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Multiobjective Optimal Filtering and Control.

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MULTIOBJECTIVE OPTIMAL FILTERING AND CONTROL

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To My Mother

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NOTATIONS

The notations used in this dissertation are standard and most of them are defined before they are used. Some widely recognized notations used without definitions are listed here:

I always denotes the identity matrix with dimensions determined in context.

$E\{\cdot\}$ is the expectation operator.

$exp\{\cdot\}$ is the (matrix) exponential function.

If A is a matrix or a vector, then A^T is its transpose and A^* is its conjugate transpose.

If x is a complex number, then $Re(x)$ and $Im(x)$ are, respectively, its real and imaginary parts.

$\mathbf{R}$ is the set of real numbers.

$\mathcal{RH}_\infty$ is the space of all proper and real rational stable matrix transfer functions.

If A is a square matrix, then $trace(A)$ is the trace of A.

Abstract

A fundamental question in feedback control design is how to achieve desired performance under system uncertainties and external disturbances. The well-known LQG and $\mathcal{H}_2$ control design techniques are well suited for achieving some well-defined optimal transient performance under certain classes of stochastic external disturbances such as white noise. However, these optimal control design techniques are highly model dependent and may be very sensitive to parameter variations and system uncertainties. The $\mathcal{H}_\infty$ control theory, on the other hand, was developed precisely because of the desire to overcome these deficiencies. One potential shortfall of the existing $\mathcal{H}_\infty$ control design method is that it is very hard to handle transient performance naturally. Thus it is desirable to develop a systematic design technique that combines the good aspects of both LQG (or $\mathcal{H}_2$) and $\mathcal{H}_\infty$ design techniques. This is precisely the motivation for the multiobjective design framework developed in this dissertation.

Motivated by the development of a time domain game approach for $\mathcal{H}_2/\mathcal{H}_\infty$ control, three multiobjective design problems related to filtering and control are formulated on time domain in this dissertation. Based on a new constrained optimization result proved in this dissertation and $\mathcal{H}_\infty$ control design, the multiobjective filtering problem has been solved by an optimally designed filter while the multiobjective control problems have been solved by combining an optimally designed filter and a feedback gain. It is shown that all the results can be obtained by solving the corresponding set of coupled Riccati equations.

Chapter 1

Introduction

This chapter gives a brief introduction to and motivation for the multiobjective optimal control theory developed in this dissertation. Some previous work in the literature are reviewed and a brief overview of this dissertation is given at the end of the chapter.

1.1 Multiobjective Optimal Design

It is probably fair to say that the most important objective of any control design is to achieve certain desired performance specifications in spite of external disturbances and noises, system parameter variations, and variations of system operating conditions. The desired performance specifications are usually measured in terms of the behavior of the system's steady state response and the behavior of the system's transient response, respectively. For example, the requirements on the steady state error with respect to some desired tracking signals and the requirements on overshoot, rise time, and settling time with respect to a step reference signal are typical steady state and transient performance specifications. Designing a controller to satisfy these performance specifications for an *exactly* known linear system is in general not very hard. Many optimal design methods can be used to achieve (at least approximately) the goal, for instance, some well-known modern state space control techniques such as pole placement, LQG, and $\mathcal{H}_2$ design methods. The most notorious problem associated with those design techniques is the lack of guaranteed robustness with respect to external disturbances and model uncertainties. That is,

the performance of these control laws may be very sensitive to inevitable external disturbances and system uncertainties.

It is precisely the robustness consideration for which feedback control was originally developed. But merely feedback is, however, not sufficient to guarantee the robustness of a control system just as pole placement, LQG, and $\mathcal{H}_2$ control laws are all feedback control laws. In fact, it is in general impossible to design a feedback control law that will perform well in all aspects. For example, it is well known from classical control theory that, to have a good tracking for signals with large bandwidth, one needs large bandwidth for the closed-loop transfer function; on the other hand, to have a good disturbance rejection to measurement noise (usually in a high frequency range), one needs to roll off the high frequency response as much as possible; that is, one would prefer to have low bandwidth for the closed-loop transfer function. These objectives clearly contradict each other. What a judicially designed feedback can usually achieve is to improve the system performance in one aspect by sacrificing the system performance in another aspect. Thus a feedback control design is a process of making tradeoffs between conflicting objectives.

Two prominent conflicting objectives in most feedback control designs are good transient response and robustness with respect to disturbances and system uncertainties. Usually a very robust control law tends to make the system's transient response poor. On the other hand, a system with an extremely good transient response for a nominal operation condition (or model) may be very sensitive to external disturbances and parameter variations. This can be easily understood by considering, for example, the manoeuvrability and the ability to handle the road roughness between driving a tank and riding a bike. In this case a good control design should be a compromise between good transient performance and robustness.

It is generally agreed that an LQG or $\mathcal{H}_2$ criterion can be a good measure for transient performance, while the $\mathcal{H}_\infty$ optimal control design framework is developed primarily because of the robustness consideration. Thus it is natural to consider a design framework that can systematically make the design tradeoffs between these two design objectives. The development of this multiobjective design framework is the main topic of this dissertation.

1.2 Overview on Multiobjective Filtering and Control

The multiobjective control problem has received a lot of attention from the control research community in the past decade [4, 5, 7, 10, 11, 14, 17, 22, 26, 30, 32, 34, 33, 36, 44, 48]. Many different formulations have been proposed in the literature (see also [40], though it is the $\mathcal{H}_2/\mathcal{H}_\infty$ approach has a better physical interpretation and clearer motivation as discussed in the last section, and attracts a great number of researchers. It should be pointed out that the term $\mathcal{H}_2/\mathcal{H}_\infty$ usually is assigned to any multiobjective optimal design of which the performance measures have both $\mathcal{H}_2(LQG)$ and $\mathcal{H}_\infty$ interpretations.

Some major results about multiobjective control with an $\mathcal{H}_2/\mathcal{H}_\infty$ interpretation are discussed as follows.

- Fixed-order controller design by minimizing an auxiliary cost functional [4, 17, 18]: This formulation minimizes an upper bound on the $\mathcal{H}_2$ norm of the closed-loop transfer function subject to an $\mathcal{H}_\infty$ norm constraint.

- Convex optimization using LMI [22, 14, 19, 11, 7, 5]: The advantage of the convex optimization approach is that there exist effective and powerful algo-

rithms for the solutions of these problems. However, it is difficult to generalize this approach to nonlinear system.

- Optimizing an entropy cost functional [29, 30, 15]: This approach designs a controller to minimize the so-called closed-loop entropy which provides an upper bound of $\mathcal{H}_2$ cost, while guarantee the $\mathcal{H}_\infty$ performance. It turns out that this approach is equivalent to the approach of minimizing an auxiliary cost functional in [4] for the single external input case[29].

- Power signal characterization[10, 48]: This approach can treat systems with both white noise and bounded power disturbances. The design objective is to minimize the power of the output error signal. It is essentially a time domain approach and has a close relationship with the problems considered in the subsequent chapters.

- Nash game approach [26, 34]: This approach uses the Nash equilibrium strategy as performance measure to characterize the problem with a very clear $\mathcal{H}_2/\mathcal{H}_\infty$ interpretation. It is also possible to generalize this approach to nonlinear system [27]. Another benefit we can get from this approach is that it allows us to define $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems on finite time horizon, which, clearly, could not be done by frequency-domain approaches. However, only a state feedback problem was solved in [26] and the output feedback problem turns out to be very difficult [34].

Another multiobjective optimization topic is the so-called multiobjective filtering design problem. This problem is interesting because it can either provide solutions for multiobjective optimal control as shown in this dissertation or for robust signal

processing. One significant work was done by Khargonekar and Rotea [23] using the frequency domain approach and Kalman filter structure.

Motivated by the bounded power approach, game approach and the work in [23], three new multiobjective filtering and control problems, that is, multiobjective filtering, multiobjective output feedback control and $\mathcal{H}_\infty$ Gaussian control, are formulated in this dissertation. It turns out that, comparing with the existing $\mathcal{H}_2/\mathcal{H}_\infty$ formulations stated above, the formulations in this dissertation have clearer motivation and natural $\mathcal{H}_2/\mathcal{H}_\infty$ interpretations, and more important, the solutions to these problems are given in simple and computable forms.

1.3 Dissertation Overview

The purpose of this dissertation is to provide a systematic, self-contained and, in most cases, rigorous presentation for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering and control theory in time domain. The dissertation is organized as follows: After Introduction, Chapter 2 provides some preliminary results from signals, $\mathcal{H}_2/\mathcal{H}_\infty$ control theory and proves a new constrained optimization problem which is used throughout the subsequent chapters. Chapters 3, 4, 5 are devoted to solve multiobjective optimal filtering, control problems and $\mathcal{H}_\infty$ Gaussian control design. Finally, conclusions can be found in Chapter 6.

Chapter 2

Preliminary Results

In this chapter, some important preliminary results are presented. Complete proofs for most of these results, though well-known, are still given bearing in mind two purposes: first, it is the author's desire to make this dissertation self-contained; second, some new proofs for old results reflect new point of view on the problem (e.g. output feedback $\mathcal{H}_\infty$ control), which provides the motivation for the design results obtained in Chapters 3, 4, and 5. Some results in control theory (e.g. transition matrix for a time-varying system), which could be easily found in standard textbooks, are used directly without citations and proofs.

In Section 2.1, we first introduce $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms of a matrix transfer function, then we discuss the properties of bounded power signals and white noise signals. In Section 2.2, a new constrained optimization problem is solved and the results will be used throughout this dissertation. In Section 2.3, $\mathcal{H}_2(LQG)$ control design is presented while, in Section 2.4, $\mathcal{H}_\infty$ control design is addressed.

2.1 Norms and Signals

In this section, we shall give the definitions of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norm of a matrix transfer function. These norms will be used as performance measures for multiobjective filtering and control designs. There are two classes of stochastic signals which are of interests in this dissertation for the development of multiobjective filtering and control design. The first class is called the bounded power signal and the second is the well-known white noise signal. The basic definitions and properties of bounded

power signals will be given in Section 2.1.2, while those of white noise signals are summarized in Section 2.1.3. We shall also present the relations between these signals and norms of a matrix transfer function. It should be pointed out that a deterministic version of bounded power signals can also be defined (see [48] and [13]) and the results obtained in this dissertation can, then, be derived correspondingly in a deterministic framework.

2.1.1 $\mathcal{H}_\infty$ AND $\mathcal{H}_2$ NORMS

A norm is a real-valued functional $||\cdot||$ defined on some vector space X (of signals or systems) if it satisfies the following properties:

1. $||x|| \geq 0$,
2. $||x|| = 0$ if and only if $x = 0$,
3. $||\alpha x|| = |\alpha|||x||$, for any scalar α,
4. $||x + y|| \leq ||x|| + ||y||$,

for any $x \in X$ and $y \in X$. A real-valued functional $||\cdot||$ is called a semi-norm on X if it satisfies properties 1, 3 and 4 but not necessarily 2.

Given a $G(s) \in \mathcal{RH}_\infty$ with a state space realization (A, B, C, D), we denote

$$G(s) = D + C(sI - A)^{-1}B := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

The $\mathcal{H}_\infty$ norm of a stable $G(s)$ is defined as:

$$||G(s)||_\infty := \sup_\omega \bar{\sigma}\{G(j\omega)\}$$

where $\bar{\sigma}(s)$ is the largest singular value of $G(s)$.

The $\mathcal{H}_2$ norm of a stable $G(s)$ with $D = 0$ is defined as

$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} trace[G^*(j\omega)G(j\omega)]d\omega.}$$

2.1.2 Bounded Power Signals

Given a real stochastic signal u: $u = [u_1(t) \quad u_2(t) \quad \cdots \quad u_m(t)]^T \in \mathbf{R}^m$, where $u_i(t), \quad i = 1, \cdots, m$ are real stationary random processes, define the **mean** and **autocorrelation matrix** of u, if they exist, respectively, as follows:

$$E\{u\} := \begin{bmatrix} E\{u_1(t)\} \\ E\{u_2(t)\} \\ \vdots \\ E\{u_m(t)\} \end{bmatrix}, \quad R_{uu}(\tau) := \lim_{T\to\infty}\frac{1}{T}\int_0^T E\{u(t+\tau)u^T(t)\}dt.$$

The Fourier transform of $R_{uu}(\tau)$, if it exists, is as follows:

$$S_{uu} := \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{uu}(\tau)e^{-j\omega\tau}d\tau.$$

We shall be interested in the set of signals u for which both R_{uu} and S_{uu} exist. The so-called bounded power signal is defined as follows:

Definition 2.1 *A vector stationary stochastic signal u is said to have bounded power if*

1. *both R_{uu} and S_{uu} exist,*

2. $$\lim_{T\to\infty}\frac{1}{T}\int_0^T E\{\|u\|^2\}dt < \infty.$$

Let $\mathcal{P}$ be the space of all signals with bounded power. A seminorm can be defined on $\mathcal{P}$:

$$\|u\|_{\mathcal{P}} := \sqrt{\lim_{T\to\infty} \frac{1}{T}\int_0^T E\{\|u\|^2\}dt} = \sqrt{trace[R_{uu}(0)]}, \quad \forall u \in \mathcal{P}.$$

An important property of bounded power signals is that we can use them to induce the $\mathcal{H}_\infty$ norm of a system. For a stable system shown in Figure 2.1:

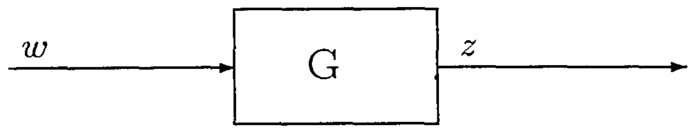

Figure 2.1: A Stable System Driven by A Bounded Power Signal

Let w be a bounded power signal, we have

$$\|G(s)\|_\infty = \sup_w \frac{\sqrt{\lim_{T\to\infty} \frac{1}{T}\int_0^T E\{\|z\|^2\}dt}}{\sqrt{\lim_{T\to\infty} \frac{1}{T}\int_0^T E\{\|w\|^2\}dt}},$$

where '*sup*' is taken over all bounded power signals w.

2.1.3 Gaussian White Noise Signals

Many sources of noise signals in engineering are normally modeled by the well-known Gaussian white noise. Mathematically, a Gaussian white noise $w_0(t)$ is a stationary random process that satisfies:

W1) $E\{w_0(t)\} \equiv 0$,

W2) $E\{w_0(t)w_0^T(\tau)\} = Q(t)\delta(t-\tau)$,

where $\delta(t)$ is Dirac δ function and $Q(t)$ is a positive definite matrix. In this dissertation, we shall assume, without loss of generality, $Q(t) = I$, where I is an identity matrix, i.e., $w_0(t)$ is a zero mean stationary process with an identity power

spectrum. A more rigorous description of the white noise process can be found in standard textbooks for stochastic processes (see, e.g., [41]).

White noise signals have a natural relation with the $\mathcal{H}_2$ norm of a stable system as shown in Figure 2.2:

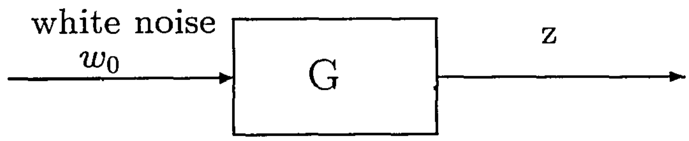

Figure 2.2: A Stable System Driven by a White Noise

For this system, we have

$$\|G(s)\|_2 := \sqrt{\lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T \|z\|^2 dt\right\}}.$$

Indeed, let G be a system described by

$$\begin{aligned} \dot{x} &= Ax + Bw_0, \quad x(0) = 0, \\ z &= Cx, \end{aligned}$$

then the impulse response of this system is $g(t) = Cexp(At)B$. By Parseval Theorem, we have

$$\|G(s)\|_2 = \sqrt{\int_0^\infty trace[g^*(t)g(t)]dt} = \sqrt{trace(B^T QB)}$$

where $Q = \int_0^\infty exp(A^T t)C^T Cexp(At)dt \geq 0$ solves the following Lyapunov equation

$$A^T Q + QA + C^T C = 0.$$

Now, since $z(t) = \int_0^t Cexp[A(t-\tau)]Bw_0(\tau)d\tau$, we have

$$\lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T \|z\|^2 dt\right\}$$

$$= \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T\int_0^t\int_0^t w_0^T(\tau)B^T exp(A^T\tau)C^T C exp(As)Bw_0(s)ds\ d\tau\ dt\right\}$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^t\int_0^t trace\{B^T exp(A^T\tau)C^T C exp(As)BE[w_0(s)w_0^T(\tau)]\}dsd\tau dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^t\int_0^t trace\{B^T exp(A^T\tau)C^T C exp(As)B\delta(s-\tau)\}ds\ d\tau\ dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^t trace\{B^T exp(A^T\tau)C^T C exp(A\tau)B\}d\tau\ dt$$

$$= \int_0^\infty trace\{B^T exp(A^T t)C^T C exp(At)\}Bdt = trace\{B^T QB\} = \|G(s)\|_2^2.$$

An important relation between stochastic signals is the so-called independence of signals which is defined as follows:

Definition 2.2 *Two vector stationary stochastic signals $w_1(t)$ and $w_2(t)$ are said to be (mutually) independent if for any $t_1 \geq 0$ and $t_2 \geq 0$*

$$E\{w_1(t_1)w_2^T(t_2)\} = E\{w_1(t_1)\}E^T\{w_2(t_2)\}.$$

2.2 Constrained Optimization

A new constrained optimization problem is solved in this section. The results are used in the subsequent chapters to prove results of multiobjective optimal filtering and control design. Proofs of both sufficient and necessary conditions for the constrained optimization are given.

Given $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times r}$, $C \in \mathbf{R}^{p\times n}$, $D \in \mathbf{R}^{p\times r}$ and $R = DD^T > 0$, define, respectively, the following index functionals:

$$J_1(L(t), P(t)) = trace\left(\int_0^T P(t)dt\right),\quad T \geq 0,$$

where $L(t)$ and $P(t) \geq 0$ on $[0, T]$ with $P(0) = 0$ satisfies:

$$(A + L(t)C)P(t) + P(t)(A + L(t)C)^T + (B + L(t)D)(B + L(t)D)^T = \dot{P}(t), \quad (2.1)$$

and

$$J_2(L,P) = trace(P),$$

where $A + LC$ is Hurwitz and $L,\ \ P \geq 0$ satisfies:

$$(A + LC)P + P(A + LC)^T + (B + LD)(B + LD)^T = 0. \tag{2.2}$$

The constrained optimization problem is defined as:

Problem 1: finding $(L_*(t), P_*(t))$ such that $J_1(L(t), P(t))$ is minimized at $(L_*(t), P_*(t))$, i.e.

$$J_1(L_*(t), P_*(t)) = \min_{L,P} trace\left(\int_0^T P(t)dt\right),$$

where $(L(t), P(t))$ and $(L_*(t), P_*(t))$ are all subject to the constraint (2.1).

Problem 2: finding (L_*, P_*), where $A + L_*C$ is Hurwitz, such that $J_2(L,P)$ is minimized at (L_*, P_*), i.e.

$$J_2(L_*, P_*) = \min_{L,P} trace(P),$$

where (L,P) and (L_*, P_*) are all subject to the constraint (2.2).

Since the values of $J_1(J_2)$ are determined only by $L(t)(L)$ or $P(t)(P)$, for simplicity, $J_1(L(t), P(t))$ and $J_2(L,P)$ will be denoted by $J_1(L(t))$ and $J_2(L)$ hereafter.

The sufficient conditions for this optimization problem are proved in Section 2.2.1, while the necessary conditions are presented in Section 2.2.2.

2.2.1 Sufficient Conditions

The sufficient conditions for the constrained optimization problem are summarized in the next two theorems, reflecting finite time horizon case and infinite time horizon case, respectively.

Theorem 2.1 *Consider* **Problem 1** *defined in finite time horizon. If there is a solution $P_*(t) \geq 0$ on $[0,T]$ with $P_*(0) = 0$ to*

$$(A - BD^TR^{-1}C)P_*(t) + P_*(t)(A - BD^TR^{-1}C)^T - P_*(t)C^TR^{-1}CP_*(t)$$

$$+B(I - D^TR^{-1}D)B^T = \dot{P}_*(t),$$

then $J_1(L(t))$ achieves the minimum value at $L_*(t) = -(P_*(t)C^T + BD^T)R^{-1}$

PROOF. Take $\Delta P(t) = P(t) - P_*(t)$. Then

$$\Delta\dot{P}(t) = (A + LC)\Delta P(t) + \Delta P(t)(A + LC)^T + (L - L_*)R(L - L_*)^T,$$

where $L_*(t) = -(P_*(t)C^T + BD^T)R^{-1}$. Now, let $\Phi(t,0)$ be the transition matrix of $A + L(t)C$. Then

$$\Delta P(t) = \int_0^t \Phi(t,s)(L(s) - L_*(s))R(L(s) - L_*(s))^T\Phi^T(t,s)ds \geq 0$$

for any $L(t)$ and $\Delta P(t) = 0$ if $L(t) = L_*(t)$. Therefore $J(L) - J(L_*) \geq 0$ for any $L(t)$ which means that $J(L(t))$ achieves the minimum value at $L_*(t)$. □

Theorem 2.2 *Consider* **Problem 2** *defined in infinite time horizon case. If there is a stabilizing solution $P_* \geq 0$ for*

$$(A - BD^TR^{-1}C)P_* + P_*(A - BD^TR^{-1}C)^T - P_*C^TR^{-1}CP_*$$

$$+B(I - D^TR^{-1}D)B^T = 0,$$

i.e., $A - BD^TR^{-1}C - P_*C^TR^{-1}C$ is stable, then $J_2(L)$ achieves the minimum value at $L_* = -(P_*C^T + BD^T)R^{-1}$

PROOF. Since P_* is a stabilizing solution, so $A + L_*C$ is Hurwitz, where $L_* = -(P_*C^T + BD^T)R^{-1}$. Now for any L for which $A + LC$ is stable, we have $P \geq 0$ solving

$$(A + LC)P + P(A + LC)^T + (B + LD)(B + LD)^T = 0.$$

Now take $\Delta P = P - P_*$. Then

$$(A + LC)\Delta P + \Delta P(A + LC)^T + (L - L_*)R(L - L_*)^T = 0.$$

By a standard property of Lyapunov equation, we have $\Delta P \geq 0$ and $\Delta P = 0$ if and only if $L = L_*$. Hence $J(L) - J(L_*) \geq 0$ for any L, which means that $J(L)$ achieves the minimum value at L_*. □

2.2.2 Necessary Conditions

The following theorems provide the necessary conditions for finite time horizon and the infinite time horizon case.

Theorem 2.3 *Consider* **Problem 1** *defined in finite time horizon. If there are $L(t)$ and $P(t)$ satisfying*

$$(A + L(t)C)P(t) + P(t)(A + L(t)C)^T + (B + L(t)D)(B + L(t)D)^T = \dot{P}(t),$$

and $L(t)$ is the argument that minimizes $J_1(L(t))$, then there is a solution $P_*(t) \geq 0$ on $[0, T]$ with $P_*(0) = 0$ for

$$(A - BD^TR^{-1}C)P_* + P_*(A - BD^TR^{-1}C)^T - P_*C^TR^{-1}CP_*$$

$$+B(I - D^TR^{-1}D)B^T = \dot{P}_*,$$

and the minimum value of $J_1(L(t))$ is also achieved at $L_* = -(P_*C^T + BD^T)R^{-1}$

Theorem 2.4 *Consider* **Problem 2** *defined in infinite time horizon. Suppose (C, A) is detectable. If there are L_1 and $P_1 \geq 0$, where $A + L_1C$ is Hurwitz and P_1 solves*

$$(A + L_1C)P_1 + P_1(A + L_1C)^T + (B + L_1D)(B + L_1D)^T = 0,$$

such that $J_2(L)$ is minimized at L_1, then there is a $P_* \geq 0$ solving

$$(A - BD^TR^{-1}C)P_* + P_*(A - BD^TR^{-1}C)^T - P_*C^TR^{-1}CP_*$$

$$+B(I - D^TR^{-1}D)B^T = 0.$$

Moreover, an optimal L_* can be obtained as $L_* = -(P_*C^T + BD^T)R^{-1}$ if $A + L_*C$ is Hurwitz.

The proof will be given only for Theorem 2.4 since the proof of Theorem 2.3 is (a little bit) easier and follows from that of Theorem 2.4 closely, as long as we drop the requirement that $A + L_1C$ is Hurwitz.

Before proving Theorem 2.4, we need to establish some preliminary definitions and propositions.

First, two sets S_L and S_P are defined as follows:

Definition 2.2 *Define $S_L \subset \mathbf{R}^{n\times p}$ as*

$$S_L = \{L : L \in \mathbf{R}^{n\times p}, A + LC \ \ \textit{is Hurwitz}\}$$

and $S_P \subset \mathbf{R}^{n\times n}$ as

$$S_P = \{P : P = P^T \in \mathbf{R}^{n\times n},$$

$$(A + LC)P + P(A + LC)^T + (B + LD)(B + LD)^T = 0, \ \ \textit{for some } L \in S_L\}.$$

Clearly, given A, B, C, and D, S_L and S_P are not empty since (C, A) is detectable. A direct conclusion from the above definition is $P \geq 0$, $\forall P \in S_P$.

Proposition 2.5 *For any $L \in S_L$, there is one and only one $P \in S_P$ solving*

$$(A + LC)P + P(A + LC)^T + (B + LD)(B + LD)^T = 0.$$

In addition, $A - (PC^T + BD^T)R^{-1}C$ is Hurwitz.

PROOF. For any $L \in S_L$, it is obvious that there is a $P \geq 0$ solving

$$(A + LC)P + P(A + LC)^T + (B + LD)(B + LD)^T = 0.$$

Hence $P \in S_P$. Now let $P_1 \geq 0 \in S_P$ also solve the above equation, i.e.,

$$(A + LC)P_1 + P_1(A + LC)^T + (B + LD)(B + LD)^T = 0.$$

Define $\Delta P = P - P_1$, then

$$(A + LC)\Delta P + \Delta P(A + LC)^T = 0.$$

This gives $\Delta P = 0$ or $P = P_1$.

Next, if $A - (PC^T + BD^T)R^{-1}C$ is not Hurwitz, then it has (at least) one eigenvalue λ on the closed right-half plane, thus $Re(\lambda) \geq 0$. Rewrite the Lyapunov equation

$$(A + LC)P + P(A + LC)^T + (B + LD)(B + LD)^T = 0$$

as

$$[A - (PC^T + BD^T)R^{-1}C]P + P[A - (PC^T + BD^T)R^{-1}C]^T + B(I - D^T R^{-1}D)B^T$$

$$+PC^T R^{-1}CP + [L + (PC^T + BD^T)R^{-1}]R[L + (PC^T + BD^T)R^{-1}]^T = 0.$$

Let x be a left eigenvector corresponding to λ, i.e., $x^T[A - (PC^T + BD^T)R^{-1}C] = \lambda x^T$, then:

$$x^T[A - (PC^T + BD^T)R^{-1}C]Px + x^T P[A - (PC^T + BD^T)R^{-1}C]^T x$$

$$+x^T PC^T R^{-1} CPx + x^T B(I - D^T R^{-1} D)B^T x$$

$$+x^T[L + (PC^T + BD^T)R^{-1}]R[L + (PC^T + BD^T)R^{-1}]^T x = 0,$$

or

$$2Re(\lambda)x^T Px + x^T PC^T R^{-1} CPx + x^T B(I - D^T R^{-1} D)B^T x$$

$$+x^T[L + (PC^T + BD^T)R^{-1}]R[L + (PC^T + BD^T)R^{-1}]^T x = 0.$$

Since $I - D^T R^{-1} D \geq 0$, we have:

$$CPx = 0, \quad (I - D^T R^{-1} D)B^T x = 0, \quad [L + (PC^T + BD^T)R^{-1}]^T x = 0,$$

or $x^T L = -x^T(PC^T + BD^T)R^{-1}$ which implies:

$$x^T(A + LC) = x^T[A - (PC^T + BD^T)R^{-1}C] = \lambda x^T,$$

i.e., $A + LC$ is not Hurwitz, a contradiction. Hence $A - (PC^T + BD^T)R^{-1}C$ is Hurwitz. □

Let $\{P_i, \quad i = 1, 2, \cdots\}$ be a sequence in $\mathbf{R}^{n\times n}$. Correspondingly, we define a sequence $\{L_i, \quad i = 2, 3 \cdots\}$ in $\mathbf{R}^{n\times p}$ with $L_{i+1} = -(P_i C^T + BD^T)R^{-1}$. The limits of $\{P_i\}$ and $\{L_i\}$ are defined as follows:

Definition 2.3 *P_* and L_* are said to be the limits of sequences $\{P_i\}$ and $\{L_i\}$ if for any $x \in \mathbf{R}^n$,*

$$x^T P_* x = \lim_{i\to\infty} x^T P_i x, \quad L_* = -(P_* C^T + BD^T)R^{-1}.$$

If these limits exist, we denote

$$P_* = \lim_{i\to\infty} P_i, \quad L_* = \lim_{i\to\infty} L_{i+1} = -\lim_{i\to\infty}(P_iC^T + BD^T)R^{-1}.$$

It is easy to see that L_i has a limit if P_i does.

Proposition 2.6 *A sequence $\{P_i\}$ converges to some P_* if and only if the convergence is entry-wise, that is, if p^i_{kj} and p_{kj*} are entries of P_i and P_*, then*

$$p_{kj*} = \lim_{i\to\infty} p^i_{kj}, \qquad k, j = 1, 2, \cdots, n.$$

PROOF. If the convergence is entry-wise, i.e.,

$$p_{kj*} = \lim_{i\to\infty} p^i_{kj}, \quad k, j = 1, 2, \cdots, n,$$

then, for any $x \in \mathbf{R}^n$, we have

$$\lim_{i\to\infty} x^T P_i x = \lim_{i\to\infty} \sum_{k,j} p^i_{kj} x_k x_j = \sum_{k,j} \lim_{i\to\infty} p^i_{kj} x_k x_j = \sum_{k,j} p_{kj*} x_k x_j = x^T P_* x.$$

So

$$P_* = \lim_{i\to\infty} P_i.$$

Conversely, if P_i converges to P_*, i.e. $\forall x \in \mathbf{R}^n$

$$x^T P_* x = \lim_{i\to\infty} x^T P_i x, \quad \forall x \in \mathbf{R}^n,$$

or

$$\sum_{j,q} p_{jq*} x_j x_q = \lim_{i\to\infty} \sum_{j,q} p^i_{jq} x_j x_q = \sum_{j,q} \lim_{i\to\infty} p^i_{jq} x_j x_q.$$

Comparing coefficients on both sides and considering that x is arbitrary, we obtain:

$$p_{jq*} = \lim_{i\to\infty} p^i_{jq},$$

that is, P_i converges to P_* entry-wisely. □

We are interested in a pair of special sequences $\{P_i\}$ and $\{L_i\}$, which are generated by the following procedures:

Procedures:

1. Choose L_1 from S_L,

2. Solve $P_i, i = 1, 2, \cdots$, from:

$$(A + L_iC)P_i + P_i(A + L_iC)^T + (B + L_iD)(B + L_iD)^T = 0,$$

3. Set $L_{i+1} = -(P_iC^T + BD^T)R^{-1}$, $i = 1, 2, \cdots$.

A direct consequence from this construction is, by Proposition 2.5, that $A + L_1C$ and $A + L_{i+1}C = A - (P_iC^T + BD^T)R^{-1}C$, $i = 1, 2, \cdots$, are all Hurwitz.

Proposition 2.7 *Sequences P_i and L_i generated by the above Procedures 1–3 always have limits P_* and L_*.*

PROOF. We only need to prove that P_i has a limit P_*. Note that we have, for $i = 1, 2, \cdots$,

$$(A + L_iC)P_i + P_i(A + L_iC)^T + (B + L_iD)(B + L_iD)^T = 0,$$

$$(A + L_{i+1}C)P_{i+1} + P_{i+1}(A + L_{i+1}C)^T + (B + L_{i+1}D)(B + L_{i+1}D)^T = 0.$$

Define $\Delta P_i = P_{i+1} - P_i$ and $\Delta L_i = L_{i+1} - L_i$, then

$$(A + L_{i+1}C)\Delta P_i + \Delta P_i(A + L_{i+1}C)^T - \Delta L_iR\Delta L_i^T = 0,$$

which gives that $\Delta P_i \leq 0$. This means that for any $x \in \mathbf{R}^n$

$$0 \leq \cdots \leq x^TP_{i+1}x \leq x^TP_ix \leq \cdots \leq x^TP_1x.$$

Hence $\lim_{i\to\infty} x^T P_i x$ exists and

$$\lim_{i\to\infty} x^T P_i x = \lim_{i\to\infty} \sum_{k,j} p_{kj}^i x_k x_j = \sum_{k,j} \lim_{i\to\infty} p_{kj}^i x_k x_j = \sum_{k,j} p_{kj*} x_k x_j = x^T P_* x,$$

where $p_{kj*} = \lim_{i\to\infty} p_{kj}^i$ and $P_* = [p_{kj*}]$. Therefore P_i has a limit and so does L_i with

$$L_* = \lim_{i\to\infty} L_{i+1} = -\lim_{i\to\infty} (P_i C^T + BD^T)R^{-1} = -(P_* C^T + BD^T)R^{-1}.$$

□

Lemma 2.1 *For sequences P_i and L_i generated by the Procedures 1-3, if P_* and L_* are the limit points of these sequences, then $P_* \geq 0$ solves,*

$$(A + L_* C)P_* + P_*(A + L_* C)^T + (B + L_* D)(B + L_* D)^T = 0,$$

where $L_* = -(P_* C^T + BD^T)R^{-1}$.

PROOF. Suppose P_* and L_* are the limit points of sequences P_i and L_i, with $L_1 \in S_L$. Let p_{kj}^i and p_{kj*} be entries of P_i and P_*. Let l_{mq}^i and l_{mq*} be entries of L_i and L_*. By Proposition 2.6, we have entry-wise convergence:

$$p_{kj*} = \lim_{i\to\infty} p_{kj}^i \qquad k, j = 1, 2, \cdots, n.$$

and consequently, for any $m = 1, \cdots, n$ and $q = 1, \cdots, p$,

$$l_{mq*} = \lim_{i\to\infty} l_{mq}^i(p_{kj}^i,\ k, j = 1, 2, \cdots, n) = l_{mq}^i(\lim_{i\to\infty} p_{kj}^i,\ k, j = 1, 2, \cdots, n),$$

since l_{mq}^i is a continuous function of p_{kj}^i, $k, j = 1, 2, \cdots, n$.

Next, we define

$$F(P_i, L_i) = (A + L_i C)P_i + P_i(A + L_i C)^T + (B + L_i D)(B + L_i D)^T.$$

Obviously, $F(P_i, L_i) = 0, \ \forall i = 1, 2, \cdots$. Let $f^i_{kj}, \quad k, j = 1, \cdots, n$ be entries of $F(P_i, L_i)$, then they are continuous functions of all p^i_{kj} and l^i_{mq}. Therefore

$$f_{kj*} = \lim_{i\to\infty} f^i_{kj}(p^i_{kj}, l^i_{mq}) = 0, \quad k, j = 1, \cdots, n.$$

This shows that $F(P_*, L_*) = 0$ or

$$(A + L_*C)P_* + P_*(A + L_*C)^T + (B + L_*D)(B + L_*D)^T = 0,$$

where $L_* = -(P_*C^T + BD^T)R^{-1}$. □

It is worth pointing out that $A + L_*C$ may not be Hurwitz. Actually, its eigenvalues are on the close left-half plane. The reason is as follows: recall that eigenvalues $\lambda_{1i}, \cdots, \lambda_{ni}$ of $A + L_iC$ are all on the open left-half plane, for $i = 1, 2, \cdots$, and clearly $\lambda_{ji}, \ j = 1, \cdots, n$ are continuous functions of $l^i_{mq}, \ m = 1, \cdots, n \ \ q = 1, \cdots, p$. If $\lambda_{1*}, \cdots, \lambda_{n*}$ are eigenvalues of $A + L_*C$, then

$$Re(\lambda_{j*}) = \lim_{i\to\infty} Re[\lambda_{ji}(l^i_{mq})] = Re[\lambda_{j*}(\lim_{i\to\infty} l^i_{mq})] \geq 0, \quad j = 1, \cdots, n.$$

So λ_{j*} lie on the close left-half plane.

Now we are in the position to prove Theorem 2.4.

PROOF. If there is an $L_1 \in S_L$ and a $P_1 \in S_P$ such that

$$(A + L_1C)P_1 + P_1(A + L_1C)^T + (B + L_1D)(B + L_1D)^T = 0,$$

and $J_2(L)$ achieves the minimum value at L_1, take L_1 as the initial value and generate the sequences P_i and L_i using the Procedures 1–3. Then the following claims can be made (see Proposition 2.7):

1. $0 \leq \cdots \leq P_{i+1} \leq P_i \leq \cdots \leq P_1,$

2. $\{P_i, \ i = 1, 2, \cdots,\}$ and $\{L_i, \ i = 1, 2, \cdots,\}$ have limit points P_* and L_* and $P_* \leq P_1$.

Hence by Lemma 2.1, P_* and $L_* = -(P_*C^T + BD^T)R^{-1}$ solve

$$(A + L_*C)P_* + P_*(A + L_*C)^T + (B + L_*D)(B + L_*D)^T = 0.$$

If $A + L_*C = A - (P_*C^T + BD^T)R^{-1}C$ is Hurwitz, then

$$J_2(L_*) = trace(P_*) \leq trace(P_1) = J_2(L_1) \leq J_2(L_*),$$

thus $J_2(L_*) = J_2(L_1)$, i.e., $J_2(L)$ achieves the minimum value at $L_* = -(P_*C^T + BD^T)R^{-1}$

□

The next corollary tells when we have a stabilizing solution.

Corollary 2.1 *Suppose (A, C) is detectable and*

$$\begin{bmatrix} A - j\omega & B \\ C & D \end{bmatrix} \text{ has full row rank for all } \omega,$$

then $A + L_*C$ is stable, where $L_* = -(P_*C^T + BD^T)R^{-1}$ and P_* solves

$$(A + L_*C)P_* + P_*(A + L_*C)^T + (B + L_*D)(B + L_*D)^T = 0.$$

2.3 $\mathcal{H}_2(LQG)$ Control

We shall revisit the standard $\mathcal{H}_2$ control in this section from a different point of view in terms of the optimization problem solved in the last section.

Consider a dynamical system:

$$\dot{x} = Ax + B_1w + B_2u, \quad x(0) = 0, \tag{2.3}$$

$$z = C_1x + D_{12}u, \tag{2.4}$$

$$y = C_2x + D_{21}w. \tag{2.5}$$

We shall make the following standard assumptions:

(A1) (A, B_2) is stabilizable and (C_2, A) is detectable,

(A2) $R_1 := D_{12}^T D_{12} > 0$, $R_2 := D_{21} D_{21}^T > 0$,

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω,

(A4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

Let T_{zw} denote the matrix transfer function from w to z.

$\mathcal{H}_2$ **Control Problem:** find a control law $u = K(s)y$ that stabilizes the closed-loop system and minimizes $||T_{zw}||_2$, where

$$||T_{zw}||_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} trace[T_{zw}^*(j\omega) T_{zw}(j\omega)] d\omega}.$$

LQG **Control Problem:** Let w be a Gaussian white noise signal with unit power spectrum: $E\{w(t)\} = 0$, $E\{w(t)w^T(\tau)\} = I\delta(t - \tau)$. An *LQG* control problem is to find a control law $u = K(s)y$ that stabilizes the closed-loop system and minimizes (in infinite time horizon)

$$J = \lim_{T \to \infty} \frac{1}{T} \int_0^T E\{||z||^2\} dt.$$

Define

$$A_x := A - B_2 R_1^{-1} D_{12}^T C_1, \quad A_y := A - B_1 D_{21}^T R_2^{-1} C_2,$$

$$P := B_1(I - D_{21}^T R_2^{-1} D_{21}) B_1^T, \quad Q := C_1^T(I - D_{12} R_1^{-1} D_{12}^T) C_1.$$

It is well-known that the $\mathcal{H}_2$ and LQG problems are equivalent so we have the following theorem:

Theorem 2.8 *There exists an $\mathcal{H}_2(LQG)$ controller in the form of*

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B_2 u + L_2(C_2\hat{x} - y), \quad \hat{x}(0) = 0, \\ u &= F_2\hat{x}, \end{aligned}$$

where $F_2 = -R_1^{-1}(D_{12}^T C_1 + B_2^T X_2)$, $L_2 = -(B_1 D_{21}^T + Y_2 C_2^T) R_2^{-1}$ and $X_2 \geq 0$, $Y_2 \geq 0$ are stabilizing solutions to

$$A_x^T X_2 + X_2 A_x - X_2 B_2 R_1^{-1} B_2^T X_2 + Q = 0,$$

$$A_y Y_2 + Y_2 A_y^T - Y_2 C_2^T R_2^{-1} C_2 Y_2 + P = 0.$$

We have shown in Section 1.1.2 that if a system $G(s)$ is driven by a white noise signal w_0, then

$$\|G(s)\|_2 := \sqrt{\lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T \|z\|^2 dt\right\}},$$

where z is the system response to w_0. We will use this result in the proof of Theorem 2.8. We shall also need the following lemma:

Lemma 2.2 *Consider the system described by equations* $(2.3)-(2.5)$ *where $w = w_0$ is a white noise. Suppose the controller $K(s) = \bar{B}(sI - \bar{A})^{-1}\bar{C}$. Then we have*

$$E\{x(t)w_0^T(s)\} = (e_{11}B_1 + e_{12}\bar{B}D_{21})/2, \quad s \leq t \quad or \quad E\{x(t)w_0^T(s)\} = 0, \quad s > t$$

where $\hat{A} = \begin{bmatrix} A & B_2\bar{C} \\ \bar{B}C_2 & \bar{A} \end{bmatrix}$ *and* $exp[\hat{A}(t-s)] = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$.

Proof. Since $K(s) = \bar{B}(sI - \bar{A})^{-1}\bar{C}$, the closed-loop system becomes

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}w_0.$$

where $\hat{x} = [\ x^T \quad \bar{x}^T\ ]^T$ and $\hat{B} = [\ B_1^T \quad (\bar{B}D_{21})^T\ ]^T$. Hence

$$\hat{x} = \int_0^t exp[\hat{A}(t-\tau)]\hat{B}w_0(\tau)d\tau$$

and

$$E[\ w_0(s)x^T(t) \quad w_0(s)\bar{x}^T(t)\ ]^T = E\left\{\int_0^t exp[\hat{A}(t-\tau)]\hat{B}w_0(\tau)w_0^T(s)d\tau\right\}$$

$$= \int_0^t exp[\hat{A}(t-\tau)]\hat{B}E\{w_0(\tau)w_0^T(s)\}d\tau = \int_0^t exp[\hat{A}(t-\tau)]\hat{B}\delta(\tau - s)d\tau$$

which gives $E\{x(t)w_0^T(s)\} = (e_{11}B_1 + e_{12}\bar{B}D_{21})/2$, for $s \le t$; or $E\{x(t)w_0^T(s)\} = 0$, for $s > t$. $\square$

Corollary 2.2 *Consider the system:*

$$\begin{aligned} \dot{x} &= Ax + B_0w_0 + B_1w + B_2u, \quad x(0) = 0, \\ y &= C_2x + D_{20}w_0. \end{aligned}$$

where w_0 is a white noise and w is a stationary signal. Suppose w_0 and w are (mutually) independent and all notations are same as those in Lemma 2.2, then, we have

$$E\{x(t)w_0^T(s)\} = (e_{11}B_0 + e_{12}\bar{B}D_{20})/2, \quad s \le t \quad or \quad E\{x(t)w_0^T(s)\} = 0, \quad s > t$$

Now we prove Theorem 2.8.

PROOF. As we have pointed out, $\mathcal{H}_2$ minimization problem is equivalent to minimize $\lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T \|z\|^2 dt\right\}$, where z is the system response to white noise $w = w_0$. Hence, we can prove the theorem as follows: Let $X_2 \geq 0$ and $Y_2 \geq 0$ be stabilizing solutions to

$$A_x^T X_2 + X_2 A_x - X_2 B_2 R_1^{-1} B_2^T X_2 + Q = 0$$

$$A_y Y_2 + Y_2 A_y^T - Y_2 C_2^T R_2^{-1} C_2 Y_2 + P = 0.$$

We can use the first Riccati equation to get

$$\lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T \|z\|^2 dt\right\} = \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T (\|z\|^2 + \frac{d}{dt}x^T X_2 x) dt\right\}$$

$$= \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T (\|z(t)\|^2 + 2x^T X_2 \dot{x}) dt\right\}$$

$$= \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T (\|C_1 x + D_{12} u\|^2 + 2x^T X_2 (Ax + B_1 w_0 + B_2 u)) dt\right\}$$

$$= \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T ((u - F_2 x)^T R_1 (u - F_2 x) + 2x^T X_2 B_1 w_0) dt\right\}$$

$$= \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T ((u - F_2 x)^T R_1 (u - F_2 x) dt\right\}$$

$$+ \lim_{T\to\infty} \frac{1}{T}\int_0^T trace[2X_2 B_1 E\{w_0 x^T\}] dt$$

$$= \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T ((u - F_2 x)^T R_1 (u - F_2 x) dt\right\} + trace(B_1^T X_2 B_1).$$

Clearly, if the system states are available, then $\mathcal{H}_2$ control law would be $u = F_2 x$. If the states are not available, we can design a standard observer-based controller:

$$\dot{\hat{x}} = A\hat{x} + B_2 u + L(C_2\hat{x} - y), \quad u = F_2\hat{x}.$$

Defining $e = x - \hat{x}$, we get

$$\dot{e} = (A + LC_2)e + (B_1 + LD_{21})w_0 := A_L e + B_L w_0, \quad u - F_2 x = -F_2 e.$$

So we can solve $e(t) = \int_0^t exp[A_L(t-\tau)]B_L w_0(\tau)d\tau$ and

$$\lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T \|z(t)\|^2 dt\right\}$$

$$= \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T e^T F_2^T R_1 F_2 e dt\right\} + trace(B_1^T X_2 B_1)$$

$$= trace\{F_2^T R_1 F_2 Y\} + trace(B_1^T X_2 B_1),$$

where $Y = \int_0^\infty exp(A_L t)B_L B_L^T exp(A_L^T t)dt$ satisfies:

$$(A + LC_2)Y + Y(A + LC_2)^T + (B_1 + LD_{21})(B_1 + LD_{21})^T = 0.$$

Therefore,

$$(A + LC_2)(Y - Y_2) + (Y - Y_2)(A + LC_2)^T + (L - L_2)R_2(L - L_2)^T = 0,$$

which shows that $\|T_{zw}\|_2^2 = E\{\int_0^\infty \|z(t)\|^2 dt\}$ is minimized by $L = L_2$.

□

2.4 $\mathcal{H}_\infty$ Control Design

In this section, some results from $\mathcal{H}_\infty$ control design are given. Our interest here is to show that the output feedback controller structures for $\mathcal{H}_\infty$ and $\mathcal{H}_2$ design could have very similar details though essentially they are different as pointed out in [16]. The significances of those similarities are that they provide clear motivation, especially, for the research conducted in the multiobjective control design, which shall be addressed in detail in the subsequent chapters.

Consider the control system shown in Figure 2.3:

The system equations are as follows:

$$\dot{x} = Ax + B_1 w + B_2 u, \quad x(0) = 0, \tag{2.6}$$

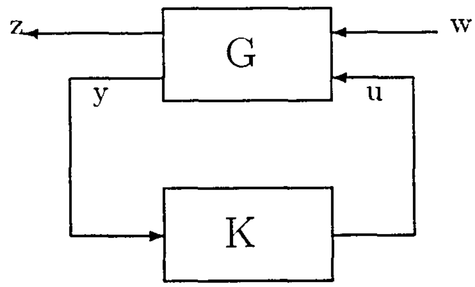

Figure 2.3: Linear Control System with Disturbance w

$$z = C_1 x + D_{12} u, \tag{2.7}$$

$$y = C_2 x + D_{21} w, \tag{2.8}$$

where $x \in \mathbf{R}^n$, $y \in \mathbf{R}^p$, $z \in \mathbf{R}^{q_1}$ and $w \in \mathbf{R}^{r_1}$ is a disturbance signal. Let T_{zw} be the closed-loop matrix transfer function from w to z.

$\mathcal{H}_\infty$ **Control Problem:** find a control law $u = K(s)y$ such that the closed-loop system is stable and

$$||T_{zw}||_\infty < \gamma, \ \ where \ ||T_{zw}||_\infty = \sup_\omega \bar{\sigma}\{T_{zw}(j\omega)\}$$

for some prespecified $\gamma > 0$.

2.4.1 State Feedback Design

For state feedback design, the following assumptions are made:

(A1) (A, B_2) is stabilizable,

(A2) $R_1 := D_{12}^T D_{12} > 0$,

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω.

Theorem 2.9 *There exists a state feedback controller such that* $\|T_{zw}\|_\infty < \gamma$ *if and only if there is a stabilizing solution* $X_\infty \geq 0$ *solving*

$$A_x^T X_\infty + X_\infty A_x + X_\infty (B_1 B_1^T/\gamma^2 - B_2 R_1^{-1} B_2^T) X_\infty + Q = 0,$$

where $A_x := A - B_2 R_1^{-1} D_{12}^T C_1$ *and* $Q := C_1^T (I - D_{12} R_1^{-1} D_{12}^T) C_1$.

If X_∞ *exists, then the state feedback control can be taken as:* $u = F_\infty x$, *where* $F_\infty = -R_1^{-1}(D_{12}^T C_1 + B_2^T X_\infty)$, *and the worst disturbance signal is* $w_* = \gamma^{-2} B_1^T X_\infty x$.

2.4.2 Output Feedback Design

We shall make the following standard assumptions:

(A1) (A, B_2) is stabilizable and (C_2, A) is detectable,

(A2) $R_1 := D_{12}^T D_{12} > 0$, $R_2 := D_{21} D_{21}^T > 0$,

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω,

(A4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

For simplicity, the following orthogonalities are assumed:

(A5) $D_{12}^T C_1 = 0$ and $B_1 D_{21}^T = 0$.

So $A_x = A$ and $Q = C_1^T C_1$.

Clearly, state feedback design should have a solution if we want to consider the output feedback design. Hence, there exists a stabilizing solution $X_\infty \geq 0$ for:

$$A^T X_\infty + X_\infty A + X_\infty (B_1 B_1^T/\gamma^2 - B_2 R_1^{-1} B_2^T) X_\infty + C_1^T C_1 = 0$$

and $\mathcal{H}_\infty$ state feedback gain becomes $F_\infty = -R_1^{-1}B_2^T X_\infty$. It is well known that there may exist many $\mathcal{H}_\infty$ output feedback controllers. However, as we have mentioned before, tracking the similarities between $\mathcal{H}_\infty$ and $\mathcal{H}_2$ controllers leads us to being especially interested in the existence of the follow special structure of $\mathcal{H}_\infty$ output feedback control u:

$$\begin{aligned} \dot{\hat{x}} &= (A + \gamma^{-2}B_1B_1^T X_\infty)\hat{x} + B_2u + L(C_2\hat{x} - y), \quad \hat{x}(0) = 0, \\ u &= F_\infty \hat{x}. \end{aligned}$$

Initially, this controller looks very similar to an $\mathcal{H}_2(LQG)$ controller if we treat F_∞ as the optimal LQR feedback gain, but a further observation could tell the essential difference, that is, we can no longer separate the state estimation and feedback control. On the other hand, the term $\gamma^{-2}B_1^T X_\infty \hat{x}$ can be thought as the estimation of the worst possible disturbance signal w_*, hence, basically, this controller is constructed based on the worst possible disturbance case and that is the very reason to cause the big difference between this controller and an $\mathcal{H}_2(LQG)$ controller.

Now we state the following theorem:

Theorem 2.10 *Suppose the state feedback $\mathcal{H}_\infty$ control design has a solution. That is, there is a stabilizing solution $X_\infty \geq 0$ solving*

$$A^T X_\infty + X_\infty A + X_\infty(B_1B_1^T/\gamma^2 - B_2R_1^{-1}B_2^T)X_\infty + C_1^T C_1 = 0,$$

then, there exists an output feedback $\mathcal{H}_\infty$ controller in the form of

$$\begin{aligned} \dot{\hat{x}} &= (A + \gamma^{-2}B_1B_1^T X_\infty)\hat{x} + B_2u + L(C_2\hat{x} - y), \quad \hat{x}(0) = 0, \\ u &= F_\infty \hat{x}, \quad F_\infty = -R_1^{-1}B_2^T X_\infty, \end{aligned}$$

if and only if there is a stabilizing solution $Y_\infty \geq 0$ solving

$$Y_\infty(A+\gamma^{-2}B_1B_1^TX_\infty+LC_2)+(A+\gamma^{-2}B_1B_1^TX_\infty+LC_2)^TY_\infty$$

$$+\gamma^{-2}Y_\infty(B_1+LD_{21})(B_1+LD_{21})^TY_\infty+X_\infty B_2R_1^{-1}B_2^TX_\infty=0,$$

where L is chosen such that $A+\gamma^{-2}B_1B_1^TX_\infty+LC_2$ is stable.

PROOF. First, it is noted that $\|T_{zw}\|_\infty<\gamma$ is equivalent to

$$0<J(u,w)=\int_0^\infty(\gamma^2\|w\|^2-\|z\|^2)dt, \quad \forall u, \quad w\neq 0.$$

Next, by using the Riccati equation:

$$A^TX_\infty+X_\infty A+X_\infty(B_1B_1^T/\gamma^2-B_2R_1^{-1}B_2^T)X_\infty+C_1^TC_1=0,$$

it is easy to complete square for $J(u,w)$:

$$J(u,w)=\int_0^\infty(\gamma^2\|w-\tilde{w}_*\|^2-\|D_{12}(u-\tilde{u}_*)\|^2)dt,$$

where $\tilde{w}_*=\gamma^{-2}B_1^TX_\infty x$ and $\tilde{u}_*=-R_1^{-1}B_2^TX_\infty x=F_\infty x$. Clearly, for state feedback, we have $u=\tilde{u}_*$ and $w=\tilde{w}_*$ as the optimal solutions which proves a special case of Theorem 2.9.

Define $r:=w-\gamma^{-2}B_1^TX_\infty x$ and $v:=D_{12}[u+R_1^{-1}B_2^TX_\infty]x$. The system equations are converted into:

$$\begin{aligned}
\dot{x} &= (A+\gamma^{-2}B_1B_1^TX_\infty)x+B_1r+B_2u, \quad x(0)=0,\\
v &= D_{12}[R_1^{-1}B_2^TX_\infty x+u]=-D_{12}F_\infty x+D_{12}u,\\
y &= C_2x+D_{21}w=C_2x+D_{21}r,
\end{aligned}$$

and the performance index becomes:

$$J(u,w)=\int_0^\infty(\gamma^2\|r\|^2-\|v\|^2)dt.$$

Now consider an observer-based controller for the above system and performance index:

$$\dot{\hat{x}} = (A + \gamma^{-2}B_1B_1^TX_\infty)\hat{x} + B_2u + L(C_2\hat{x} - y), \quad \hat{x}(0) = 0,$$
$$u = F_\infty\hat{x},$$

where L is chosen such that $A + \gamma^{-2}B_1B_1^TX_\infty + LC_2$ is stable. Define $e = x - \hat{x}$, the system equations can be further simplified to

$$\dot{e} = (A + \gamma^{-2}B_1B_1^TX_\infty + LC_2)e + (B_1 + LD_{21})r, \quad e(0) = 0,$$
$$v = D_{12}R_1^{-1}B_2^TX_\infty e = -D_{12}F_\infty e.$$

Now if there is a stabilizing solution $Y_\infty \geq 0$ solving

$$Y_\infty(A + \gamma^{-2}B_1B_1^TX_\infty + LC_2) + (A + \gamma^{-2}B_1B_1^TX_\infty + LC_2)^TY_\infty$$
$$+\gamma^{-2}Y_\infty(B_1 + LD_{21})(B_1 + LD_{21})^TY_\infty + X_\infty B_2R_1^{-1}B_2^TX_\infty = 0,$$

then using this equation, we can complete square again for $J(u, w)$:

$$J(u,w) = \int_0^\infty (\gamma^2\|r\|^2 - \|v\|^2)dt = \int_0^\infty \gamma^2\|r - \gamma^{-2}B_1^TY_\infty e\|^2 dt.$$

Hence $J(u, w) > 0, \quad \forall r \neq \gamma^{-2}B_1^TY_\infty e$, or, $\|T_{zw}\|_\infty < \gamma$. Clearly, the worst disturbance signal is $r_* = \gamma^{-2}B_1^TY_\infty e$ or $w_* = \gamma^{-2}B_1^T(X_\infty x + Y_\infty e)$ which is not achievable in this case.

On the other hand, if the controller:

$$\dot{\hat{x}} = (A + \gamma^{-2}B_1B_1^TX_\infty)\hat{x} + B_2u + L(C_2\hat{x} - y), \quad \hat{x}(0) = 0,$$
$$u = F_\infty\hat{x}$$

guarantees the $\mathcal{H}_\infty$ performance from w to z, or, equivalently, from r to v of the system

$$
\begin{aligned}
\dot{e} &= (A + \gamma^{-2} B_1 B_1^T X_\infty + LC_2)e + (B_1 + LD_{21})r, \quad e(0) = 0, \\
v &= D_{12} R_1^{-1} B_2^T X_\infty e = -D_{12} F_\infty e,
\end{aligned}
$$

then by the Bounded-Real Lemma, there exists a stabilizing solution Y_∞ for

$$
\begin{aligned}
& Y_\infty(A + \gamma^{-2} B_1 B_1^T X_\infty + LC_2) + (A + \gamma^{-2} B_1 B_1^T X_\infty + LC_2)^T Y_\infty \\
& + \gamma^{-2} Y_\infty (B_1 + LD_{21})(B_1 + LD_{21})^T Y_\infty + X_\infty B_2 R_1^{-1} B_2^T X_\infty = 0.
\end{aligned}
$$

This concludes the proof. □

A lemma from [26](see Lemma 2 in [26]) is given as follows, which will be used in the subsequent chapters.

Lemma 2.3 *Suppose*

$$
\begin{aligned}
\dot{x} &= Ax + Bw, \quad x(0) = 0, \\
z &= Cx
\end{aligned}
$$

describes a linear operator R_{zw}. Define $\|R_{zw}\|_{\infty,[0,T]} = \sup_w \sqrt{\frac{\int_0^T \|z\|^2 dt}{\int_0^T \|w\|^2 dt}}$. Then the following two statements are equivalent:

1. $\|R_{zw}\|_{\infty,[0,T]} < \gamma$, *i.e.,* $0 < \int_0^T (\gamma^2 \|w\|^2 - \|z\|) dt, \quad \forall w \neq 0$

2. *The Riccati equation*

$$
-\dot{P} = A^T P + PA + \gamma^{-2} PBB^T P + C^T C
$$

has a solution $P(t) \geq 0, P(T) = 0$ with no finite escape time on $[0, T]$.

Chapter 3

Multiobjective Optimal Filtering

This chapter is dedicated to multiobjective optimal filtering design. The filter problem is formulated in Section 3.1. In Section 3.2, the results for finite time horizon are presented while the results for infinite time horizon are given in Section 3.3.

3.1 Problem Formulation

Consider the following plant G:

$$\begin{aligned} \dot{x} &= Ax + B_0 w_0 + B_1 w, \qquad x(0) = 0, && (3.1) \\ z &= C_1 x, && (3.2) \\ z_0 &= C_0 x, && (3.3) \\ y &= C_2 x + D_{20} w_0, \quad R_0 := D_{20} D_{20}^T > 0, && (3.4) \end{aligned}$$

where $x \in \mathbf{R}^n$, $y \in \mathbf{R}^p$, $z \in \mathbf{R}^{q_1}$, $z_0 \in \mathbf{R}^{q_2}$, $w \in \mathbf{R}^{r_1}$ is a bounded power disturbance signal and $w_0 \in \mathbf{R}^{r_2}$ is a white noise signal with $E\{w_0(t)\} = 0$ and $E\{w_0(t) w_0^T(\tau)\} = I\delta(t-\tau)$. w_0 and w are (mutually) independent. Let the filter be denoted as:

$$F : y \rightarrow \begin{bmatrix} \hat{z} \\ \\ \hat{z}_0 \end{bmatrix},$$

where $\hat{z}$ and $\hat{z}_0$ are estimates of z and z_0.

The multiobjective optimal filtering problem considered in this chapter is to find a filter F in the following given form:

$$\dot{\hat{x}} = A\hat{x} + L(C_2\hat{x} - y) = (A + LC_2)\hat{x} - Ly, \quad \hat{x}(0) = 0, \tag{3.5}$$

$$\hat{z} = C_1\hat{x}, \tag{3.6}$$

$$\hat{z}_0 = C_0\hat{x}, \tag{3.7}$$

where L is the filter gain, such that $\hat{z}$ and $\hat{z}_0$ are made as close to z and z_0 as possible in some sense. This problem is graphically illustrated in Figure 3.1.

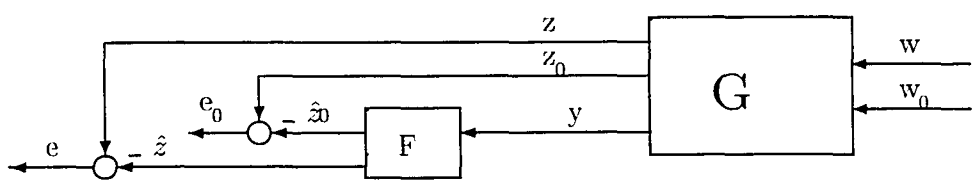

Figure 3.1: Multiobjective Optimal Filtering

Let $e = z - \hat{z}$ and $e_0 = z_0 - \hat{z}_0$. For optimization purpose, two cost functionals are defined as:

$$J_1(F, w, w_0) = E\int_0^T (\gamma^2\|w\|^2 - \|e\|^2)dt, \quad J_2(F, w, w_0) = E\int_0^T \|e_0\|^2 dt$$

for finite time horizon, and

$$J_3(F, w, w_0) = \lim_{T\to\infty}\frac{1}{T}\int_0^T E(\gamma^2\|w\|^2 - \|e\|^2)dt, \quad J_4(F, w, w_0) = \lim_{T\to\infty}\frac{1}{T}\int_0^T E\|e_0\|^2 dt$$

for infinite time horizon.

Note that for the case of finite time horizon, the resulted F is a time -varying filter and there is no stability requirement imposed on it, while for the case of infinite time horizon the resulted F is a time-invariant filter and is indeed expected to be

stable. So in the case of infinite time horizon, we have the transfer function $F(s)$ of the desired filter F and we say that F is admissible if $F(s) \in \mathcal{RH}_\infty$.

Hence the multiobjective optimal filtering problem discussed in this chapter can be defined as follows:

Filtering Problem 1 (finite time horizon):

Find a filter F_* in the given form of (3.5) – (3.7) and a worst disturbance signal w_* under white noise such that:

$$J_1(F_*, w_*, w_0) \le J_1(F_*, w, w_0), \quad J_2(F_*, w_*, w_0) \le J_2(F, w_*, w_0)$$

hold for all F and all w.

Filtering Problem 2 (infinite time horizon):

Find an admissible filter F_* in the given form of (3.5) – (3.7) and a worst disturbance signal w_* under white noise such that:

$$J_3(F_*, w_*, w_0) \le J_3(F_*, w, w_0), \quad J_4(F_*, w_*, w_0) \le J_4(F, w_*, w_0)$$

hold for all F and all w.

3.2 Multiobjective Filtering Design—Finite Time Horizon

A design approach is presented in this section for **Filtering Problem 1** formulated in Section 3.1.

Note that by defining $e_x := x - \hat{x}$, the plant-filter system can be characterized by the following equations called the Plant-Filter Equations (PFEs):

$$\dot{e}_x = (A + LC_2)e_x + (B_0 + LD_{20})w_0 + B_1 w, \quad e_x(0) = 0, \tag{3.8}$$

$$e = C_1 e_x, \tag{3.9}$$

$$e_0 = C_0 e_x. \tag{3.10}$$

The filter design results are summarized in the following theorem:

Theorem 3.1 *For the given plant G and given cost functionals J_1 and J_2, there exist a filter F_* and the worst disturbance signal w_* under white noise such that*

$$J_1(F_*, w_*, w_0) \le J_1(F_*, w, w_0), \quad J_2(F_*, w_*, w_0) \le J_2(F, w_*, w_0)$$

for all F and w, if there are $P_1(t) \ge 0$ and $P_2(t) \ge 0$ on $[0, T]$ with $P_1(T) = 0$ and $P_2(0) = 0$ solving

$$(A - P_2C_2^TR_0^{-1}C_2 - B_0D_{20}^TR_0^{-1}C_2)^TP_1 + P_1(A - P_2C_2^TR_0^{-1}C_2 - B_0D_{20}^TR_0^{-1}C_2)$$

$$+\gamma^{-2}P_1B_1B_1^TP_1 + C_1^TC_1 = -\dot{P}_1(t),$$

$$(A - B_0D_{20}^TR_0^{-1}C_2 + \gamma^{-2}B_1B_1^TP_1)P_2 + P_2(A - B_0D_{20}^TR_0^{-1}C_2 + \gamma^{-2}B_1B_1^TP_1)^T$$

$$-P_2C_2^TR_0^{-1}C_2P_2 + B_0(I - D_{20}^TR_0^{-1}D_{20})B_0^T = \dot{P}_2(t).$$

Moreover, if the above solutions exist, then an optimal filter F_* can be obtained as:

$$\begin{aligned}
\dot{\hat{x}} &= (A - P_2C_2^TR_0^{-1}C_2 - B_0D_{20}^TR_0^{-1}C_2)\hat{x} + (P_2C_2^T + B_0D_{20}^T)R_0^{-1}y, \quad \hat{x}(0) = 0 \\
\hat{z} &= C_1\hat{x} \\
\hat{z}_0 &= C_0\hat{x}
\end{aligned}$$

i.e., $L_* = -(P_2C_2^T + B_0D_{20}^T)R_0^{-1}$, and the worst disturbance signal under white noise is $w_* = \gamma^{-2}B_1^TP_1e_x$.

Furthermore, the optimal filter F_* and the worst signal $w'_* = \gamma^{-2}B_1^TP_1e'_x$ achieve

$$0 \le J_1(F_*, w'_*, 0) \le J_1(F_*, w, 0)$$

for all $w \ne w'_*$, where e'_x satisfies:

$$\dot{e}'_x = (A + LC_2)e'_x + B_1w$$

PROOF. Suppose there exist solutions $P_1(t) \geq 0$ and $P_2(t) \geq 0$ on $[0,T]$ with $P_1(T) = 0$ and $P_2(0) = 0$ for the corresponding Riccati equations. Taking $L_* = -(P_2C_2^T + B_0D_{20}^T)R_0^{-1}$, we have

$$J_1(F_*, w, w_0) = E\int_0^T (\gamma^2\|w\|^2 - \|e\|^2)dt = E\int_0^T (\gamma^2\|w\|^2 - \|e\|^2 - \frac{d}{dt}e_x^T P_1(t)e_x)dt$$

$$= E\int_0^T (\gamma^2\|w\|^2 - \|e\|^2 - 2e_x^T P_1(t)[(A + L_*C_2)e_x + (B_0 + L_*D_{20})w_0 + B_1 w]$$

$$-e_x^T \dot{P}_1(t)e_x)dt$$

$$= E\int_0^T \gamma^2\|w - w_*\|^2 dt - \int_0^T trace\{(B_0 + L_*D_{20})^T P_1(t)(B_0 + L_*D_{20})\}dt,$$

where $w_* = \gamma^{-2}B_1^T P_1 e_x$. Note that Corollary 2.2 and the Riccati equation about P_1 are used to derive this result. It is easy to see that: $J_1(F_*, w_*, w_0) \leq J_1(F_*, w, w_0)$ for all w.

Next, it is shown that J_2 does achieve the minimum value at L_*. For any L, substituting w_* into the PFEs, we get:

$$\begin{aligned} \dot{e}_x &= (A + LC_2 + \gamma^{-2}B_1B_1^T P_1)e_x + (B_0 + LD_{20})w_0, \\ e_0 &= C_0 e_x. \end{aligned}$$

Now, let $\Phi(t,0)$, $\Phi(0,0) = I$ be the transition matrix of $A + LC_2 + \gamma^{-2}B_1B_1^T P_1$, then $e_0 = \int_0^t C_0\Phi(t,s)(B_0 + LD_{20})w_0(s)ds$. Hence

$$J_2(F, w_*, w_0) = E\int_0^T \|e_0\|^2 dt$$

$$= E\left\{\int_0^T\int_0^t\int_0^t w^T(\tau)(B_0 + LD_{20})^T\Phi^T(t,\tau)C_0^T C_0\Phi(t,s)(B_0 + LD_{20})w(s)d\tau\, ds\, dt\right\}$$

$$= trace\left\{\int_0^T\int_0^t\int_0^t \delta(s-\tau)(B_0 + LD_{20})^T\Phi^T(t,\tau)C_0^T C_0\Phi(t,s)(B_0 + LD_{20})d\tau\, ds\, dt\right\}$$

$$= trace\left\{\int_0^T C_0^T C_0 \int_0^t \Phi(t,s)(B_0 + LD_{20})(B_0 + LD_{20})^T \Phi^T(t,s)ds\ dt\right\}$$

$$= trace\left\{C_0 \int_0^T Y(t)dt C_0^T\right\},$$

where $Y(t) = \int_0^t \Phi(t,s)(B_0 + LD_{20})(B_0 + LD_{20})^T \Phi^T(t,s)ds$ satisfies:

$$(A + LC_2 + \gamma^{-2}B_1B_1^TP_1)Y(t) + Y(t)(A + LC_2 + \gamma^{-2}B_1B_1^TP_1)^T$$

$$+(B_0 + LD_{20})(B_0 + LD_{20})^T = \dot{Y}(t).$$

This is the constrained optimization problem solved in Chapter 2, thus, by Theorem 2.1 in Chapter 2, J_2 achieves the minimum value at $L_* = -(P_2C_2^T + B_0D_{20}^T)R_0^{-1}$. Therefore, the filter F_*:

$$\begin{aligned}
\dot{\hat{x}} &= (A - P_2C_2^TR_0^{-1}C_2 - B_0D_{20}^TR_0^{-1}C_2)\hat{x} + (P_2C_2^T + B_0D_{20}^T)R_0^{-1}y, \\
\hat{z} &= C_1\hat{x}, \\
\hat{z}_0 &= C_0\hat{x}
\end{aligned}$$

and $w_* = \gamma^{-2}B_1^TP_1e_x$ achieve:

$$J_1(F_*, w_*, w_0) \leq J_1(F_*, w, w_0), \qquad J_2(F_*, w_*, w_0) \leq J_2(F, w_*, w_0).$$

It is easy to show that if there is no white noise, then $w'_* = \gamma^{-2}B_1^TP_1e'_x$ and the filter F_* achieve: $0 \leq J_1(F_*, w_*, 0) \leq J_1(F_*, w, 0)$, where e'_x satisfies:

$$\dot{e}'_x = (A + LC_2)e'_x + B_1w$$

□

Theorem 3.2 *For the given plant G and cost functionals J_1 and J_2, if there exists a a filter F_* in the given form with a worst disturbance signal w'_* such that:*

$$0 < J_1(F_*, w'_*, 0) \leq J_1(F_*, w, 0)$$

for $w \neq w'_*$ and a worst disturbance signal w_* under white noise such that

$$J_1(F_*, w_*, w_0) \leq J_1(F_*, w, w_0), \qquad J_2(F_*, w_*, w_0) \leq J_2(F, w_*, w_0)$$

for all w, then, there are $P_1(t) \geq 0$ and $P_2(t) \geq 0$ on $[0, T]$ with $P_1(T) = 0$ and $P_2(0) = 0$ solving

$$(A - P_2 C_2^T R_0^{-1} C_2 - B_0 D_{20}^T R_0^{-1} C_2)^T P_1 + P_1 (A - P_2 C_2^T R_0^{-1} C_2 - B_0 D_{20}^T R_0^{-1} C_2)$$

$$+\gamma^{-2} P_1 B_1 B_1^T P_1 + C_1^T C_1 = -\dot{P}_1(t),$$

$$(A - B_0 D_{20}^T R_0^{-1} C_2 + \gamma^{-2} B_1 B_1^T P_1) P_2 + P_2 (A - B_0 D_{20}^T R_0^{-1} C_2 + \gamma^{-2} B_1 B_1^T P_1)^T$$

$$-P_2 C_2^T R_0^{-1} C_2 P_2 + B_0 (I - D_{20}^T R_0^{-1} D_{20}) B_0^T = \dot{P}_2(t).$$

PROOF. Suppose there exist a filter F_* in the given form (hence, there exists an L_*) and a w'_* such that

$$0 < J_1(F_*, w'_*, 0) \leq J_1(F_*, w, 0) \text{ for all } w \neq w'_*.$$

This tells that $\|R_{ew}\|_{\infty,[0,T]} < \gamma$, where R_{ew} is a linear operator defined by (note that we take $w_0 = 0$):

$$\begin{aligned} \dot{e}'_x &= (A + L_* C_2) e'_x + B_1 w, \quad e'_x(0) = 0, \\ e &= C_1 e'_x. \end{aligned}$$

By Lemma 2.3, there exists a $P_1(t) \geq 0$ with $P_1(T) = 0$ solving

$$-\dot{P}_1 = (A + L_* C_2)^T P_1 + P_1 (A + L_* C_2) + \gamma^{-2} P_1 B_1 B_1^T P_1 + C_1^T C_1$$

and $w_*' = \gamma^{-2}B_1^T P_1 e_x'$. It is easy to show that $w_* = \gamma^{-2}B_1^T P_1 e_x$ is the worst disturbance signal under white noise, i.e., we have: $J_1(F_*, w_*, w_0) \leq J_1(F_*, w, w_0)$. Now substituting w_* into the PFEs, we get

$$\begin{aligned} \dot{e}_x &= (A + L_* C_2 + \gamma^{-2}B_1 B_1^T P_1)e_x + (B_0 + L_* D_{20})w_0, \\ e_0 &= C_0 e_x, \end{aligned}$$

and $J_2(F, w_*, w_0) = E\int_0^T \|e_0\|^2 dt = trace\left\{C_0 \int_0^T Y(t)dt C_0^T\right\}$, where $Y(t) = \int_0^t \Phi(t,s)(B_0 + LD_{20})(B_0 + LD_{20})^T \Phi^T(t,s)ds$ satisfies:

$$\begin{aligned} &(A + LC_2 + \gamma^{-2}B_1 B_1^T P_1)Y(t) + Y(t)(A + LC_2 + \gamma^{-2}B_1 B_1^T P_1)^T \\ &+(B_0 + LD_{20})(B_0 + LD_{20})^T = \dot{Y}(t). \end{aligned}$$

Since F_* (or L_*) achieves the minimum value of J_2, by Theorem 2.3 in Chapter 2, there exists a $P_2(t) \geq 0$ with $P_2(0) = 0$ solving

$$\begin{aligned} &(A - B_0 D_{20}^T R_0^{-1} C_2 + \gamma^{-2}B_1 B_1^T P_1)P_2 + P_2(A - B_0 D_{20}^T R_0^{-1} C_2 + \gamma^{-2}B_1 B_1^T P_1)^T \\ &-P_2 C_2^T R_0^{-1} C_2 P_2 + B_0(I - D_{20}^T R_0^{-1} D_{20})B_0^T = \dot{P}_2(t). \end{aligned}$$

Moreover, $L_* = -(P_2 C_2^T + B_0 D_{20}^T)R_0^{-1}$ and hence $P_1 \geq 0$ solves:

$$\begin{aligned} &(A - P_2 C_2^T R_0^{-1} C_2 - B_0 D_{20}^T R_0^{-1} C_2)^T P_1 + P_1(A - P_2 C_2^T R_0^{-1} C_2 - B_0 D_{20}^T R_0^{-1} C_2) \\ &+\gamma^{-2}P_1 B_1 B_1^T P_1 + C_1^T C_1 = -\dot{P}_1(t). \end{aligned}$$

This concludes the proof. $\square$

3.3 Multiobjective Filtering Design—Infinite Time Horizon

In this section, the design results for **Filtering Problem 2** formulated in Section 3.1 are presented.

The following standard assumptions are made:

1). (C_2, A) is detectable,

2). $\begin{bmatrix} A - j\omega I & B_0 \\ C_2 & D_{20} \end{bmatrix}$ has full row rank for all ω.

The filter design results are summarized in the next theorem.

Theorem 3.3 *For the given system G and cost functionals J_3 and J_4, let (C, A) be detectable. If there are stabilizing solutions $P_1 \geq 0$ and $P_2 \geq 0$ for:*

$$(A - P_2 C_2^T R_0^{-1} C_2 - B_0 D_{20}^T R_0^{-1} C_2)^T P_1 + P_1 (A - P_2 C_2^T R_0^{-1} C_2 - B_0 D_{20}^T R_0^{-1} C_2)$$

$$+\gamma^{-2} P_1 B_1 B_1^T P_1 + C_1^T C_1 = 0,$$

$$(A - B_0 D_{20}^T R_0^{-1} C_2 + \gamma^{-2} B_1 B_1^T P_1) P_2 + P_2 (A - B_0 D_{20}^T R_0^{-1} C_2 + \gamma^{-2} B_1 B_1^T P_1)^T$$

$$-P_2 C_2^T R_0^{-1} C_2 P_2 + B_0 (I - D_{20}^T R_0^{-1} D_{20}) B_0^T = 0,$$

then by choosing $L_* = -(P_2 C_2^T + B_0 D_{20}^T) R_0^{-1}$, the filter F_*:

$$\begin{aligned} \dot{\hat{x}} &= (A - P_2 C_2^T R_0^{-1} C_2 - B_0 D_{20}^T R_0^{-1} C_2)\hat{x} + (P_2 C_2^T + B_0 D_{20}^T) R_0^{-1} y, \quad \hat{x}(0) = 0, \\ \hat{z} &= C_1 \hat{x}, \\ \hat{z}_0 &= C_0 \hat{x}, \end{aligned}$$

and the worst disturbance signal $w_* = \gamma^{-2} B_1^T P_1 e_x$ *under white noise achieve:*

$$J_3(F_*, w_*, w_0) \leq J_3(F_*, w, w_0), \quad J_4(F_*, w_*, w_0) \leq J_4(F, w_*, w_0).$$

Furthermore, the worst disturbance signal $w'_* = \gamma^{-2} B_1^T P_1 e'_x$ *and filter* F_* *achieve:*

$$0 < J_3(F_*, w'_*, 0) \leq J_3(F_*, w, 0)$$

for all $w \neq w'_*$*, where* e'_x *satisfies:*

$$\dot{e}'_x = (A + LC_2)e'_x + B_1 w.$$

Conversely, if there exists a filter F_* *(hence an* L_**) with a worst disturbance signal* w'_* *such that :*

$$0 < J_3(F_*, w'_*, 0) \leq J_3(F_*, w, 0) \text{ for all } w \neq w'_*,$$

and a worst disturbance signal w_* *under white noise such that*

$$J_3(F_*, w_*, w_0) \leq J_3(F_*, w, w_0), \quad J_4(F_*, w_*, w_0) \leq J_4(F, w_*, w_0),$$

then, there are stabilizing solutions $P_1 \geq 0$ *and* $P_2 \geq 0$ *solving:*

$$(A + L_* C_2)^T P_1 + P_1(A + L_* C_2) + \gamma^{-2} P_1 B_1 B_1^T P_1 + C_1^T C_1 = 0,$$

$$(A - B_0 D_{20}^T R_0^{-1} C_2 + \gamma^{-2} B_1 B_1^T P_1) P_2 + P_2 (A - B_0 D_{20}^T R_0^{-1} C_2 + \gamma^{-2} B_1 B_1^T P_1)^T$$

$$-P_2 C_2^T R_0^{-1} C_2 P_2 + B_0 (I - D_{20}^T R_0^{-1} D_{20}) B_0^T = 0.$$

Moreover, if $A + \gamma^{-2} B_1 B_1^T P_1 - (P_2 C_2^T + B_0 D_{20}^T) R_0^{-1} C_2$ *is stable, then* L_* *can be chosen as* $L_* = -(P_2 C_2^T + B_0 D_{20}^T) R_0^{-1}$*. In this case, there is a stabilizing solution* $P_1 \geq 0$ *for*

$$(A - P_2 C_2^T R_0^{-1} C_2 - B_0 D_{20}^T R_0^{-1} C_2)^T P_1 + P_1 (A - P_2 C_2^T R_0^{-1} C_2 - B_0 D_{20}^T R_0^{-1} C_2)$$

$$+\gamma^{-2} P_1 B_1 B_1^T P_1 + C_1^T C_1 = 0.$$

PROOF. (Sufficiency) Note that $A - B_0 D_{20}^T R_0^{-1} C_2 - P_2 C_2^T R_0^{-1} C_2 + B_1 B_1^T P_1/\gamma^2$ is stable since P_1 and P_2 are stabilizing solutions.

First it is shown that $A + L_* C_2$ is stable where $L_* = -(P_2 C_2^T + B_0 D_{20}^T) R_0^{-1}$. Suppose that $A + L_* C_2$ is not stable, then there is an x and a λ with $\mathrm{Re}(\lambda) \geq 0$ such that $(A + L_* C_2)x = \lambda x$. Multiply x^T from the left and x from the right of the P_1 equation, we get

$$(\lambda + \overline{\lambda}) x^T P_1 x + x^T P_1 B_1 B_1^T P_1 x/\gamma^2 + x^T C_1^T C_1 x = 0.$$

The above equation implies that $B_1^T P_1 x = 0, \quad C_1 x = 0$. Hence

$$(A + L_* C_2 + \gamma^{-2} B_1 B_1^T P_1)x = (A - B_0 D_{20}^T R_0^{-1} C_2 - P_2 C_2^T R_0^{-1} C_2 + B_1 B_1^T P_1/\gamma^2)x$$

$$= (A + L_* C_2)x = \lambda x,$$

i.e., $A - B_0 D_{20}^T R_0^{-1} C_2 - P_2 C_2^T R_0^{-1} C_2 + B_1 B_1^T P_1/\gamma^2$ is not stable either, a contradiction. Thus $A + L_* C_2$ is stable.

Now

$$J_3(F_*, w, w_0) = \left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T E(\gamma^2 \|w\|^2 - \|e\|^2) dt \right\}$$

$$= \left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T E \left(\gamma^2 \|w\|^2 - \|e\|^2 - \frac{d}{dt} e_x^T(t) P_1 e_x(t) \right) dt \right\}$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_0^T \gamma^2 E \|w - w_*\|^2 dt - trace\{(B_0 + L_* D_{20})^T P_1 (B_0 + L_* D_{20})\}$$

where $w_* = \gamma^{-2} B_1^T P_1 e_x$ is the worst signal under white noise and it is bounded since

$$A + L_* C_2 + B_1 \gamma^{-2} B_1^T P_1 = A - B_0 D_{20}^T R_0^{-1} C_2 - P_2 C_2^T R_0^{-1} C_2 + B_1 B_1^T P_1/\gamma^2$$

is stable. So L_*(or F_*) and w_* achieve $J_3(F_*, w_*, w_0) \leq J_3(F_*, w, w_0)$.

Next it is shown that J_4 achieves the minimum value at L_* under w_*. Let L be any filter gain such that both $A + LC$ and $A + LC_2 + \gamma^{-2} B_1 B_1^T P_1$ are stable (we

know that L_* is one of them so such an L does exist). Substituting w_* into the PFE to get :

$$\dot{e}_x = (A + LC_2 + \gamma^{-2}B_1B_1^TP_1)e_x + (B_0 + LD_{20})w_0, \quad (3.11)$$

$$e_0 = C_0e_x. \quad (3.12)$$

We can solve e_0 as $e_0(t) = \int_0^t C_0exp\{A_L(t-\tau)\}B_Lw_0(\tau)d\tau$, where $A_L = A + LC_2 + \gamma^{-2}B_1B_1^TP_1$, $\quad B_L = B_0 + LD_{20}$. Define $Q_L := B_L^Texp[A_L^T(t-\tau)]C_0^TC_0exp[A_L(t-s)]B_L$. So

$$J_4(F, w_*, w_0) = \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T \|e_0(t)\|^2\, dt\right\}$$

$$= \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T \int_0^t \int_0^t w_0^T(\tau)Q_Lw_0(s)\ d\tau\ ds\ dt\right\}$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_0^t \int_0^t \text{trace}\left\{Q_LE[w_0(s)w_0^T(\tau)]\right\} d\tau ds dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_0^t \text{trace}\left\{C_0^TC_0exp[A_L(t-s)]B_LB_L^Texp[A_L^T(t-s)]\right\}\ ds\ dt$$

$$= \text{trace}\left\{C_0^TC_0 \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_0^t exp(As)B_LB_L^Texp(A_L^Ts)ds\ dt\ \right\} = \text{trace}(C_0^TC_0\tilde{P}_2),$$

where $\tilde{P}_2 = \int_0^\infty exp(As)B_LB_L^Texp(A_L^Ts)ds$ satisfies

$$A_L\tilde{P}_2 + \tilde{P}_2A_L^T + B_LB_L^T = 0.$$

Since there exists a stabilizing solution $P_2 \geq 0$ to

$$(A - B_0D_{20}^TR_0^{-1}C_2 + \gamma^{-2}B_1B_1^TP_1)P_2 + P_2(A - B_0D_{20}^TR_0^{-1}C_2 + \gamma^{-2}B_1B_1^TP_1)^T$$

$$-P_2C_2^TR_0^{-1}C_2P_2 + B_0(I - D_{20}^TR_0^{-1}D_{20})B_0^T = 0.$$

Hence by Theorem 2.2, $L_* = -(P_2C_2^T + B_0D_{20}^T)R_0^{-1}$ will minimize $J_4(F, w_*, w_0)$, i.e., $J_4(F_*, w_*, w_0) \leq J_4(F, w_*, w_0)$ for all L (or F).

It is easy to show that $w'_* = \gamma^{-2} B_1^T P_1 e'_x$ achieves $0 \le J_3(F_*, w'_*, 0) \le J_3(F_*, w, 0)$, where e'_x satisfies $\dot{e}'_x = (A + LC_2)e'_x + B_1 w$.

(Necessity) Suppose there exist a filter F_* (hence an L_*) in the given form and a w'_* such that: $0 < J_3(F_*, w'_*, 0) \le J_3(F_*, w, 0)$ for all $w \ne w'_*$. This tells that $\|R_{ew}\|_\infty < \gamma$, where R_{ew} is a linear operator defined by (note that $w_0 = 0$ in this case):

$$\begin{aligned} \dot{e}'_x &= (A + L_* C_2)e'_x + B_1 w, \quad e_x(0) = 0, \\ e &= C_1 e'_x. \end{aligned}$$

By the well-known Bounded-Real Lemma [46], we have a stabilizing solution $P_1 \ge 0$ solving

$$(A + L_* C_2)^T P_1 + P_1(A + L_* C_2) + \gamma^{-2} P_1 B_1 B_1^T P_1 + C_1^T C_1 = 0,$$

and the worst disturbance signal $w'_* = \gamma^{-2} B_1^T P_1 e'_x$. Moreover, $A + L_* C_2 + \gamma^{-2} B_1 B_1^T P_1$ is stable, so $(C_2, A + \gamma^{-2} B_1 B_1^T P_1)$ is detectable. It is easy to see that $w_* = \gamma^{-2} B_1^T P_1 e_x$ achieves $J_3(F_*, w_*, w_0) \le J_3(F_*, w, w_0)$, i.e., w_* is the worst disturbance signal under white noise. Now substituting w_* into the PFEs, we have:

$$\begin{aligned} \dot{e}_x &= (A + L_* C_2 + \gamma^{-2} B_1 B_1^T P_1)e_x + (B_0 + L_* D_{20})w_0 = A_L e_x + B_L w_0, \\ e_0 &= C_0 e_x, \end{aligned}$$

and by the assumption, $J_4(F, w_*, w_0) \ge \mathrm{trace}(C_0^T C_0 \tilde{P}_2) = J_4(F_*, w_*, w_0)$ holds for all F (or L) in the given form, where $\tilde{P}_2$ satisfies:

$$A_{L_*} \tilde{P}_2 + \tilde{P}_2 A_{L_*}^T + B_{L_*} B_{L_*}^T = 0.$$

By Theorem 2.4, there exists a solution $P_2 \geq 0$ and $\tilde{P}_2 \geq P_2$ solving

$$(A - B_0 D_{20}^T R_0^{-1} C_2 + \gamma^{-2} B_1 B_1^T P_1) P_2 + P_2 (A - B_0 D_{20}^T R_0^{-1} C_2 + \gamma^{-2} B_1 B_1^T P_1)^T$$

$$-P_2 C_2^T R_0^{-1} C_2 P_2 + B_0 (I - D_{20}^T R_0^{-1} D_{20}) B_0^T = 0.$$

If $A + \gamma^{-2} B_1 B_1^T P_1 - (P_2 C_2^T + B_0 D_{20}^T) R_0^{-1} C_2$ is stable, then L_* can be chosen as $L_* = -(P_2 C_2^T + B_0 D_{20}^T) R_0^{-1}$ since $trace(C_0^T C_0 P_2) = J_4(F_*, w_*, w_0)$. Consequently, there is a stabilizing solution $P_1 \geq 0$ solving

$$(A - P_2 C_2^T R_0^{-1} C_2 - B_0 D_{20}^T R_0^{-1} C_2)^T P_1 + P_1 (A - P_2 C_2^T R_0^{-1} C_2 - B_0 D_{20}^T R_0^{-1} C_2)$$

$$+\gamma^{-2} P_1 B_1 B_1^T P_1 + C_1^T C_1 = 0.$$

This concludes the proof. $\square$

CHAPTER 4

MULTIOBJECTIVE OPTIMAL CONTROL

This chapter addresses the output feedback multiobjective optimal control problem. The design approach is mostly motivated by the state feedback results in [26] which will be presented first without proof in Section 4.2. The filtering design results in Chapter 3 also provide some guidance for us to look for a controller structure. The design results are presented in Section 4.3 and 4.4, for finite time horizon and infinite time horizon, respectively.

4.1 PROBLEM FORMULATION

The motivation for the multiobjective optimal control problem considered in this chapter can be nicely presented as follows:

First, consider the standard LQG control in Figure 4.1, where w_0 is a white noise signal. It is well known that the LQG design treats system performance under exter-

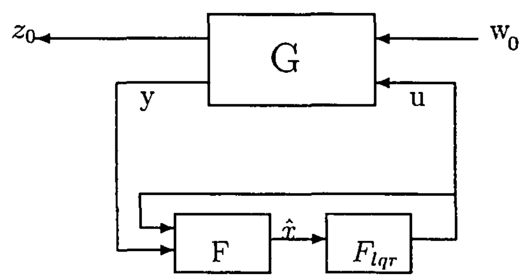

Figure 4.1: Classical LQG Control

nal stochastic disturbance without model uncertainties. Hence, a natural question is: can we take the robustness problem into account when we try to achieve the quadratic optimization performance? This question is shown in Figure 4.2.

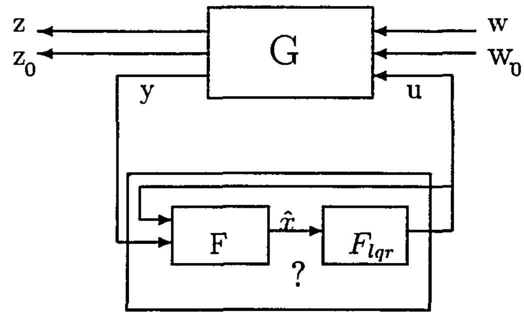

Figure 4.2: A Possible $\mathcal{H}_2/\mathcal{H}_\infty$ Controller

This problem can be formulated as follows: consider a linear control system G in Figure 4.3.

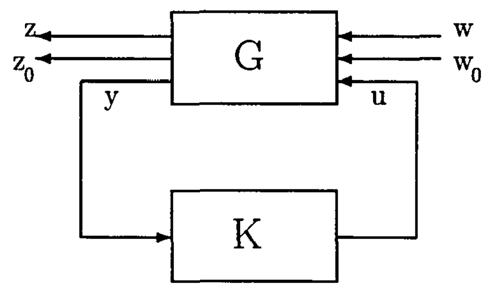

Figure 4.3: Control System with Bounded Power Disturbance and White Noise

The system equations are given as follows:

$$\dot{x} = Ax + B_0 w_0 + B_1 w + B_2 u, \quad x(0) = 0, \tag{4.1}$$

$$y = C_2 x + D_{20} w_0, \quad R_{20} := D_{20} D_{20}^T > 0, \tag{4.2}$$

$$z = C_1 x + D_{12} u, \quad R_{12} = D_{12}^T D_{12}, \tag{4.3}$$

$$z_0 = C_0 x + D_{02} u, \quad R_{02} := D_{02}^T D_{02} > 0, \tag{4.4}$$

where the disturbance w is assumed to have bounded power and w_0 is assumed to be a white noise with $E\{w_0(t)\} = 0$ and $E\{w_0(t) w_0^T(\tau)\} = I\delta(t-\tau)$, where I is an identity matrix .

Define the following cost functionals:

$$J_1(u,w,w_0) := E\left\{\int_0^T (\gamma^2 \|w\|^2 - \|z\|^2)dt\right\}, \tag{4.5}$$

$$J_2(u,w,w_0) := E\left\{\int_0^T \|z_0\|^2 \, dt\right\} \tag{4.6}$$

for finite time horizon case and

$$J_3(u,w,w_0) := \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T (\gamma^2 \|w\|^2 - \|z\|^2)dt\right\}, \tag{4.7}$$

$$J_4(u,w,w_0) := \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T \|z_0\|^2 \, dt\right\} \tag{4.8}$$

for infinite time horizon case, where u is the output feedback control law to be designed.

Then the **Multiobjective Optimal Control Problem** is defined as:

finding an output feedback control law u_* and the worst disturbance signal w_* such that:

$$J_1(u_*,w_*,w_0) \leq J_1(u_*,w,w_0), \tag{4.9}$$

$$J_2(u_*,w_*,w_0) \leq J_2(u,w_*,w_0) \tag{4.10}$$

for the finite time horizon case; and

$$J_3(u_*,w_*,w_0) \leq J_3(u_*,w,w_0), \tag{4.11}$$

$$J_4(u_*,w_*,w_0) \leq J_4(u,w_*,w_0) \tag{4.12}$$

for the infinite time horizon case.

4.2 Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control—State Feedback

Clearly, in order to let the output feedback control problem formulated in the last section have a solution, the corresponding state feedback control problem must be solvable first. In this section, a state feedback approach for mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design is presented without proof. A simplified case with its proof can be found in [26].

Note that to simplify the notations, the following abbreviations are used:

$$A_s = A - B_2 R_{02}^{-1} D_{02}^T C_0, \quad A_f = A - B_0 D_{20}^T R_{20}^{-1} C_2,$$

$$P := B_0(I - D_{20}^T R_{20}^{-1} D_{20}) B_0^T, \quad Q := C_0^T (I - D_{02} R_{02}^{-1} D_{02}^T) C_0.$$

Theorem 4.1 *For system G described by equations* $(4.1) - (4.4)$ *and the associated cost functionals J_1 and J_2, there exist linear memoryless state feedback strategies (Nash equilibrium strategies) u_* and w_* such that:*

$$\begin{aligned} 0 \le J_1(u_*, w_*, 0) &\le J_1(u_*, w, 0), \\ J_2(u_*, w_*, 0) &\le J_2(u, w_*, 0), \end{aligned}$$

if and only if the following coupled Riccati differential equations:

$$-\dot{P}_1 = (A_s - B_2 R_{02}^{-1} B_2^T P_2)^T P_1 + P_1 (A_s - B_2 R_{02}^{-1} B_2^T P_2) + \gamma^{-2} P_1 B_1 B_1^T P_1$$

$$+[C_1 - D_{12} R_{02}^{-1} (D_{02}^T C_0 + B_2^T P_2)]^T [C_1 - D_{12} R_{02}^{-1} (D_{02}^T C_0 + B_2^T P_2)],$$

$$-\dot{P}_2 = (A_s + \gamma^{-2} B_1 B_1^T P_1)^T P_2 + P_2 (A_s + \gamma^{-2} B_1 B_1^T P_1) - P_2 B_2 R_{02}^{-1} B_2^T P_2 + Q$$

have solutions $P_1 \ge 0$ and $P_2 \ge 0$ on $[0, T]$ with $P_1(T) = 0$ and $P_2(T) = 0$. Furthermore, if the solutions exist, we have $w_* = \gamma^{-2} B_1^T P_1 x$ and $u_* = F_* x$ with $F_* := -R_{02}^{-1} (D_{02}^T C_0 + B_2^T P_2)$.

For the case of infinite time horizon, we shall make the following standard assumptions:

(A1) (A, B_2) is stabilizable and (C_2, A) is detectable.

Theorem 4.2 *There exist linear memoryless state feedback strategies (Nash equilibrium strategies) u_* and w_* such that:*

$$\begin{aligned} 0 \le J_3(u_*, w_*, 0) &\le J_3(u_*, w, 0), \\ J_4(u_*, w_*, 0) &\le J_4(u, w_*, 0), \end{aligned}$$

if and only if the following coupled Riccati equations:

$$(A_s - B_2 R_{02}^{-1} B_2^T P_2)^T P_1 + P_1 (A_s - B_2 R_{02}^{-1} B_2^T P_2) + \gamma^{-2} P_1 B_1 B_1^T P_1$$

$$+[C_1 - D_{12} R_{02}^{-1}(D_{02}^T C_0 + B_2^T P_2)]^T [C_1 - D_{12} R_{02}^{-1}(D_{02}^T C_0 + B_2^T P_2)] = 0,$$

$$(A_s + \gamma^{-2} B_1 B_1^T P_1)^T P_2 + P_2 (A_s + \gamma^{-2} B_1 B_1^T P_1) - P_2 B_2 R_{02}^{-1} B_2^T P_2 + Q = 0$$

have stabilizing solutions $P_1 \ge 0$ and $P_2 \ge 0$, i.e., $A_s + \gamma^{-2} B_1 B_1^T P_1 - B_2 R_{02}^{-1} B_2^T P_2$ is stable.

Furthermore, if the solutions exist, we have $w_* = \gamma^{-2} B_1^T P_1 x$ and $u_* = F_* x$ with $F_* := -R_{02}^{-1}(D_{02}^T C_0 + B_2^T P_2)$.

4.3 Output Feedback Multiobjective Control—Finite Time Horizon

Output feedback design results are established in the following theorem for multi-objective optimal control in finite time horizon.

Theorem 4.3 *Suppose the state feedback control problem in Section 4.2 is solvable, i.e., there are $P_1 \geq 0$ and $P_2 \geq 0$ with $P_1(T) = 0$ and $P_2(T) = 0$ solving*

$$-\dot{P}_1 = (A_s - B_2 R_{02}^{-1} B_2^T P_2)^T P_1 + P_1(A_s - B_2 R_{02}^{-1} B_2^T P_2) + \gamma^{-2} P_1 B_1 B_1^T P_1$$

$$+[C_1 - D_{12} R_{02}^{-1}(D_{02}^T C_0 + B_2^T P_2)]^T [C_1 - D_{12} R_{02}^{-1}(D_{02}^T C_0 + B_2^T P_2)],$$

$$-\dot{P}_2 = (A_s + \gamma^{-2} B_1 B_1^T P_1)^T P_2 + P_2(A_s + \gamma^{-2} B_1 B_1^T P_1) - P_2 B_2 R_{02}^{-1} B_2^T P_2 + Q.$$

There exist a $w_* = \gamma^{-2} B_1^T P_1 x$ and an output feedback control law u_* in the form of:

$$\begin{aligned} \dot{\hat{x}} &= (A + \gamma^{-2} B_1 B_1^T P_1 + B_2 F_*)\hat{x} + L_*(C_2 \hat{x} - y), \\ u_* &= F_* \hat{x}, \end{aligned}$$

where $F_* = -R_{02}^{-1}(D_{02}^T C_0 + B_2^T P_2)$, such that

$$\begin{aligned} J_1(u_*, w_*, w_0) &\leq J_1(u_*, w, w_0), \\ J_2(u_*, w_*, w_0) &\leq J_2(u, w_*, w_0) \end{aligned}$$

if and only if there is a $P_3 \geq 0$ on $[0, T]$ with $P_3(0) = 0$ solving

$$\dot{P}_3 = (A_f + \gamma^{-2} B_1 B_1^T P_1) P_3 + P_3 (A_f + \gamma^{-2} B_1 B_1^T P_1)^T - P_3 C_2^T R_{20}^{-1} C_2 P_3 + P.$$

Moreover, if the solution exists, then L_* can be chosen as $L_* = -(B_0 D_{20}^T + P_3 C_2^T) R_{20}^{-1}$.

PROOF. Since there exist solutions $P_1 \geq 0$ and $P_2 \geq 0$ on $[0, T]$ solving

$$-\dot{P}_1 = (A_s - B_2 R_{02}^{-1} B_2^T P_2)^T P_1 + P_1(A_s - B_2 R_{02}^{-1} B_2^T P_2) + \gamma^{-2} P_1 B_1 B_1^T P_1$$

$$+[C_1 - D_{12} R_{02}^{-1}(D_{02}^T C_0 + B_2^T P_2)]^T [C_1 - D_{12} R_{02}^{-1}(D_{02}^T C_0 + B_2^T P_2)],$$

$$-\dot{P}_2 = (A_s + \gamma^{-2} B_1 B_1^T P_1)^T P_2 + P_2 (A_s + \gamma^{-2} B_1 B_1^T P_1) - P_2 B_2 R_{02}^{-1} B_2^T P_2 + Q,$$

we have:

$$\begin{aligned}
&E\left\{\int_0^T (\gamma^2 \|w\|^2 - \|z\|^2) dt\right\} \\
&= E\left\{\int_0^T [\gamma^2 \|w\|^2 - \|z\|^2 - \frac{d}{dt}(x^T P_1 x)] dt\right\} \\
&= E\left\{\int_0^T [\gamma^2 \|w\|^2 - \|z\|^2 - \dot{x}^T P_1 x - x^T \dot{P}_1 x - x^T P_1 \dot{x}] dt\right\}.
\end{aligned}$$

Now, using the equation for $\dot{P}_1$, we get

$$\begin{aligned}
&E\left\{\int_0^T (\gamma^2 \|w\|^2 - \|z\|^2) dt\right\} \\
&= E\{\int_0^T [\gamma^2 \|w - w_*\|^2 - u^T R_{12} u + \bar{u}_*^T R_{12} \bar{u}_* \\
&-2x^T (P_1 B_2 + C_1^T D_{12})(u - \bar{u}_*) - 2x^T P_1 B_0 w_0] dt\},
\end{aligned}$$

where $R_{12} := D_{12}^T D_{12}$, $w_* = \gamma^{-2} B_1^T P_1 x$, and $\bar{u}_* = -R_{02}^{-1}(D_{02}^T C_0 + B_2^T P_2)x$, i.e., the optimal strategies for state feedback case. Clearly, if we take $w = w_*$, then for any u, hence for the optimally designed output feedback control law u_*, we have

$$J_1(u_*, w_*, w_0) \leq J_1(u_*, w, w_0).$$

Now we prove the theorem:

(Sufficiency) If there is a $P_3 \geq 0$ on $[0, T]$ with $P_3(0) = 0$ solving

$$\dot{P}_3 = (A_f + \gamma^{-2} B_1 B_1^T P_1) P_3 + P_3 (A_f + \gamma^{-2} B_1 B_1^T P_1)^T - P_3 C_2^T R_{20}^{-1} C_2 P_3 + P,$$

by substituting the w_* into the system equation, we get

$$\begin{aligned}
\dot{x} &= (A + \gamma^{-2} B_1 B_1^T P_1) x + B_0 w_0 + B_2 u, \quad x(0) = 0, && (4.13) \\
y &= C_2 x + D_{20} w_0, \quad R_{20} := D_{20} D_{20}^T > 0, && (4.14) \\
z_0 &= C_0 x + D_{02} u, \quad R_{02} := D_{02}^T D_{02} > 0. && (4.15)
\end{aligned}$$

This is a standard LQG problem [1, 16]. Thus the optimal control law is given by

$$\dot{\hat{x}} = (A+\gamma^{-2}B_1B_1^TP_1)\hat{x} + B_2u_* + L_*(C_2\hat{x}-y), \tag{4.16}$$

$$u_* = F_*\hat{x}, \tag{4.17}$$

where $F_* = -R_{02}^{-1}(D_{02}^TC_0 + B_2^TP_2)$, $L_* = -(B_0D_{20}^T + P_3C_2^T)R_{20}^{-1}$, P_2 and P_3 solve:

$$-\dot{P}_2 = (A_s+\gamma^{-2}B_1B_1^TP_1)^TP_2 + P_2(A_s+\gamma^{-2}B_1B_1^TP_1) - P_2B_2R_{02}^{-1}B_2^TP_2 + Q,$$

$$\dot{P}_3 = (A_f+\gamma^{-2}B_1B_1^TP_1)P_3 + P_3(A_f+\gamma^{-2}B_1B_1^TP_1)^T - P_3C_2^TR_{20}^{-1}C_2P_3 + P,$$

with $P_2(T)=0$ and $P_3(0)=0$, and u_* achieves:

$$J_2(u_*,w_*,w_0) \le J_2(u,w_*,w_0).$$

Note that it can be calculated that

$$J_2(u_*,w_*,w_0) = trace\left\{B_0B_0^T\int_0^T P_2dt\right\} + trace\left\{F_*^TR_{02}F_*\int_0^T P_3(t)dt\right\}.$$

(Necessity) Suppose that $w_* = \gamma^{-2}B_1^TP_1x$ and u_*:

$$\dot{\hat{x}} = (A+\gamma^{-2}B_1B_1^TP_1 + B_2F_*)\hat{x} + L_*(C_2\hat{x}-y),$$

$$u_* = F_*\hat{x},$$

where $F_* = -R_{02}^{-1}(D_{02}^TC_0 + B_2^TP_2)$ such that:

$$J_1(u_*,w_*,w_0) \le J_1(u_*,w,w_0),$$

$$J_2(u_*,w_*,w_0) \le J_2(u,w_*,w_0).$$

Substituting w_* into the system equation, we get

$$\dot{x} = (A+\gamma^{-2}B_1B_1^TP_1)x + B_0w_0 + B_2u_*, \quad x(0)=0,$$

$$\begin{aligned} y &= C_2x + D_{20}w_0, \quad R_{20} := D_{20}D_{20}^T > 0, \\ z_0 &= C_0x + D_{02}u_*, \quad R_{02} := D_{02}^TD_{02} > 0. \end{aligned}$$

Thus

$$J_2(u_*, w_*, w_0) = E\int_0^T \|z_0\|^2 dt$$

$$= E\int_0^T [x^TC_0^TC_0x + 2x^TC_0^TD_{02}u_* + u_*^TR_{02}u_* + \frac{d}{dt}(x^TP_2x)]dt$$

$$= E\int_0^T [x^TC_0^TC_0x + 2x^TC_0^TD_{02}u_* + u_*^TR_{02}u_* + \dot{x}^TP_2x + x^T\dot{P}_2x + x^TP_2\dot{x}]dt.$$

Now using the equation about P_2, we get

$$J_2(u_*, w_*, w_0) = E\int_0^T (u_* - \bar{u}_*)^TR_{02}(u_* - \bar{u}_*)dt + 2E\int_0^T x^TP_2B_0w_0dt$$

$$= E\int_0^T (u_* - \bar{u}_*)^TR_{02}(u_* - \bar{u}_*)dt + 2trace\int_0^T P_2B_0E\{w_0x^T\}dt.$$

By Lemma 2.2, we have $E\{xw_0^T\} = B_0/2$. Hence

$$J_2(u_*, w_*, w_0) = E\int_0^T (u_* - \bar{u}_*)^TR_{02}(u_* - \bar{u}_*)dt + trace\int_0^T P_2B_0B_0^Tdt$$

$$= E\int_0^T (u_* - \bar{u}_*)^TR_{02}(u_* - \bar{u}_*)dt + trace\int_0^T B_0^TP_2B_0dt.$$

We only need to consider the first term. Define $e = x - \hat{x}$, then

$$E\int_0^T (u_* - \bar{u}_*)^TR_{02}(u_* - \bar{u}_*)dt = E\int_0^T e^TF_*^TR_{02}F_*edt.$$

With u_* in the form:

$$\begin{aligned} \dot{\hat{x}} &= (A + \gamma^{-2}B_1B_1^TP_1 + B_2F_*)\hat{x} + L_*(C_2\hat{x} - y), \\ u_* &= F_*\hat{x}, \end{aligned}$$

e satisfies:

$$\dot{e} = \dot{x} - \dot{\hat{x}} = (A + \gamma^{-2}B_1B_1^TP_1 + L_*C_2)e + (B_0 + L_*D_{20})w_0 = A_{L_*}e + B_{L_*}w_0.$$

Let $\Phi(t,0)$ be the transition matrix of $A + \gamma^{-2}B_1B_1^TP_1 + L_*C_2$, then

$$e = \int_0^t \Phi(t,\tau)(B_0 + LD_{20})w_0 d\tau.$$

This gives:

$$\begin{aligned}
&E\int_0^T e^TF_*^TR_{02}F_*edt \\
&= E\left\{\int_0^T\int_0^t\int_0^t w^T(\tau)B_{L_*}^T\Phi^T(t,\tau)F_*^TR_{02}F_*\Phi(t,\tau)B_{L_*}w(s)d\tau\ ds\ dt\right\} \\
&= trace\left\{\int_0^T\int_0^t\int_0^t F_*^TR_{02}F_*\Phi(t,\tau)B_{L_*}E\{w(s)w(\tau)\}B_{L_*}^T\Phi^T(t,\tau)d\tau\ ds\ dt\right\} \\
&= trace\left\{\int_0^T\int_0^t\int_0^t F_*^TR_{02}F_*\Phi(t,\tau)B_{L_*}\delta(\tau - s)B_{L_*}^T\Phi^T(t,\tau)d\tau\ ds\ dt\right\} \\
&= trace\left\{F_*^TR_{02}F_*\int_0^T Y(t)dt\right\},
\end{aligned}$$

where $Y(t) = \int_0^t \Phi(t,s)B_{L_*}B_{L_*}^T\Phi^T(t,s)ds \geq 0$ satisfies:

$$\dot{Y} = A_{L_*}Y + YA_{L_*}^T + B_{L_*}B_{L_*}^T, \quad Y(0) = 0.$$

Since

$$J_2(u_*, w_*, w_0) = trace\left\{F_*^TR_{02}F_*\int_0^T Y(t)dt\right\} + trace\int_0^T B_0^TP_2B_0dt$$

is the minimum value, by Theorem 2.3 in Chapter 2, there is a $P_3 \geq 0$, $P_3 \leq Y$ with $P_3(0) = 0$ solving

$$\dot{P}_3 = (A_f + \gamma^{-2}B_1B_1^TP_1)P_3 + P_3(A_f + \gamma^{-2}B_1B_1^TP_1)^T - P_3C_2^TR_{20}^{-1}C_2P_3 + P,$$

and L_* can be chosen as $L_* = -(B_0D_{20}^T + P_3C_2^T)R_{20}^{-1}$ since

$$J_2(u_*, w_*, w_0) = trace\left\{F_*^TR_{02}F_*\int_0^T P_3dt\right\} + trace\int_0^T B_0^TP_2B_0dt.$$

This concludes the proof.

□

4.4 Output Feedback Multiobjective Control—Infinite Time Horizon

The design for the infinite time horizon case is more complicated than that for finite time horizon case. The main concern is the stability requirement since asymptotic properties are considered here. Some additional assumptions are made as well as the Assumption $(A1)$ made in Section 4.1:

(A2) $\begin{bmatrix} A - j\omega I & B_2 \\ C_0 & D_{02} \end{bmatrix}$ has full column rank for all ω,

(A3) $\begin{bmatrix} A - j\omega I & B_0 \\ C_2 & D_{20} \end{bmatrix}$ has full row rank for all ω.

These two assumptions guarantee that the corresponding $\mathcal{H}_2$ Riccati equations have stabilizing solutions when $\gamma \to \infty$.

The design results are summarized in the next theorem, addressing both sufficient and necessary conditions.

Theorem 4.4 *There exist a w_* and an output feedback control law u_* such that*

$$\begin{aligned} J_3(u_*, w_*, w_0) &\leq J_3(u_*, w, w_0), \\ J_4(u_*, w_*, w_0) &\leq J_4(u, w_*, w_0), \end{aligned}$$

if the coupled Riccati algebraic equations:

$$(A_s - B_2 R_{02}^{-1} B_2^T P_2)^T P_1 + P_1 (A_s - B_2 R_{02}^{-1} B_2^T P_2) + \gamma^{-2} P_1 B_1 B_1^T P_1$$

$$+ [C_1 - D_{12} R_{02}^{-1} (D_{02}^T C_0 + B_2^T P_2)]^T [C_1 - D_{12} R_{02}^{-1} (D_{02}^T C_0 + B_2^T P_2)] = 0,$$

$$(A_s + \gamma^{-2}B_1B_1^TP_1)^TP_2 + P_2(A_s + \gamma^{-2}B_1B_1^TP_1) - P_2B_2R_{02}^{-1}B_2^TP_2 + Q = 0,$$

$$(A_f + \gamma^{-2}B_1B_1^TP_1)P_3 + P_3(A_f + \gamma^{-2}B_1B_1^TP_1)^T - P_3C_2^TR_{20}^{-1}C_2P_3 + P = 0$$

have stabilizing solutions $P_1 \geq 0$, $P_2 \geq 0$, *and* $P_3 \geq 0$, *i.e., both* $A_s + \gamma^{-2}B_1B_1^TP_1 - B_2R_{02}^{-1}B_2^TP_2$ *and* $A_f + \gamma^{-2}B_1B_1^TP_1 - P_3C_2^TR_{20}^{-1}C_2$ *are stable.*

If the solutions exist, we have: $w_* = \gamma^{-2}B_1^TP_1x$ *and* u_* *is of the following form:*

$$\dot{\hat{x}} = (A + \gamma^{-2}B_1B_1^TP_1 + B_2F_*)\hat{x} + L_*(C_2\hat{x} - y), \tag{4.18}$$

$$u_* = F_*\hat{x}, \tag{4.19}$$

where $F_* = -R_{02}^{-1}(D_{02}^TC_0 + B_2^TP_2)$ *and* $L_* = -(B_0D_{20}^T + P_3C_2^T)R_{20}^{-1}$.

Conversely, if the state feedback control problem is solvable, i.e., there are stabilizing solutions $P_1 \geq 0$ *and* $P_2 \geq 0$ *for:*

$$(A_s - B_2R_{02}^{-1}B_2^TP_2)^TP_1 + P_1(A_s - B_2R_{02}^{-1}B_2^TP_2) + \gamma^{-2}P_1B_1B_1^TP_1$$

$$+[C_1 - D_{12}R_{02}^{-1}(D_{02}^TC_0 + B_2^TP_2)]^T[C_1 - D_{12}R_{02}^{-1}(D_{02}^TC_0 + B_2^TP_2)] = 0,$$

$$(A_s + \gamma^{-2}B_1B_1^TP_1)^TP_2 + P_2(A_s + \gamma^{-2}B_1B_1^TP_1) - P_2B_2R_{02}^{-1}B_2^TP_2 + Q = 0,$$

and there is an optimal control u_* *in the form of:*

$$\dot{\hat{x}} = (A + \gamma^{-2}B_1B_1^TP_1 + B_2F_*)\hat{x} + L_*(C_2\hat{x} - y), \tag{4.20}$$

$$u_* = F_*\hat{x}, \tag{4.21}$$

where $F_* = -R_{02}^{-1}(D_{02}^TC_0 + B_2^TP_2)$ *and a* w_* *such that:*

$$J_3(u_*, w_*, w_0) \leq J_3(u_*, w, w_0),$$

$$J_4(u_*, w_*, w_0) \leq J_4(u, w_*, w_0),$$

then, there is a $P_3 \geq 0$ *solving*

$$(A_f + \gamma^{-2} B_1 B_1^T P_1) P_3 + P_3 (A_f + \gamma^{-2} B_1 B_1^T P_1)^T - P_3 C_2^T R_{20}^{-1} C_2 P_3 + P = 0.$$

Moreover, if $A + \gamma^{-2} B_1 B_1^T P_1 - (B_0 D_{20}^T + P_3 C_2^T) R_{20}^{-1} C_2$ *is stable, then* $L_* = -(B_0 D_{20}^T + P_3 C_2^T) R_{20}^{-1}$.

Proof. Suppose there exist stabilizing solutions $P_1 \geq 0$, $P_2 \geq 0$ and $P_3 \geq 0$, to the following Riccati equations:

$$(A_s - B_2 R_{02}^{-1} B_2^T P_2)^T P_1 + P_1 (A_s - B_2 R_{02}^{-1} B_2^T P_2) + \gamma^{-2} P_1 B_1 B_1^T P_1$$

$$+ [C_1 - D_{12} R_{02}^{-1} (D_{02}^T C_0 + B_2^T P_2)]^T [C_1 - D_{12} R_{02}^{-1} (D_{02}^T C_0 + B_2^T P_2)] = 0,$$

$$(A_s + \gamma^{-2} B_1 B_1^T P_1)^T P_2 + P_2 (A_s + \gamma^{-2} B_1 B_1^T P_1) - P_2 B_2 R_{02}^{-1} B_2^T P_2 + Q = 0,$$

$$(A_f + \gamma^{-2} B_1 B_1^T P_1) P_3 + P_3 (A_f + \gamma^{-2} B_1 B_1^T P_1)^T - P_3 C_2^T R_{20}^{-1} C_2 P_3 + P = 0.$$

Let u be any stabilizing control law. Since

$$\lim_{T \to \infty} E \left\{ \frac{1}{T} \int_0^T (\gamma^2 \|w\|^2 - \|z\|^2) dt \right\}$$

$$= \lim_{T \to \infty} E \left\{ \frac{1}{T} \int_0^T (\gamma^2 \|w\|^2 - x^T C_1^T C_1 x - 2 x^T C_1^T D_{12} u - u^T R_{12} u) dt \right\}.$$

Now, using the equation for P_1, we get:

$$\lim_{T \to \infty} E \left\{ \frac{1}{T} \int_0^T (\gamma^2 \|w\|^2 - \|z\|^2) dt \right\}$$

$$= E\{ \lim_{T \to \infty} \int_0^T [\gamma^2 \|w - w_*\|^2 - u^T R_{12} u + \bar{u}_*^T R_{12} \bar{u}_*$$

$$- 2 x^T (P_1 B_2 + C_1^T D_{12})(u - \bar{u}_*) - 2 x^T P_1 B_0 w_0] dt \},$$

where $R_{12} := D_{12}^T D_{12}$, $w_* = \gamma^{-2} B_1^T P_1 x$, and $\bar{u}_* = -R_{02}^{-1}(D_{02}^T C_0 + B_2^T P_2)x$, i.e., the optimal strategies for state feedback case. Clearly, if we take $w = w_*$, then for any u, hence for the optimally designed output feedback control law u_*, we have

$$J_3(u_*, w_*, w_0) \leq J_3(u_*, w, w_0).$$

Now, we design u_* to minimize J_4. By substituting the w_* into the system equation, we get:

$$\begin{aligned} \dot{x} &= (A + \gamma^{-2} B_1 B_1^T P_1)x + B_0 w_0 + B_2 u, \qquad x(0) = 0, && (4.22) \\ y &= C_2 x + D_{20} w_0, \quad R_{20} := D_{20} D_{20}^T > 0, && (4.23) \\ z_0 &= C_0 x + D_{02} u, \quad R_{02} := D_{02}^T D_{02} > 0. && (4.24) \end{aligned}$$

Since the index functional J_4 is taken into account, clearly, this is a standard LQG problem [1, 16]. Thus the optimal control law is given by

$$\begin{aligned} \dot{\hat{x}} &= (A + \gamma^{-2} B_1 B_1^T P_1)\hat{x} + B_2 u_* + L_*(C_2 \hat{x} - y), && (4.25) \\ u_* &= F_* \hat{x}, && (4.26) \end{aligned}$$

where $F_* = -R_{02}^{-1}(D_{02}^T C_0 + B_2^T P_2)$, $L_* = -(B_0 D_{20}^T + P_3 C_2^T) R_{20}^{-1}$ and u_* achieves:

$$J_4(u_*, w_*, w_0) \leq J_4(u, w_*, w_0).$$

Note that it can be calculated:

$$J_4(u_*, w_*, w_0) = trace\left\{B_0 B_0^T P_2\right\} + trace\left\{F_*^T R_{02} F_* P_3\right\}.$$

Conversely, suppose the state feedback control problem is solvable, i.e., there are stabilizing solutions to:

$$(A_s - B_2 R_{02}^{-1} B_2^T P_2)^T P_1 + P_1(A_s - B_2 R_{02}^{-1} B_2^T P_2) + \gamma^{-2} P_1 B_1 B_1^T P_1$$

$$+[C_1 - D_{12}R_{02}^{-1}(D_{02}^T C_0 + B_2^T P_2)]^T [C_1 - D_{12}R_{02}^{-1}(D_{02}^T C_0 + B_2^T P_2)] = 0,$$

$$(A_s + \gamma^{-2}B_1B_1^TP_1)^T P_2 + P_2(A_s + \gamma^{-2}B_1B_1^TP_1) - P_2B_2R_{02}^{-1}B_2^TP_2 + Q = 0,$$

and let u_* be in the form of:

$$\begin{aligned} \dot{\hat{x}} &= (A + \gamma^{-2}B_1B_1^TP_1 + B_2F_*)\hat{x} + L_*(C_2\hat{x} - y), & (4.27) \\ u_* &= F_*\hat{x}, & (4.28) \end{aligned}$$

where $F_* = -R_{02}^{-1}(D_{02}^TC_0 + B_2^TP_2)$. Let u_* and w_* achieve:

$$\begin{aligned} J_3(u_*, w_*, w_0) &\leq J_3(u_*, w, w_0), \\ J_4(u_*, w_*, w_0) &\leq J_4(u, w_*, w_0). \end{aligned}$$

Note that, from the proof above, $w_* = \gamma^{-2}B_1^TP_1x$. Now substitute w_* and u_* into the system equations, we get:

$$\begin{aligned} \dot{x} &= (A + \gamma^{-2}B_1B_1^TP_1)x + B_0w_0 + B_2u_*, \qquad x(0) = 0, \\ y &= C_2x + D_{20}w_0, \quad R_{20} := D_{20}D_{20}^T > 0, \\ z_0 &= C_0x + D_{02}u_*, \quad R_{02} := D_{02}^TD_{02} > 0, \end{aligned}$$

and

$$J_4(u_*, w_*, w_0) = \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T \|z_0\|^2 dt\right\}$$

$$= \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T [x^TC_0^TC_0x + 2x^TC_0^TD_{02}u_* + u_*^TR_{02}u_*]dt\right\}.$$

Now using the equation about P_2, we get:

$$J_4(u_*, w_*, w_0)$$

$$= \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T (u_* - \bar{u}_*)^T R_{02}(u_* - \bar{u}_*)dt\right\} + 2\lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T x^T P_2 B_0 w_0 dt\right\}$$

$$= \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T (u_* - \bar{u}_*)^T R_{02}(u_* - \bar{u}_*)dt\right\} + 2trace\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T P_2 B_0 E\{w_0 x^T\}dt\right\}$$

$$= \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T (u_* - \bar{u}_*)^T R_{02}(u_* - \bar{u}_*)dt\right\} + trace(P_2 B_0 B_0^T),$$

where $\bar{u}_* = -R_{02}^{-1}(D_{02}^T C_0 + B_2^T P_2)x$. We only need to consider the first term. Define $e = x - \hat{x}$, then

$$\lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T (u_* - \bar{u}_*)^T R_{02}(u_* - \bar{u}_*)dt\right\} = \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T e^T F_*^T R_{02} F_* e dt\right\},$$

where e satisfies:

$$\dot{e} = \dot{x} - \dot{\hat{x}} = (A + \gamma^{-2}B_1 B_1^T P_1 + L_* C_2)e + (B_0 + L_* D_{20})w_0 = A_{L_*}e + B_{L_*}w_0.$$

Hence $e = \int_0^t exp[A_{L_*}(t-\tau)]B_{L_*}w_0 d\tau$. This gives:

$$\lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T e^T F_*^T R_{02} F_* e dt\right\} = trace(F_*^T R_{02} F_* Y),$$

where $Y = \int_0^\infty exp(A_{L_*}s)B_{L_*}B_{L_*}^T exp(A_{L_*}^T s)ds \geq 0$ satisfies:

$$A_{L_*}Y + YA_{L_*}^T + B_{L_*}B_{L_*}^T = 0.$$

Since $J_4(u_*, w_*, w_0) = trace(F_*^T R_{02} F_* Y) + trace(B_0^T P_2 B_0)$ is the minimum value, by Theorem 2.3 in Chapter 2, there is a $P_3 \geq 0$, $P_3 \leq Y$ solving

$$(A_f + \gamma^{-2}B_1 B_1^T P_1)P_3 + P_3(A_f + \gamma^{-2}B_1 B_1^T P_1)^T - P_3 C_2^T R_{20}^{-1} C_2 P_3 + P = 0,$$

and if $A + \gamma^{-2}B_1 B_1^T P_1 - (B_0 D_{20}^T + P_3 C_2^T)R_{20}^{-1}C_2$ is stable, then, L_* can be chosen as $L_* = -(B_0 D_{20}^T + P_3 C_2^T)R_{20}^{-1}$ since

$$J_4(u_*, w_*, w_0) = trace(F_*^T R_{02} F_* P_3) + trace(B_0^T P_2 B_0).$$

This concludes the proof.

□

Chapter 5

$\mathcal{H}_\infty$ Gaussian Control

In this chapter, a new multiobjective control problem is formulated which will be called the $\mathcal{H}_\infty$ Gaussian control. The reason for using this name is that the design is naturally motivated by $\mathcal{H}_\infty$ and LQG control and hence presents a trade-off between $\mathcal{H}_\infty$ performances and LQG performances. It turns out that this $\mathcal{H}_\infty$ Gaussian problem can be solved in a very similar way to that in which the classical LQG control problem is solved: combining an optimally designed filter and a state feedback control. But here we use the $\mathcal{H}_\infty$ state feedback gain instead of LQR feedback gain. It should be pointed out that unlike LQG design, the Separation Theorem does not hold for $\mathcal{H}_\infty$ Gaussian control design, i.e., the filtering design must be designed together with state feedback control.

In Section 5.1, the proposed $\mathcal{H}_\infty$ Gaussian problem is formulated. The motivation for this problem can also be found in this section. The $\mathcal{H}_\infty$ Gaussian design are the main contents in Section 5.2 for finite time horizon case and in Section 5.3 for infinite time horizon case.

5.1 Motivation and Problem Formulation

To motivate the problem, let us consider a feedback control system shown in Figure 5.1, where $\Delta \in \mathcal{RH}_\infty$ is the modeling error with $\|\Delta\|_\infty < \frac{1}{\gamma}$ for some prespecified $\gamma > 0$ and w_0 is the measurement noise.

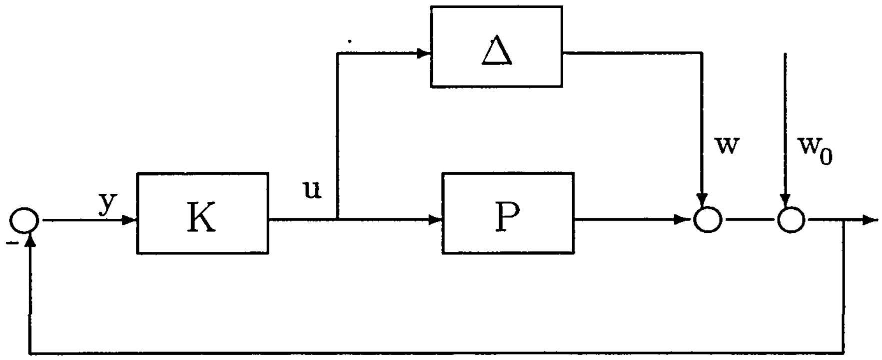

Figure 5.1: A Feedback Control System

We want to ask this question:

Can we find a K such that the system is robustly stable with respect to a set of model uncertainties and the effect of w_0 on the system performance is minimized?

To answer this question, let us separate the problem into two parts:

1. If there is no w_0, i.e., the measurement is noise-free, this turns out to be a typical $\mathcal{H}_\infty$ control design problem.

2. If there is no model uncertainty Δ, i.e., we know model perfectly, then it is well known that an *LQG* approach provides a good design to minimize the effect of w_0 on the system performance while guarantee the closed-loop stability.

The question is how one can take into account both model uncertainties and external disturbances in feedback design. As it is pointed out in Chapter 1, a design trade-off has to be made, i.e., to do the trade-off between $\mathcal{H}_\infty$ design and *LQG* design.

Note that, without noise w_0, the feedback control system in Figure 5.1 can be put into the general $\mathcal{H}_\infty$ framework as shown in Figure 5.2

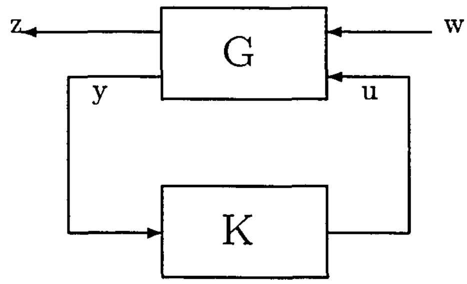

Figure 5.2: $\mathcal{H}_\infty$ Control System with Disturbance w

If the plant G is subject to some additional white noise disturbance w_0, naturally, we have an extended system framework as shown in Figure 5.3.

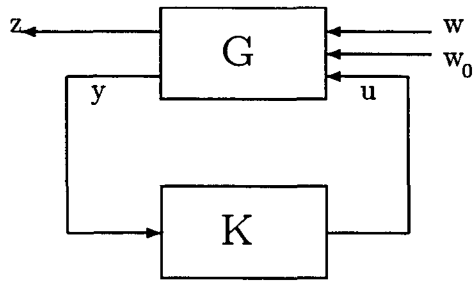

Figure 5.3: A System with Both Model Uncertainty and White Noise

To provide more hints for problem formulation, let us consider $\mathcal{H}_\infty$ design and LQG design in more details:

- Hint from $\mathcal{H}_\infty$ **Control**: Consider the structure of a central $\mathcal{H}_\infty$ Controller shown in the Figure 5.4. Clearly, this controller consists of a state feedback $\mathcal{H}_\infty$ control F_∞ and a state estimator $F(s)$.

- Hint from LQG **Control**: Consider the structure of an LQG controller shown in Figure 5.5. This controller consists of a Kalman filter and LQR feedback gain F_{lqr}.

Obviously, these designs suggest that we may achieve design trade-off between $\mathcal{H}_\infty$ and LQG performances by adopting the controller structure shown in Figure 5.6.

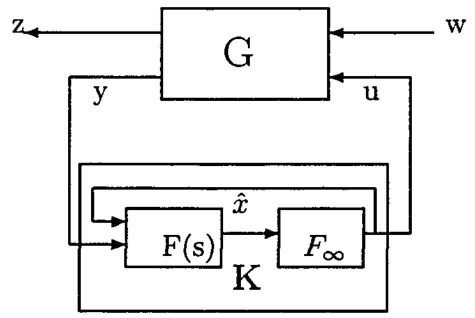

Figure 5.4: A Central $\mathcal{H}_\infty$ Controller

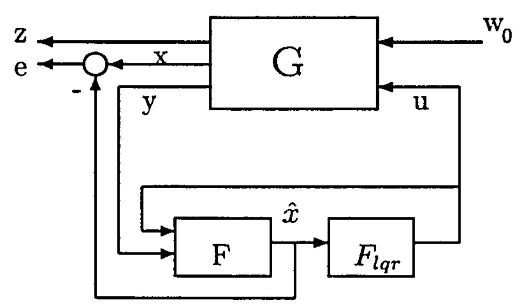

Figure 5.5: An LQG Controller

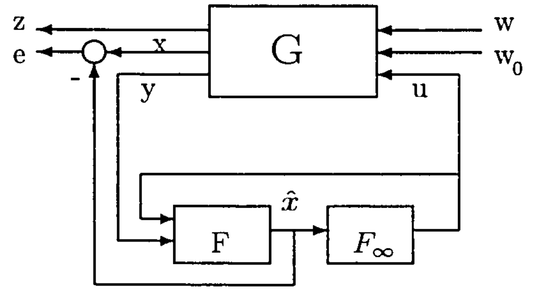

Figure 5.6: A Possible H_∞ Gaussian Controller

Based on the above consideration, the $\mathcal{H}_\infty$ Gaussian control problem is formulated as follows:

Consider a generalized system G with the following equations:

$$\dot{x} = Ax + B_0 w_0 + B_1 w + B_2 u, \quad x(0) = 0, \tag{5.1}$$

$$z = C_1 x + D_{12} u, \quad R_1 = D_{12}^T D_{12} > 0, \tag{5.2}$$

$$y = C_2 x + D_{20} w_0, \quad R_0 = D_{20} D_{20}^T > 0, \tag{5.3}$$

where $x \in \mathbf{R}^n$, $y \in \mathbf{R}^p$, $z \in \mathbf{R}^{q_1}$; $w \in \mathbf{R}^{r_1}$ is a bounded power signal and $w_0 \in \mathbf{R}^{r_2}$ is a white noise signal. w_0 and w are (mutually) independent.

Motivated by $\mathcal{H}_\infty$ and LQG designs, it is assumed that the controller has the following structure:

$$\dot{\hat{x}} = \hat{A}\hat{x} + B_2 u - Ly, \quad \hat{x}(0) = 0, \tag{5.4}$$

$$u = \hat{F}\hat{x}, \tag{5.5}$$

where $\hat{x} \in \mathbf{R}^n$ is an optimal estimate of the state in some sense and $\hat{A}$, $\hat{F}$ and L are design parameters to be chosen.

Let $e = x - \hat{x}$. Define the following cost functionals:

$$J_1(u, w, w_0) := E\left\{\int_0^T (\gamma^2\|w\|^2 - \|z\|^2)dt\right\}, \tag{5.6}$$

$$J_2(u, w, w_0) := E\left\{\int_0^T \|e\|^2 dt\right\}, \tag{5.7}$$

for finite time horizon case and

$$J_3(u, w, w_0) := \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T (\gamma^2\|w\|^2 - \|z\|^2)dt\right\}, \tag{5.8}$$

$$J_4(u, w, w_0) \quad := \quad \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T \|e\|^2 dt\right\}, \tag{5.9}$$

for infinite time horizon case. Then the $\mathcal{H}_\infty$ Gaussian control design problem is defined as follows:

$\mathcal{H}_\infty$ Gaussian Control: Find an optimal output feedback control law u_* and a worst disturbance signal w_* under white noise such that:

$$J_1(u_*, w_*, w_0) \quad \leq \quad J_1(u_*, w, w_0) \tag{5.10}$$

$$J_2(u_*, w_*, w_0) \quad \leq \quad J_2(u, w_*, w_0), \tag{5.11}$$

or

$$J_3(u_*, w_*, w_0) \quad \leq \quad J_3(u_*, w, w_0) \tag{5.12}$$

$$J_4(u_*, w_*, w_0) \quad \leq \quad J_4(u, w_*, w_0), \tag{5.13}$$

hold for all w and u.

5.2 $\mathcal{H}_\infty$ Gaussian Control Design–Finite Time Horizon

In finite time horizon, there is no stability requirement so what is concerned is just the control regulation. The results are summarized in the following theorem.

To simplify notations, we shall introduce:

$$A_x := A - B_2R_1^{-1}D_{12}^TC_1, \quad A_y := A - B_0D_{20}^TR_0^{-1}C_2,$$

$$P := B_0(I - D_{20}^TR_0^{-1}D_{20})B_0^T, \quad Q := C_1^T(I - D_{12}R_1^{-1}D_{12}^T)C_1.$$

Theorem 5.1 *Let the dynamical system G be described by equations (5.1)–(5.3). If there are solutions $P_1 \geq 0$, $P_2 \geq 0$ and $P_3 \geq 0$ with $P_1(T) = 0$, $P_2(T) = 0$ and $P_3(0) = 0$ solving the following differential Riccati equations:*

$$A_x^T P_1 + P_1 A_x + P_1(B_1 B_1^T/\gamma^2 - B_2 R_1^{-1} B_2^T)P_1 + Q = -\dot{P}_1,$$

$$P_2(A_y + \gamma^{-2} B_1 B_1^T P_1 - P_3 C_2^T R_0^{-1} C_2) + (A_y + \gamma^{-2} B_1 B_1^T P_1 - P_3 C_2^T R_0^{-1} C_2)^T P_2$$

$$+\gamma^{-2} P_2 B_1 B_1^T P_2 + (D_{12}^T C_1 + B_2^T P_1)^T R_1^{-1} (D_{12}^T C_1 + B_2^T P_1) = -\dot{P}_2,$$

$$[A_y + \gamma^{-2} B_1 B_1^T (P_1 + P_2)] P_3 + P_3 [A_y + \gamma^{-2} B_1 B_1^T (P_1 + P_2)]^T$$

$$-P_3 C_2^T R_0^{-1} C_2 P_3 + P = \dot{P}_3,$$

then, there exist an optimal control law u_* and a worst disturbance signal w_* under white noise such that:

$$J_1(u_*, w_*, w_0) \leq J_1(u_*, w, w_0), \quad J_2(u_*, w_*, w_0) \leq J_2(u, w_*, w_0).$$

If the solutions exist, we have $w_* = \gamma^{-2} B_1^T (P_1 x + P_2 e)$ and the optimal controller is given by:

$$\begin{aligned} \dot{\hat{x}} &= (A + \gamma^{-2} B_1 B_1^T P_1 + L_* C_2 + B_2 F_*)\hat{x} - L_* y, \quad \hat{x}(0) = 0, \\ u_* &= F_* \hat{x}, \end{aligned}$$

where $F_* := -R_1^{-1}(D_{12}^T C_1 + B_2^T P_1)$ and $L_* = -(B_0 D_{20}^T + P_3 C_2^T) R_0^{-1}$. Furthermore, u_* and $w'_* = \gamma^{-2} B_1^T (P_1 x' + P_2 e')$ achieve $0 < J_1(u_*, w'_*, 0) \leq J_1(u_*, w, 0)$ for all $w \neq w'_*$, where

$$\dot{x}' = A x' + B_1 w + B_2 u, \quad e' = x' - \hat{x}.$$

Conversely, let $P_1 \geq 0$ *with* $P_1(T) = 0$ *solve*

$$A_x^T P_1 + P_1 A_x + P_1(B_1 B_1^T/\gamma^2 - B_2 R_1^{-1} B_2^T)P_1 + Q = -\dot{P}_1.$$

and let controller u_*

$$\begin{aligned} \dot{\hat{x}} &= (A + \gamma^{-2} B_1 B_1^T P_1 + L_* C_2 + B_2 F_*)\hat{x} - L_* y, \quad \hat{x}(0) = 0, \\ u_* &= F_* \hat{x}, \quad F_* := -R_1^{-1}(D_{12}^T C_1 + B_2^T P_1 \end{aligned}$$

with a worst disturbance signal w'_* *achieve*

$$0 < J_1(u_*, w'_*, 0) \leq J_1(u_*, w, 0), \textit{for all } w \neq w'_*.$$

If there exists a worst disturbance signal w_* *under white noise such that*

$$J_1(u_*, w_*, w_0) \leq J_1(u_*, w, w_0), \quad J_2(u_*, w_*, w_0) \leq J_2(u, w_*, w_0),$$

then there are solutions $P_2 \geq 0$ *and* $P_3 \geq 0$ *with* $P_2(T) = 0$ *and* $P_3(0) = 0$ *solving the following differential Riccati equations:*

$$P_2(A_y + \gamma^{-2} B_1 B_1^T P_1 - P_3 C_2^T R_0^{-1} C_2) + (A_y + \gamma^{-2} B_1 B_1^T P_1 - P_3 C_2^T R_0^{-1} C_2)^T P_2$$

$$+\gamma^{-2} P_2 B_1 B_1^T P_2 + (D_{12}^T C_1 + B_2^T P_1)^T R_1^{-1} (D_{12}^T C_1 + B_2^T P_1) = -\dot{P}_2,$$

$$[A_y + \gamma^{-2} B_1 B_1^T (P_1 + P_2)] P_3 + P_3 [A_y + \gamma^{-2} B_1 B_1^T (P_1 + P_2)]^T$$

$$-P_3 C_2^T R_0^{-1} C_2 P_3 + P = \dot{P}_3.$$

Moreover, L_* *can be chosen as* $L_* = -(B_0 D_{20}^T + P_3 C_2^T) R_0^{-1}$ *and* $J_2(L)$ *is minimized at* L_*.

PROOF. (Sufficiency) Suppose there exist solutions $P_1(t) \geq 0$, $P_2(t) \geq 0$ and $P_3(t) \geq 0$, $\forall t \in [0,T]$, with $P_1(T) = 0$, $P_2(T) = 0$ and $P_3(0) = 0$ solving those three differential Riccati equations.

Define $r := w - \gamma^{-2}B_1^T P_1 x$, $\quad v := D_{12}\left\{u + R_1^{-1}(D_{12}^T C_1 + B_2^T P_1)x\right\}$. Then the system equations can be rewritten as

$$\begin{aligned} \dot{x} &= (A + \gamma^{-2}B_1 B_1^T P_1)x + B_0 w_0 + B_1 r + B_2 u, \quad x(0) = 0, \\ v &= D_{12}\left\{R_1^{-1}(D_{12}^T C_1 + B_2^T P_1)x + u\right\}, \\ y &= C_2 x + D_{20} w_0, \end{aligned}$$

and the performance index J_1 becomes:

$$\begin{aligned} J_1(u, w, w_0) &= E\left\{\int_0^T (\gamma^2 \|w\|^2 - \|z\|^2)dt\right\} \\ &= E\left\{\int_0^T (\gamma^2 \|r\|^2 - \|v\|^2)dt\right\} - \int_0^T trace\{B_0^T P_1(t) B_0\}dt. \end{aligned}$$

Note that the first Riccati equation and Corollary 2.2 are used to get the result above. Using $L_* = -R_1^{-1}(D_{12}^T C_1 + B_2^T P_1)$ to construct a standard state estimator:

$$\dot{\hat{x}} = (A + \gamma^{-2}B_1 B_1^T P_1)\hat{x} + B_2 u + L_*(C_2\hat{x} - y), \quad \hat{x}(0) = 0,$$

then, a natural choice of the optimal control law $u = u_*$ would be $u_* = -R_1^{-1}(D_{12}^T C_1 + B_2^T P_1)\hat{x}$. Accordingly, the system can be further simplified into

$$\begin{aligned} \dot{e} &= (A + \gamma^{-2}B_1 B_1^T P_1 + L_* C_2)e + (B_0 + L_* D_{20})w_0 + B_1 r, \quad e(0) = 0, \\ v &= D_{12} R_1^{-1}(D_{12}^T C_1 + B_2^T P_1)e, \end{aligned}$$

and J_1 becomes(by using the second Riccati equation and Corollary 2.2):

$$J_1(u_*, w, w_0) = E\left\{\int_0^T \gamma^2 \|r - \gamma^{-2}B_1^T P_2 e\|^2 dt\right\} - \int_0^T trace\{B_0^T P_1(t) B_0\}dt$$

$$-2E\int_0^T e^T P_2(B_0 + L_* D_{20}) w_0 dt$$

$$= E\left\{\int_0^T \gamma^2 \|r - \gamma^{-2} B_1^T P_2 e\|^2 dt\right\} - \int_0^T trace\{B_0^T P_1(t) B_0\} dt$$

$$- \int_0^T trace\{(B_0 + L_* D_{20})^T P_2(t)(B_0 + L_* D_{20})\} dt$$

$$+\gamma^{-2}\int_0^T \int_0^t E\{x^T(\tau) P_1(\tau) B_1 B_1^T w_0(t)\} d\tau dt$$

$$= E\left\{\int_0^T \gamma^2 \|r - \gamma^{-2} B_1^T P_2 e\|^2 dt\right\} - \int_0^T trace\{B_0^T P_1(t) B_0\} dt$$

$$- \int_0^T trace\{(B_0 + L_* D_{20})^T P_2(t)(B_0 + L_* D_{20})\} dt$$

Hence we have $J_1(u_*, w_*, w_0) \le J_1(u_*, w, w_0)$, where $r_* = \gamma^{-2} B_1^T P_2 e \quad \rightarrow \quad w_* = r_* + \gamma^{-2} B_1^T P_1 x = \gamma^{-2} B_1^T (P_1 x + P_2 e)$.

Next, it is shown that u_* does minimize the index J_2 under the worst disturbance w_*. Let L be any filter gain. Substitute w_* (or r_*) into the system equations, we get:

$$\begin{aligned} \dot{e} &= (A + \gamma^{-2} B_1 B_1^T P_1 + LC_2 + \gamma^{-2} B_1 B_1^T P_2) e + (B_0 + LD_{20}) w_0, \quad e(0) = 0, \\ &:= A_L e + B_L w_0. \end{aligned}$$

Let $\Phi(t,0)$ be the transition matrix for A_L, then $e = \int_0^t \Phi(t,\tau) B_L w_0(\tau) d\tau$ and

$$J_2(u, w_*, w_0) = E\left\{\int_0^T \|e\|^2 dt\right\}$$

$$= E\left\{\int_0^T \int_0^t \int_0^t w_0^T(\tau) B_L^T \Phi^T(t,\tau) \Phi(t,s) B_L w_0(s) d\tau\, ds\, dt\right\}$$

$$= trace\left\{\int_0^T \int_0^t \int_0^t \Phi(t,s) B_L E\{w_0(s) w_0^T(\tau)\} B_L^T \Phi^T(t,\tau) d\tau\, ds\, dt\right\}$$

$$= trace\left\{\int_0^T \int_0^t \int_0^t \Phi(t,s)B_L\delta(\tau - s)B_L^T\Phi^T(t,\tau)d\tau\, ds\, dt\right\}$$

$$= trace\left\{\int_0^T \int_0^t \Phi(t,s)B_LB_L^T\Phi^T(t,s)ds\, dt\right\} = trace\{\int_0^T Y dt\},$$

where $Y = \int_0^t \Phi(t,s)B_LB_L^T\Phi^T(t,s)ds \geq 0$ satisfies:

$$A_LY + YA_L^T + B_LB_L^T = \dot{Y}.$$

By Theorem 2.1 in Chapter 2 and using the third Riccati equation, J_2 achieves the minimum value at $L = L_*$, which means that u_* is the desired optimal control. Thus u_* and w_* achieve:

$$J_1(u_*, w_*, w_0) \leq J_1(u_*, w, w_0), \quad J_2(u_*, w_*, w_0) \leq J_2(u, w_*, w_0).$$

It is easy to verify that $w_*' = \gamma^{-2}B_1^T(P_1x' + P_2e')$ and u_* achieve

$$0 < J_1(u_*, w_*', 0) \leq J_1(u_*, w, 0),$$

where x' satisfies

$$\dot{x}' = Ax' + B_1w + B_2u, \quad e' = x' - \hat{x}$$

(Necessity) Suppose $P_1(t) \geq 0$, $\forall t \in [0,T]$ with $P_1(T) = 0$ solves

$$A_x^TP_1 + P_1A_x + P_1(B_1B_1^T/\gamma^2 - B_2R_1^{-1}B_2^T)P_1 + Q = -\dot{P}_1$$

and the controller u_*

$$\begin{aligned} \dot{\hat{x}} &= (A + \gamma^{-2}B_1B_1^TP_1 + L_*C_2 + B_2F_*)\hat{x} - L_*y, \quad \hat{x}(0) = 0, \\ u_* &= F_*\hat{x}, \quad F_* := -R_1^{-1}(D_{12}^TC_1 + B_2^TP_1) \end{aligned}$$

with a w_*' achieves

$$0 < J_1(u_*, w_*', 0) \leq J_1(u_*, w, 0).$$

Hence for the noise-free system

$$\dot{x}' = Ax' + B_1 w + B_2 u_*, \quad x(0) = 0, \tag{5.14}$$

$$z = C_1 x' + D_{12} u_*, \quad R_1 = D_{12}^T D_{12} > 0, \tag{5.15}$$

$$y = C_2 x', \tag{5.16}$$

we have $\|R_{zw}\|_{\infty,[0,T]} < \gamma$. A completing square procedure can be done for $J_1(u_*, w, 0)$ by using the Riccati equation above:

$$J_1(u_*, w, 0) = E\{\int_0^T (\gamma^2 \|w - \tilde{w}_*\|^2 - \|D_{12}(u_* - \tilde{u}_*)\|^2) dt\},$$

where $\tilde{w}_* = \gamma^{-2} B_1^T P_1 x'$ and $\tilde{u}_* = -R_1^{-1}(D_{12}^T C_1 + B_2^T P_1)x'$. Define

$$r' := w - \gamma^{-2} B_1^T P_1 x', \quad v_* := D_{12}\left\{u_* + R_1^{-1}(D_{12}^T C_1 + B_2^T P_1)x'\right\}, \quad e' = x' - \hat{x}.$$

Then the system can be converted into another equivalent one:

$$\begin{aligned} \dot{e}' &= (A + \gamma^{-2} B_1 B_1^T P_1 + L_* C_2)e' + B_1 r', \quad e(0) = 0, \\ v_* &= D_{12}\left\{R_1^{-1}(D_{12}^T C_1 + B_2^T P_1)e'\right\}. \end{aligned}$$

which defines an equivalent operator $R_{v_* r'} = R_{zw}$. Thus $\|R_{v_* r'}\|_{\infty,[0,T]} < \gamma$. By Lemma 2.3, there is a $P_2(t) \geq 0$, $\forall t \in [0, T]$, with $P_2(T) = 0$ solving:

$$P_2(A + \gamma^{-2} B_1 B_1^T P_1 + L_* C_2) + (A_y + \gamma^{-2} B_1 B_1^T P_1 + L_* C_2)^T P_2$$

$$+\gamma^{-2} P_2 B_1 B_1^T P_2 + (D_{12}^T C_1 + B_2^T P_1)^T R_1^{-1}(D_{12}^T C_1 + B_2^T P_1) = -\dot{P}_2,$$

and the worst disturbance r'_* (thus w'_*) is:

$$r'_* = \gamma^{-2} B_1^T P_2 e', \quad w'_* = r_* + \gamma^{-2} B_1^T P_1 x' = \gamma^{-2} B_1^T (P_1 x' + P_2 e').$$

Now for the system with white noise

$$\begin{aligned}
\dot{x} &= Ax + B_0 w_0 + B_1 w + B_2 u, \quad x(0) = 0, \\
z &= C_1 x + D_{12} u, \quad R_1 = D_{12}^T D_{12} > 0, \\
y &= C_2 x + D_{20} w_0, \quad R_0 = D_{20} D_{20}^T > 0,
\end{aligned}$$

it is easy to verify that $w_* = \gamma^{-2} B_1^T (P_1 x + P_2 e)$ is the worst disturbance signal under white noise, that is, $J_1(u_*, w_*, w_0) \leq J_1(u_*, w, w_0)$. On the other hand, by assumption we have $J_2(u_*, w_*, w_0) \leq J_2(u, w_*, w_0)$, which means that J_2 achieves the minimum value on the support of $[0, T]$ through the optimal control u_* under w_*. By substituting w_* into the system equations, we get:

$$\begin{aligned}
\dot{e} &= (A + \gamma^{-2} B_1 B_1^T P_1 + L_* C_2 + \gamma^{-2} B_1 B_1^T P_2) e + (B_0 + L_* D_{20}) w_0, \quad e(0) = 0 \\
&:= A_{L_*} e + B_{L_*} w_0.
\end{aligned}$$

Again, let $\Phi(t, 0)$ be the transition matrix of A_{L_*}, then $e = \int_0^t \Phi^T(t, \tau) B_{L_*} w_0(\tau) d\tau$ and

$$J_2(u_*, w_*, w_0) = trace\{\int_0^T Y dt\}$$

is the minimum value, where $Y = \int_0^t \Phi(t, s) B_{L_*} B_{L_*}^T \Phi^T(t, s) ds \geq 0, Y(0) = 0$ satisfies:

$$A_{L_*} Y + Y A_{L_*}^T + B_{L_*} B_{L_*}^T = \dot{Y}.$$

Thus by Theorem 2.3 in Chapter 2, there is a $P_3(t) \geq 0$, $\forall t \in [0, T]$, with $P_3(0) = 0$ solving:

$$[A_y + \gamma^{-2} B_1 B_1^T (P_1 + P_2)] P_3 + P_3 [A_y + \gamma^{-2} B_1 B_1^T (P_1 + P_2)]^T$$

$$-P_3 C_2^T R_0^{-1} C_2 P_3 + P = \dot{P}_3$$

and L_* can be chosen as $L_* = -(B_0 D_{20}^T + P_3 C_2^T) R_0^{-1}$. Substituting L_* back into the Riccati equation about P_2, clearly $P_2(t) \geq 0$, $\forall t \in [0, T]$ with $P_2(T) = 0$ solves:

$$P_2(A_y + \gamma^{-2} B_1 B_1^T P_1 - P_3 C_2^T R_0^{-1} C_2) + (A_y + \gamma^{-2} B_1 B_1^T P_1 - P_3 C_2^T R_0^{-1} C_2)^T P_2$$

$$+\gamma^{-2} P_2 B_1 B_1^T P_2 + (D_{12}^T C_1 + B_2^T P_1)^T R_1^{-1} (D_{12}^T C_1 + B_2^T P_1) = -\dot{P}_2.$$

$\square$

5.3 $\mathcal{H}_\infty$ Gaussian Control Design—Infinite Time Horizon

We shall make the following standard assumptions for infinite time horizon case:

(A1) (A, B_2) is stabilizable and (C_2, A) is detectable,

(A2) $R_0 := D_{20} D_{20}^T > 0$ and $R_1 := D_{12}^T D_{12} > 0$,

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω,

(A4) $\begin{bmatrix} A - j\omega I & B_0 \\ C_2 & D_{20} \end{bmatrix}$ has full row rank for all ω.

Note that these assumptions guarantee that the Riccati equations corresponding to the standard $\mathcal{H}_2$ control problem have stabilizing solutions.

To simplify notations, we shall use the same abbreviations introduced in the last section. The infinite time $\mathcal{H}_\infty$ Gaussian control design is presented in the next theorem.

Theorem 5.2 *Let the dynamical system G be described by equations (5.1)-(5.3) and assume that assumptions (A1)-(A4) are satisfied.*

If there are stabilizing solutions $P_1 \geq 0$, $P_2 \geq 0$ and $P_3 \geq 0$ solving the following Riccati equations:

$$A_x^T P_1 + P_1 A_x + P_1(B_1 B_1^T/\gamma^2 - B_2 R_1^{-1} B_2^T)P_1 + Q = 0,$$

$$P_2(A_y + \gamma^{-2} B_1 B_1^T P_1 - P_3 C_2^T R_0^{-1} C_2) + (A_y + \gamma^{-2} B_1 B_1^T P_1 - P_3 C_2^T R_0^{-1} C_2)^T P_2$$

$$+\gamma^{-2} P_2 B_1 B_1^T P_2 + (D_{12}^T C_1 + B_2^T P_1)^T R_1^{-1} (D_{12}^T C_1 + B_2^T P_1) = 0,$$

$$[A_y + \gamma^{-2} B_1 B_1^T (P_1 + P_2)] P_3 + P_3 [A_y + \gamma^{-2} B_1 B_1^T (P_1 + P_2)]^T$$

$$-P_3 C_2^T R_0^{-1} C_2 P_3 + P = 0,$$

i.e., $A_x + (B_1 B_1^T/\gamma^2 - B_2 R_1^{-1} B_2^T)P_1$, and $A_y + \gamma^{-2} B_1 B_1^T (P_1 + P_2) - P_3 C_2^T R_0^{-1} C_2$ are both stable. Then, there exist an optimal control law u_* and a worst disturbance signal w_* under white noise such that:

$$J_3(u_*, w_*, w_0) \leq J_3(u_*, w, w_0), \quad J_4(u_*, w_*, w_0) \leq J_4(u, w_*, w_0).$$

If the solutions exist, we have $w_* = \gamma^{-2} B_1^T (P_1 x + P_2 e)$ and the optimal controller is given by:

$$\begin{aligned} \dot{\hat{x}} &= (A + \gamma^{-2} B_1 B_1^T P_1 + L_* C_2 + B_2 F_*)\hat{x} - L_* y, \quad \hat{x}(0) = 0, \\ u_* &= F_* \hat{x}, \end{aligned}$$

where $F_* := -R_1^{-1}(D_{12}^T C_1 + B_2^T P_1)$ and $L_* = -(B_0 D_{20}^T + P_3 C_2^T) R_0^{-1}$. Furthermore, u_* and $w'_* = \gamma^{-2} B_1^T (P_1 x' + P_2 e')$ achieve $0 < J_3(u_*, w_*, 0) \leq J_3(u_*, w, 0)$, where

$$\dot{x}' = Ax' + B_1 w + B_2 u, \quad e' = x' - \hat{x}.$$

Conversely, let $P_1 \geq 0$ be a stabilizing solution to

$$A_x^T P_1 + P_1 A_x + P_1(B_1 B_1^T/\gamma^2 - B_2 R_1^{-1} B_2^T) P_1 + Q = 0,$$

and let controller u_*:

$$\begin{aligned} \dot{\hat{x}} &= (A + \gamma^{-2} B_1 B_1^T P_1 + L_* C_2 + B_2 F_*)\hat{x} - L_* y, \quad \hat{x}(0) = 0, \\ u_* &= F_* \hat{x}, \quad F_* := -R_1^{-1}(D_{12}^T C_1 + B_2^T P_1) \end{aligned}$$

with a worst disturbance signal w'_* achieve

$$0 < J_3(u_*, w'_*, 0) \leq J_3(u_*, w, 0), \quad \textit{for all } w \neq w'_*.$$

If there exists a worst disturbance signal w_* under white noise such that

$$J_1(u_*, w_*, w_0) \leq J_1(u_*, w, w_0), \qquad J_2(u_*, w_*, w_0) \leq J_2(u, w_*, w_0),$$

then there are solutions $P_2 \geq 0$ and $P_3 \geq 0$ solving the following Riccati equations:

$$P_2(A + \gamma^{-2} B_1 B_1^T P_1 + L_* C_2) + (A + \gamma^{-2} B_1 B_1^T P_1 + L_* C_2)^T P_2$$

$$+\gamma^{-2} P_2 B_1 B_1^T P_2 + (D_{12}^T C_1 + B_2^T P_1)^T R_1^{-1} (D_{12}^T C_1 + B_2^T P_1) = 0,$$

$$[A_y + \gamma^{-2} B_1 B_1^T (P_1 + P_2)] P_3 + P_3 [A_y + \gamma^{-2} B_1 B_1^T (P_1 + P_2)]^T$$

$$-P_3 C_2^T R_0^{-1} C_2 P_3 + P = 0.$$

Moreover, if $A + \gamma^{-2} B_1 B_1^T (P_1 + P_2) - (B_0 D_{20}^T + P_3 C_2^T) R_0^{-1} C_2$ is stable, then we can choose $L_* = -(B_0 D_{20}^T + P_3 C_2^T) R_0^{-1}$ and there is a stabilizing solution $P_2 \geq 0$ solving:

$$P_2(A_y + \gamma^{-2} B_1 B_1^T P_1 - P_3 C_2^T R_0^{-1} C_2) + (A_y + \gamma^{-2} B_1 B_1^T P_1 - P_3 C_2^T R_0^{-1} C_2)^T P_2$$

$$+\gamma^{-2} P_2 B_1 B_1^T P_2 + (D_{12}^T C_1 + B_2^T P_1)^T R_1^{-1} (D_{12}^T C_1 + B_2^T P_1) = 0.$$

PROOF. (Sufficiency) Suppose that there are $P_1 \geq 0$, $P_2 \geq 0$ and $P_3 \geq 0$ solving:

$$A_x^T P_1 + P_1 A_x + P_1(B_1 B_1^T/\gamma^2 - B_2 R_1^{-1} B_2^T)P_1 + Q = 0,$$

$$P_2(A_y + \gamma^{-2} B_1 B_1^T P_1 - P_3 C_2^T R_0^{-1} C_2) + (A_y + \gamma^{-2} B_1 B_1^T P_1 - P_3 C_2^T R_0^{-1} C_2)^T P_2$$

$$+\gamma^{-2} P_2 B_1 B_1^T P_2 + (D_{12}^T C_1 + B_2^T P_1)^T R_1^{-1}(D_{12}^T C_1 + B_2^T P_1) = 0,$$

$$[A_y + \gamma^{-2} B_1 B_1^T (P_1 + P_2)]P_3 + P_3[A_y + \gamma^{-2} B_1 B_1^T (P_1 + P_2)]^T$$

$$-P_3 C_2^T R_0^{-1} C_2 P_3 + P = 0.$$

First, it is claimed that $A_y + \gamma^{-2} B_1 B_1^T P_1 - P_3 C_2^T R_0^{-1} C_2 = A + \gamma^{-2} B_1 B_1^T P_1 + L_* C_2$ is stable, where $L_* = -(B_0 D_{20}^T + P_3 C_2^T) R_0^{-1}$. The reason is as follows: if $A + \gamma^{-2} B_1 B_1^T P_1 + L_* C_2$ is not stable, then at least one of its eigenvalue λ is on the closed right-half plane, i.e., $Re(\lambda) \geq 0$. Let x be the eigenvector corresponding to λ, then

$$x^T P_2(A_y + \gamma^{-2} B_1 B_1^T P_1 - P_3 C_2^T R_0^{-1} C_2)x$$

$$+x^T(A_y + \gamma^{-2} B_1 B_1^T P_1 - P_3 C_2^T R_0^{-1} C_2)^T P_2 x$$

$$+\gamma^{-2} x^T P_2 B_1 B_1^T P_2 x + x^T(D_{12}^T C_1 + B_2^T P_1)^T R_1^{-1}(D_{12}^T C_1 + B_2^T P_1)x = 0,$$

or

$$2Re(\lambda) x^T P_2 x + x^T(D_{12}^T C_1 + B_2^T P_1)^T R_1^{-1}(D_{12}^T C_1 + B_2^T P_1)x$$

$$+\gamma^{-2} x^T P_2 B_1 B_1^T P_2 x = 0,$$

which gives $B_1^T P_2 x = 0$ and $(D_{12}^T C_1 + B_2^T P_1)x = 0$. Thus

$$[A_y + \gamma^{-2} B_1 B_1^T (P_1 + P_2) - P_3 C_2^T R_0^{-1} C_2]x = [A + \gamma^{-2} B_1 B_1^T P_1 + L_* C_2]x = \lambda x,$$

which means that $A_y+\gamma^{-2}B_1B_1^T(P_1+P_2)-P_3C_2^TR_0^{-1}C_2$ is not stable, a contradiction.

Now consider the index J_3. Let u be any stabilizing control law. Define

$$r := w - \gamma^{-2}B_1^TP_1x, \quad v := D_{12}\left\{u + R_1^{-1}(D_{12}^TC_1 + B_2^TP_1)x\right\}.$$

Then the system equations can be rewritten as

$$\begin{aligned}
\dot{x} &= (A+\gamma^{-2}B_1B_1^TP_1)x + B_0w_0 + B_1r + B_2u, \qquad x(0)=0, \\
v &= D_{12}\left\{R_1^{-1}(D_{12}^TC_1 + B_2^TP_1)x + u\right\}, \\
y &= C_2x + D_{20}w_0,
\end{aligned}$$

and the performance index J_3 becomes:

$$J_3(u,w,w_0) = \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T(\gamma^2\|w\|^2 - \|z\|^2)dt\right\}$$

$$= \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T(\gamma^2\|r\|^2 - \|v\|^2)dt\right\} - trace\{B_0^TP_1B_0\}.$$

Note that the first Riccati equation and Corollary 2.2 are used here.

Using L_* to construct a standard state estimator:

$$\dot{\hat{x}} = (A+\gamma^{-2}B_1B_1^TP_1)\hat{x} + B_2u + L_*(C_2\hat{x} - y), \quad \hat{x}(0) = 0,$$

then, a natural choice of the optimal control $u = u_*$ would be

$$u_* = -R_1^{-1}(D_{12}^TC_1 + B_2^TP_1)\hat{x}.$$

The system can be further simplified into

$$\begin{aligned}
\dot{e} &= (A+\gamma^{-2}B_1B_1^TP_1 + L_*C_2)e + (B_0 + L_*D_{20})w_0 + B_1r, \qquad e(0)=0, \\
v &= D_{12}R_1^{-1}(D_{12}^TC_1 + B_2^TP_1)e.
\end{aligned}$$

Note that $A+\gamma^{-2}B_1B_1^TP_1+L_*C_2$ is stable. Now J_3 becomes (by using the second Riccati equation and Corollary 2.2):

$$J_3(u_*,w,w_0)=\lim_{T\to\infty}E\left\{\frac{1}{T}\int_0^T\gamma^2\|r-\gamma^{-2}B_1^TP_2e\|^2dt\right\}-trace\{B_0^TP_1B_0\}$$

$$-2\lim_{T\to\infty}E\left\{\frac{1}{T}\int_0^T e^TP_2(B_0+L_*D_{20})w_0dt\right\}$$

$$=\lim_{T\to\infty}E\left\{\frac{1}{T}\int_0^T\gamma^2\|r-\gamma^{-2}B_1^TP_2e\|^2dt\right\}-trace\{B_0^TP_1B_0\}$$

$$-trace\{(B_0+L_*D_{20})^TP_2(B_0+L_*D_{20})\}$$

Hence we have:

$$J_3(u_*,w_*,w_0)\leq J_3(u_*,w,w_0),$$

where $r_*=\gamma^{-2}B_1^TP_2e \;\rightarrow\; w_*=r_*+\gamma^{-2}B_1^TP_1x=\gamma^{-2}B_1^T(P_1x+P_2e)$. Next, it is shown that u_* does minimize the index J_4 under the worst disturbance w_*. Let L be any filter gain such that both $A+\gamma^{-2}B_1B_1^TP_1+LC_2$ and $A+\gamma^{-2}B_1B_1^TP_1+LC_2+\gamma^{-2}B_1B_1^TP_2$ are stable. Substitute w_* (or r_*) into the system equations, we get:

$$\begin{aligned}\dot{e}&=(A+\gamma^{-2}B_1B_1^TP_1+LC_2+\gamma^{-2}B_1B_1^TP_2)e+(B_0+LD_{20})w_0,\quad e(0)=0,\\ &:=A_Le+B_Lw_0.\end{aligned}$$

Note that $e=\int_0^t exp[A_L(t-\tau)]B_Lw_0(\tau)d\tau$ and

$$J_4(u,w_*,w_0)=\lim_{T\to\infty}E\left\{\frac{1}{T}\int_0^T\|e\|^2dt\right\}$$

$$=\lim_{T\to\infty}E\left\{\frac{1}{T}\int_0^T\int_0^t\int_0^t w_0^T(\tau)B_L^Texp[A_L^T(t-\tau)]exp[A_L(t-s)]B_Lw_0(s)d\tau\,ds\,dt\right\}$$

$$=trace\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^t\int_0^t exp[A_L(t-s)]B_L\delta(\tau-s)B_L^Texp[A_L^T(t-\tau)]d\tau\,ds\,dt\right\}$$

$$= trace\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^t exp[A_L(t-s)]B_LB_L^Texp[A_L^T(t-s)]ds\ dt\right\} = trace\{Y\}$$

where $Y = \int_0^\infty exp(A_Ls)B_LB_L^Texp(A_L^Ts)ds \geq 0$ satisfies:

$$A_LY + YA_L^T + B_LB_L^T = 0.$$

By Theorem 2.2 and using the third Riccati equation, J_4 achieves the minimum value at $L = L_*$ where $L_* = -(B_0D_{20}^T + P_3C_2^T)R_0^{-1}$, which means that u_* is the desired optimal control. Thus u_* and w_* achieve:

$$J_3(u_*, w_*, w_0) \leq J_3(u_*, w, w_0), \quad J_4(u_*, w_*, w_0) \leq J_4(u, w_*, w_0).$$

It is trivial to show that that $w_*' = \gamma^{-2}B_1^T(P_1x' + P_2e')$ and u_* achieve

$$0 < J_3(u_*, w_*', 0) \leq J_3(u_*, w, 0),$$

where x' satisfies

$$\dot{x}' = Ax' + B_1w + B_2u, \quad e' = x' - \hat{x}$$

(Necessity) Suppose $P_1 \geq 0$ solves

$$A_x^TP_1 + P_1A_x + P_1(B_1B_1^T/\gamma^2 - B_2R_1^{-1}B_2^T)P_1 + Q = 0$$

and the controller u_*

$$\begin{aligned}\dot{\hat{x}} &= (A + \gamma^{-2}B_1B_1^TP_1 + L_*C_2 + B_2F_*)\hat{x} - L_*y, \quad \hat{x}(0) = 0,\\ u_* &= F_*\hat{x}, \quad F_* := -R_1^{-1}(D_{12}^TC_1 + B_2^TP_1)\end{aligned}$$

with a w_*' achieves $0 < J_3(u_*, w_*', 0) \leq J_3(u_*, w, 0)$. Hence for the noise-free system

$$\dot{x}' = Ax' + B_1w + B_2u_*, \quad x(0) = 0, \tag{5.17}$$

$$z = C_1x' + D_{12}u_*, \quad R_1 = D_{12}^TD_{12} > 0, \tag{5.18}$$

$$y = C_2x', \tag{5.19}$$

we have $\|R_{zw}\|_\infty < \gamma$. A completing square procedure can be done for $J_3(u_*, w, 0)$ by using the Riccati equation above:

$$J_3(u_*, w, 0) = \lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T (\gamma^2\|w - \tilde{w}_*\|^2 - \|D_{12}(u_* - \tilde{u}_*)\|^2)dt\right\},$$

where $\tilde{w}_* = \gamma^{-2}B_1^T P_1 x'$ and $\tilde{u}_* = -R_1^{-1}(D_{12}^T C_1 + B_2^T P_1)x'$. Define

$$r' := w - \gamma^{-2}B_1^T P_1 x', \quad v_* := D_{12}\left\{u_* + R_1^{-1}(D_{12}^T C_1 + B_2^T P_1)x'\right\}, \quad e' = x' - \hat{x}.$$

Then the system can be converted into another equivalent one:

$$\begin{aligned} \dot{e}' &= (A + \gamma^{-2}B_1 B_1^T P_1 + L_* C_2)e' + B_1 r', \quad e(0) = 0, \\ v_* &= D_{12}\left\{R_1^{-1}(D_{12}^T C_1 + B_2^T P_1)e'\right\}. \end{aligned}$$

which defines an equivalent operator $R_{v_* r'} = R_{zw}$. Thus $\|R_{v_* r'}\|_\infty < \gamma$. By Bounded-Real Lemma, there is a $P_2 \geq 0$ solving:

$$P_2(A + \gamma^{-2}B_1 B_1^T P_1 + L_* C_2) + (A_y + \gamma^{-2}B_1 B_1^T P_1 + L_* C_2)^T P_2$$

$$+\gamma^{-2}P_2 B_1 B_1^T P_2 + (D_{12}^T C_1 + B_2^T P_1)^T R_1^{-1}(D_{12}^T C_1 + B_2^T P_1) = 0,$$

and the worst disturbance r'_* (thus w'_*) is:

$$r'_* = \gamma^{-2}B_1^T P_2 e', \;\; w'_* = r_* + \gamma^{-2}B_1^T P_1 x' = \gamma^{-2}B_1^T(P_1 x' + P_2 e').$$

Now for the system with white noise

$$\begin{aligned} \dot{x} &= Ax + B_0 w_0 + B_1 w + B_2 u, \quad x(0) = 0, \\ z &= C_1 x + D_{12}u, \quad R_1 = D_{12}^T D_{12} > 0, \\ y &= C_2 x + D_{20}w_0, \quad R_0 = D_{20}D_{20}^T > 0, \end{aligned}$$

it is easy to verify that $w_* = \gamma^{-2}B_1^T(P_1 x + P_2 e)$ is the worst disturbance signal under white noise, that is, $J_3(u_*, w_*, w_0) \leq J_3(u_*, w, w_0)$. On the other hand, by

assumption we have $J_2(u_*, w_*, w_0) \leq J_2(u, w_*, w_0)$, which means that J_2 achieves the minimum value through the optimal control u_* under w_*. By substituting w_* into the system equations, we get:

$$\begin{aligned} \dot{e} &= (A + \gamma^{-2} B_1 B_1^T P_1 + L_* C_2 + \gamma^{-2} B_1 B_1^T P_2) e + (B_0 + L_* D_{20}) w_0, \quad e(0) = 0 \\ &:= A_{L_*} e + B_{L_*} w_0. \end{aligned}$$

Note that $e = \int_0^t exp[A_{L_*}(t - \tau)] B_{L_*} w_0(\tau) d\tau$ and $J_4(u_*, w_*, w_0) = trace\{Y\}$ is the minimum value, where $Y = \int_0^\infty exp(A_{L_*} s) B_{L_*} B_{L_*}^T exp(A_{L_*}^T s) ds \geq 0$ satisfies:

$$A_{L_*} Y + Y A_{L_*}^T + B_{L_*} B_{L_*}^T = 0.$$

Thus by Theorem 2.3 in Chapter 2, there is a $P_3 \geq 0$ and $P_3 \leq Y$ solving:

$$\begin{aligned} &[A_y + \gamma^{-2} B_1 B_1^T (P_1 + P_2)] P_3 + P_3 [A_y + \gamma^{-2} B_1 B_1^T (P_1 + P_2)]^T \\ &- P_3 C_2^T R_0^{-1} C_2 P_3 + P = 0. \end{aligned}$$

Furthermore, if $A + \gamma^{-2} B_1 B_1^T (P_1 + P_2) - (B_0 D_{20}^T + P_3 C_2^T) R_0^{-1} C_2$ is stable, then L_* can be chosen as $L_* = -(B_0 D_{20}^T + P_3 C_2^T) R_0^{-1}$. Substituting L_* back into the Riccati equation about P_2, clearly P_2 solves:

$$\begin{aligned} &P_2 (A_y + \gamma^{-2} B_1 B_1^T P_1 - P_3 C_2^T R_0^{-1} C_2) + (A_y + \gamma^{-2} B_1 B_1^T P_1 - P_3 C_2^T R_0^{-1} C_2)^T P_2 \\ &+ \gamma^{-2} P_2 B_1 B_1^T P_2 + (D_{12}^T C_1 + B_2^T P_1)^T R_1^{-1} (D_{12}^T C_1 + B_2^T P_1) = 0. \end{aligned}$$

The proof is complete. $\square$

Chapter 6

Conclusions

Control theory as engineering science has experienced tremendous progresses in 20th century. The distinct features of control are that it is both engineering-oriented and mathematics-oriented. However solving engineering problems in simple and understandable mathematical ways is not always an easy thing. There is a trade-off here, that is, the trade-off between simplicity and complexity. Hence, the achievements in control should stick closely with problems which have solid engineering backgrounds while at same time the results should be presented in simple and clear mathematical languages. Like Einstein said, "Everything should be made as simple as possible, but not simpler."

In the last decade, researchers in control have dedicated a lot of efforts to multiobjective control in hoping to design trade-off controllers between different performance requirements. This dissertation is a continuation of such efforts and contributes mainly to developing time domain game approach for multiobjective optimal filtering and control, aiming at providing design trade-off between robust performances and $\mathcal{H}_2(LQG)$ performances. From filtering to control, the results obtained provide systematic design approaches with clear interpretations for practical dynamical systems. Hence, the theory developed in this dissertation is closer to engineering rather than an 'artificial theory', which strongly entitles the possibility of potential engineering applications of the results obtained.

During the process of developing multiobjective optimal filtering and control theory in this dissertation, significant difficulties were met when trying to generalize

game approach to output feedback design which must be adopted in engineering since the system states are usually difficult to obtain. The difficulties are mainly caused by the desired consistency between theoretic framework and the performance of dynamical systems as well as the computation complexity. Motivated by Kalman filter design and its application to LQG control, a similar filter structure is naturally adopted for filtering and control design in this dissertation. It is, then, very clear that the trade-off design between robust and LQG performance should be the optimization target of a multiobjective design. This formulation finally returns simple, graceful and computable results as presented in this dissertation. It is also noted that, in practice, a trade-off between robustness and transient (LQG) performances is of more engineering interests than artificial theoretic interests. All results in this dissertation could be obtained through solving some set of coupled Riccati equations and these coupled Riccati equations are solvable by standard numerical integration.

It is worth pointing out that, though discrete time systems are not treated in this dissertation, all results of filtering and control can be extended to discrete time systems through a corresponding formulation. An example of such a generalization can be found in [6].

Yet there are a lot of problems remaining to be solved which may be of further research interests, though significant progresses have been achieved in this dissertation. Let me end this dissertation by pointing out some of these problems.

1. It is still of theoretic interest to show if the filter structure used for multiobjective optimal filtering and control in Chapter 3, 4 and 5 is a globally necessary optimal choice, which will provide a rigorous answer to the question: why do we want to choose this special filter structure?

2. Kalman filter does have a recursive algorithm and it is used broadly in the fields of both control, estimation and signal processing. Hence it will be of great interests to develop the recursive algorithms for the multiobjective filtering and control results obtained in this dissertation.

3. Sampled-data control systems are another important class of engineering systems. How to generalize the results obtained to multiobjective filtering and control for sampled-data control problems is a challenging and very interesting problem.

The dissertation ends here but, definitely, control research is endless.

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VITA

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DOCTORAL EXAMINATION AND DISSERTATION REPORT

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Title of Dissertation: Multiobjective Optimal Filtering and Control

Approved:

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EXAMINING COMMITTEE:

Date of Examination:

June 15, 1998

IMAGE EVALUATION
TEST TARGET (QA-3)

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