57mm), at the other locations, as shown in Fig. 11. The average stress over the gage length at the locations can be calculated in terms of the measured strains by the following relation:

\[
\begin{bmatrix}
\sigma_1 \\ \\ \\
\sigma_2
\end{bmatrix} = \frac{E_{11}}{1 - \nu_{12} \nu_{21}} \begin{bmatrix}
E_{11} & \nu_{21} E_{11} \\ \\ \\
\nu_{12} E_{22} & E_{22}
\end{bmatrix} \begin{bmatrix}
\varepsilon_1 \\ \\ \\
\varepsilon_2
\end{bmatrix}
\]

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CHAPTER 7. RESULTS AND DISCUSSIONS

7.1 Stress Concentration Factor

Stress concentration factors for unidirectional and crossply laminates with various hole sizes and specimen lengths can be predicted by the developed theoretical models. A comparison between the proposed model and finite element analysis as well as prediction based on Tan’s model is provided in Table 8 and Table 9 for unidirectional and crossply laminates, respectively. The model results are based on the finite width model, as discussed in Chapter 4. Although the strain gages were used in the experimental work for measuring the strain distribution, because of the small size of the holes (0.4" and 0.8") and specimen width (2"), the strains measured at the hole edge were actually the average strains of the gage length (0.125"). The stress distribution curves from the proposed model show that for the case of $d/W = 0.2$, the normal stress along the loading direction will decrease from its maximum value at the hole edge, $\sigma = 3.96 \rho$, to $\sigma = 1.97 \rho$ at the point which is 0.125" away from the hole edge, where $\rho$ is the applied tensile stress. In other words, 50% of the peak value will be lost after a strain gage length. In addition, the actual size of a strain gage is measured by its “matrix size” which is larger than the active size. This means that the active area of a strain gage will never actually at the hole edge. Therefore, using strain gage to measure the stress concentration factor for such a small size of specimen is meaningless. It is for this reason, the experimental work conducted in this research focused on the determination of laminate strength rather than stress concentration factors. Thus the comparison in Table 8 and Table 9 does not include experimental data.
Table 8. Hole Size Effect on SCF (Unidirectional Laminates)

<table>
<thead>
<tr>
<th>d/W</th>
<th>Proposed Model</th>
<th>FEA*</th>
<th>Tan's Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>3.82</td>
<td>3.89</td>
<td>3.79</td>
</tr>
<tr>
<td>0.2</td>
<td>3.96</td>
<td>4.03</td>
<td>3.85</td>
</tr>
<tr>
<td>0.3</td>
<td>4.23</td>
<td>4.29</td>
<td>3.97</td>
</tr>
<tr>
<td>0.4</td>
<td>4.71</td>
<td>4.72</td>
<td>4.15</td>
</tr>
<tr>
<td>0.5</td>
<td>5.44</td>
<td>5.39</td>
<td>4.45</td>
</tr>
</tbody>
</table>

* L/W = 2.5 is used in FEA modeling.

Table 9. Hole Size Effect on SCF (Crossply Laminates)

<table>
<thead>
<tr>
<th>d/W</th>
<th>Proposed Model</th>
<th>FEA</th>
<th>Tan’s Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>3.60</td>
<td>3.67</td>
<td>3.59</td>
</tr>
<tr>
<td>0.2</td>
<td>3.74</td>
<td>3.81</td>
<td>3.64</td>
</tr>
<tr>
<td>0.3</td>
<td>3.99</td>
<td>4.05</td>
<td>3.75</td>
</tr>
<tr>
<td>0.4</td>
<td>4.45</td>
<td>4.46</td>
<td>3.93</td>
</tr>
<tr>
<td>0.5</td>
<td>5.14</td>
<td>5.09</td>
<td>4.21</td>
</tr>
</tbody>
</table>

* L/W = 2.5 is used in FEA modeling.

The finite width of laminate, or size of hole, has considerable effect on stress concentration at the hole edge. Compared with an infinite laminate containing a circular hole for which SCF = 3.78 and 3.57 for unidirectional and crossply Scotchply laminate, respectively. For unidirectional laminates, when d/W = 0.1, there is only a 1.06% increase in SCF value. As d/W ratio increases, the SCF increases by 4.76%, 11.90%, 32.96%, and 43.92% for d/W = 0.2, 0.3, 0.4, and 0.5, respectively. For crossply laminates, there are increases of 0.84%, 4.76%, 11.76%, 24.65%, and 43.98% for the cases d/W = 0.1, 0.2, 0.3, 0.4, and 0.5, respectively.
From Table 8 and Table 9 it can be seen that the deviation between the proposed model and Tan's model increases as the $d/W$ ratio increases. When $d/W = 0.1$, the differences are 0.78% and 0.3%, for unidirectional and crossply laminate, respectively. The deviations become 18.2% and 18.1% for the two types of laminates when $d/W = 0.5$. The results on both Tables 8 and 9 show that the proposed model is much more agreeable with the FEA results than the Tan's model is. Actually, the differences between the model predictions and FEA results are within 2% for both unidirectional and crossply laminates. Figures 12 and 13 plot the same results provided in the two tables.

Table 10 and Table 11 show the effect of specimen gage length on stress concentration factors. The stress concentration factors for the specimens with 2" width, 1" hole diameter and five different gage lengths, $L/W = 1.0, 1.5, 2.0, 3.0, \text{ and } 4.0$, are evaluated. It can be seen that for both unidirectional and crossply laminates, the length of specimen does not affect the value of stress concentration factor noticeably when $L/W \geq 3.0$. Using the stress concentration factor at $L/W = 4.0$ as reference, the differences of SCFs for $L/W = 1.0, 1.5, 2.0, \text{ and } 3.0$ are 68.31%, 27.89%, 3.23%, and 0.38%, respectively, for unidirectional laminates, and are 68.47%, 27.71%, 3.21%, and 0.62%, respectively, for crossply laminates. These predictions are based on the second theoretical model proposed in this study. The model predictions are consistent with the results from finite element analysis. On the other hand, since Tan did not consider the effect of plate length, his model cannot be used to evaluate the effect of plate length on stress concentration factors. Figures 14 and 15 plot the results of Tables 10 and 11.
Fig. 12  Effect of Finite Width on SCF (Unidirectional Scotchply)
Fig. 13  Effect of Finite Width on SCF (Crossply Scotchply)
### Table 10. Effect of Specimen Gage Length on SCF
(Unidirectional Laminates, d/W = 0.5)

<table>
<thead>
<tr>
<th>L/W</th>
<th>Proposed Model</th>
<th>FEA</th>
<th>Tan's Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>8.87</td>
<td>8.72</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>6.74</td>
<td>6.38</td>
<td>4.45</td>
</tr>
<tr>
<td>2.0</td>
<td>5.44</td>
<td>5.39</td>
<td></td>
</tr>
<tr>
<td>3.0</td>
<td>5.29</td>
<td>5.20</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>5.27</td>
<td>5.19</td>
<td></td>
</tr>
</tbody>
</table>

### Table 11. Effect of Specimen Gage Length on SCF
(Crossply Laminates, d/W = 0.5)

<table>
<thead>
<tr>
<th>L/W</th>
<th>Proposed Model</th>
<th>FEA</th>
<th>Tan's Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>8.39</td>
<td>8.23</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>6.36</td>
<td>6.03</td>
<td>4.21</td>
</tr>
<tr>
<td>2.0</td>
<td>5.14</td>
<td>5.09</td>
<td></td>
</tr>
<tr>
<td>3.0</td>
<td>5.01</td>
<td>4.91</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>4.98</td>
<td>4.90</td>
<td></td>
</tr>
</tbody>
</table>
Fig. 14  Effect of Finite Length on SCF (Unidirectional Scotchply)
Fig. 15  Effect of Finite Length on SCF (Crossply Scotchply)
7.2 Normal Stress Distribution

Similar to the case of isotropic plates, the profile of the stress distribution in a finite width unidirectional and cross ply laminate containing a circular hole is determined by two coupled factors. One is the existence of the circular hole, which results in the highest stress at the hole edge (stress concentration). The other is the finite width effect (or hole size effect, or straight edge effect) which changes not only the stress concentration level but also the stress distribution profile. Tan’s model is based on the Timoshenko solution for infinite isotropic plate, while the theoretical models proposed in this study are based on the solutions for an isotropic plate with finite width or both finite width and length. Figures 16 (a) through (e) show the normal stress distribution along y-axis for unidirectional laminates when \( d/W = 0.1, 0.2, 0.3, 0.4, \) and 0.5, respectively. The stress distribution curves for crossply laminates for different \( d/W \) ratios are given in Figs. 17 (a) through (e). The predictions from the proposed model and from Tan’s model, along with finite element results, are provided in these figures. The results of model predictions are based on the finite width correction model.

As expected, Figs. (16) and (17) show that when \( d/W \) is small, such as \( d/W = 0.1 \), three stress distribution curves, from proposed model, FEA and Tan’s model, are almost identical, which suggests that the case is quite close to the infinite plate. As the \( d/W \) ratio increase, i.e., as the hole size increases, the difference between the proposed model and Tan’s model increases. This difference is characterized in three aspects: the stress value at the hole edge, which is represented by stress concentration factor; the stress value at the edge of the plate; and the shape of the curve. The stress concentration factors for various
Prediction of Stress along $x = 0$ Section
(Unidirectional Scotchply, $d/W = 0.1$)

Fig. 16 Normal Stress Distribution (Unidirectional Scotchply)
(a) $d/W = 0.1$  (b) $d/W = 0.2$  (c) $d/W = 0.3$
(d) $d/W = 0.4$  (e) $d/W = 0.5$

(Figure continued)
Prediction of Stress along $x = 0$ Section
(Unidirectional Scotchply, $d/W = 0.2$)

Fig. 16 (b)

(Figure continued)

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Prediction of Stress along $x = 0$ Section
(Unidirectional Scotchply, $d/W = 0.3$)

![Graph showing stress distribution](image)

Fig. 16 (c)

(Figure continued)
Prediction of Stress along $x = 0$ Section
(Unidirectional Scotchply, $d/W = 0.4$)

Fig. 16 (d)
Prediction of Stress along $x = 0$ Section
(Unidirectional Scotchply, $d/W = 0.5$)

![Graph of Stress Distribution](image-url)

Fig. 16 (e)
Prediction of Stress along $x = 0$ Section 
(Crossply Scotchply, $d/W = 0.1$)

Fig. 17  Normal Stress Distribution (Crossply Scotchply) 
(a) $d/W = 0.1$  (b) $d/W = 0.2$  (c) $d/W = 0.3$
(d) $d/W = 0.4$  (e) $d/W = 0.5$

(Figure continued)
Prediction of Stress along $x = 0$ Section
(Crossply Scotchply, $d/W = 0.2$)

Fig. 17 (b)
Prediction of Stress along $x = 0$ Section
(Crossply Scotchply, $d/W = 0.3$)

![Graph showing stress distribution](image)

$\sigma_x/\sigma_0$ vs. $y/(W/2)$

Fig. 17 (c)

(Figure continued)
Prediction of Stress along x = 0 Section
(Crossply Scotchply, d/W = 0.4)

Fig. 17 (d)

(Figure continued)
Prediction of Stress along $x = 0$ Section
(Crossply Scotchply, $d/W = 0.5$)

![Graph showing stress distribution](image)

- **Proposed Model**
- **FEA**
- **Tan's Model**

**Fig. 17 (e)**

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hole size have been discussed in the section 7.1. We look at the stress at the two free straight edges. For unidirectional laminates, when \(d/W = 0.1\), the difference between the proposed model and Tan’s model is only 2.0%; the differences increase to 5.2%, 19.1%, and 48.0% as the \(d/W\) ratio increases to 0.2, 0.3, and 0.4, respectively. When \(d/W = 0.5\), the difference reaches the maximum – 66.7%. For the crossply laminates, the differences between the proposed model and Tan’s model, in predicting the normal stress at the free edge, are 7.45%, 13.3%, 19.1%, 52.1%, and 76.5%, for \(d/W = 0.1, 0.2, 0.3, 0.4,\) and 0.5, respectively. It can be found from Figs. 16 and 17 that the predictions from the proposed model are very consistent with the results from finite element analysis. The deviations between the model prediction and FEA results are, for unidirectional laminates, 0.5%, 2.1%, 2.3%, 2.9% and 14.1% for \(d/W = 0.1, 0.2, 0.3, 0.4,\) and 0.5, respectively. For crossply laminates, the differences are 0.6%, 1.1%, 1.0%, 6.0% and 13.2%, for these five different \(d/W\) ratios.

### 7.3 Tensile Strength

Table 12 and Table 13 present the experimentally determined tensile strength of unidirectional and crossply laminate specimens, respectively. Both ultimate (rupture) strength and initial failure strength are given in the tables. These results are also plotted in Figs. 18 and 19 along with the initial strength prediction from the proposed model. It can be seen that the test results from the two sets of specimens, for both unidirectional laminates and crossply laminates, are quite consistent. As anticipated, both initial failure strength and ultimate strength decrease as the hole size, or \(d/W\) ratio, increases.
### Table 12. Experimental Tensile Strength of Unidirectional Laminates

<table>
<thead>
<tr>
<th>d/W</th>
<th>Ultimate Strength (ksi)</th>
<th>Initial Failure Strength (ksi)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Set I</td>
<td>Set II</td>
</tr>
<tr>
<td>0.05</td>
<td>88.47</td>
<td>90.39</td>
</tr>
<tr>
<td>0.10</td>
<td>72.34</td>
<td>79.47</td>
</tr>
<tr>
<td>0.20</td>
<td>66.39</td>
<td>64.43</td>
</tr>
<tr>
<td>0.30</td>
<td>56.75</td>
<td>53.85</td>
</tr>
<tr>
<td>0.40</td>
<td>48.08</td>
<td>47.72</td>
</tr>
<tr>
<td>0.50</td>
<td>41.35</td>
<td>37.69</td>
</tr>
<tr>
<td>0.60</td>
<td>32.70</td>
<td>31.06</td>
</tr>
<tr>
<td>0.75</td>
<td>20.20</td>
<td>19.24</td>
</tr>
</tbody>
</table>

### Table 13. Experimental Tensile Strength of Crossply Laminates

<table>
<thead>
<tr>
<th>d/W</th>
<th>Ultimate Strength (ksi)</th>
<th>Initial Failure Strength (ksi)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Set I</td>
<td>Set II</td>
</tr>
<tr>
<td>0.05</td>
<td>43.29</td>
<td>43.16</td>
</tr>
<tr>
<td>0.10</td>
<td>37.61</td>
<td>37.27</td>
</tr>
<tr>
<td>0.20</td>
<td>30.97</td>
<td>31.30</td>
</tr>
<tr>
<td>0.30</td>
<td>26.43</td>
<td>27.66</td>
</tr>
<tr>
<td>0.40</td>
<td>23.75</td>
<td>24.33</td>
</tr>
<tr>
<td>0.50</td>
<td>21.22</td>
<td>21.02</td>
</tr>
<tr>
<td>0.60</td>
<td>18.80</td>
<td>17.69</td>
</tr>
<tr>
<td>0.75</td>
<td>10.58</td>
<td>11.07</td>
</tr>
</tbody>
</table>

A comparison between the tested results and theoretical prediction is provided in Table 14 and Table 15, where the experimental initial strength values are the average of the
two specimens with the same d/W ratio. For unidirectional laminates, the differences between the model prediction and test data range from 1.23% for the case of d/W = 0.1 to 11.65% for the case of d/W = 0.60. For crossply laminates, the deviations between the model prediction and experimental results are under 12%, with a minimum value of 4.41% for the case d/W = 0.20 and a maximum value of 11.17% at d/W = 0.60. No relationship between the deviation and the d/W ratio was observed.

![Graph showing experimental and predicted tensile strength for unidirectional Scotchply.](image)

Fig. 18  Experimental and Predicted Tensile Strength (Unidirectional Scotchply)
Fig. 19  Experimental and Predicted Tensile Strength (Crossply Scotchply)
Table 14. Comparison of Model Prediction and Test Results
(Unidirectional Laminates)

<table>
<thead>
<tr>
<th>d/W</th>
<th>Initial Strength (Test) (ksi)</th>
<th>Initial Strength (Model Prediction) (ksi)</th>
<th>Deviation (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>57.89</td>
<td>54.31</td>
<td>6.18</td>
</tr>
<tr>
<td>0.10</td>
<td>54.62</td>
<td>53.95</td>
<td>1.23</td>
</tr>
<tr>
<td>0.20</td>
<td>52.89</td>
<td>51.90</td>
<td>1.87</td>
</tr>
<tr>
<td>0.30</td>
<td>44.62</td>
<td>48.66</td>
<td>9.05</td>
</tr>
<tr>
<td>0.40</td>
<td>42.12</td>
<td>43.60</td>
<td>3.51</td>
</tr>
<tr>
<td>0.50</td>
<td>35.00</td>
<td>37.84</td>
<td>8.11</td>
</tr>
<tr>
<td>0.60</td>
<td>27.12</td>
<td>30.28</td>
<td>11.65</td>
</tr>
<tr>
<td>0.75</td>
<td>18.53</td>
<td>19.82</td>
<td>6.96</td>
</tr>
</tbody>
</table>

Table 15. Comparison of Model Prediction and Test Results
(Crossply Laminates)

<table>
<thead>
<tr>
<th>d/W</th>
<th>Initial Strength (Test) (ksi)</th>
<th>Initial Strength (Model Prediction) (ksi)</th>
<th>Deviation (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>29.96</td>
<td>28.75</td>
<td>9.86</td>
</tr>
<tr>
<td>0.10</td>
<td>26.17</td>
<td>28.55</td>
<td>4.71</td>
</tr>
<tr>
<td>0.20</td>
<td>26.33</td>
<td>27.49</td>
<td>4.41</td>
</tr>
<tr>
<td>0.30</td>
<td>23.91</td>
<td>25.76</td>
<td>7.74</td>
</tr>
<tr>
<td>0.40</td>
<td>21.64</td>
<td>23.08</td>
<td>6.65</td>
</tr>
<tr>
<td>0.50</td>
<td>18.54</td>
<td>20.03</td>
<td>8.04</td>
</tr>
<tr>
<td>0.60</td>
<td>14.42</td>
<td>16.03</td>
<td>11.17</td>
</tr>
<tr>
<td>0.75</td>
<td>9.52</td>
<td>10.49</td>
<td>10.19</td>
</tr>
</tbody>
</table>
The effect of specimen gage length on the initial strength of unidirectional and crossply laminates have also been studied and the results are shown in Tables 16 and 17. It is found that the strength increases as the L/W ratio increases for both types of laminates. However, the initial strength at L/W = 3.2 is only slightly greater than that when L/W = 2.5. Specifically, there is only 2.2% (test) or 1.69% (model prediction) increase in strength for unidirectional laminate, and an increase of 1.19% (test) or 4.64% (model prediction) for crossply laminates. The differences between model prediction and test results are within 11% for unidirectional laminates and within 12% for crossply laminates.

### Table 16. Effect of Specimen Gage Length on Initial Strength
(Unidirectional Laminates, d/W = 0.5)

<table>
<thead>
<tr>
<th>L/W</th>
<th>Test (ksi)</th>
<th>Model Prediction (ksi)</th>
<th>Deviation (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Set I</td>
<td>Set II</td>
<td>Average</td>
</tr>
<tr>
<td>1.5</td>
<td>29.33</td>
<td>28.01</td>
<td>28.67</td>
</tr>
<tr>
<td>2.0</td>
<td>33.52</td>
<td>34.00</td>
<td>33.76</td>
</tr>
<tr>
<td>2.5</td>
<td>35.77</td>
<td>34.23</td>
<td>35.00</td>
</tr>
<tr>
<td>3.0</td>
<td>35.85</td>
<td>35.70</td>
<td>35.78</td>
</tr>
</tbody>
</table>

### Table 17. Effect of Specimen Gage Length on Initial Strength
(Crossply Laminates, d/W = 0.5)

<table>
<thead>
<tr>
<th>L/W</th>
<th>Test (ksi)</th>
<th>Model Prediction (ksi)</th>
<th>Deviation (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Set I</td>
<td>Set II</td>
<td>Average</td>
</tr>
<tr>
<td>1.5</td>
<td>14.02</td>
<td>13.49</td>
<td>13.76</td>
</tr>
<tr>
<td>2.0</td>
<td>16.11</td>
<td>15.75</td>
<td>15.93</td>
</tr>
<tr>
<td>2.5</td>
<td>18.10</td>
<td>18.98</td>
<td>18.54</td>
</tr>
<tr>
<td>3.0</td>
<td>18.60</td>
<td>18.92</td>
<td>18.76</td>
</tr>
</tbody>
</table>
7.4 Limitations of Proposed Models

In the development of two theoretical models, a Width Correction Function in Chapter 4, Eq. (43), and a Size Correction Function in Chapter 5, Eq. (66), were defined. These two functions are both derived from isotropic material: no anisotropic material properties were involved in the derivation. This is a major assumption in the theoretical analysis. For an anisotropic laminate with a circular hole, the stress distribution under a tensile load is determined by two coupled factors: one is the plate geometry, the other is the material properties and laminate configuration. These are two coupled factors. But the assumption used in this analysis actually decoupled these two factors. It assumes that the profile (shape) of the stress distribution curve is determined by the plate geometry; while value of the stress is decided by the material properties and laminate configuration. From the experimental work conducted in this study on unidirectional and crossply Scotchply laminate specimens and the finite element analysis, the theoretical models based on this simplification work quite well for these two types of laminates in predicting stress concentration factor, normal stress distribution, and initial tensile strength.

Both theoretical models proposed in this study are single-stress models. That is to say, only one stress component, the normal stress along the loading direction, is evaluated due to the extreme difficulties in the stress analysis. As a result, the failure criterion used in the strength analysis is also a single component criterion—the maximum stress criterion. It is known that the maximum stress criterion works only in the applications where the failure is single stress dominated. The experiments conducted in this study show that for both unidirectional and crossply laminated specimens, the failures are fiber dominated, i.e.,
the fiber breakage. Since the fiber breakage occurs at the cross-section perpendicular to the loading direction, the implication of tensile stress dominant is evident. This observation is consistent with other published experimental results. But for the laminates other than unidirectional or crossply, this may not be the case. For example, experiments by Wass et al. (1989) and Tan et al. (1994) showed that [±45°]_m laminates experienced matrix cracking along ±45° while [±30°/±60°]_m laminates exhibited mixed failure modes along 30° and/or 60°. For these cases, transverse normal stress and shear stress have to be considered and, correspondingly, the interactive failure criterion should be used. Therefore, the proposed theoretical models can be used to predict the initial tensile strength of unidirectional and crossply laminates containing a circular hole, but they cannot be directly used for general composite materials.
CHAPTER 8. CONCLUSIONS

8.1 Summary

A systematic study has been conducted in this research on a laminate containing a circular hole subjected to tension. Efforts have been made in three aspects: theoretical model development, computer simulation using finite element method, and experimental evaluation using unidirectional and crossply Scotchply laminates.

Theoretically, two models have been proposed to describe the normal stress distribution of the laminate with finite width and finite length. By defining a width correction function and a size correction function, the normal stress in a laminate with finite width and finite size can be predicted in terms of the normal stress in an infinite laminate. The introduction of both the width correction function and the size correction function is based upon the elastic solution of finite and infinite isotropic plates containing a circular hole under uniform tension.

Experimental work has been carried out on laminated specimens made of unidirectional and crossply Scotchply laminates. Width effect, or hole size effect, has been studied using 2" wide specimens with hole size varying from 0.1" to 1.5" (the corresponding d/W ratio from 0.05 to 0.75). Specimens with different lengths have also been tested. Both initial tensile strength and ultimate (rupture) strength have been obtained for specimens with various hole sizes and gage lengths. Strain gages have also been used in several specimens to check the theoretical prediction of normal stress distribution.

Finite element method has been extensively used in the stress analysis of the unidirectional and crossply laminates in this study using COSMOS software, which has the
capability to model laminated composites. Composite quadrilateral plate elements were used for the laminates. A total of 29 cases have been studied for laminates with various hole sizes and gage lengths.

8.2 Conclusions

Based on the results from the theoretical, experimental, and numerical (FEA) efforts, the following conclusions have been reached:

The proposed theoretical models can be used to provide predictions for unidirectional and crossply laminated composite containing a circular hole subjected to tensile load. These predictions include the stress concentration factor at the hole edge, the normal stress distribution along the \( x = 0 \) cross section, and initial tensile strength based on maximum stress failure criterion. For the Scotchply laminates used in this study, the predictions from the models have been consistent with experimental data as well as the results from finite element analysis. When predicting the stress concentration factors, the differences between the model predictions and FEA results are within 2% for both unidirectional and crossply laminates with \( d/W \) ratios up to 0.5, compared with 18% difference between the prediction from Tan’s model and FEA results. In predicting the initial tensile strength, the differences between the model prediction and test data range from 1.23% to 11.65% for unidirectional laminates. For crossply laminates, the deviations between the model prediction and experimental results are under 12%.

The finite width of laminate, or size of hole, has considerable effect on stress concentration at the hole edge. Using stress concentration factors of infinite laminate for finite with laminated plate will result in significant deviation unless the hole size is
relatively small. When \( d/W = 0.5 \), the difference between the two stress concentration factors is about 45\% for both unidirectional and crossply Scotchply laminates. However, if the ratio \( d/W \) is smaller than 0.2, the deviation will be less than 5\% for both unidirectional and crossply laminates.

The finite width of laminate, or size of hole, also has noticeable effect on normal stress distribution which is characterized by the stress increase at hole edge and decrease at the free straight edges. The proposed theoretical models can describe the profile of normal stress distribution which is very consistent with finite element results. In contrast, the stress change near the free straight is not exhibited by Tan's model which is based on infinite plate solution.

The finite width of laminate or size of hole, has profound effect on the initial and ultimate (rupture) strength of the laminates. Both strengths reduce as the \( d/W \) ratio increases. When \( d/W \) is small, the ultimate (rupture) strength of laminate is distinctively greater than the initial tensile strength. The difference between the two strengths decreases as the \( d/W \) ratio increases.

Specimen gage lengths also affect stress concentration factors. However, when the ratio \( L/W \) larger than 3, the effect becomes insignificant. Using stress concentration factor at \( L/W = 4.0 \) as reference, the differences of SCFs for \( L/W \) between 1.0 and 3.0 ranges from 0.4\% to 68\% for unidirectional laminates, and from 0.6\% to 69\% for crossply laminates. Therefore, for tensile specimens with \( L/W \) greater than 3, the effect of specimen gage length is negligible.
Since there is only one normal stress component which is described in the proposed models, the prediction of initial strength is based on the maximum stress criterion. Therefore, the validity of the proposed models in predicting initial strength is limited to the cases where failure is dominated by a single stress, such as a unidirectional or crossply laminate containing a circular hole subjected to uniaxial tension. For general angle ply composites, or unidirectional and crossply laminates subjected to loads other than uniform tension, interactive failure criteria should be used.

8.3 Extension of This Study

This research could be extended in the following aspects:

(1) Effect of stacking sequence While the theoretical models proposed in this study are claimed to be applicable to unidirectional and crossply laminates, the configuration of the crossply laminate which has been studied is limited to [0/6(90/0)]. It is not clear whether the stacking sequence will affect the stress concentration factor, the stress distribution at different layers, and the initial and ultimate tensile strength.

(2) Laminates subjected to biaxial tension The theoretical model discussed in Chapter 5 can be extended to the case of biaxial tension through some modifications. However, the biaxial tensile test has been a challenging task. Not only a biaxial tensile tester is required, but also the grip fixture needs to be carefully designed such that the biaxial specimen fails at the hole edge rather than the grip area. Researchers at University of Houston has been working on the topic for years. They designed a special grips through optimization analysis. These grips work well...
for biaxial specimens with large hole size (such as $d/W > 0.5$) but not for small to medium hole size in which failure frequently occurs at the gripping areas.

(3) Laminates under bending  Laminated composites subjected to bending is very common in engineering applications. Obviously, from theoretical point of view, this would be a much difficult topic compared with the case of tension. A theoretical solution to the problem would be, if not impossible, extremely difficult to achieve. One major reason is that for a laminate under bending, not only the normal stress, but also the interlaminar shear stress have to be considered. A few studies have been reported on the subject including the work by the author of this dissertation. A common assumption used in these studies is the exclusion of shear stress. It should be understood that such an assumption does not reflect the negligence of the author on the nature of the subject, but rather a pursuit for a first trial to the problem. It is for this reason, the details of the work on this subject by the author have been placed in the Appendix A of this dissertation in stead of a regular chapter.

(4) In finite element analysis conducted in this study, composite laminated elements were used which are based on the classical laminated theory. Improved results can be achieved by using bending and stretching elements based on refined theory for thick composite plates (Voyiadjis and Pecquet, 1988; Voyiadjis and Baluch, 1988). With the bending and stretching elements, the influence of transverse normal strain and stress, as well as the transverse shear effects can be considered.
REFERENCES

References regarding the researches on the subject by others


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References regarding the author's work on composite materials


APPENDIX A. BENDING STRENGTH PREDICTION
– A PRELIMINARY MODEL

Despite the fact that composite laminated plates with cutouts under bending are very common in engineering design, no studies have been reported on this topic. It has been found from the literature survey that all of the reported work are limited to the in-plane loadings, especially tension. The main reason is the relative easiness in stress analysis. It is known that a failure strength analysis requires the knowledge of the finite stress distribution in the vicinity of the notch. In the case of in-plane loadings, an approach based on the overall laminate properties or an approach based on in-plane properties of each ply could be valid depending on the laminate configurations. But both approaches may no longer be valid in the case of bending, in which the bending stress varies in the thickness direction.

As a first trial, an attempt was made to extend the Whitney and Nuismer’s point stress model and average stress model to the case of bending. In this preliminary work, an orthotropic laminated composite containing an elliptical hole subjected to pure bending was studied.

A.1 Extended Point Stress and Average Stress Models

For an isotropic laminate under in-plane tension, the normal stress is uniformly distributed across the laminate thickness. But when the laminate is subjected to a bending load, the stress varies in the thickness direction. This difference also exists in the laminated plates. As an example, consider a cross-ply laminated plate. Under the in-plane tensile load, all of the 0° plies are in the same stress level, and rest all the 90° plies are also in the
same stress level. If the stress in one 0° ply (or 90° ply) reaches the ultimate strength, all
the 0° plies (or 90° plies) will fail simultaneously. But for the cross-ply laminates under
bending, each 0° ply or 90° ply is in the different stress level. The stress at one point or
the average stress over a length may not represent the stress level or stress concentration
of the laminate. Therefore, these criteria may not be directly applied to the case of bending.
It is more reasonable to include the stress variation across the laminate thickness in the
failure models. By this consideration, the following failure models were presented to
evaluate the notched strength of laminates under bending.

A.1.1 Modified Point Stress Failure Model

This model extends Whitney and Nuismer’s point stress model by defining the
"point stress" as the average stress of the plies with same fiber angle across the laminate
thickness. For the laminates other than on-axis unidirectional or cross-ply, the maximum
stresses usually will not occur along a radial line parallel to the applied bending moment
but at some angles. Thus the modified point stress criterion should be applied along a path
which encloses the elliptical opening. The Point Stress Criterion becomes

\[ \frac{2}{h_a} \int_{0}^{h_2/2} \frac{1}{ha} \frac{z^2}{(a \cdot a)^2} \frac{\nu^2}{(b \cdot a)^2} dz = \sigma_f \]  \hspace{1cm} (A.1)

where \( \sigma_a \) is the stress of the ply with fiber angle \( \alpha \), \( h_a \) is the total thickness of the plies with
fiber angle \( \alpha \), \( h \) is the total thickness of the laminate, \( \sigma_f \) is the failure strength of the
material along or perpendicular to the fiber direction, and \( a_i \) is a characteristic parameter
to evaluate the failure strength of the laminate.

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A.1.2 Modified Average Stress Failure Model

In the case of in-plane loadings, the average stress criterion includes the contribution of the stress distribution along the radial direction in the vicinity of the hole edge. For notched laminates under bending, the contribution of stress distribution along the thickness direction should also be considered. Mathematically, this modified model can be written as

\[
\frac{2}{a_2h_a} \int_{R}^{2\cdot a_2 h/2} \int_{0}^{R} \sigma_\text{e}(\theta, r, z) dr dz = \sigma_f
\]  

(A.2)

where \( R \) is the radial distance from the edge to the center of the hole, \( \theta \) is the location of the section where maximum stress occurs, and \( a_2 \) is another characteristic parameter to measure the failure strength of the laminates under bending.

A.1.3 Approach of Strength Evaluation

Similar to the case of tension, application of point stress model and average stress model in the case of bending requires the experimental determination of the parametric lengths \( a_1 \) and \( a_2 \). In addition, unlike the case of tension, there is no analytical form of stress distribution function available for the case of bending; therefore, finite element method has be to used to provide the necessary stress information. Using this 3-in-1 approach, i.e., a combination of model, experiments, and finite element analysis, the parametric lengths \( a_1 \) and \( a_2 \) can be obtained. The results are quite expected: the parameter \( a_1 \) and \( a_2 \) vary with the hole size, and they are not material constants. Since these models cannot be used to
predict the bending strength of notched laminates, only a brief description of the related work is provided here.

A four-point bending test was performed to determine the failure load of laminated composites containing an elliptical hole under bending. The test was conducted on an Instron servohydraulic testing machine. The notched laminate samples were simply supported on both ends. Two loading points were 4 inches apart. In order to provide the deflection readings during the test, a dial gauge was attached to the bottom surface of the sample at the hole edge. The deflection at two loading points were also recorded by a personal computer.

Scotchply laminated samples containing an elliptical hole were tested. Each sample was a 9"× 3"× 1/8" [(0/90)_3/0]_s laminated plate with a central elliptical hole. The hole size along y-direction was set at 0.5", while the hole size along x-direction varied from case to case. Five aspect ratios were used: b/a = 1.0, 1.5, 2.0, 2.5, and 3.0. For each aspect ratio, two samples were tested. Two unnotched laminated plates with the same size were also tested. All the experimental data presented later were the average values of the two samples with the same configuration. Table A.1 gives the strength reduction factor which in the case of bending is defined as

\[ SR = \frac{M}{M_o} \]

where M and M_o are failure moments of notched and unnotched laminates under bending, respectively. Table A.2 provides the values and locations maximum circumferential stress.
at the hole edge based on finite element analysis. These results were obtained under an applied bending moment of 100 in-lb. The final determined parameters $a_i$ and $a_j$ are shown in Table A.3. The parameters $b$ and $a$ denote the major axis and minor axis of the elliptical opening, respectively.

Table A.1 Strength Reduction Factors

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
</tr>
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<tbody>
<tr>
<td>SR factor</td>
<td>0.8756</td>
<td>0.8500</td>
<td>0.8205</td>
<td>0.8039</td>
<td>0.7906</td>
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</table>

Table A.2 Maximum Stress at Edge of Elliptical Hole (FEA Result)

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>$\sigma_{x,max}$ at 0-ply</th>
<th>$\theta$</th>
<th>$\sigma_{x,max}$ at 90-ply</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>169.6 MPa (24.60 ksi)</td>
<td>90°</td>
<td>62.3 MPa (9.03 ksi)</td>
<td>56.25°</td>
</tr>
<tr>
<td>1.5</td>
<td>198.6 MPa (28.80 ksi)</td>
<td>90°</td>
<td>73.8 MPa (10.70 ksi)</td>
<td>65.79°</td>
</tr>
<tr>
<td>2.0</td>
<td>235.0 MPa (34.08 ksi)</td>
<td>90°</td>
<td>88.5 MPa (12.84 ksi)</td>
<td>78.07°</td>
</tr>
<tr>
<td>2.5</td>
<td>271.6 MPa (39.38 ksi)</td>
<td>90°</td>
<td>104.1 MPa (15.10 ksi)</td>
<td>80.41°</td>
</tr>
<tr>
<td>3.0</td>
<td>305.9 MPa (44.36 ksi)</td>
<td>90°</td>
<td>122.1 MPa (17.70 ksi)</td>
<td>86.08°</td>
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</tbody>
</table>

Table A.3 Parametric Lengths

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>9.17 mm (0.361&quot;)</td>
<td>9.12 mm (0.359&quot;)</td>
<td>7.16 mm (0.282&quot;)</td>
<td>6.78 mm (0.267&quot;)</td>
<td>2.03 mm (0.08&quot;)</td>
</tr>
<tr>
<td>$a_2$</td>
<td>32.79 mm (1.291&quot;)</td>
<td>26.14 mm (1.029&quot;)</td>
<td>23.65 mm (0.931&quot;)</td>
<td>23.62 mm (0.930&quot;)</td>
<td>20.34 mm (0.801&quot;)</td>
</tr>
</tbody>
</table>
A.2 Correlation of Tensile and Bending Strengths

A finite laminated plate containing a circular hole subjected to bending is a three dimensional problem. An analytical solution to the problem would be extremely difficult to achieve. With this concern, the second approach focuses on the correlation between the tensile and bending strengths of composites containing a circular hole of various sizes. Some investigations have been reported on the tensile-flexural strength ratio on composite materials such as the work done by Weibull (1951), Bullock (1974), Whitney and Knight (1980), and Wisnom (1992). However, these studies were primarily based on the unnotched sample experiments and did not concern the inclusion of a hole. Therefore, the corresponding results cannot be simply applied to the composites containing cutouts. The current study is to develop a tensile-bending strength relationship for composite plates containing a circular hole. Such a correlation would make it possible to extend the existing work on notched composites under in-plane loadings to the case of bending.

A.2.1 Basic Concept

Considering the case of an orthotropic plate containing a circular hole subjected to tensile loading. If we denote $\sigma_T$ as the maximum stress along the hole edge of an infinite width orthotropic plate, the maximum stress at the hole edge of a finite width plate is assumed to be in the form

$$\sigma_T = f_T \sigma_T^{\infty}$$  \hspace{1cm} (A.3)

where $f_T$ is a function representing edge effect. Similarly, for the case that an orthotropic plate containing a circular hole subjected to bending, we have the relation
where $\sigma_\infty$ and $\sigma_\infty^*$ are the maximum in-plane normal stresses at the hole edge of the finite and infinite width plates, respectively, and $f_B$ is the function describing the edge effect. The stresses $\sigma_T$ and $\sigma_B^*$ are related to the applied tensile load, $T$, and bending moment, $M$, respectively, i.e.,

$$\sigma_T^* = g_T T, \quad \sigma_B^* = g_B M$$

(A.5)

where $g_T$ and $g_B$ are functions of material properties and plate geometry. If the notched composite is considered to be failed when $\sigma_T^*$ or $\sigma_B^*$ reaches the failure strength of the composites, that is:

$$\sigma_f = \sigma_T = f_T \sigma_T^* = f_T g_T T_f$$

(A.6) or

$$\sigma_f = \sigma_B = f_B \sigma_B^* = f_B g_B M_f$$

(A.7)

then the failure uniaxial load $T_f$ and failure bending moment $M_f$ can simply be related as

$$f_T g_T T_f = f_B g_B M_f$$

(A.8)

Due to the complex nature of the problem, the expressions for $f_T$ and $f_B$, which represent the edge effect, are extremely difficult to obtain. In this study, we assume that $f_T = f_B$, that is, a notched composites under tensile loading is subjected to the same edge effect (or hole size effect) as it is under bending. Finite element analysis has been conducted to justify this
assumption and the results are shown in later sections. Under this assumption, the correlation between $T_f$ and $M_f$ can be simplified as

$$T_f = \left( \frac{g_b}{g_T} \right) M_f \quad (A.9)$$

It should be noted that functions $g_T$ and $g_B$ are related to the infinite width composite plate. Thus the correlation between tensile and bending strength of finite composite plate with a circular hole can be established approximately in terms of maximum stresses corresponding to infinite width plates.

### A.2.2 Orthotropic plate with a circular hole under tension and bending

For an infinite width orthotropic plate containing a circular hole under tension in the principal direction of the material, the hoop stress at the hole edge is (Lekhnitskii, 1968)

$$\sigma_\theta = T \left( \frac{E_\theta}{E_1} \right) \left\{ T \left[ \sqrt{E_1/E_2} \cos^2 \theta \right] + \left[ 1 + \frac{1}{2} \left( \sqrt{E_1/E_2} - v_{12} \right) + E_1/G_{12} \right] \sin^2 \theta \right\} \quad (A.10)$$

where $T$ is the applied tensile load expressed in force per unit area and $E_\theta$ is the modulus of elasticity along $\theta$ direction which can be evaluated in terms of engineering constants $E_1$, $E_2$, $G_{12}$, and $v_{12}$ (or $v_{21}$) from the following equation:

$$E_\theta = \frac{E_1}{\sin^4 \theta + \left( \frac{E_1}{G_{12}} - 2v_{12} \right) \sin^2 \theta \cos^2 \theta + \frac{E_1}{E_2} \cos^4 \theta} \quad (A.11)$$

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The governing equation for the deflection of an anisotropic plate is:

\[ D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4} = 0 \quad (A.12) \]

where \( D_{11} \) and \( D_{22} \) are the bending rigidity about the y- and x- axes; \( D_{66} \) is the twisting rigidity; and \( D_{16}, D_{26} \) are additional rigidities. \( D_{12} \) is related to the bending rigidities through \( D_{12} = v_{12}D_{22} = v_{21}D_{11} \). The general expression of \( w \) depends on the roots of the characteristic equation:

\[ D_{22} \mu^4 + 4D_{26} \mu^3 + 2(D_{12} + 2D_{66}) \mu^2 + 4D_{16} \mu + D_{11} = 0 \quad (A.13) \]

For orthotropic material, \( D_{16} = D_{26} = 0 \), and we define \( D_1 = D_{11}, D_2 = D_{22}, D_6 = D_{66}, \) and \( D_3 = v_{21}D_1 + 2D_6 = v_{12}D_2 + 2D_6 \). Then, the rigidities can be expressed in terms of engineering constants:

\[ D_2 = \frac{E_1 h^3}{12(1 - v_{12}v_{21})}, \quad D_3 = \frac{E_2 h^3}{12(1 - v_{12}v_{21})}, \quad D_4 = v_{21}D_1 + 2D_6 = \frac{v_{21}E_1 h^3}{12(1 - v_{12}v_{21})} + \frac{G_1}{(A.14)} \]

The characteristic equation can then be reduced to

\[ D_2 \mu^4 + 2D_3 \mu^2 + D_1 = 0 \quad (A.15) \]

or

\[ \mu^4 + 2\left( v_{12} + 2(1 - v_{12}v_{21}) \frac{G_1}{E_2} \right) \mu^2 + \frac{E_1}{E_2} = 0 \quad (A.16) \]

For the case of pure bending in the principal direction, the hoop stress at the hole edge can be obtained based on Lekhnitskii's solution (1968)
\[ \sigma_\theta = \frac{12M_\theta z}{h^3} \quad \text{(A.17)} \]

where \(h\) is the thickness of the plate and

\[ M_\theta = M \left( 1 + \frac{\sqrt{D_1 D_2}}{D(\beta + 4\lambda)} (\eta_1 \sin^4 \theta + \eta_2 \sin^2 \theta \cos^2 \theta + \eta_3 \cos^4 \theta) \right) \quad \text{(A.18)} \]

\(M\) is the applied bending load expressed as moment per unit width and

\[ D_\theta = D_1 \cos^4 \theta + 2D_3 \sin^2 \theta \cos^2 \theta + D_2 \sin^4 \theta \quad \text{(A.19)} \]

\[ \eta_1 = \alpha, \quad \eta_2 = \beta - \alpha^2 - 1 + 4\lambda(1 + \nu_{21})(1 + \alpha), \quad \eta_3 = \beta \left[ 1 - \beta - 4\lambda(1 + \nu_{21}) \right] \quad \text{(A.20)} \]

\[ \alpha = -i(\mu_1 + \mu_2), \quad \beta = -\mu_1 \mu_2 = \sqrt{D_1/D_2}, \quad \lambda = G_{12}/E_2 \quad \text{(A.21)} \]

### A.2.3 Tension-bending correlation

For an infinite width orthotropic plate under tensile load, the maximum hoop stress occurs at either \( \theta = 0^\circ, 180^\circ \) or \( \theta = \pm 90^\circ \), depending on material properties (Lekhnitskii, 1968). At \( \theta = 0^\circ, 180^\circ \),

\[ (\sigma_\theta)_{\theta=0,180^\circ} = \sigma_T^o = -T\sqrt{E_2/E_1} \quad \text{(A.22)} \]

and at \( \theta = \pm 90^\circ \),

\[ (\sigma_\theta)_{\theta=\pm90^\circ} = \sigma_T^o = T \left[ 1 + \sqrt{2(E_1/E_2 - \nu_{12}) + E_1/G_{12}} \right] \quad \text{(A.23)} \]
Failure occurs when either of the two stresses reaches the corresponding strength:

$$\sigma_y^T = \sigma_{f_2}, \quad \text{or} \quad \sigma_x^T = \sigma_{f_1}$$ \hspace{1cm} (A.24)

where $\sigma_{f_1}$ and $\sigma_{f_2}$ are the strength of materials at $x$ and $y$ directions.

For an orthotropic plate under pure bending, the maximum value of $\sigma_\theta$ occurs at $z = \pm h/2$, with either $\theta = 0^\circ, 180^\circ$ or $\theta = \pm 90^\circ$ (Lekhnitskii, 1968), depending on material properties. At $\theta = 0^\circ, 180^\circ$,

$$\left(\sigma_\theta\right)_{\theta=0,180^\circ} = \sigma_y^B = \left(1 - 4v_{21}\lambda\right)\frac{6M}{h^2} \hspace{1cm} (A.25)$$

and at $\theta = \pm 90^\circ$,

$$\left(\sigma_\theta\right)_{\theta=\pm90^\circ} = \sigma_x^B = \left(1 + \frac{\alpha\beta}{\beta+4\lambda}\right)\frac{6M}{h^2} \hspace{1cm} (A.26)$$

Failure occurs when either of the two stresses reaches the corresponding strength:

$$\sigma_y^B = \sigma_{f_2}, \quad \text{or} \quad \sigma_x^B = \sigma_{f_1}$$ \hspace{1cm} (A.27)

The relation between the tensile load and bending moment at failure, $T_f$ and $M_f$, can thus be related. If both tensile and bending failures occur at $\theta = 0^\circ, 180^\circ$,

$$T_f = \frac{6}{h^2} \sqrt{E_1/E_2} \left(1 - 4v_{21}\lambda\right) M_f \hspace{1cm} (A.28)$$

and for tensile and bending failures occurring at $\theta = \pm 90^\circ$,
\[ T_f = \frac{6 \left( 1 + \frac{\alpha \beta}{\beta + 4 \lambda} \right)}{r^2 \left( 1 + \sqrt{2 \left( \frac{E_i}{E} - v_{12} \right) + E_i/G_{12}} \right)} M_f \]  
(A.29)

To determine which equation should be used, one must check the critical stress first by comparing Eqs. (A.22) with (A.23), and Eqs. (A.25) with (A.26). For the cases that tensile and bending failures occur at different locations, the correlation can be established by selecting the related stress expressions. Generally, when \( E_1 \geq E_2 \), the maximum stress at the edge of the circular hole occurs at \( \theta = \pm 90^\circ \) for both tension and bending at principal direction. This is also true for \( E_1 \leq E_2 \) as long as the orthotropy remains moderate. As the orthotropy ratio \( E_2/E_1 \) increases, the tensile failure or bending failure, or both tensile and bending failures may occur at the location \( \theta = 0, 180^\circ \).

A.2.4 Experimental Verification

It should be noted that the relation obtained in the preceding section is developed with the assumption \( f_T = f_B \). To justify this theoretical relation, both finite element analysis and sample experiments have been performed.

In the FEA modeling, 24 cases have been analyzed with width-to-diameter ratio being 2, 3, 4, 6, 8, and 12 for each of the following material groups:

- Group 1: \( E_1 = E_2 \) (\( E_i/E_2 = 1, E_i/G_{12} = 6, v_{12} = 0.2 \))
- Group 2: Isotropic (\( E_1 = E_2 = E, G = E/2(1+v), v = 0.2 \))
- Group 3: \( E_1 > E_2 \) (\( E_i/E_2 = 4, E_i/G_{12} = 10, v_{12} = 0.2 \))
- Group 4: \( E_1 < E_2 \) (\( E_i/E_2 = 0.25, E_2/G_{12} = 10, v_{12} = 0.05 \))

In order to study the effect of orthotropy ratio on the proposed correlation,
additional 10 cases were analyzed with the ratio $E/E_2$ varying from $1/20$ to $20$. COSMOS software was used in the finite element analysis. Due to symmetry, only one fourth of the plate was modeled. In FEA analysis, orthotropic plates were modeled as single layer plates with homogeneously orthotropic properties. That is, the plate possesses the homogeneous constants $E_1$, $E_2$, $G_{12}$, and $v_{12}$ (or $v_{21}$) throughout the thickness.

Uniaxial tension and 4-point bending tests have been conducted using unidirectional and crossply Scotchply plates with a circular hole of various sizes. The dimension of the samples are $9" \times 1.5" \times 1/4"$. A total of 32 samples were used with width-to-diameter ratio being 3, 4, 6, and 12, and two samples for each configuration. An MTS testing machine

![Composite materials used in theoretical and FE analysis](image)

Fig. A.1  Composite Materials Used in Theoretical and FE Analysis
was used in conducting both tests. In the tensile test, hydraulic grips were utilized to assure the slip-free tightening at the ends of the samples. In the four-point bending test, the samples were simply supported on both ends with a span of 8 inches. Two loading points were 4 inches apart.

A.2.5 Preliminary results and discussion

A major assumption used in the theoretical analysis is that $f_T = f_B$, i.e., the width correction factor for an orthotropic plate containing a central circular hole subjected to tensile loading is assumed to be the same as that of the identical plate subjected to pure bending. To see how much deviation this assumption may cause, 28 cases have been analyzed using finite element method with width-to-diameter ratio being 2, 3, 4, 6, 8, 12, and 20 for each of the four material groups. In the FEA modelling, the case $W/d = 30$ was used as the reference of "infinite" width plate. Again, the plates were treated as homogeneously orthotropic ones. Table 5.4 lists the FEA results for the four material groups. The results are interesting: while both $f_T$ and $f_B$ vary with different width-to-diameter ratios $W/d$, the ratio of the two factors $f_T/f_B$ does not vary with $W/d$ noticeably. Further, the ratio $f_T/f_B$ is very close to 1. That is to say, the assumption $f_T = f_B$ made in the theoretical analysis is very close to the actual cases based on the FEA results for the four types of materials considered. It can be seen from Table A.4 that the larger deviation occurs at smaller $W/d$ ratios. The deviations are within 6% with the largest difference occurring at $W/d = 2$ for material group 2. As the ratio $W/d$ increases, the ratio of $f_T/f_B$ approaches 1.
Table A.4 Plate Width Correction Factors

(a) Material Group I

<table>
<thead>
<tr>
<th>W/d</th>
<th>$f_T$</th>
<th>$f_B$</th>
<th>$f_T/f_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.33</td>
<td>1.36</td>
<td>0.98</td>
</tr>
<tr>
<td>3</td>
<td>1.10</td>
<td>1.14</td>
<td>0.96</td>
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<tr>
<td>4</td>
<td>1.04</td>
<td>1.08</td>
<td>0.96</td>
</tr>
<tr>
<td>6</td>
<td>1.02</td>
<td>1.04</td>
<td>0.98</td>
</tr>
<tr>
<td>8</td>
<td>1.01</td>
<td>1.03</td>
<td>0.98</td>
</tr>
<tr>
<td>12</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>20</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
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</table>

(b) Material Group II

<table>
<thead>
<tr>
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<th>$f_B$</th>
<th>$f_T/f_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.42</td>
<td>1.34</td>
<td>1.06</td>
</tr>
<tr>
<td>3</td>
<td>1.16</td>
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<tr>
<td>6</td>
<td>1.01</td>
<td>1.01</td>
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<td>1.00</td>
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<tr>
<td>20</td>
<td>1.00</td>
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(c) Material Group III

<table>
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<tr>
<th>W/d</th>
<th>$f_T$</th>
<th>$f_B$</th>
<th>$f_T/f_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.25</td>
<td>1.27</td>
<td>0.98</td>
</tr>
<tr>
<td>3</td>
<td>1.06</td>
<td>1.10</td>
<td>0.96</td>
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<tr>
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<td>1.02</td>
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<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>12</td>
<td>1.00</td>
<td>1.00</td>
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<tr>
<td>20</td>
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(d) Material Group IV

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<th>$f_T/f_B$</th>
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<td>2</td>
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<td>0.99</td>
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<td>1.01</td>
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<td>1.00</td>
<td>1.01</td>
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<tr>
<td>20</td>
<td>1.00</td>
<td>1.01</td>
<td>1.00</td>
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</tbody>
</table>

As the result of the assumption $f_T = f_B$, the correlation between tensile and bending strengths of a finite width orthotropic plate with a central circular hole becomes the same as that of the infinite width plate. In other words, the hole size or plate width has no influence on this strength ratio. Comparisons of theoretical and finite element analysis results for four groups of materials, as addressed in the preceding section, with varying
diameter-to-width ratios, are given in Tables A5(a) through (d), where the dimensionless parameter \( f \) is defined by \( T_f = \frac{f M}{h^2} \). It can be seen that the hole size or plate width does not affect the correlation of the two strengths significantly. The maximum deviation was found in the material group 1, i.e., the material which has the identical properties in the two orthotropic principal directions. The corresponding deviation ranges from 8.3% to 10.5%. Figure A.2 shows the hole size effect on the parameter \( f \) based on the theoretical and finite element analyses.

Comparisons of the parameter \( f \) among the theoretical, finite element analysis, and experimental results are provided in Tables A.5(a) and (b), for unidirectional and crossply composites, respectively. Reasonably good agreement has been observed among these results with maximum difference being 15.5% between the theoretical prediction and experimental data for crossply composites, and 14.8% maximum difference for unidirectional composites.

**Table A.5 Comparison of Parameter \( f \)**

<table>
<thead>
<tr>
<th>(a) Material Group 1</th>
<th>(b) Material Group 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>W/d</td>
<td>FEA</td>
</tr>
<tr>
<td>2</td>
<td>3.588</td>
</tr>
<tr>
<td>3</td>
<td>3.641</td>
</tr>
<tr>
<td>4</td>
<td>3.650</td>
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<td>3.599</td>
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<td>3.566</td>
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<tr>
<td>12</td>
<td>3.563</td>
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<tr>
<td>3</td>
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<tr>
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<td>3.667</td>
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<tr>
<td>12</td>
<td>3.670</td>
</tr>
</tbody>
</table>

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Fig. A.2 Hole Size Effect on Parameter $f$
Table A.5 (Continued)

<table>
<thead>
<tr>
<th>W/d</th>
<th>FEA</th>
<th>Theory Dev.(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3.194</td>
<td>5.48</td>
</tr>
<tr>
<td>3</td>
<td>3.258</td>
<td>7.34</td>
</tr>
<tr>
<td>4</td>
<td>3.264</td>
<td>7.51</td>
</tr>
<tr>
<td>6</td>
<td>3.261</td>
<td>7.41</td>
</tr>
<tr>
<td>8</td>
<td>3.176</td>
<td>4.94</td>
</tr>
<tr>
<td>12</td>
<td>3.148</td>
<td>4.10</td>
</tr>
</tbody>
</table>

Table A.6 Comparison of Various Results

(a) Parameter f for Unidirectional Composite

<table>
<thead>
<tr>
<th>W/d</th>
<th>Theory</th>
<th>Test</th>
<th>FEA</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3.019</td>
<td>--</td>
<td>3.194 (5.48%)</td>
</tr>
<tr>
<td>3</td>
<td>3.417 (11.65%)</td>
<td>3.258 (7.34%)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3.405 (11.36%)</td>
<td>3.264 (7.51%)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>3.545 (14.84%)</td>
<td>3.261 (7.41%)</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>--</td>
<td>3.176 (4.94%)</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>3.331 (9.37%)</td>
<td>3.148 (4.10%)</td>
<td></td>
</tr>
</tbody>
</table>

(b) Parameter f for Crossply Composites

<table>
<thead>
<tr>
<th>W/d</th>
<th>Theory</th>
<th>Test</th>
<th>FEA</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3.268</td>
<td>--</td>
<td>3.588 (8.92%)</td>
</tr>
<tr>
<td>3</td>
<td>2.830 (15.48%)</td>
<td>3.641 (10.24%)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3.208 (1.87%)</td>
<td>3.650 (10.47%)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>3.016 (8.36%)</td>
<td>3.599 (9.10%)</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>--</td>
<td>3.566 (8.36%)</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>3.219 (1.52%)</td>
<td>3.563 (8.28%)</td>
<td></td>
</tr>
</tbody>
</table>
To study the effect of the degree of orthotropy on the correlation of the tensile and bending strengths, finite element analysis was performed to analyze the cases with the ratio $E_1/E_2$ varying from 1/20 to 20. The results are shown in Table A.7, Fig. A.3 and Fig. A.4 which were generated from the plate with $W/d = 4$. It was found that for $E_1/E_2 \geq 1/6$, all the failures occur at $\theta = \pm 90^\circ$. For $E_1/E_2 = 1/8$ and 1/10, both theoretical model and finite element analysis predict that tensile failure occurs at $\theta = 0, 180^\circ$ while bending failure occurs at $\theta = \pm 90^\circ$. For $E_1/E_2 = 1/15$ and 1/20, both theoretical model and finite element result predict that maximum tensile and bending stresses occur at $\theta = 0, 180^\circ$. It can be found from Table A.7 that for orthotropy ratio within the range, $1/6 \leq E_1/E_2 \leq 8$, the differences between the theoretical and finite element results are within 10%. As the ratio $E_1/E_2$ further increases, the differences between the two results increase. The maximum deviation reaches 15.34% for the case $E_1/E_2 = 20$. It is worth noting that for the cases $E_1/E_2 = 1/8$ and 1/10, the differences between theoretical and finite element results are also noticeable. In these two cases, the notched plates are subjected to compressive failure under tension (at $\theta = 0, 180^\circ$), and tensile failure under bending (at $\theta = \pm 90^\circ$). It should also be pointed out that for the cases in which the maximum tensile and bending stresses occur at different locations, the correlation between tensile and bending stress will include the ratio of material yield strength in the two corresponding directions, $\sigma_n/\sigma_{\alpha\beta}$, as explained in Eq. (A.24). For the purpose of illustration, the results shown in Table A.7, Fig. A.3 and Fig. A.4 were produced by assuming $\sigma_n/\sigma_{\alpha\beta}$ to be unit for the cases $E_1/E_2 = 1/8$ and 1/10.
### Table A.7 Effect of Orthotropy Ratio

<table>
<thead>
<tr>
<th>$E_1/E_2$</th>
<th>Theory</th>
<th>FEA</th>
<th>Deviation (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/20</td>
<td>2.147</td>
<td>2.284</td>
<td>6.00</td>
</tr>
<tr>
<td>1/15</td>
<td>2.344</td>
<td>2.570</td>
<td>8.79</td>
</tr>
<tr>
<td>1/10</td>
<td>2.764</td>
<td>3.230</td>
<td>14.43</td>
</tr>
<tr>
<td>1/8</td>
<td>3.191</td>
<td>3.601</td>
<td>11.39</td>
</tr>
<tr>
<td>1/6</td>
<td>3.691</td>
<td>3.625</td>
<td>1.82</td>
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<tr>
<td>1/4</td>
<td>3.543</td>
<td>3.523</td>
<td>0.57</td>
</tr>
<tr>
<td>1/2</td>
<td>3.272</td>
<td>3.364</td>
<td>2.73</td>
</tr>
<tr>
<td>1</td>
<td>2.985</td>
<td>3.225</td>
<td>7.44</td>
</tr>
<tr>
<td>2</td>
<td>3.022</td>
<td>3.254</td>
<td>7.13</td>
</tr>
<tr>
<td>4</td>
<td>3.019</td>
<td>3.264</td>
<td>7.51</td>
</tr>
<tr>
<td>6</td>
<td>2.995</td>
<td>3.289</td>
<td>8.94</td>
</tr>
<tr>
<td>8</td>
<td>2.967</td>
<td>3.295</td>
<td>9.95</td>
</tr>
<tr>
<td>10</td>
<td>2.909</td>
<td>3.338</td>
<td>12.85</td>
</tr>
<tr>
<td>15</td>
<td>2.875</td>
<td>3.341</td>
<td>13.95</td>
</tr>
<tr>
<td>20</td>
<td>2.819</td>
<td>3.330</td>
<td>15.34</td>
</tr>
</tbody>
</table>
Fig. A.3  Effect of Orthotropy Ratio on Parameter $f$ ($E_1 > E_2$)
Fig. A.4  Effect of Orthotropy Ratio on Parameter $f (E_1 < E_2)$
APPENDIX B: DEMONSTRATION OF FINITE ELEMENT ANALYSIS RESULTS

Finite Element Mesh for Isotropic Plate, d/W = 0.4, L/W = 2.5

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Distribution of Normal Stress, $\sigma_x$, Unidirectional Laminate, $d/W = 0.2$, $L/W = 2.5$
Distribution of Normal Stress, $\sigma_n$, Unidirectional Laminate, $d/W = 0.3$, $L/W = 2.5$
Distribution of Normal Stress, $\sigma_0$, Unidirectional Laminate, $d/W = 0.5, L/W = 2.5$
Distribution of Normal Stress, \(\sigma_{xx}\), Unidirectional Laminate, \(d/W = 0.5, L/W = 3.0\)
APPENDIX C. SUPPORTIVE DOCUMENTS

Command File of Finite Element Analysis

Finite element method has been used throughout this research work. A total of 29 cases have been carried out to generate desired information. Due to the large volume of the finite element files, it is less likely that all these files be included in this dissertation. Nevertheless, to demonstrate the procedure of the analysis, one executable input file is included here as reference.

```
C* This is an executable file which is to perform a static analysis for a [±45°] laminate subject to uniform tension.
C* COSMOS/M Geostar V1.75
C* Problem : D:\PHD\BASE Date : 6-29-98 Time : 11:25:46
---------
VIEW,0,0,1,0,
PLAN.E,Z,0,1,
ACTDMESH,PH,1
PT,1,0,0,0,
PT,2,5,0,0,
PT,3,7.5,0,0,
PT,4,0,7.5,0,
PT,5,7.5,7.5,0,
CRPCIRCLE,1,1,2,5,90,1,
CRLINE,2,3,5,
CRLINE,3,4,5,
CRBRK,1,1,1,2,0,
CRPTBRK,2,7,0,
CRPTBRK,3,7,0,
CRLINE,7,7,8,
CRLINE,8,7,9,
CRBRK,3,3,1,2,0,
CRBRK,2,2,1,2,0,
CRBRK,2,2,1,2,0,
CRBRK,10,10,1,2,0,
CRBRK,9,9,1,2,0,
CREXTR,10,10,1,Y,-5,
CREXTR,14,14,1,Y,-5,
CREXTR,13,13,1,X,-5,
SCALE,0,
PTINTCC,16,1,1,0.00005,
PTINTCC,4,14,15,1,0.00005,
```

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CRPTBRK, 64, 65, 0,
CREXTR, 64, 65, 1, Y, -1,
PTINTCC, 58, 66, 67, 1, 0.00005,
CRPTBRK, 66, 71, 0,
CRPTBRK, 67, 72, 0,
CRDEL, 68, 69, 1,
CRBK, 20, 20, 1, 2, 0,
CREXTR, 73, 73, 1, X, 1,
PTINTCC, 69, 66, 66, 1, 0.00005,
CRPTBRK, 67, 75, 0,
CRPTBRK, 69, 75, 0,
CRDEL, 71, 71, 1,
CRBK, 69, 69, 1, 2, 0,
CREXTR, 77, 77, 1, Y, -0.2,
PTINTCC, 72, 58, 58, 1, 0.00005,
CRPTBRK, 72, 79, 0,
CRDEL, 72, 72, 1,
CRDEL, 73, 73, 1,
CREXTR, 77, 77, 1, Y, -0.3,
PTINTCC, 72, 58, 58, 1, 0.00005,
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SF4PT, 2, 73, 77, 64, 27, 0,
SF4PT, 3, 79, 71, 64, 77, 0,
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SF4PT, 5, 65, 59, 63, 72, 0,
SF4PT, 6, 59, 32, 19, 63, 0,
EGROUP, 1, SHELL4L, 1, 1, 0, 0, 0, 0, 0, 0,
RCONST, 1, 1, 1, 5, 0, 1, 0, 0, 2, 1, 0,
SF4PT, 7, 27, 64, 67, 25, 0,
SF4PT, 8, 64, 65, 68, 67, 0,
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SF4PT, 10, 67, 68, 60, 25, 0,
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SF4PT, 15, 7, 52, 56, 41, 0,
SF4PT, 16, 32, 33, 34, 36, 0,
SF4PT, 17, 33, 41, 42, 34, 0,
SF4PT, 18, 36, 42, 9, 10, 0,
SF4PT, 19, 42, 57, 56, 41, 0,
SF4PT, 20, 52, 8, 38, 57, 0,
SF4PT, 21, 42, 57, 54, 9, 0,
SF4PT, 22, 57, 38, 5, 54, 0,
SF4PT, 23, 18, 13, 8, 7, 0,
SF4PT, 24, 23, 11, 13, 18, 0,
SF4PT, 25, 24, 12, 11, 23, 0,
SF4PT, 26, 2, 3, 12, 24, 0,
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M_SF, 13, 17, 1, 4, 1, 1, 1, 1,
M_SF, 19, 19, 1, 4, 1, 1, 1, 1,
M_SF, 12, 12, 1, 4, 2, 2, 1, 1,
M_SF, 18, 18, 1, 4, 2, 2, 1, 1,
M_SF, 21, 21, 1, 4, 1, 2, 1, 1,
M_SF, 20, 20, 1, 4, 2, 2, 1, 1,
M_SF, 22, 22, 1, 4, 2, 2, 1, 1,
M_SF, 23, 26, 1, 4, 3, 1, 1, 1,
M_SF, 21, 21, 1, 4, 2, 2, 1, 1,
M_SF, 20, 20, 1, 4, 2, 2, 1, 1,
M_SF, 22, 22, 1, 4, 2, 2, 1, 1,
M_SF, 23, 26, 1, 4, 3, 1, 1, 1,

MPROP, 1, EX, 8.8E6, EY, 3.6E6, GXY, 1.74E6, NUU, 0.23,
NMERGE, 1, 142, 1, 0.0001, 0, 0, 0,
NCOMPRESS, 1, 138,
C* R_CHECK, STATIC,
C*
ACTDMESH, PH, I
DND, 62, SY, 0.65, 1,
DND, 1, SX, 0.3, 2,
DND, 5, SX, 0.5, 1,
DND, 14, SX, 0.14, 1,
DND, 28, SX, 0.28, 1,
DND, 31, SX, 0.31, 1,
P, 29, -1, 2, 29, 1, 4,
P, 31, -1, 35, 2, 4,
P, 38, -1, 47, 3, 4,
NMERGE, 1, 65, 1, 0.0001, 0, 0, 0,
NCOMPRESS, 1, 65,
ACTDMESH, PH, I
C* R STATIC,
C*
PT, 80, 10, 0, 0,
PT, 81, 10, 7.5, 0,
PT, 82, 10, 10, 0,
PT, 83, 0, 10, 0,
SCALE, 0,
CRLINE, 128, 80, 81,
CRPTBRK, 128, 8, 0,
CRPTBRK, 129, 38, 0,
CRLINE, 131, 83, 82,
CRPTBRK, 131, 5, 0,
CRPTBRK, 131, 9, 0,
SF4PT, 27, 4, 9, 87, 83, 0,
SF4PT, 28, 9, 5, 86, 87, 0,
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SF4PT, 32, 38, 85, 81, 5, 0,
M_SF, 27, 27, 1, 4, 2, 1, 1,
M_SF, 28, 28, 1, 4, 3, 2, 1, 1,
M_SF, 29, 29, 1, 4, 2, 2, 1, 1,
M_SF, 32, 32, 1, 4, 2, 2, 1, 1,
M_SF, 31, 31, 1, 4, 2, 2, 1, 1,
M_SF, 30, 30, 1, 4, 2, 4, 1, 1,
NMERGE, 1, 134, 1, 0.0001, 0, 0, 0,
NCOMPRESS, 1, 131,
PEDEL, 1, 2, 100, 1,
ACTDMESH, PH, 1
DND, 66, SX, 0.66, 1,
DND, 71, SX, 0.71, 1,
DND, 96, SY, 0.96, 1,
DND, 97, SY, 0.97, 1,
PEL, 63, -1.2, 81, 2.4,
NMERGE, 1, 103, 1, 0.0001, 0, 0, 0,
NCOMPRESS, 1, 103,
ACTDMESH, PH, 1
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C*
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PT, 89, 20, 10, 0,
SCALE, 0,
CRLINE, 145, 88, 89,
CRPTBRK, 145, 84, 0,
CRPTBRK, 146, 85, 0,
CRPTBRK, 147, 81, 0,
PEDEL, 1, 2, 100, 1,
SF4PT, 33, 80, 88, 90, 84, 0,
SF4PT, 34, 84, 90, 91, 85, 0,
SF4PT, 35, 85, 91, 92, 81, 0,
SF4PT, 36, 81, 92, 89, 82, 0,
M_SF, 33, 33, 1, 4, 6, 4, 1, 1,
M_SF, 34, 36, 1, 4, 6, 2, 1, 1,
NMERGE, 1, 201, 1, 0.0001, 0, 0, 0,
NCOMPRESS, 1, 201,
ACTDMESH, PH, 1
DND, 104, SY, 0, 109, 1,
PEL, 87, -1.2, 141, 6, 4,
ACTDMESH, PH, 1
C* R_STATIC,
C*
PT, 93, 0, 12.5, 0,
PT, 94, 25, 0, 0,
SCALE, 0,
PT, 95, 25, 12.5, 0,
CRLINE, 154, 94, 95,
CRLINE, 155, 93, 95,
CRPTBRK, 154, 89, 0,
CRPTBRK, 155, 89, 0,
CRPTBRK, 155, 82, 0,
CRPTBRK, 155, 86, 0,
CRPTBRK, 155, 87, 0,
CRPTBRK, 154, 90, 0,
CRPTBRK, 161, 91, 0,
CRPTBRK, 162, 92, 0,
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M_SF,38,41,1,4,3,2,1,1,
M_SF,42,42,1,4,6,2,1,1,
M_SF,43,43,1,4,2,2,1,1,
M_SF,44,44,1,4,3,2,1,1,
M_SF,45,45,1,4,4,2,1,1,
NMERGE,1,294,1,0.0001,0,0,0,
ACTDMESH,PH,1
PEDEL,1,2,210,1,
PEL,144,-1,2,177,3,4,
DND,285,SX,0,285,1,
DND,290,SX,0,290,1,
DND,171,SY,0,173,1,
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NCOMPRESS,1,293,
ACTDMESH,PH,1
C* R_STATIC,
C*
ACTDMESH,PH,1
EGROUP,1,SHELL4L,1,4,0,0,0,0,0,0,0,
RCDEL,1,1,1,
RCONST,1,1,1,10,0,1,0,0,0,5,1,45,0,05,1,,-45,0,05,1,
RCONST,1,1,1,1,4,-45,0,05,1,45,
C*
ELSYM,1,207,1,X,1,0,
DNDL,1,AL,10000,1,
ELSYM,1,414,1,Y,1,0,
SCALE,0,
NMERGE,1,960,1,0.0001,0,0,0,
NCOMPRESS,1,960,
ACTDMESH,PH,1
ACTDMESH,PH,1
PEL,558,1,1,558,1,4,
PEDEL,558,1,558,1,
PEL,558,1,2,558,1,4,
PEDEL,558,2,558,1,
PEL,558,1,4,558,1,4,
PEDEL,558,4,558,1,
PEL,558,-1,4,591,3,4,
ACTDMESH,PH,1
DND,442,UX,0,442,1,
DND,438,UX,0,438,1,
DND,434,UX,0,434,1,
VITA

Yi Zhao was born in Shanghai, China, in October 1959. He received a bachelor of science degree in Engineering Mechanics in 1982 from Shanghai Jiao Tong University (SJTU). Then he worked as an Instructor at East China University of Science and Technology (ECUST). He received a master of engineering degree in Process Equipment from ECUST in 1987. In 1990, Yi Zhao entered the master’s program at Louisiana State University (L.S.U.). He received a master of science in Mechanical Engineering from L.S.U. in 1991. Since 1992, Yi Zhao has been working as a Research Associate in the Composite Materials Research Laboratory of Mechanical Engineering Department at L.S.U. He began his doctoral studies in Mechanical Engineering in 1994 with Civil Engineering as his minor field. During his advanced study, Yi Zhao has published ten research papers in technical journals, most of them in the composite materials field. He has also co-authored more than twenty papers during the period, which have been presented at various technical conferences. In May 1998, Yi Zhao won the second prize of the Best Student Paper Award sponsored by Association of American Chinese Professionals. In July 1998, he won the first prize in the Graduate Student Paper Competition in the 1998 ASME/JSME Joint Pressure Vessels and Piping Conference in San Diego, California. Yi Zhao has maintained a 4.0 grade point average during his graduate study at L.S.U. Yi Zhao is now a candidate for the degree of Doctor of Philosophy and expects to receive the degree in December, 1998.
Candidate: Yi Zhao

Major Field: Mechanical Engineering

Title of Dissertation: Stress and Strength of Laminated Composite Containing A Circular Hole

Approved:

Su-Sang Pang
Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

Date of Examination:

October 23, 1998