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On the Value Functions of Some Singular Optimal Control Problems.

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ON THE VALUE FUNCTIONS OF SOME SINGULAR
OPTIMAL CONTROL PROBLEMS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
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in

The Department of Mathematics

by

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Abstract

Infinite horizon singular optimal control problems with control taking values in a closed cone $U$ in $\mathbb{R}^n$ lead to a dynamic programming equation of the form:

$$\max \left[ F_2(x, v, v', v''), F_1(x, v, v') \right] = 0, \quad \text{for all } x \in \Theta,$$

where $\Theta$, the state space of the control problem, is some nonempty connected open subset of $\mathbb{R}^n$, and $F_1, F_2$ are continuous real-valued functions on $\Theta \times \mathbb{R}^2$ and $\Theta \times \mathbb{R}^3$ respectively, with the coercivity assumption for $F_2$, that is, the function $r \mapsto F_2(x, u, p, r)$ is nonincreasing on $\mathbb{R}$. A major concern is to determine how smooth the value function $v$ is across the free boundary of the problem. Linear-convex deterministic and stochastic singular control problems in dimension one are considered. We present the analysis of the above equation together with smoothness of the value function across the free boundary. The interest here is the explicitness of the results and the fact that the smooth fit principle depends on the parameters of the control problem. In particular, the value function for the stochastic control problem is founded explicitly, and at the same time, optimal controls are identified using a verification theorem.
Chapter 1
Introduction

This thesis refers to deterministic and stochastic singular optimal control problems. Sections 1.1 - 1.5, contain an introduction to these two types of optimal control problems and to the terminology used throughout the thesis. Section 1.6 is an overview of the original research exposed in this dissertation and section 1.7 contains some examples from the literature to provide the reader with some applied problems closely related to ours. The original research exposed in this thesis is contained in chapters 2 and 3.

1.1 The Deterministic Optimal Control Problem

The idea of control can be expressed as the process of influencing at will, within certain constraints, the behavior of a dynamical system to achieve a desired goal. In an important class of models the controlled dynamical system is represented by the vector differential equation

\[ \dot{x} = f(t, x(t), u(t)), \quad x(t_0) = x, \]  

(1.1)

under suitable conditions on the function \( f \) and for a class \( \mathcal{U} \) of admissible controls \( u(\cdot) \) taking values in a set \( U \). For instance, the state \( x(\cdot) \) could be the position and velocity of an aircraft, and the control \( u(\cdot) \) the forcing term or thrust.

The optimal control problem consists in optimizing some payoff function (or cost function) which depends on the trajectories and control inputs to the system. Specifically, we want to minimize over all admissible controls \( u(\cdot) \in \mathcal{U} \) a cost functional of the form

\[ J^{u(\cdot)}(x) = \ell(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t))dt, \]  

(1.2)
under suitable conditions for the real-valued functions \( L \) and \( \ell \). Here \( L \) represents a cost for displacement \( x(-) \) and control effort \( u(-) \), while \( \ell \) penalizes deviation from some desired final state at final time \( t_f \).

A useful tool for the study of optimal control problems is the value function which is a function of the initial state \( x \) defined as the infimum over \( \mathcal{U} \) of the costs, that is,

\[
v(x) = \inf_{u \in \mathcal{U}} J^u(x).
\]

If \( u^*(-) \in \mathcal{U} \) is such that it minimizes the above cost functional, then \( u^*(-) \) is called an optimal control and the associated trajectory \( x^*(-) \) an optimal trajectory. During the 1950's and 1960's optimization problems of space flight dynamics such as minimum fuel and minimum time, and those of high performance aircraft near the end of the World War II such as maximum range of aircraft for a given quantity of fuel and minimum time to climb greatly stimulated the development of deterministic optimal control theory.

1.1.1 The Pontryagin Maximum Principle (PMP)

In 1959, L.S. Pontryagin et al. presented necessary conditions for optimality in their Maximum Principle which consolidated the theory for constrained problems. The Pontryagin Maximum Principle (PMP) for the optimization problem (1.1), (1.2) states that if \( u^*(-) \) is an optimal control, then there exists a nonvanishing and smooth enough vector function \( \phi \) together with \( \lambda \leq 0 \) such that

\[
-\dot{\phi} = H_z(x^*,u^*,\phi),
\]

\[
\dot{x}^*(t) = H_\phi(x^*,u^*,\phi),
\]

\[
H(x^*(t),u^*(t),\phi(t)) = \max_{u \in \mathcal{U}} H(x^*(t),u,\phi),
\]
where the Hamiltonian $H$ is given by

$$H(x, u, \phi) = \lambda L(x, u) + \phi \cdot f(x, u).$$

But at that time the mathematical theory was not sufficient for some special problems for which the Pontryagin Maximum Principle (PMP) gave no additional information. These problems were described as singular optimal control problems and they arise in many engineering and economics applications. An optimal control is said to be *singular* if the determinant $\det(H_{uu})$ vanishes at any point along the optimal trajectory. Otherwise, it is said to be nonsingular. In particular, if the Hamiltonian $H$ is linear with respect to one or more components of some optimal control $u^*$, then the optimal control is singular. From the 1960's to the 1980's much effort was put into the development of new theory to deal with these singular problems. New necessary conditions, sufficient conditions and necessary and sufficient conditions have been found for singular extremals to be optimal. Today deterministic optimal control theory provides tools of much wider applicability to problems from diverse areas of engineering, life sciences, economics, finance and management science.

### 1.2 The Stochastic Optimal Control Problem

The study of optimal control for continuous time Markov processes began in the early 1960's with the stochastic linear regulator problem which applies to a large number of design problems in engineering. Until that time some imperfections, noise or disturbances affecting control systems had been ignored in the study of deterministic control theory. Optimal stochastic control theory deals with control systems in which random system disturbances are allowed.

Consider a system model in which the state process $x$ is a finite dimensional diffusion which evolves according to an autonomous system of stochastic differential
equations. The dynamics of the state process $x(t)$ being controlled are governed by a stochastic differential equation of the form

$$dx = f(x(t), u(t))dt + \sigma(x(t), u(t))d\omega(t), \quad x(0) = x, \quad (1.4)$$

where $f$ and $\sigma$ satisfy the usual conditions under which this s.d.e. has a unique solution. See [13, IV, and Appendix D]. $\omega$ is a $d$-dimensional standard Brownian motion and introduces random disturbances into the dynamics. The time evolution of $x(t)$ is actively influenced by another stochastic process $u(t)$, called a control process. Let $\mathcal{O} \subset \mathbb{R}^n$ be open, with either $\mathcal{O} = \mathbb{R}^n$ or $\partial \mathcal{O}$ a compact manifold of class $C^3$. Let $\tau$ denote the exit time of $x(t)$ from $\mathcal{O}$, or $\tau = \infty$, if $x(t) \notin \mathcal{O}$ for all $t \geq 0$. Given any initial data $x$ and any admissible progressively measurable control process $u(\cdot)$, the infinite horizon discounted cost functional is given by

$$J^u(x) = \mathbb{E}^x \left\{ \int_0^\tau e^{-\beta t} L(x(t), u(t)) \, dt + \chi_{\{\tau < \infty\}} e^{-\beta \tau} g(x(\tau)) \right\}, \quad (1.5)$$

where $\mathbb{E}^x$ refers to the conditional expectation given $x(0) = x$, $\beta \geq 0$, the function $g$ is continuous on $\mathbb{R}^n$ and the function $L$ is continuous on $\mathbb{R}^n \times U$ satisfying some polynomial growth condition.

The optimal control problem is to determine some admissible control process $u^*(\cdot)$ that minimizes the cost functional $J^u(x)$. As in section 1.1, the value function is a function of the initial state $x$ and it is defined as the infimum over the set of admissible controls $\mathcal{U}$ of the cost functional, that is

$$v(x) = \inf_u J^u(x). \quad (1.6)$$

1.3 The Dynamic Programming Approach

In 1957 Richard Bellman introduced the dynamic programming method to solving optimal control problems. The dynamic programming method produces a par-
tial differential equation, called the dynamic programming equation or Hamilton-Jacobi-Bellman (HJB) equation that the value function \( v \) must satisfy. For a period of 20 years Bellman's result was superceded in deterministic optimal control theory by the methods of the Pontryagin Maximum Principle (PMP), since this is valid under much weaker conditions than those required for the validity of the dynamic programming equation. For instance, Bellman's assumption on the continuous differentiability of the value function does not hold even in simple cases. On the other hand, the PMP version for stochastic optimal control theory is not very useful, at least from the computational point of view. The dynamical programming method has been successfully in applied stochastic control in the last fifteen years. In fact, Fleming and Rishel [12] study the relation between the value function for the optimal stochastic control problem and the corresponding dynamic programming equation. If the value function is smooth enough, say \( C^2 \), then it solves the HJB equation in the classical or usual sense. But the dynamic programming approach has the difficulty that the value function \( v \) frequently fails to be \( C^1 \) in the deterministic case, and to be \( C^2 \) in the stochastic case, which means that the value function may not satisfy the dynamic programming equation in the classical sense. Crandall and Lions [6] introduced a weaker notion of solution of this equation, called viscosity solution. With this notion the dynamic programming method can be pursued in most cases of interest. It has been proved in great generality that the value function is a viscosity solution of the corresponding dynamic programming equation. Then the dynamic programming equation is a necessary condition that the value function must satisfy. This is the fact that we use in this thesis to carry out a complete analysis of the value function for a deterministic singular optimal control problem and to find explicitly the value function for an infinite horizon singular stochastic control problem.
1.3.1 Deterministic Case

The dynamic programming approach to solving the deterministic optimal control problem (1.1), (1.2), (1.3) replaces the original minimization over control functions by a nonlinear partial differential equation involving pointwise minimization over the set of control values, usually a subset of \( \mathbb{R}^n \). Computation of the minimal cost is reduced to solving this first order nonlinear partial differential equation (PDE) called the Hamilton-Jacobi-Bellman (HJB) equation. In fact, under suitable conditions, the value function is a classical solution of the HJB equation

\[
\frac{\partial v}{\partial t}(t, x) + \sup_{u \in U} \left\{ -\frac{\partial v}{\partial x}(t, x) f(x, u) - L(x, u) \right\} = 0. \tag{1.7}
\]

1.3.2 Stochastic Case

As in the deterministic case, the dynamic programming approach to solving the stochastic optimal control problem (1.4), (1.5), (1.6) reduces to solving a second order nonlinear PDE also called the Hamilton-Jacobi-Bellman (HJB) equation. In fact, the value function for the optimal control problem (1.4), (1.5), (1.6), under suitable conditions is a classical solution of the HJB equation

\[
\beta v(x) = \min_{u \in U} \left\{ -\mathcal{L}^u v(x) + L(x, u) \right\}, \tag{1.8}
\]

where

\[
-\mathcal{L}^u v(x) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x, u)v_{x_i x_j} + \sum_{i=1}^{n} f_i(x, u)v_{x_i},
\]

with \( a = (a_{ij}) = \sigma \sigma' \).

1.4 The Free Boundary for Singular Optimal Control Problems

We consider the special case of infinite horizon singular deterministic and stochastic control problems.
1.4.1 Stochastic Case

We set the stochastic control system as

\[ \dot{x} = (f(x) + u)dt + \sigma(x)dW(t), \quad x(0) = x \in \mathbb{R}^n, \]

where \( W \) is a standard Brownian motion, \( f, \sigma \in C^1(\mathbb{R}^n) \) with bounded first order partial derivatives, and the nonanticipating admissible controls \( u(\cdot) \) are in the family

\[ \mathcal{U} = L^\infty([0, \infty), U), \]

where \( U \subset \mathbb{R}^n \) is a closed cone in \( \mathbb{R}^n \), i.e.,

\[ u \in U, \lambda \geq 0 \implies \lambda u \in U. \]

We set the cost functional as

\[ J^u(x) = E^x \int_0^\infty e^{-\beta t}[L(x(t)) + \theta(u(t))]dt, \]

where \( \beta > 0 \) and \( L, \theta \in C(\mathbb{R}^n) \) satisfying

\[ \theta(\lambda v) = \lambda \theta(v), \quad \forall \lambda \geq 0. \]

Here the control set is not bounded, there is no coercivity condition satisfied by \( f \) and \( L \), and the dynamics and the running cost are linear in the control. According to [13, chapter VIII], the dynamic programming equation (1.8) has to be interpreted carefully. In fact (1.8) can be rewritten as

\[ \max[F(x, u(x), Dv(x), D^2v(x)), H(Dv(x))] = 0, \quad x \in \mathbb{R}^n, \]

where \( F, \) and \( H \) are given by

\[ F(x, u(x), Dv(x), D^2v(x)) = \beta v(x) - \frac{1}{2}tr(\hat{a}(x)D^2v(x)) - f(x) \cdot Dv(x) - L(x), \]

(1.12)
with \( \dot{a} = \sigma \dot{r} \) and

\[
H(p) = \sup_{|u|=1} (-p \cdot u - \theta(u)).
\]  

(1.13)

Note that at any point of the state at least one of the expressions (1.12), (1.13) equals zero. Then the state space splits into two regions separated by the subset \( B \) of \( \mathbb{R}^n \), called the free boundary, where

\[
F(x, v(x), Dv(x), D^2 v(x)) = H(Dv(x)) = 0.
\]

The free boundary is the subset where there is a switch between the conditions

\[
F(x, v(x), Dv(x), D^2 v(x)) \leq 0, \quad H(Dv(x)) = 0,
\]

and

\[
F(x, v(x), Dv(x), D^2 v(x)) = 0, \quad H(Dv(x)) \leq 0.
\]

Nonsmoothness of the value function often occurs only along the free boundary \( B \).

The property of smooth fit, see [3], is said to hold for a particular control problem if the value function is smooth enough along the free boundary \( B \) so that the HJB equation is solved in the classical sense. Therefore, the dynamic programming equation is also called a free boundary problem, since the crucial step in solving it is to locate the subset \( B \).

### 1.4.2 Deterministic Case

Now we consider the control system obtained by dropping the diffusion term in (1.9),

\[
\dot{x} = (f(x) + u)dt, \quad x(0) = x \in \mathbb{R}^n,
\]

(1.14)

with analogue conditions for \( f \) and the controls \( u(\cdot) \). We set the cost functional as

\[
J^u(x) = \int_0^\infty e^{-\beta t}[L(x(t)) + \theta(u(t))]dt,
\]

(1.15)
with $\beta, L, \theta$ as in (1.10). As in the stochastic case, the dynamic programming equation (1.7) has to be rewritten and reasoning as in [8], [10], [11], the dynamic programming equation for this optimal control problem is of the form

$$\max [F(x, v(x), Dv(x)) - H(Dv(x))] = 0, \quad x \in \mathbb{R}^n,$$  \hspace{1cm} (1.16)

where $F$, and $H$ are given by

$$F(x, v(x), Dv(x)) = \beta v(x) - f(x) \cdot Dv(x) - L(x),$$  \hspace{1cm} (1.17)

and

$$H(p) = \sup_{|w|=1} (-p \cdot u - \theta(u)).$$  \hspace{1cm} (1.18)

Then the HJB equation induces a free boundary problem which is defined as in the stochastic case.

### 1.5 Viscosity Solutions of the Singular HJB Equation

We will now give the definition of a viscosity solution of the dynamic programming equation that arises in singular control problems, see [13, p. 66]. For more about equivalent definitions see [5].

Let $\mathcal{S}_n$ denote the set of all symmetric real-valued $n \times n$-matrices. Infinite horizon singular optimal control problems with the control taking values in a closed cone $U \subset \mathbb{R}^d$ lead to a dynamic programming equation of the form, see [13],

$$\max [F^1(x, v(x), Dv(x), D^2v(x)), F^2(x, v(x), Dv(x))] = 0, \quad \text{for} \quad x \in \mathcal{O},$$  \hspace{1cm} (1.19)

where $\mathcal{O}$, the state space of the control problem, is some nonempty connected open subset of $\mathbb{R}^n$, $F^1$ is a continuous real-valued function on $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n$ and $F^2$ is a continuous real-valued function on $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^n$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
A continuous function \( v \) on the open set \( \mathcal{O} \subset \mathbb{R}^n \) is a \textit{viscosity subsolution} of (1.19) if for each \( x_0 \in \mathcal{O} \)
\[
\max \left[ F^1(x_0, v(x_0), D\phi(x_0), D^2\phi(x_0)), F^2(x_0, v(x_0), D\phi(x_0)) \right] \leq 0,
\]
for all test functions \( \phi \in C^2(\mathcal{O}) \) such that \( v(x_0) = \phi(x_0) \), and \( v - \phi \) attains a local maximum at \( x_0 \).

A continuous function \( v \) on the open set \( \mathcal{O} \subset \mathbb{R}^n \) is a \textit{viscosity supersolution} of (1.19) if for each \( x_0 \in \mathcal{O} \)
\[
\max \left[ F^1(x_0, v(x_0), D\phi(x_0), D^2\phi(x_0)), F^2(x_0, v(x_0), D\phi(x_0)) \right] \geq 0,
\]
for all test functions \( \phi \in C^2(\mathcal{O}) \) such that \( v(x_0) = \phi(x_0) \), and \( v - \phi \) attains a local minimum at \( x_0 \).

Finally, \( v \) is a \textit{viscosity solution} of (1.19) if it is both a viscosity subsolution and a viscosity supersolution. Clearly then, any classical solution (hence \( C^2 \)) of (1.19) is a viscosity solution. So, the theory of viscosity solutions is consistent with the notion of classical solutions. Also it can be proved that if a viscosity solution is smooth enough, then it turns out to be classical solution, see Shreve and Soner [22].

The dynamic programming equation for infinite horizon singular deterministic optimal control problems is simply the special case, see [10], [11], [8]
\[
\max[ F^1(x, v(x), Dv(x)), F^2(x, v(x), Dv(x))] = 0, \quad \text{for} \quad x \in \mathcal{O},
\]
since there is no diffusion term in the control system; so, a viscosity solution is defined in a similar way.

Uniqueness results for these partial differential equations represent an important achievement of the theory of viscosity solutions. Jensen [16], Crandall and Lions
[6], Ishii [15], have given uniqueness results for HJB equations. In such a case, the value function is perfectly determined once a viscosity solution of the HJB equation is obtained. But still in the second order case, the stochastic one, something else has to be added to obtain uniqueness results. In fact, in chapter 3 we present an optimal control problem for which the value function is not smooth enough and the corresponding dynamic programming equation has more than one viscosity solution.

1.6 Brief Overview

The organization of this dissertation is as follows. In chapter 2 we present an extension of the results in the paper "Smooth Fit for Some Bellman Equations" written by J. Hijab and G. Ferreyra, see [10]. We consider the one dimensional deterministic singular optimal control problem (2.1), (2.2), (2.3) allowing the controls $u$ take values in the whole real line while [10] dealt with $u(\cdot) \geq 0$ only. We use the dynamic programming approach to do a complete analysis of this control problem. We find that the free boundary is a pair of points in $\mathbb{R}$, say $\{\alpha^-, \alpha^+\}$, which are determined by the parameters of the control problem. The value function $v$ is a classical solution of the HJB equation. Moreover, $v$ is in $C^1(\mathbb{R})$, in $C^2(\mathbb{R} \setminus \{\alpha^-, \alpha^+\})$, but is never $C^2$ on $\mathbb{R}$. Smoothness of $v$ depends on the parameters of the problem. We find that feedback optimal controls depend on the location of the free boundary

(i) If $\alpha^- \leq 0 \leq \alpha^+$ the control $u^*(\cdot) \equiv 0$ is optimal on $[\alpha^-, \alpha^+]$, and outside this interval the optimal control is impulsive.

(ii) If $\alpha^+ < 0$, then the previous description of the optimal control holds, except that the control $u^*(\cdot) \equiv -\beta \alpha^+$ is optimal at $\alpha^+$,
(iii) If $\alpha^- > 0$, then the description (i) of the optimal control holds except that the control $u^*(\cdot) \equiv -\beta \alpha^-$ is optimal at $\alpha^-$. 

In chapter 3 we consider the one dimensional infinite horizon singular stochastic control problem (3.1), (3.2), (3.3). We apply the dynamic programming approach to explicitly solve this control problem. This method involves using differential equation methods to piece together a candidate value function, checking that this candidate value function is indeed a viscosity solution of the dynamic programming equation, and finally identifying candidate optimal controls and evaluating the cost of these in order to prove a verification theorem.

In section 3.6, Lemma 3.1 we prove that the value function of our control problem is of quadratic growth. In section 3.7, using the HJB equation (3.4) as a necessary condition, we find that the free boundary is a single point in $\mathbb{R}$, say $\alpha$, and construct a nonnegative convex polynomially growing solution of the HJB equation (3.4). Then we get one candidate value function for each one of the following cases

(i) $K < \frac{1-b}{2}$,

(ii) $K > \frac{1-b}{2}$,

(iii) $K = \frac{1-b}{2}$.

In section 3.8, Lemmas 3.7, 3.8, 3.9 prove that the candidate value function defined in Lemma 3.7 for the case (i) is in $C^2(\mathbb{R})$, positive, convex, and is a classical solution of the HJB equation (3.4).

In section 3.9, Lemmas 3.10, 3.11, 3.12, prove that the candidate value function defined in Lemma 3.10 for the case (ii) is in $C^2(\mathbb{R})$, positive, convex, and is a classical solution of the HJB equation (3.4).
In section 3.10, Lemmas 3.13, 3.14 prove that the candidate value function defined in Lemma 3.13 for the case (iii) is convex, positive, in $C^2(\mathbb{R} \setminus \{0\}), \ C^1$ at zero, but not $C^2$ at zero, and it is a viscosity solution of the HJB equation (3.4). This example shows that the principle of smooth fit fails for this control problem. In Lemma 3.15 we define an affine function which is a convex, $C^\infty$, (therefore, viscosity) classical solution of the HJB equation (3.4). Therefore, this shows that viscosity solutions of the dynamic programming equation, in general, are not unique.

In section 3.11, we prove a verification theorem 3.18 using part of the verification theorem 3.17 from [13, Chapter VIII], finding optimal controls given by (3.29), (3.30) to prove that the candidate value function for the case (iii) is in fact the value function.

### 1.7 Examples

In this section we present some applied problems whose solution can be given using the general theory of singular optimal control problems.

#### 1.7.1 Singular control in Space Navigation

Practical problems involving singular controls first arose in the study of optimal trajectories for space manoeuvres. Trajectories for rocket propelled vehicles in which the thrust magnitude is bounded exhibit singularity in the rate of fuel consumption. We consider the problem of a spacecraft attempting to make a soft landing on the moon using a minimum amount of fuel. To define a simplified version of this problem, let $m$ denote the mass, $h$ the height, and $v$ the vertical velocity of the spacecraft above the moon. Let $u$ denote the thrust of the spacecraft’s engine. Let $M$ denote the mass of the spacecraft without fuel, $h_0$ and $v_0$ the initial height and initial vertical velocity of the spacecraft. Let $F$ be the initial amount of fuel, $\alpha$ the maximum thrust attainable by the spacecraft’s engine, $k$ a constant, and $g$
the gravitational acceleration of the moon. The gravitational acceleration $g$ may be considered constant near the moon. The equations of motion of the spacecraft are

\[
\begin{align*}
\dot{h} &= v, \\
\dot{v} &= -g + m^{-1} u, \\
\dot{m} &= -ku.
\end{align*}
\]

The thrust $u(t)$ of the spacecraft’s engine is the control for the problem. Suppose the class $\mathcal{U}$ of control functions is all piecewise continuous functions $u(t)$ defined on an interval $[a, b]$ such that

\[0 \leq u(t) \leq \alpha.\]

We take initial time $t_0 = 0$, and terminal time $t_f$ equal to the first time the spacecraft reaches the surface of the moon. Terminal conditions which must be satisfied at the initial time and terminal time are

\[
\begin{align*}
h(0) - h_0 &= 0, & h(t_f) &= 0, \\
v(0) - v_0 &= 0, & v(t_f) &= 0, \\
m(0) - M - F &= 0.
\end{align*}
\]

The problem is to land using a minimum amount of fuel or equivalently to minimize

\[
\inf_{w \in \mathcal{U}} -m(t_f).
\]

In this example the maximization of the Hamiltonian gives little information about the value of the control. For a complete solution of this problem, see [12]. For a three dimensional version of this problem see [2].
1.7.2 The Calculus of Variations

The major concern of calculus of variations is to choose an absolutely continuous function \( x : [a, b] \rightarrow \mathbb{R}^n \) such that it minimizes
\[
\int_a^b L(t, x(t), \dot{x}(t)),
\]
subject to the endpoint constraints
\[
x(a) = A, \ x(b) = B.
\]

The time interval \([a, b]\), the integrand \( L \), called "Lagrangian", and the endpoint constraints are all given as part of the problem's statement. The calculus of variations was born in June 1696 when Johann Bernoulli posed the Brachistochrone problem. "Find the shape of a wire joining two given points in a vertical plane \( \alpha = (a, A), \beta = (b, B) \), with \( A < B \), such that a bead falling along the wire under the influence of gravity, (neglect frictional effects), will travel from \( \alpha \) to \( \beta \) in least time". Labelling the horizontal axis with \( t \) and the vertical axis with \( x \), and taking \( x \) to be positive downward, it is easy to recognize this as an instance of the above general formulation of a problem of calculus of variations. Here the Lagrangian \( L \) is given by
\[
L(t, x, v) = \frac{\sqrt{1 + v^2}}{\sqrt{x - A}}.
\]

The choices \( U = \mathbb{R}^n \) and \( f(t, x, u) = u \) allow us to recognise this problem as a special case of an optimal control problem. The solution is the unique cycloid with a vertical cusp at \( \alpha \) and passing through \( \beta \). For a complete solution see [23].

1.7.3 An Optimal Portfolio Selection Problem

During the last fifteen years powerful tools from Stochastic Analysis and from Stochastic Control have invaded almost all aspects of Mathematical Finance to
study consumption/investment optimization, arbitrage, hedging, pricing, portfolio optimization problems, and so on. Now we illustrate some of those techniques by applying them to a simple case, taken from [20], of optimal portfolio diversification. This problem has been considered in more general settings by many authors, see for example [18]. Let $X(t)$ denote the wealth of a person at time $t$. Suppose that the person has the choice of two different investments. The price $p_1(t)$ at time $t$ of one of the assets is assumed to satisfy the equation

$$\frac{dp_1}{dt} = p_1(a + \alpha W(t)),$$

where $\omega(t)$ denotes a standard brownian motion and $a, \alpha > 0$ are constants measuring the average relative rate of change of $p_1$ and the size of the noise, respectively. We interpret equation (1.22) as the stochastic differential equation

$$dp_1 = p_1 a \, dt + p_1 \alpha \, dW(t).$$

(1.22)

This investment is called "risky", since $\alpha > 0$, which means that it is subject to systematic risk. We shall refer to them as "stocks". We assume that the price $p_2$ of the other asset satisfies a similar equation, but with no noise, i.e. without systematic risk.

$$dp_2 = p_2 b \, dt.$$  

(1.23)

These assets might be "bonds", "certificates of deposits" and so on. This investment is called "safe". So it is natural to assume that $b < a$. At each instant the person can choose how big a fraction $u$ (control) of his wealth he will invest in the risky asset, thereby investing the fraction $1 - u$ in the safe one. This gives the following stochastic differential equation for the wealth $X(t)$

$$dX(t) = uX(t) \, adt + uX(t) \alpha dW(t) + (1 - u)X(t) \, b \, dt$$

$$= X(t)(au + b(1 - u)) dt + auX(t) dW(t).$$

(1.24)
Suppose that starting with the wealth \(X(s) = x > 0\) at time \(s\), the person wants to maximize the expected utility of the wealth at some future time \(t_0 > s\). If we allow no borrowing, i.e. we require \(X \geq 0\) and we are giving an utility function \(N : [0, \infty) \rightarrow [0, \infty)\), with \(N(0) = 0\) (usually assumed to be increasing and concave), the problem is to find \(V(s, x)\) and a (Markov) control \(u^* = u^*(s, X(s))\), \(0 \leq u^* \leq 1\), such that

\[
V(s, x) = \sup_{0 \leq u \leq 1} J^u(s, x) = J^{u^*}(s, x),
\]

where

\[
J^u(s, x) = E^{\pi(0)}[N(X(T))],
\]

and \(T\) is the first exit time from the region

\[
G = \{(r, z) : r < t_0, z > 0\}.
\]

This is a performance criterion of the form (1.5). The HJB equation becomes

\[
\frac{\partial V}{\partial t}(t, x) + \sup_{0 \leq u \leq 1} (\mathcal{L}^u V)(t, x) = 0, \quad \text{for } (t, x) \in G,
\]

with

\[
\mathcal{L}^u V(t, x) = x(au + b(1 - u)) \frac{\partial V}{\partial x}(t, x) + \frac{1}{2} \alpha^2 u^2 x^2 \frac{\partial^2 V}{\partial x^2}(t, x),
\]

and

\[
V(t, x) = N(x), \text{ for } t = t_0,
\]

\[
V(t, 0) = N(0), \text{ for } t < t_0.
\]

Therefore, for each \((t, x)\) we try to find the value \(u = u(t, x)\) which maximizes the function

\[
\phi(u) = \frac{\partial V}{\partial t} + x(au + b(1 - u)) \frac{\partial V}{\partial x} + \frac{1}{2} \alpha^2 u^2 x^2 \frac{\partial^2 V}{\partial x^2}.
\]
If \( V_x := \frac{\partial V}{\partial x} > 0 \) and \( V_{xx} := \frac{\partial^2 V}{\partial x^2} < 0 \), the solution is

\[
\begin{align*}
  u &= u(t, x) = -\frac{(a - b)^2 V_x}{x a^2 V_{xx}}. 
\end{align*}
\] (1.29)

If we substitute this into the HJB equation (1.27) we get the following nonlinear boundary value problem for \( V \)

\[
\begin{align*}
  V_t + b x V_x - &\frac{(a - b)^2 V_x^2}{2 a^2 V_{xx}} = 0, \quad \text{for } t < t_0, \ x > 0, \\
  V(t, x) &= N(x), \quad \text{for } t = t_0 \text{ or } x = 0. 
\end{align*}
\] (1.30) (1.31)

The problem (1.30), (1.31) is hard to solve for a general function \( N \). So we choose a power function which is an increasing and concave function, say

\[
N(x) = x^r, \quad \text{with } 0 < r < 1.
\]

We try to find a solution of (1.30), (1.31) of the form

\[
V(t, x) = f(t)x^r.
\]

Substituting, we obtain

\[
V(t, x) = e^{\lambda(t-t_0)}x^r,
\]

where

\[
\lambda = b r + \frac{(a - b)^2 r}{2 a^2 (1 - r)}.
\]

Using (1.29), we obtain the optimal control

\[
u^* = u^*(t, x) = \frac{a - b}{a^2 (1 - r)}.
\]

If \( \frac{a - b}{a^2 (1 - r)} \in (0, 1) \), then this is the solution to the problem. Note that \( u^* \) is constant.
Chapter 2
A One Dimensional Deterministic Control Problem

2.1 Introduction

This chapter refers to a class of infinite horizon singular optimal control problems which are optimal control problems whose set of control values is unbounded and the controls appear linearly in the dynamic and in the running cost. We consider the scalar control system

\[ \dot{x} = f(x) + u, \quad x(0) = x \in \mathbb{R}, \quad (2.1) \]

where \( f \) is a differentiable function with bounded derivatives and the control \( u(\cdot) \) is a measurable function of time in the family

\[ \mathcal{U} = L^\infty([0, \infty), \mathbb{R}). \]

The optimal control problem consists of minimizing over all controls \( u(\cdot) \in \mathcal{U} \) the infinite horizon discounted cost functional

\[ J^u(x) = \int_0^\infty e^{-t}[L(x(t)) + |u(t)|]dt. \quad (2.2) \]

The value function for this optimal control problem is a function of the initial state \( x \) defined as the infimum of the costs, that is,

\[ v(x) = \inf\{v^u(x) : u(\cdot) \in \mathcal{U}\}, \quad (2.3) \]

and the optimal control \( u^*(\cdot) \), if it exists, is the argument that minimizes the cost functional.

The dynamic programming equation, also called the Hamilton-Jacobi-Bellman (HJB) equation, for a deterministic optimal control problem is in general a first
order nonlinear partial differential equation (PDE) that provides an approach to solving optimal control problems. It is well known, see [12], that if the value function is smooth enough, then it is the classical solution of the HJB equation. But also by using the weaker notion, called viscosity solution, introduced by Crandall and Lions [6], the dynamic programming method can be pursued when the value function is not smooth enough. In fact, the HJB equation is a necessary condition, in the viscosity sense, that the value function must satisfy. The dynamic programming equation for the above deterministic optimal control problem is of the form

\[ \max \left[ F^1(x, v(x), v'(x)), F^2(x, v(x), v'(x)) \right] = 0, \quad -\infty < x < \infty, \]

for suitable continuous functions \( F^1 \), \( F^2 \). The subset \( B \) of \( \mathbb{R} \) where \( F^1(x, v(x), v'(x)) = F^2(x, v(x), v'(x)) = 0 \) is called the free boundary. Our variational problem is homogeneous of degree 1 in the control, thus we expect the optimal control to be extreme or to be singular. Moreover, since our running cost is nonnegative we expect optimal controls to equal zero, plus or minus infinity, or to be singular. By the control being plus or minus infinity we mean that it is an impulse. The free boundary separates the null region (where the optimal control is zero) and the jump region (where the optimal control is impulsive). Nonsmoothness of the value function often occurs only along the free boundary \( B \). The property of smooth fit is said to hold for a particular optimal control problem if the value function is smooth enough, \( C^1 \) in our case, along the free boundary \( B \) so that it solves the HJB equation in the classical sense. The dynamic programming equation gives rise to a free boundary problem since the crucial step in solving it is to locate the subset \( B \) where there is a switch between the conditions

\[ F^1(x, v(x), v'(x)) \leq 0, \quad F^2(x, v(x), v'(x)) = 0, \]
and

\[ F^1(x, v(x), v'(x)) = 0, \quad F^2(x, v(x), v'(x)) \leq 0. \]

Ferreyra and Hijab [10] studied the optimal control problem (2.1), (2.2), (2.3), assuming linearity of the function \( f \) and convexity of the function \( L \), with controls taking values in \([0, \infty)\). This enables them to present a complete analysis of the solution of the control problem. They used the dynamic programming method and proved that the free boundary is just a single point giving its location in terms of the parameters of the problem. Also, they found that additional smoothness of \( v \) depends on the parameters of the problem. We consider this optimal control problem with the same assumptions as in [10], but allowing the controls take values in the whole real line. We use the dynamic programming method to prove that the free boundary is a pair of points in \( \mathbb{R} \), locating them in terms of the parameters of the problem. We also see that \( C^2 \)-fit is a property that depends on the parameters of the problem.

2.2 Ferreyra and Hijab's Result

Theorem 2.1 (Ferreyra and Hijab). Let's consider the optimal control problem (2.1), (2.2), (2.3) with the following assumptions

(i) \( L \) is \( C^2 \) and \( L(x) \geq 0 \),

(ii) \(|L'(x)| \leq C_1(1 + L(x))\),

(iii) \( 0 < \mu \leq L''(x) \leq C_2(1 + L(x)) \),

(iv) \( f(x) \) is linear and \( f'(x) < 0 \),

(v) the control \( u(\cdot) \) is a nonnegative measurable function, \( u(\cdot) \in L^\infty([0, \infty), [0, \infty)) \).
Then the value function $v$ is a classical solution of the Hamilton-Jacobi-Bellman equation
\[
\max\{v(x) - f(x)v'(x) - L(x), -v'(x) - 1\} = 0, \quad -\infty < x < \infty,
\]
and there is a point $\alpha$ in $\mathbb{R}$ such that
\[
-v' - 1 = 0 \quad \text{on} \quad J = (-\infty, \alpha],
\]
\[
v - bxv' - f = 0 \quad \text{on} \quad N = [\alpha, \infty).
\]
Moreover, $v \in C^2(\mathbb{R} \setminus \{\alpha\})$ and $v \in C^2(\mathbb{R})$ iff $f(\alpha) < 0$. The quantity $\alpha$ can be computed in terms of the data of the problem. Even if $L$ is $C^\infty$, $v$ is never $C^3$ at $\alpha$.

2.3 The Main Result

We consider the optimal control problem (2.1), (2.2), (2.3) with the same assumptions as in Theorem 2.1, but dropping assumption (v) by allowing the controls to take also negative values. For clarity we set $f(x) = \beta x$, with $\beta < 0$. In the remainder of this chapter we prove Theorem 2.2 below along with some needed lemmas.

Theorem 2.2. The value function $v$ for the control problem (2.1), (2.2), (2.3) with assumptions (i), (ii), (iii), (iv) as in Theorem 2.1 is a classical $C^1$-solution of the Hamilton-Jacobi-Bellman equation
\[
\max\{v(x) - \beta xv'(x) - L(x), |v'(x)| - 1\} = 0, \quad -\infty < x < \infty. \quad (2.4)
\]
Moreover, there exist $\alpha^-, \alpha^+ \in \mathbb{R}$ such that
\[
-v'(x) - 1 = 0, \quad \forall x \in J^- = (-\infty, \alpha^-],
\]
\[
v(x) - \beta xv'(x) - L(x) = 0, \quad \forall x \in N = [\alpha^-, \alpha^+],
\]
\[
v'(x) - 1 = 0, \quad \forall x \in J^+ = [\alpha^+, +\infty).
\]
The value function \( v \) is never \( C^2 \) but

\[
v \in C^2(\mathbb{R} \setminus \{ \alpha^-, \alpha^+ \}), \quad \text{and} \quad v \in C^2 \text{ at } \alpha^- \iff 0 < \alpha^- , \quad \text{and} \quad v \in C^2 \text{ at } \alpha^+ \iff \alpha^+ < 0.
\]

The quantities \( \alpha^- \) and \( \alpha^+ \) can be computed in terms of the parameters of the problem.

2.4 Convexity and Differentiability of the Value Function

Lemma 2.3. The value function \( v \) is convex, \( C^1 \), and a classical solution of the Hamilton-Jacobi-Bellman (HJB) equation (2.4). Moreover, \( v'' \) exists almost everywhere and

(i) \( 0 \leq v(x) \leq L^*(x) \),

(ii) \( |v'(x)| \leq C_1(1 + L^*(x)) \),

(iii) \( 0 \leq v''(x) \leq C_2(1 + L^*(x)) \) for almost every \( x \),

where \( L^*(x) \) denotes the maximum value of the function \( L \) over the line segment joining \( x \) and the origin.

Proof.

Note that since \( L \) is convex we have

\[
L^*(x) =: \max\{L(y) : 0 \leq y \leq x\} = \max(L(x), L(0)).
\]

It is clear that \( v(x) \geq 0 \), \( \forall x \in \mathbb{R} \). Let's show that \( v \) is convex. Let \( x_0^0, x_1^0 \in \mathbb{R} \), and \( s \in [0, 1] \). Given \( \varepsilon > 0 \), there exist \( u_0, u_1 \in \mathcal{U} \) such that

\[
\nu^u(x_0^0) \leq v(x_0^0) + \varepsilon \quad \text{and} \quad \nu^{u_1}(x_1^0) \leq v(x_1^0) + \varepsilon.
\]
Let \( u = (1 - s)u_0 + su_1 \). It is clear that \( u \) is a measurable function, hence \( u \in \mathcal{U} \).

Let \( x_0 = (1 - s)x_0^0 + sx_0^1 \). Let \( x_i(t) \) be the solution of \( \dot{x} = f(x) + u \), with initial value \( x(0) = x_i^0, \ i = 1, 2 \). Then, \( x(t) = (1 - s)x_0(t) + sx_1(t) \) is the solution of \( \dot{x} = \beta x + u \), with initial value \( x(0) = (1 - s)x_0^0 + sx_0^1 = x_0 \). In fact, since \( f \) is a linear function

\[
\frac{d}{dt}[x(t)] = (1 - s)(f(x_0(t)) + u) + s(f(x_1(t)) + u) = \beta[(1 - s)x_0(t) + sx_1(t)] + u = \beta(x(t)) + u.
\]

Now we need to prove that

\[
v[(1 - s)x_0^0 + sx_0^1] \leq (1 - s)v(x_0^0) + sv(x_0^1).
\]

By definition of \( v \), convexity of \( L \) and using the triangle inequality, we have

\[
v[(1 - s)x_0^0 + sx_0^1] = v(x_0) \\
\leq v^u(x_0) \\
= \int_0^\infty e^{-t}[L(x(t)) + |u(t)|]dt \\
= \int_0^\infty e^{-t}[L((1 - s)x_0(t) + sx_1(t)) + |(1 - s)u_0 + su_1|]dt \\
\leq \int_0^\infty e^{-t}[(1 - s)L(x_0(t)) + sL(x_1(t)) + (1 - s)|u_0| + s|u_1|]dt \\
= (1 - s)\int_0^\infty e^{-t}[L(x_0(t)) + |u_0(t)|]dt + s\int_0^\infty e^{-t}[L(x_1(t)) + |u_1(t)|]dt \\
= (1 - s)v^u(x_0^0) + sv^u_1(x_0^1) \\
\leq (1 - s)[v(x_0^0) + \varepsilon] + s[v(x_0^1) + \varepsilon] \\
= (1 - s)v(x_0^0) + sv(x_0^1) + \varepsilon.
\]

Since \( \varepsilon \) was arbitrary, this implies \( v \) is convex.

To conclude the proof of (i) note that when \( u(\cdot) \equiv 0 \), \( x(t) \) lies on the line segment

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joining $x$ to 0 because $\beta < 0$. This implies

$$v(x) = \inf\{v''(x) : u(\cdot) \in U\}$$

$$\leq v^0(x)$$

$$= \int_0^\infty e^{-t}L(x(t))dt$$

$$\leq \int_0^\infty e^{-t}L^*(x)dt$$

$$= L^*(x).$$

Then we need only to consider controls $u(\cdot)$ in (2.3) satisfying $v''(x) \leq L^*(x)$.

Now, using $\nabla$ to mean first derivative with respect to $x$,

$$|\nabla v''(x)| \leq \int_0^\infty e^{-t}|\nabla L(x(t))|dt$$

$$\leq \int_0^\infty e^{-t}[C_1(1 + L(x(t)))]dt$$

$$= C_1[1 + \int_0^\infty e^{-t}L(x(t))dt]$$

$$= C_1[1 + v''(x)]$$

$$= C_1[1 + f^*(x)].$$

Similarly,

$$|\nabla^2 v''(x)| \leq \int_0^\infty e^{-t}|\nabla^2 f(x(t))|dt \leq C_2[1 + f^*(x)].$$

Since the right hand side of this last inequality is bounded on every compact interval, we conclude that for each $a, b \in \mathbb{R}, a < b$ there exists a $k(a, b) > 0$, independent of $u$, such that $k(a, b)x^2 - v''(x)$ is convex on $[a,b]$. Taking the supremum over all $u$ it follows that $k(a, b)x^2 - v(x)$ is convex on $[a, b]$. Thus, $v$ is semiconcave. Since $v$ is also convex, then $v$ is $C^1$ and $v''$ exists almost everywhere.

Finally, the estimates on $v'$ and $v''$ follow from the above estimates for $\nabla v''$, $\nabla^2 v''$.

Then, reasoning as in Fleming-Soner [13, VIII], [10], [11], and [8], the value function
\( v \) is a viscosity solution of the HJB equation, hence \( v \) is classical solution of the dynamic programming equation

\[
\max [u(x) - \beta xv'(x) - L(x), \ H(v'(x))] = 0, \ -\infty < x < \infty,
\]

where

\[
H(p) = \sup_{|u|=1} (-pu - |u|) = \sup_{|u|=1} (-pu - 1) = -\inf_{|u|=1} (pu + 1) = |p| - 1.
\]

Therefore,

\[
\max [u(x) - \beta xv'(x) - L(x), |v'| - 1] = 0, \ -\infty < x < \infty. \quad \square
\]

2.5 The Cost of Using the Control Zero

In the next lemma we consider the cost of the control \( u(\cdot) \equiv 0 \) which we define as

\[
\omega(x) = v^0(x).
\]

Lemma 2.4. The function \( \omega \) is in \( C^2(\mathbb{R}) \), it is strictly convex and satisfies

(i) \( 0 \leq \omega(x) \leq L^*(x) \),

(ii) \( |\omega'(x)| \leq C_1(1 + L^*(x)) \),

(iii) \( 0 < \mu \leq \omega''(x) \leq C_2(1 + L^*(x)) \),

(iv) \( \omega(x) - \beta x\omega'(x) - L(x) = 0, \ -\infty < x < \infty. \)

Proof of (i).
By definition \( \omega(x) = u^0(x) = \int_0^\infty e^{-t}L(x(t))dt \), with \( x(t) = xe^{\beta t} \). Then by differentiating under the integral sign it follows that \( \omega \) is in \( C^2(\mathbb{R}) \), and \( 0 \leq \omega(x) \leq L^*(x) \).

**Proof of (ii).**

Let \( z \in \mathbb{R} \) and let \( x(t) \) be the solution of (2.1) for the control \( u(\cdot) \equiv 0 \), with initial data \( x(0) = z \). Then

\[
|\omega'(z)| \leq \int_0^\infty e^{-t} \left| L'(x(t)) \frac{dx(t)}{dx} \right| dt,
\]

where \( x(t) = ze^{\beta t} \), hence \( \frac{dx(t)}{dx} = e^{\beta t} \). Thus,

\[
|\omega'(z)| \leq \int_0^\infty e^{(\beta-1)t} |L'(ze^{\beta t})| dt \\
\leq \int_0^\infty e^{(\beta-1)t} C_1[1 + L^*(z)] dt \\
= C_1[1 + L^*(z)].
\]

**Proof of (iii).**

Similarly,

\[
\omega''(z) = \int_0^\infty e^{-t}L''(ze^{\beta t})e^{\beta t} dt = \int_0^\infty e^{(2\beta-1)t}L''(ze^{\beta t}) dt.
\]

Using the bounds on \( L'' \)

\[ 0 < \mu \leq \omega''(z) \leq C_2(1 + L^*(z)). \]

**Proof of (iv).**

Let \( x \in \mathbb{R} \). Then, integrating by parts

\[
\omega(x) - \beta x \omega'(x) - L(x) = \int_0^\infty e^{-t}L(xe^{\beta t}) dt - \beta x \int_0^\infty e^{-t}L'(xe^{\beta t})e^{\beta t} dt - L(x) \\
= \int_0^\infty e^{-t}L(xe^{\beta t}) dt - [e^{-t}L(xe^{\beta t})]_0^\infty - \int_0^\infty e^{-t}L(xe^{\beta t}) dt - L(x) \\
= 0. \quad \square
\]
2.6 The Free Boundary \( B = \{ \alpha^-, \alpha^+ \} \)

In this section we find the free boundary of our control problem (2.1), (2.2), (2.3), (2.4) which is a pair of points \( \alpha^-, \alpha^+ \in \mathbb{R} \). We will prove that \( \alpha^-, \alpha^+ \) are finite in Lemmas 2.8, 2.9.

**Lemma 2.5.** There exist \( \alpha^-, \alpha^+ \) with \(-\infty < \alpha^- < \alpha^+ < \infty\), such that

\[
\begin{align*}
-v'(x) - 1 &= 0, \quad \forall x \in J^- = (-\infty, \alpha^-], \\
v(x) - \beta xv'(x) - L(x) &= 0, \quad \forall x \in N = [\alpha^-, \alpha^+], \\
v'(x) - 1 &= 0, \quad \forall x \in J^+ = [\alpha^+, +\infty).
\end{align*}
\]

**Proof.**

By the Lemma 2.4 (iii), the function \( u' : \mathbb{R} \rightarrow \mathbb{R} \) is increasing and onto \( \mathbb{R} \). Thus, we can define \( \alpha^- \) and \( \alpha^+ \) by

\[
\omega'(\alpha^-) = -1 \quad \text{and} \quad \omega'(\alpha^+) = 1. \tag{2.5}
\]

Similarly, by hypothesis the function \( L' : \mathbb{R} \rightarrow \mathbb{R} \) is increasing and onto \( \mathbb{R} \) so we can define \( b^- \) and \( b^+ \) by

\[
L'(b^-) = \beta - 1 \quad \text{and} \quad L'(b^+) = 1 - \beta. \tag{2.6}
\]

We set

\[
A^+ = \{ x : v'(x) - 1 < 0 \},
\]

\( A^+ \) is not empty. Otherwise, \( v'(x) - 1 \geq 0, \quad \forall x \in \mathbb{R} \), and by the HJB equation (2.4), \( v'(x) = 1, \quad \forall x \in \mathbb{R} \). Then, the value function \( v \) would be affine. Hence \( v \) would not be bounded below which is a contradiction since we know that \( v \geq 0 \). Then we define

\[
\alpha^+ = \sup A^+.
\]
Hence $\alpha^+ = \sup A^+ > -\infty$. Similarly, we set

$$A^- = \{x : -v'(x) - 1 < 0\}.$$  

As before, $v$ bounded below and the HJB equation (2.4) imply $A^-$ is not empty.

Then we define

$$\alpha^- = \inf A^-.$$  

Hence $\alpha^- = \inf A^- < +\infty$. Since the function $v'$ is increasing, $v'(x) - 1 \geq 0$, $\forall x \geq \alpha^+$. Thus, by the HJB equation (2.4)

$$v'(x) - 1 = 0, \quad \forall x \geq \alpha^+.$$  

Likewise, since the function $-v'$ is decreasing, $-v'(x) - 1 \geq 0$, $\forall x \leq \alpha^-$. Thus, by the HJB equation (2.4)

$$-v'(x) - 1 = 0, \quad \forall x \leq \alpha^-.$$  

Since $v'$ is increasing and continuous, then $\alpha^- < \alpha^+$ and

$$-1 < v'(x) < 1, \quad \forall x \in (\alpha^-, \alpha^+).$$  

Hence,

$$|v'(x)| < 1, \quad \forall x \in (\alpha^-, \alpha^+).$$  

Thus, by the HJB equation (2.4), and since $|v'(x)| - 1 < 0$, $\forall x \in (\alpha^-, \alpha^+)$

$$v(x) - \beta xv'(x) - L(x) = 0; \quad \forall x \in (\alpha^-, \alpha^+).$$  

(2.7)

Notice that if $\alpha^-, \alpha^+$ are finite then

$$-v'(x) - 1 = 0, \quad \forall x \in J^- = (-\infty, \alpha^-),$$

$$v(x) - \beta xv'(x) - L(x) = 0, \quad \forall x \in N = [\alpha^-, \alpha^+],$$

$$v'(x) - 1 = 0, \quad \forall x \in J^+ = [\alpha^+, +\infty).$$

In particular, $v(\alpha^-) = L(\alpha^-) - \beta(\alpha^-)$, and $v(\alpha^+) = L(\alpha^+) - \beta(\alpha^+)$. Also, $v'(\alpha^-) = -1$, and $v'(\alpha^+) = 1.$

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2.7 The Control Zero on $(\alpha^-, \alpha^+)$

Proposition 2.6. We consider the optimal control problem (2.1), (2.2), (2.3). Let $x \in (\alpha^-, \alpha^+)$. Let $x(t)$ be the solution of $\dot{x} = \beta x$, $x(0) = x$, for the control $u(\cdot) \equiv 0$. Let's suppose that there exists $T > 0$ such that $x(t) \in (\alpha^-, \alpha^+), \forall t \in [0, T)$.

Then

$$v(x) = e^{-T}v(x(T)) + \int_0^T e^{-t}L(x(t))dt. \quad (2.8)$$

Proof.

Let $x \in (\alpha^-, \alpha^+)$, let $x(t)$ be the solution of $\dot{x} = \beta x$, $x(0) = x$, for the control $u(\cdot) \equiv 0$, and let $T > 0$ be such that $x(t) \in (\alpha^-, \alpha^+), \forall t \in [0, T)$.

By (2.7)

$$v(x(t)) - \beta x(t)v'(x(t)) = L(x(t)), \forall t \in [0, T).$$

Therefore, differentiating the function $t \rightarrow e^{-t}v(x(t))$,

$$\frac{d}{dt}[e^{-t}v(x(t))] = -e^{-t}[v(x(t)) - \beta x(t)v'(x(t))] = -e^{-t}L(x(t)), \forall t \geq 0.$$

Integrating, over the interval $[0, T]$

$$v(x) = e^{-T}v(x(T)) + \int_0^T e^{-t}L(x(t))dt. \quad \square$$

Proposition 2.7. We consider the optimal control problem (2.1), (2.2), (2.3).

Suppose $\alpha^- \leq 0 \leq \alpha^+$, then on $(\alpha^-, \alpha^+)$ the control $u(\cdot) \equiv 0$ is optimal. Hence $v = \omega$ on $(\alpha^-, \alpha^+)$, where $\omega$ is the cost of the control $u(\cdot) \equiv 0$ studied in Lemma 2.4.
Proof.
Let \( x \in (\alpha^-, \alpha^+) \) and let \( x(t) \) be the solution of \( \dot{x} = \beta x, \quad x(0) = x \), for the control \( u(\cdot) \equiv 0 \). Since \( 0 \in (\alpha^-, \alpha^+) \) and \( \beta < 0 \), then \( x(t) \in (\alpha^-, \alpha^+) \), \( \forall t \geq 0 \). Hence, by Proposition 2.6 the equation (2.8) holds for all \( T > 0 \). That is,
\[
v(x) = e^{-T}v(x(T)) + \int_0^T e^{-t}L(x(t))\,dt, \quad \forall T > 0.
\]
Letting \( T \to \infty \), yields
\[
v(x) = \int_0^\infty e^{-t}L(x(t))\,dt = v^0(x) = \omega(x). \quad \Box
\]

2.8 \( \alpha^-, \alpha^+ \) Are Finite

Lemma 2.8. \( \alpha^- \) is finite.

Proof.
We know that \( -\infty < \alpha^- < \alpha^+ < +\infty \), let's suppose that \( \alpha^- = -\infty \).

Case (i) \( \alpha^+ \geq 0 \).
Then \( \alpha^- \leq 0 \leq \alpha^+ \). Therefore, by Proposition 2.7 the control \( u(\cdot) \equiv 0 \) is optimal in \( (\alpha^-, \alpha^+) \) and
\[
v(x) = v^0(x) = \omega(x), \quad \forall x \in (\alpha^-, \alpha^+).
\]
Then,
\[
v'(x) = \omega'(x); \quad \forall x \in (\alpha^-, \alpha^+).
\]
In particular, by continuity of \( v' \) and \( \omega' \), and by (2.5), \( v'(a^-) = \omega'(a^-) = -1 \). This means that \( a^- \leq \alpha^- = -\infty \). This is a contradiction, since \( a^- \in \mathbb{R} \).

Case (ii) \( \alpha^+ < 0 \).
Let \( x \in (\alpha^-, \alpha^+) \). Let \( x(t) \) be the solution of \( \dot{x} = \beta x, \quad x(0) = x \), for the control \( u(\cdot) \equiv 0 \). Since \( \dot{x}(t) > \beta \alpha^+ \), there exists \( T > 0 \) such that
\[
x(T) = \alpha^+, \quad \text{and} \quad x(t) \in (\alpha^-, \alpha^+), \quad \forall t \in [0, T).
\]
Therefore, by Proposition 2.6 the equation (2.8) holds. So,

\[ v(x) = e^{-T}v(\alpha^+) + \int_0^T e^{-t}L(x(t))dt. \]

To compute \( v'(x) \) and \( v''(x) \) we need to express \( T \) as a function of \( x \). But \( xe^{\beta T} = \alpha^+ \).

Solving for \( T \) and replacing above we get

\[ v(x) = (\frac{\alpha^+}{x})^{-\frac{1}{\beta}}v(\alpha^+) + \int_0^{\varphi(x)} e^{-t}L(xe^{\beta t})e^{\beta t}dt, \]

where

\[ \varphi(x) = \frac{1}{\beta} \log(\frac{\alpha^+}{x}). \]

Therefore,

\[
\begin{align*}
v'(x) &= v(\alpha^+)(-\frac{1}{\beta})(\frac{\alpha^+}{x})^{-\frac{1}{\beta} - 1}(-\frac{\alpha^+}{x^2}) + \int_0^{\varphi(x)} e^{-t}L'(xe^{\beta t})e^{\beta t}dt \\
&\quad + e^{-\varphi(x)}L(xe^{\beta \varphi(x)})v'(x) \\
&= v(\alpha^+)(\frac{\alpha^+}{x})^{-\frac{1}{\beta}} + \int_0^{\varphi(x)} e^{(\beta-1)t}L'(xe^{\beta t})dt \\
&\quad + (\frac{\alpha^+}{x})^{-\frac{1}{\beta}}L(\alpha^+)(-\frac{1}{\beta x}) \\
&= [v(\alpha^+)-L(\alpha^+)](\frac{\alpha^+}{x})^{-\frac{1}{\beta}} \frac{1}{\beta x} + \int_0^{\varphi(x)} e^{(\beta-1)t}L'(xe^{\beta t})dt \\
&= \beta \alpha^+(\frac{\alpha^+}{x})^{-\frac{1}{\beta}} \frac{1}{\beta x} + \int_0^{\varphi(x)} e^{(\beta-1)t}L'(xe^{\beta t})dt \\
&= (\frac{\alpha^+}{x})^{\frac{\beta - 1}{\beta}} \int_0^{\varphi(x)} e^{(\beta-1)t}L'(xe^{\beta t})dt.
\end{align*}
\]

Now, let's compute the second derivative at \( x \)

\[
\begin{align*}
v''(x) &= \frac{\beta - 1}{\beta} (\frac{\alpha^+}{x})^{-\frac{1}{\beta}}(-\frac{\alpha^+}{x^2}) + \int_0^{\varphi(x)} e^{-t}L''(xe^{\beta t})e^{2\beta t}dt \\
&\quad + e^{-\varphi(x)}L'(xe^{\beta \varphi(x)})e^{\beta \varphi(x)} \varphi'(x) \\
&= \frac{-(\beta - 1)}{\beta x} (\frac{\alpha^+}{x})^{\frac{\beta - 1}{\beta}} + \int_0^{\varphi(x)} e^{-t}L''(xe^{\beta t})e^{2\beta t}dt \\
&\quad + (\frac{\alpha^+}{x})^{-\frac{1}{\beta}}L'(\alpha^+)(\frac{\alpha^+}{x})(-\frac{1}{\beta x}) \\
&= \left[ -\frac{1}{\beta x} (\frac{\alpha^+}{x})^{\frac{\beta - 1}{\beta}} (\beta - 1 + L'(\alpha^+)) \right] + \int_0^{\varphi(x)} e^{-t}L''(xe^{\beta t})e^{2\beta t}dt.
\end{align*}
\]
Let
\[ \psi(x) = -\frac{1}{\beta x} \left( \frac{\alpha^+}{x} \right)^{\frac{2\beta-1}{\beta}} (\beta - 1 + L'(\alpha^+)). \]

It is clear that \( \psi(x) \to 0 \) and \( \left( \frac{\alpha^+}{x} \right)^{\frac{2\beta-1}{\beta}} \to 0 \) as \( x \to 0 \). Then, given \( \varepsilon > 0 \) there exists \( K < 0 \) such that for \( x < K \) we have \( 0 < \left( \frac{\alpha^+}{x} \right)^{\frac{2\beta-1}{\beta}} < 1 \) and
\[
\begin{align*}
v''(x) &> \int_0^{\varphi(\varepsilon)} e^{-t} \mu e^{2\beta t} dt - \varepsilon \\
&= \mu \left[ \frac{1}{2\beta - 1} (e^{(2\beta-1)x} - 1) \right] - \varepsilon \\
&= \mu \left[ \frac{1}{2\beta - 1} \left( \left( \frac{\alpha^+}{x} \right)^{\frac{2\beta-1}{\beta}} - 1 \right) \right] - \varepsilon \\
&> \mu \left[ \frac{1}{2\beta - 1} (1 - 1) \right] - \varepsilon.
\end{align*}
\]

Thus, taking \( \varepsilon > 0 \) small, and the corresponding \( K < 0 \)
\[ v''(x) \geq \gamma > 0; \forall x \in (\infty, K). \]

Now, integrating over the interval \([x, K]\), for \( -\infty < x < K \) yields
\[ v'(K) - v'(x) \geq \gamma (K - x). \]

Thus,
\[ v'(x) \leq \gamma(x - K) + v'(K). \]

Therefore,
\[ v'(x) \to -\infty, \quad \text{as} \quad x \to -\infty. \]

This is a contradiction since the function \( v' \) can never be less than \(-1\). Case (i) and (ii) imply \( \alpha^- \neq -\infty \). Thus \( -\infty < \alpha^- < +\infty \). \( \square \)

**Lemma 2.9.** \( \alpha^+ \) is finite.
Proof.

We know that $-\infty \leq \alpha^- < \alpha^+ \leq \infty$. Let's suppose that $\alpha^+ = +\infty$.

Case (i) $\alpha^- \leq 0$.

Then $\alpha^- \leq 0 \leq \alpha^+$. Therefore, by Proposition 2.7 the control $u(\cdot) \equiv 0$ is optimal and

$$v(x) = v^0(x) = \omega(x), \quad \text{for } x \in (\alpha^-, \alpha^+).$$

Then,

$$v'(x) = \omega'(x), \forall x \in (\alpha^-, \alpha^+).$$

In particular, by continuity of $v'$, $\omega'$ and by (2.5) $v'(\alpha^+) = 1 = \omega'(\alpha^+)$. This means that $\alpha^+ \geq \alpha^+ = +\infty$. This is a contradiction, since $\alpha^+ \in \mathbb{R}$.

Case (ii) $\alpha^- > 0$.

Let $x \in (\alpha^-, \alpha^+)$ and let $x(t)$ be the solution of $\dot{x} = \beta x, \quad x(0) = x$, for the control $u(\cdot) \equiv 0$. Then, there exists $T > 0$ such that

$$x(T) = \alpha^-, \quad \text{and} \quad x(t) \in (\alpha^-, \alpha^+), \quad \forall t \in [0, T).$$

Therefore, by Proposition 2.6 the equation (2.8) holds for $T > 0$. So,

$$v(x) = e^{-T}v(\alpha^-) + \int_0^T e^{-t}f(x(t))dt. \quad (2.9)$$

To compute $v'(x)$ and $v''(x)$ we need to express $T$ as a function of $x$. But $xe^{\beta T} = \alpha^+$.

Solving for $T$ and replacing above we get

$$v(x) = (\frac{\alpha^-}{x})^{-\frac{1}{\beta}}v(\alpha^-) + \int_0^{\varphi(x)} e^{-t}L(xe^{\beta t})dt.$$
Therefore,

\[
\begin{align*}
    v'(x) &= v(\alpha^-)\left(-\frac{1}{\beta}\right)(\frac{\alpha^-}{x})^{-\frac{1}{2} - 1}(-\frac{\alpha^-}{x^2}) + \int_0^{\varphi(x)} e^{-t} L'(xe^{\beta t})e^{\beta t}dt \\
    &\quad + e^{-\varphi(x)} L(xe^{\beta \varphi(x)}) \varphi'(x) \\
    &= \frac{v(\alpha^-)}{\beta x}(\frac{\alpha^-}{x})^{-\frac{1}{2}} + \int_0^{\varphi(x)} e^{(\beta - 1)t} L'(xe^{\beta t})dt \\
    &\quad + (\frac{\alpha^-}{x})^{-\frac{1}{2}} L(\alpha^-)(-\frac{1}{\beta x}) \\
    &= [v(\alpha^-) - L(\alpha^-)](\frac{\alpha^-}{x})^{-\frac{1}{2}} + \frac{1}{\beta x} + \int_0^{\varphi(x)} e^{(\beta - 1)t} L'(xe^{\beta t})dt \\
    &= -\beta \alpha^- (\frac{\alpha^-}{x})^{-\frac{1}{2}} + \frac{1}{\beta x} + \int_0^{\varphi(x)} e^{(\beta - 1)t} L'(xe^{\beta t})dt.
\end{align*}
\]

So,

\[
v'(x) = -\left(\frac{\alpha^-}{x}\right)^{-\frac{1}{2}} + \int_0^{\varphi(x)} e^{(\beta - 1)t} L'(xe^{\beta t})dt. \quad (2.10)
\]

Now, let's compute the second derivative at \( x \)

\[
\begin{align*}
    v''(x) &= -\frac{\beta - 1}{\beta x} \left(\frac{\alpha^-}{x}\right)^{-\frac{1}{2}}(-\frac{\alpha^-}{x^2}) + \int_0^{\varphi(x)} e^{(\beta - 1)t} L''(xe^{\beta t})e^{\beta t}dt \\
    &\quad + e^{-\varphi(x)} L'(xe^{\beta \varphi(x)})e^{\beta \varphi(x)} \varphi'(x) \\
    &= \beta - 1 \left(\frac{\alpha^-}{x}\right)^{-\frac{2}{\beta}} + \int_0^{\varphi(x)} e^{(2\beta - 1)t} L''(xe^{\beta t})dt \\
    &\quad + \left(\frac{\alpha^-}{x}\right)^{-\frac{1}{2}} L'(\alpha^-)(-\frac{1}{\beta x}).
\end{align*}
\]

Then,

\[
v''(x) = \left[\frac{1}{\beta x}\left(\frac{\alpha^-}{x}\right)^{-\frac{2}{\beta}} (\beta - 1 - L'(\alpha^-))\right] + \int_0^{\varphi(x)} e^{(2\beta - 1)t} L''(xe^{\beta t})dt. \quad (2.11)
\]

Let

\[
\psi(x) = \frac{1}{\beta x}\left(\frac{\alpha^-}{x}\right)^{-\frac{2}{\beta}} (\beta - 1 - L'(\alpha^-)).
\]

It is clear that \( \psi(x) \to 0 \) and \( \left(\frac{\alpha^-}{x}\right)^{\frac{2\beta - 1}{\beta}} \to 0 \) as \( x \to +\infty \). Then, given \( \varepsilon > 0 \) there exists \( K < 0 \) such that for \( x > K \) we have \( 0 < \left(\frac{\alpha^-}{x}\right)^{\frac{2\beta - 1}{\beta}} < 1 \) and
Thus, taking \( \varepsilon > 0 \) small, and the corresponding \( K > 0 \)

\[
v''(x) > \int_{0}^{\nu(x)} e^{(2\beta-1)t} \mu dt - \varepsilon
\]

\[
= \frac{1}{2\beta-1} \left( e^{(2\beta-1)\nu(x)} - 1 \right) - \varepsilon
\]

\[
= \frac{1}{2\beta-1} \left( \left( \frac{\alpha^-}{x} \right)^{\frac{2\beta-1}{\beta}} - 1 \right) - \varepsilon
\]

\[
> \frac{1}{2\beta-1}(-1) - \varepsilon.
\]

Thus, taking \( \varepsilon > 0 \) small, and the corresponding \( K > 0 \)

\[
v''(x) \geq \gamma > 0; \forall x \in [K, +\infty).
\]

Now, integrating over the interval \([K, x]\), for \( K < x < +\infty \) we have

\[-v'(K) + v'(x) \geq \gamma(x - K).
\]

Thus,

\[v'(x) \geq \gamma(x - K) + v'(K).
\]

Therefore,

\[v'(x) \to +\infty, \quad \text{as} \quad x \to +\infty.
\]

This is a contradiction since the function \( v' \) can never be greater than 1. Case (i) and (ii) imply \( \alpha^+ \neq +\infty \). Thus \(-\infty < \alpha^+ < +\infty\). \( \square \)

2.9 The Second Order Derivative of the Value Function

Proposition 2.10. The value function \( v \) is in \( C^2(\mathbb{R} \setminus \{\alpha^-, \alpha^+\}) \). Also, the \( C^2 \) condition at \( \{\alpha^-, \alpha^+\} \) for the value function \( v \) is as follows:

(i)

\[v \text{ is } C^2 \text{ at } \alpha^- \iff 0 < \alpha^- \iff 0 < \alpha^+.
\]
(ii) \[ v \text{ is } C^2 \text{ at } \alpha^+ \iff \alpha^+ < 0 \iff a^+ < 0. \]

As a consequence \( v \) is never \( C^2 \) on IR. Moreover, in any case the free boundary set \( \{\alpha^-, \alpha^+\} \) is determined in terms of the parameters of the control problem as

\[ \alpha^- = \min(a^-, b^-) \quad \text{and} \quad \alpha^+ = \max(a^+, b^+). \]

Proof.

Case \( \alpha^- \leq 0 \leq \alpha^+ \).

By Proposition 2.7 the control \( u(\cdot) = 0 \) is optimal on \((a^-, \alpha^+)\). Hence \( v = \omega \) in \((\alpha^-, \alpha^+), \) where \( \omega \) is the cost of the control \( u(\cdot) = 0 \) studied in Lemma 2.4 Thus,

\[ v'(x) = \omega'(x), \quad \forall x \in (\alpha^-, \alpha^+). \]

In particular, by continuity of \( v' \) and \( \omega' \),

\[ v'(\alpha^-) = -1 = \omega'(\alpha^-) \quad \text{and} \quad v'(\alpha^+) = 1 = \omega'(\alpha^+). \]

Therefore, since \( \omega'(\alpha^-) = -1 \), and \( \omega'(\alpha^+) = 1 \), and since \( \omega' \) is strictly increasing

\[ \alpha^- = a^- \quad \text{and} \quad \alpha^+ = a^+. \]

Also since the function \( \omega \) is strictly convex, and since the value function \( v \) is an affine function to the left of \( \alpha^- \) and to the right of \( \alpha^+ \), we have

\[ 0 = v''(\alpha^-) = v''(\alpha^+). \]

But

\[ 0 < \omega''_+(\alpha^-) = v''_+(\alpha^-) \quad \text{and} \quad 0 < \omega''_-(\alpha^+) = v''_-(\alpha^+). \]

Therefore,

\[ v \in C^2(\mathbb{R} \setminus \{\alpha^-, \alpha^+\}) \]
and $v$ is $C^2$ neither at $\alpha^-$ nor at $\alpha^+$.

Now, let's show that $a^- < b^-$. By Lemma 2.4 (iv)

$$\omega(x) - \beta x \omega'(x) - L(x) = 0, \quad \forall x \in (\alpha^-, \alpha^+) = (a^-, a^+).$$

Thus, differentiating,

$$\omega'(x) - \beta x \omega''(x) - \beta \omega'(x) - L'(x) = 0, \quad \forall x \in (\alpha^-, \alpha^+) = (a^-, a^+), \quad (2.12)$$

and inserting $x = a^- = \alpha^-$, yields

$$-1 + \beta - \beta a^- \omega''(a^-) - L'(a^-) = 0.$$ 

Therefore,

$$L'(a^-) - \beta = -1 - \beta a^- \omega''(a^-)$$

$$< -1$$

$$= L'(b^-) - \beta.$$ 

Thus,

$$L'(a^-) < L'(b^-), \quad \text{hence} \quad a^- < b^-.$$ 

So,

$$\alpha^- = a^- = \min(a^-, b^-).$$

Now, let's show that $b^+ \leq a^+$. By Lemma 2.4 (iv) inserting $x = b^+$ yields

$$\omega'(b^+) - \beta \omega'(b^+) - \beta b^+ \omega''(b^+) - L'(b^+) = 0.$$ 

Thus,

$$\omega'(b^+)(1 - \beta) - (1 - \beta) = \beta b^+ \omega''(b^+).$$

So,

$$(\omega'(b^+) - 1)(1 - \beta) = \beta b^+ \omega''(b^+).$$
Let’s suppose that \( a^+ < b^+ \). Thus, \( b^+ > 0 \), since \( a^+ \geq 0 \). So, \( \beta b^+ \omega''(b^+) < 0 \), which implies \( \omega'(b^+) - 1 < 0 \). Therefore,

\[
\omega'(a^+) < \omega'(b^+) < 1,
\]

since the function \( \omega' \) is increasing. This is a contradiction since we know that \( \omega'(a^+) = 1 \). Therefore, \( b^+ \leq a^+ \). Hence

\[
a^+ = \max(a^+, b^+).
\]

Case \( 0 < a^- < a^+ \).

Let’s prove that the control \( u^*(t) \equiv -\beta \alpha^- \), \( \forall t \geq 0 \) is optimal at \( \alpha^- \). According to (2.7) and inserting \( x = \alpha^- \) yields

\[
u(\alpha^-) = L(\alpha^-) - \beta \alpha^-.
\]

On the other hand, note that \( x(t) = \alpha^- \) is the solution of

\[
\dot{x} = \beta(x - \alpha^-), \quad x(0) = \alpha^-.
\]

Therefore,

\[
v^u^*(\alpha^-) = \int_0^\infty e^{-t}[L(x(t)) + |u^*(t)|]dt
\]

\[
= \int_0^\infty e^{-t}[L(\alpha^-) + (-\beta \alpha^-)]dt
\]

\[
= [L(\alpha^-) - \beta \alpha^-] \int_0^\infty e^{-t}dt
\]

\[
= L(\alpha^-) - \beta \alpha^-.
\]

Thus, \( u^*(t) \equiv -\beta \alpha^- \), \( \forall t \geq 0 \) is optimal at \( \alpha^- \).

Now we try to pin down the parameter \( \alpha^- \). For any bounded control \( u \) consider the family of bounded controls defined by

\[
u_\varepsilon(t) = -\beta \alpha^- + \varepsilon u(t), \quad \forall t \geq 0, \quad \varepsilon > 0.
\]
Let $v^*(\alpha^-)$ be the corresponding cost starting at $\alpha^-$. Then

$$v^*(\alpha^-) \leq v^*(\alpha^-), \text{ for all } \varepsilon > 0,$$

since $u^*$ is the optimal control at $\alpha^-$. Then

$$\frac{d}{d\varepsilon}(v^*(\alpha^-))|_{\varepsilon=0} = 0.$$

Given $\varepsilon > 0$, and the control $u_\varepsilon$, let $x_\varepsilon(t)$ be the solution of

$$\dot{x} = \beta x + [-\beta \alpha^- + \varepsilon u], \quad x_\varepsilon(0) = \alpha^-.$$

Interchanging $\frac{d}{dt}$ and $\frac{d}{d\varepsilon}$ we see that $\frac{d}{d\varepsilon}[x_\varepsilon(t)]$ is the solution of

$$\dot{z} = F(z, u), \quad z(0) = 0,$$

where

$$F(z, u) = \beta z + u.$$

Then, the variation of the constants formula gives us that

$$\frac{d}{d\varepsilon}[x_\varepsilon(t)] = e^{\beta t} \left[ \int_0^t e^{-\beta s} u(s) ds \right]$$

$$= \int_0^t e^{\beta (t-s)} u(s) ds.$$

Thus,

$$0 = \frac{d}{d\varepsilon}(v^*(\alpha^-))|_{\varepsilon=0} = \int_0^\infty e^{-t} \{ L'(x_\varepsilon(t)) \frac{d}{d\varepsilon}[x_\varepsilon(t)] + u(t) \}|_{\varepsilon=0} dt$$

$$= \int_0^\infty e^{-t} \{ L'(\alpha^-) \left[ \int_0^t e^{\beta (t-s)} u(s) ds \right] + u(t) \} dt$$

$$= L'(\alpha^-) \int_0^\infty e^{-t} [ \int_0^t e^{\beta (t-s)} u(s) ds ] dt + \int_0^\infty e^{-t} u(t) dt.$$

Now, integrating by parts

$$0 = L'(\alpha^-) \left[ \int_0^t e^{\beta s} u(s) ds \left( \frac{1}{\beta - 1} e^{(\beta-1)t} \right) \right]_0^\infty$$

$$- L'(\alpha^-) \int_0^\infty \frac{1}{\beta - 1} e^{(\beta-1)t} (e^{-\beta t} u(t)) dt$$

$$+ \int_0^\infty e^{-t} u(t) dt.$$
Therefore,
\[
0 = \frac{L'(\alpha^-)}{1 - \beta} \int_0^\infty e^{-t} u(t) dt + \int_0^\infty e^{-t} u(t) dt = \left[ \frac{L'(\alpha^-)}{1 - \beta} + 1 \right] \int_0^\infty e^{-t} u(t) dt.
\]

Since \( u \) was arbitrary
\[
\frac{L'(\alpha^-)}{1 - \beta} + 1 = 0, \quad \text{hence} \quad L'(\alpha^-) = \beta - 1.
\]

By definition \( L'(b^-) = \beta - 1 \). Therefore,
\[
\alpha^- = b^-.
\]

Now, let's show that \( b^- < a^- \). Using Lemma 2.4 (iv) for \( x = \alpha^- = b^- \) yields
\[
\omega'(b^-) - \beta \omega'(b^-) - \beta b^- \omega''(b^-) - L'(b^-) = 0.
\]

Since \( \beta b^- < 0 \) and \( \omega \) is strictly convex
\[
\omega'(b^-)(1 - \beta) + (1 - \beta) = \beta b^- \omega''(b^-) < 0.
\]

So,
\[
(\omega'(b^-) + 1)(1 - \beta) = \beta b^- \omega''(b^-) < 0.
\]

Then,
\[
\omega'(b^-) + 1 < 0, \quad \text{hence} \quad \omega'(b^-) < -1.
\]

Therefore,
\[
b^- < a^-,
\]

since the function \( \omega' \) is increasing and since \( \omega'(a^-) = -1 \). So,
\[
\alpha^- = b^- = \min(a^-, b^-).
\]
Now, let's show that $b^+ < a^+$. Using Lemma 2.4 (iv) and inserting $x = b^+$ yields

$$
\omega'(b^+) - \beta \omega'(b^+) - \beta b^+ \omega''(b^+) - L'(b^+) = 0.
$$

Thus,

$$
\omega'(b^+)(1 - \beta) - (1 - \beta) = \beta b^+ \omega''(b^+) < 0.
$$

So,

$$
(\omega'(b^+ - 1)(1 - \beta) = \beta b^+ \omega''(b^+) < 0.
$$

Then,

$$
\omega'(b^+) - 1 < 0, \quad \text{hence } \omega'(b^+) < 1.
$$

Therefore,

$$
b^+ < a^+,
$$

since the function $\omega'$ is increasing and since $\omega'(a^+) = 1$. Now, let's show that

$$
\alpha^+ = a^+ = \max(a^+, b^+).
$$

In fact,

$$
1 = \nu'(\alpha^+) = \omega'(\alpha^+) = \omega'(a^+).
$$

Thus,

$$
\alpha^+ = a^+,
$$

since the function $\omega'$ is strictly increasing.

Now, let's prove that the value function $\nu$ is $C^2$ at $\alpha^-$ but not at $\alpha^+$. By (2.7)

$$
\nu(x) - \beta x \nu'(x) - L(x) = 0, \quad \forall x \in (\alpha^-, \alpha^+).
$$

Thus, differentiating on the right hand side of $\alpha^-$, and since

$$
\nu'(\alpha^-) = -1, \quad \text{and } L'_+ (\alpha^-) = L'(b^-) = \beta - 1,
$$

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we have

\[
\begin{align*}
0 &= v'_+(\alpha^-) - \beta \alpha^- v''_+(\alpha^-) - \beta v'_+(\alpha^-) - L'_+(\alpha^-) \\
&= -1 + \beta - \beta \alpha^- v''_+(\alpha^-) - (\beta - 1) \\
&= -\beta \alpha^- v''_+(\alpha^-).
\end{align*}
\]

Thus,

\[v''_+(\alpha^-) = 0.\]

On the other hand, by the Lemma 2.5 \(v'(x) = -1, \ \forall x \in (\alpha^-, \infty].\)

So,

\[v''_+(\alpha^-) = 0.\]

Therefore,

\[v''_+(\alpha^-) = 0.\]

Hence, the value function \(v\) is \(C^2\) at \(\alpha^-\).

Now, let's prove that the value function \(v\) is not \(C^2\) at \(\alpha^+\). By the Lemma 2.5 \(v'(x) = 1, \ \forall x \in [\alpha^+, \infty).\)

So,

\[v''_+(\alpha^+) = 0.\]

It suffices to show that \(v''_+(\alpha^+) \neq 0\). Given \(x \in (\alpha^-, \alpha^+)\), note that \(0 < \alpha^-\), then there exists \(T > 0\) such that

\[x(T) = \alpha^+, \ and \ x(t) \in (\alpha^-, \alpha^+), \forall t \in [0, T).\]

Therefore, by Proposition 2.6 equation (2.8) holds. So,

\[v(x) = e^{-T}v(\alpha^-) + \int_0^T e^{-t}L(x(t))dt.\]

Then

\[v'(x) = -\left(\frac{\alpha^-}{x}\right)^{\frac{\beta-1}{\beta}} + \int_0^{v(x)} e^{(\beta-1)t}L'(xe^{\delta t})dt.\]
So

\[ v''(x) = \left[ \frac{1}{\beta x} \left( \frac{\alpha^-}{x} \right)^{\frac{1}{T+1}} (\beta - 1 - L'(\alpha^-)) \right] + \int_0^{\varphi(x)} e^{(2\beta-1)t} L''(xe^t) dt. \]

Note that \( \alpha^- = b^- \). Then \( \beta - 1 - L'(\alpha^-) = 0 \). So, if \( T > 0 \) is such that \( \varphi(x) = T \), then

\[
\begin{align*}
v''(x) &= \int_0^{\varphi(x)} e^{(2\beta-1)t} L''(xe^t) dt \\
&> \int_0^T e^{(2\beta-1)t} \mu dt \\
&> Te^{(2\beta-1)t} \mu > 0.
\end{align*}
\]

Letting \( x \uparrow \alpha^+ \), we get

\[ v''(\alpha^+) \geq Te^{(2\beta-1)t} \mu > 0. \]

Therefore,

\[ v''(\alpha^+) = 0. \]

Hence the value function \( v \) is not \( C^2 \) at \( \alpha^+ \).

Case \( \alpha^- < \alpha^+ < 0 \).

This case is similar to the previous one where \( 0 < \alpha^- < \alpha^+ \).

Claim: The control \( u^*(t) \equiv -\beta \alpha^+, \forall t \geq 0 \) is optimal at \( \alpha^+ \).

According to (2.7) and inserting \( x = \alpha^+ \) yields

\[ v(\alpha^+) = L(\alpha^+) + \beta \alpha^+. \]

On the other hand, note that \( x(t) \equiv \alpha^+ \) is the solution of

\[ \dot{x} = \beta(x - \alpha^+), \quad x(0) = \alpha^+. \]
Therefore,

\[ v^*(\alpha^+) = \int_0^\infty e^{-t}[L(x(t)) + |u^*(t)|]dt \]

\[ = \int_0^\infty e^{-t}[L(\alpha^+) + \beta \alpha^+]dt \]

\[ = [L(\alpha^-) + \beta \alpha^+] \int_0^\infty e^{-t}dt \]

\[ = L(\alpha^-) + \beta \alpha^+. \]

Thus, \( u^*(t) \equiv -\beta \alpha^+, \quad \forall t \geq 0 \) is optimal at \( \alpha^+. \)

Now we try to pin down the parameter \( \alpha^+. \) For any bounded control \( u \) consider the family of bounded controls defined by

\[ u_\varepsilon(t) = -b \alpha^+ - \varepsilon u(t), \quad \forall t \geq 0. \]

where \( \varepsilon > 0 \) is small enough so that \( u_\varepsilon(t) < 0. \) Let \( v_{u^*}(\alpha^+) \) be the corresponding cost starting from \( \alpha^+. \) Then

\[ v_{u^*}(\alpha^+) \leq v_{u^*}(\alpha^+), \quad \text{for all } \varepsilon > 0, \]

since \( u^* \) is the optimal control at \( \alpha^+. \) Then

\[ \frac{d}{d\varepsilon}(v_{u^*}(\alpha^+))|_{\varepsilon = 0} = 0. \]

Given \( \varepsilon > 0, \) and the control \( u_\varepsilon, \) let \( x_\varepsilon(t) \) be the solution of

\[ \dot{x} = \beta x + [-\beta \alpha^+ - \varepsilon u], \quad x_\varepsilon(0) = \alpha^+. \]

Interchanging \( \frac{d}{dt} \) and \( \frac{d}{d\varepsilon} \) we see that \( \frac{d}{d\varepsilon}[x_\varepsilon(t)] \) is the solution of

\[ \dot{z} = F(z, u), \quad z(0) = 0, \]

where

\[ F(z, u) = \beta z - u. \]
Then, the variation of the constants formula gives us that

\[
\frac{d}{d\varepsilon}[x_\varepsilon(t)] = e^{\beta t} \left[ \int_0^t e^{-\beta s}(-u(s))ds \right] = -\int_0^t e^{\beta(t-s)}u(s)ds.
\]

Thus,

\[
0 = \frac{d}{d\varepsilon}(v^{\varepsilon_0}(\alpha^+))|_{\varepsilon=0} = \int_0^\infty e^{-t}\left\{ L'(x_\varepsilon(t))\frac{d}{d\varepsilon}[x_\varepsilon(t)] + u(t) \right\}|_{\varepsilon=0}dt
\]

\[
= \int_0^t e^{-t}\left\{ L'(\alpha^+)[-\int_0^t e^{\beta(t-s)}u(s)ds] + u(t) \right\} dt
\]

\[
= -L'(\alpha^+) \int_0^\infty e^{-t}\left[\int_0^t e^{\beta(t-s)}u(s)ds \right] dt + \int_0^\infty e^{-t}u(t)dt.
\]

Now, integrating by parts

\[
0 = -L'(\alpha^+)\left\{\int_0^t e^{-\beta s}u(s)ds\left(\frac{1}{\beta - 1}e^{(\beta - 1)t}\right)\right\}_0^\infty
\]

\[
+ L'(\alpha^+) \int_0^\infty \frac{1}{\beta - 1}e^{(\beta - 1)t}(e^{-\beta t}u(t))dt
\]

\[
+ \int_0^\infty e^{-t}u(t)dt.
\]

But

\[
\{\left(\int_0^t e^{-\beta s}u(s)ds\left(\frac{1}{\beta - 1}e^{(\beta - 1)t}\right)\right)\}_0^\infty = 0.
\]

Therefore,

\[
0 = -\frac{L'(\alpha^+)}{1 - \beta} \int_0^\infty e^{-t}u(t)dt + \int_0^\infty e^{-t}u(t)dt
\]

\[
= \left[-\frac{L'(\alpha^+)}{1 - \beta} + 1\right] \int_0^\infty e^{-t}u(t)dt.
\]

Since \( u \) was arbitrary

\[
\frac{L'(\alpha^+)}{1 - \beta} + 1 = 0.
\]

Hence, by definition

\[
L'(\alpha^+) = 1 - \beta = L'(b^+).
\]
Thus,
\[ \alpha^+ = b^+ . \]

since the function \( L' \) is strictly increasing. Let's prove that
\[ b^+ > a^+ . \] Hence \( \alpha^+ = \max(b^+, a^+) \).

Using Lemma 2.4 (iv) and inserting \( x = b^+ = \alpha^+ \), yields
\[ \omega'(b^+) - \beta \omega'(b^+) - \beta b^+ \omega''(b^+) - L'(b^+) = 0. \]

But \( L'(b^+) = L'(\alpha^+) = 1 - \beta \), and \( \omega \) is strictly convex; so,
\[ \omega'(b^+)(1 - \beta) - (1 - \beta) = \beta b^+ \omega''(b^+) > 0, \]

since \( b^+ = \alpha^+ < 0 \), then
\[ (\omega'(b^+) - 1)(1 - \beta) = \beta b^+ \omega''(b^+) > 0, \]

so,
\[ \omega'(b^+) - 1 > 0, \] hence \( \omega'(b^+) > 1. \)

Therefore,
\[ b^+ > a^+ , \]

since the function \( \omega' \) is increasing and since \( \omega'(\alpha^+) = 1 \). Let's prove that
\[ b^- > a^- . \] Hence \( \alpha^- = \min(b^-, a^-) \).

By Lemma 2.4 (iv), differentiating and inserting \( x = b^- \), yields
\[ \omega'(b^-) - \beta \omega'(b^-) - \beta b^- \omega''(b^-) - L'(b^-) = 0. \]

But \( L'(b^-) = \beta - 1 \), then
\[ \omega'(b^-)(1 - \beta) + (1 - \beta) = \beta b^- \omega''(b^-) > 0, \]
since \( b^- < b^+ = \alpha^+ < 0 \), thus
\[
(\omega'(b^-) + 1)(1 - \beta) = \beta b^- \omega''(b^-) > 0.
\]
Then,
\[
\omega'(b^-) + 1 > 0, \quad \text{hence } \omega'(b^-) > -1.
\]
Therefore,
\[
b^- > a^-,
\]
since \( \omega' \) is increasing and since \( \omega'(a^-) = -1 \). On the other hand, note that
\[
-1 = v'(\alpha^-) = \omega'(\alpha^-) = \omega'(a^-).
\]
Then
\[
\alpha^- = a^-,
\]
since \( \omega' \) is strictly increasing. Therefore,
\[
\alpha^- = \min(a^-, b^-) = a^-.
\]
Now, let's prove that the value function \( v \) is \( C^2 \) at \( \alpha^+ \). But \( v \) is not \( C^2 \) at \( \alpha^+ \).
We recall (2.7)
\[
v(x) - bxv'(x) - f(x) = 0, \quad \forall x \in (\alpha^-, \alpha^+).
\]
Thus, differentiating on the left hand side of \( \alpha^+ \), and since
\[
v'_-(\alpha^+) = 1, \quad \text{and } L'_-(\alpha^+) = L'(b^+) = 1 - \beta,
\]
yields
\[
0 = v'_-(\alpha^+) - \beta \alpha^+ v''(\alpha^+) - \beta \alpha^+ v'_-(\alpha^+) - L'_-(\alpha^+)
= 1 - \beta - \beta \alpha^+ v''(\alpha^+) - (1 - \beta)
= -\beta \alpha^+ v''(\alpha^+).
\]
Thus,

\[ v''(\alpha^+) = 0. \]

On the other hand, by Lemma 2.5 \( v'(x) = 1, \ \forall x \in (\alpha^+, \infty) \). So,

\[ v''(\alpha^+) = 0. \]

Therefore,

\[ v''(\alpha^+) = 0. \]

Hence, the value function \( v \) is \( C^2 \) at \( \alpha^+ \). Now, let's prove that the value function \( v \) is not \( C^2 \) at \( \alpha^- \). By Lemma 2.5 \( v'(x) = -1, \ \forall x \in (-\infty, \alpha^-) \). So,

\[ v''(\alpha^-) = 0. \]

It suffices to show that \( v''(\alpha^-) \neq 0 \). Given \( x \in (\alpha^-, \alpha^+) \), note that \( \alpha^+ < 0 \), then there exists \( T > 0 \) such that

\[ x(T) = \alpha^+, \text{ and } x(t) \in (\alpha^-, \alpha^+), \forall t \in [0, T). \]

Therefore, by Proposition 2.6 equation (2.8) holds. So,

\[ v(x) = e^{-T}v(\alpha^+) + \int_0^T e^{-t}L(x(t))dt. \]

So,

\[ v'(x) = (\frac{\alpha^+}{x})^{\frac{\beta - 1}{\beta}} + \int_0^{\varphi(x)} e^{(\beta - 1)t}L'(xe^{\beta t})dt. \]

Then

\[ v''(x) = [-\frac{1}{\beta x}(\frac{\alpha^+}{x})^{\frac{\beta - 1}{\beta}}(\beta - 1 + L'(\alpha^+))] + \int_0^{\varphi(x)} e^{(2\beta - 1)t}L''(xe^{\beta t})dt. \]
Note that \( \alpha^+ = b^+ \). Then \( \beta - 1 + L'(\alpha^+) = 0 \). So, if \( T > 0 \) is such that \( \varphi(x) = T \), then

\[
v''(x) = \int_0^{\varphi(x)} e^{(2\beta-1)t} L''(xe^{\beta t}) dt
\]

\[
> \int_0^T e^{(2\beta-1)t} \mu dt
\]

\[
> Te^{(2\beta-1)t} \mu > 0.
\]

Letting \( x \downarrow \alpha^- \), yields

\[
v''_+(\alpha^-) \geq Te^{(2\beta-1)T} \mu > 0.
\]

Therefore,

\[
v''_+(\alpha^-) > v''_-(\alpha^-) = 0.
\]

Hence the value function \( v \) is not \( C^2 \) at \( \alpha^- \).
Chapter 3
The Explicit Solution of a Stochastic Control Problem

3.1 Introduction

This chapter refers to the class of infinite horizon singular optimal control problems. The dynamics of the state process \( x(\cdot) \) being controlled are governed by a stochastic differential equation (SDE) of the form

\[
dx = f(x(t), u(t))dt + \sigma(x(t), u(t))dW(t), \quad x(0) = x,
\]

where \( f \) and \( \sigma \) satisfy the usual conditions under which this SDE has a unique solution. See [13, IV, and Appendix D]. The time evolution of \( x(\cdot) \) is actively influenced by another stochastic process \( u(t) \), called the control process. In classical control problems the displacement of the state process \( x(\cdot) \) due to the control effort is absolutely continuous. But in singular control problems this displacement is allowed to be discontinuous. Our optimal control problem consists in minimizing over all control processes \( u(\cdot) \) in an appropriate class \( \mathcal{U} \) an infinite horizon discounted cost functional

\[
J^u(x) = E^x \int_0^\infty e^{-\beta t} L(x(t), u(t))dt.
\]

The functions \( f, \sigma \) and \( L \) will be specified in section 3.2. The value function for this stochastic control problem is defined to be the infimum over \( \mathcal{U} \) of the costs and it is a function of the initial state \( x \), that is,

\[
v(x) = \inf_{u(\cdot)} J^u(x).
\]

An optimal control process \( u^*(\cdot) \), if it exists, is an argument that minimizes the cost functional. The dynamic programming equation for a stochastic control problem is a second order nonlinear partial differential equation (PDE) that provides an
approach to solving optimal control problems. The value function is a solution of
this PDE in an appropriate sense to be discussed later. The dynamic programming
equation for a singular stochastic control problem consists of a pair of differential
inequalities, called a variational inequality, of the form

$$\max [F^1(x, v(x), v'(x), v''(x)), F^2(x, v(x), v'(x))] = 0; \text{ for all } x \in \mathbb{R}.$$ 

3.2 A Singular Stochastic Control Problem

Let's consider the one dimensional stochastic differential equation (SDE)

$$dx(t) = [bx(t) + u(t)]dt + \epsilon x(t)dW(t),$$

where \(x(0) = x\) is the initial state, \(W(t)\) is a Brownian motion, the control \(u(.)\)
is a nonanticipating nonnegative measurable function of time, i.e.

$$u(.) \in \mathcal{U} = \{u(.) \in L^\infty([0, \infty), [0, \infty)) : u(.) \text{ is nonanticipating}\},$$

\(b\) and \(\epsilon\) are constants such that \(b < 0\) and \(\epsilon > 0\) is small enough. We consider the
cost functional

$$J^u(x) = E^x \int_0^\infty e^{-t}[(x(t) - K)^2 + u(t)]dt,$$

where \(E^x\) is the conditional expectation given the initial state \(x\), \(K\) is a constant,
and \(x(.)\) is the solution of the above SDE, given \(x(0) = x\) and the control \(u(.) \in \mathcal{U}\).
The value function for this control problem is defined in the usual way as

$$v(x) = \inf_{u(.)} J^u(x).$$

3.3 Control Processes of Bounded Variation

Due to the linear dependence of the running cost on the control \(u\), in general there
does not exist optimal controls in the class \(\mathcal{U}\). In small periods of time optimal
controls take arbitrarily large values. For this reason we reformulate the above
control problem following exactly Fleming and Soner's work [13, page 315-319]. Let's rewrite the state dynamic as follows, let

\[ \hat{u}(x) = \begin{cases} |u(s)|^{-1}u(s), & \text{if } u(s) \neq 0, \\ 0, & \text{if } u(s) = 0, \end{cases} \]

and

\[ \xi(t) = \int_0^t |u(s)| ds. \]

Then the stochastic differential equation for the above control problem becomes

\[ dx(t) = bx(t) \, dt + \hat{u}(t) \, d\xi(t) + \varepsilon x(t) \, dW(t), \quad t > 0, \tag{3.1} \]

with the usual hypotheses on the Brownian process \( W(t) \). We now regard

\[ z(t) = \int_{[0,t]} \hat{u}(s) \, d\xi(s), \]

as the control variable at time \( t \). However, in order to obtain optimal controls, we must enlarge the class of controls to admit \( z(t) \) which may not be an absolutely continuous function of \( t \). But we assume that \( z(t) \) is a function of bounded variation on every finite interval \([0, t]\); namely, the function \( z(t) \) is the difference of two monotone functions of \( t \). The total variation of \( z(\cdot) \) on \([0, t] \) is given by

\[ \xi(t) = \sup\left\{ \sum_{i=1}^n |z(t_i) - z(t_{i-1})| : 0 = t_0 < t_1 < \cdots < t_n = t \right\}. \]

Let \( \mu(\cdot) \) be the total variation measure of \( z(\cdot) \), that is

\[ \xi(t) = \int_{[0,t]} d\mu(s). \]

Then

(i) \( \xi(\cdot) \) is nondecreasing, real-valued, left continuous with \( \xi(0) = 0 \).
Here we identify the process $z(s)$ with the pair $(\xi(\cdot), \tilde{u}(\cdot))$, uniquely determined, with $z(\cdot)$ and $\tilde{u}(\cdot)$ progressively measurable. See [13, Appendix D].

Also we assume that

(ii) $\tilde{u}(s) \in [0, \infty)$, for $\mu$-almost every $s \geq 0$,

(iii) $E|z(t)|^m < \infty$, $m = 1, 2, \ldots$,

where $E$ stands for expectation. Let $\hat{A}$ be the set of all progressively measurable $z(\cdot) = (\xi(\cdot), \tilde{u}(\cdot))$ satisfying the above conditions (i), (ii), (iii). Thus, for a given $x \in \mathbb{R}$, the usual Picard iteration gives $x_n(t), \ n = 1, 2, \ldots$, such that

$$x_n(t) - z(t) \to x(t) - z(t), \quad \text{as} \quad n \to \infty,$$

with probability 1, uniformly for bounded $t$. The process $x(\cdot)$ is the unique, left continuous solution to

$$x(t) = x + z(t) + \int_0^t bx(s)ds + \int_0^t \varepsilon x(s)d\omega(s), \quad t \geq 0,$$

with

$$x_+(t) - x(t) = z_+(t) - z(t).$$

Note that $x(t)$ is not in general continuous. In fact, at every $t \geq 0$,

$$x_+(t) = \lim_{s \to t} x(s) = x(t) + z_+(t) - z(t)$$

$$= x(t) + \tilde{u}(t)(\xi_+(t) - \xi(t)).$$

We want to minimize over all $z(\cdot) = (\xi(\cdot), \tilde{u}(\cdot)) \in \hat{A}$ the cost functional which becomes

$$J^{\xi, \tilde{u}}(x) = E^x \int_0^\infty e^{-t}[(x(t) - K)^2 dt + \tilde{u}(t) d\xi(t)], \quad (3.2)$$

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where $E^x$ is the conditional expectation given the initial state $x$, $K$ is a constant, and $x(t)$ is the solution of (3.1), given $x(0) = x$ and the control process $z(\cdot) = (\xi(\cdot), \hat{u}(\cdot))$. The value function $v$ is defined as usual

$$v(x) = \inf_{\mathcal{A}} J^{K, \hat{u}}(x).$$

(3.3)

The interpretation, roughly speaking, is that we are attempting to 'steer' $|x(t) - K|$ to zero with minimum 'use of fuel'.

### 3.4 The Dynamic Programming Equation

According to [13, page 318-338] the value function $v(x)$ satisfies the Dynamic Programming Principle, which says that $\forall x \in \mathbb{R}, \delta > 0$

$$v(x) = \inf_{\mathcal{A}} \left\{ E^x \int_0^\delta e^{-\delta t}[(x(t) - K)^2 \, dt + \hat{u}(t) \, d\xi(t)] + v(x(\delta)) \right\}.$$ 

Also, from the above reference $v(x)$ is a viscosity solution of the Hamilton-Jacobi-Bellman (HJB) equation

$$\max \{Lv(x) - (x - K)^2, H(v'(x))\} = 0, \quad -\infty < x < \infty,$$

(3.4)

where $L$ is given by

$$Lv(x) = v(x) - \frac{1}{2} \varepsilon^2 x^2 v''(x) - bxv'(x),$$

and

$$H(v'(x)) = \sup_{|u|=1} (-v'(x)u - u) = -v'(x) - 1,$$

provided that $v$ is a polynomially growing real-valued continuous function on $\mathbb{R}$. In fact, we will show in Lemma 3.1 that the value function $v$ for our stochastic control problem satisfies a polynomial growth condition.

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3.5 The Free Boundary

The HJB equation (3.4) is equivalent to a set of differential inequalities

\[ \mathcal{L}(x) - (x - K)^2 \leq 0, \quad -v'(x) - 1 \leq 0, \quad \text{for all } x \in \mathbb{R}, \]

and

\[ \mathcal{L}(x) - (x - K)^2 \geq 0, \quad -v'(x) - 1 \leq 0, \quad \text{for all } x \in \mathbb{R}. \]

Therefore,

if \( \mathcal{L}(x) - (x - K)^2 < 0 \), then \(-v'(x) - 1 = 0\), 

(3.5)

and

if \(-v'(x) - 1 < 0\), then \(\mathcal{L}(x) - (x - K)^2 = 0\). 

(3.6)

The state space splits into two regions, the "no-action" region where (3.6) holds, and the "push" or "active" region where (3.5) holds. These two regions are separated by the so-called free boundary \( B \) where both \(-v'(x) - 1 = 0\) and \(\mathcal{L}(x) - (x - K)^2 = 0\) hold. Nonsmoothness of the value function often occurs only along the free boundary \( B \). The property of smooth fit is said to hold for a particular control problem if the value function is smooth enough along the free boundary \( B \) so that it solves the HJB equation (3.4) in the classical sense. Therefore, the dynamic programming equation is also called a free boundary problem, since the crucial step in solving it is to locate the subset \( B \), the free boundary, where there is a switch between the conditions (3.5) and (3.6). Then we have the following three subsets of the state space \( \mathbb{R} \). The jump or push region is

\[ J = \{ x \in \mathbb{R} : -v'(x) - 1 = 0 \text{ and } \mathcal{L}(x) - (x - K)^2 < 0 \}. \]

The null or no-action region is

\[ N = \{ x \in \mathbb{R} : -v'(x) - 1 < 0 \text{ and } \mathcal{L}(x) - (x - K)^2 = 0 \}. \]
The free boundary is

\[ F = \{ x \in \mathbb{R} : -v'(x) - 1 = \mathcal{L}(x) - (x - K)^2 = 0 \}. \]

In our research we've found that the smooth fit property depends on the parameters of the problem. We will prove that the free boundary for the stochastic control problem (3.1), (3.2), (3.3), (3.4), is just one point, say \( \alpha \), and that for the special case of \( K = \frac{1 - b}{2} \) the value function \( v \in C^2(\mathbb{R} \setminus \{ \alpha \}) \) and it is not \( C^2 \) at \( \alpha \). Also we will discuss in this chapter the other two cases \( K < \frac{1 - b}{2} \), and \( K > \frac{1 - b}{2} \), but their complete solution is not presented in this thesis.

### 3.6 Polynomial Growth Condition of the Value Function

**Lemma 3.1.** The value function \( v \) for the stochastic control problem (3.1), (3.2), (3.3), is majorized by a quadratic function. More precisely, there exist \( M > 0 \), and \( N \in \mathbb{R} \), such that,

\[ 0 \leq v(x) \leq Mx^2 + Nx + K^2, \quad \forall x \in \mathbb{R}. \]

**Proof.**

The state equation (3.1) with \( \bar{u}(\cdot) \equiv 0, \xi(\cdot) \equiv 0 \), becomes

\[ dx(t) = bx(t)dt + ex(t)dW(t); \quad \text{with} \quad x(0) = x, \]

which is a linear stochastic differential equation whose solution, according to [7], is given by

\[ x(t) = x \exp [(b - \epsilon^2/2)t + \epsilon W(t)] \]

\[ = x \exp [(b - \epsilon^2/2)t] \exp [\epsilon W(t)]. \]

It is easy to check this using Itô's rule. Since \( W \) is a Brownian Motion, then

\[ \mu = EW(t) = 0, \quad \sigma^2 = E[W(t)]^2 = t, \quad \text{and} \quad W(t) \quad \text{is Gaussian,} \]
Therefore, the mean of the process $x(t)$ is:

\[
E_x(t) = x \exp \left[ (b - \epsilon^2/2)t \right] E \left[ \exp \left( \epsilon W(t) \right) \right]
\]

\[
= x \exp \left[ (b - \epsilon^2/2)t \right] \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp \left( \epsilon x - \frac{x^2}{2t} \right) dx
\]

\[
= x \exp \left[ (b - \epsilon^2/2)t \right] \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp \left[ -\left( \frac{x^2}{2t} - \epsilon x + \frac{te^2}{2} \right) \right] dx
\]

\[
= x \exp \left[ (b - \epsilon^2/2)t \right] \frac{1}{\sqrt{2\pi t}} \exp \left[ \left( \frac{te^2}{2} \right) \right] \int_{-\infty}^{\infty} \exp \left[ -\left( \frac{x^2}{2t} - \epsilon \frac{\sqrt{t}}{\sqrt{2}} \right) \right] \frac{dx}{\sqrt{2t}}
\]

\[
= x \exp \left[ (b - \epsilon^2/2)t \right] \frac{1}{\sqrt{\pi}} \exp \left( \frac{te^2}{2} \right) \sqrt{\pi}.
\]

Therefore,

\[
E_x(t) = x \exp \left( bt \right). \quad (3.7)
\]

On the other hand,

\[
E(x(t))^2 = E\left[ x^2 \exp \left( 2(b - \epsilon^2/2)t \right) \exp \left( 2\epsilon W(t) \right) \right]
\]

\[
= x^2 \exp \left( 2(b - \epsilon^2/2)t \right) E\left[ \exp \left( 2\epsilon W(t) \right) \right]
\]

\[
= x^2 \exp \left( (2b - \epsilon^2)t \right) \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp \left( 2\epsilon x - \frac{x^2}{2t} \right) dx
\]

\[
= x^2 \exp \left( 2bt \right) \exp \left( -\epsilon^2 t \right) \frac{1}{\sqrt{\pi}} \exp \left( 2te^2 \right) \int_{-\infty}^{\infty} \exp \left[ -\left( \frac{x}{\sqrt{2t}} - \sqrt{2te^2} \right) \right] \frac{dx}{\sqrt{2t}}
\]

\[
= x^2 \exp \left( 2bt \right) \exp \left( -\epsilon^2 t \right) \exp \left( 2te^2 \right) \frac{1}{\sqrt{\pi}} \sqrt{\pi}.
\]

Therefore,

\[
E(x(t))^2 = x^2 \exp \left( 2bt \right) \exp \left( te^2 \right). \quad (3.8)
\]
Now, using (3.7), (3.8) after taking the control process \( \hat{u}(\cdot) \equiv 0, \xi(\cdot) \equiv 0 \), for all \( x \in \mathbb{R} \):

\[
J^{0,0}(x) = E^x \int_0^\infty e^{-t}E(x(t))^2 dt - 2K \int_0^\infty e^{-t}Ex(t) dt + K^2 \\
= x^2 \int_0^\infty \exp [(2b + \epsilon^2 - 1)t] dt - 2Kx \int_0^\infty \exp [(b - 1)t] dt + K^2 \\
= \frac{1}{1 - 2b - \epsilon^2} - \frac{2Kx}{1 - b} + K^2.
\]  

(3.9)

Therefore, by definition of the value function \( v \)

\[
0 \leq v(x) \leq Mx^2 + Nx + K^2,
\]

(3.10)

where

\[
M = \frac{1}{1 - 2b - \epsilon^2}, \quad \text{and} \quad N = \frac{-2K}{1 - b}.
\]

### 3.7 Constructing the Candidate Value Function

The running cost (3.2) is quadratic and convex, thus \( v \) is convex as a function of \( x \), see [4, page 134-185], and [13, Chapter VIII]. Therefore, \( v \) is continuous on \( \mathbb{R} \) and locally Lipschitz, see [21]. Also \( v \) turns out to be almost everywhere twice differentiable, see [13, Appendix E]. We will first construct a nonnegative convex polynomially growing solution \( V \) of (3.4) and then using the Verification Theorem [13, Theorem VIII 4.1] we will show that \( V = v \). In order to construct a candidate value function we look in this section for a solution of (3.4) which is \( C^1 \) in \( \mathbb{R} \) and \( C^2 \) in \( \mathbb{R} \setminus \{\alpha\} \), with \( \alpha \) to be determined.

**Theorem 3.2.** The value function \( v \) for the stochastic control problem (3.1), (3.2), (3.3) is convex. Moreover, the free boundary is just a single point, say \( \alpha \). Also the free boundary \( \alpha \) is determined by the data of the problem. If \( K = \frac{1-b}{2} \), then \( v \) satisfies \( v \in C^2(\mathbb{R} \setminus \{\alpha\}) \), and \( v \) is not \( C^2 \) at \( \alpha \), and \( \alpha = 0 \).
**Proof.**

We set

\[ \alpha = \inf \{ x : -\nu'(x) - 1 < 0 \} . \]

Notice that the set \( \{ x : -\nu'(x) - 1 < 0 \} \) is nonempty; otherwise, (3.4) would imply \( \nu'(x) = -1 \) for almost every (a.e.) \( x \in \mathbb{R} \), so \( \nu \) would take negative values in \( \mathbb{R} \). In fact, by the Fundamental Theorem of Calculus, given \( a < x \)

\[ v(x) - v(a) = \int_a^x v'(s) ds = \int_a^x (-1) ds = x - a . \]

Hence, \( v(x) < 0 \) for large \( x \). This is a contradiction. Then \( \alpha < +\infty \). By convexity \( \nu' \) is increasing; therefore, \( -\nu'(x) - 1 < 0 \), for a.e. \( x > \alpha \). Then using (3.6)

\[ \nu(x) - \frac{1}{2} \epsilon^2 x^2 \nu''(x) - bx \nu'(x) - (x - K)^2 = 0, \quad \text{for a.e.} \quad x > \alpha . \]  

(3.11)

On the other hand, using (3.4) again and by convexity of \( v \)

\[ \nu'(x) = -1, \quad \text{for a.e.} \quad x \leq \alpha . \]  

(3.12)

We'll see later in Corollary 3.5 that \( -\infty < \alpha \). We now solve (3.11) and (3.12) finding out where \( \alpha \) is located in terms of the data of the problem. Let \( \omega_p \) be a particular solution of the nonhomogeneous equation (3.11)

\[ \omega_p(x) = Ax^2 + Bx + C, \]

where

\[ A = (1 - \epsilon^2 - 2b)^{-1}, \quad B = \frac{-2K}{(1 - b)}, \quad \text{and} \quad C = K^2 . \]  

(3.13)

Let \( \omega_g \) be the general solution of the homogeneous Euler equation

\[ \omega(x) - \frac{1}{2} \epsilon^2 x^2 \omega''(x) - bx \omega'(x) = 0 , \quad x > \alpha , \]  

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thus
\[ w_g(x) = C_1|x|^\gamma + C_2|x|^\gamma, \]
for any \( x > \alpha \) in any interval not containing the origin, where \( r_1 \) and \( r_2 \) are the solutions of the quadratic equation
\[
 r^2 + \left( \frac{2b}{\varepsilon^2} - 1 \right)r - \frac{2}{\varepsilon^2} = 0. \tag{3.14}
\]
Thus,
\[
 r_{1,2} = \frac{1}{2} \left[ (1 - \frac{2b}{\varepsilon^2}) \pm \left( (1 - \frac{2b}{\varepsilon^2})^2 + \frac{8}{\varepsilon^2} \right)^{\frac{1}{2}} \right].
\]
So \( r_1 > 2 \) for \( \varepsilon > 0 \) small enough, and \( r_2 < 0 \). Then the general solution \( \omega \) of (3.11) can be expressed either as
\[
 \omega_+(x) = \omega_g + \omega_p = C_1|x|^\gamma + C_2|x|^\gamma + Ax^2 + Bx + K^2, \text{ for } x > \alpha \geq 0, \tag{3.15}
\]
or as
\[
 \omega_-(x) = \begin{cases} 
 C_1^-|x|^\gamma + C_2^-|x|^\gamma + Ax^2 + Bx + K^2, & \text{if } \alpha < x < 0, \\
 C_1^+|x|^\gamma + C_2^+|x|^\gamma + Ax^2 + Bx + K^2, & \text{if } \alpha < 0 < x.
 \end{cases} \tag{3.16}
\]
Note that in case that \( \alpha < 0 \), we need two expressions of the form of (3.15) and we have four constants, two for each expression of \( \omega(x) \) on each side of the origin.

With the next two lemmas we eliminate some of these terms.

Lemma 3.3.
\[
 C_1 = C_1^+ = 0.
\]

Proof.

Note that for \( \varepsilon > 0 \) small enough, \( r_1 > 2 \), then by the quadratic bound in (3.10) we must have \( C_1 = C_1^+ = 0. \)

Lemma 3.4.
\[
 -\infty \leq \alpha < 0 \implies C_2 = C_2^- = C_2^+ = 0.
\]

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Proof.

Let's consider the control system (3.1) with control $\tilde{u}(\cdot) \equiv 0$, $\xi(\cdot) \equiv 0$, and initial value $x(0) = 0$. Thus, (3.1) becomes

$$\frac{dx(t)}{dt} = bx(t)dt + ex(t)du(t), \text{ with } x(0) = 0.$$ 

So $x(t) \equiv 0$ is the solution. Therefore,

$$0 \leq v(0) = E^0 \int_0^{\infty} e^{-t}K^2 dt = K^2 < \infty. \quad (3.17)$$

On the other hand, if $C^-_2 \neq 0$, then $|\omega_{-}(x)| \to \infty$, as $x \to 0^-$. This contradicts (3.17). Therefore, we must have $C^-_2 = 0$. Similarly, $C^+_2 = C_2 = 0$. \qed

Corollary 3.5. $\alpha > -\infty$. That is, $\alpha$ is a real number.

Proof.

Let's suppose that $\alpha = -\infty$. By Lemmas 3.3, 3.4 we have $C^-_2 = C^+_2 = C_1^+ = 0$. Thus, (3.16) becomes

$$\omega_{-}(x) = \begin{cases} Ax^2 + Bx + K^2 + C^-_1|x|^{-1}, & \text{if } x < 0, \\ Ax^2 + Bx + K^2, & \text{if } x > 0. \end{cases}$$

So, $\omega_{-}(x) = 2Ax \to -\infty$, as $x \to -\infty$ since $A > 0$, for $\varepsilon > 0$ small enough. But this is impossible because (3.4) yields $v'(x) \geq -1$, for a.e. $x \in \mathbb{R}$. Therefore, $\alpha > -\infty$, and hence $\alpha$ is a real number. \qed

Corollary 3.6. If $\alpha = 0$, and if $v = \omega_{+}$, then $v$ is not $C^2$ at $\alpha$. Moreover, $\alpha = 0$ if and only if $K = \frac{1-b}{2}$.

Proof.

We show that if $\alpha = 0$ then $v''(\alpha)$ does not exist. By Lemmas 3.3, 3.4 the constants $C_1, C_2$ in (3.15) verify $C_1 = C_2 = 0$. Hence $\omega_{+}(x) = Ax^2 + Bx +...
$K^2$, for all $x > \alpha = 0$, thus, $\omega_+''(0^+) = 2A \neq 0 = \omega''(0^-)$. Moreover, $\nu'(\alpha) = -1$ implies $2A\alpha + B = -1$, and using (3.13)

$$\alpha = (K - \frac{1-b}{2}) \frac{1-2b-e^2}{1-b} = 0,$$

if and only if $K = \frac{1-b}{2}$.

Finally, the candidate value function $V$ can be written as

Case $\alpha < 0$.

$$V(x) = \begin{cases} 
\alpha - x + Q(\alpha) + C_1^{-}\lvert x \rvert^{r_1}, & \text{if } x < \alpha, \\
Q(x) + C_1^{-}\lvert x \rvert^{r_1}, & \text{if } \alpha \leq x \leq 0, \\
Q(x), & \text{if } x > 0,
\end{cases}$$

where

$$Q(x) = Ax^2 + Bx + K^2, \quad C_1 \neq 0,$$

and

$$\alpha = (K - \frac{1-b}{2}) \frac{(1-2b-e^2)(r_1-1)}{(1-b)(r_1-2)} < 0, \quad \text{iff } K < \frac{1-b}{2}.$$ 

Case $\alpha > 0$.

$$V(x) = \begin{cases} 
\alpha - x + Q(\alpha) + C_2\lvert x \rvert^{r_2}, & \text{if } x < \alpha, \\
Q(x) + C_2\lvert x \rvert^{r_2}, & \text{if } x \geq \alpha,
\end{cases}$$

where $C_2 \neq 0$, and

$$\alpha = (K - \frac{1-b}{2}) \frac{(1-2b-e^2)(r_2-1)}{(1-b)(r_2-2)} > 0, \quad \text{iff } K > \frac{1-b}{2}.$$ 

Case $\alpha = 0$.

$$V(x) = \begin{cases} 
-x + K^2, & \text{if } x \leq 0, \\
Q(x), & \text{if } x > 0.
\end{cases}$$

The constants $C_1^{-}, C_2$ will be determined below.
3.8 The Candidate Value Function for the Case 
\( K < \frac{1-b}{2} \)

Lemma 3.7. Suppose that \( K < \frac{1-b}{2} \). Consider the function \( V \) defined by

\[
V(x) = \begin{cases} 
\alpha - x + Q(\alpha) + C_1^-|\alpha|^\gamma, & \text{if } x < \alpha, \\
Q(x) + C_1^-|x|^\gamma, & \text{if } \alpha \leq x \leq 0, \\
Q(x), & \text{if } x > 0,
\end{cases}
\]

where \( \gamma > 2 \) is the positive solution of the quadratic equation (3.14) for \( \epsilon > 0 \) small enough and

\[ Q(x) = Ax^2 + Bx + K^2, \quad \text{and} \quad C_1^- \neq 0, \]

with \( A \) and \( B \) given by (3.13). Then \( V \) is in \( C^2(\mathbb{R}) \) if and only if

\[
\alpha = (K - \frac{1-b}{2}) \frac{(1-2b-\epsilon^2)(\gamma-1)}{(1-b)(\gamma-2)} < 0, \tag{3.18}
\]

and

\[
C_1^- = \frac{-2A|\alpha|^{2-\gamma}}{\gamma(\gamma-1)}. \tag{3.20}
\]

Proof.

Note that from the definition, \( V \in C^\infty(\mathbb{R} \setminus \{\alpha, 0\}) \). It is easy to check that \( V \) is \( C^2 \) at 0 since \( \gamma > 2 \). Let’s see when \( V \) is \( C^2 \) at \( \alpha \). From the definition

\[
\begin{align*}
V_-'(\alpha) &= -1, \\
V_+'(\alpha) &= 2A\alpha + B + C_1^-\gamma|\alpha|^{\gamma-1}\text{sgn}(\alpha), \\
V_''(\alpha) &= 0, \\
V_+''(\alpha) &= 2A + C_1^-\gamma(\gamma-1)|\alpha|^{\gamma-2},
\end{align*}
\]

where \( \text{sgn}(\cdot) \) is the signum function. Therefore, the function \( V \) is \( C^2 \) at \( \alpha \) if and only if

\[
\begin{align*}
2A\alpha + B + C_1^-\gamma|\alpha|^{\gamma-1}\text{sgn}(\alpha) &= -1, \tag{3.19} \\
2A + C_1^-\gamma(\gamma-1)|\alpha|^{\gamma-2} &= 0. \tag{3.20}
\end{align*}
\]

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From (3.20)
\[ C_1^{-\gamma} |\alpha|^{\gamma-1} = -\frac{2A|\alpha|}{(\gamma - 1)}. \]

From (3.19)
\[ 2A\alpha - B + \left[ -\frac{2A|\alpha|}{(\gamma - 1)} \right] \text{sgn}(\alpha) = -1. \]

Then
\[ 2A\alpha (1 - \frac{1}{\gamma - 1}) = -(B + 1). \]

Thus \( V \) is \( C^2 \) at \( \alpha \) if and only if
\[
\alpha = \frac{-(B + 1) (\gamma - 1)}{2A (\gamma - 2)} = (K - \frac{1-b}{2} \frac{(1-2b-\epsilon^2)(\gamma - 1)}{(1-b)(\gamma - 2)}) < 0,
\]
and \( C_1^- \) solves (3.20).

**Lemma 3.8.** Suppose that \( K < \frac{1-b}{2} \), and that \( \alpha \) satisfies (3.18). Then the function \( V \) defined in Lemma 3.7, with \( C_1^- \) satisfying (3.20), is convex and positive.

**Proof.**

First, let's see that the function \( V \) is convex. Note that from Lemma 3.7 the function \( V \) is in \( C^2(\mathbb{R}) \).

\[
V''(x) = \begin{cases}
0, & \text{if } x < \alpha, \\
2A + C_1^- \gamma(\gamma - 1)|x|^{\gamma-2}, & \text{if } \alpha \leq x \leq 0, \\
2A > 0, & \text{if } x > 0.
\end{cases}
\]

But from (3.20)
\[
C_1^- = -\frac{2A|\alpha|^{2-\gamma}}{\gamma(\gamma - 1)},
\]

\[
V''(x) = 2A + \left[ -\frac{2A|\alpha|^{2-\gamma}}{\gamma(\gamma - 1)} \right] \gamma(\gamma - 1)|x|^{\gamma-2}, \quad \forall x \in [\alpha, 0]
= 2A - 2A|\alpha|^{2-\gamma}|x|^{\gamma-2}, \quad \forall x \in [\alpha, 0]
= 2A[1 - (\frac{x}{\alpha})^{\gamma-2}] \geq 0, \quad \forall x \in [\alpha, 0].
\]
Therefore,

\[ V''(x) \geq 0, \quad \forall x \in \mathbb{R}. \]

Hence, the function \( V \) is convex. Now, let's prove that the function \( V \) is positive.

(i) Case of: \( 0 < K < \frac{1-b}{2} \).

Since \( V'(0) = B = \frac{-2K}{1-b} < 0 \) and \( V \) is convex, then it must attain its minimum value at a point \( x_0 > 0 \). Thus, the function \( V \) attains its minimum when \( Q'(x_0) = 0 \), that is when \( x_0 = -\frac{B}{2A} > 0 \). Therefore, \( \forall x \in \mathbb{R} \) we have

\[
V(x) \geq A\left(-\frac{B}{2A}\right)^2 + B\left(-\frac{B}{2A}\right) + K^2 \\
= \frac{B^2}{4A} - \frac{B^2}{2A} + K^2 \\
= \frac{B^2}{4A} + K^2 \\
= -\frac{4K^2}{(1-b)^2} \left(1 - 2b - \varepsilon^2\right) + K^2 \\
= -K^2 \left(1 - \varepsilon^2\right) \\
= K^2 \left[1 - \frac{(1 - 2b - \varepsilon^2)}{(1-b)^2}\right] \\
= K^2 \left(1 - \frac{(1-b)^2[(1-b)^2 - (1-2b-\varepsilon^2)]}{(1-b)^2}\right) \\
= K^2 \frac{1}{(1-b)^2}(b^2 + \varepsilon^2) > 0.
\]

(ii) Case of: \( K < 0 < \frac{1-b}{2} \).

Since \( V'(0) = B = \frac{-2K}{1-b} > 0 \), then \( V \) attains its minimum at a point \( \alpha < x_0 < 0 \).

Then

\[
V'(x_0) = 2Ax_0 + B - C_1 \gamma |x_0|^{(\gamma-1)} = 0.
\]

Therefore,

\[
V(x) \geq Ax_0^2 + Bx_0 + K^2 + C_1|x_0|^\gamma, \quad \forall x \in \mathbb{R}. \quad (3.21)
\]
Now, by convexity

$$V''(x) = 2A + C_l(\gamma - 1)\gamma|x|^{\gamma - 2} \geq 0,$$

then

$$C_l|x|^{\gamma} \geq -\frac{2A}{\gamma(\gamma - 1)}x^2. \quad (3.22)$$

Now we retake (3.21) using inequality (3.22)

$$V(x) \geq A\varepsilon_0^2 + B\varepsilon_0 + K^2 - \frac{2A}{\gamma(\gamma - 1)}\varepsilon_0^2$$

$$= A[1 - \frac{2}{\gamma(\gamma - 1)}]\varepsilon_0^2 + B\varepsilon_0 + K^2$$

$$= A\frac{(\gamma - 2)(\gamma + 1)}{\gamma(\gamma - 1)}\varepsilon_0^2 + B\varepsilon_0 + K^2.$$

Now finding the minimum value of this quadratic expression.

$$V(x) \geq A\frac{(\gamma - 2)(\gamma + 1)}{\gamma(\gamma - 1)}[\frac{-B\gamma(\gamma - 1)}{2A(\gamma - 2)(\gamma + 1)}]^2$$

$$+ B[\frac{-B\gamma(\gamma - 1)}{2A(\gamma - 2)(\gamma + 1)}] + K^2$$

$$= \frac{B^2}{4A(\gamma - 2)(\gamma + 1)} - \frac{B^2}{2A(\gamma - 2)(\gamma + 1)} + K^2$$

$$= \frac{B^2}{4A(\gamma - 2)(\gamma + 1)} + K^2$$

$$= -\frac{4K^2}{(1 - b)^2 A(\gamma - 2)(\gamma + 1)} + K^2$$

$$= K^2[1 - \frac{(1 - 2b - \epsilon^2)}{(1 - b)^2 (\gamma - 2)(\gamma + 1)}]$$

$$= K^2[(1 - b)^2(\gamma - 2)(\gamma + 1) - (1 - 2b - \epsilon^2)\gamma(\gamma - 1)]$$

$$= K^2[1 - \frac{(1 - 2b + \epsilon^2)(\gamma^2 - \gamma - 2) + 4b + (\epsilon^2\gamma - \epsilon^2\gamma - 2)}{(1 - b)^2(\gamma - 2)(\gamma + 1)}].$$

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Now, $\gamma$ is one of the roots of the quadratic equation (3.14), so

$$\varepsilon^2 \gamma^2 - \varepsilon^2 \gamma + 2b\gamma - 2 = 0.$$ 

Then, since $\gamma > 2$ and since $b < 0$,

$$V(x) \geq \frac{K^2[b^2(\gamma^2 - \gamma - 2) + 4b - 2b\gamma]}{(1 - b)^2(\gamma - 2)(\gamma + 1)}$$

$$= \frac{K^2[b^2(\gamma - 2)(\gamma + 1) - 2b(\gamma - 2)]}{(1 - b)^2(\gamma - 2)(\gamma + 1)}$$

$$= \frac{K^2(\gamma - 2)[b^2(\gamma + 1) - 2b]}{(1 - b)^2(\gamma - 2)(\gamma + 1)}$$

$$= \frac{K^2[b^2(\gamma + 1) - 2b]}{(1 - b)^2(\gamma + 1)} > 0. \quad \square$$

**Lemma 3.9.** Suppose that $K < \frac{1 - b}{2}$, and $\alpha$ satisfies (3.18). Then the function $V$ defined in Lemma 3.7 is a classical solution of the Hamilton-Jacobi-Bellman equation

$$\max[F_2(x, v, v', v''), F_1(x, v, v')] = 0, \quad -\infty < x < \infty,$$

where

$$F_2(x, v, v', v'') = v(x) - \frac{1}{2} \varepsilon^2 x^2 v''(x) - bxv'(x) - (x - K)^2, \quad \forall x \in \mathbb{R},$$

$$F_1(x, v, v') = -v' - 1, \quad \forall x \in \mathbb{R}.$$ 

**Proof.**

Since $V$ is $C^2$, it suffices to prove that

(i) $F_1(x, V, V') \leq 0, \quad \forall x \geq \alpha,$

(ii) $F_2(x, V, V', V'') = 0, \quad \forall x \geq \alpha,$

(iii) $F_1(x, V, V') = 0, \quad \forall x \leq \alpha,$

(iv) $F_2(x, V, V', V'') \leq 0, \quad \forall x \leq \alpha.$
Proof of (i).

Let

\[ V_2(x) = \begin{cases} 
    Q(x) + C_1^{-1}|x|^\gamma, & \forall \alpha \leq x \leq 0, \\
    Q(x), & \forall x > 0.
\end{cases} \]

Then for \( \alpha \leq x \leq 0 \)

\[ F_1(x, V_2(x), V'_2(x)) = -V'_2(x) - 1. \]

So, by convexity of \( V_2 \)

\[ \frac{d}{dx} F_1(x, V_2(x), V'_2(x)) = -V''_2(x) \leq 0. \]

Then \( F_1(\cdot, V_2(\cdot), V'_2(\cdot)) \) is nonincreasing and it attains its maximum at \( x = \alpha \). Therefore, by (3.19)

\[ F_1(x, V_2(x), V'_2(x)) \leq F_1(\alpha, V_2(\alpha), V'_2(\alpha)) = -2A\alpha - B + C_1^{-1} \gamma |\alpha|^{\gamma-1} - 1 = 0. \]

Proof of (ii).

Let \( V_2(x) = Q(x) + C_1^{-1}|x|^\gamma, \ \forall \alpha \leq x \leq 0. \) Then

\[ V_2(x) = Ax^2 + Bx + K^2 + C_1^{-1}|x|^\gamma, \]

\[ V'_2(x) = 2Ax + B + C_1^{-1} \gamma |x|^{\gamma-1} \text{sgn}(x), \]

\[ V''_2(x)) = 2A + C_1^{-1}(\gamma - 1) \gamma |x|^{\gamma-2}. \]
Then
\[
F_2(x, V_2(x), V'_2(x), V''_2(x)) = V_2(x) - \frac{1}{2} \varepsilon^2 x^2 V''_2(x) - bx V'_2(x) - (x - K)^2
\]
\[
= Ax^2 + Bx + K^2 + C_1 |x|^7
- \frac{1}{2} \varepsilon^2 x^2 [2A + C_1 (\gamma - 1) \gamma |x|^7 - 2]
- bx [2Ax + B + C_1 \gamma |x|^7 \text{sgn}(x)] - (x - K)^2
\]
\[
= [A(1 - 2b - \varepsilon^2) - 1]x^2 + [B(1 - b) + 2K]x + [1 - \frac{\varepsilon^2}{2} \gamma (\gamma - 1) - b \gamma] C_1 |x|^7
= [1 - 1]x^2 + [-2K + 2K]x + \frac{1}{2} [2 - \varepsilon^2 \gamma^2 + \varepsilon^2 \gamma - 2b \gamma] C_1 |x|^7
= 0.
\]

Since \( \gamma \) is the positive solution of the quadratic equation (3.14). The calculation is the same for \( 0 < x \), except that \( C_1 = 0 \). \( \square \)

**Proof of (iii).**

Let \( V_1(x) = \alpha - x + Q(\alpha) + C_1 |\alpha|^\gamma, \quad \forall x \leq \alpha \). Then it is clear that
\[
F_1(x, V_1(x), V'_1(x)) = -V'_1(x) - 1 = 1 - 1 = 0. \quad \square
\]

**Proof of (iv).**

Let \( V_1(x) = \alpha - x + Q(\alpha) + C_1 |\alpha|^\gamma, \quad \forall x \leq \alpha \). Note that \( V'_1(x) = -1 \), and \( V''_1(x) = 0 \).

We recall
\[
C_1 |\alpha|^\gamma = -\frac{2A|\alpha|^2}{\gamma(\gamma - 1)},
\]
and define
\[
\beta = K - \frac{1-b}{2}.
\]

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Then

\[ B + 1 = \frac{-2\beta}{1 - b}, \]
\[ \alpha = \beta \frac{(1 - 2b - \varepsilon^2)(\gamma - 1)}{(1 - b)(\gamma - 2)}. \]

Then,

\[
F_2(x, V_1(x), V'_1(x), V''_1(x)) = V_1(x) - \frac{1}{2} \varepsilon^2 x^2 V''_1(x) - bx V'_1(x) - (x - K)^2
\]
\[ = \alpha - x + Q(\alpha) + C_1|\alpha|^2 + bx - x^2 + 2Kx - K^2
\]
\[ = -x^2 - (1 - b - 2K)x + (B + 1)\alpha + A\alpha^2 + C_1|\alpha|^2
\]
\[ = -x^2 + 2\beta x - \frac{2\beta}{(1 - b)}\alpha + A\alpha^2 - \frac{2A|\alpha|^2}{\gamma(\gamma - 1)}
\]
\[ = -x^2 + 2\beta x - 2\beta^2 \frac{(1 - 2b - \varepsilon^2)(\gamma - 1)}{(1 - b)^2 (\gamma - 2)}
\]
\[ + \left[ 1 - \frac{2}{\gamma(\gamma - 1)} \right] A|\alpha|^2
\]
\[ = -x^2 + 2\beta x - 2\beta^2 \frac{(1 - 2b - \varepsilon^2)(\gamma - 1)}{(1 - b)^2 (\gamma - 2)}
\]
\[ + \beta^2 \left[ \frac{(\gamma + 1)(\gamma - 2)}{\gamma(\gamma - 1)} \frac{(1 - 2b - \varepsilon^2)(\gamma - 1)^2}{(1 - b)^2 (\gamma - 2)^2} \right].
\]

Therefore,

\[
F_2(x, V_1(x), V'_1(x), V''_1(x)) = -x^2 + 2\beta x - \beta^2 \frac{(1 - 2b - \varepsilon^2)(\gamma - 1)^2}{(1 - b)^2 \gamma(\gamma - 2)}.
\]

The roots of the quadratic expression on the right hand side are:

\[ x_{1,2} = \beta \pm |\beta| \sqrt{1 - \frac{(1 - 2b - \varepsilon^2)(\gamma - 1)^2}{(1 - b)^2 \gamma(\gamma - 2)}}
\]
\[ = \beta \left( 1 \mp \sqrt{1 - \frac{(1 - 2b - \varepsilon^2)(\gamma - 1)^2}{(1 - b)^2 \gamma(\gamma - 2)}} \right),
\]

since \( \beta = (K - \frac{1 - b}{2}) < 0 \). To show \( F_2 \leq 0, \forall x \leq \alpha \), it is enough to prove that

\[ \alpha \leq \min(x_1, x_2). \]

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In fact, we shall prove that
\[ \alpha = \min (x_1, x_2) = x_2. \]  
(3.23)

Note that if \( \frac{(1-2b-\varepsilon^2)(\gamma-1)^2}{(1-b)^2 \gamma(\gamma-2)} > 1 \). There is nothing to prove. Let's assume that
\[ 0 \leq \frac{(1-2b-\varepsilon^2)(\gamma-1)^2}{(1-b)^2 \gamma(\gamma-2)} \leq 1. \]

Note that since
\[ \alpha = \beta \frac{(1-2b-\varepsilon^2)(\gamma-1)}{(1-b)(\gamma-2)}, \]
\[ \alpha = x_2 \iff \frac{\alpha}{\beta} = 1 + \sqrt{1 - \frac{(1-2b-\varepsilon^2)(\gamma-1)^2}{(1-b)^2 \gamma(\gamma-2)}} \]
\[ \iff \frac{\alpha}{\beta} - 1 = \sqrt{1 - \frac{\alpha (\gamma-1)}{\beta \gamma(1-b)}} \]
\[ \iff (\frac{\alpha}{\beta} - 1)^2 = 1 - \frac{\alpha (\gamma-1)}{\beta \gamma(1-b)} \]
\[ \iff (\frac{\alpha}{\beta})^2 - 2 \frac{\alpha}{\beta} = \frac{\alpha (\gamma-1)}{\beta \gamma(1-b)} \]
\[ \iff \frac{\alpha}{\beta} - 2 = -\frac{(\gamma-1)}{\gamma(1-b)}. \]

On the other hand,
\[ \frac{\alpha}{\beta} - 2 = \frac{(1-2b-\varepsilon^2)(\gamma-1)}{(1-b)(\gamma-2)} - 2 \]
\[ = \frac{(1-2b-\varepsilon^2)(\gamma-1)-2(1-b)(\gamma-2)}{(1-b)(\gamma-2)} \]
\[ = \frac{\gamma-1 - 2b\gamma + 2b - \varepsilon^2\gamma + \varepsilon^2 - 2\gamma + 4 + 2b\gamma - 4b}{(1-b)(\gamma-2)} \]
\[ = \frac{-\gamma + 3 - 2b - \varepsilon^2\gamma + \varepsilon^2}{(1-b)(\gamma-2)} \]
\[ = \frac{-\gamma^2 + 3\gamma - 2b\gamma - \varepsilon^2\gamma^2 + \varepsilon^2\gamma}{(1-b)\gamma(\gamma-2)}. \]

But \( \gamma \) is one of the roots of the quadratic equation (3.14). So,
\[ \varepsilon^2\gamma^2 + 2b\gamma - \varepsilon^2\gamma - 2 = 0. \]
Therefore,

\[ \frac{\alpha}{\beta} - 2 = \frac{-\gamma^2 + 3\gamma - 2}{(1 - b)\gamma(\gamma - 2)} = \frac{-(\gamma - 2)(\gamma - 1)}{(1 - b)\gamma(\gamma - 2)} = \frac{-(\gamma - 1)}{\gamma(1 - b)}. \]

Thus, (3.23) holds. \(\square\)

### 3.9 The Candidate Value Function for the Case \(K > \frac{1-b}{2}\)

**Lemma 3.10.** Suppose that \(K > \frac{1-b}{2}\), we consider the function \(V\) defined by

\[
V(x) = \begin{cases} 
\alpha - x + Q(x), & \text{if } x < \alpha, \\
Q(x) + C_2|x|^{\rho}, & \text{if } \alpha \leq x,
\end{cases}
\]

where \(\rho < 0\) is the negative solution of the quadratic equation (3.14), with \(\epsilon > 0\) small enough, and

\[Q(x) = Ax^2 + Bx + K^2, \quad \text{and} \quad C_2 \neq 0,\]

with \(A\) and \(B\) given by (3.13). Then \(V\) is in \(C^2(\mathbb{R})\) if and only if

\[
\alpha = (K - \frac{1-b}{2}) \frac{(1-2b-\epsilon^2)(\rho-1)}{(1-b)(\rho-2)} > 0, \quad (3.24)
\]

and

\[C_2 = -\frac{2A|\alpha|^{2-\rho}}{\rho(\rho-1)}.\]

**Proof.**
Note that from the definition, \( V \in C^\infty(\mathbb{R} \setminus \{a\}) \). Let’s see that \( V \) is \( C^2 \) at \( a \).

From the definition,

\[
V'_-(a) = -1, \\
V'_+(a) = 2A\alpha + B + C_2|\alpha|^{\rho-1}\text{sgn}(\alpha), \\
V''(a) = 0, \\
V''_+(a) = 2A + C_2(\rho - 1)|\alpha|^{\rho-2},
\]

where \( \text{sgn}(\cdot) \) is the signum function. Therefore, the function \( V \) is \( C^2 \) at \( a \) if and only if

\[
2A\alpha + B + C_2|\alpha|^{\rho-1}\text{sgn}(\alpha) = -1, \\
2A + C_2(\rho - 1)|\alpha|^{\rho-2} = 0. 
\]

From (3.26)

\[
C_2|\alpha|^{\rho-1} = -\frac{2A|\alpha|}{(\rho - 1)}. 
\]

From (3.25)

\[
2A\alpha + B + \left[-\frac{2A|\alpha|}{(\rho - 1)}\right]\text{sgn}(\alpha) = -1. 
\]

Then

\[
2A\alpha(1 - \frac{1}{\rho - 1}) = -(B + 1). 
\]

Thus \( V \) is \( C^2 \) at \( a \) if and only if

\[
\alpha = \frac{-(B + 1)(\rho - 1)}{2A \cdot (\rho - 2)} \\
= \frac{(K - \frac{1-b}{2})(1 - 2b - \varepsilon^2)(\rho - 1)}{(1 - b)(\rho - 2)} > 0,
\]

and \( C_2 \) solves (3.26). \( \square \)

**Lemma 3.11.** Suppose that \( K > \frac{1-b}{2} \) and that \( \alpha \) satisfies (3.24), then the function \( V \) defined in Lemma 3.10, with \( C_2 \) satisfying (3.26) is convex and positive.
Proof.

First, let’s see that the function $V$ is convex. Note that from Lemma 3.10 the function $V$ is in $C^2(\mathbb{R})$.

$$V''(x) = \begin{cases} 
0, & \text{if } x < \alpha, \\
2A + C_2(p - 1)|x|^{\rho - 2}, & \text{if } \alpha \leq x.
\end{cases}$$

But from (3.26)

$$C_2 = -\frac{2A|\alpha|^{2-\rho}}{\rho(\rho - 1)},$$

$$V''(x) = 2A + \left[ -\frac{2A|\alpha|^{2-\rho}}{\rho(\rho - 1)} \right] \rho(\rho - 1)|x|^{\rho - 2}, \quad \forall x \geq \alpha$$

$$= 2A - 2A|\alpha|^{2-\rho}|x|^{\rho - 2}, \quad \forall x \geq \alpha$$

$$= 2A[1 - (|x|^{\rho})^{2-\rho}] \geq 0, \quad \forall x \geq \alpha.$$ 

Therefore,

$$V''(x) \geq 0, \quad \forall x \in \mathbb{R}.$$ 

Hence, the function $V$ is convex. Now, let’s prove that the function $V$ is positive. Since $V'(x) = -1 < 0, \quad \forall x \leq \alpha$, then by convexity the function $V$ attains its minimum at some $x_0 > \alpha$. Then

$$V'(x_0) = 2Ax_0 + B - C_2\rho|x_0|^\rho = 0.$$ 

Thus,

$$C_2|x_0|^\rho = \frac{2A}{\rho}x_0^2 - \frac{B}{\rho}x_0.$$ 

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Therefore, \( \forall x \in \mathbb{R} \)

\[
V(x) \geq Ax_0^2 + Bx_0 + K^2 + C_2|x_0|^p \\
= Ax_0^2 + Bx_0 + K^2 + \left(-\frac{2A}{\rho}x_0^2 - \frac{B}{\rho}x_0\right) \\
= A(1 - \frac{2}{\rho})x_0^2 + B(1 - \frac{1}{\rho})x_0 + K^2 \\
= A\left(\frac{\rho-2}{\rho}\right)x_0^2 + B\left(\frac{\rho-1}{\rho}\right)x_0 + K^2.
\]

Now, finding the minimum value of this quadratic expression

\[
V(x) \geq A\left(\frac{\rho-2}{\rho}\right)^2 \left[ -\frac{B(\rho-1)}{\rho} \frac{\rho}{2A(\rho-2)} \right]^2 \\
+ B\left(\frac{\rho-1}{\rho}\right)^2 \left[ -\frac{B(\rho-1)}{\rho} \frac{\rho}{2A(\rho-2)} \right] + K^2 \\
= \frac{B^2(\rho-1)^2}{4A\rho(\rho-2)} - \frac{B^2(\rho-1)^2}{2A\rho(\rho-2)} + K^2 \\
= \frac{B^2(\rho-1)^2}{4A\rho(\rho-2)} + K^2 \\
= \frac{4K^2}{(1-b)^2} \frac{1}{4A\rho(\rho-2)} + K^2 \\
= K^2[1 - \frac{(1-2b-\epsilon^2)(\rho-1)^2}{(1-b)^2\rho(\rho-2)}] \\
= K^2[\frac{(1-b)^2\rho(\rho-2) - (1-2b-\epsilon^2)(\rho-1)^2}{(1-b)^2\rho(\rho-2)}] \\
= K^2[\frac{(1-2b+b^2)(\rho^2-2\rho) - (1-2b-\epsilon^2)(\rho^2-2\rho+1)}{(1-b)^2\rho(\rho-2)}] \\
= K^2[\frac{(b^2\rho^2-2b^2\rho) - 1 + 2b + (\epsilon^2\rho^2-2\epsilon^2\rho+\epsilon^2)}{(1-b)^2\rho(\rho-2)}] \\
= K^2[\frac{b^2(\rho^2-2\rho) - 1 + 2b + (\epsilon^2\rho^2-2\epsilon^2\rho+\epsilon^2)}{(1-b)^2\rho(\rho-2)}].
\]

Now, \( \rho \) is one of the roots of the quadratic equation (3.14), so

\[
\epsilon^2\rho^2 - \epsilon^2\rho + 2b\rho - 2 = 0.
\]
Therefore,

\[ V(x) \geq \frac{K^2[b^2\rho(\rho - 2) - 1 + 2b + (2 - 2b\rho) + (e^2 - e^2\rho)]}{(1 - b)^2\rho(\rho - 2)} \]

\[ = \frac{K^2[b^2\rho(\rho - 2) + 1 + (e^2 - e^2\rho) + 2b - 2b\rho]}{(1 - b)^2\rho(\rho - 2)} \]

\[ = \frac{K^2[b^2\rho^2(\rho - 2) + \rho + (e^2\rho - e^2\rho^2) + 2b\rho - 2b\rho^2]}{(1 - b)^2\rho^2(\rho - 2)} \]

\[ = \frac{K^2[b^2\rho^2(\rho - 2) + \rho + (-2 + 2b\rho) + 2b\rho - 2b\rho^2]}{(1 - b)^2\rho^2(\rho - 2)} \]

\[ = \frac{K^2[b^2\rho^2(\rho - 2) + (\rho - 2) + 4b\rho - 2b\rho^2]}{(1 - b)^2\rho^2(\rho - 2)} \]

\[ = \frac{K^2[b^2\rho^2(\rho - 2) + (\rho - 2) - 2b\rho(\rho - 2)]}{(1 - b)^2\rho^2(\rho - 2)} \]

\[ = \frac{K^2(\rho - 2)[b^2\rho^2 + 1 - 2b\rho]}{(1 - b)^2\rho^2(\rho - 2)} \]

\[ = \frac{K^2(b\rho - 1)^2}{(1 - b)^2\rho^2} > 0. \quad \square \]

**Lemma 3.12.** Suppose that \( K > \frac{1 - b}{2} \), and \( \alpha \) satisfies (3.24). Then the function \( V \) defined in Lemma 3.10 is a classical solution of the Hamilton-Jacobi-Bellman equation

\[ \max[F_2(x, v, v', v''), F_1(x, v, v')] = 0, \quad -\infty < x < \infty, \]

where

\[ F_2(x, v, v', v'') = v(x) - \frac{1}{2}e^2x^2v''(x) - bxv'(x) - (x - K)^2, \quad \forall x \in \mathbb{R}, \]

\[ F_1(x, v, v') = -v' - 1, \quad \forall x \in \mathbb{R}. \]

**Proof.**

By Lemma 3.10, \( V \) is \( C^2 \), then it suffices to prove that

(i) \( F_1(x, V, V') \leq 0, \quad \forall x \geq \alpha, \)

(ii) \( F_2(x, V, V', V'') = 0, \quad \forall x \geq \alpha, \)

(iii) \( F_1(x, V, V') = 0, \quad \forall x \leq \alpha, \)
Proof of (i).

Let

\[ V_2(x) = Q(x) + C_2|x|^\rho, \quad \forall x \geq \alpha, \]

with \( Q(x) = Ax^2 + Bx + K^2 \). Then

\[ F_1(x, V_2(x), V'_2(x)) = -V'_2(x) - 1. \]

So, by convexity of \( V_2 \)

\[ \frac{d}{dx} F_1(x, V_2(x), V'_2(x)) = -V''_2(x) \leq 0. \]

Then \( F_1(\cdot, V_2(\cdot), V'_2(\cdot)) \) is nonincreasing and it attains its maximum at \( x = \alpha \). Therefore, by (3.25)

\[ F_1(x, V_2(x), V'_2(x)) \leq F_1(\alpha, V_2(\alpha), V'_2(\alpha)) \]

\[ = -2Ax - B - C_2|\alpha|^\rho - 1 = 0. \quad \square \]

Proof of (ii).

Let \( V_2(x) = Q(x) + C_2|x|^\rho, \quad \forall x \geq \alpha. \) Then

\[ V_2(x) = Ax^2 + Bx + K^2 + C_2|x|^\rho, \]

\[ V'_2(x) = 2Ax + B + C_2\rho|x|^\rho - 1, \]

\[ V''_2(x) = 2A + C_2\rho(\rho - 1)|x|^\rho - 2. \]
Then using that \( \rho \) is the negative solution of the quadratic equation (3.14)

\[
F_2(x, V_2(x), V'_2(x), V''_2(x)) = V_2(x) - \frac{1}{2} \epsilon^2 x^2 V''_2(x) - bx V'_2(x) - (x - K)^2
\]

\[
= A x^2 + B x + K^2 + C_2|\alpha|\rho
- \frac{1}{2} \epsilon^2 x^2 [2A + C_2 \rho (\rho - 1)] |\alpha|^{\rho - 2}
- bx [2Ax + B + C_2 \rho |\alpha|^{\rho - 1}] - (x - K)^2
= [A(1 - 2b - \epsilon^2) - 1] x^2
+ [B(1 - b) + 2K] x
+ [1 - \frac{\epsilon^2}{2} \rho (\rho - 1) - b \rho] C_2 |\alpha| \rho
= [1 - 1] x^2 + [-2K + 2K] x
+ \frac{1}{2} [2 - \epsilon^2 \rho^2 + \epsilon^2 \rho - 2b \rho] C_2 |\alpha| \rho
= 0. \quad \square
\]

Proof of (iii).

Let \( V_1(x) = \alpha - x + Q(\alpha) + C_2 |\alpha|^\rho \), \( \forall x \leq \alpha \). Then it is clear that

\[
F_1(x, V_1(x), V'_1(x)) = -V'_1(x) - 1 = 1 - 1 = 0, \quad \forall x \leq \alpha. \quad \square
\]

Proof of (iv).

Let \( V_1(x) = \alpha - x + Q(\alpha) + C_2 |\alpha|^\rho \) \( \forall x \leq \alpha \). Note that \( V'_1(x) = -1 \), and \( V''_1(x) = 0 \). We recall

\[
C_2 |\alpha|^\rho = -\frac{2A |\alpha|^2}{\rho (\rho - 1)},
\]

and define

\[
\beta = K - \frac{1 - b}{2}.
\]

Then

\[
B + 1 = -\frac{2\beta}{1 - b}
\]

\[
\alpha = \beta \frac{(1 - 2b - \epsilon^2) (\rho - 1)}{(1 - b) (\rho - 2)}.
\]
Then,

\[ F_2(x, V_1(x), V'_1(x), V''_1(x)) = V_1(x) - \frac{1}{2} \varepsilon^2 x^2 V''_1(x) - bx V'_1(x) - (x - K)^2 \]

\[ = V_1(x) + bx - (x - K)^2 \]

\[ = \alpha - x + Q(\alpha) + C_2|\alpha|\rho + bx - x^2 + 2Kx - K^2 \]

\[ = -x^2 - (1 - b - 2K)x + (B + 1)\alpha + A\alpha^2 + C_2|\alpha|\rho \]

\[ = -x^2 + 2\beta x - \frac{2\beta}{(1 - b)} \alpha + A\alpha^2 - \frac{2A|\alpha|^2}{\rho(\rho - 1)} \]

\[ = -x^2 + 2\beta x - 2\beta^2 \frac{(1 - 2b - \varepsilon^2)(\rho - 1)}{(1 - b)^2} \frac{(\rho - 1)}{(\rho - 2)} \]

\[ + \left[ 1 - \frac{2}{\rho(\rho - 1)} \right] A|\alpha|^2 \]

\[ = -x^2 + 2\beta x - 2\beta^2 \frac{(1 - 2b - \varepsilon^2)(\rho - 1)}{(1 - b)^2} \frac{(\rho - 1)}{(\rho - 2)} \]

\[ + \beta^2 \left[ \frac{(\rho + 1)(\rho - 2)}{\rho(\rho - 1)} \right] \frac{(1 - 2b - \varepsilon^2)(\rho - 1)^2}{(1 - b)^2} \frac{(\rho - 2)^2}{(\rho - 2)^2}. \]

Therefore,

\[ F_2(x, V_1(x), V'_1(x), V''_1(x)) = -x^2 + 2\beta x - \beta^2 \frac{(1 - 2b - \varepsilon^2)(\rho - 1)^2}{(1 - b)^2} \frac{(\rho - 1)}{\rho(\rho - 2)}. \]

The roots of the quadratic expression of the right hand side are:

\[ x_{1,2} = \beta \pm |\beta| \sqrt{1 - \frac{(1 - 2b - \varepsilon^2)(\rho - 1)^2}{(1 - b)^2} \frac{(\rho - 1)}{\rho(\rho - 2)}} \]

\[ = \beta \left\{ 1 \pm \sqrt{1 - \frac{(1 - 2b - \varepsilon^2)(\rho - 1)^2}{(1 - b)^2} \frac{(\rho - 1)}{\rho(\rho - 2)}} \right\}, \]

since \( \beta = (K - \frac{1 - b}{2}) > 0 \). To show \( F_2 \leq 0 \), \( \forall x \leq \alpha \), it is enough to prove that \( \alpha \leq \min(x_1, x_2) \).

In fact, we shall prove that

\[ \alpha = \min(x_1, x_2) = x_2. \]  

(3.27)
Note that if \( \frac{(1-2b-\varepsilon^2)(\rho-1)^2}{(1-b)^3 \rho(\rho-2)} > 1 \). There is nothing to prove. Let's assume that

\[
0 \leq \frac{(1-2b-\varepsilon^2)(\rho-1)^2}{(1-b)^2 \rho(\rho-2)} \leq 1.
\]

Note that since

\[
\alpha = \beta \frac{(1-2b-\varepsilon^2)(\rho-1)}{(1-b)(\rho-2)},
\]

\[
\alpha = x_2 \iff \frac{\alpha}{\beta} = 1 - \sqrt{1 - \frac{(1-2b-\varepsilon^2)(\rho-1)^2}{(1-b)^2 \rho(\rho-2)}}
\]

\[
\iff \frac{\alpha}{\beta} - 1 = -\sqrt{1 - \frac{\alpha (\rho-1)}{\beta \rho(1-b)}}
\]

\[
\iff \left( \frac{\alpha}{\beta} - 1 \right)^2 = 1 - \frac{\alpha (\rho-1)}{\beta \rho(1-b)}
\]

\[
\iff \left( \frac{\alpha}{\beta} - 2 \right)\frac{2\alpha}{\beta} = -\frac{\alpha (\rho-1)}{\beta \rho(1-b)}
\]

\[
\iff \frac{\alpha}{\beta} - 2 = -\frac{(\rho-1)}{\rho(1-b)}.
\]

On the other hand,

\[
\frac{\alpha}{\beta} - 2 = \frac{(1-2b-\varepsilon^2)(\rho-1)}{(1-b)(\rho-2)} - 2
\]

\[
= \frac{(1-2b-\varepsilon^2)(\rho-1) - 2(1-b)(\rho-2)}{(1-b)(\rho-2)}
\]

\[
= \frac{\rho - 1 - 2bp + 2b - \varepsilon^2 \rho + \varepsilon^2 - 2\rho + 4 + 2b\rho - 4b}{(1-b)(\rho-2)}
\]

\[
= \frac{-\rho + 3 - 2b - \varepsilon^2 \rho + \varepsilon^2}{(1-b)(\rho-2)}
\]

\[
= \frac{-\rho^2 + 3\rho - 2b\rho - \varepsilon^2 \rho^2 + \varepsilon^2 \rho}{(1-b)\rho(\rho-2)}.
\]

But \( \rho \) is one of the roots of the quadratic equation (3.14)

\[
\varepsilon^2 \rho^2 + 2b\rho - \varepsilon^2 \rho - 2 = 0.
\]
Therefore,

\[
\frac{\alpha}{\beta} - 2 = \frac{-\rho^2 + 3\rho - 2}{(1 - b)\rho(\rho - 2)} = \frac{-(\rho - 2)(\rho - 1)}{(1 - b)\rho(\rho - 2)} = \frac{-\rho - 1}{\rho(1 - b)}.
\]

Thus, (3.27) holds. \(\square\)

### 3.10 The Candidate Value Function for the Case \(K = \frac{1-b}{2}\)

**Lemma 3.13.** Suppose that \(K = \frac{1-b}{2}\), we consider the function \(V\) defined by

\[
V(x) = \begin{cases} 
-x + K^2, & \text{if } x < 0, \\
Ax^2 + Bx + K^2, & \text{if } 0 \leq x,
\end{cases}
\]

with \(A\) and \(B\) given by (3.13). Then,

(i) \(V\) is continuous,

(ii) \(V\) is in \(C^\infty(\mathbb{R} \setminus \{0\})\),

(iii) \(V\) is not \(C^2\) at zero,

(iv) \(V\) is convex,

(v) \(V\) is \(C^1\) at zero,

(vi) \(V\) is positive.

**Proof of (i), (ii) and (iii).**

Note that the function \(V\) is an affine function on the left hand side of 0 and a quadratic function on the right hand side of 0. Thus, \(V\) is in \(C^\infty(\mathbb{R} \setminus \{0\})\). On the other hand,

\[
\lim_{x \to 0} V(x) = V(0).
\]
So, $V$ is continuous. Furthermore, $V''(0^-) = 0 < 2A = V''(0^+)$. So $V$ is not $C^2$ at 0.

Proof of (iv).

Note that

$$V''(x) = \begin{cases} 
0, & \text{for } x < 0, \\
2A, & \text{for } x > 0.
\end{cases}$$

So, $V''(x) \geq 0, \forall x \in \mathbb{R} \setminus \{0\}$. Therefore, $V$ is convex.

Proof of (v).

Clearly $V'(0) = -1$. So we only need to see that $V'(0) = -1$. In fact,

$$V'(0) = 2A(0) + B = \frac{-2K}{1-b} = \frac{-2}{1-b}\left(\frac{1-b}{2}\right) = -1.$$ 

Therefore, $V$ is $C^1$ at 0, and hence $V \in C^1(\mathbb{R})$.

Proof of (vi).

Since $V'(0) = -1 < 0$ then $V$ is decreasing in a neighborhood of 0. So $V$ does not reach its minimum at 0. Then $V$ attains its minimum value at

$$x_0 = -\frac{B}{2A} = -\frac{(-2K)}{(1-b)}\frac{1}{2A} = \frac{1}{2A}.$$ 

Thus, it will be enough to prove that $V(x_0) > 0$.

$$V(x_0) = Ax_0^2 - \frac{2K}{1-b}x_0 + K^2$$

$$= A\left(\frac{1}{2A}\right)^2 - \frac{1}{2A} + \left(\frac{1-b}{2}\right)^2$$

$$= -\frac{1}{4A} + \frac{(1-b)^2}{4}$$

$$= \frac{(1-2b-\varepsilon^2)}{4} + \frac{(1-b)^2}{4}$$

$$= \frac{\varepsilon^2 + b^2}{4} > 0.$$ 

Lemma 3.14. Suppose that $K = \frac{1+b}{2}$, then the function $V$ defined in Lemma 3.13 is a viscosity solution of the Hamilton-Jacobi-Bellman equation (3.4)

$$\max[F_2(x,v,v',v''), F_1(x,v,v')] = 0, \quad -\infty < x < \infty,$$
where

\[ F_2(x, v, v', v'') = v(x) - \frac{1}{2} e^2 x^2 v''(x) - bxv'(x) - (x - K)^2, \quad \forall x \in \mathbb{R}, \]
\[ F_1(x, v, v') = -v' - 1, \quad \forall x \in \mathbb{R}. \]

**Proof.**

It suffices to prove that

(i) \( F_1(x, V, V') \leq 0, \quad \forall x \geq 0, \)

(ii) \( F_2(x, V, V', V'') = 0, \quad \forall x > 0, \)

(iii) \( F_1(x, V, V') = 0, \quad \forall x \leq 0, \)

(iv) \( F_2(x, V, V', V'') \leq 0, \quad \forall x < 0, \)

and then [9, Theorem 1] implies that the function \( V \) is a viscosity solution of (3.4).

**Proof of (i).**

For \( x \geq 0 \) let \( V^x(x) = Ax^2 + Bx + K^2 \). Then \( V''^x(x) = 2Ax + B, \forall x \geq 0 \). So

\[ F_1(x, V^x(x), V'^x(x)) = -V'^x(x) - 1 \]
\[ = -2Ax - B - 1. \]

This affine function is decreasing and \( V''^x(0) = -1 \) as shown in Lemma 3.13.

**Proof of (ii).**

Since

\[ V''^x(x) = 2Ax + B \quad \text{and} \quad V'^x(x) = 2A, \]

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then

\[
F_2(x, V_2(x), V_2'(x), V_2''(x)) = V_2(x) - \frac{1}{2} \varepsilon^2 x^2 V_2''(x) - bx V_2'(x) - (x - K)^2
\]

\[
= Ax^2 + Bx + K^2 - \frac{1}{2} \varepsilon^2 x^2 (2A)
\]

\[
- bx(2Ax + B) - (x - K)^2
\]

\[
= [A(1 - 2b - \varepsilon^2) - 1]x^2 + [B(1 - b) + 2K]x
\]

\[
= [1 - 1]x^2 + [-2K + 2K]x
\]

\[
= 0.
\]

Proof of (iii).

Let \( V_1(x) = -x + K^2 \) \( \forall x < 0 \). Then it is clear that

\[
F_1(x, V_1(x), V_1'(x)) = -V_1'(x) - 1 = 1 - 1 = 0, \ \forall x < 0.
\]

Proof of (iv).

Note that \( V_1'(x) = -1 \), and \( V_1''(x) = 0 \). So,

\[
F_2(x, V_1(x), V_1'(x), V_1''(x)) = V_1(x) - \frac{1}{2} \varepsilon^2 x^2 V_1''(x) - bx V_1'(x) - (x - K)^2
\]

\[
= V_1(x) + bx - (x - K)^2
\]

\[
= -x + K^2 + bx - x^2 + 2Kx - K^2
\]

\[
= -x^2 - (1 - b - 2K)x
\]

\[
= -x^2 + 2(K - \frac{1 - b}{2})x
\]

\[
= -x^2 < 0. \ \Box
\]

Lemma 3.15. Suppose that \( K = \frac{1 - b}{2} \), and consider the function \( \omega \) defined by

\[
\omega(x) = K^2 - x, \ \forall x \in \mathbb{R},
\]

then \( \omega \) is a convex \( C^\infty \) (therefore, viscosity) solution of (3.4).
Proof.

The function \( \omega \) is affine, so \( \omega \in C^\infty(\mathbb{R}) \). We prove that

(i) \( F_1(x, \omega, \omega') = 0, \ \forall x \in \mathbb{R}, \)

(ii) \( F_2(x, \omega, \omega', \omega'') \leq 0, \ \forall x \in \mathbb{R}. \)

Proof of (i).

It is clear that

\[
F_1(x, \omega(x), \omega'(x)) = -\omega'(x) - 1 = 1 - 1 = 0, \ \forall x \in \mathbb{R}.
\]

Proof of (ii).

Note that \( \omega'(x) = -1 \), and \( \omega''(x) = 0 \). So,

\[
F_2(x, \omega(x), \omega'(x), \omega''(x)) = \omega(x) - \frac{1}{2} \epsilon^2 x^2 \omega''(x) - bx \omega'(x) - (x - K)^2
\]

\[
= \omega(x) + bx - (x - K)^2
\]

\[
= -x + K^2 + bx - x^2 + 2Kx - K^2
\]

\[
= -x^2 - (1 - b - 2K)x
\]

\[
= -x^2 + 2(K - \frac{1-b}{2})x
\]

\[
= -x^2 \leq 0. \ \Box
\]

3.11 Applying a Verification Theorem

We start with the definition of classical solution of (3.4) in the sense of Fleming and Soner, see [13, VIII, Definition 4.1]. We specialize this definition to our control problem (3.1), (3.2), (3.3).

Let \( \mathcal{O} \subset \mathbb{R}^k \). Let \( V_{loc}^{1,\infty}(\mathcal{O}, \mathbb{R}^k) \) be the set of all \( \mathbb{R}^k \)-valued functions of \( \mathcal{O} \) which are Lipschitz continuous on every bounded subset of \( \mathcal{O} \), let \( C_p(\mathcal{O}) \) be the set of all polynomial growing functions in \( C(\mathcal{O}) \), and \( C^k(\mathcal{O}) \) be the set of all \( k \)-times continuously differentiable functions on \( \mathcal{O} \).
Definition 3.16. Let \( V \in C_{p}(\mathbb{R}) \cap C^{1}(\mathbb{R}) \) with \( DV \in V_{loc}^{1,\infty}(\mathbb{R}) \) be given. Define

\[
P = \{ x \in \mathbb{R} : H(V'(x)) < 0 \},
\]

with \( H \) as in (3.4). The function \( V \) is a classical solution of (3.4) if \( V \in C^{2}(P) \),

\[
\mathcal{L}V(x) = (x - K)^2, \quad \forall x \in P,
\]

\[
H(V'(x)) = -V'(x) - 1 \leq 0, \quad \forall x \in \mathbb{R},
\]

and

\[
\mathcal{L}V(x) \leq (x - K)^2, \quad \text{for a.e.} \ x \in \mathbb{R},
\]

where \( \mathcal{L} \) is defined as in (3.4).

Now, we recall part of the verification theorem from [13, VIII, Theorem 4.1].

Theorem 3.17 (Verification). Let \( V \) be a classical solution (in the above sense) of (3.4). Then for every \( x \in \mathbb{R} \)

\[
V(x) \leq J^{\bar{u},\bar{\xi}}(x),
\]

for any \( (\bar{u}(.),\bar{\xi}(.)) \in \bar{A} \), such that

\[
\lim_{t \to \infty} \inf E[e^{-t}E[V(x(t))]] = 0. \tag{3.28}
\]

Theorem 3.18. Let's consider the stochastic control problem (3.1), (3.2), (3.3), and let's suppose that \( K = \frac{1-b}{2} \). Then the candidate value function \( V \) defined in Lemma 3.13 is the value function \( v \), that is

\[
V(x) = v(x), \forall x \in \mathbb{R}.
\]

The control \((u^*,\xi^*)\) given by (3.29), (3.30) below is optimal.
Proof.

By Lemmas 3.13 and 3.14 the candidate value function $V$ defined in Lemma 3.13 is a classical solution of (3.4) in the sense of Definition 3.16. In fact, by Lemma 3.13 this candidate value function $V$ satisfies $V \in C_p(\mathbb{R}) \cap C^1(\mathbb{R})$ with $V' \in V_{loc}^{1,\infty}(\mathbb{R})$. Also, since $\alpha = 0$ is the free boundary for the stochastic control problem in the case $K = \frac{1-b}{2}$

$$
P = \{ x \in \mathbb{R} : H(V'(x)) < 0 \}
= \{ x \in \mathbb{R} : -V'(x) - 1 < 0 \}
= (0, \infty),
$$

with $V \in C^2(P) = C^2((0, \infty))$. Finally, by Lemma 4.17 (i),(ii),(iii),(iv),

$$
\mathcal{L}V(x) = (x - K)^2, \quad \forall x \in P,
$$

$$
H(V'(x)) = -V'(x) - 1 \leq 0, \quad \forall x \in \mathbb{R},
$$

$$
\mathcal{L}V(x) \leq (x - K)^2, \quad \text{for a.e. } x \in \mathbb{R}.
$$

Here $H$ and $\mathcal{L}$ are defined as in (3.4). Now we check that the candidate value function $V$ satisfies all the hypothesis of the verification theorem 3.17. By (3.10) the function $V$ is of quadratic growth, that is,

$$
V(x) \leq Ax^2 + Bx + K^2,
$$

and any control variable $z(t) = (\hat{u}(\cdot), \xi(\cdot)) \in \hat{A}$ is a function of bounded variation that satisfies our assumption (iii) for the control processes, that is

$$
E|z(t)|^m < \infty, \quad m = 1, 2, \cdots,
$$

Therefore, for any $(\hat{u}(\cdot), \xi(\cdot)) \in \hat{A}$, the condition (3.28), that is

$$
\lim_{t \to \infty} \inf e^{-t}E^x[V(x(t))] = 0,
$$

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holds. It remains to prove that $V$ can be attained as a cost for some feedback control processes. We define these feedback control processes in two separated cases.

Case of $x \geq 0$.

Let $x \geq 0$, we claim that the control process $z^*(\cdot) = (u^*(\cdot), \xi^*(\cdot))$, with $u^*(\cdot) \equiv 0$ and $\xi^*(\cdot) \equiv 0$ is optimal. In fact by the Verification Theorem 3.17

$$V(x) \leq J^{u,\xi}(x), \forall (u(\cdot), \xi(\cdot)) \in \hat{A},$$

Now, taking this control process $z^*(\cdot) = (0, 0)$, by (3.9)

$$J^{u^*,\xi^*}(x) = J^{0,0}(x) = Ax^2 + Bx + K^2 = V(x),$$

since $A, B$ are given by (3.13). Then the control $z^*(\cdot) = (0, 0)$ is optimal, and $V(x) = v(x), \forall x \geq 0$.

Case of $x < 0$.

Let $x < 0$. Take the impulsive control process that produces a jump of the state from $x(0) = x$ to $x(0^+) = 0$

$$u^*(t) = \begin{cases} 1, & t = 0, \\ 0, & \text{otherwise}, \end{cases} \quad (3.29)$$

$$\xi^*(t) = \begin{cases} 0, & t = 0, \\ -x, & t > 0. \end{cases} \quad (3.30)$$

Let $x^*(\cdot)$ be the corresponding state process. Note that

$$x^*(t^+) - x^*(t) = z(t^+) - z(t)$$

$$= \int_{t}^{t^+} u^*(s)d\xi^*(s) - \int_{t}^{t^+} u^*(s)d\xi^*(s)$$

$$= \int_{t}^{t^+} u^*(s)d\xi^*(s) - \int_{0}^{t} u^*(s)d\xi^*(s)$$

$$= \int_{[t, t^+]} u^*(s)d\xi^*(s).$$
Then
\[
x^*(t^+) - x^*(t) = \begin{cases} 
-x \times (\xi^*(0^+) - \xi^*(0)) = -x, & t = 0, \\
0, & \text{otherwise},
\end{cases}
\] (3.31)

where '×' means the scalar multiplication operation. Thus, \(x^*(\cdot)\) has only one jump, which occurs at \(t = 0\). Since \(x^*(0^+) - x^*(0) = -x\), then
\[
x^*(0^+) = -x + x^*(0) = -x + x = 0.
\]
Therefore,
\[
x^*(t) = \begin{cases} 
x, & t = 0, \\
0, & \text{otherwise}.
\end{cases}
\] (3.32)

Now we evaluate the associated cost
\[
J^{u^*, \xi^*}(x) = \mathbb{E}_x \left[ \int_0^\infty e^{-t} \left[ (x^*(t) - K)^2 dt + u^*(t) d\xi^*(t) \right] \right]
\]
\[
= \mathbb{E}_x \left[ \int_0^{0^+} e^{-t} \left[ (x - K)^2 dt + (-x) d\xi^*(t) \right] \right]
+ \mathbb{E}_x \left[ \int_{0^+}^{\infty} e^{-t} \left[ (0 - K)^2 dt + (0) d\xi^*(t) \right] \right]
\]
\[
= -x(\xi^*(0^+) - \xi^*(0)) + K^2 = -x + K^2.
\]

Therefore, \(J^{u^*, \xi^*}(x) = V(x)\). Hence \(V(x) = v(x), \forall x \in \mathbb{R}\).

### 3.12 Conclusions

The stochastic control problem (3.1), (3.2), (3.3), in the case that \(K = \frac{1-a}{2}\) is an interesting example. It shows that viscosity solutions of the dynamic programming equation (3.4) are not unique. In fact, the functions defined in Lemmas 3.13, 3.15 are both viscosity solutions of the HJB equation (3.4). This example also shows that the principle of smooth fit can fail even for convex problems, see [13, below]
Theorem 4.2 in chapter VIII]. In fact, we use a verification theorem from [13] to prove that the candidate value function $V$ defined in Lemma 3.13 is indeed the value function for this stochastic control problem. By that lemma then, the value function is not $C^2$. This unusual feature of the value function is due to the fact that the diffusion (3.4) is degenerate at $x = 0$ and that is precisely where the free boundary lies.

The problem of finding optimal control processes for the stochastic control problem (3.1), (3.2), (3.3), in the case that $K \neq \frac{1-k}{2}$ is harder. So, the verification theorems for the two nonnegative $C^2(\mathbb{R})$ functions defined in Lemmas 3.7, 3.10, which are classical solutions of the HJB equation (3.4), are left for further research.
References


Vita

Jesus Pascal was born in Cabimas, Zulia State, Venezuela. He earned a bachelor of science degree with a major in Mathematics and a Magister en Matematicas Degree at La Universidad del Zulia, Maracaibo, Venezuela. He also earned a master of science degree at Louisiana State University, Baton Rouge, Louisiana. He has been teaching for several years at L.U.Z., Venezuela. In 1993, on leave from his university, he began a doctoral course of study at L.S.U., and will receive his degree of Doctor of Philosophy in December, 1998.
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