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Spikes in Matroid Theory.

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SPIKES IN MATROID THEORY

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

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For an integer \( n \geq 3 \), a rank-\( n \) matroid is called an \( n \)-spike if it consists of \( n \) three-point lines through a common point \( t \) such that, for all \( k \) in \( \{1, 2, \ldots, n - 1\} \), the union of every set of \( k \) of these lines has rank \( k + 1 \). The point \( t \) is called the tip of the \( n \)-spike. Ding, Oporowski, Oxley, and Vertigan proved that, for all \( n \geq 3 \), there is an integer \( N(n) \) such that every 3-connected matroid with at least \( N(n) \) elements has a minor isomorphic to a wheel or whirl of rank \( n \), \( M(K_3,n) \) or its dual, \( U_{2,n+2} \) or its dual, or a rank-\( n \) spike. In the first chapter of this dissertation, we characterize each of these classes of unavoidable matroids in terms of an extremal connectivity condition. In particular, it is proved in this chapter that if \( M \) is a 3-connected matroid of rank at least seven for which every single-element deletion or contraction is 3-connected but no 2-element deletion or contraction is, then \( M \) is a spike with its tip deleted. It is further proved that if \( M \) is a 3-connected matroid of rank at least four for which every single-element deletion is 3-connected but no 1-element contraction or 2-element deletion is, then \( M \cong M^*(K_3,n) \).

The second chapter of this dissertation evaluates the number of \( n \)-spikes representable over finite fields. It is well known that there is a unique binary \( n \)-spike for each integer \( n \geq 3 \). In this chapter, we first prove that, for each integer \( n \geq 3 \), there are exactly two distinct ternary \( n \)-spikes, and there are exactly \( \lfloor n^2 + 6n + 24 \rfloor \) quaternary \( n \)-spikes. Then we prove that, for each integer \( n \geq 4 \), there are exactly \( n + 2 + \lfloor \frac{n}{2} \rfloor \) quaternary \( n \)-spikes and, for each integer \( n \geq 18 \), the number of \( n \)-spikes representable over \( GF(7) \) is \( \lfloor \frac{2n^2 + 6n + 6}{3} \rfloor \). Finally, for each \( q > 7 \), we find the asymptotic value of the number of distinct rank-\( n \) spikes that are representable over \( GF(q) \).
INTRODUCTION

This dissertation consists of two parts.

The first part, Chapter I, characterizes each of the classes of unavoidable 3-connected matroids noted in [5] in terms of an extremal connectivity condition. One of the most important results in this chapter is that the class of 2-minimally, 2-cominimally, 3-connected rank-\(n\) matroids is exactly the class of \(n\)-spikes with their tips deleted, provided \(n \geq 7\).

The second chapter investigates the number of \(n\)-spikes representable over finite fields. For each integer \(q \leq 5\), the number of \(GF(q)\)-representable \(n\)-spikes is determined, and the number of \(GF(7)\)-representable \(n\)-spikes is also determined for \(n \geq 18\). Moreover, for each integer \(q > 7\), an asymptotic formula for the number of \(GF(q)\)-representable \(n\)-spikes is provided.
CHAPTER I

EXTREMAL CONNECTIVITY PROPERTIES OF UNAVOIDABLE MATROIDS*

1.1 Introduction

A matroid $M$ is said to be $k$-minimally $n$-connected if, for each $X \subseteq E(M)$ with $|X| < k$, the matroid $M \setminus X$ is $n$-connected, but, for each $X \subseteq E(M)$ with $|X| = k$, $M \setminus X$ is not $n$-connected. A matroid is said to be $m$-cominimally $n$-connected if its dual is $m$-minimally $n$-connected. We shall be primarily interested here in the case when $n$ is 2 or 3. Usually, 1-minimally $n$-connected matroids are called simply minimally $n$-connected, and $k$-minimally 2-connected matroids are called $k$-minimally connected matroids. Minimally connected matroids have been investigated by several authors including Murty [8], Seymour [15], White [19], and Oxley [10], [11], [12]. Moreover, Akkari [1], [2], Akkari and Oxley [3], and Oxley [9] examined $k$-minimally 3-connected matroids when $k$ is 1 or 2.

Ding, Oporowski, Oxley, and Vertigan [5] identified certain rank-$r$ 3-connected matroids as being unavoidable in the sense that every sufficiently large 3-connected matroid has one of the specified matroids as a minor. Included among these unavoidable matroids are the wheels and whirls, whose fundamental role within the class of 3-connected matroids is well known. Perhaps the primary contributor to the notoriety of wheels and whirls is Tutte’s Wheels and Whirls Theorem [18], which asserts that the

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class of minimally, cominimally 3-connected matroids coincides exactly with the class of wheels and whirls of rank exceeding two. This chapter shows that each of the classes of unavoidable 3-connected matroids noted in [5] can be characterized in terms of an extremal connectivity condition. This fact helps to explain the exact composition of the list of unavoidable matroids.

For \( n \geq 3 \), a matroid \( M \) is called a \( n \)-spike with tip \( p \) [5] if it satisfies the following three conditions:

(i) the ground set is the union of \( n \) lines, \( L_1, L_2, \ldots, L_n \), all having three points and passing through a common point \( p \);

(ii) for all \( k \in \{1, 2, \ldots, n-1\} \), the union of any \( k \) of \( L_1, L_2, \ldots, L_n \) has rank \( k+1 \);

and

(iii) \( r(L_1 \cup L_2 \cup \ldots \cup L_n) = n \).

\( M \setminus p \) is called a spike without tip. In this chapter, we will only be concerned with spikes without tips and we shall call them simply spikes.

The well-known matroid \( R_{10} \) is a regular matroid represented by the following matrix over every field:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \\
\end{bmatrix}
\]

The matroid \( H_{10} \) is a quartenary matroid represented by the following matrix over the four-element field \( \{0, 1, \omega, 1+\omega\} \):
The matroid $H_{12}$ is a binary matroid with 12 elements, represented by:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\end{bmatrix}.
$$

The matroids $H_{10}$ and $H_{12}$ are newly discovered matroids. Both of them are self-dual, but not identically self-dual, 2-minimally, 2-cominimally, 3-connected matroids.

In the rest of this chapter, the notation and terminology will follow Oxley [13]. Seymour [15] proved that the 4-point line is the unique 2-minimally, 2-cominimally connected matroid. The following are the main results of this chapter. The first theorem is the analogue of Seymour's result for 3-connected matroids.

(1.1.1) Theorem. If $M$ is a 2-minimally, 2-cominimally 3-connected matroid with rank greater than or equal to 5, then $M$ is a spike, or $M$ is isomorphic to one of the matroids $H_{10}$, $R_{10}$, and $H_{12}$. Conversely, if $M$ is a spike with $r(M) \geq 4$, then $M$ is 2-minimally, 2-cominimally 3-connected.

(1.1.2) Theorem. A matroid is 2-minimally, 1-cominimally 3-connected if and only if it is isomorphic to $F_7$, $F_7^-$, or $M^*(K_{3,n})$ for some $n \geq 3$. 

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Theorem 1.1.1 and 1.1.2 will be proved in sections 1.3 and 1.4, respectively. Section 1.2 contains some preliminary results that will be needed in these proofs, while section 1.5 examines some of the properties of 2-minimally 3-connected matroids. Finally, section 1.6 establishes that lines can be characterized in terms of an extremal connectivity condition. On combining that result with Theorem 1.1.1 and 1.1.2 and the main theorem of [5], we obtain the following result.

(1.1.3) Theorem. For every integer $r$ exceeding six, there is an integer $N(r)$ such that every 3-connected matroid with at least $N(r)$ elements has a minor $M$ such that $M$ or $M^*$ is isomorphic to a rank-$r$, $j$-minimally, $k$-cominimally 3-connected matroid for some $(j, k)$ in $\{(1, 1), (1, 2), (2, 2), (1, r)\}$.

1.2 Preliminaries

In this section, we recall some results from [9] and [13], and then prove some new results which will be used to establish Theorem 1.1.1 and 1.1.2.

(1.2.1) Proposition. [13, Section 2.1.11] If $C$ is a circuit and $C^*$ is a cocircuit of a matroid $M$, then $|C \cap C^*| \neq 1$.

The last property of matroids is often referred to as orthogonality.

(1.2.2) Proposition. [13, Section 8.1.6] If $M$ is an $n$-connected matroid and $|E(M)| \geq 2(n - 1)$, then all circuits and all cocircuits of $M$ have at least $n$ elements.

(1.2.3) Corollary. Let $M$ be a 2-minimally, 2-cominimally 3-connected matroid with $|E(M)| > 4$, then all circuits and all cocircuits of $M$ have at least 4 elements.

Proof. Apply (1.2.2) to $M\setminus e$ and $M/e$ for some $e \in E(M)$. □
(1.2.4) **Theorem.** [9, 2.5] If $C$ is a circuit of a minimally 3-connected matroid $M$ with $|E(M)| \geq 4$, then $M$ has at least two distinct triads intersecting $C$.

(1.2.5) **Corollary.** Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid with $|E(M)| > 4$, $C$ is a 4-circuit of $M$, and $e \notin C$. Then $C$ intersects at least two distinct 4-cocircuits containing $e$.

(1.2.6) **Lemma.** [9, 2.6] Suppose that $x$ and $y$ are distinct elements of an $n$-connected matroid $M$ where $n \geq 2$ and $|E(M)| \geq 2(n - 1)$. Assume that $M \backslash x/y$ is $n$-connected but that $M \backslash x$ is not $n$-connected. Then $M$ has a cocircuit of size $n$ containing both $x$ and $y$.

(1.2.7) **Corollary.** Suppose that $x$ and $y$ are distinct elements of a 3-connected matroid $M$, and $|E(M)| \geq 4$. Assume that $M \backslash x/y$ is 3-connected but that $M \backslash x$ is not. Then $M$ has a triangle containing both $x$ and $y$.

(1.2.8) **Corollary.** Suppose that $M$ is a 2-cominimally 3-connected matroid with $|E(M)| > 4$, and $x_1, x_2$, and $y$ are distinct elements of $M$. Assume that $M \backslash x_1, x_2 \backslash y$ is 3-connected. Then $M$ has a 4-circuit containing $x_1, x_2$, and $y$.

(1.2.9) **Lemma.** [9, 2.10] Let $M$ be a minimally 3-connected matroid having at least four elements, and let $U$ be the set of elements of $M$ which are not contained in a triad. If $V$ is a subset of $U$, then $M/V$ is minimally 3-connected.

(1.2.10) **Proposition.** Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid, $C_1$ and $C_2$ are distinct 4-circuits of $M$, and $|C_1 \cap C_2| = 3$. Then $M|(C_1 \cup C_2) \cong U_{3,5}$.
Proof. Since $|E(M)| > |C_1| = 4$, it follows by (1.2.3) that all circuits and all cocircuits of $M$ have at least four elements. Let $e$ be an element of $C_1 \cap C_2$. By circuit elimination, and the fact that $|(C_1 \cup C_2) - e| = 4$, we deduce that $(C_1 \cup C_2) - e$ is a 4-circuit of $M$. Hence every 4-element subset of $C_1 \cup C_2$ is a circuit of $M$; that is, $M|(C_1 \cup C_2)$ is isomorphic to $U_{3,5}$.

(1.2.11) Proposition. Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid, and that $r(M) = 3$ or $r^*(M) = 3$. Then $M \cong U_{3,6}$.

Proof. It is easy to check that $|E(M)| > 4$. If $r(M) = 3$, then every subset of $M$ of size four is dependent, and hence is a circuit. Thus $M$ is isomorphic to $U_{3,|E(M)|}$. Since $U_{3,6}$ is clearly the only rank-3 uniform matroid which is 2-minimally, 2-cominimally 3-connected, we have the required conclusion for this case. In the case when $r^*(M) = 3$, the result follows by duality.

(1.2.12) Proposition. Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid, and that $E_1$ is a subset of $E(M)$ such that $M|E_1 \cong U_{3,6}$. Then $M \cong U_{3,6}$.

Proof. Let $C$ be a 4-circuit of $M|E_1$. Suppose that there is an $x \in E(M) - E_1$. Then, by (1.2.5), there is a 4-cocircuit containing $x$ and intersecting $C$. By the assumption that $M|E_1 \cong U_{3,6}$, this 4-cocircuit will intersect some 4-circuit of $M|E_1$ in exactly one element, contradicting orthogonality. Thus $E(M) = E_1$, and $M \cong U_{3,6}$.

Define $N_M(e) = \{x \in E(M) - e : \text{there is no 4-cocircuit containing both } x \text{ and } e\}$. Then, we have the following.
(1.2.13) **Lemma.** Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid with $|E(M)| > 4$. Then, for each $e \in E(M)$, the set $N_M(e)$ has cardinality at most two.

**Proof.** We argue by contradiction. Suppose that we have $x_1, x_2, x_3 \in N(e)$. Then, since $M$ is 2-cominimally 3-connected, none of $M/x_1, x_2$, $M/x_1, x_3$, and $M/x_2, x_3$ is 3-connected. However, since $M\backslash e$ is minimally 3-connected, by (1.2.9), each of $M/x_1, x_2\backslash e$, $M/x_1, x_3\backslash e$, and $M/x_2, x_3\backslash e$ is 3-connected. Hence, by (1.2.8), there is a 4-circuit $C_1$ containing $x_2, x_3, e$, a 4-circuit $C_2$ containing $x_1, x_3, e$, and a 4-circuit $C_3$ containing $x_1, x_2, e$. By (1.2.5), there are at least two 4-cocircuits containing $e$. Let $C^*$ be one of such 4-cocircuit. By assumption, $C^*$ does not contain any of $x_1, x_2, x_3$. But $C^*$ meets each of $C_1, C_2$, and $C_3$. Since $|(C_1 \cup C_2 \cup C_3) - \{x_1, x_2, x_3, e\}| \leq 3$, it follows by orthogonality that $C^*$ must contain all elements in the set $(C_1 \cup C_2 \cup C_3) - \{x_1, x_2, x_3, e\}$. If this set has cardinality 3, then $C^*$ is the unique 4-cocircuit passing through $e$, a contradiction to (1.2.5). Therefore, we may assume that $C_1$ and $C_2$ have a common element $f$ other than $x_3$ and $e$. Then, by circuit elimination, $\{x_1, x_2, x_3, e\}$ is a 4-circuit. Thus, by orthogonality, every 4-cocircuit containing $e$ must contain $x_1, x_2,$ or $x_3$. This contradicts the choice of $x_1, x_2,$ and $x_3$. □

(1.2.14) **Theorem.** [9, 4.7, 5.2, 5.6] Let $M$ be a minimally 3-connected matroid of rank $r$ with $3 \leq r \leq 6$. Then $|E(M)| \leq 2r$. If $M$ has precisely $2r$ elements, then $M$ is isomorphic to $M(W_r)$ or $W^r$, or $r(M) = 6$ and $M$ is a disjoint union of four triads.
(1.2.15) Corollary. Let $M$ be a 2-minimally, 2-cominimally 3-connected matroid of rank $r$. If $3 \leq r \leq 6$, then

$$|E(M)| = 2r = 2r^*.$$ 

Moreover, if $r \geq 7$, then $|E(M)| \geq 14$.

Proof. Firstly, suppose that $3 \leq r \leq 6$. Let $e$ be an element of $M$. Since $M\backslash e$ is minimally 3-connected, it follows by (1.2.14) that $|E(M\backslash e)| \leq 2r$; that is:

$$r + r^*(M) - 1 \leq 2r,$$

or $r^* \leq r + 1$.

Since $M$ has no triangles, $M\backslash e$ is not a wheel or a whirl. Since $M$ has a 4-circuit passing through $e$ but has no triads, it follows by orthogonality that $M\backslash e$ cannot have four disjoint triads. Therefore, by (1.2.14),

$$|E(M\backslash e)| \neq 2r,$$

and hence, $r^* < r + 1$.

Using $M^*$ in place of $M$ in the above argument, we deduce that $r < r^* + 1$. Thus $r = r^*$ and $|E(M)| = 2r = 2r^*$. Finally, if $r \geq 7$ but $r^* \leq 6$, then $|E(M)| = 2r^* = 2r$; a contradiction. Thus $r^* \geq 7$, and $|E(M)| \geq 14$. □

The next lemma sharpens the bound on $N_M(e)$ given in Lemma 1.2.13.

(1.2.16) Lemma. Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid with $|E(M)| > 4$. Then, for each $e \in E(M)$, the sets $N_M(e)$ has cardinality at most one. Moreover, if $3 \leq r(M) \leq 5$, then $N_M(e)$ is empty.

Proof. Suppose that $r(M) = 3$. Then it follows by (1.2.11) that $M \cong U_{3,6}$. Thus $N_M(e)$ is empty.
Suppose that \( r(M) = 4 \), and \( x \in E(M) - e \). By (1.2.3), every circuit and every cocircuit of \( M \) has size at least four. Thus \( M \setminus e \) has no loops, coloops, or 2-circuits.

Since \( M \) is 2-minimally 3-connected, \( M \setminus e \) is not 3-connected. Combining this with the fact that \( r(M \setminus e, x) = 2 \), we deduce that \( M \setminus e, x \) has 2-cocircuits; that is, there is a 4-cocircuit containing both \( e \) and \( x \). Thus \( x \notin N_M(e) \), and \( N_M(e) \) is empty.

Suppose that \( r(M) = 5 \), and \( x \in N_M(e) \). Then \( x \) is not in any triad of \( M \setminus e \).

Since \( M \setminus e \) is minimally 3-connected, it follows by (1.2.9) that \( M / x \setminus e \) is minimally 3-connected. By (1.2.15), \(|E(M)| = 10\). Hence \(|E(M / x \setminus e)| = 8\). Since \( r(M / x \setminus e) = 4 \), it follows by (1.2.14) that \( M / x \setminus e \) is a wheel or a whirl. Therefore, we may assume that \( D_1^* = \{a_1, a_2, a_3\} \), \( D_2^* = \{a_3, a_4, a_5\} \), \( D_3^* = \{a_5, a_6, a_7\} \), and \( D_4^* = \{a_7, a_8, a_1\} \) are the only triads of \( M \setminus e \), while \( \{e, x, a_1, a_2, \ldots, a_8\} \) is the ground set of \( M \). Since \( x \in N_M(e) \), the matroid \( M \setminus e, x \) has no 2-cocircuits. By the relations among the triads of \( M \setminus e \), the geometrical representation of \((M \setminus e, x)^*\) is not the union of two lines. Therefore, \((M \setminus e, x)^*\) is 3-connected, as it has rank three. This contradicts the assumption that \( M \) is 2-minimally 3-connected. Thus \( N_M(e) \) is empty.

Now we suppose that \( r(M) \geq 6 \). By (1.2.15), \(|E(M)| \geq 12\). By (1.2.13), we have \(|N_M(e)| \leq 2\). Suppose that \( N_M(e) = \{x_1, x_2\} \). Then, since \( M \) is 2-cominimally 3-connected, \( M / x_1, x_2 \) is not 3-connected. Since \( M \setminus e \) is minimally 3-connected, and \( x_1 \) and \( x_2 \) are in \( N(e) \), it follows by (1.2.9) that \( M / x_1, x_2 \setminus e \) is 3-connected. Thus, by (1.2.8), there is a 4-circuit \( C \) containing \( x_1, x_2, \) and \( e \). Let \( C = \{x_1, x_2, e, f\} \). Then, by the choice of \( x_1 \) and \( x_2 \), all 4-cocircuits containing \( e \) must contain \( f \). If two of these 4-cocircuits meet in exactly three elements, then, by (1.2.10), \( M^* \) restricted to their union will be isomorphic to \( U_{3,5} \), and hence \( M \) will have a 4-cocircuit containing only one of
$e$ and $f$, a contradiction. Therefore, no two 4-cocircuits containing $e$ meet in exactly three elements. Thus the set $E(M) - \{x_1, x_2, e, f\}$ can be labeled $\{a_1, b_1, \ldots, a_n, b_n\}$ such that $\{e, f, a_i, b_i\}$ is a 4-cocircuit for each $i$ in $\{1, 2, \ldots, n\}$. By circuit elimination, for each pair of distinct elements $i$ and $j$ in $\{1, 2, \ldots, n\}$, the set $\{e, a_i, b_i, a_j, b_j\}$ contains a cocircuit. Since $C$ is a circuit, by orthogonality, we deduce that $\{a_i, b_i, a_j, b_j\}$ is a cocircuit. Since $|E(M)| \geq 12$, we deduce that $n \geq 4$. It follows by orthogonality that every 4-circuit containing $a_i$ contains $b_i$ for each $i$ in $\{1, 2, \ldots, n\}$.

If there is a 4-circuit containing both $a_i$ and $a_j$, then, by orthogonality, we deduce that this 4-circuit must be $\{a_i, b_i, a_j, b_j\}$. By (1.2.13), $N_{M^*}(a_1) \leq 2$. Therefore, we may assume that $\{a_1, b_1, a_i, b_i\}$ is a 4-circuit of $M$ for each $i$ in $\{2, 3, \ldots, n-1\}$. If $\{a_1, b_1, a_n, b_n\}$ is not a 4-circuit, it follows by (1.2.13) that $\{a_2, b_2, a_n, b_n\}$ is a 4-circuit. By applying circuit elimination to $\{a_1, b_1, a_2, b_2\}$ and $\{a_2, b_2, a_n, b_n\}$, we obtain that $\{a_1, b_1, a_n, b_n, a_2\}$ contains a circuit. Orthogonality now implies that $\{a_1, b_1, a_n, b_n\}$ must be a circuit, a contradiction. Therefore, we conclude that $\{a_i, b_i, a_j, b_j\}$ is a 4-circuit for each pair of distinct elements $i$ and $j$ in $\{1, 2, \ldots, n\}$.

By (1.2.13), we may assume that there is a 4-cocircuit containing both $x_1$ and $a_1$. It follows by orthogonality that this 4-cocircuit must contain $b_1$ and either $f$ or $x_2$. Since this 4-cocircuit cannot meet the 4-cocircuit $\{e, f, a_1, b_1\}$ in exactly three elements, it must be $\{x_1, x_2, a_1, b_1\}$. By circuit elimination, $\{e, f, x_1, x_2, a_1\}$ contains a cocircuit. By orthogonality, $a_1$ is not in this cocircuit. Thus we deduce that $\{e, f, x_1, x_2\}$ is a 4-cocircuit, a contradiction to the assumption that $x_1 \in N_1(e)$. We deduce that $N_M(e)$ has cardinality at most one.

□
If $M$ is 2–minimally, 2–cominimally 3–connected, then so is its dual. Hence the last lemma implies that $N_M(e)$ also has cardinality at most one.

(1.2.17) Theorem. Suppose that $M$ is a 2–minimally, 2–cominimally 3–connected matroid with $|E(M)| > 4$. If $C_1$ and $C_2$ are two 4–circuits such that $|C_1 \cap C_2| = 3$, then $M \cong U_{3,6}$.

Proof. We argue by contradiction. Suppose that $M \not\cong U_{3,6}$. Then, by (1.2.11), $r(M) > 3$. By (1.2.10), $M|(C_1 \cup C_2) \cong U_{3,5}$. Let $C_1 \cup C_2 = \{e_1, e_2, e_3, e_4, e_5\}$. If $r(M) \geq 6$, then, by (1.2.14), $|E(M)| \geq 12$. There are at least seven elements of $M$ not in $C_1 \cup C_2$. By (1.2.16), at least six of them have the property that they lie in a 4–cocircuit with $e_1$. However, if a 4–cocircuit intersects $C_1 \cup C_2$, then, by orthogonality, it has at least three elements in $C_1 \cup C_2$ since $M|(C_1 \cup C_2) \cong U_{3,5}$. Hence, $M$ has at least six distinct 4–cocircuits containing $e_1$. If there are exactly six, then, there is an element, say $x$, of $E(M) – (C_1 \cup C_2)$, such that there is no 4–cocircuit containing both $x$ and $e_1$. Thus, by (1.2.16), for each $e_i$ with $i$ in $\{2, 3, 4, 5\}$, there is a 4–cocircuit containing $x$ and $e_i$. As each 4–cocircuit intersecting $C_1 \cup C_2$ intersects it in at least three elements, there is an $e_i$, say $e_2$, such that $\{x, e_2\}$ is contained in at least two 4–cocircuits. Since there are at least six elements other than $x$ and the $e_i$'s, at least five of these elements lie in some 4–cocircuit with $e_2$. Moreover, none of these 4–cocircuits contains $x$. Therefore, we have at least seven distinct 4–cocircuits that contain $e_2$ and meet $E(M) – (C_1 \cup C_2)$. Since the number of 3–element subsets of $\{e_1, e_2, e_3, e_4, e_5\}$ containing $e_2$ is six, among these seven 4–cocircuits, there are at least two that have the same 3–element intersection with $C_1 \cup C_2$. Thus, by circuit elimination, there is
a 4–cocircuit intersecting $C_1 \cup C_2$ in exactly two elements. This contradicts the fact that $M| (C_1 \cup C_2) \cong U_{3,5}$. Therefore, there are at least seven distinct 4–cocircuits that contain $e_1$ and meet $E(M) - (C_1 \cup C_2)$. In this case, an argument similar to the above produces the same contradiction. Thus $r(M) \leq 5$.

If $r(M) = 4$, then, by (1.2.15), $E(M) = 8$, so there are three elements not in $C_1 \cup C_2$. By (1.2.16), there is a 4–cocircuit containing at least two elements of these three. But, this 4–cocircuit intersects $C_1 \cup C_2$ in one or two elements, a contradiction to orthogonality.

If $r(M) = 5$, then, $|E(M)| = 10$. Let $E(M) - (C_1 \cup C_2) = F = \{f_1, f_2, f_3, f_4, f_5\}$. By orthogonality and (1.2.16), it is easy to show that $M^*|\{f_1, f_2, f_3, f_4, f_5\} \cong U_{3,5}$. By orthogonality, every 4–circuit intersecting $F$ intersects it in at least three elements. By (1.2.16) and the fact that $|F| = 5$, it follows that there are at least two distinct 4–circuits passing through $e_1$ and intersecting $F$. We may assume that $\{e_1, f_1, f_2, f_3\}$ is a 4–circuit. Moreover, by (1.2.16), there is a 4–cocircuit containing $e_1$ and $f_4$. This 4–cocircuit must have three elements in $C_1 \cup C_2$. Hence it intersects the 4–circuit $\{e_1, f_1, f_2, f_3\}$ in exactly one element, a contradiction to orthogonality. \[\square\]

(1.2.18) Corollary. Suppose that $M$ is a 2–minimally, 2–cominimally 3–connected matroid with $|E(M)| > 4$. If $M \not\cong U_{3,5}$, and $C_1$ and $C_2$ are two 4–circuits of $M$, then $|C_1 \cap C_2| \neq 3$. Hence if $x \in E(M)$, and $T_1$ and $T_2$ are two distinct triangles of $M/x$, then $|T_1 \cap T_2| \leq 1$. 

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1.3 Proof of Theorem 1.1.1

(1.3.1) Lemma. Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid, and that for some element $x$ of $E(M)$, the matroid $M/x$ has four triangles sharing a common element $y$. Then $M$ is a spike.

Proof. If $M$ is isomorphic to $U_{3,6}$, the result holds, since $U_{3,6}$ is a spike. Otherwise, by (1.2.18), each pair of triangles of $M/x$ intersect in at most one element. Thus each two of the four triangles containing $y$ have no other common elements. Thus $M$ has four 4-circuits that contain $\{x, y\}$ but are otherwise disjoint. A 4-cocircuit passing through $x$ intersects all four of these 4-circuits. Thus, by orthogonality, such a 4-cocircuit must contain $y$. Hence every 4-cocircuit containing $x$ also contains $y$. Similarly, each 4-cocircuit containing $y$ also contains $x$. Hence, by (1.2.16), if $z \in E(M) - \{x, y\}$, then $z$ is in a 4-cocircuit meeting $\{x, y\}$, so $z$ is in a 4-cocircuit containing $\{x, y\}$.

Thus, by (1.2.18) and (1.2.16), we can denote the elements of $E(M) - \{x, y\}$ by $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ for some $n > 3$, such that, for each $i$ in $\{1, 2, \ldots, n\}$, the set $\{x, y, a_i, b_i\}$ is a 4-cocircuit. If $i$ and $j$ are distinct elements of $\{1, 2, \ldots, n\}$, then, by (1.2.16), there is a 4-circuit containing $a_i$ and either $a_j$ or $b_j$, say $a_j$. By orthogonality, this 4-circuit either contains both $x$ and $y$, or contains neither of these elements. In the latter case, this 4-circuit is $\{a_i, b_i, a_j, b_j\}$. In the former case, $\{x, y, a_i, a_j\}$ is a 4-circuit and, by (1.2.16) again, there is a 4-circuit containing $b_j$ and $x$ or $y$. By orthogonality, this 4-circuit must contain both $x$ and $y$, and, by circuit elimination and orthogonality, $b_i$ must be in this 4-circuit and $\{a_i, b_i, a_j, b_j\}$ is a 4-circuit. We conclude that in both cases, $\{a_i, b_i, a_j, b_j\}$ is a 4-circuit.
Now consider 4-cocircuits. By (1.2.16), there is a 4-cocircuit containing \( a_i \) and \( a_j \) or \( b_j \). We may assume that this 4-cocircuit contains \( a_i \) and \( a_j \). Since, by (1.2.18), \( \{x, y, a_i, a_j\} \) cannot be a 4-cocircuit, and all sets of the form \( \{a_k, b_k, a_i, b_j\} \) are 4-circuits, we deduce that \( \{a_i, b_i, a_j, b_j\} \) is a 4-cocircuit for each pair of elements \( i, j \) of \( \{1, 2, \ldots, n\} \). Denote \( x \) by \( a_{n+1} \), and \( y \) by \( b_{n+1} \). Then, it follows by the above results that, for each pair of elements \( i, j \) of \( \{1, 2, \ldots, n+1\} \), the set \( \{a_i, b_i, a_j, b_j\} \) is a 4-cocircuit.

Similarly, it is easy to deduce that the set \( \{a_i, b_i, a_j, b_j\} \) is also a 4-circuit for each pair of elements \( i, j \) of \( \{1, 2, \ldots, n+1\} \). Let \( E_{i,j} = \{a_i, b_i, a_j, b_j\} \) for each pair of elements \( i, j \) of \( \{1, 2, \ldots, n+1\} \). Then each \( E_{i,j} \) is both a 4-circuit and a 4-cocircuit. If a circuit \( C \) meets three of the sets \( E_1 = \{a_1, b_1\}, E_2 = \{a_2, b_2\}, \ldots, \) and \( E_{n+1} = \{a_{n+1}, b_{n+1}\} \), then it cannot be any of the \( E_{i,j} \)’s. Since all \( E_{i,j} \)’s are cocircuits, it follows by orthogonality that \( C \) must meet all the \( E_i \)’s. Thus, for each non-empty set \( J \subseteq \{1, 2, \ldots, n+1\} \) such that \( |J| \neq n \), the set \( F_J = \cup_{i \in J} E_i \) is a flat of \( M \). Let \( \mathcal{M} \) be the collection of such \( F_J \)’s.

It is easily checked that \( \mathcal{M} \) is a modular cut of \( M \). Let \( p \) be an element not in \( E(M) \). By [11, 7.2.2], the unique extension \( N \) of \( M \) on \( E(M) \cup p \) such that \( \mathcal{M} \) consists of those flats \( F \) of \( M \) for which \( F \cup p \) is a flat of \( N \) is an \((n+1)\)-spike with tip \( p \). Thus \( M \) is a spike.

The next six results deal with the case when \( M \) has rank 5.

(1.3.2) Proposition. Suppose that \( M \) is a 2-minimally, 2-cominimally 3-connected matroid with \( r(M) = 5 \). Then, for each pair of elements \( x, y \) of \( E(M) \), there is at least one 4-circuit containing both.
Proof. Since $M^*$ is also 2-minimally, 2-cominimally 3-connected, it follows by (1.2.16) that $N_{M^*}(x)$ is empty for each $x$ in $E(M)$; that is, each pair of elements $x, y$ of $E(M)$ is in at least one 4-circuit of $M$. □

(1.3.3) Proposition. Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid with $r(M) = 5$, and $x \in E(M)$. Then $M/x$ has at least four triangles.

Proof. By (1.2.15), $|E(M)| = 10$. By (1.3.2), each element is in some triangle of $M/x$. Since $|E(M/x)| = 9$, there are at least three triangles in $M/x$. If there are exactly three of them, then they are disjoint. We denote them by $T_1 = \{a_1, b_1, c_1\}$, $T_2 = \{a_2, b_2, c_2\}$, and $T_3 = \{a_3, b_3, c_3\}$. There are three 2-element subsets of $T_1$, three of $T_2$, and three of $T_3$. By the dual of (1.3.2), each of these subsets is contained in a 4-cocircuit of $M$. By orthogonality, each of these 4-cocircuits must contain another 2-element subset of this kind. Since there is an odd number of subsets of this kind, at least one of them, say $\{b_1, c_1\}$, is in at least two 4-cocircuits of $M$. By (1.2.18), we may assume that these two 4-cocircuits are $\{b_1, c_1, b_2, c_2\}$ and $\{b_1, c_1, b_3, c_3\}$. By applying the circuit elimination axiom to these two cocircuits and using orthogonality, we deduce that $\{b_2, c_2, b_3, c_3\}$ is also a 4-cocircuit of $M$. By (1.3.2), there is a 4-circuit of $M$ containing $a_1$ and $a_2$. To avoid a contradiction to orthogonality and (1.2.18), this 4-circuit has to be $\{a_1, a_2, b_3, c_3\}$. In $M/x$, apply circuit elimination to this 4-circuit and $T_3$ to obtain that $\{a_1, a_2, a_3, b_3\}$ contains a circuit of $M/x$. By orthogonality and the fact that $\{b_1, c_1, b_3, c_3\}$ is a cocircuit of $M/x$, we deduce that $\{a_1, a_2, a_3\}$ is a circuit of $M/x$, a contradiction to the original assumption. □
(1.3.4) Proposition. Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid with $r(M) = 5$. If $M$ is not a spike, and there is a pair of elements of $E(M)$ contained in three distinct 4-circuits, then $M \cong H_{10}$.

Proof. By (1.2.15), $|E(M)| = 10$. Since $M$ is not a spike, by (1.3.1), for each $x \in E(M)$, the matroid $M/x$ has at most three triangles sharing a common element. Suppose that $\{x, y\}$ is contained in three distinct 4-circuits. Let $T_1 = \{y, a_1, b_1\}$, $T_2 = \{y, a_2, b_2\}$, and $T_3 = \{y, a_3, b_3\}$ be the three corresponding triangles of $M/x$, and the remaining elements of $M/x$ be $a_4$ and $b_4$.

If there is no triangle of $M/x$ containing both $a_4$ and $b_4$, then, by (1.2.18), we may assume that $T_4 = \{b_1, b_2, a_4\}$ is a triangle of $M/x$, and $b_4$ is in a triangle $T_5$ of $M/x$. By the dual of (1.3.2), there is a 4-cocircuit of $M$ containing both $a_1$ and $a_4$. By orthogonality, this 4-cocircuit must be either $\{a_4, a_1, x, y\}$, or $\{a_4, a_1, b_1, b_4\}$. In the first case, it follows by orthogonality that $T_5$ contains $\{b_4, a_1\}$. Since at least one of $a_3$ and $b_3$, say $a_3$, is not in $T_5$, a 4-cocircuit containing both $b_4$ and $a_3$ will intersect some 4-circuit of $M$ in exactly one element, a contradiction to orthogonality. The second case also results in a contradiction in a similar way. Thus $M/x$ has a triangle containing both $a_4$ and $b_4$. By (1.3.1) and the assumption that $M$ is not a spike, we may assume that this triangle is $T_4 = \{b_1, a_4, b_4\}$.

By the dual of (1.3.2), each of the sets $\{y, a_3\}$, $\{y, b_3\}$, $\{y, a_2\}$, and $\{y, b_2\}$ is in a 4-cocircuit of $M$. If such a 4-cocircuit contains $x$, then, by orthogonality, it must contain either $a_4$ or $b_4$. Thus, by (1.2.18), there are at most two of these 4-cocircuits containing $x$. Therefore, we may assume that there is a 4-cocircuit containing $\{y, a_2\}$ and avoiding $x$. Since $T_1 \cup x$, $T_2 \cup x$, $T_3 \cup x$, and $T_4 \cup x$ are 4-circuits of $M$, it follows by orthogonality...
that, up to relabeling on \(\{a_3, b_3\}\), this 4-cocircuit is \(D^*_1 = \{y, a_1, a_2, a_3\}\). If a 4-cocircuit containing \(\{y, b_2\}\) contains \(x\), then, up to relabeling on \(\{a_4, b_4\}\), it follows by orthogonality that this cocircuit is \(\{x, y, b_2, a_4\}\). If a 4-cocircuit containing \(\{y, b_3\}\) also contains \(x\), then, by orthogonality and (1.2.18), this 4-cocircuit must be \(\{x, y, b_3, b_4\}\). By circuit elimination, the set \(\{x, b_2, b_3, a_4, b_4\}\) contains a cocircuit. By orthogonality, \(x\) is not in this cocircuit. Thus \(\{b_2, b_3, a_4, b_4\}\) is a 4-cocircuit. This 4-cocircuit, which is also a 4-cocircuit of \(M/x\), intersects the set \(\{a_1, b_1, a_2, b_2\}\), which by circuit elimination and (1.2.18) is a circuit of \(M/x\), in exactly one element, a contradiction to orthogonality. Therefore, we may assume that \(M\) has a 4-cocircuit containing \(\{y, b_2\}\) and avoiding \(x\).

By orthogonality and (1.2.18), this 4-cocircuit is \(D^*_2 = \{y, a_1, b_2, b_3\}\). Since \(M\) is not a spike, (1.3.1) and (1.3.3) imply that there is a 4-cocircuit of \(M\) containing \(a_1\) but avoiding \(y\). By orthogonality, it must be \(D^*_3 = \{a_1, b_1, a_2, b_2\}\). Applying circuit elimination to \(D^*_1\) and \(D^*_2\), we have, by orthogonality and the fact \(M\) does not have any cocircuit of size less than four, that \(D^*_4 = \{a_2, b_2, a_3, b_3\}\) is also a 4-cocircuit of \(M\). Since there is a 4-cocircuit containing both \(a_4\) and \(a_2\), by orthogonality, it is either \(\{a_4, a_2, x, y\}\) or \(\{a_4, a_2, b_4, b_2\}\). In the first case, consider the 4-cocircuit containing both \(a_4\) and \(a_3\). By (1.2.18), this 4-cocircuit must be \(D^*_5 = \{a_4, a_3, b_4, b_3\}\). Applying circuit elimination to \(D^*_4\) and \(D^*_5\), we have, by orthogonality and the fact that \(M\) does not have any cocircuit of size less than four, \(D^*_6 = \{a_4, a_2, b_4, b_2\}\). The second case also implies that the same sets \(D^*_6\) and \(D^*_5\) are cocircuits. Hence these two sets are indeed 4-cocircuits of \(M\). By the dual of (1.3.2), there is a 4-cocircuit of \(M\) containing both \(a_2\) and \(b_1\). By orthogonality, it must be one of \(\{a_2, b_1, x, y\}\), \(\{a_2, b_1, x, a_3\}\), and \(\{a_2, b_1, x, b_3\}\). If \(\{a_2, b_1, x, y\}\) is a 4-cocircuit, consider the 4-cocircuit of \(M\) containing both \(b_2\) and \(b_1\). By (1.2.18),
this 4-cocircuit must be either \( \{b_2, b_1, x, b_3\} \), or \( \{b_2, b_1, x, a_3\} \). Therefore, by symmetry, we may assume that either

(i) \( \{a_2, b_1, x, a_3\} \), or

(ii) \( \{a_2, b_1, x, b_3\} \).

is a 4-cocircuit.

In case (i), \( D_7^{a} = \{x, b_1, a_2, a_3\} \) is a 4-cocircuit of \( M \). Consider the 4-circuit of \( M \) containing both \( a_4 \) and \( a_2 \). By orthogonality and the existence of the 4-cocircuits \( D_1^{a}, D_2^{a}, \ldots, D_7^{a} \), this 4-circuit must be either \( \{a_4, a_2, a_3, b_1\} \) or \( \{a_4, a_2, a_3, b_4\} \). In the former case, from considering the 4-circuit of \( M \) containing both \( b_4 \) and \( a_2 \), we obtain a contradiction to (1.2.18). Hence \( C = \{a_4, b_4, a_2, a_3\} \) is a 4-circuit of \( M \). Similarly, \( C' = \{a_4, b_4, b_2, b_3\} \) is also a 4-circuit of \( M \). By the dual of (1.3.2), there is a 4-cocircuit \( D_{5}^{a} \) of \( M \) containing both \( a_4 \) and \( x \). By orthogonality and the fact that \( T_1 \cup x, T_2 \cup x, T_3 \cup x, \) and \( T_4 \cup x \) are 4-circuits of \( M \), we conclude that \( y \) is an element of \( D_{5}^{a} \). Since \( D_{5}^{a} \) intersects both \( C \) and \( C' \), and it already contains \( \{x, y, a_4\} \), by orthogonality, the fourth element must be \( b_4 \); that is, \( D_{6}^{a} = \{a_4, b_4, x, y\} \) is a 4-cocircuit of \( M \). Therefore, the 4-cocircuits \( D_3^{a}, D_5^{a}, D_6^{a}, \) and \( D_{8}^{a} \) all share two common elements \( a_4 \) and \( b_4 \). By (1.3.1), \( M \) is a spike, a contradiction to the assumption.

In case (ii), \( D_7^{a} = \{x, b_1, a_2, b_3\} \) is a 4-cocircuit. Consider the 4-cocircuit containing both \( a_3 \) and \( b_1 \). By orthogonality and (1.2.18), it must be either \( \{a_3, b_1, x, y\} \), or \( \{a_3, b_1, x, b_2\} \). If the former case occurs, consider the 4-cocircuit containing both \( b_2 \) and \( b_1 \), it follows by orthogonality that \( D_{8}^{a} = \{x, b_1, b_2, a_3\} \) or \( \{x, b_1, b_2, b_3\} \), a contradiction to (1.2.18). Therefore, \( D_{8}^{a} = \{x, b_1, b_2, a_3\} \) is a 4-cocircuit of \( M \). Consider the 4-
cocircuit containing \( \{y, b_1\} \). By orthogonality, it contains \( x \). By (1.2.18) and the existence of \( D_7^* \) and \( D_8^* \), this 4-cocircuit must contain either \( a_4 \) or \( b_4 \). By symmetry, we may assume that \( D_8^* = \{y, b_1, x, a_4\} \) is this 4-cocircuit. Similarly, consider the 4-cocircuit containing \( \{x, a_1\} \). By (1.2.18), this 4-cocircuit is \( D_{10}^* = \{x, a_1, y, b_4\} \).

Using the obtained information about 4-circuits and 4-cocircuits, we argue similarly to the above and obtain ten 4-circuits of \( M \). Applying orthogonality and (1.2.18), it is routine to show that there are no other 4-circuits and no other 4-cocircuits. It is now straightforward to find all other circuits of \( M \) and check that \( M \cong H_{10} \).

(1.3.5) Proposition. Suppose that \( M \) is a 2-minimally, 2-cominimally 3-connected matroid with \( r(M) = 5 \). If \( M \) is not a spike, and there is an \( x \in E(M) \) such that \( M/x \) has exactly four triangles, then \( M \cong H_{10} \).

Proof. If there are three triangles sharing a common element, then, by (1.3.4), \( M \cong H_{10} \). Thus we may suppose that each pair of elements of \( M \) is in at most two distinct 4-circuits.

Choose two disjoint triangles of \( M/x \) and denote them by \( T_1 = \{a_1, b_1, c_1\} \) and \( T_2 = \{a_2, b_2, c_2\} \). Denote the remaining three elements of \( E(M/x) \) by \( a_3, b_3, \) and \( c_3 \). Suppose that \( T_3 \) is a triangle of \( M/x \) containing \( a_3 \). If it meets \( T_1 \) but not \( T_2 \), by (1.2.18), we may assume it is \( \{a_3, b_3, a_1\} \). The element \( c_3 \) is in the remaining triangle of \( M/x \). Up to relabeling, this triangle is one of (i) \( \{c_3, c_1, b_3\} \), (ii) \( \{a_2, b_3, c_3\} \), and (iii) \( \{c_1, a_2, c_3\} \). In these three cases, we consider the 4-cocircuits of \( M \) containing \( \{a_2, c_2\} \), \( \{a_3, c_3\} \), and \( \{c_3, b_1\} \), respectively. In each case, we can find a 4-circuit of \( M \) meeting the chosen 4-cocircuit in exactly one element; a contradiction to orthogonality.
We conclude that the triangles meeting \{a_3, b_3, c_3\} will be disjoint from \(T_1\) and \(T_2\), or will intersect both of them. By assumption, there are exactly four triangles, so, one of the remaining two triangles intersects both \(T_1\) and \(T_2\). By (1.3.2) and the above argument, the last triangle must be disjoint from both \(T_1\) and \(T_2\). We may assume that \(T_3 = \{a_3, b_3, c_3\}\), and \(T_4 = \{a_1, a_2, a_3\}\).

Consider a 4-cocircuit containing \(\{b_1, c_1\}\). By orthogonality, it does not contain \(x\) and does not intersect \(T_4\). Thus we may assume that \(D_1^* = \{b_1, c_1, b_2, c_2\}\). Consider a 4-cocircuit containing \(\{b_3, c_3\}\). Similarly, we may assume that \(D_2^* = \{b_1, c_1, b_3, c_3\}\) is the 4-cocircuit of \(M\). By circuit elimination and orthogonality, \(D_3^* = \{b_2, c_2, b_3, c_3\}\) is also a 4-cocircuit. By the dual of (1.3.2), each of the sets \(\{a_1, b_1\}\), \(\{a_1, c_1\}\), \(\{a_2, b_2\}\), \(\{a_2, c_2\}\), \(\{a_3, b_3\}\), and \(\{a_3, c_3\}\) is contained in a 4-cocircuit. By orthogonality, (1.2.18), and the existence of the \(T_i\)'s, each of these 4-cocircuits consists of two such 2-element sets. Suppose that some of these 2-element sets are contained in two such 4-cocircuits. Then, by circuit elimination, three of these 2-element sets will occur in two such 4-cocircuits. Thus we may assume that \(D_4^* = \{a_1, b_1, a_2, b_2\}\), \(D_5^* = \{a_1, b_1, a_3, b_3\}\), and \(D_6^* = \{a_2, b_2, a_3, b_3\}\) are 4-cocircuits. This implies that the 4-circuit containing \(\{c_1, c_2\}\) has to be \(C = \{c_1, c_2, a_3, b_3\}\). Applying circuit elimination to \(C\) and \(T_3\) in \(M/\!\!/x\), we deduce that \(C' = \{c_1, c_2, a_3, c_3\}\) contains a circuit of \(M/\!\!/x\). By orthogonality, \(a_3\) is not in this circuit. Thus \(\{c_1, c_2, c_3\}\) is a triangle of \(M/\!\!/x\). This contradiction implies that each of these six 2-element sets occurs in exactly one 4-cocircuit, and hence we may assume that \(D_4^* = \{a_1, b_1, a_2, b_2\}\), \(D_5^* = \{a_1, c_1, a_3, c_3\}\), and \(D_6^* = \{a_2, c_2, a_3, b_3\}\) are 4-cocircuits of \(M\).
Consider a 4-circuit of $M$ containing $\{b_2, c_3\}$. By orthogonality, it contains two of $a_1, b_1$ and $c_1$. Suppose that $c_1$ is in this circuit. Then, the remaining element is either $b_1$ or $a_1$. Consider a 4-circuit of $M$ containing $\{b_1, c_3\}$. By orthogonality, it contains $b_2$ and one of $a_1$ and $c_1$. By (1.2.18), this implies that $C = \{b_1, c_1, b_2, c_3\}$ is a 4-circuit of $M$. Similarly, assuming that $b_1$ is in this 4-circuit, we consider the 4-cocircuit containing $\{c_1, b_2\}$ to draw the same conclusion. Thus, $C$ is indeed a 4-circuit of $M$. Consider a 4-circuit $C'$ containing both $c_1$ and $b_3$. By orthogonality, it is easy to show that it contains $c_2$ and one of $c_3$ and $a_3$. Now consider a 4-cocircuit containing both $x$ and $b_3$. By orthogonality, this cocircuit contains one element of each of $T_1, T_2, T_3$. Among these elements, one is $a_i$ for some $i \in \{1, 2, 3\}$. It follows that this cocircuit does not meet $C$, but meets $C'$ in two elements. Thus, it must be $D^*_i = \{x, a_1, c_2, b_3\}$.

We now find that $D^*_3, D^*_5, \text{ and } D^*_7$ all share two common elements $c_2$ and $b_3$. By (1.3.4), $M^* \cong H_{10}$. Since $H_{10}$ is self dual, we conclude that $M \cong H_{10}$. □

(1.3.6) Proposition. Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid with $r(M) = 5$. If, for each pair of elements $x, y$ of $E(M)$, there are at most two 4-circuits containing both, then $M \cong R_{10}$.

Proof. Let $x \in E(M)$. By (1.3.4) and (1.3.5), $M/x$ has at least five triangles. By mimicking the first part of the proof of (1.3.5), we may assume that $T_1 = \{a_1, b_1, c_1\}$, $T_2 = \{a_2, b_2, c_2\}$, $T_3 = \{a_3, b_3, c_3\}$, $T_4 = \{a_1, a_2, a_3\}$, and $T_5 = \{c_1, c_2, c_3\}$ are five of the triangles of $M/x$. By the dual of (1.3.2), $M$ has a 4-cocircuit containing both $a_1$ and $b_1$. By orthogonality, we may assume that $D_1^* = \{a_1, b_1, a_2, b_2\}$. Consider a 4-cocircuit containing $a_3$ and $b_3$. By orthogonality, we may assume that it is $D_2^* = \{a_1, b_1, a_3, b_3\}$. By circuit elimination, $D_3^* = \{a_2, b_2, a_3, b_3\}$ is also a 4-cocircuit. Similarly, $D_4^* = \{a_2, b_2, a_3, b_3\}$. By orthogonality, it contains $b_2$ and one of $a_1$ and $c_1$. By (1.2.18), this implies that $C = \{b_1, c_1, b_2, c_3\}$ is a 4-circuit of $M$. Similarly, assuming that $b_1$ is in this 4-circuit, we consider the 4-cocircuit containing $\{c_1, b_2\}$ to draw the same conclusion. Thus, $C$ is indeed a 4-circuit of $M$. Consider a 4-circuit $C'$ containing both $c_1$ and $b_3$. By orthogonality, it is easy to show that it contains $c_2$ and one of $c_3$ and $a_3$. Now consider a 4-cocircuit containing both $x$ and $b_3$. By orthogonality, this cocircuit contains one element of each of $T_1, T_2,$ $T_3$. Among these elements, one is $a_i$ for some $i \in \{1, 2, 3\}$. It follows that this cocircuit does not meet $C$, but meets $C'$ in two elements. Thus, it must be $D^*_i = \{x, a_1, c_2, b_3\}$.

We now find that $D^*_3, D^*_5, \text{ and } D^*_7$ all share two common elements $c_2$ and $b_3$. By (1.3.4), $M^* \cong H_{10}$. Since $H_{10}$ is self dual, we conclude that $M \cong H_{10}$. □

(1.3.6) Proposition. Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid with $r(M) = 5$. If, for each pair of elements $x, y$ of $E(M)$, there are at most two 4-circuits containing both, then $M \cong R_{10}$.

Proof. Let $x \in E(M)$. By (1.3.4) and (1.3.5), $M/x$ has at least five triangles. By mimicking the first part of the proof of (1.3.5), we may assume that $T_1 = \{a_1, b_1, c_1\}$, $T_2 = \{a_2, b_2, c_2\}$, $T_3 = \{a_3, b_3, c_3\}$, $T_4 = \{a_1, a_2, a_3\}$, and $T_5 = \{c_1, c_2, c_3\}$ are five of the triangles of $M/x$. By the dual of (1.3.2), $M$ has a 4-cocircuit containing both $a_1$ and $b_1$. By orthogonality, we may assume that $D_1^* = \{a_1, b_1, a_2, b_2\}$. Consider a 4-cocircuit containing $a_3$ and $b_3$. By orthogonality, we may assume that it is $D_2^* = \{a_1, b_1, a_3, b_3\}$. By circuit elimination, $D_3^* = \{a_2, b_2, a_3, b_3\}$ is also a 4-cocircuit. Similarly, $D_4^* = \{a_2, b_2, a_3, b_3\}$.
\{b_1, c_1, b_2, c_2\}, \ D_5 \ = \ \{b_1, c_1, b_3, c_3\}, \ and \ D_6 \ = \ \{b_2, c_2, b_3, c_3\} \ are \ all \ 4\text{-cocircuits} \ of \ M. \n
Consider \ a \ 4\text{-cocircuit} \ containing \ both \ a_1 \ and \ c_1. \ By \ \text{(1.2.18)} \ and \ orthogonality, \ this \ 4\text{-cocircuit} \ must \ be \ D_7 \ = \ \{a_1, c_1, a_2, c_2\} \ or \ D_5 \ = \ \{a_1, c_1, a_3, c_3\}. \ We \ may \ assume \ that \ D_7 \ occurs. \ Consider \ the \ 4\text{-cocircuit} \ containing \ a_3 \ and \ c_3. \ By \ \text{(1.2.18)} \ and \ orthogonality, \ it \ must \ be \ D_6 \ or \ D_5 \ = \ \{a_2, c_2, a_3, c_3\}. \ By \ circuit \ elimination, \ we \ conclude \ that \ D_7, \ D_8, \ and \ D_6 \ are \ all \ 4\text{-cocircuits} \ of \ M. \ If \ \{b_1, b_2, b_3\} \ is \ not \ a \ triangle \ of \ M/x, \ then, \ consider \ a \ 4\text{-circuit} \ containing \ both \ b_1 \ and \ b_2. \ By \ orthogonality, \ the \ nine \ 4\text{-cocircuits} \ D_1, D_2, \ldots, D_6 \ force \ the \ 4\text{-circuit} \ to \ be \ \{b_1, b_2, a_3, c_3\}. \ Applying \ circuit \ elimination \ to \ this \ circuit \ and \ T_3 \ in \ M/x, \ we \ conclude \ that \ \{b_1, b_2, a_3, b_3\} \ contains \ a \ circuit \ of \ M/x. \n
Since \ D_7 \ is \ a \ cocircuit \ of \ M/x, \ by \ orthogonality, \ a_3 \ is \ not \ in \ this \ circuit \ of \ M/x. \ Hence \ \{b_1, b_2, b_3\} \ is \ a \ triangle \ of \ M/x. \ This \ implies \ that \ M/x \ is \ isomorphic \ M^*(K_{3,3}) \ for \ each \ x \ \in \ E(M). \ It \ is \ routine \ to \ check \ that \ M \simeq R_{10}. \ \square \n
On \ combining \ (1.3.4), \ (1.3.5) \ and \ (1.3.6), \ we \ immediately \ obtain \ the \ following: \n
(1.3.7) Lemma. \ Suppose \ that \ M \ is \ a \ 2\text{-minimally}, \ 2\text{-cominimally} \ 3\text{-connected} \ matroid \ with \ r(M) = 5. \ Then \ M \ is \ a \ spike, \ or \ M \ is \ isomorphic \ to \ either \ H_{10} \ or \ R_{10}. \n
(1.3.8) Proposition. \ Suppose \ that \ M \ is \ a \ 2\text{-minimally}, \ 2\text{-cominimally} \ 3\text{-connected} \ matroid \ with \ r(M) \ \geq \ 6. \ Then, \ for \ each \ e \ \in \ E(M), \ the \ matroid \ M\setminus e \ has \ at \ least \ five \ triads. \n
Proof. \ As \ r(M) \ \geq \ 6, \ by \ (1.2.15), \ |E(M\setminus e)| \ \geq \ 11. \ By \ (1.2.16), \ M\setminus e \ has \ at \ least \ four \ triads. \ If \ the \ union \ of \ the \ triads \ of \ M\setminus e \ has \ at \ least \ eleven \ elements, \ and \ M\setminus e \ has \ exactly \ four \ triads, \ then \ M\setminus e \ has \ three \ disjoint \ triads. \ By \ (1.2.16), \ there \ is \ a \ 4\text{-circuit} \ of \ M \ containing \ e \ and \ some \ element \ not \ in \ any \ of \ these \ three \ triads \ of \ M\setminus e. \ This
4–circuit will intersect some 4–cocircuit of $M$ in exactly one element, a contradiction to orthogonality. Hence if $M \setminus e$ has at least eleven elements that are in triads, then $M \setminus e$ has at least five triads. We now consider the case that $M \setminus e$ has at most ten elements that are in triads. By (1.2.16), $|E(M \setminus e)| \leq 11$. Thus, by (1.2.16), $r(M) \leq 6$. Hence $r(M) = 6$, and $|E(M)| = 12$. Let $f$ be the element not in any triad of $M \setminus e$. Then, by (1.2.9) and (1.2.14), $M \setminus e/f$ is isomorphic to $W_5$ or $M(W_5)$, and again $M \setminus e$ has at least five triads. □

(1.3.9) Proposition. Suppose that $M$ is a 2–minimally, 2–cominimally 3–connected matroid with $r(M) \geq 6$. Then $M$ has a pair of elements $e$ and $f$ such that there are at least three 4–circuits of $M$ containing both.

Proof. Assume the contrary. Then, for each pair of elements of $M$, there are at most two 4–circuits of $M$ containing both. By (1.2.15), as $r(M) \geq 6$, we have $|E(M)| \geq 12$. Let $x$ be an element of $E(M)$. Since every element of $E(M/x)$ is in at most two triangles, if there are at least seven triangles, any 4–cocircuit of $M$ containing $x$ will intersect some 4–circuit of $M$ containing $x$ in exactly one element, a contradiction. If there are exactly six triangles in $M/x$, then, as $|E(M/x)| \geq 11$, there are at least two elements of $M/x$ such that each is in at most one triangle of $M/x$. Hence, by (1.2.16), there is a 4–cocircuit of $M$ containing $x$ and one such element. This 4–cocircuit in turn will intersect some 4–circuit of $M$ containing $x$ in exactly one element, a contradiction. Thus, by (1.3.8), for every element $x \in E(M)$, the matroid $M/x$ has exactly five triangles.

If $r(M) > 6$, then, by (1.2.15), $|E(M)| > 13$. By (1.2.16), the union of the triangles of $M/x$ has cardinality greater than or equal to 12. As $M/x$ has exactly five triangles, it
is easy to find three disjoint triangles. Thus a 4-cocircuit containing \( x \) and an element not in these three disjoint triangles will intersect some 4-circuit in exactly one element, a contradiction. Therefore, we have that \( r(M) = 6 \), and \( M/x \) has exactly five triangles.

If each element of \( M/x \) is in at least one triangle, then, by the fact that \( |E(M/x)| = 11 \), there are exactly four elements of \( E(M/x) \) such that each is in exactly two triangles. Suppose there is a triangle such that each of its elements is in only one triangle. Then, by (1.2.16), there is an element \( y \) in exactly one of the other four triangles such that \( y \) is in a 4-cocircuit of \( M \) containing \( x \). It follows that this 4-cocircuit intersects some 4-circuit of \( M \) in exactly one element, a contradiction. We conclude that every triangle of \( M/x \) intersects some other triangle of \( M/x \). It follows by (1.2.18) that there are four triangles, \( T_1, T_2, T_3, T_4 \), such that \( |T_1 \cap T_2| = 1 \), \( |T_3 \cap T_4| = 1 \), and \( |(T_1 \cup T_2) \cap (T_3 \cup T_4)| = 0 \). Up to relabeling, the remaining triangle will intersect \( T_2 \) and either \( T_1 \) or \( T_3 \). In the former case, by (1.2.16), one element in \( T_3 - T_4 \) is in a 4-cocircuit of \( M \) containing \( x \), and this contradicts orthogonality. In the latter case, by (1.2.16) again, \( M \) has a 4-cocircuit containing \( x \) and an element in \( (T_2 - T_1) \cup (T_3 - T_4) \) that is only in one triangle of \( M/x \). This 4-cocircuit will intersect some 4-circuit in exactly one element, a contradiction.

We may now assume that \( r(M) = 6 \), and that, for each \( x \in E(M) \), the matroid \( M/x \) has exactly five triangles and has an element \( y(x) \) such that \( y(x) \) is not in any triangle of \( M/x \). By (1.2.14), \( M/x \setminus y(x) \) is a wheel or a whirl. Thus \( M/x, y(x) \) is isomorphic to \( M(K_5) \). This contradicts the assumption that \( M \) is 2-cominimally 3-connected and hence proves the proposition. \( \square \)
Lemma. Let $M$ be a 2-minimally, 2-cominimally 3-connected matroid with $r(M) \geq 6$. Suppose that, for some element $a_1$ in $E(M)$, the matroid $M/a_1$ has three triangles sharing a common element $a_2$, and that $M$ is not a spike. Then $r(M) = 6$ and $M \cong H_{12}$.

Proof. Since $M$ is not a spike, by (1.3.8) and (1.3.1), for each pair of elements $x, y$ in $E(M)$, there are at least two 4-circuits and two 4-cocircuits of $M$ containing $x$ and avoiding $y$.

Denote the three triangles of $M/a_1$ by $T_1 = \{a_2, a_3, a_4\}$, $T_2 = \{a_2, b_1, b_3\}$, and $T_3 = \{a_2, b_2, b_4\}$. Then, by (1.3.1), there are no other triangles of $M/a_1$ containing $a_2$. Since there are two 4-cocircuits $D_1^*$ and $D_2^*$ of $M$ that contain $a_2$ and avoid $a_1$, by orthogonality, we may assume that $D_1^* = \{a_2, a_4, b_3, b_4\}$ is a 4-cocircuit of $M$. If $D_2^* = \{a_2, a_3, b_1, b_2\}$, then, by orthogonality, every other triangle of $M/a_1$ will either intersect the set $\{a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$ in at least two elements or avoid this set. The first case is impossible since it leads to the conclusion that every 4-cocircuit of $M$ containing $a_1$ will contain $a_2$ which implies that $M$ is a spike. The second case is also not possible as it forces the matroid $M/a_1$ to have rank four, contradicting the fact that $r(M) \geq 6$. Therefore, by (1.2.18), $D_2^* = \{a_2, a_4, b_1, b_2\}$. Applying circuit elimination to $D_1^*$ and $D_2^*$, we conclude by orthogonality that $\{a_2, b_1, b_2, b_3, b_4\}$ contains a cocircuit.

By orthogonality and the fact that $T_1 \cup a_1$ is a circuit of $M$, this circuit cannot contain $a_2$. Hence $D_3^* = \{b_1, b_2, b_3, b_4\}$ is also a 4-cocircuit of $M$.

If $M/a_1$ has a triangle disjoint from $T_1 \cup T_2 \cup T_3$, then, by orthogonality, every 4-cocircuit containing $a_1$ must contain $a_2$. Hence $M^*$ is a spike. This contradiction implies that every triangle of $M/a_1$ must intersect the set $\{a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$. Moreover,
by orthogonality, every triangle of $M/a_1$ intersecting $\{a_4, b_1, b_2, b_3, b_4\}$ intersects the set in at least two elements. As $r(M/a_1) \geq 5$, there is at least one triangle of $M/a_1$ which contains $a_3$, and avoids $\{a_4, b_1, b_2, b_3, b_4\}$. Let $T_4 = \{a_3, c_1, c_2\}$ be this triangle.

If $r(M) > 6$, then, by (1.2.15), $|E(M)| \geq 14$. Since $M$ is not a spike, there are at most three triangles of $M/a_1$ containing $a_3$. Thus there are at least two elements $s, t$ in $E(M/a_1) - (T_1 \cup T_2 \cup T_3)$ that are not in triangles of $M/a_1$ containing $a_3$. By (1.2.16), we may assume that $s$ is in a triangle $T'$ of $M/a_1$. Consider a 4-cocircuit of $M$ containing $t$ and one of $a_1$ and $a_2$. By orthogonality, it must contain $a_1$, $a_2$, and one element of $T_4$. Hence this cocircuit intersects the 4-circuit $T' \cup a_1$ of $M$ in exactly one element, a contradiction. Therefore, $r(M) = 6$, and so $|E(M)| = 12$.

Let $E(M) - \{a_1, a_2, a_3, c_1, c_2\} = \{c_3, c_4\}$. If $c_3$ is in a triangle containing $a_3$, then, by orthogonality and the existence of the cocircuits $D_1^*$ and $D_2^*$, it follows that $c_4$ is the remaining element of this triangle. If $c_3$ is not in a triangle containing $a_3$, then neither is $c_4$. If this is the case, then, by (1.2.16), we may assume that $c_3$ is in a triangle $T'$ avoiding $a_3$, and $M$ has a 4-cocircuit $C^*$ containing $c_4$ and either $a_1$ or $a_2$. It follows by orthogonality that $C^*$ contains both $a_1$ and $a_2$, and one element of $T_4$. Hence $C^*$ intersects the 4-circuit $T' \cup a_1$ in exactly one element. This contradiction shows that $T_5 = \{a_3, c_3, c_4\}$ is a triangle of $M/a_1$.

Since $M^*$ is not a spike, it follows by (1.3.8) and (1.3.1), there are at least two 4-cocircuits containing $a_3$ and avoiding $a_1$. By (1.2.18), orthogonality and relabeling on $\{c_1, c_2\}$ and $\{c_3, c_4\}$, we may assume that these 4-cocircuits are $D_4^* = \{a_3, a_4, c_1, c_3\}$ and $D_5^* = \{a_3, a_4, c_2, c_4\}$. Applying circuit elimination to $D_4^*$ and $D_5^*$, it follows by orthogonality that $D_6^* = \{c_1, c_2, c_3, c_4\}$ is another 4-cocircuit of $M$. By (1.2.16), there
is a 4-cocircuit of $M$ containing $b_1$ and either $c_1$ or $c_2$. By orthogonality, this cocircuit is $D^*_{17} = \{b_1, b_3, c_1, c_2\}$. Similarly, $D^*_{18} = \{b_1, b_3, c_3, c_4\}$, $D^*_{19} = \{b_2, b_4, c_1, c_2\}$, and $D^*_{10} = \{b_2, b_4, c_3, c_4\}$ are all 4-cocircuits of $M/a_1$. By (1.2.16), there is a 4-circuit $C$ of $M$ containing $b_4$ and either $c_3$ or $c_4$. We may assume that $c_3 \in C$. Then, by orthogonality and the existing 4-cocircuits, we conclude that either $C = \{b_3, b_4, c_1, c_3\}$, or $C = \{b_2, b_4, c_3, c_4\}$. By (1.2.16), there is a 4-cocircuit containing $a_3$ and one of $b_1$ and $b_2$. By orthogonality, this circuit contains $\{a_1, a_3\}$, one of $\{b_2, b_4\}$ and one of $\{b_1, b_3\}$. This implies that $C = \{b_3, b_4, c_1, c_3\}$ and $D^*_{11} = \{a_1, a_3, b_1, b_2\}$ is a 4-cocircuit of $M$. Similarly, $D^*_{12} = \{a_1, a_3, b_3, b_4\}$, $D^*_{13} = \{a_1, a_2, c_1, c_3\}$, and $D^*_{14} = \{a_1, a_2, c_2, c_4\}$ are also 4-cocircuits of $M$. Applying circuit elimination to $D^*_{1}$ and $D^*_{13}$, we conclude that $\{a_1, a_2, a_3, a_4, c_1\}$ contains a cocircuit of $M$. Since this cocircuit does not meet $C$ in exactly one element, $c_1$ is not contained in this cocircuit. Thus this cocircuit must be $D^*_{15} = \{a_1, a_2, a_3, a_4\}$. Arguing with $M^*$, we will also obtain fifteen 4-circuits of $M$. Moreover, by orthogonality and the existence of the $D^*_i$'s, it is now straightforward to check that $M$ has no 5-circuits and no 7-circuits. Therefore, $M$ is binary, and it is routine to check that the matroid $M$ is isomorphic to $H_{12}$. \qed

**Proof of Theorem 1.1.1.** The first part of the theorem follows immediately on combining (1.3.1), (1.3.7) and (1.3.10). The check that each spike of rank at least four is 2-minimally, 2-cominimally 3-connected is straightforward and is omitted. \qed

The last theorem shows that a matroid of rank at least seven is 2-minimally, 2-cominimally 3-connected if and only if it is a spike. Although there are only three 2-minimally, 2-cominimally, 3-connected matroids of rank at least five that are not
spikes, there are more than thirty 2-minimally, 2-cominimally 3-connected matroids of rank four that are not spikes.

1.4 2-minimally, 1-cominimally 3-connected matroids

This section identifies all 2-minimally, 1-cominimally 3-connected matroids by proving Theorem 1.1.2.

(1.4.1) Proposition. Suppose that $M$ is a 2-minimally 3-connected matroid with $|E(M)| \geq 6$. Then no 4-cocircuit of $M$ contains a triangle of $M$.

Proof. Suppose that $T = \{a, b, c\}$ is a triangle of $M$, and that $\{a, b, c, d\}$ is a 4-cocircuit of $M$. Then $T$ is both a triangle and a triad of $M \setminus d$. Let $r$ be the rank function of $M \setminus d$. Then, as $|E(M \setminus d)| \geq 5$ and $r(T) + r^*(T) - |T| = 1$, it follows that $(T, E(M \setminus d) - T)$ is a 2-separation of $M \setminus d$. This contradicts the assumption that $M$ is 2-minimally 3-connected. □

(1.4.2) Proposition. Suppose that $M$ is a 2-minimally, 1-cominimally 3-connected matroid with $|E(M)| \geq 6$. Then $M$ has at most one element not contained in a triangle.

Proof. Suppose that each of $x$, $y$ is an element of $M$ not contained in a triangle. Since $M$ is 1-cominimally 3-connected, it follows by the dual of (1.2.9) that $M \setminus x, y$ is 3-connected. This contradicts the assumption that $M$ is 2-minimally 3-connected. □

(1.4.3) Proposition. Let $M$ be a 2-minimally 3-connected matroid with $|E(M)| \geq 6$. Suppose that $T_1$ and $T_2$ are two distinct triangles of $M$. Then $|T_1 \cap T_2| \leq 1$.

Proof. Suppose that $|T_1 \cap T_2| = 2$. Then, by circuit elimination and (1.2.2), $M|(T_1 \cup T_2) \cong U_{2,4}$. Let $x \in T_1$. Then $M \setminus x$ is minimally 3-connected. It follows by (1.2.4)
that $M \setminus x$ has a triad. Thus $M$ has a 4-cocircuit $D^*$ containing $x$. By orthogonality and the fact that $M|(T_1 \cup T_2) \cong U_{2,4}$, $D^*$ contains at least three elements of $T_1 \cup T_2$, a contradiction to (1.4.1). □

(1.4.4) Proposition. Let $M$ be a 2-minimally 3-connected matroid with $|E(M)| \geq 6$. Suppose that $D_1^*$ and $D_2^*$ are two distinct 4-cocircuits of $M$ and $T$ is a triangle meeting $D_i^*$. Then $|D_1^* \cap D_2^*| \neq 3$.

Proof. Suppose that $D_1^* = \{e_1, e_2, e_3, e_4\}$, and $D_2^* = \{e_1, e_2, e_3, e_5\}$. It follows by (1.2.2) and circuit elimination that $M^*|(D_1^* \cup D_2^*) \cong U_{3,5}$. We may assume that $e_1$ is in $T$. By (1.4.1), $T \not\subseteq D_1^* \cup D_2^*$. Thus $T$ meets some 4-cocircuit in $D_1^* \cup D_2^*$ in exactly one element, a contradiction to orthogonality. □

(1.4.5) Proposition. Let $M$ be a 2-minimally 3-connected matroid with $|E(M)| \geq 6$. Suppose that $T$ is a triangle of $M$, $e \in T$, and $x \in E(M) - T$. Then $\{e, x\}$ is contained in a 4-cocircuit of $M$.

Proof. By (1.2.4), $T$ meets at least two triads of $M \setminus x$. By orthogonality and (1.4.1), each of these triads contains exactly two elements of $T$. It follows by (1.4.4) that every element of $T$ is contained in at least one of these triads of $M \setminus x$. □

(1.4.6) Lemma. Let $M$ be a 2-minimally, 1-cominimally 3-connected matroid with $|E(M)| \geq 6$. Suppose that $M$ has three triangles sharing a common element. Then $M$ is isomorphic to either $F_7$ or $F_7^{-}$.

Proof. By (1.4.3), we may assume that these three triangles are $T_1 = \{e_1, e_2, e_3\}$, $T_2 = \{e_1, e_4, e_5\}$, and $T_3 = \{e_1, e_6, e_7\}$. If there is an element $x$ in $E(M) - \{e_1, e_2, \ldots, e_7\}$, then, by (1.4.5), $e_1$ is contained in a triad of $M \setminus x$. It follows that this triad meets
one of $T_1$, $T_2$, and $T_3$ in exactly one element, a contradiction. Therefore, $E(M) = \{e_1, e_2, \ldots, e_7\}$. Moreover, clearly $r(M) = 3$.

Since the matroid $M \setminus e_1$ is minimally 3-connected, it is not isomorphic to $U_{3,6}$, and hence has at least one triangle. By (1.4.3), we may assume that $T_4 = \{e_3, e_5, e_7\}$ is a triangle of $M$. By (1.4.1), orthogonality, and the fact that $T_4$ meets at least two triads of $M \setminus e_1$, we conclude that at least two of the sets $C_1^* = \{e_2, e_5, e_7\}$, $C_2^* = \{e_3, e_4, e_7\}$, and $C_3^* = \{e_3, e_5, e_6\}$ are triads of $M \setminus e_1$. As $r(M \setminus e_1) = 3$ and $|E(M)| = 7$, $M \setminus e_1$ has corank 3. By (1.4.4), every triad of $M \setminus e_1$ is a cohyperplane of $M \setminus e_1$. Hence the complement of a triad of $M \setminus e_1$ is a triangle of $M \setminus e_1$. Therefore, at least two of the sets $E(M \setminus e_1) - C_1^*$, $E(M \setminus e_1) - C_2^*$, and $E(M \setminus e_1) - C_3^*$ are circuits of $M \setminus e_1$; that is, at least two of $\{e_3, e_4, e_6\}$, $\{e_2, e_5, e_6\}$, and $\{e_2, e_4, e_7\}$ are triangles of $M$. Thus $M$ is isomorphic to either $F_7$ or $F_7^\perp$.

(1.4.7) Proposition. Let $M$ be a 2-minimally, 1-cominimally 3-connected matroid with $|E(M)| \geq 6$. Suppose that $T_1$, $T_2$, and $T_3$ are distinct triangles of $M$ such that $|T_1 \cap T_2| = 1$, and $M$ is not isomorphic to $F_7$ or $F_7^\perp$. Then $T_3$ meets exactly one of $T_1$ and $T_2$.

Proof. Suppose that $T_1 = \{e_1, e_2, e_3\}$ and $T_2 = \{e_1, e_4, e_5\}$. If $T_3$ is disjoint from $T_1 \cup T_2$, then, by (1.2.4), $T_3$ meets two distinct triads of $M \setminus e_2$. Thus $T_3$ meets two 4-cocircuits of $M$ containing $e_2$. As $|T_3| = 3$, it follows by orthogonality that some element of $T_3$ is contained in both 4-cocircuits. By (1.4.4) and orthogonality, one of these two 4-cocircuits must contain $e_1$ and two elements of $T_3$. This implies that this 4-cocircuit meets $T_2$ in exactly one element, a contradiction. Therefore, $T_3$ meets at least one of $T_1$ and $T_2$. 

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Suppose that $T_3$ meets both $T_1$ and $T_2$. By (1.4.3) and the assumption that $M \not\cong F_7$ or $F_7^-$, we may assume that $T_3 = \{e_3, e_5, e_6\}$. Since $M$ is 2-minimally 3-connected, $M \setminus e_6$ is 3-connected. Since $M \setminus (T_1 \cup T_2)$ is not 3-connected, there is an element $e_7$ in $E(M) - (T_1 \cup T_2 \cup T_3)$. Suppose that $x \in E(M) - (T_1 \cup T_2 \cup T_3)$. By (1.4.5), $M$ has a 4-cocircuit containing both $x$ and $e_2$. By orthogonality, this 4-cocircuit must be either $C_1^* = \{x, e_1, e_2, e_4\}$ or $C_2^* = \{x, e_4, e_5, e_6\}$. By (1.4.5) again, $M$ has a 4-cocircuit containing $x$ and $e_6$. By orthogonality, this 4-cocircuit must be either $C_2^*$ or $C_3^* = \{x, e_2, e_3, e_6\}$. In other words, at least two of $C_1^*, C_2^*$ and $C_3^*$ are 4-cocircuits of $M$.

It follows by (1.4.4) that $|E(M) - (T_1 \cup T_2 \cup T_3)| \leq 12$. Therefore, $E(M) = \{e_1, e_2, \ldots, e_7\}$.

By (1.4.4), it is clear that $r^*(M) \geq 4$. Thus $r(M) \leq 3$. We conclude by (1.4.3) that $r(M) = 3$. By (1.4.5), there is a 4-cocircuit $D^*$ containing both $e_4$ and $e_3$. By (1.4.1) and orthogonality, $D^*$ is either $\{e_1, e_3, e_5, e_6\}$ or $\{e_2, e_3, e_4, e_5\}$. Therefore, either $\{e_2, e_5, e_7\}$ or $\{e_1, e_6, e_7\}$ is a hyperplane of $M$. As $r(M) = 3$, this hyperplane is a triangle of $M$. Therefore, either $e_1$ or $e_5$ is contained in three distinct triangles. By (1.4.6), $M$ is isomorphic to either $F_7$ or $F_7^-$, a contradiction. Therefore, $T_3$ meets exactly one of $T_1$ and $T_2$. □

(1.4.8) Lemma. Let $M$ be a 2-minimally, 1-cominimially 3-connected matroid with $|E(M)| \geq 6$. Suppose that $T_1, T_2$ are triangles of $M$ such that $|T_1 \cap T_2| = 1$, and $M$ is not isomorphic to $F_7$ or $F_7^-$. Then $M \cong M^*(K_{3,3})$.

Proof. Let $T_1 = \{e_1, e_2, e_3\}$, $T_2 = \{e_1, e_4, e_5\}$, and $x \in E(M) - (T_1 \cup T_2)$. Since $M \setminus x$ is 3-connected but $M \setminus (T_1 \cup T_2)$ is not, $|E(M)| \geq 7$. By (1.4.5), $M$ has a 4-cocircuit containing both $x$ and $e_1$. By orthogonality, this 4-cocircuit contains one
element of \{e_2, e_3\} and one element of \{e_4, e_5\}. As \(4^2 = 4\), it follows by (1.4.4) that 
\[ |E(M) - (T_1 \cup T_2)| \leq 4. \]
Thus \(|E(M)| \leq 9\).

Since \(|E(M)| \geq 7\), by (1.4.2), there is an element \(e_6\) in \(E(M) - (T_1 \cup T_2)\) which is contained in a triangle \(T_3\). By (1.4.7), we may assume that \(T_3 = \{e_2, e_6, e_7\}\). Since \(M \setminus e_3\) is 3-connected but \(M \setminus (T_2 \cup T_3)\) is not, \(M\) has an element, say \(e_8\), that is not contained in \(T_1 \cup T_2 \cup T_3\). By (1.4.5), \(M\) has a 4-cocircuit \(C^*\) containing both \(e_1\) and \(e_8\). By orthogonality, we may assume that \(C^* = \{e_1, e_3, e_4, e_8\}\). By (1.4.5), \(M\) has a 4-cocircuit \(D^*\) containing both \(e_5\) and \(e_8\). By orthogonality and (1.4.4), \(D^*\) contains \(\{e_4, e_5, e_8\}\) and one element not in \(T_1 \cup T_2 \cup T_3\), say \(e_9\). Therefore, \(|E(M)| = 9\).

By (1.4.2), we may assume that \(e_8\) is contained in a triangle \(T_4\). Applying (1.4.7) to the triangles \(T_1, T_2,\) and \(T_4\), we conclude that \(T_4\) meets exactly one of \(T_1\) and \(T_2\). Applying (1.4.7) again, this time to \(T_1, T_3,\) and \(T_4\), we conclude that \(T_4\) meets exactly one of \(T_1\) and \(T_3\). Therefore, \(T_4\) either meets both \(T_2\) and \(T_3\), or meets \(T_1\) but avoids \(T_2 \cup T_3\). Thus, we may assume that \(T_4\) is either \(\{e_4, e_6, e_7\}\) or \(\{e_3, e_8, e_9\}\).

In the former case, consider the set \(B = \{e_1, e_2, e_3, e_4, e_5\}\). Since \(M\) is 2-minimally 3-connected, it follows by (1.2.2) that all cocircuits of \(M\) have at least four elements. Thus, by orthogonality, it is easy to check that \(B\) contains no cocircuits. Therefore, \(r^*(M) \geq 5\). By (1.4.5), \(M\) has a 4-cocircuit \(D_1^*\) containing both \(e_1\) and \(e_7\). By orthogonality, \(D_1^* = \{e_1, e_2, e_5, e_7\}\). Similarly, \(M\) has a 4-cocircuit \(D_2^*\) containing both \(e_4\) and \(e_7\). By orthogonality and (1.4.4), \(D_2^* = \{e_4, e_5, e_6, e_7\}\). By (1.4.5), \(M\) has a 4-cocircuit \(D_3^*\) containing both \(e_1\) and \(e_6\). By orthogonality and (1.4.4), \(D_3^* = \{e_1, e_2, e_4, e_6\}\). Let \(H = \{e_1, e_2, e_4, e_5, e_6, e_7\}\). Then \(H = D_1^* \cup D_2^* \cup D_3^*\) and \(r^*(H) \leq 4\). Thus \(T_5 = E(M) - H = \{e_3, e_8, e_9\}\) is dependent and so \(T_5\) is a triangle of \(M\). By the
fact that \( r^*(M) \geq 5 \), we conclude that \( r^*(M) = 5 \). Hence \( r(M) = 4 \). By a similar argument to the above, we conclude that \( T_6 = \{e_5, e_7, e_9\} \) is also a triangle. Therefore, \( M \cong M^*(K_{3,3}) \).

It remains to consider the case when \( T_4 = \{e_3, e_8, e_9\} \). In that case, we can apply a similar argument to the above to draw the same conclusion. \( \square \)

Let \( h \) be an integer exceeding one. An \( h \)-raft [3] is a matroid of rank \( 2h - 2 \) whose ground set is the union of \( h \) disjoint triangles such that, for all \( k < h \), the union of every set of \( k \) of these triangles has rank \( 2k \). Thus, for example, \( M^*(K_{3,3}) \) is a 3-raft.

(1.4.9) Lemma. Let \( M \) be a 2-minimally, 1-cominimally 3-connected matroid with \( |E(M)| \geq 6 \). Suppose that each pair of distinct triangles of \( M \) are disjoint. Then \( M \) is a binary raft.

Proof. Suppose that \( T = \{a, b, c\} \) is a triangle of \( M \), and \( x \in E(M) - T \). By (1.4.5), \( M \) has a 4-cocircuit \( D^* \) containing \( x \) and meeting \( T \). By (1.4.1) and orthogonality, we may assume that \( D^* = \{x, y, a, b\} \), while \( y \) is not an element of \( T \). By (1.4.2), \( M \) has a triangle \( T' \) containing one of \( x \) and \( y \). By assumption, \( T \cap T' = \emptyset \). It follows by orthogonality that \( T' \cap D^* = \{x, y\} \). Thus, every element of \( M \) is contained in a triangle. We conclude that there is a positive integer \( n \) such that \( E(M) = 3n \), and the ground set of \( M \) is the union of \( n \) disjoint triangles.

Since \( M \) is 2-minimally 3-connected, \( M \) cannot be one of \( U_{2,6}, R_6, \) or \( U_{2,3} \odot U_{2,3} \). Thus, \( n \geq 3 \). Denote the \( n \) triangles of \( M \) by \( T_1 = \{a_1, b_1, c_1\} \), \( T_2 = \{a_2, b_2, c_2\} \), \ldots, \( T_n = \{a_n, b_n, c_n\} \). By (1.2.4), the matroid \( M \setminus a_2 \) has two triads meeting \( T_1 \). Thus \( M \) has two 4-cocircuits containing \( a_2 \) and meeting \( T_1 \). By orthogonality and (1.4.4), we may assume that these two 4-cocircuits are \( C^*_{1,2} = \{a_1, b_1, a_2, b_2\} \) and \( D^*_{1,2} = \{a_1, c_1, a_2, c_2\} \).
Similarly, $M$ has two 4-cocircuits containing $c_2$ and meeting $T_1$. By orthogonality and (1.4.4), these two 4-cocircuits are $D_{1,2}^*$ and $E_{1,2}^* = \{b_1, c_1, b_2, c_2\}$. Therefore, up to relabeling, we may assume that, for each pair of distinct integers $i, j$ in $\{1, 2, \ldots, n\}$, $C_{i,j}^* = \{a_i, b_i, a_j, b_j\}$, $D_{i,j}^* = \{a_i, c_i, a_j, c_j\}$, and $E_{i,j}^* = \{b_i, c_i, b_j, c_j\}$ are all 4-cocircuits of $M$.

Since $M$ is 2-minimally 3-connected and $|E(M)| > 4$, every cocircuit of $M$ has at least four elements. Thus every cocircuit of $M$ meets at least two triangles. If a set contains a triangle and two elements of another triangle, it contains a 4-cocircuit. Thus a cocircuit cannot contain a triangle. Therefore, a cocircuit is either disjoint from a triangle or meets that triangle in two elements. If $X$ is the union of four 2-element sets, each of which is a subset of distinct triangles, then it is clear that $X$ contains a 4-cocircuit. Thus we deduce that $M$ has only 4-cocircuits and 6-cocircuits. Applying circuit elimination to $C_{i,j}^*$ and $D_{j,k}^*$, it follows by orthogonality that $C_{i,j}^* \Delta D_{j,k}^*$ is a 6-cocircuit of $M$. It is now straightforward that for each pair of distinct cocircuits of $M$, their symmetric difference is a disjoint union of cocircuits. Therefore, $M$ is binary.

By orthogonality, the set $\{a_1, a_2, \ldots, a_n\} \cup \{b_1, c_1\}$ contains no cocircuit. Thus $r^*(M) \geq n + 2$. By orthogonality, the set $\cup_{2 \leq j \leq n} C_{1,j}^*$ is a coflat. Thus the set $C = \{c_1, c_2, \ldots, c_n\}$ is dependent. By orthogonality, $C$ must be a circuit. Therefore, $\cup_{2 \leq j \leq n} C_{1,j}^*$ is a cohyperplane, and $r^*(M) = r^*(\cup_{2 \leq j \leq n} C_{1,j}^*) + 1 \leq (n + 1) + 1$.

We deduce that $r^*(M) = n + 2$, and hence $r(M) = 3n - (n + 2) = 2n - 2$. Moreover, by arguing as for $C$, we deduce that $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ are circuits of $M$. By orthogonality, every circuit having more than three elements must meet all triangles. On combining this observation with the fact that $A$, $B$, and $C$ are...
circuits, we conclude that, for each \( k < n \), the union of \( k \) distinct triangles has rank \( 2k \). Thus \( M \) is a raft. \( \Box \)

**Proof of Theorem 1.1.2.** It is easy to check that there is no 2-minimally, 1-cominimally 3-connected matroid \( M \) with \( |E(M)| < 6 \). Moreover, it is proved in [4] that, for all \( n \geq 3 \), the only binary \( n \)-raft is the matroid \( M^*(K_{3,n}) \); and the last matroid is easily shown to be 2-minimally, 1-cominimally 3-connected. On combining these observations with (1.4.6), (1.4.8), and (1.4.9), we obtain (1.1.2). \( \Box \)

### 1.5 2-minimally 3-connected matroids

In the preceding two sections, we showed that both 2-minimally, 2-cominimally 3-connected matroids and 2-minimally, 1-cominimally 3-connected matroids have a familiar structure. The combination of (1.2.16), Theorem 1.1.1, and Theorem 1.1.2 implies the following theorem about their 4-cocircuits.

(1.5.1) **Theorem.** Let \( M \) be a 2-minimally, \( k \)-cominimally 3-connected matroid with \( |E(M)| \geq 5 \) and \( k \in \{1,2\} \). Then each pair of distinct elements of \( M \) is contained in a 4-cocircuit of \( M \).

In [3], Akkari and Oxley proved:

(1.5.2) **Theorem.** Let \( M \) be a matroid with \( |E(M)| \geq 4 \). Then \( M \) is 2-minimally connected if and only if each pair of distinct elements of \( M \) is contained in a triad.

By analogy with Theorem 1.5.2, one may hope that Theorem 1.5.1 can be extended to give that in all 2-minimally 3-connected matroids with at least five elements, every
pair of distinct elements is contained in a 4-cocircuit. The following example shows that this is false.

(1.5.3) Example. Let $A$ be the matrix over $GF(11)$ shown below and let $M$ be the matroid represented by $A$. Then every 2-element subset of $E(M)$ except $\{1, 2\}$ is in a 4-circuit. Using this, it is not difficult to check that $M$ is 2-cominimally 3-connected.

$$
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 4 & 0 & 3 \\
0 & 0 & 1 & 0 & 1 & 0 & 4 & 2 & 6 & 2 & 5 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 4 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 3 & 3
\end{bmatrix}
$$

In spite of this example, we do have the following result.

(1.5.4) Theorem. Let $M$ be a 2-minimally 3-connected matroid with $|E(M)| \geq 5$. If $M$ has a triangle, then there is at most one pair of distinct elements of $M$ that is not contained in a 4-cocircuit of $M$.

Proof. It is easy to check that there is no 5-element 2-minimally 3-connected matroid. Thus we may assume that $|E(M)| \geq 6$. Suppose that $T$ is a triangle of $M$ and $x, y \in E(M) - T$. We shall show that $x$ and $y$ are contained in a 4-cocircuit. By (1.2.4), the matroid $M \backslash x$ has two triads meeting $T$. Since $|E(M \backslash x)| \geq 5$, $T$ cannot be a triad of $M \backslash x$, otherwise $\{T, E(M \backslash x) - T\}$ is a 2-separation of the 3-connected matroid $M \backslash x$. Thus we deduce that $M$ has two 4-cocircuits, say $D_1^*$ and $D_2^*$, each of which contains $x$ and meets $T$ in exactly two elements. Similarly, $M$ has two 4-cocircuits, say $D_3^*$ and $D_4^*$, each of which contains $y$ and meets $T$ in exactly two elements. If $y$ is not contained in $D_1^* \cup D_2^*$, and $x$ is not contained in $D_3^* \cup D_4^*$, then, as $T$ has only three distinct
2-element subsets, we may assume that $D_1^* \cap T = D_2^* \cap T$. Applying circuit elimination to $D_1^*$ and $D_2^*$, it follows by orthogonality that the 4-element set $(D_1^* \cup D_2^*) - T$, which contains both $x$ and $y$, is a cocircuit of $M$. Therefore, each pair of distinct elements of $E(M) - T$ is contained in a 4-cocircuit of $M$.

By (1.4.4), $D_1^* \cap T \neq D_2^* \cap T$. Thus at most one 2-element subset of $T$ is not contained in some 4-cocircuit. Moreover, by (1.4.5), every 2-element subset of $E(M)$ that meets $T$ in a single element is contained in some 4-cocircuit. We conclude that $M$ has at most one pair of distinct elements that is not contained in a 4-cocircuit, and when such pair exists, it is a subset of $T$. □

By combining the last sentence of the proof of the preceding theorem with (1.4.3), we immediately obtain the following:

(1.5.5) Corollary. Let $M$ be a 2-minimally 3-connected matroid with $|E(M)| \geq 5$. If $M$ has two distinct triangles, then every pair of distinct elements of $M$ is contained in a 4-cocircuit of $M$.

1.6 Unavoidable matroids

In [5], Ding, Oporowski, Oxley, and Vertigan proved the following:

(1.6.1) Theorem. For every integer $n$ exceeding two, there is an integer $N(n)$ such that every 3-connected matroid with at least $N(n)$ elements has a minor isomorphic to $U_{n,n+2}$, $U_{2,n+2}$, $M(K_{3,n})$, $M^*(K_{3,n})$, the cycle matroid of a wheel with $n$ spokes, the whirl of rank $n$, or an $n$-spike.
By Tutte’s Wheels and Whirls Theorem [18], the minimally, cominimally 3-connected matroids are exactly wheels and whirls. By Theorem 1.1.1, the 2-minimally, 2-cominimally 3-connected matroids of rank more than six are exactly spikes with their tips deleted. By the dual of Theorem 1.1.2, the minimally, 2-cominimally 3-connected matroids of rank more than four are exactly the cycle matroids of $K_{3,n}$ with $n \geq 3$. In this section, we prove that, for each $n \geq 3$, the only $n$-minimally, 1-cominimally 3-connected matroid is $U_{2,n+2}$. Using all these results, Theorem 1.1.3 is just a restatement of Theorem 1.6.1.

(1.6.2) Proposition. Let $k$ be an integer exceeding two and $M$ be a $k$-minimally, 1-cominimally 3-connected matroid. If $|E(M)| \geq k + 4$, then no $(k+2)$-cocircuit of $M$ contains a triangle of $M$.

Proof. Suppose that $D^*$ is a $(k+2)$-cocircuit of $M$ and $T$ is a triangle contained in $D^*$. Let $X = D^* - T$. Then $|X| = k - 1$ and the matroid $M \setminus X$ is minimally 3-connected. Clearly, $T$ is both a triad and triangle of $M \setminus X$. As $|E(M)| \geq k + 4$, it follows that $|E(M \setminus X)| \geq 5$. Thus $(T, E(M \setminus X) - T)$ is a 2-separation of $M \setminus X$, a contradiction. □

(1.6.3) Proposition. Let $k$ be an integer exceeding two and $M$ be a $k$-minimally, 1-cominimally 3-connected matroid. Then $|E(M)| \leq k + 3$.

Proof. We argue by contradiction. Hence assume that $|E(M)| \geq k + 4$.

Suppose first that $k \geq 4$. Since $M^*$ is minimally 3-connected and $|E(M)| \geq 4$, it follows by (1.2.4) that $M$ has a triangle $T$. Let $X$ be a subset of $E(M)$ such that $T \subseteq X$ and $|X| = k - 1$. Since the matroid $M \setminus X$ is minimally 3-connected and $|E(M)| \geq 5$,
by (1.2.4), \( M \setminus X \) has a triad \( C^* \). Clearly, \( X \cup C^* \) is a \((k + 2)\)-cocircuit of \( M \) that contains a triangle, a contradiction to (1.6.2).

We may now suppose that \( k = 3 \). By (1.2.4), we may assume that \( T_1 \) and \( T_2 \) are distinct triangles of \( M \). If \( |T_1 \cap T_2| = 2 \), by circuit elimination, it is easy to show that \( M|(T_1 \cup T_2) \cong U_{2,4} \). Let \( e, f \) be distinct elements of \( E(M) - (T_1 \cup T_2) \). Then \( M \setminus e, f \) is minimally 3-connected. By (1.2.4), \( M \setminus e, f \) has a triad \( C^* \) meeting \( T_1 \). By orthogonality and the fact that \( M|(T_1 \cup T_2) \cong U_{2,4} \), \( C^* \) must be a subset of \( T_1 \cup T_2 \). Thus \( M \) has a 5-cocircuit \( C^* \cup \{e, f\} \) that contains a triangle, a contradiction. Therefore, \( |T_1 \cap T_2| \leq 1 \).

If \( |T_1 \cap T_2| = 1 \), let \( e \) be the element in \( T_1 \cap T_2 \) and \( X = T_1 - e \). Since \( M \setminus X \) is minimally 3-connected, it follows by (1.2.4) that \( M \setminus X \) has two distinct triads \( C_1^* \) and \( C_2^* \) meeting \( T_2 \). Since both \( C_1^* \cup X \) and \( C_2^* \cup X \) are 5-cocircuits of \( M \), it follows by (1.6.2) that \( e \notin C_1^* \cup C_2^* \). Thus, by orthogonality, \( |C_1^* \cap T_2| = |C_2^* \cap T_2| = 2 \), and hence \( C_1^* \cap T_2 = C_2^* \cap T_2 \). Let \( x \) be an element of \( C_1^* \cap T_2 \). Applying circuit elimination to \( C_1^* \) and \( C_2^* \), we deduce that \((C_1^* \cup C_2^*) - x\) contains a cocircuit of \( M \setminus X \). By orthogonality, \((C_1^* \cup C_2^*) - T_2\) contains a cocircuit of \( M \setminus X \); that is, \( M \setminus X \) has a cocircuit of size at most two, a contradiction. We conclude that no two distinct triangles of \( M \) meet.

Let \( U \) be the set of elements \( e \) of \( M \) for which \( e \) is not contained in a triangle. By the dual of (1.2.9), the matroid \( M \setminus V \) is 3-connected for every \( V \subseteq U \). Since \( k = 3 \), \( M \) is 3-minimally 3-connected. Thus \( |U| \leq 2 \). If \( |U| = 2 \), consider the matroid \( M \setminus U \).

Suppose that \( C^* \) is a triad of \( M \setminus U \). Since every element of \( C^* \) is in a triangle of \( M \setminus U \), it follows by orthogonality that \( C^* \) is a triad, a contradiction. Therefore, \( |U| \leq 1 \).

Let \( T_1 = \{a_1, b_1, c_1\} \) and \( T_2 \) be two distinct triangles and \( X = \{a_1, b_1\} \). Since the matroid \( M \setminus X \) is minimally 3-connected, it has two triads \( C_1^* \) and \( C_2^* \) meeting \( T_2 \). By
(1.6.2), neither $X \cup C_1^*$ nor $X \cup C_2^*$ contains either $T_1$ or $T_2$. By orthogonality, both $C_1^*$ and $C_2^*$ contain two elements of $T_2$ and one element that is not contained in any triangle. Since $|U| \leq 1$, we conclude that $U = \{e\}$, and $e \in C_1^* \cap C_2^*$. Applying circuit elimination to $C_1^*$ and $C_2^*$, we deduce that the set $D^* = (C_1^* \cup C_2^*) - e$ contains a cocircuit of $M \setminus X$. By the fact that $|D^*| = 5$ and $T_2 \subseteq D^*$, it follows that $T_2$ is a triad of $M \setminus X$, a contradiction. □

(1.6.4) Theorem. Let $k$ be an integer exceeding two and $M$ be a $k$-minimally, 1-cominimally 3-connected matroid. Then $M \cong U_{2,k+2}$.

Proof. By (1.6.3), $|E(M)| \leq k + 3$. Let $X$ be a subset of $E(M)$ such that $|X| = k - 1$. Then $|E(M \setminus X)| \leq 4$. But $M \setminus X$ is minimally 3-connected. Hence $M \setminus X \cong U_{2,3}$. Therefore, each 3-element set of $E(M)$ is a triangle. Thus $M \cong U_{2,k+2}$. □
CHAPTER II
THE NUMBER OF n–SPIKES OVER FINITE FIELDS

2.1 Introduction

Spikes are appearing with increasing frequency in the matroid theory literature. Long before the name “spike” was introduced, the Fano and non-Fano matroids, two examples of 3–spikes, had already appeared in almost every corner of matroid theory [13]. Oxley [13, Section 11.2] showed that all rank–n, 3–connected binary matroids without a 4–wheel minor can be obtained from a binary n–spike by deleting at most two elements. Oxley, Vertigan, and Whittle [14] used spikes and one other class of matroids to show that, for all \( q \geq 7 \), there is no fixed bound on the number of inequivalent \( GF(q) \)-representations of a 3–connected matroid, thereby disproving a conjecture of Kahn [6].

Ding, Oporowski, Oxley, and Vertigan [5] showed that every sufficiently large 3–connected matroid has, as a minor, \( U_{2,n+2}, U_{n,n+2} \), a wheel or whirl of rank \( n \), \( M(K_{3,n}) \), \( M^*(K_{3,n}) \), or an \( n \)-spike. Moreover, Wu [20] showed that spikes, like wheels and whirls, can be characterized in terms of a natural extremal connectivity condition. This chapter studies the representability of spikes over finite fields.

For \( n \geq 3 \), a matroid \( M \) is an \( n \)-spike with tip \( t \) [5] if it satisfies the following three conditions:
(i) the ground set is the union of \( n \) lines, \( L_1, L_2, \ldots, L_n \), all having three points and passing through a common point \( t \);

(ii) for all \( k \) in \( \{1, 2, \ldots, n-1\} \), the union of any \( k \) of \( L_1, L_2, \ldots, L_n \) has rank \( k+1 \);

and

(iii) \( r(L_1 \cup L_2 \cup \ldots \cup L_n) = n \).

In this chapter, an \( n \)-spike with tip \( t \) will be simply called an \( n \)-spike.

Some \( 3 \)-spikes have the property that more than one element may be viewed as the tip of the spike. However, it is clear that the tip is unique for an \( n \)-spike when \( n \geq 4 \).

Since there are only six \( 3 \)-spikes, and it is easy to verify all our results for the case \( n = 3 \), we will assume that \( n \) is at least four in the proofs of our theorems so that we can fix the tip.

For an \( n \)-spike \( M \) representable over a field \( F \), if we choose a base \( \{1, 2, \ldots, n\} \) containing exactly one element from each of the lines \( L_i \), then \( M \) can be represented in the form

\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 1 & 1 + x_1 & 1 & 1 & \cdots & 1 \\
2 & 0 & 1 & 0 & \cdots & 0 & 1 & 1 & 1 + x_2 & 1 & \cdots & 1 \\
3 & 0 & 0 & 1 & \cdots & 0 & 1 & 1 & 1 + x_3 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
n & 0 & 0 & 0 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 + x_n
\end{bmatrix},
\]

where the tip of \( M \) corresponds to column \( n + 1 \). We shall call this matrix a special standard representation of \( M \) and \( \{1, 2, \ldots, n\} \) the distinguished basis associated with the representation. Clearly, this matrix is uniquely determined by the vector \((x_1, x_2, \ldots, x_n)\). We shall call this vector the diagonal of the representation.
Two matrix representations $A_1$ and $A_2$ are \textit{equivalent} if $A_1$ can be obtained from $A_2$ by a sequence of the following six operations. (For details, see [13, Section 6.3].)

(i) \textit{Interchange two rows.}

(ii) \textit{Scale a row, that is, multiply it by a non-zero member of $F$.}

(iii) \textit{Replace a row by the sum of that row and another.}

(iv) \textit{Interchange two columns (moving their labels with the columns).}

(v) \textit{Scale a column, that is, multiply it by a non-zero member of $F$.}

(vi) \textit{Replace each entry of the matrix by its image under some automorphism of $F$.}

$A_1$ and $A_2$ are \textit{weakly equivalent} if we are also allowed to relabel the matroid, that is, $A_1$ can be obtained from $A_2$ by a sequence of operations (i) - (vii) where the last of these operations is the following:

(vii) \textit{Relabel the columns.}

Since our main purpose is to count the number of distinct non-isomorphic spikes, we will often consider unlabeled matroids. Thus, we will frequently ignore the labels on elements of matroids, and consider weak equivalence.

If two special standard representations are weakly equivalent, their corresponding diagonals will also be said to be \textit{weakly equivalent}. Two diagonals are \textit{distinct} if they are not weakly equivalent. Two elements of an $n$-spike are \textit{conjugate} if they lie on the same line $L_i$ and neither of them is the tip. In a special standard representation of a given spike, if we interchange some base elements with their conjugates, and standardize
the resulting matrix, we obtain another special standard representation of the spike. Moreover, all possible special standard representations of the spike are obtainable in this way. In the rest of the chapter, we shall call this interchanging-standardizing procedure \textit{swapping}. For two special standard representations $A_1$ and $A_2$ of an $n$-spike, the distinguished bases of $M[A_1]$ and $M[A_2]$ are $n$-element subsets intersecting all the lines $L_i$. Since the tip is fixed and is in neither distinguished basis, $A_1$ and $A_2$ are weakly equivalent if and only if we can obtain the distinguished basis of $M[A_1]$ from that of $M[A_2]$ by swappings. Therefore, $A_1$ and $A_2$ are weakly equivalent if and only if $A_1$ can be obtained from $A_2$ by a sequence of swappings, and replacing each entry of the resulting matrix by its image under some automorphism of the field $F$.

In the rest of this chapter, the matroid notation and terminology will follow Oxley [13]. Quaternary and quinternary matroids are those representable over $GF(4)$ and $GF(5)$, respectively. We denote by $p_k(n)$ the number of partitions of the integer $n$ into exactly $k$ parts, and by $p_{\leq k}(n)$ the number of partitions of $n$ into at most $k$ parts. We also use $p_k^{(\neq)}(n)$ to denote the number of partitions of $n$ into exactly $k$ parts all of which are distinct. The following are the main results of this chapter. They will be proved in Sections 2.3 and 2.4. The first three theorems determine the exact numbers of non-isomorphic $n$-spikes representable over $GF(3)$, $GF(4)$, and $GF(5)$, respectively.

(2.1.1) \textbf{Theorem.} \textit{For each integer }$n \geq 3$, \textit{there are exactly two distinct ternary }$n$-\textit{spikes.}

(2.1.2) \textbf{Theorem.} \textit{For each integer }$n \geq 3$, \textit{the number of distinct quaternary }$n$-\textit{spikes is }$\left\lfloor \frac{n^2 + 6n + 24}{12} \right\rfloor$. 

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(2.1.3) Theorem. For each integer \( n \geq 4 \), the number of distinct quaternary \( n \)-spikes is \( n + 2 + \lfloor \frac{n}{3} \rfloor \), while there are five distinct quaternary 3-spikes.

Over \( GF(7) \), we can also count the number of non-isomorphic \( n \)-spikes exactly, provided \( n \geq 18 \).

(2.1.4) Theorem. For each integer \( n \geq 18 \), the number of distinct \( n \)-spikes representable over \( GF(7) \) is \( \lceil \frac{2n^2+6n+8}{3} \rceil \).

The next three theorems give asymptotic values for the number \( N(n, q) \) of non-isomorphic \( GF(q) \) representable \( n \)-spikes when \( q \) is an odd prime, a power of an odd prime, and a power of two, respectively.

(2.1.5) Theorem. Let \( q \) be a prime number greater than five, and \( n \) be an integer greater than or equal to \( \frac{(q-1)^2}{2} \). Then

\[
(i) \quad N(n, q) \geq \left( \frac{n+1}{2} \right) \left( \frac{q-3}{2} \right)! p_{\frac{n-1}{2}} \left( n - \frac{(q-1)(q-3)}{8} \right);
\]

\[
(ii) \quad N(n, q) \geq \left( \frac{n+1}{q-1} \right) \left( \frac{q-3}{2} \right)! \left( n - 1 - \frac{(q-1)(q-3)}{8} \right) \left( \frac{q-3}{2} \right) \left( \frac{q}{2} \right);
\]

\[
(iii) \quad N(n, q) \leq \left( \frac{n+1}{2} \right) \left( \frac{q-3}{2} \right)! p_{\frac{n-1}{2}} \left( n + \frac{q-1}{2} \right);
\]

\[
(iv) \quad \lim_{n \to \infty} \frac{N(n, q)}{n(q-3)^{\frac{1}{2}}} = \left( \frac{q+1}{2} \right)^{\frac{1}{2}} \left( \frac{(q-1)/2)!}{(q-1/2)!} \right).
\]

(2.1.6) Theorem. Let \( n \) be a positive integer, \( p \) be an odd prime, and \( q = p^a \). Then

\[
(i) \quad N(n, q) \geq \frac{1}{\prod_{i=1}^{a} (q-p^i)} \left( \frac{q-3}{2} \right)! p_{\frac{n-1}{2}} \left( n - \frac{(q-1)(q-3)}{2} - \frac{(q-1)(q-1)}{8} \right);
\]

\[
(ii) \quad \lim_{n \to \infty} \frac{N(n, q)}{n(q-3)^{\frac{1}{2}}} = \frac{1}{\prod_{i=1}^{a-1} (q-p^i)} \left( \frac{q+1}{2} \right)^{\frac{1}{2}} \left( \frac{(q-1)/2)!}{(q-1/2)!} \right).
\]

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(2.1.7) Theorem. Let $n$ be a positive integer, and $q = 2^s$. Then

(i) $N(n, q) \geq \frac{(q-2)!}{\prod_{i=1}^{s-1}(q-2^i)} f_{q-1} (n - \frac{(q-1)(q-2)}{2})$;

(ii) $\lim_{n \to \infty} \frac{N(n, q)}{n^{q-2}} = \frac{1}{(q-1)! \prod_{i=1}^{s-1}(q-2^i)}$.

2.2 Preliminaries

In the remainder of this chapter, let $F$ be the finite field $GF(q)$, and $d$ be an element of $F \setminus \{0, -1\}$. Let

$$f_d : F \to F$$

be defined by $f_d(x) = (1 + d^{-1})x$ for each $x \in F$. Moreover, let $f_0$ denote the identity mapping on $F$, and $\mathcal{F}$ denote the set $\{f_0\} \cup \{f_d : d \in F \setminus \{0, -1\}\}$.

The following lemma will play an important part in the proofs of the theorems. Its straightforward proof is omitted.

(2.2.1) Lemma. For each $d$ in $F \setminus \{0, -1\}$, the function $f_d$ satisfies the following:

(i) $f_d$ is a bijection that fixes 0;

(ii) $f_d(d) = d + 1$;

(iii) if $x + y = 0$, then $f_d(x) + f_d(y) = 0$;

(iv) if $f_{d_1}(x) = f_{d_2}(x)$ for some $x \neq 0$, then $d_1 = d_2$;

(v) $(f_d)^{-1} = f_{-(1+d)}$;

(vi) if $d_1 + d_2 \neq -1$, then $f_{d_1} \circ f_{d_2} = f_{\frac{d_1+d_2}{d_1+d_2+1}}$.
The next lemma is not difficult to prove by using Lemma 2.2.1 and induction. Again we omit the proof.

(2.2.2) Lemma. Suppose that \( d_1, d_2, \ldots, d_m \in F \setminus \{0, -1\} \), and \( g : F \to F \) is defined by \( g = f_{d_m} \circ f_{d_{m-1}} \circ \cdots \circ f_{d_1} \). If there is an \( x \in F \setminus \{0\} \) such that \( g(x) = x \), then \( g \) is the identity mapping on \( F \). Otherwise, there is an element \( d \) of \( F \setminus \{0, -1\} \) such that \( g = f_d \).

(2.2.3) Lemma. Suppose that \( A \) is a special standard representation of an \( n \)-spike, and let its diagonal \( \tilde{v} \) be \((x_1, x_2, \ldots, x_n)\). Suppose that \( x_1 \neq -1 \). Then the diagonal of the representation obtained by swapping the element corresponding to \( x_1 \) and its conjugate is

\[
\tilde{v}' = (f_{x_1}(-x_1), f_{x_1}(x_2), \ldots, f_{x_1}(x_n)).
\]

Proof. Note that by the assumption that \( x_1 \neq -1 \) and the fact that 0 is not in any diagonal, both \( 1 + x_1 \) and \( -x_1 \) are invertible. To achieve the desired swapping of columns \( n + 2 \) and 1 in the representation, we pivot on \( 1 + x_1 \), the first entry of column \( n + 2 \). Recall that the pivot operation includes the natural column interchange to maintain a representation in standard form, we deduce that this pivot produces the representation

\[
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
-(1 + x_1)^{-1} & 1 + x_2 - (1 + x_1)^{-1} & \cdots & x_1(1 + x_1)^{-1} \\
x_1(1 + x_1)^{-1} & -(1 + x_1)^{-1} & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
x_1(1 + x_1)^{-1} & -(1 + x_1)^{-1} & x_1(1 + x_1)^{-1} & \cdots & 1 + x_n - (1 + x_1)^{-1}
\end{bmatrix}
\]

Next we put this representation in special standard form. To achieve this, first multiply
all but the first row by the inverse of $1 - (1 + x_1)^{-1}$. Then scale each of the first $n$
columns to normalize it. The resulting matrix is

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & \cdots & 1 \\
1 & -x_1^{-1} & 1 + f_{x_1}(x_2) & 1 & \cdots & 1 \\
I_n & 1 & -x_1^{-1} & 1 & 1 + f_{x_1}(x_3) & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & -x_1^{-1} & 1 & 1 & \cdots & 1 & 1 + f_{x_1}(x_n)
\end{bmatrix}
$$

Since $f_{x_1}(-x_1) = -(1 + x_1)$, by multiplying column $(n + 2)$ by $-x_1$, we obtain

$$
\begin{bmatrix}
1 & 1 + f_{x_1}(-x_1) & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 + f_{x_1}(x_2) & 1 & \cdots & 1 \\
I_n & 1 & 1 & 1 + f_{x_1}(x_3) & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & 1 & \cdots & 1 + f_{x_1}(x_n)
\end{bmatrix}
$$

This matrix is in special standard form and the resulting diagonal is indeed $\bar{v}'$. □

(2.2.4) Corollary. If $x_1 \neq -1$, then the special standard representation corresponding
to the diagonal

$$\bar{v} = (x_1, x_2, \ldots, x_n)$$

is weakly equivalent to the special standard representation corresponding to the diagonal

$$\bar{v}' = (f_{x_1}(-x_1), f_{x_1}(x_2), \ldots, f_{x_1}(x_n)).$$

(2.2.5) Proposition. The special standard representation corresponding to the dia-

$$\bar{v} = (-1, -1, x_3, \ldots, x_n)$$
is weakly equivalent to the special standard representation corresponding to the diagonal

\[ \tilde{v}' = (-1, -1, -x_3, \ldots, -x_n). \]

**Proof.** When \( x_1 = x_2 = -1 \), both \( 1 + x_1 \) and \( 1 + x_2 \) are zero so we cannot pivot on these entries in the matrix. In this case, we perform two successive pivots, the first on the first entry of column \( n+3 \), and the second on the second entry of column \( n+2 \). The effect of these pivots, with their included natural column interchanges, is to interchange columns \( n+2 \) and \( n+3 \) with their conjugates, columns 2 and 1, respectively. By scaling the matrix obtained from these two pivots to put it in special standard form, we obtain a special standard representation for which the corresponding diagonal is \( \tilde{v}' \). \( \square \)

From now on, we will call the swapping in (2.2.3) a 1-swapping and denote it by \( s_{x_1} \), and call the swapping in (2.2.5) a 2-swapping and denote it by \( s_{-1} \). Moreover, we denote by \( f_{-1} \) the mapping over \( GF(q) \) satisfying \( f_{-1}(d) = -d \) for each \( d \) in \( GF(q) \).

Suppose \( F\{0\} = \{d_1, d_2, \ldots, d_{q-1}\} \). Let \( d^{(k)}_i \) denote a \( k \)-tuple every entry of which equals \( d_i \). Since changing the order of the components of a diagonal will result in a diagonal of a weakly equivalent standard representation, every diagonal is weakly equivalent to one of the form

\[ (d^{(k_1)}_1, d^{(k_2)}_2, \ldots, d^{(k_{q-1})}_{q-1}), \]

where \( k_i \) is the number of \( d_i \)'s appearing in the diagonal. We shall call the tuple in the first place, namely \( d^{(k_1)}_1 \), the *first tuple of the diagonal*, and the tuple in the second place the *second tuple of the diagonal*, and so on.
Since we will use the tuple notation in the remainder of the chapter, we shall re-interpret (2.2.3), (2.2.4), and (2.2.5) in terms of the tuple notation. Suppose first that \( q = 2^s \), where \( s > 1 \). In this case, (2.2.3) asserts that the \( 1 \)-swapping \( s_{d_1} \) will change the diagonal
\[
(d_1^{(k_1)}, d_2^{(k_2)}, \ldots, d_{q-1}^{(k_{q-1})})
\]
to the diagonal
\[
(f_{d_1}(d_1)^{(k_1)}, f_{d_1}(d_2)^{(k_2)}, \ldots, f_{d_1}(d_{q-1})^{(k_{q-1})}).
\]
Thus these two diagonals are weakly equivalent. Since \( GF(2^s) \) has characteristic two, we have \( a = -a \) for each \( a \) in the field. Therefore, (2.2.5) says that the \( 2 \)-swapping \( s_{-1} \) has no impact on diagonals.

Now we consider the case that \( q = p^s \) where \( p \) is an odd prime. Since changing the order of components of a diagonal results in a weakly equivalent diagonal, we may assume that, in this case, \( d_{x^{-1}+i} = -d_i \) for each \( i \) in \( \{1, 2, \ldots, \frac{q-1}{2}\} \), and the general form of a diagonal is
\[
\vec{v} = (d_1^{(k_1)}, (-d_1)^{(l_1)}, d_2^{(k_2)}, (-d_2)^{(l_2)}, \ldots, (d_{\frac{q-1}{2}})^{(k_{\frac{q-1}{2}})}, (-d_{\frac{q-1}{2}})^{(l_{\frac{q-1}{2}})}).
\]
Suppose that \( d_1 \neq -1 \), and \( k_1 \geq 1 \). Then (2.2.3) says that the \( 1 \)-swapping \( s_{d_1} \) changes \( \vec{v} \) to the diagonal
\[
(f_{d_1}(d_1)^{(k_1-1)}, f_{d_1}(-d_1)^{(l_1+1)}, f_{d_1}(d_2)^{(k_2)}, f_{d_1}(-d_2)^{(l_2)}, \ldots, f_{d_1}(-d_{\frac{q-1}{2}})^{(l_{\frac{q-1}{2}})}),
\]
thereby showing that the last diagonal is weakly equivalent to \( \vec{v} \).

On the other hand, if \( d_1 = -1 \) and \( k_1 \geq 2 \), then (2.2.5) says that the \( 2 \)-swapping \( s_{-1} \) changes \( \vec{v} \) to the diagonal

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thereby showing that the last diagonal is weakly equivalent to \( \vec{v} \).

The following proposition is not hard to prove by induction. We shall omit the proof.

(2.2.6) Proposition. Suppose that \( n \) is a positive integer and that \( x_i \neq 0 \) for all \( i \in \{1, 2, \ldots, n\} \). Then the determinant of the matrix

\[
\det \begin{pmatrix}
1 + x_1 & 1 & 1 & \cdots & 1 \\
1 & 1 + x_2 & 1 & \cdots & 1 \\
1 & 1 & 1 + x_3 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 1 + x_n
\end{pmatrix} = [1 + \sum_{i=1}^{n} x_i^{-1}] \cdot \prod_{i=1}^{n} x_i.
\]

(2.2.7) Proposition. Let \( q \) be an odd prime and \( F = GF(q) \). Let \( g_0 = f_0 \) and, for all \( k \in \{1, 2, \ldots, q - 2\} \), let \( g_k \) be the mapping \( f_k \circ g_{k-1} \) on \( F \). Moreover, for each \( k \) in \( F \setminus \{q - 1\} \), let \( h_k \) be \( g_{q-k-2} \circ f_{q-1}^2 \). Then

(i) \( g_k = f_{k-1} \) for each \( k \) in \( \{1, 2, \ldots, q - 2\} \);

(ii) \( \{g_0, g_1, \ldots, g_{q-2}\} = F \); and

(iii) \( h_k = g_k \) for each \( k \) in \( \{0, 1, \ldots, q - 2\} \).

Proof. By definition, it is easy to see that \( g_1 = f_1 \). Suppose that \( g_k = f_{k-1} \) for some integer \( k \geq 2 \). Then,

\[
g_{k+1} = f_{k+1} \circ g_k = f_{k+1} \circ f_{k-1}.
\]
By (2.2.1)(vi),
\[ f_{k+1} \circ f_{k-1} = f_{\frac{k+1-k-1}{k+1-k-1+1}} = f_{(k+1)-1}. \]

Part (i) follows by induction. Moreover, by (i), \( g_k = g_l \) if and only if \( k = l \). Part (ii) follows immediately.

By (2.2.1)(v), \( f_{-1} \) is the inverse of itself. Since \( f_1^2(x) = -x = f_{-1}(x) \) for all \( x \) in \( F \), we conclude that \( f_1^2 = f_{-1} \). Therefore,
\[ h_0 = g_{-2} \circ f_{-\frac{1}{2}} = f_{-\frac{1}{2}} \circ f_{-\frac{1}{2}} = f_0. \]

Moreover, for each \( k \) in \( \{1, 2, \ldots, q-2\} \), by part (i) and (2.2.1)(vi),
\[ h_k = g_{q-k-2} \circ f_{-\frac{1}{2}} = f_{-\frac{1}{k+1}} \circ f_{-\frac{1}{2}} = f_{k-1} = g_k. \]

(2.2.8) Proposition. Suppose that \( p \) is an odd prime, \( q = p^s \), and
\[ \bar{v} = (d_1^{(k_1)}, (-d_1)^{(l_1)}, d_2^{(k_2)}, (-d_2)^{(l_2)}, \ldots, (d_{\frac{q-1}{2}})^{(k_{\frac{q-1}{2}})}, (-d_{\frac{q-1}{2}})^{(l_{\frac{q-1}{2}})}) \]
is the diagonal of a special standard representation of an \( n \)-spike over \( GF(q) \). Suppose that \( k_1 \geq p \). Then \( \bar{v} \) is weakly equivalent to
\[ \bar{v}' = (d_1^{(k_1-p)}, (-d_1)^{(l_1+p)}, d_2^{(k_2)}, (-d_2)^{(l_2)}, \ldots, (d_{\frac{q-1}{2}})^{(k_{\frac{q-1}{2}})}, (-d_{\frac{q-1}{2}})^{(l_{\frac{q-1}{2}})}). \]

Proof. First consider the case that \( d_1 \in GF(q) \backslash GF(p) \). Since \( k_1 \geq 1 \), we can do the 1-swapping \( s_{d_1} \) which swaps an element of the first tuple with its conjugate. By (2.2.3), we deduce that the diagonal resulting from the last swapping is
\[ (f_{d_1}(d_1)^{(k_1-1)}, f_{d_1}(-d_1)^{(l_1+1)}, f_{d_1}(d_2)^{(k_2)}, f_{d_1}(-d_2)^{(l_2)}, \ldots, f_{d_1}(d_{\frac{q-1}{2}})^{(k_{\frac{q-1}{2}})}, f_{d_1}(-d_{\frac{q-1}{2}})^{(l_{\frac{q-1}{2}})}). \]
By (2.2.1)(ii), we deduce that \( f_{d_1}(d_1) = d_1 + 1 \). We conclude that:

\[ (2.2.8.1) \text{ a 1-swapping corresponding to the first tuple increases the base of that tuple by one and decreases the length of that tuple by one.} \]

We continue to swap elements of the first tuple with their conjugates, one at a time, until the total number of 1-swappings we have done is \( p \). In other words, we do the sequence of 1-swappings \( s_{d_1}, s_{d_1+1}, \ldots, s_{d_1+p-1} \). Let \( f = f_{d_1+p-1} \circ f_{d_1+p-2} \circ \ldots \circ f_{d_1+1} \circ f_{d_1} \). By (2.2.8.1), we deduce that the diagonal resulting from the last sequence of swappings is

\[ (f(d_1)^{(k_1-p)}, f(-d_1)^{(l_1+p)}, f(d_2)^{(k_2)}, f(-d_2)^{(l_2)}, \ldots, f(d_{2^k-1})^{(k_{2^k-1})}, f(-d_{2^k-1})^{(l_{2^k-1})}). \]

By (2.2.1)(vi), we deduce that \( f \in \mathcal{F} \). By (2.2.1)(ii), it follows that \( f(d_1) = d_1 + p = d_1 \). Therefore, by (2.2.2), \( f = f_0 \), and we conclude that the last diagonal is indeed \( u' \).

Now consider the case that \( d_1 \in GF(p) \setminus \{0\} \). From (2.2.8.1), we know the effect of doing a 1-swapping corresponding to the first tuple. However, at a certain point after doing a sequence of such 1-swappings, the base of the first tuple will become \( p - 1 \). In this situation, since the first entry of the corresponding column of the matrix representation is \( 1 + p - 1 = 0 \), we cannot do a 1-swapping on the first tuple. When this occurs, we first do the sequence of 1-swappings \( s_{d_1}, s_{d_1+1}, \ldots, s_{p-2} \). Clearly, each of the swappings of the last sequence swaps one element of the first tuple with its conjugate. Let \( f = f_{p-2} \circ f_{p-3} \circ \ldots \circ f_{d_1+1} \circ f_{d_1} \). Then it follows by (2.2.1)(ii) that \( f(d_1) = p - 1 \). By (2.2.3), the diagonal resulting from the last sequence of 1-swappings is

\[ (f(d_1)^{(k_1-(p-d_1-1))}, f(-d_1)^{(l_1+(p-d_1-1))}, f(d_2)^{(k_2)}, f(-d_2)^{(l_2)}, \ldots, f(-d_{2^k-1})^{(l_{2^k-1})}). \]

Since \( f(d_1) = -1 \), we can now do the 2-swapping \( s_{-1} \) which swaps two elements of the first tuple with their conjugates. Let \( g = f_{-1} \circ f \). It follows by (2.2.5) that the
diagonal resulting from this 2-swapping is

\[(g(d_1)^{k_1-p+d_1-1}, g(d_1)^{(l_1+p-d_1+1)}, g(d_2)^{(k_2)}, g(-d_2)^{(l_2)}, \ldots, g(d_{x-1})^{(k_{x-1})}, g(-d_{x-1})^{(l_{x-1})})\].

Since \(g(d_1) = f_{-1}(-1) = 1\), we are now able to continue to do 1-swappings corresponding to the first tuple. Clearly, the sequence of 1-swappings \(s_1, s_2, \ldots, s_{d_1-1}\) swaps \(d_1 - 1\) elements of the first tuple with their conjugates. Let \(h = f_{d_1-1} \circ f_{d_2-2} \circ \ldots \circ f_2 \circ f_1 \circ g\).

Then, by (2.2.3), the last sequence of 1-swappings changes the last diagonal to

\[(h(d_1)^{k_1-p}), h(-d_1)^{(l_1+p)}, h(d_2)^{(k_2)}, h(-d_2)^{(l_2)}, \ldots, h(d_{x-1})^{(k_{x-1})}, h(-d_{x-1})^{(l_{x-1})})\].

By the definition of \(f_{-1}\), it is easy to see that \(f_{-1} = f_{-\frac{1}{2}}\). Therefore, \(h\) is a composition of mappings of \(\mathcal{F}\). By (2.2.2), we deduce that \(h \in \mathcal{F}\). By (2.2.1)(ii), \(h(d_1) = d_1\).

We deduce by (2.2.2) that \(h = f_0\). Therefore, the last diagonal is indeed \(v'\) and the proposition follows immediately. \(\square\)

(2.2.9) Lemma. Suppose that both \(n\) and \(k\) are positive integers. Then

(i) \(p_k(n) \geq \frac{1}{k!} \binom{n-1}{k-1}\);  

(ii) \(p_{\leq k}(n) = p_k(n+k)\);  

(iii) \(p_{\leq k-1}(n) = p_k(n - \frac{k(k-1)}{2})\);  

(iv) \(p_k(n) = \frac{1}{(k-1)!} n^{k-1} + c_{k-2} n^{k-2} + \ldots + c_1 n + c_0\), where \(c_0, c_1, \ldots, c_{k-2}\) depend only on \(k\) and the congruence class (modulo \(k!\)) of \(n\).

Proof. Suppose that we have \(n\) balls arranged as a sequence, and we have \(k-1\) separators. There are \(n-1\) places between two consecutive balls that are allowable
places for the separators. Therefore, there are \( \binom{n-1}{k-1} \) ways of placing the separators.

This proves that there are \( \binom{n-1}{k-1} \) ways of writing \( n \) as an ordered sum of exactly \( k \) positive integers. Part (i) follows easily.

Parts (ii) and (iii) are well known and part (iv) can be proved by induction. For details, see [7, Section 8.6].

\[\square\]

2.3 Proofs of Theorems 2.1.1 through 2.1.5

Recall that two diagonals are distinct exactly when they are not weakly equivalent. This terminology will be used repeatedly throughout this section.

Proof of Theorem 2.1.1. Let \( \vec{v} = (1^{(k)}, (-1)^{(n-k)}) \) be the diagonal of a special standard representation. Suppose that \( k \geq 3 \). By (2.2.4), \( \vec{v} \) is weakly equivalent to \( ((-1)^{(k-1)}, 1^{(n-k+1)}) \) under the 1-swapping \( s_1 \). Moreover, by (2.2.5), the 2-swapping \( s_{-1} \) shows that \( ((-1)^{(k-1)}, 1^{(n-k+1)}) \) is weakly equivalent to \( (1^{(k-3)}, (-1)^{(n-k+3)}) \). We shall call the series of swappings \( s_1, s_{-1} \) a 3-shift. By applying 3-shifts, it is clear that there are at most three different special standard representations of ternary \( n \)-spikes for each \( n \), namely those corresponding to \( (1^{(0)}, (-1)^{(n)}), (1^{(1)}, (-1)^{(n-1)}), \) and \( (1^{(2)}, (-1)^{(n-2)}) \).

If \( n - k \geq 3 \), then, by applying the swapping \( s_{-1} \) followed by \( s_1 \), we change the diagonal \( (1^{(k)}, (-1)^{(n-k)}) \) to \( (1^{(k+3)}, (-1)^{(n-k-3)}) \). We shall call this composition a 3-back-shift. It is easy to see that a diagonal will not change under either two consecutive applications of \( s_1 \) or under two consecutive applications of \( s_{-1} \). Therefore, a sequence of
1-swappings and 2-swappings is equivalent to either a series of 3-shifts or 3-back-shifts, or a 1-swapping or 2-swapping followed by a series of 3-shifts or 3-back-shifts.

Suppose that $m$ is an integer and $n = 3m$. Then the diagonal $(1^{(0)}, (-1)^{(3m)})$ is weakly equivalent to the diagonal $(1^{(3m)}, (-1)^{(0)})$ by a series of $m$ 3-back-shifts. Moreover, a 1-swapping shows that $(1^{(3m)}, (-1)^{(0)})$ is weakly equivalent to $((-1)^{(3m-1)}, 1^{(1)})$. Therefore, the diagonals $(1^{(0)}, (-1)^{(3m)})$ and $(1^{(1)}, (-1)^{(3m-1)})$ are weakly equivalent. On the other hand, a 1-swapping changes $(1^{(2)}, (-1)^{(3m-2)})$ to $((-1)^{(1)}, 1^{(3m-1)})$, and a 2-swapping changes $(1^{(2)}, (-1)^{(3m-2)})$ to $((-1)^{(4)}, 1^{(3m-4)})$. Since $3m - 4 \equiv 3m - 1 \equiv 2 \text{ (mod 3)}$, the discussion in the last paragraph proves that if $(1^{(2)}, (-1)^{(3m-2)})$ is weakly equivalent to $(1^{(k)}, (-1)^{(3m-k)})$, then $k \equiv 2 \text{ (mod 3)}$. Therefore, there are exactly two distinct diagonals for each $n = 3m$. Since there is no non-trivial automorphism of $GF(3)$, there are exactly two distinct special standard representations when $n = 3m$. Similarly, there are exactly two distinct special standard representations for each of the cases $n = 3m + 1$ and $n = 3m + 2$. By the unique representability of ternary matroids over $GF(3)$ [13, Section 10.1], we conclude that there are exactly two ternary $n$-spikes for each integer $n \geq 3$. $\Box$

Proof of Theorem 2.1.2. Let $\bar{v} = (1^{(k)}, \omega^{(l)}, (\omega + 1)^{(m)})$ be the diagonal of a special standard representation of a quaternary $n$-spike. By (2.2.5) and the fact that the field $GF(4)$ is of characteristic two, 2-swappings have no impact on the diagonals. Moreover, in the case that $l = m = 0$, no 1-swapping is defined. Thus the diagonal $(1^{(n)}, \omega^{(0)}, (\omega + 1)^{(0)})$ is not weakly equivalent to any other diagonal. On the other hand, in the case that $k = l = 0$, since the only 1-swapping is $f_{\omega+1}$, it follows by
(2.2.1)(ii) and (iii) and the fact that there is a unique non-trivial automorphism of $GF(4)$, that the only diagonal to which $(\mathbf{1}^{(0)}, \omega^{(0)}, (\omega + 1)^{(n)})$ is weakly equivalent is $(\mathbf{1}^{(0)}, \omega^{(n)}, (\omega + 1)^{(0)})$. Therefore, the three cases in which exactly one of $k, l,$ and $m$ is positive produce a total of two distinct diagonals.

We now show that if at least two of $k, l,$ and $m$ are positive, and \{a, b, c\} = \{k, l, m\}, then $\bar{v}$ is weakly equivalent to $((\mathbf{1}^{(a)}, \omega^{(b)}, (\omega + 1)^{(c)}))$. Suppose that both $k$ and $l$ are positive. Since $l > 0$, we can do the $1$-swapping $f_\omega$ to the diagonal $\bar{v}$. By (2.2.4), $\bar{v}$ is weakly equivalent to $(\omega^{(k)}, (\omega + 1)^{(l)}, 1^{(m)})$ under this $1$-swapping. Since $k > 0$, the $1$-swapping $f_\omega$ shows that $(\omega^{(k)}, (\omega + 1)^{(l)}, 1^{(m)})$ is weakly equivalent to $((\omega + 1)^{(k)}, 1^{(l)}, \omega^{(m)})$. Therefore, $(1^{(k)}, \omega^{(l)}, (\omega + 1)^{(m)}), (1^{(m)}, \omega^{(k)}, (\omega + 1)^{(l)}), (1^{(l)}, \omega^{(m)}, (\omega + 1)^{(k)})$ are weakly equivalent to each other. Under the only non-trivial automorphism of $GF(4)$, these three diagonals are weakly equivalent to $(1^{(k)}, \omega^{(m)}, (\omega + 1)^{(l)}), (1^{(m)}, \omega^{(l)}, (\omega + 1)^{(k)}), (1^{(l)}, \omega^{(k)}, (\omega + 1)^{(m)})$, respectively. Therefore, the last six diagonals are weakly equivalent to each other. In the case that $k$ or $l$ is zero, a similar argument to the above will yield the same conclusion. By (2.2.1)(i), (iii), and the fact that the characteristic of $GF(4)$ is two, no other diagonal is weakly equivalent to any of the above six diagonals. Therefore, the total number of different diagonals is $2 + p_2(n) + p_3(n)$. Calculating this number for each of the cases $n \equiv k(mod 6)$ for $k \in \{0, 1, 2, 3, 4, 5\}$, or using a result of [7, Section 8.6], we obtain that $2 + p_2(n) + p_3(n)$ is equal to $[n^2 + 6n + 24 \over 12]$. Since spikes are $3$-connected and $3$-connected quaternary matroids are uniquely representable over $GF(4)$ [6], it follows that the number of quaternary $n$–spikes is indeed $[n^2 + 6n + 24 \over 12]$. □

In order to prove Theorems 2.1.3, 2.1.4, and 2.1.5, we require some preliminaries. In the remainder of this section, we assume that $g$ is a prime number greater than three.
where $F = GF(q)$. In the following argument, we first show that if $n$ is large enough, then a diagonal of an $n$–spike is weakly equivalent to a diagonal in normal form, a form which will be defined later. Then we show that $n$–spikes representable over a prime field are uniquely representable over that field, provided that $n$ is large enough. Combining these, we conclude that the number of $GF(q)$–representable $n$–spikes is the number of distinct diagonals of total length $n$. Finally, we calculate the number of such diagonals, and hence prove the theorems. For our convenience, we shall often use $q$ to replace the element $0$ of $GF(q)$. Therefore, (2.2.1)(iii) becomes

$$(2.2.1) (iii)' \text{ if } x + y = q, \text{ then } f_{d}(x) + f_{d}(y) = q.$$  

(2.3.1) Proposition. Suppose that

$$\bar{v} = (1^{(k_1)}, (q-1)^{(l_1)}, 2^{(k_2)}, (q-2)^{(l_2)}, \ldots, (\frac{q-1}{2})^{(l_{\frac{q-1}{2}})}, (\frac{q+1}{2})^{(l_{\frac{q+1}{2}})})$$

is the diagonal of a special standard representation of a spike over $GF(q)$. Suppose that there is an element $d$ in $\{1, 2, \ldots, \frac{q-1}{2}\}$ such that $k_d + l_d \geq q - 1$. Then there is an integer $m$ and a mapping $f \in F$ such that $f(d) = 1$, and $\bar{v}$ is weakly equivalent to

$$(f(1)^{(k_1)}, f(q-1)^{(l_1)}, \ldots, f(d)^{(k_d-m)}, f(q-d)^{(l_d+m)}), \ldots, f(\frac{q-1}{2})^{(l_{\frac{q-1}{2}})}, f(\frac{q+1}{2})^{(l_{\frac{q+1}{2}})}).$$

Proof. If $d = 1$, the proposition holds by taking $m = 0$ and $f = f_0$. Thus we assume that $d > 1$. Suppose that $k_d + d \geq q + 1$. If $d < q - 1$, then, since $k_d \geq q + 1 - d \geq 1$, we can swap an element of the tuple $d^{(k_d)}$ with its conjugate. By taking $f = f_d$, it follows by (2.2.3) that the last swapping, which is $s_d$, changes $\bar{v}$ to the diagonal

$$(f(1)^{(k_1)}, f(q-1)^{(l_1)}, \ldots, f(d)^{(k_d-1)}, f(q-d)^{(l_d+1)}), \ldots, f(\frac{q-1}{2})^{(l_{\frac{q-1}{2}})}, f(\frac{q+1}{2})^{(l_{\frac{q+1}{2}})}).$$

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If \( f(d) < q - 1 \), then we can do another 1-swapping corresponding to the tuple \( f(d)(k_d-1) \). Since \( f_d(d) = d + 1 \), this 1-swapping is \( s_{d+1} \). By taking \( f_{d+1} \circ f \) to be our new \( f \), it follows by (2.2.3) that the diagonal resulting from the last swapping is
\[
(f(1)^{(k_1)}, f(q-1)^{(l_1)}, \ldots, f(d)^{(k_d-2)}, f(q-d)^{(l_{d+2})}), \ldots, f\left(\frac{q-1}{2}\right)^{(k_{\frac{q-1}{2}}-1)}, f\left(\frac{q+1}{2}\right)^{(l_{\frac{q-1}{2}}-1)}).
\]

From the above argument, it is not hard to see that the sequence of 1-swappings \( s_d, s_{d+1}, \ldots, s_{q-2} \) swaps \( q - d - 1 \) elements corresponding to the tuple \( d(k_d) \) of \( \mathcal{U} \) with their conjugates. Let \( f = f_{q-2} \circ f_{q-3} \circ \ldots \circ f_{d+1} \circ f_d \). It follows that the last sequence of 1-swappings changes \( \mathcal{U} \) to the diagonal
\[
(f(1)^{(k_1)}, f(q-1)^{(l_1)}, \ldots, f(d)^{(k_d-q+d+1)}, f(q-d)^{(l_{d+q-d-1})}), \ldots, f\left(\frac{q-1}{2}\right)^{(k_{\frac{q-1}{2}}-1)}, f\left(\frac{q+1}{2}\right)^{(l_{\frac{q-1}{2}}-1)}).
\]

By (2.2.1)(ii), we deduce that \( f(d) = q - 1 \). Since \( k_d + d \geq q + 1 \), we deduce that
\[
k_d - (q - d - 1) \geq 2.
\]
Therefore, we can do the 2-swapping \( s_{-1} \) which swaps two elements of the \( d \)-th tuple of the last diagonal with their conjugates. Let \( f_{-1} \circ f \) be our new \( f \). Then it follows by (2.2.5) that the diagonal resulting from the 2-swapping is
\[
(f(1)^{(k_1)}, f(q-1)^{(l_1)}, \ldots, f(d)^{(k_d-q+d+1)}, f(q-d)^{(l_{d+q-d-1})}), \ldots, f\left(\frac{q-1}{2}\right)^{(k_{\frac{q-1}{2}}-1)}, f\left(\frac{q+1}{2}\right)^{(l_{\frac{q-1}{2}}-1)}).
\]

By the definition of \( f_{-1} \), it is easy to see that \( f_{-1} = f_{-\frac{1}{2}} \). Therefore, \( f \) equals a composition of mappings in \( \mathcal{F} \). It follows by (2.2.2) that \( f \in \mathcal{F} \). Therefore, the lemma holds by taking \( m = q - d + 1 \), and \( f \) as defined.

We may now suppose that \( k_d + d < q + 1 \). By the assumption that \( k_d + l_d \geq q - 1 \), we deduce that \( l_d \geq d - 1 \). By a similar argument to the above, it follows that the sequence of swappings \( s_{d-1}, s_{d+1}, \ldots, s_{q-2} \) swaps \( d - 1 \) elements corresponding to the tuple \( (q - d)^{(l_d)} \) with their conjugates. Let \( f = f_{q-d} \circ f_{q-d+1} \ldots \circ f_{q-2} \). Then the last sequence of 1-swappings changes \( \mathcal{U} \) to the diagonal
\[
(f(1)^{(k_1)}, f(q-1)^{(l_1)}, \ldots, f(d)^{(k_d+(d-1))}, f(q-d)^{(l_d-(d-1))}), \ldots, f\left(\frac{q-1}{2}\right)^{(k_{\frac{q-1}{2}}-1)}, f\left(\frac{q+1}{2}\right)^{(l_{\frac{q-1}{2}}-1)}).
\]

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By (2.2.1)(ii), we deduce that $f(q-d) = q-1$. By (2.2.1)(iii)', it follows that $f(d) = 1$.

Therefore, the proposition holds by taking $m = 1 - d$, and $f$ as defined. \hfill \Box

(2.3.2) Lemma. Suppose that

$$
\bar{v} = (1^{(k_1)}, (q-1)^{(l_1)}, 2^{(k_2)}, (q-2)^{(l_2)}, \ldots, \left(\frac{q-1}{2}\right)^{(k_{s-1})}, \left(\frac{q+1}{2}\right)^{(l_{s-1})})
$$

is the diagonal of a special standard representation of a spike over $GF(q)$. Suppose that

$k_1 + l_1 \geq q - 1$. Then there is an integer $m$ such that $\bar{v}$ is weakly equivalent to

$$
(1^{(k_1-m)}, (q-1)^{(l_1+m)}, 2^{(0)}, (q-2)^{(k_2+l_2)}, \ldots, \left(\frac{q-1}{2}\right)^{(0)}, \left(\frac{q+1}{2}\right)^{(k_{s-1}+l_{s-1})}).
$$

Proof. Initialize $f$ as the identity mapping on $F$ and $m$ as zero. The strategy of the proof is to perform swappings in order to reduce the length of the third tuple in $\bar{v}$ to zero. The parameter $m$ measures the difference between $k_1$ and the length of the first tuple in the resulting diagonal. If $k_2$ is positive, we do the 1-swapping $s_2$ and let $f_2 \circ f$ be the new $f$. By (2.2.4), $\bar{v}$ is weakly equivalent to

$$
(f(1)^{(k_1)}, f(q-1)^{(l_1)}, f(2)^{(k_2-1)}, f(q-2)^{(l_2+1)}, \ldots, f\left(\frac{q-1}{2}\right)^{(k_{s-1})}, f\left(\frac{q+1}{2}\right)^{(l_{s-1})}).
$$

By (2.2.1)(ii), $f_2(2) = 3$. If $k_2 - 1$ is positive, we then do the 1-swapping $s_3$, and let $f_3 \circ f$ be our new $f$. By (2.2.2), $f \in F$. By (2.2.4) again, $\bar{v}$ is, in turn, weakly equivalent to

$$
(f(1)^{(k_1)}, f(q-1)^{(l_1)}, f(2)^{(k_2-2)}, f(q-2)^{(l_2+2)}, \ldots, f\left(\frac{q-1}{2}\right)^{(k_{s-1})}, f\left(\frac{q+1}{2}\right)^{(l_{s-1})}).
$$

Continue doing this until either

(i) the length of the third tuple is reduced to zero, as desired, or

(ii) $f(2) = q - 1$. 

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Suppose that (ii) occurs. Then neither \( f(1) \) nor \( f(q-1) \) is \( q-1 \) for \( f \) is a bijection. Since \( k_1 + l_1 \geq q-1 \), either \( k_1 \) or \( l_1 \) is positive. Suppose that \( k_1 \) is positive. Then we can do the 1-swapping \( s_{f(1)} \). Let \( h \) be the mapping \( f_{f(1)} \). By (2.2.8.1), \( h \) maps \( q-1 \) to some other member of \( F\setminus\{0\} \) while \( s_{f(1)} \) decreases the length of the first tuple by one, that is, increases \( m \) by one. On the other hand, if \( k_1 \) is zero, then \( l_1 \) is positive, and we can do the 1-swapping \( s_{f(q-1)} \). Let \( h \) be the mapping \( f_{f(q-1)} \). This mapping, in turn, maps \( q-1 \) to some other member of \( F\setminus\{0\} \) while \( s_{f(q-1)} \) decreases \( m \) by one. Thus, in both cases, we can find a 1-swapping such that its underlying mapping \( h \) alters \( m \) and changes \( q-1 \) to some other member of \( F\setminus\{0\} \). Therefore, after taking \( h \circ f \) to be our new \( f \), we can now resume doing 1-swappings corresponding to the third tuple. Continue doing 1-swappings and renewing \( f \) and \( m \) in the above fashion until case (i) occurs, that is, the length of the third tuple is reduced to zero. This shows that \( \bar{v} \) is weakly equivalent to

\[
(f(1)^{k_1}, f(q-1)^{l_1}, f(2)^{0}, f(q-2)^{k_2+l_2}, \ldots, f\left(\frac{q-1}{2}\right)^{k_{\frac{q-1}{2}}}, f\left(\frac{q+1}{2}\right)^{l_{\frac{q-1}{2}}} )
\]

for some \( f \in \mathcal{F} \) and some integer \( m \) satisfying \(-l_1 \leq m \leq k_1\).

Having reduced the length of the third tuple to zero, we now shift to do 1-swappings corresponding to the fifth tuple to reduce its length to zero. Then we do 1-swappings corresponding to the seventh tuple, and so on. By following the same method as above, we can find an \( f \) in \( \mathcal{F} \), and an integer \( m \) satisfying \(-k_{q-1} \leq m \leq k_1 \), such that \( \bar{v} \) is weakly equivalent to

\[
(f(1)^{k_1-m}, f(q-1)^{l_1+m}, f(2)^{0}, f(q-2)^{k_2+l_2}, \ldots, f\left(\frac{q-1}{2}\right)^{(k_{\frac{q-1}{2}}+l_{\frac{q-1}{2}})} , f\left(\frac{q+1}{2}\right)^{(l_{\frac{q-1}{2}}+k_{\frac{q-1}{2}})} )
\]

The lemma follows immediately by (2.3.1). \( \square \)
Now we introduce a special type of diagonal. A diagonal

\[(1^{(k_1)}, (q-1)^{(l_1)}, 2^{(k_2)}, (q-2)^{(l_2)}, \ldots, (\frac{q-1}{2})^{(k_{\frac{q-1}{2}})}, (\frac{q+1}{2})^{(l_{\frac{q-1}{2}})})\]

of a special standard representation of an \(n\)-spike over \(GF(q)\) is said to be normal if it satisfies the following three conditions:

(i) \(k_1 \in \{0, 1, 2, \ldots, q-1\}\);

(ii) \(k_2 = k_3 = \ldots = k_{(q-1)/2} = 0\); and

(iii) \(k_1 + l_1 \geq l_d\) for each \(d\) in \(\{2, 3, \ldots, \frac{q-1}{2}\}\).

Such diagonals are important in counting the number of distinct spikes. Suppose that \(m, a_1, a_2, \ldots, a_{\frac{q-1}{2}}\) are non-negative integers which satisfy the following:

(i) \(m \leq q - 1\);

(ii) \(a_1 + a_2 + \ldots + a_{(q-1)/2} = n\); and

(iii) \(a_1 \geq a_i\) for all \(i \geq 2\).

Corresponding to such a sequence \(m, a_1, a_2, \ldots, a_{\frac{q-1}{2}}\), there is a unique normal diagonal

\[(2.3.3) \quad (1^{(m)}, (q-1)^{(a_1-m)}, 2^{(0)}, (q-2)^{(a_2)}, \ldots, (\frac{q-1}{2})^{(0)}, (\frac{q+1}{2})^{(a_{\frac{q-1}{2}})})\]

Each diagonal of the above form is said to be a normal diagonal corresponding to the sequence \(a_1, a_2, \ldots, a_{\frac{q-1}{2}}\). Obviously, there are \(q\) normal diagonals corresponding to such a sequence. In the remainder of this chapter, when we refer to the normal form of a diagonal, we shall think of it as being expressed in the form of \((2.3.3)\).
(2.3.4) Lemma. Suppose that $n > \frac{(q-1)(q-2)}{2}$, and $v$ is the diagonal of a special standard representation of an $n$-spike over $GF(q)$. Then $v$ is weakly equivalent to some normal diagonal.

Proof. Suppose that $v$ is the diagonal

$$(1^{(k_1)}, (q-1)^{(l_1)}, 2^{(k_2)}, (q-2)^{(l_2)}, \ldots, \left(\frac{q-1}{2}\right)^{(k_{\frac{q-1}{2}})}, \left(\frac{q+1}{2}\right)^{(l_{\frac{q-1}{2}}})).$$

Among the sums $k_1 + l_1, k_2 + l_2, \ldots, k_{\frac{q-1}{2}} + l_{\frac{q-1}{2}}$, let $k_d + l_d$ attain the largest value. Since $n > \frac{(q-1)(q-2)}{2}$, it follows by the pigeonhole principle that $k_d + l_d \geq q - 1$. By (2.3.1) and the fact that changing the order of a diagonal results in a weakly equivalent diagonal, we may assume that $d = 1$. Let $a_i = k_i + l_i$ for each $i$ in $\{1, 2, \ldots, \frac{q-1}{2}\}$. By (2.3.2), there is a non-negative integer $m$ such that $v$ is weakly equivalent to

$$(1^{(m)}, (q-1)^{(a_1-m)}, 2^{(0)}, (q-2)^{(a_2)}, \ldots, \left(\frac{q-1}{2}\right)^{(0)}, \left(\frac{q+1}{2}\right)^{(a_{\frac{q-1}{2}}})).$$

If $m \geq q$, then it follows by (2.2.8) that the last diagonal is weakly equivalent to

$$(1^{(m-q)}, (q-1)^{(a_1-m+q)}, 2^{(0)}, (q-2)^{(a_2)}, \ldots, \left(\frac{q-1}{2}\right)^{(0)}, \left(\frac{q+1}{2}\right)^{(a_{\frac{q-1}{2}}})).$$

Therefore, we can reduce the integer $m$ by $q$ if $m \geq q$, and the lemma follows. \(\Box\)

(2.3.5) Lemma. Suppose that $n \geq q - 1$, and

$$
\begin{bmatrix}
e_1 & e_2 & e_3 & \ldots & e_n & t & f_1 & f_2 & f_3 & \ldots & f_n \\
1 & 0 & 0 & \ldots & 0 & 1 & x_1 + 1 & 1 & 1 & \ldots & 1 \\
0 & 1 & 0 & \ldots & 0 & 1 & 1 & x_2 + 1 & 1 & \ldots & 1 \\
0 & 0 & 1 & \ldots & 0 & 1 & 1 & 1 & x_3 + 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 1 & 1 & 1 & 1 & \ldots & x_n + 1
\end{bmatrix}
$$

is a special standard representation of $n$-spike $M$ over $GF(q)$. Suppose that $x_i = -1$ for all $i$ in $\{1, 2, \ldots, q-2\}$. Then $M$ is uniquely representable over $GF(q)$.

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Proof. For each integer \( m \) in \( \{1, 2, 3, \ldots, q-1\} \setminus \{2\} \). Let \( X(m) \) be the set
\[
\{f_1, f_2, \ldots, f_{m-1}, e_m, e_{m+1}, \ldots, e_{q-2}, f_{q-1}, e_q, e_{q+1}, \ldots, e_n\},
\]
and \( D(m) \) be the determinant of the \( m \times m \) matrix
\[
\begin{bmatrix}
0 & 1 & 1 & \ldots & 1 & 1 \\
1 & 0 & 1 & \ldots & 1 & 1 \\
1 & 1 & 0 & \ldots & 1 & 1 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & 1 & 1 & \ldots & 0 & 1 \\
1 & 1 & 1 & \ldots & 1 & x_{q-1} + 1
\end{bmatrix}.
\]
By (2.2.6), \( D(m) = (-1)^{m-1}[1 - x_{q-1}(m - 2)] \). If, for some \( m \) in \( \{1, 3, 4, \ldots, q-1\} \), the set \( X(m) \) is a circuit-hyperplane of \( M \), then \( D(m) = 0 \), and \( x_{q-1} = (m - 2)^{-1} \).

Thus \( x_{q-1} \) is uniquely determined by the circuit-hyperplane \( X(m) \). On the other hand, if there is no \( m \) in \( \{1, 3, 4, \ldots, q-1\} \) such that \( X(m) \) is a circuit-hyperplane, then \( x_{q-1} \neq (m - 2)^{-1} \) for any \( m \) in \( \{1, 3, 4, \ldots, q-1\} \). We deduce that \( x_{q-1} = (q - 2)^{-1} \).

Therefore, \( x_{q-1} \) is uniquely determined by the structure of \( M \). Similarly, each of \( x_q, x_{q+1}, \ldots, x_n \) is uniquely determined be the structure of \( M \). We conclude that \( M \) is uniquely representable over \( GF(q) \).

\( (2.3.6) \) Lemma. Suppose that \( n \geq \frac{(q-1)^2}{2} \), and \( M \) is a \( GF(q) \)-representable \( n \)-spike.

Then \( M \) is uniquely representable over \( GF(q) \).

Proof. Because unique representability is a property of labeled matroids, throughout this proof, we shall think of the diagonal of a special standard representation as carrying the labeling of the corresponding matrix. Therefore, by swapping elements with their conjugates, we mean we interchange the corresponding columns, moving the labels with their columns, and then standardize the resulting matrix to special standard form.
Similarly, by changing the order of a diagonal, we mean that, in the special standard representation, we change the order of the corresponding columns and their conjugates, moving labels with their columns, and then standardize the resulting matrix.

By (2.3.4), we may assume that the diagonal of a special standard representation of $M$ is normal. By the pigeonhole principle, we deduce that $a_1 \geq q - 1$. We first suppose that $a_1 \geq \max\{q + 1, 2q - 7\}$. If $a_1 - m \geq q - 2$, then the tuple $(q - 1)^{(a_1 - m)}$ in the diagonal of $M$ has length at least $q - 2$, and the lemma follows directly by (2.3.5).

Otherwise, $a_1 - m < q - 2$. By the assumption that $a_1 \geq 2q - 7$, we deduce that $m \geq q - 4$. Since $m \leq q - 1$ and $a_1 \geq q + 1$, it follows that $a_1 - m \geq 2$. Therefore, we can swap two columns corresponding to the second tuple with their conjugates. This 2-swapping $s_{-1}$ changes the diagonal to

\[(q - 1)^{(m+2)}, (a_1 - m - 2), (q - 2)^{(0)}, 2^{(a_1)}, \ldots, (q + 1)^{(0)}, (q - 1)^{(a_1 - 1)}).

In this diagonal, there are at least $q - 1$ occurrences of $q - 1$. Therefore, by (2.3.5) again, $M$ is uniquely representable over $GF(q)$.

Now suppose that $a_1 < \max\{2q - 7, q + 1\}$. Since $q \geq 5$, we deduce that $a_1 \leq 2q - 5$.

Since $a_2 + a_3 + \ldots + a_{(q-1)/2} \geq \frac{(q-1)^2}{2} - (2q - 5)$, it follows by the pigeonhole principle that there is an $l$ in \(\{2, 3, \ldots, \frac{q-1}{2}\}\) such that $a_l \geq q - 2$. Consider the corresponding tuple $(q - l)^{(a_l)}$. By (2.2.1)(i) and (2.2.7), there is a $k$ in \(\{1, 2, \ldots, q - 2\}\) such that $g_k(q - l) = q - 1$.

Suppose that $m \geq k$. Then, since $g_k = f_k \circ f_{k-1} \ldots \circ f_1$, by doing the sequence of 1-swappings $s_1, s_2, \ldots, s_k$, which swaps $k$ columns corresponding to the first tuple with their conjugates, we change the tuple $(q - l)^{(a_l)}$ to $g_k(q - l)^{(a_l)}$, that is, $(q - 1)^{(a_l)}$. By
(2.3.5) again, we deduce that $M$ is uniquely representable over $GF(q)$. We may now assume that $m < k$. Then, since $a_1 \geq q - 1$, we deduce that $a_1 - m \geq q - k$, that is, the length of the second tuple, which is $(q - 1)^{(a_1-m)}$, is at least $q - k$. Therefore, we can do the sequence of swappings $s_{-1}, s_1, s_2, \ldots, s_{q-k-2}$ which swaps $q - k$ columns corresponding to the second tuple with their conjugates. Let $h = f_{-1} \circ f_1 \circ f_2 \cdots \circ f_{q-k-2}$.

By (2.2.5) and (2.2.3), the last sequence of swappings will change the diagonal $v$ to

$$
(h(1)^{(m+q-k)}, h(q-1)^{(a_1-m-q+k)}, h(2)^{(0)}, h(q-2)^{(a_2)}, \ldots, h(\frac{q-1}{2})^{(0)}, h(\frac{q+1}{2})^{(a_{q-2})}).
$$

By (2.2.7), we have $h = g_k$. Therefore, the last sequence of swappings changes the tuple $(q-1)^{(a_1)}$ to $(q-1)^{(a_1)}$ and the lemma follows as before. □

The last lemma shows that, if $n \geq \frac{(q-1)^2}{2}$, two labeled $GF(q)$-representable $n$-spikes are isomorphic if and only if they are equivalent up to relabeling. Therefore, two such $n$-spikes are not isomorphic if and only if they are not weakly equivalent.

We conclude that the number of $GF(q)$-representable $n$-spikes equals the number of distinct diagonals, provided that $n \geq \frac{(q-1)^2}{2}$. Thus, in order to prove the remaining theorems, we concentrate on counting the number of distinct diagonals.

The following proposition is straightforward, and we shall omit the proof.

(2.3.7) Proposition. Suppose that $q$ is an odd integer greater than three and $k$ is an element of $\mathbb{Z}_q$. Let $R$ be a relation on $\mathbb{Z}_q$ such that $xRy$ if and only if $y = k - x$ or $y = x$. Then $R$ is an equivalence relation and $\mathbb{Z}_q$ has exactly $\frac{q+1}{2}$ equivalent classes under $R$.

(2.3.8) Proposition. Suppose that $n \geq \frac{(q-1)^2}{2}$, and $a_1, a_2, \ldots, a_{\frac{q-1}{2}}$ is a sequence of non-negative integers for which (i) $a_1 > \max\{a_2, a_3, \ldots, a_{\frac{q-1}{2}}\}$ and (ii) $a_1 + a_2 + \ldots + \frac{q-1}{2}$
$a_{s-1} = n$. Then there are exactly $\frac{s+1}{2}$ distinct normal diagonals corresponding to the sequence $a_1, a_2, \ldots, a_{s-1}$.

Proof. Suppose that

$$\vec{v} = (1^{(m)}, (q-1)^{(a_1-m)}, 2^{(0)}, (q-2)^{(a_2)}, \ldots, \left(\frac{q-1}{2}\right)^{(0)}, \left(\frac{q+1}{2}\right)^{(a_{s-1})}).$$

is a normal diagonal of a special standard representation of an $n$-spike over $GF(q)$. Since $n \geq \frac{(q-1)^2}{2}$, it follows by the pigeonhole principle that $a_1 \geq q - 1$. By (2.3.1), there is an integer $m_1$ and a mapping $f$ in $F$ such that $f(q-1) = 1$, and $\vec{v}$ is weakly equivalent to

$$(f(1)^{(m+m_1)}, f(q-1)^{(a_1-m-m_1)}, f(2)^{(0)}, f(q-2)^{(a_2)}, \ldots, f\left(\frac{q-1}{2}\right)^{(0)}, f\left(\frac{q+1}{2}\right)^{(a_{s-1})}).$$

Since $f \in F$, and $f(1) = -1$, it follows by (2.2.1)(iv) that $f = f_{-\frac{1}{2}}$, and the last diagonal is

$$( (q-1)^{(m+m_1)}, 1^{(a_1-m-m_1)}, (q-2)^{(0)}, 2^{(a_2)}, \ldots, \left(\frac{q+1}{2}\right)^{(0)}, \left(\frac{q-1}{2}\right)^{(a_{s-1})}).$$

By (2.3.2), there is an integer $m_2$ such that the last diagonal is weakly equivalent to

$$( (q-1)^{(m+m_1+m_2)}, 1^{(a_1-m-m_1-m_2)}, (q-2)^{(a_2)}, 2^{(0)}, \ldots, \left(\frac{q+1}{2}\right)^{(a_{(q-1)/2})}, \left(\frac{q-1}{2}\right)^{(0)}).$$

Let $k$ be an integer for which $0 \leq k - m \leq q - 1$, and $k \equiv a_1 - m_1 - m_2 (mod q)$. The above argument shows that $\vec{v}$ is weakly equivalent to

$$(1^{(k-m)}, (q-1)^{(a_1-k+m)}, 2^{(0)}, (q-2)^{(a_2)}, \ldots, \left(\frac{q-1}{2}\right)^{(0)}, \left(\frac{q+1}{2}\right)^{(a_{(q-1)/2})}).$$

which is also a normal diagonal corresponding to the sequence $a_1, a_2, \ldots, a_{s-1}$. Since $a_1 > \max\{a_2, a_3, \ldots, a_{s-1}\}$, if a sequence of swappings changes the first tuple to a tuple
of x's where $x \notin \{1, q-1\}$, then the resulting diagonal is not in normal form. Therefore, by the fact that there is no non-trivial automorphism of $GF(q)$, it follows that the above weak equivalence is the only possible weak equivalence on the set of normal diagonals corresponding to the sequence $a_1, a_2, \ldots, a_{2^{-1}}$. The proposition follows by (2.3.7). □

(2.3.9) Corollary. Suppose that $n \geq \frac{(q-1)^2}{2}$, and $a_1, a_2, \ldots, a_{2^{-1}}$ is a sequence of non-negative integers for which (i) $a_1 \geq \max\{a_2, a_3, \ldots, a_{2^{-1}}\}$ and (ii) $a_1 + a_2 + \ldots + a_{2^{-1}} = n$. Then there are at most $\frac{a_1+1}{2}$ distinct normal diagonals corresponding to the sequence $a_1, a_2, \ldots, a_{2^{-1}}$.

Proof of Theorem 2.1.3. By (2.3.6), quinernary $n$-spikes are uniquely representable over $GF(5)$ if $n \geq 8$. Therefore, for $n \geq 8$, the number of distinct quinernary diagonals is the number of quinernary $n$-spikes. By (2.3.4), we only need to count the number of distinct normal diagonals.

Let $a_1, a_2$ be a pair of non-negative integers such that $a_1 + a_2 = n$. Over the field $GF(5)$, there are exactly five normal diagonals corresponding to the sequence $a_1, a_2$, namely, $(1^{(0)}, 4^{(a_1)}, 2^{(0)}, 3^{(a_2)})$, $(1^{(1)}, 4^{(a_1-1)}, 2^{(0)}, 3^{(a_2)})$, $(1^{(2)}, 4^{(a_1-2)}, 2^{(0)}, 3^{(a_2)})$, $(1^{(3)}, 4^{(a_1-3)}, 2^{(0)}, 3^{(a_2)})$, and $(1^{(4)}, 4^{(a_1-4)}, 2^{(0)}, 3^{(a_2)})$. By (2.3.8), if $a_1 > a_2$, then there are exactly three distinct quinernary diagonals corresponding to the sequence $a_1, a_2$.

Now suppose that $a_1 = a_2$. Then the roles of $a_1$ and $a_2$ are interchangeable. For $\overline{v} = (1^{(m)}, 4^{(a_1-m)}, 2^{(0)}, 3^{(a_2)})$, which is a normal diagonal corresponding to the sequence $a_1, a_2$, we may do a sequence of swappings to change any of the second, third, and fourth tuples to a tuple of ones, thereby creating a normal diagonal that corresponds to the sequence $a_2, a_1$ and is weakly equivalent to $\overline{v}$. 

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We assert that, in the case that $a_1 = a_2$, the five normal diagonals listed above fall into exactly two weak equivalence classes. To prove this assertion, we break the argument into five cases depending on the congruence class of $a_i$ modulo 5. First we consider the case that $a_1 = a_2 = 5m + 1$ for some integer $m$. Consider the diagonal $v_0 = (1^{(0)}, 4^{(5m+1)}, 2^{(0)}, 3^{(5m+1)})$. We first do swappings changing the second tuple in $v_0$ to a tuple of ones. This can be done by the 2-swapping $s_{-1}$. By (2.2.5), $s_{-1}$ changes $v_0$ to the diagonal $(f_{-1}(1)^{(2)}, f_{-1}(4)^{(5m-1)}, f_{-1}(2)^{(0)}, f_{-1}(3)^{(5m+1)})$. By the definition of $f_{-1}$, we deduce that the last diagonal is $(4^{(2)}, 1^{(5m-1)}, 3^{(0)}, 2^{(5m+1)})$. By reordering the tuples of the last diagonal, we deduce that $v_0$ is weakly equivalent to $(1^{(5m-1)}, 4^{(2)}, 2^{(5m+1)}, 3^{(0)})$. By (2.2.8), the last diagonal is weakly equivalent to $(1^{(4)}, 4^{(5m-3)}, 2^{(1)}, 3^{(5m)})$. We now normalize the last diagonal by doing a sequence of swappings. We first do the 1-swapping $s_2$. By (2.2.3), this swapping changes the last diagonal to $(4^{(4)}, 1^{(5m-3)}, 3^{(0)}, 2^{(5m+1)})$. By doing the 2-swapping $s_{-1}$, we change the last diagonal to the normal diagonal $v_2 = (1^{(2)}, 4^{(5m-1)}, 2^{(0)}, 3^{(5m+1)})$. Therefore, $v_0$ is weakly equivalent to $v_2$.

Now, we do swappings to change the third tuple in $v_0$ to a tuple of ones. This can be done by doing the 1-swapping $s_3$. By (2.2.3), $s_3$ changes $v_0$ to the diagonal $(3^{(0)}, 2^{(5m+1)}, 1^{(1)}, 4^{(5m)})$. By reordering the tuples of the last diagonal, we deduce that $v_0$ is weakly equivalent to the diagonal $(1^{(1)}, 4^{(5m)}, 2^{(5m+1)}, 3^{(0)})$. To normalize the last diagonal, we first do the 1-swapping $s_2$. This 1-swapping changes the last diagonal to $(4^{(1)}, 1^{(5m)}, 3^{(0)}, 2^{(1)})$. By (2.2.8), the last diagonal is weakly equivalent to $(4^{(1)}, 1^{(5m)}, 3^{(0)}, 2^{(5m+1)})$. Now do the sequence of 1-swappings $s_1$, $s_2$, and $s_3$. By (2.2.8.1), this sequence of 1-swappings increases the base of the second tuple.
by three, and reduces the length of that tuple by three. Since $f_3 \circ f_2 \circ f_1 = f_{-1}$, we conclude that the last diagonal is weakly equivalent to the normal diagonal $\tilde{v}_4 = (1^{(4)}, 4^{(5m-3)}, 2^{(0)}, 3^{(5m+1)})$. Therefore, $\tilde{v}_0$ is weakly equivalent to $\tilde{v}_4$.

We can also do swappings to change the last tuple in $\tilde{v}_0$ to a tuple of ones. First, we deduce by (2.2.8) that $\tilde{v}_0$ is weakly equivalent to $(1^{(0)}, 4^{(5m+1)}, 2^{(5m)}, 3^{(1)})$. We then do the $1$-swapping $s_2$, followed by $s_3$. It follows by (2.2.3) that the diagonal resulting from this sequence of swappings is $(2^{(0)}, 3^{(5m+1)}, 4^{(5m-2)}, 1^{(3)})$. Therefore, $\tilde{v}_0$ is weakly equivalent to the diagonal $\tilde{v}_3 = (1^{(3)}, 4^{(5m-2)}, 2^{(0)}, 3^{(5m+1)})$.

We have proved that $\tilde{v}_0$ is in a weak equivalence class with at least four elements. On the other hand, since $q = 5$, $\tilde{v}_0$ has exactly four tuples. Each of these four tuples may be changed to a tuple of ones by a sequence of swappings. We conclude that $\tilde{v}_0$ can be weakly equivalent to at most three other normal diagonals. By the fact that $\tilde{v}_0$ is weakly equivalent to $\tilde{v}_2$, $\tilde{v}_3$, and $\tilde{v}_4$, we conclude that the weak equivalence class of $\tilde{v}_0$ has exactly four members, and the remaining diagonal $(1^{(1)}, 4^{(5m)}, 2^{(0)}, 3^{(5m+1)})$ is the only member of the other weak equivalence class. Therefore, we have proved the assertion in the case that $a_1 = 5m + 1$. For each of the cases that $a_1 = 5m$, $a_1 = 5m + 2$, $a_1 = 5m + 3$, and $a_1 = 5m + 4$, the assertion can be proved by an argument similar to the above. We omit the routine details here noting that, in each case, one shows that there are exactly two weak equivalence classes of normal diagonals, one of which has just one member. Since $a_2$ may have the value zero, we conclude that the number of distinct quaternary $n$-spikes is $3p_{\leq 2}(n) - \frac{1+(-1)^n}{2}$, provided $n \geq 8$. This number is exactly $n + \lfloor \frac{n}{2} \rfloor + 2$. 

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It is easy to check that there are five distinct quaternary 3-spikes. For \( n = 4, 5, 6, \) and 7, we can follow the ideas of Lemmas 2.3.2 and 2.3.4 to check that the number of distinct quaternary \( n \)-spikes is also exactly \( n + \lceil \frac{3}{2} \rceil + 2 \) as asserted. \( \square \)

**Proof of Theorem 2.1.4.** By (2.3.6), \( GF(7) \)-representable \( n \)-spikes are uniquely representable over \( GF(7) \) if \( n \geq 18 \). Therefore, for \( n \geq 18 \), the number of distinct \( n \)-spikes representable over \( GF(7) \) is the number of distinct diagonals. By (2.3.4), we only need to count the number of distinct normal diagonals.

Suppose that \( l \) is a positive integer, \( n = 6l \), and \( a_1, a_2, a_3 \) is a sequence of non-negative integers for which \( a_1 + a_2 + a_3 = n \) and \( a_1 \geq \max\{a_2, a_3\} \). To count the number of distinct normal diagonals, we count the number of them for each of the following six cases:

(i) \( a_1 > a_2 > a_3 \);

(ii) \( a_1 > a_3 > a_2 \);

(iii) \( a_1 > a_2 \), and \( a_2 = a_3 \);

(iv) \( a_1 > a_2 \), and \( a_1 = a_3 \);

(v) \( a_1 > a_3 \), and \( a_1 = a_2 \);

(vi) \( a_1 = a_2 = a_3 \).

Let

\[ \vec{v} = (1^{(m)}, 6^{(a_1 - m)}, 2^{(0)}, 5^{(a_2)}, 3^{(0)}, 4^{(0)}) \]

be a normal diagonal corresponding to the sequence \( a_1, a_2, a_3 \). By (2.3.8), there are exactly four distinct normal diagonals for each sequence in case (i). Clearly, the number
of sequences $a_1, a_2, a_3$ in case (i) is $p_2^{(0)}(n) + p_3^{(0)}(n)$. By direct calculation, or by using a result of [7, Section 8.6], we deduce that $p_2^{(0)}(n) + p_3^{(0)}(n) = 3n^2$. Therefore, there are $12l^2$ distinct normal diagonals for case (i). Similarly, the number of distinct normal diagonals for case (ii) is also $12l^2$.

For case (iii), since $a_1 > \max\{a_2, a_3\}$, it follows by (2.3.8) that there are again exactly four distinct normal diagonals for each sequence $a_1, a_2, a_3$. Since $a_2$ can be any member of $\{0, 1, 2, \ldots, 2l - 1\}$, we deduce that there are $2l$ distinct sequences $a_1, a_2, a_3$ in this case. Therefore, there are $8l$ distinct normal diagonals for case (iii).

Now consider case (iv). By (2.3.1), there is an integer $m_1$ and a mapping $f \in \mathcal{F}$ satisfying $f(4) = 1$, such that $\tilde{v}$ is weakly equivalent to

$$ (f(1)^{m_1}, f(6)^{(a_1-m_1)}, f(2)^{(0)}, f(5)^{(a_2)}, f(3)^{(m_1)}, f(4)^{(a_3-m_1)}). $$

Since $f(4) = 1$, we deduce that $f = f_1$, and the last diagonal is

$$ (2^{(m_1)}, 5^{(a_1-m_1)}, 4^{(0)}, 3^{(a_2)}, 6^{(m_1)}, 1^{(a_3-m_1)}). $$

By (2.3.2), there is an integer $m_2$ such that the last diagonal is weakly equivalent to

$$ (2^{(0)}, 5^{(a_1)}, 4^{(a_2)}, 3^{(0)}, 6^{(m_1+m_2)}, 1^{(a_3-m_1-m_2)}). $$

Since $a_1 = a_3$ in this case, we conclude that $\tilde{v}$ is weakly equivalent to a normal diagonal corresponding to the sequence $a_3, a_1, a_2$. Therefore, by changing the sixth tuple of $\tilde{v}$ to a tuple of ones, a normal diagonal in case (iv) is weakly equivalent to a normal diagonal in case (v). Similarly, $\tilde{v}$ is again weakly equivalent to a normal diagonal corresponding to a sequence in case (v) by changing the fifth tuple of $\tilde{v}$ to a tuple of ones. By the symmetry between cases (iv) and (v), we conclude that the set of distinct diagonals
of case (iv) is the same as that of case (v). Therefore, we will count the number of distinct diagonals for case (iv) and ignore case (v). In case (iv), \( a_2 \) can be any member of \( \{0, 2, 4, \ldots, 2l - 2\} \). We deduce that, in this case, there are \( l \) distinct sequences, and so the number of distinct diagonals is \( 4l \).

For case (vi), since \( a_1 = a_2 = a_3 \), each permutation on \( a_1, a_2, a_3 \) produces the same sequence. Therefore, for each \( d \) in \( \{1, 2, \ldots, 6\} \), changing the tuple of \( d \)'s in \( \vec{v} \) into a tuple of ones by a series of swappings will create a weak equivalence on the set of normal diagonals corresponding to the sequence \( a_1, a_2, a_3 \). Arguing on \( a_1 \) modulo 7, we need seven straightforward cases to show that there are exactly two distinct normal diagonals corresponding to the sequence. The argument here is similar to that used for the case \( a_1 = a_2 \) in the proof of (2.1.3), and the details are omitted.

Overall, for the integer \( n = 6l \), we conclude that the number of distinct normal diagonals is \( 12l^2 + 12l^2 + 8l + 4l + 1 \). This number is exactly \( \left\lfloor \frac{2n^2 + 6n + 6}{3} \right\rfloor \). For the cases \( n = 6l + 1, n = 6l + 2, n = 6l + 3, n = 6l + 4, \) and \( n = 6l + 5 \), we deduce, by similar arguments to those used above for the case \( n = 6l \), that the numbers of distinct normal diagonals are \( 24l^2 + 20l + 4, 24l^2 + 28l + 8, 24l^2 + 36l + 14, 24l^2 + 44l + 20, \) and \( 24l^2 + 52l + 28 \), respectively. It is routine to check that each of these numbers equals \( \left\lfloor \frac{2n^2 + 6n + 6}{3} \right\rfloor \).

Proof of Theorem 2.1.5. Suppose that \( b_1 > b_2 > \ldots > b_{\frac{n-1}{2}} \) is a partition of \( n \) into \( \frac{n-1}{2} \) distinct parts. Let \( a_1 = b_1, a_2, a_3, \ldots, a_{\frac{n-1}{2}} \) be a permutation of \( b_2, b_3, \ldots, b_{\frac{n-1}{2}} \). Let \( n = a_1 + a_2 + \ldots + a_{\frac{n-1}{2}} \). By (2.3.8), there are \( \frac{n+1}{2} \) normal diagonals corresponding to each sequence \( a_1, a_2, \ldots, a_{\frac{n-1}{2}} \). Since all \( a_i \)'s are distinct, there are \( (\frac{n+3}{2})! \) distinct
permutations of $b_2, b_3, \ldots, b_{2^2-1}$. We conclude that there are at least \( \left( \frac{2^2+1}{2} \right) \left( \frac{2^3-3}{2} \right)! p_{\frac{2^2-1}{2}}(n) \) distinct \( GF(q) \)-representable \( n \)-spikes. Parts (i) and (ii) follow immediately by (2.2.9).

On the other hand, suppose that \( \{ a_1, a_2, \ldots, a_{2^2-1} \} \) is a set of non-negative integers such that \( a_1 \geq a_i \) for all \( i \). Since there are at most \( \left( \frac{2^3-3}{2} \right)! \) distinguishable permutations of \( a_2, a_3, \ldots, a_{2^2-1} \), and there are at most \( \frac{2^2+1}{2} \) distinct normal diagonals for each permutation, the number of distinct \( GF(q) \)-representable \( n \)-spikes is less than or equal to \( \left( \frac{2^2+1}{2} \right) \left( \frac{2^3-3}{2} \right)! p_{\frac{2^2-1}{2}}(n) \). Part (iii) follows easily by (2.2.9)(ii). Part (iv) follows by (i), (iii), and (2.2.9)(iv).

\[ \square \]

2.4 Proofs of Theorems 2.1.6 and 2.1.7

Suppose that \( p \) is an odd prime, and \( q = p^s \). Suppose that \( \omega \) is a root of a \( GF(p) \)-irreducible polynomial of degree \( s \). Then \( GF(q) \) can be represented by \( \{ \Sigma_{i=0}^{s-1} a_i \omega^i \mid a_i \in GF(p) \} \). We will use this representation of \( GF(q) \) throughout this section unless specified otherwise. Let \( \{ d_1, d_2, \ldots, d_{2^2-1} \} \) be a subset \( D \) of \( GF(q) \setminus \{0\} \). Define \( D^- = \{ -d \mid d \in D \} \). In the following discussion, we choose a fixed set \( D \) which satisfies the following two conditions:

(i) \( D \cup D^- = GF(q) \setminus \{0\} \); and

(ii) \( d_i = \omega^{i-1} \) for all \( i \) in \( \{1, 2, \ldots, s\} \).

Clearly, a diagonal of a special standard representation of an \( n \)-spike representable over \( GF(q) \) can be written as

\[ (d_1^{(k_1)}, (-d_1)^{(l_1)}, d_2^{(k_2)}, (-d_2)^{(l_2)}, \ldots, d_{2^2-1}^{(k_{2^2-1})}, (-d_{2^2-1})^{(l_{2^2-1})}) . \]

We will use the last form as the general form of a diagonal.
Two diagonals are said to be **quasi-equivalent** if one can be obtained from the other by taking a series of swappings. Two diagonals are **quasi-distinct** if they are not quasi-equivalent. Two special standard representations of $n$-spikes $A_1$ and $A_2$ are quasi-equivalent if their corresponding diagonals are quasi-equivalent. Clearly, $A_1$ and $A_2$ are quasi-equivalent if $A_1$ can be obtained from $A_2$ by a sequence of operations (i)-(v) and (vii) where the operations are those given in Section 2.1. Over prime field, quasi-equivalence is the same as weak equivalence, since there are no non-trivial field automorphisms over prime fields. By (2.3.6), $n$-spikes representable over a prime field are uniquely representable over that field when $n$ is large. Therefore, in the last section, to count the number of non-isomorphic $n$-spikes over a prime field, we only needed to count the number of distinct diagonals over that field. However, since $q$ is not a prime number in this section, we will see that an $n$-spike can have other quasi-distinct special standard representations apart from that are related by non-trivial field automorphisms. Therefore, we shall first consider quasi-equivalence, and then find the number of quasi-distinct special standard representations of an $n$-spike, thereby calculating the number of non-isomorphic $n$-spikes.

Suppose that $a_1, a_2, \ldots, a_{\frac{q-1}{2}}$ is a sequence of non-negative integers, and $\overline{v}$ is the diagonal

$$
(d_1^{(k_1)}, (-d_1)^{(l_1)}, d_2^{(k_2)}, (-d_2)^{(l_2)}, \ldots, (d_{\frac{q-1}{2}})^{(k_{\frac{q-1}{2}})}, (-d_{\frac{q-1}{2}})^{(l_{\frac{q-1}{2}})}).
$$

We say that $\overline{v}$ is a **diagonal associated with the sequence** $a_1, a_2, \ldots, a_{\frac{q-1}{2}}$ if $k_i + l_i = a_i$ for all $i$ in $\{1, 2, \ldots, \frac{q-1}{2}\}$. 

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A sequence of integers \(a_1,a_2,\ldots,a_{\frac{q-1}{2}}\) is said to be a *special sequence* if it satisfies the following conditions:

(i) \(a_i > p\) for all \(i\) in \(\{1,2,\ldots,\frac{q-1}{2}\}\);

(ii) \(a_1 > a_i\) for all \(i\) in \(\{2,3,\ldots,\frac{q-1}{2}\}\); and

(iii) \(a_i \neq a_j\) if \(i \neq j\).

Diagonals associated with a special sequence are very important in counting the number of quasi-distinct diagonals.

**Proposition.** Let \(a_1,a_2,\ldots,a_{\frac{q-1}{2}}\) be a special sequence, and \(b_1,b_2,\ldots,b_{\frac{q-1}{2}}\) be a permutation of \(a_1,a_2,\ldots,a_{\frac{q-1}{2}}\). Suppose that the diagonal \(\tilde{v}\) is

\[\left(\left(d_{1}^{(k_1)}, (-d_{1})^{(l_1)}, d_{2}^{(k_2)}, (-d_{2})^{(l_2)}, \ldots, (d_{\frac{q-1}{2}})^{\left(k_{\frac{q-1}{2}}\right)}, (-d_{\frac{q-1}{2}})^{\left(l_{\frac{q-1}{2}}\right)}\right)\]

where \(k_i + l_i = b_i\) for all \(i\) in \(\{1,2,\ldots,\frac{q-1}{2}\}\). Then \(\tilde{v}\) is quasi-equivalent to a diagonal associated with a special sequence.

**Proof.** If \(b_1 = a_1\), then \(\tilde{v}\) itself is a diagonal associated with a special sequence. In the following argument, we suppose that \(b_j = a_i\) where \(j \neq 1\). By definition, we have \(d_1 = 1\). Therefore, \(d_j \neq 1\) and \(1 - d_j \neq 0\). Let \(x = \frac{d_j}{1-d_j}\). Then, there is an \(i\) in \(\{1,2,\ldots,\frac{q-1}{2}\}\), such that \(x = d_i\) or \(x = -d_i\). Assume first that \(x = d_i\). Since \(b_i > p\), it follows by (2.2.8) that we may assume that \(k_i > 0\). Therefore, we can do the 1-swapping \(s_{d_i}\). This 1-swapping swaps an element of the tuple \(d_i^{(k_i)}\) in \(\tilde{v}\) with its conjugate, and changes \(\tilde{v}\) to the diagonal

\[(f_{d_i}(d_1)^{(k_1)}, \ldots, f_{d_i}(-d_{i-1})^{(l_{i-1})}, f_{d_i}(d_i)^{(k_i-1)}, f_{d_i}(-d_i)^{(l_{i+1})}, \ldots, f_{d_i}(-d_{\frac{q-1}{2}})^{\left(l_{\frac{q-1}{2}}\right)}).\]

By (2.2.1), we deduce that \(f_{d_i}(d_j) = (1 + d_i)d_j = (1 + \frac{1-d_j}{d_j})d_j = 1 = d_1\). We conclude
that the last diagonal is a diagonal associated with a special sequence. For the case that \( x = -d_i \), a similar argument shows that there is a 1-swapping that changes the tuple \( d_j^{(k_j)} \) in \( \bar{v} \) to a tuple of ones, thereby completing the proof of the proposition. \( \square \)

Suppose that \( a_1, a_2, \ldots, a_{2^{s-1}} \) is a special sequence. The diagonal

\[
(d_1^{(k_1)}, (-d_1)^{(l_1)}, d_2^{(k_2)}, (-d_2)^{(l_2)}, \ldots, (d_{2^{s-1}})^{(k_{2^{s-1}})}, (-d_{2^{s-1}})^{(l_{2^{s-1}})})
\]

is said to be a normal diagonal associated with the special sequence \( a_1, a_2, \ldots, a_{2^{s-1}} \) if it satisfies the following two conditions:

(i) \( 0 \leq k_i \leq p - 1 \) for all \( i \) in \( \{1, 2, \ldots, s\} \); and

(ii) \( k_i = 0 \) for all \( i \) in \( \{s + 1, s + 2, \ldots, 2^{s-1}\} \).

Clearly, there are \( p^s = q \) normal diagonals associated with each special sequence \( a_1, a_2, \ldots, a_{2^{s-1}} \).

Recall that, in the last section, we defined normal diagonals over a prime field. The definition here is a generalization of that in the last section except that we require that all \( a_i \)'s be greater than \( p \). Just as in the last section, normal diagonals will serve as the critical tool in this section for counting the number of quasi-distinct diagonals.

(2.4.2) Proposition. Suppose that \( \bar{v} \) is a diagonal associated with the special sequence \( a_1, a_2, \ldots, a_{2^{s-1}} \). Then \( \bar{v} \) is quasi-equivalent to a normal diagonal associated with a special sequence that is a permutation of \( a_1, a_2, \ldots, a_{2^{s-1}} \).

Proof. Let \( \bar{v} \) be

\[
(d_1^{(k_1)}, (-d_1)^{(l_1)}, d_2^{(k_2)}, (-d_2)^{(l_2)}, \ldots, (d_{2^{s-1}})^{(k_{2^{s-1}})}, (-d_{2^{s-1}})^{(l_{2^{s-1}})})
\]

where \( k_i + l_i = a_i \) for all \( i \) in \( \{1, 2, \ldots, 2^{s-1}\} \). Since \( a_1 \) is the largest member of a sequence
of distinct integers each of which is greater than $p$, we deduce that $a_1 > p + \frac{q-1}{2} > p + 1$.

Our strategy is first to do swappings corresponding to the tuple $d_{s+1}^{(k_{s+1})}$, that is, the $(2s + 1)$-th tuple, and thereby reduce its length to zero. Then we apply the same procedure to reduce the length of the $(2s + 3)$-th tuple to zero. Next we reduce the length of the $(2s + 5)$-th tuple to zero and so on. This procedure is the same as that used in the proof of (2.3.2), so we shall omit the details. By the procedure just described, we deduce that there is an integer $m$ and a mapping $f$ in $\mathcal{F}$, such that $\overline{u}$ is quasi-equivalent to the diagonal

$$((f(d_1))^{(k_1-m)}, f(-d_1)^{(l_1+m)}, \ldots, f(d_{s+1})^{(l_0)}, f(-d_{s+1})^{(a_{s+1})}, \ldots, f(d_{\frac{s-1}{2}})^{(a_{s-1})}, f(-d_{\frac{s-1}{2}})^{(a_{s-1})}).$$

If $f(d_1) = d_1$, the proposition will follow directly by (2.2.8). If $f(d_1) = -d_1$, then, since $a_1 = k_1 + l_1 > p + 1$, we may assume by (2.2.8) that $l_1 + m > 1$. By the definition of $D$, we have $-d_1 = -1$, so we can do the 2-swapping $s_1$. This swapping will change the last diagonal to

$$(d_1^{(k_1-m+2)}, (-d_1)^{(l_1+m-2)}, d_2^{(k_2)}, (-d_2)^{(l_2)}, \ldots, (d_{\frac{s-1}{2}})^{(k_{s-1})}, (-d_{\frac{s-1}{2}})^{(l_{s-1})}),$$

thereby proving the proposition. Therefore, we may assume that $f(d_1) \neq -1$. Assume that $f(d_1) = c_0 + c_1 \omega^{-1} + \ldots + c_{s-1} \omega^{-(s-1)}$. If $c_0 = 0$, then, since $a_1 > p$, we may assume by (2.2.8) that the length of the first tuple if not zero, and hence we can do a 1-swapping corresponding to the first tuple. By (2.2.8.1), the last 1-swapping increases the base of the first tuple by one, and decreases its length by one. Therefore, we may assume that $c_0 \neq 0$. Since $f \in \mathcal{F}$, we deduce, by the definition of mappings of $\mathcal{F}$, that $f(d_s) = f(\omega^{s-1}) = f(1)\omega^{s-1} = c_0\omega^{s-1} + c_1\omega^{s-2} + \ldots + c_{s-1}$. By (2.2.8) and the fact that $a_s > p$, we may assume that we can do 1-swappings corresponding to the tuple of $f(d_s)$'s whenever we desire to do so. Suppose that $c_{s-1} \neq 0$. Then we do
the 1-swapping corresponding to the tuple of $f(d_a)$'s. By (2.2.8.1), the last swapping increases the base of that tuple by one, and reduces its length by one. Continue doing 1-swappings corresponding to this tuple until the constant term of its base is increased to $p$. Since $GF(q)$ has characteristic $p$, the above sequence of 1-swappings changes the base of the $(2s - 1)$-th tuple to an element of $GF(q)$ whose constant term is zero. Therefore, we may assume that $c_{s-1} = 0$. Since $f(d_{s-1}) = f(1)d_{s-1}$, we deduce that $f(d_{s-1}) = c_0 \omega^{s-2} + c_1 \omega^{s-3} + \ldots + c_{s-2}$. By a similar argument to that just given, a sequence of 1-swappings corresponding to the $(2s - 3)$-th tuple will change the constant term of the base of that tuple to zero. Next we shift to doing 1-swappings corresponding to the $(2s - 5)$-th tuple, and so on. We conclude that we can reduce each of the coefficients $c_{s-1}, c_{s-2}, \ldots, c_1$ to zero. Therefore, we deduce that there is a mapping $f$ in $\mathcal{F}$ such that $f(d_1) = c_0$, and $\mathcal{U}$ is quasi-equivalent to

$$(f(d_1)^{(k'_i)}, f(-d_1)^{(l'_i)}, \ldots, f(d_{s+1})^{(0)}, f(-d_{s+1})^{(a_{s+1})}, \ldots, f(d_{2s-1})^{(0)}, f(-d_{2s-1})^{(a_{2s-1})}).$$

Since 0 is not in any diagonal, we deduce that $c_0 \neq 0$. If $c_0 = 1$, then the proposition follows by (2.2.8). Therefore, we assume that $c_0 \neq 1$. Then $f(-d_1) = -f(d_1) \neq -1$, so we can do 1-swappings corresponding to the second tuple. Again, by (2.2.8) and the fact that $a_1 > p$, we may assume that we can do a 1-swapping corresponding to the second tuple if we choose to do so. Since such a 1-swapping increases the base of that tuple by one, there is a sequence of 1-swappings corresponding to the second tuple that will increase its base to $p - 1$. After the last sequence of 1-swappings, the resulting
The proposition now follows by (2.2.8). □

By (2.4.1), a diagonal associated with a sequence that is a permutation of a special sequence is quasi-equivalent to a diagonal associated to a special sequence. By Proposition 2.4.2, we conclude that, in order to count the number of such diagonals, we only need to count the number of quasi-distinct normal diagonals associated with a special sequence. The next lemma determines the number of such diagonals.

(2.4.3) Lemma. Suppose that \( \mathcal{V} \) is the collection of all normal diagonals associated with a special sequence \( a_1, a_2, \ldots, a_{s-1} \). Then \( \mathcal{V} \) falls into exactly \( \frac{s+1}{2} \) quasi-equivalence classes.

Proof. Suppose that \( \bar{v} \) is the normal diagonal

\[
(d_1^{(m_1)}, (-d_1)^{(a_1-m_1)}, \ldots, d_s^{(m_s)}, (-d_s)^{(a_s-m_s)}, d_s^{(0)}, (-d_{s+1})^{(a_{s+1})}, \ldots, d_{\frac{s-1}{2}}^{(0)}, (-d_{\frac{s+1}{2}})^{(a_{s-1})}).
\]

Since \( a_1 > a_2 > p \), we deduce that \( a_1 > p + 1 \). By (2.2.8), we may assume that \( a_1 - m_1 > 1 \). Therefore, we may do the 2-swapping \( s_1 \). This 2-swapping swaps two elements of the second tuple with their conjugates, and changes \( \bar{v} \) to

\[
((-d_1)^{(m_1+1)}, d_1^{(a_1-m_1-1)}, \ldots, (-d_s)^{(m_s)}, d_s^{(a_s-m_s)}, (-d_{s+1})^{(0)}, d_{s+1}^{(a_{s+1})}, \ldots, (-d_{\frac{s-1}{2}})^{(0)}, d_{\frac{s+1}{2}}^{(a_{s-1})}).
\]

By doing the normalization procedure used in the proof of the last proposition, we deduce that there are integers \( m'_1, m'_2, \ldots, m'_s \), all having values between 0 and \( p - 1 \), such that \( \bar{v} \) is quasi-equivalent to

\[
((-d_1)^{(a_1-m'_1)}, d_1^{(m'_1)}, \ldots, (-d_s)^{(a_s-m'_s)}, d_s^{(m'_s)}, (-d_{s+1})^{(a_{s+1})}, d_{s+1}^{(0)}, \ldots, (-d_{\frac{s-1}{2}})^{(a_{s-1})}, d_{\frac{s+1}{2}}^{(0)}).
\]
Recall that $a_1 > a_i$ for all $i$ in $\{2, 3, \ldots, \frac{q-1}{2} \}$. If, by swappings, we change some tuple other than the first two to a tuple of ones, the resulting diagonal is not in normal form. Therefore, it follows by (2.3.7) that $\mathcal{V}$ falls into exactly $\frac{q+1}{2}$ quasi-equivalent classes. □

In this section, we deal with the field $GF(q)$ where $q$ is a non-trivial power of a prime. Therefore, there are non-trivial field automorphisms, and spikes need not be uniquely representable over $GF(q)$. Thus, quasi-distinct diagonals may represent the same spike. In the following discussion, two special standard representations are said to be different if their corresponding diagonals are not quasi-equivalent. The next lemma determines the number of different special standard representations of an $n$-spike, provided certain conditions are satisfied.

(2.4.4) Lemma. Suppose that $M$ is a $GF(q)$-representable $n$-spike, and

$$
\begin{bmatrix}
e_1 & e_2 & e_3 & \ldots & e_n & t & f_1 & f_2 & f_3 & \ldots & f_n \\
1 & 0 & 0 & \ldots & 0 & 1 & 1 + x_1 & 1 & 1 & \ldots & 1 \\
0 & 1 & 0 & \ldots & 0 & 1 & 1 & 1 + x_2 & 1 & \ldots & 1 \\
0 & 0 & 1 & \ldots & 0 & 1 & 1 & 1 & 1 + x_3 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 1 & 1 & 1 & 1 & \ldots & 1 + x_n
\end{bmatrix}
$$

is a special standard representation of $M$ over $GF(q)$. Suppose that the multi-set \{ $x_1, x_2, \ldots, x_n$ \} contains $GF(q) \setminus \{0\}$. Then $M$ has exactly $\prod_{i=1}^{\frac{q-1}{2}} (q - p^i)$ different special standard representations over $GF(q)$.
Proof. Let $M' = M[A']$ be an $n$-spike on the same ground set of $M$, where

$$
A' = \begin{bmatrix}
e_1 & e_2 & e_3 & \ldots & e_n & t & f_1 & f_2 & f_3 & \ldots & f_n \\
1 & 0 & 0 & \ldots & 0 & 1 & 1+y_1 & 1 & 1 & \ldots & 1 \\
0 & 1 & 0 & \ldots & 0 & 1 & 1 & 1+y_2 & 1 & \ldots & 1 \\
0 & 0 & 1 & \ldots & 0 & 1 & 1 & 1+y_3 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 1 & 1 & 1 & 1 & \ldots & 1+y_n 
\end{bmatrix}
$$

is a special standard representation of $M'$ over $GF(q)$. Let $B = \{e_1, e_2, \ldots, e_n\}$ be the distinguished basis. For a set $S$ which is a subset of the set $\{1, 2, \ldots, n\}$, let $H(S)$ denote the set $(B \cup \cup_{i \in S} e_i) \cup (\cup_{i \in S} f_i)$.

Suppose that $M' = M$. Then $M$ and $M'$ have the same circuit-hyperplanes. Clearly, $H(S)$ is a circuit-hyperplane of $M$ if and only if the sub-matrix of $A$ whose columns are labeled by $\cup_{i \in S} e_i$ and whose rows are labeled by $\cup_{i \in S} f_i$ has zero determinant. Therefore, $H(\{i\})$ is a circuit-hyperplane of $M$ if and only if $1 + x_i = 0$. Since $M = M'$, we deduce that

(i) $1+y_i = 0$ if and only if $1+x_i = 0$.

Since the multi-set $\{x_1, x_2, \ldots, x_n\}$ contains $GF(q) \setminus \{0\}$, we deduce that there is a pair of elements $i$ and $j$ in $\{1, 2, \ldots, n\}$ such that $H(\{i, j\})$ is a circuit-hyperplane of $M$. It is easily seen that $H(\{i, j\})$ is a circuit-hyperplane if and only if $1 + x_i = (1 + x_j)^{-1}$. We deduce that

(ii) $1+y_i = (1+y_j)^{-1}$ if and only if $1+x_i = (1+x_j)^{-1}$. 

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Moreover, if both $H({i, j})$ and $H({i, k})$ are circuit-hyperplanes of $M$, then $1 + x_j = 1 + x_k$, and $1 + y_j = 1 + y_k$. We conclude that

(iii) $1 + y_i = 1 + y_j$ if and only if $1 + x_i = 1 + x_j$.

Suppose that $f$ is a one-to-one mapping on $GF(q)$ satisfying $f(1) = 1$ and $f(0) = 0$. Let $f(A)$ denote the matrix obtained by replacing each entry of $A$ by its image under $f$. Then $f(A)$ is a special standard representation of some $n$-spike. We say that $f$ preserves the circuit-hyperplane $H$ if $H$ is a circuit-hyperplane of both $M[A]$ and $M[f(A)]$. Moreover, $f$ is said to preserve all circuit-hyperplanes of $M[A]$ if $M[A]$ and $M[f(A)]$ have the same circuit-hyperplanes.

Since $A'$ is a special standard representation of $M[A]$, it follows by (i) and (iii) that there is a unique one-to-one mapping on $GF(q)$ such that $f(A) = A'$, and

(a) $f(0) = 0$, and $f(1) = 1$.

Clearly, the number of different special standard representations of $M[A]$ equals the number of one-to-one mappings over $GF(q)$ which satisfy (a) and preserve all circuit-hyperplanes of $M[A]$. In order to count the number of different special standard representations of $M[A]$, we count the number of such mappings.

Suppose that $f$ is a one-to-one mapping on $GF(q)$ which satisfies (a) and preserves all circuit-hyperplanes of $M[A]$. By (ii), we deduce that

(b) $f(1 + x) = 1 + y$ if and only if $f(\frac{1}{1 + x}) = \frac{1}{1 + y}$.

For an $x$ in $GF(q) \setminus \{0\}$, by the assumption that the multi-set $\{x_1, x_2, \ldots, x_n\}$ contains the set $GF(q) \setminus \{0\}$, we deduce that there is a set $\{i, j, k\} \subseteq \{1, 2, \ldots, n\}$, such
that \( x_i = 0, x_j = 1 + x, \) and \( x_k = 1 - x. \) By calculating the corresponding determinant, we deduce that \( H(\{i, j, k\}) \) is a circuit-hyperplane of \( M[A]. \) It follows by (a) that

\( (c) \quad f(1+x) = 1+y \) if and only if \( f(1-x) = 1-y. \)

Recall that \( \omega \) is a root of a \( GF(p) \)-irreducible polynomial of degree \( s. \) Then

\[
GF(q) \setminus \{1\} = \{1 + \frac{\omega^{s-1}}{\sum_{j=0}^{s-1} a_j \omega^j} | a_j \in GF(p), \text{ and } \{a_1, a_2, \ldots, a_{s-1}\} \neq \{0\} \}.
\]

Let \( \sigma_1, \sigma_2, \ldots, \sigma_{s-1} \) be elements in \( GF(q) \setminus GF(p), \) such that

\[
f(1+\omega^i) = 1+\prod_{j=1}^i \sigma_j \quad \text{for all } j \in \{1, 2, \ldots, s-1\}.
\]

We assert that

\( (d) \quad f(1+\frac{\omega^i}{\sum_{j=0}^{s-1} a_j \omega^j}) = 1+\frac{\prod_{j=1}^i \sigma_j}{\sum_{j=0}^{s-1} a_j \prod_{k=1}^j \sigma_k} \) for all \( i \in \{0, 1, \ldots, s-1\}. \)

To prove (d), we first prove, by induction, that

\( (e) \quad f(1+\frac{1}{1+k}) = 1+\frac{1}{1+k} \) for all \( k \in \{0, 1, \ldots, p-2\}. \)

By (a), we have \( f(1-1) = 0 = 1-1. \) It follows by (c) that \( f(1+1) = 1+1. \) Therefore, (e) holds for \( k = 0. \) Suppose that (e) holds for some \( k \geq 0. \) Then it follows by (b) that

\[
f(1-\frac{1}{2+k}) = f(\frac{1+k}{2+k}) = f((1+\frac{1}{1+k})^{-1}) = (1+\frac{1}{1+k})^{-1} = 1-\frac{1}{2+k}.
\]

By (c), we deduce that

\[
f(1+\frac{1}{2+k}) = 1+\frac{1}{2+k},
\]

thereby proving (e) by induction.
Next we prove that, for each \( i \) in \( \{1, 2, \ldots, s-1\} \), and for all \( a, b \in GF(p) \) such that \( \{a, b\} \neq \{0\} \),

\[
(f) \quad f(1 + \frac{\omega^i}{a + b\omega^i}) = 1 + \frac{\prod_{j=1}^{i-1} \sigma_j}{a + b\prod_{j=1}^{i} \sigma_j}.
\]

This proof is also by induction. Note that \((f)\) follows by \((e)\) if \( a = 0 \). We first show that \((f)\) holds for \( a = 1 \). Since \( f(1 + \omega^i) = 1 + \prod_{j=1}^{i} \sigma_j \), we see that \((f)\) holds when \( a = 1 \) and \( b = 0 \). Suppose that \((f)\) holds when \( a = 1 \) and \( b = k \) for some \( k \geq 0 \). Thus,

\[
(1 + \frac{\omega^i}{1 + k\omega^i}) = 1 + \frac{\prod_{j=1}^{i} \sigma_j}{1 + k\prod_{j=1}^{i} \sigma_j}.
\]

By \((b)\), we have

\[
f((1 + \frac{\omega^i}{1 + k\omega^i})^{-1}) = (1 + \frac{\prod_{j=1}^{i} \sigma_j}{1 + k\prod_{j=1}^{i} \sigma_j})^{-1},
\]

that is,

\[
f(1 - \frac{\omega^i}{1 + (1 + k)\omega^i}) = 1 - \frac{\prod_{j=1}^{i} \sigma_j}{1 + (1 + k)\prod_{j=1}^{i} \sigma_j}.
\]

It follows by \((c)\) that

\[
f(1 + \frac{\omega^i}{1 + (1 + k)\omega^i}) = 1 + \frac{\prod_{j=1}^{i} \sigma_j}{1 + (1 + k)\prod_{j=1}^{i} \sigma_j}.
\]

Therefore, by induction, we conclude that \((f)\) holds for the case that \( a = 1 \).

Suppose that, for some \( k \geq 1 \), \((f)\) holds for all \( a \leq k \). By the assumption that the multi-set \( \{x_1, x_2, \ldots, x_n\} \) contains \( GF(q) \backslash \{0\} \), we deduce that there are \( u, v, \) and \( w \) in \( \{1, 2, \ldots, n\} \), such that

\[
1 + x_u = 1 - \frac{\omega^i}{k + bw^i}, \quad 1 + x_v = 1 - \frac{\omega^i}{1 + \omega^i}, \quad \text{and} \quad 1 + x_w = 1 + \frac{\omega^i}{(1 + k) + bw^i}.
\]
By (2.2.6), the determinant corresponding to $H(\{u, v, w\})$ is zero. Therefore, the set $H(\{u, v, w\})$ is a circuit-hyperplane of $M[A]$. By the induction assumption, we have

$$f(1 + x_u) = 1 - \frac{\prod_{j=1}^{i} \sigma_j}{k + b \prod_{j=1}^{i} \sigma_j}, \quad \text{and}$$

$$f(1 + x_v) = 1 - \frac{\prod_{j=1}^{i} \sigma_j}{1 + \prod_{j=1}^{i} \sigma_j}.$$  

Since $f$ preserves all circuit-hyperplanes, it follows by (2.2.6) that

$$f(1 + x_w) = 1 + \frac{\prod_{j=1}^{i} \sigma_j}{(1 + k) + b \prod_{j=1}^{i} \sigma_j}.$$  

Equation (f) follows by induction.

Using (e) and (f), we now prove (d). It follows by (e) that (d) certainly holds for $i = 0$. Suppose that, for some $k \geq 0$, (d) holds for all $i \leq k$. Consider

$$f(1 + \frac{\omega^{k+1}}{\sum_{j=0}^{k+1} a_j \omega^j}).$$  

If $a_0 = 0$, then

$$1 + \frac{\omega^{k+1}}{\sum_{j=0}^{k+1} a_j \omega^j} = 1 + \frac{\omega^{k}}{\sum_{j=0}^{k} a_j \omega^j},$$  

and (d) follows by the induction assumption. Thus, we assume that $a_0 \neq 0$. Since the multi-set $\{x_1, x_2, \ldots, x_n\}$ contains $GF(q)\{0\}$, we deduce that there are $u, v$, and $w$ in $\{1, 2, \ldots, n\}$, such that

$$1 + x_u = 1 - \frac{\omega^{k}}{\sum_{j=0}^{k-1} a_j \omega^j + \omega^k}, \quad 1 + x_v = 1 - \frac{\omega^{k+1}}{a_0 + a_{k+1} \omega^{k+1}}, \quad \text{and} \quad 1 + x_w = 1 + \frac{\omega^{k+1}}{\sum_{j=0}^{k+1} a_j \omega^j}.$$  

By (2.2.6), the determinant of the matrix corresponding $H(\{u, v, w\})$ is zero. Therefore, $H(\{u, v, w\})$ is a circuit-hyperplane of $M[A]$. By (c), (f), and the induction assumption, we deduce that

$$f(1 + x_u) = 1 - \frac{\prod_{j=1}^{k} \sigma_j}{\sum_{j=0}^{k} a_j \prod_{j=1}^{k} \sigma_j}, \quad \text{and}$$

$$f(1 + x_v) = 1 - \frac{\prod_{j=1}^{k} \sigma_j}{\sum_{j=0}^{k} a_j \prod_{j=1}^{k} \sigma_j}, \quad \text{and}$$

$$f(1 + x_w) = 1 + \frac{\prod_{j=1}^{k} \sigma_j}{(1 + k) + b \prod_{j=1}^{k} \sigma_j}.$$  

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\[ f(1 + x_w) = 1 - \frac{\prod_{j=1}^{k+1} \sigma_j}{a_0 + a_{k+1} \prod_{j=1}^{k+1} \sigma_j}. \]

Since \( f \) preserves all circuit-hyperplanes, it follows by (2.2.6) that

\[ f(1 + x_w) = 1 + \frac{\prod_{j=1}^{k+1} \sigma_j}{\sum_{j=0}^{k+1} a_j \prod_{i=1}^{k+1} \sigma_i}. \]

Equation (d) follows by induction.

Overall, we have proved that if \( f \) is a one-to-one mapping over \( GF(q) \) that satisfies (a) and preserves all circuit-hyperplanes of \( M[A] \), then \( f \) satisfies (d) for certain \( \sigma_1, \sigma_2, \ldots, \sigma_{s-1} \). In particular, \( f \) satisfies

\[ (g) \quad f(1 + \frac{\omega^{s-1}}{\sum_{j=0}^{s-1} a_j \omega^j}) = 1 + \frac{\prod_{j=0}^{s-1} \sigma_j}{\sum_{j=0}^{s-1} a_j \prod_{k=1}^{s-1} \sigma_k}. \]

On the other hand, consider a one-to-one mapping on \( GF(q) \) that satisfies (a) and (g). Suppose that \( T \subseteq \{1, 2, \ldots, n\} \), and \( |T| = t \). We may assume that \( T = \{1, 2, \ldots, t\} \).

Suppose that, for all \( j \in T \),

\[ 1 + x_j = 1 + \frac{\omega^{s-1}}{\sum_{i=0}^{s-1} a_{ij} \omega^i}. \]

By (2.2.6), \( H(T) \) is a circuit-hyperplane of \( M[A] \) if and only if

\[ (h) \quad \sum_{j=1}^{t} \frac{\sum_{i=0}^{s-1} a_{ij} \omega^i}{\omega^{s-1}} + 1 = 0. \]

Since \( \omega \) is a root of a \( GF(p) \)-irreducible polynomial of degree \( s \), (h) is equivalent to the combination of the following equations:

\[ (j) \quad \sum_{j=1}^{t} a_{ij} = 0 \text{ for all } i \in \{0, 1, \ldots, s-2\}; \text{ and} \]

\[ (k) \quad \sum_{j=1}^{t} a_{ij} = -1 \text{ for } i = s - 1. \]

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Since $f$ is one-to-one, we conclude that the set \( \{ \Pi_{j=1}^{s} \sigma_i \mid i \in \{0, 1, \ldots, s-1\} \} \) is a basis of $GF(q)$ when we view $GF(q)$ as an $n$-dimensional vector space over $GF(p)$. Therefore, the combination of (j) and (k) is equivalent to
\[
(m) \quad \sum_{j=0}^{s-1} a_{ij} \Pi_{k=1}^{s} \sigma_k + 1 = 0.
\]
By (2.2.6) and (g), we deduce that (m) holds if and only if $H(T)$ is a circuit of $M[f(A)]$. Since $H(T)$ is of size $n$ and $M[f(A)]$ is an $n$-spike, $H(T)$ is a circuit if and only if it is a circuit-hyperplane. We conclude that a one-to-one mapping $f$ over $GF(q)$ satisfying (a) and (g) preserves all circuit-hyperplanes of $M[A]$.

We now count the number of one-to-one mappings $f$ that satisfy (a) and (g). Clearly, (e) is a special case of (g). Therefore, $f$ keeps elements of $GF(p)$ unchanged. We deduce that $1 + \sigma_1$, that is, $f(1 + \omega)$, is not in $GF(q) \setminus GF(p)$. Thus, there are $(q - p)$ choices for $\sigma_1$. Suppose that $\sigma_1$ has been chosen. By (d) with $i = 1$, which is another special case of (g), we conclude that
\[
f(1 + \sigma_1) = 1 + \frac{\sigma_1}{a_0 + a_1 \omega}.
\]
Therefore, $p^2$ elements are fixed after $\sigma_1$ has been chosen, and so there are exactly $(q - p^2)$ choices left for $\sigma_2 \sigma_1$. After we have chosen $\sigma_2$, we deduce, according to (d) with $i = 2$, that $p^3$ elements are fixed, and hence there are exactly $(q - p^3)$ choices left for $\sigma_3$. Continuing in this way, we conclude that the total number of one-to-one mappings $f$ of the required type is $\Pi_{i=1}^{s-1}(q - p^i)$.

By (c) of the proof of the last lemma, we conclude that, the total number of $x$'s and $-x$'s in the diagonal corresponding to $A$ equals the total number of $y$'s and $-y$'s in the diagonal corresponding to $f(A)$. Therefore, if the diagonal corresponding to
A is a diagonal associated with a special sequence \( a_1, a_2, \ldots, a_{\frac{q-1}{2}} \), then, by the fact that \( f \) keeps \( GF(p) \) unchanged, we deduce that the diagonal corresponding to \( f(A) \) is a diagonal associated to a special sequence that is a permutation of \( a_1, a_2, \ldots, a_{\frac{q-1}{2}} \). For a special sequence \( a_1, a_2, \ldots, a_{\frac{q-1}{2}} \), let \( \mathcal{W} \) be the set of all quasi-distinct normal diagonals associated with a special sequence that is a permutation of \( a_1, a_2, \ldots, a_{\frac{q-1}{2}} \). Since \( a_1 > a_i \) for all \( i > 1 \), if a permutation of \( a_1, a_2, \ldots, a_{\frac{q-1}{2}} \) is again a special sequence, we conclude, by definition, that \( a_1 \) remains as the first term of that permutation. Because all \( a_i \)'s are distinct, there are \( \left( \frac{q-3}{2} \right)! \) distinguishable permutations of \( a_2, a_3, \ldots, a_{\frac{q-1}{2}} \).

By (2.4.3), we deduce that
\[
|\mathcal{W}| = \left( \frac{q+1}{2} \right) \left( \frac{q-3}{2} \right)!,
\]
and we conclude, by (2.4.4), that the number of non-isomorphic \( GF(q) \)-representable \( n \)-spikes corresponding to diagonals in \( \mathcal{W} \) is
\[
\left( \frac{q+1}{2} \right) \left( \frac{q-3}{2} \right)! \frac{1}{\Pi_{i=1}^{\frac{q-1}{2}} (q - p^i)}.
\]

Proof of Theorem 2.1.6. Consider the diagonal
\[
(d_1^{(k_1)}, (-d_1)^{(l_1)}, d_2^{(k_2)}, (-d_2)^{(l_2)}, \ldots, (d_{\frac{q-1}{2}})^{(k_{\frac{q-1}{2}})}, (-d_{\frac{q-1}{2}})^{(l_{\frac{q-1}{2}})}).
\]
For each \( i \), let \( a_i = k_i + l_i \). Then the above diagonal is a diagonal associated with the sequence of non-negative integers \( a_1, a_2, \ldots, a_{\frac{q-1}{2}} \). By (2.2.8), we may assume that \( 0 \leq k_i \leq p - 1 \) for all \( i \) in \( \{1, 2, \ldots, \frac{q-1}{2} \} \). Therefore, there are at most \( p^{\frac{q-1}{2}} \) quasi-distinct diagonals associated with the last sequence. The total number of such sequences is \( p_{\leq \frac{q-1}{2}} (n) \). By (2.2.9)(ii), the last partition number is \( p_{\leq \frac{q-1}{2}} (n + \frac{q-1}{2}) \). By the definition of

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special sequences, it is not hard to see that the number of special sequences is \( p_{\frac{n-1}{2}}^2(n - \frac{(p-1)(q-1)}{2}) \). By (2.2.9)(iii), the last partition number equals \( p_{\frac{n-1}{2}}^2(n - \frac{(p-1)(q-1)}{2}) - \frac{(q-1)(q-3)}{8} \). Part (i) of the theorem follows directly using the discussion of the last paragraph. Let

\[
N_1 = p_{\frac{n-1}{2}}^2(n + \frac{q-1}{2}) - p_{\frac{n-1}{2}}^2(n - \frac{(p-1)(q-1)}{2}) - \frac{(q-1)(q-3)}{8}.
\]

Then \( N_1 \) is the number of non-negative sequences \( a_1, a_2, \ldots, a_{\frac{n-1}{2}} \) that are not special sequences. Therefore, the number of quasi-distinct diagonals that are not associated with some special sequence is at most \( p_{\frac{n-1}{2}}^2(n + \frac{q-1}{2})!N_1 \). Thus, we conclude that

\[
N(n, q) - p_{\frac{n-1}{2}}^2(n + \frac{q-1}{2})!N_1 \leq \frac{(q-1)(q-3)}{n-1}!p_{\frac{n-1}{2}}^2(n - \frac{(p-1)(q-1)}{2}) - \frac{(q-1)(q-3)}{8}.
\]

By (2.2.9)(iv), we deduce that

\[
p_{\frac{n-1}{2}}^2(n + \frac{q-1}{2}) = \frac{1}{(\frac{q-1}{2})!(\frac{q-3}{2})!}n^{\frac{q-3}{2}} + o(n^{\frac{q-3}{2}}); \quad \text{and}
\]

\[
p_{\frac{n-1}{2}}^2(n - \frac{(p-1)(q-1)}{2}) - \frac{(q-1)(q-3)}{8} = \frac{1}{(\frac{q-1}{2})!(\frac{q-3}{2})!}n^{\frac{q-3}{2}} + o(n^{\frac{q-3}{2}}).
\]

Therefore,

\[
\lim_{n \to \infty} \frac{N_1}{n^{\frac{q-3}{2}}} = 0.
\]

Part (ii) of the theorem follows by the last equation and (2.2.9)(iv). \( \square \)

**Proof of Theorem 2.1.7.** Since \( GF(q) \) has characteristic two, we deduce that \( d = -d \) for each \( d \) in \( GF(q) \). Let \( GF(q) \setminus \{0\} = \{d_1, d_2, \ldots, d_{q-1}\} \). Then the diagonal of a \( GF(q) \)-representable \( n \)-spike has the form

\[
(d_1^{(a_1)}, d_2^{(a_2)}, \ldots, d_{q-1}^{(a_{q-1})}).
\]

In this case, we define a special sequence to be a sequence of \( q-1 \) distinct positive integers \( a_1, a_2, \ldots, a_{q-1} \) where \( a_1 > a_i \) for all \( i > 1 \). Then, by a similar argument to
that given for (2.4.1), it is easy to see that a diagonal associated with a sequence that is a permutation of a special sequence is quasi-equivalent to a diagonal associated with some special sequence. Moreover, (2.4.4) holds in this case. For a special sequence $a_1, a_2, \ldots, a_{q-1}$, let $\mathcal{W}$ be the set of all quasi-distinct normal diagonals associated with a special sequence that is a permutation of $a_1, a_2, \ldots, a_{q-1}$. Since $a_1 > a_i$ for all $i > 1$, if a permutation of $a_1, a_2, \ldots, a_{q-1}$ is again a special sequence, we conclude, by definition, that $a_1$ remains as the first term of that permutation. Because all $a_i$'s are distinct, there are $(q - 2)!$ distinguishable permutations of $a_2, a_3, \ldots, a_{q-1}$. We deduce that $|\mathcal{W}| = (q - 2)!$, and we conclude, by (2.4.4), that the number of non-isomorphic $GF(q)$-representable $n$-spikes corresponding to diagonals in $\mathcal{W}$ is $(q - 2)! \prod_{i=1}^{q-2} \frac{1}{(q-p)}$.

The theorem follows by an argument similar to that given for the last theorem. We shall omit the details to avoid repetition. $\square$
CONCLUSION

The classes of unavoidable 3–connected matroids listed in [5] are characterized here in terms of extremal conditions. The structures and properties of wheels and whirls, and $U_{2,n+2}$ and $U_{n,n+2}$ are well known. Theorem 1.1.2 proves that the collection of 2–minimally, 1–cominimally, 3–connected matroids of rank at least four is precisely the collection of matroids $M^*(K_{3,n})$ for all $n \geq 3$. This result helps us to reach a thorough understanding of the matroids $M^*(K_{3,n})$ and $M(K_{3,n})$.

In the first chapter, Theorem 1.1.1 explores the relations between the class of 2–minimally, 2–cominimally, 3–connected matroids and the class of $n$–spikes, thereby helping us to deepen our understanding of spikes, the remaining class of matroids in the list of unavoidable 3–connected matroids.

For a prime number $p$, Lemma 2.3.6 proves that a $GF(p)$–representable $n$–spike is uniquely representable over $GF(p)$, provided $n \geq \frac{(p-1)^2}{2}$. Moreover, it can be proved using Lemmas 2.3.5 and 2.3.6 that such an $n$–spike is not representable over any field with characteristic not equal to $p$. The last result, and Theorems 2.1.1-2.1.7, which investigate the number of $n$–spikes representable over finite fields, provide us with the best understanding so far of the representability of spikes over finite fields.
REFERENCES


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May 14, 1998

File: 4286
Paper: On extremal connectivity properties of unavoidable matroids
Author: Zhaoyang Wu

Dear Dr. Wu,

Your above entitled paper has been accepted for publication in the Journal of Combinatorial Theory, Series B, subject to minor revision. We have no objection to your including this paper as a part of your Ph.D. dissertation.

Yours sincerely,

U.S.R. Murty
Editor in Chief
VITA

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