Unavoidable Minors of Graphs of Large Type.

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UNAVOIDABLE MINORS
OF GRAPHS OF LARGE TYPE

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in

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# TABLE OF CONTENTS

Acknowledgements ............................................................................................................ ii

Abstract ............................................................................................................................. iv

Chapter 1. Introduction ................................................................................................. 1
  1.1 On Type ............................................................................................................. 1
  1.2 Definitions and Notation ................................................................................. 6

Chapter 2. Some Results on the Type of Graphs ...................................................... 9
  2.1 Graphs of Very Small Type ........................................................................... 9
  2.2 Some Special Graphs of Large Type ............................................................. 12

Chapter 3. A Characterization of 3-Connected Graphs of Large Type ............... 22
  3.1 Graphs That Have a Large Fan as a Minor ................................................. 22
  3.2 3-Connected Graphs That Have Large Type ............................................. 27

Chapter 4. Unavoidable Minors of 2-Connected Graphs of Large Type .......... 36
  4.1 2-Sums and Tree Structures ....................................................................... 36
  4.2 A Long Path in a 3-Block Tree ................................................................. 46
  4.3 A Sufficient Condition for a $C_{n,n}^*$-Minor or a $P_{n,n}$-Minor ......... 57
  4.4 Results for $n$-Close Block Trees ............................................................... 66

Chapter 5. On Contraction-Type .............................................................................. 90

Chapter 6. Summary of Results .............................................................................. 94

References ...................................................................................................................... 95

Vita .................................................................................................................................. 96
ABSTRACT

In this paper, we study one measure of complexity of a graph, namely its type. The type of a graph $G$ is defined to be the minimum number $n$ such that there is a sequence of graphs $G = G_0, G_1, \ldots, G_n$, where $G_i$ is obtained by contracting one edge in or deleting one edge from each block of $G_{i-1}$, and where $G_n$ is edgeless. We show that a 3-connected graph has large type if and only if it has a minor isomorphic to a large fan. Furthermore, we show that if a graph has large type, then it has a minor isomorphic to a large fan or to a large member of one of two specified families of graphs.
CHAPTER 1
INTRODUCTION

1.1 On Type

Infinite sets of objects, such as infinite sets of graphs, arise frequently in graph theory and other related areas. Being able to describe such an infinite set in terms of a finite set of objects can sometimes give us a simple way to determine whether an object belongs to the infinite set or not. We shall state two such theorems that require the following definition. A graph $H$ is a minor of a graph $G$, denoted $H \preceq_m G$, if $H$ can be obtained by contracting a (possibly empty) set of edges in a subgraph of $G$; furthermore, if $G$ and $H$ are not isomorphic, then a minor $H$ of $G$ is proper. Below, we state without proof the following simple result which describes the infinite class of forests in terms of a single graph, namely a loop.

(1.1.1) Proposition. A graph is a forest if and only if it does not contain a loop as a minor. □

The above result is an example of an excluded-minor theorem. Another well-known excluded-minor theorem, a version of Kuratowski’s Theorem due to Wagner [W], that describes the infinite class of planar graphs in terms of two graphs is the following.

(1.1.2) Theorem (Wagner). A graph is planar if and only if it contains neither $K_5$ nor $K_{3,3}$ as a minor. □

We have just seen two examples of excluded-minor theorems, but we have not yet defined what is meant by an excluded minor. If $\mathcal{G}$ is any class of graphs, then the graph $K$ is an excluded minor for $\mathcal{G}$ if and only if $K$ is not a minor of $G$, for each $G$ in $\mathcal{G}$, and for each proper minor $K'$ of $K$, there is a graph $G_{K'}$ in $\mathcal{G}$ that contains $K'$ as a minor. It follows from the two above examples that a loop is the only excluded minor for the class of forests and that $\{K_5, K_{3,3}\}$ is the set of
excluded minors for the class of planar graphs; in fact, (1.1.1) and (1.1.2) imply that these classes of graphs are characterized by their respective sets of excluded minors. This follows from the fact that these classes are minor-closed; that is, a class \( G \) of graphs is minor-closed if \( H \in G \) whenever \( H \leq_m G \), for some \( G \in G \). In general, if \( G \) is a minor-closed class of graphs, one can show that it is characterized by its set of excluded minors.

The class of forests and the class of planar graphs, we have seen, each have a finite set of excluded minors. Moreover, it follows from the important theorem of Robertson and Seymour [RS4] stated below that the set of excluded minors for an arbitrary class of graphs is finite.

(1.1.3) **Theorem** (Robertson, Seymour). In any infinite set of graphs there are two graphs, one of which is a minor of the other. □

It is natural to try to generalize this result to a larger class of objects that, in some sense, includes all graphs. One such larger class is the class \( M \) of all matroids. (For the definition of a matroid, see [O].) It is known that (1.1.3) cannot be extended to \( M \), but it is not known if it is extendible to some of its important subclasses. One important unanswered question in matroid theory regarding the extendibility of (1.1.3) is the following.

(1.1.4) **Question.** Does every infinite subclass of the class of GF\((q)\)-representable matroids, where \( q \) is a fixed prime power, contain two matroids, one of which is a minor of the other?

Let us note that (1.1.3) is the culmination of a long sequence of papers by Robertson and Seymour. It is believed that obtaining an answer to (1.1.4) will be extremely difficult, even for \( q = 2 \) (that is, for binary matroids), since proving (1.1.3) was long and difficult; so even small results related to (1.1.4) might provide valuable insight into this problem.
One of the early steps in proving (1.1.3), which was presented in [RS2], was showing that it held, for each positive integer \( n \), when restricted to the class of graphs whose elements have tree-width at most \( n \). (Tree-width measures, in some sense, how close a graph is to a tree; for a definition of tree-width, see [RS1].) Ding, Oporowski, and Oxley have proved in [DOO] a matroid result that is similar to this early step in proving (1.1.3). Instead of graphs of bounded tree-width, they considered the class of \( GF(q) \)-representable matroids, given a prime power \( q \), whose complexity measured by type is bounded. This result of Ding, Oporowski, and Oxley is stated in (1.1.5) below, and the definition of "type" follows it.

(1.1.5) Theorem. In any infinite subclass of the class of all \( GF(q) \)-representable matroids whose elements have type at most \( n \), given a prime power \( q \) and a non-negative integer \( n \), there are two matroids, one of which is a minor of the other. □

(1.1.6) Definition. Let \( M \) be a matroid. We define the type of \( M \), denoted \( t(M) \), as follows. If \( |E(M)| = 0 \), then \( t(M) = 0 \). If \( M \) is connected and \( E(M) \) is non-empty, then \( t(M) = \min\{t(M/e), t(M \setminus e) : e \in E(M)\} + 1 \). If \( M \) is not connected, then \( t(M) = \max\{t(C) : C \text{ is a component of } M\} \). Let us note that by using properties of matroid duality (see [O]), it is straightforward to see that \( t(M) = t(M^*) \), where \( M^* \) is the dual of \( M \). □

Now, we translate the matroid definition of type into a graph-theoretic definition, since we will consider primarily graphs in this dissertation.

(1.1.7) Definition. If \( G \) is a graph, then the type of \( G \), denoted \( t(G) \) is defined as follows. If \( G \) is edgeless, then \( t(G) = 0 \). If \( G \) is a block that contains a positive number of edges, then \( t(G) = \min\{t(G/e), t(G \setminus e) : e \in E(G)\} + 1 \). If \( G \) is not a block, then \( t(G) = \max\{t(B) : B \text{ is a block of } G\} \). Intuitively, the type of a graph \( G \) is the minimum integer \( n \) such that there is a sequence of graphs
Another important early step in proving (1.1.3) was understanding what it means for a graph to have bounded tree-width. Robertson and Seymour have shown in [RS3] (see also [RST]) that graphs of bounded tree-width are characterized, in an weak sense, by the lack of a large grid as a minor, where the \((n \times n)\)-grid is the graph whose vertex set is \(\{v_{i,j} : 1 \leq i, j \leq n\}\) and whose edge set is \(\{v_{i,j}v_{i,j+1} : 1 \leq i \leq n, 1 \leq j \leq n - 1\} \cup \{v_{i,j}v_{i+1,j} : 1 \leq i \leq n - 1, 1 \leq j \leq n\}\). We state this early result of Robertson and Seymour rather informally in (1.1.8) in a way that resembles an excluded-minor theorem; immediately following (1.1.8), Figure 1.1 illustrates the \((3 \times 3)\)-grid and \((4 \times 4)\)-grid.

\[(1.1.8) \text{ Remark. A graph has small tree-width if and only if it does not contain a large grid as a minor.} \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{grid_3x3_4x4.png}
\caption{The \((3 \times 3)\)-grid (left) and the \((4 \times 4)\)-grid (right).}
\end{figure}

It can be shown that the class of graphs of tree-width at most \(n\), given a positive integer \(n\), is minor-closed. Consequently, by (1.1.3), this class is characterized by its (finite) set of excluded minors, but these sets of excluded minors are known for only very small values of \(n\). Although (1.1.8) is rather imprecise, it captures the essence that a single graph, namely a big grid, is the "excluded minor" of graphs of bounded tree-width. More precisely, for each positive integer \(n\), there is an integer \(tw_n\) so that if the \((n \times n)\)-grid is not a minor of \(G\), then the tree-width of \(G\) is at most \(tw_n\); conversely, for each positive integer \(n\), there is an integer \(g_n\) so that if the tree-width of \(G\) is at most \(n\), then the \((g_n \times g_n)\)-grid is not a minor of \(G\). Note
that we may take $g_n$ to be $n + 1$, and it can be shown that this is the best possible value for $g_n$. On the other hand, while an upper bound for $t_w n$ is given in [RST], the best possible value for $t_w n$ is not known and is believed to be much smaller than the bound from [RST].

The result in (1.1.8) provides motivation to understand what it means for a binary matroid to have bounded type in terms of excluding a large member of a specified family of matroids as a minor. Ding, Oporowski, and Oxley have asked the following question in (1.1.9) regarding 3-connected binary matroids of large type and large fans, where the $n$-fan, for a positive integer $n$, is the graph $F_n$ whose vertex set is $\{v_i : 0 \leq i \leq n\}$ and whose edge set is $\{v_0v_i : 1 \leq i \leq n\} \cup \{v_iv_{i+1} : 1 \leq i \leq n - 1\}$; the 3-fan and 5-fan are illustrated in Figure 1.2 immediately after (1.1.9).

(1.1.9) QUESTION. If a 3-connected binary matroid $M$ does not contain the cycle matroid of an $n$-fan as a minor, given a positive integer $n$, then is there an integer $t_n$, depending only on $n$, so that the type of $M$ is at most $t_n$?

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fan.png}
\caption{The 3-fan $F_3$ and the 5-fan $F_5$.}
\end{figure}

While this question remains open in general, we shall see in Section 3.2 that we can answer it affirmatively if we restrict ourselves to the classes of 3-connected graphs and 3-connected cographic matroids. This result, coupled with the main result of Section 3.1, is equivalent to a weak "excluded-minor" characterization of 3-connected graphs of bounded type, similar in nature to (1.1.8). In Chapter 4, we extend the main result of Section 3.2 to the class of all graphs. Since the class of graphs considered in Chapter 4 is larger than the class of graphs considered in
Chapter 3, we shall find it necessary to consider more than the fans; we shall also need to consider the multicycles and comulticycles, which are defined as follows. If \( m \) and \( n \) exceed 1 and 3, respectively, then the \((n, m)\)-multicycle, denoted \( C_{n,m} \), is obtained from the cycle \( C_n \) of length \( n \) by replacing each edge \( e \) of \( C_n \) with a class of \( m \) edges parallel to \( e \), and the \((n, m)\)-comulticycle, denoted \( C^*_{n,m} \), is the graphic dual of \( C_{n,m} \). Now, we state this extension of the main result of Section 3.2 rather informally below, and immediately following it we illustrate the \((4, 2)\)-multicycle and the \((5, 3)\)-comulticycle.

(1.1.10) **Remark.** If a graph \( G \) has sufficiently large type depending on an integer \( n \) exceeding 3, then an element of \( \{F_n, C_{n,n-2}, C^*_{n,n-2}\} \) is a minor of \( G \). □

![Figure 1.3. The \((4, 2)\)-multicycle \( C_{4,2} \) and the \((5, 3)\)-comulticycle \( C^*_{5,3} \).](image)

### 1.2 Definitions and Notation

Here we present the definitions and notation of some of the graph-theoretic terminology that we will use later.

When we consider graphs, we shall assume that \( V(G) \) and \( E(G) \) are finite and that \( V(G) \) is non-empty; also, we shall allow graphs to have parallel edges and loops. We shall call an edge that is not a loop a **link-edge**, and we shall call a non-empty maximal class of parallel link-edges a **multi-edge**. If a multi-edge contains at least two edges, then we say that the multi-edge is **proper**; otherwise, the multi-edge is **trivial**. We shall use the notation \( e \parallel f \) to indicate that the edges \( e \) and \( f \) are parallel edges, and \( e \parallel uv \) to indicate that the endvertices of \( e \) are \( u \) and \( v \). If \( v \)
is a vertex of a graph $G$, then the degree (or valency) of $v$ in $G$, denoted $d_G(v)$, is $|E_v| + 2|L_v|$, where $E_v$ is the set of link-edges of $G$ incident with $v$ and $L_v$ is the set of loops of $G$ incident with $v$; this extends the identity $\sum_{v \in V(G)} d_G(v) = 2|E(G)|$ for simple graphs to graphs with loops and multi-edges. If $n$ is a nonnegative integer, then an $n$-path or path of length $n$ is a graph isomorphic to the path on $n + 1$ vertices. Let us call any path of length 0 trivial; otherwise a path is proper.

If $e \in E(G)$, then we shall use the standard notation of $G\setminus e$ and $G/e$ to denote the deletion of the edge $e$ from $G$ and the contraction of the edge $e$ in $G$, respectively; also, if $v \in V(G)$, then $G - v$ denotes the deletion of $v$ (and the edges incident with $v$) from $G$. If $E' \subseteq E(G)$ and $V' \subseteq V(G)$, then $G\setminus E'$, $G/E'$, and $G - V'$ are defined in the obvious way. If $H$ is a subgraph of $G$, then let $G/H = G/E(H)$, and let $G\setminus H = (G\setminus E(H)) - V_H$, where $V_H$ is the set isolated vertices in $G\setminus E(H)$ whose elements are not isolated vertices in $G$.

If $H$ can be obtained by deleting only vertices from $G$, then it is standard to say that $H$ is an induced subgraph of $G$. If $V' \subseteq V(G)$, then $G[V']$ is the induced subgraph obtained by deleting $V(G) - V'$ from $G$. If $E' \subseteq E(G)$, then $G[E']$ is the smallest subgraph of $G$ whose set of edges is $E'$. We shall use the notation $H \subseteq G$ to denote that $H$ is a subgraph of $G$. We say that $H$ is a topological minor of $G$, denoted $H \subseteq_t G$, if some subdivision of $H$ is a subgraph of $G$.

Let us say that a graph $G$ is 2-connected if $G$ is loopless, $|V(G)| + |E(G)| \geq 4$, and $G - v$ is connected, for each $v \in V(G)$. Equivalently, a graph $G$ is 2-connected if and only if $|E(G)| \geq 2$ and each pair of edges of $G$ is contained in a cycle of $G$. Also, let us say that $G$ is 3-connected if $G$ is loopless, $|V(G)| \geq 4$, and $G - \{u, v\}$ is connected, for each pair $\{u, v\} \subseteq V(G)$. By a block of a graph $G$, we mean an isolated vertex of $G$, a loop of $G$, a cut-edge of $G$, or a maximal 2-connected subgraph of $G$. 

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Let $H$ be a subgraph of a graph $G$. Then a bridge of $H$ in $G$ is one of the following kinds of subgraphs of $G$.

(i) An edge of $E(G) - E(H)$ contained in $G[V(H)]$.

(ii) The union of a component $C$ of $G - V(H)$ and the set of edges that have one vertex in $V(C)$ and the other vertex in $V(H)$.

The vertices that $H$ and a bridge $B$ of $H$ in $G$ have in common are called the vertices of attachment of $B$.

The bridges of $H$ in $G$ can also be described in the following intuitive manner. Let us view $G$ as a topological space by associating a set $V$ of $|V(G)|$ points with $V(G)$ and by associating with each edge $e$ of $E(G)$ a topological space $I_e$ that is homeomorphic to the open unit interval and disjoint from $V$, so that the following are satisfied.

(i) The elements of $\{I_e : e \in E(G)\}$ are pairwise disjoint.

(ii) For each $e \in E(G)$, the closure $\overline{I_e}$ of $I_e$ is the union of $I_e$ and the subset of $V$ corresponding to the endvertices of $e$.

Let us also view $H$ as a topological subspace of $G$, and let $C$ be a component of the topological complement of $H$ in $G$. Then the subgraph $B$ of $G$ corresponding to the topological subspace $\overline{C}$ of $G$ is a bridge of $H$ in $G$, and the vertices corresponding to $H \cap \overline{C}$ are the vertices of attachment of $B$.

If $k$ is a positive integer, then let $[k]$ denote the set of nonnegative integers less than $k + 1$, and let $[k]_+$ denote the set of positive integers less than $k + 1$. 

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CHAPTER 2

SOME RESULTS ON THE TYPE OF GRAPHS

In this chapter we prove some simple results on the type of graphs. In the first section we characterize graphs of very small type. In the second section we find the type of the n-fan, the \((n, n - 2)\)-multicycle, and the \((n, n - 2)\)-comulticycle.

2.1 Graphs of Very Small Type

Let us recall from the definition of the type of a graph that the type of an edgeless graph is 0, and if a graph has edges, then its type exceeds 0. It follows that the graphs of type 0 have the following characterization.

\((2.1.1)\) Proposition. The type of a graph is 0 if and only if it is made up of isolated vertices. □

If the type of a graph \(G\) is at most 1, then each block \(B\) of \(G\) that contains an edge has an edge \(e\) such that one of \(B/e\) or \(B\setminus e\) is edgeless. It follows that \(B\) consists of a single edge; hence, \(B\) is a loop or a link-edge. Consequently, \(G\) is a forest with a (perhaps empty) set of loops; such a graph is illustrated in Figure 2.1 below. Conversely, if \(G\) is a forest with a set of loops, then each block of \(G\) contains at most one edge; it follows that \(t(G) \leq 1\). Thus, we obtain the characterization of the graphs of type at most 1 that is stated immediately after Figure 2.1.

\[
\begin{align*}
\text{Figure 2.1. A graph whose type is 1.}
\end{align*}
\]
(2.1.2) Proposition. The type of a graph is at most 1 if and only if it is a forest with a (perhaps empty) set of loops. □

Before we characterize the graphs of type at most 2, let us discuss the concept of expanding a vertex (to an edge) (that is, "uncontracting" an edge). Figure 2.2 below shows that expanding a vertex $v_e$ to an edge $e$ is not well-defined; $H_1$ is obtained from $G$ by expanding $v_e$ to a loop, and $H_2$ and $H_3$ are obtained from $G$ by expanding $v_e$ to a link-edge.

![Figure 2.2](image)

**Figure 2.2.** $H_1$, $H_2$, and $H_3$ are obtained from $G$ by expanding $v_e$ to $e$.

Although expanding a vertex to an edge is not well-defined, we can describe, given a graph $G$ and a vertex $v_e$ in $G$, the set of graphs, each element $H$ of which has an edge $e \in E(H)$ such that $H/e = G$ and $e$ is contracted to $v_e$. Let us call such a graph $H$ an expansion of $G$ at $v_e$.

We describe process of expanding a vertex $v_e \in V(G)$ to an edge as follows. It is clear that the only way to expand $v_e$ to a loop is to attach a loop to $G$ at $v_e$. So, it remains to consider the case of expanding $v_e$ to a link-edge $e$. Let $E_{v_e}$ be the subset of $E(G)$ each of whose elements is incident with $v_e$. Partition the link-edges of $E_{v_e}$ into two sets $E_x$ and $E_y$, and partition the loops of $E_{v_e}$ into three sets $L_x$, $L_y$, and $E_{xy}$. Let $H$ be obtained from $G - v_e$ by adding two new vertices $x$ and $y$ and an edge $e || xy$, an edge $e'_x || ux$ for each edge $f || vu_e$ in $E_x$, an edge $e'_y || vy$ for each edge $f || vu_e$ in $E_y$, a loop at $x$ for each element of $L_x$, a loop at $y$ for each element of $L_y$, and an edge $e' || xy$ for each element of $E_{xy}$. It is clear that $H/e = G$ and that $e$ is contracted to $v_e$; hence, $H$ is an expansion of $G$ at $v_e$. 

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Conversely, it follows from the definition of contracting a link-edge in a graph (which is not given here), that any expansion of $G$ at $v_e$, in which $v_e$ is expanded to a link-edge, can be obtained only in the above manner. Also, note that expanding $v_e$ to a link-edge $e$ becomes well-defined once $E_{v_e}$ has been partitioned into the sets $E_x, E_y, L_x, L_y,$ and $E_{xy}$ as described above. Now, we are ready to characterize the graphs of type at most 2.

If the type of a graph $G$ is at most 2, then each block $B$ of $G$ that contains at least two edges has an edge $e$ such that $t(B/e) = 1$ or $t(B\setminus e) = 1$. Let $B$ be such a block of $G$, and let $e$ be such an edge of $B$. It is clear that $B$ is 2-connected, that $|E(B) - e| > 0$, and that $e$ is not a loop. Since $B$ is 2-connected, it is clear that $B/e$ and $B\setminus e$ are connected. Then, it follows from (2.1.1) that $B/e$ or $B\setminus e$ consists of a tree with a nonnegative number of loops. If $B\setminus e$ is a tree with a nonnegative number of loops, then, since $B$ is a block, $B\setminus e$ is loopless and has exactly two monovalent vertices. It follows that $B\setminus e$ is a proper path, and, consequently, $B$ is a cycle of length at least 2. If $B/e$ is a tree with a nonnegative number of loops, then the definition of expanding a vertex and the assumption that $B$ is a block imply that $B/e$ has no link-edges. So, $B/e$ consists of a positive number of loops meeting at a single vertex; consequently, $B$ is a proper multi-edge. Hence, if $B$ is a block of a graph $G$ of type at most 2, then $B$ is a multi-edge, a cycle, or an isolated vertex.

Conversely, let $G$ be a graph such that if $B$ is a block of $G$, then $B$ is a multi-edge, a cycle, or an isolated vertex. If $B$ is a proper multi-edge, then $B/e$ consists of a non-empty set of loops that meet at a single vertex. It follows that $t(B/e) = 1$ and that $t(B) = 2$. If $B$ is a cycle of length at least 2, then $B\setminus e$ consists of a path of length at least 1. It follows that $t(B\setminus e) = 1$ and that $t(B) = 2$. Thus, $t(G) \leq 2$. Figure 2.3 shows a graph whose type is 2, and immediately following it is a characterization of graphs of type at most 2.
(2.1.3) Proposition. The type of a graph $G$ is at most 2 if and only if $B$ is a multi-edge, a cycle, or an isolated vertex, for each block $B$ of $G$. □

2.2 Some Special Graphs of Large Type

Many measures of complexity of graphs (tree width, for example) have the useful property of monotonicity under the taking of minors; that is, if the measure $m$ has this monotonicity property, and $G \leq_{m} H$, then $m(G) \leq m(H)$. Type, however, does not have this monotonicity property. We show below that type does not even have monotonicity under the taking of induced subgraphs.

Consider the graph $D$ in Figure 2.4 below. The induced subgraph $D - v$ is isomorphic to $C_{5,3}^{*}$. We shall see in (2.2.7) that $t(C_{5,3}^{*}) = 5$, but now we show that $t(D) \leq 4$. If the edges $e$ and $f$ are contracted in $D$, then the resulting graph consists of five 3-cycles meeting at a single vertex. Since each block of $D/\{e, f\}$ is a 3-cycle, $t(D/\{e, f\}) = 2$, by (2.1.3). Thus, $t(D) \leq t(D/\{e, f\}) + |\{e, f\}| = 2 + 2 = 4$.

![Figure 2.3](image.png)

**Figure 2.3.** A graph whose type is 2.

![Figure 2.4](image.png)

**Figure 2.4.** $D$ is the union of $C_{5,3}^{*}$, $e$, and $f$.

This lack of monotonicity under the taking of induced subgraphs is arbitrarily "bad" in the sense that there are graphs $G$ and $H$ such that $G$ is an induced
subgraph of $H$, but $t(G) - t(H)$ is arbitrarily large. For example, let $n$ be an arbitrary integer exceeding 4, and let $H$ be obtained from $G = C^*_n, n-2$ by attaching a path $P$ of length 2 to $G$ so that one endvertex of $P$ is identified with one of the vertices of $G$ of degree $n$, and the other endvertex of $P$ is identified with the other vertex of $G$ of degree $n$. (Note that $H$ is a generalization of $D$ in Figure 2.4.) Then $G$ is an induced subgraph of $H$, and, by (2.2.7), $t(G) = n$, but, since each block of $H/P$ is a cycle of length $n - 2$, it follows that $t(H) \leq 4$.

Although type does not have monotonicity under the taking of induced subgraphs, it does have some very special kinds of monotonicity that we shall describe in the two lemmas following the definitions below.

(2.2.1) **Definition.** Let $G$ be a graph. Then the *simplification* of $G$, denoted $\tilde{G}$, is obtained by deleting the loops of $G$ and by replacing each proper multi-edge of $G$ with a link-edge. Now, let $\mathcal{C}$ be the collection of cycles in $G$ each element of which has at most one vertex of degree exceeding 2 in $G$, and let $\mathcal{P}$ be the collection of proper paths $P$ in $G$ such that each internal vertex of $P$ has degree 2 in $G$. Then the *cosimplification* of $G$, denoted $\tilde{G}$, is obtained by contracting all but one edge of each element of $\mathcal{C}$ and all but one edge of each maximal element of $\mathcal{P}$ in $G$. Note that this definition of "cosimplification" is a graphic definition, rather than the more general matroid definition. □

(2.2.2) **Definition.** A graph $G$ is *simpler* than a graph $H$ if $G$ is a proper subgraph of $H$, and the simplifications of $G$ and $H$ are isomorphic. A graph $G$ is *cosimpler* than a graph $H$ if $G$ can be obtained by contracting a non-empty set of edges in $H$, and the cosimplifications of $G$ and $H$ are isomorphic; equivalently, $G$ is cosimpler than $H$ if $H$ can be obtained by subdividing each edge in a non-empty subset of $E(G)$ with at least one new vertex. Two link-edges $e$ and $f$ of a graph $G$ are *in series* if there is either a path $P$ in $G$ containing the pair $\{e, f\}$ of edges such that, for each internal vertex $v$ of $P$, the degree of $v$ in $G$ is 2, or a cycle $C$.
in $G$ containing $\{e, f\}$ that has at most one vertex whose degree in $G$ exceeds 2. Note that this is the graphic definition of "in series," rather than the more general matroid definition. □

(2.2.3) **Lemma.** If $G$ is simpler than $H$, then $t(G) \leq t(H)$.

**Proof.** We may assume that $|E(G)| = |E(H)| - 1$, since if $|E(G)| = |E(H)| - n$, for some integer $n$ exceeding 1, then there is a sequence of graphs $(G_i)_{i=0}^n$ such that $G = G_0$, $H = G_n$, $|E(G_{i-1})| = |E(G_i)| - 1$, and $G_{i-1}$ is simpler than $G_i$, for each $i \in [n]_+$. Let $e$ denote the edge in $E(H) - E(G)$; then $G = H \setminus e$. We proceed by induction on $t(H)$.

If $t(H) = 1$, then, by (2.1.2), $H$ is a forest with a set of loops. Since $G = H \setminus e$, it follows that $G$ is a forest with a set of loops. Thus, $t(G) \leq 1$.

Now, assume that the result holds for all graphs $G_0$ and $H_0$, where $t(H_0)$ is less than an integer $n$ exceeding 1, the graph $G_0$ is simpler than $H_0$, and $|E(G_0)| = |E(H_0)| - 1$. Let $G$ and $H$ be graphs such that $t(H) = n$, the graph $G$ is simpler than $H$, and $|E(G)| = |E(H)| - 1$. If $e$ is a loop in $H$, then $t(G) \leq t(H)$ since each block of $G$ is also a block of $H$. So we may assume that $e$ is not a loop; it follows that $e$ is parallel to some edge $e_G$ of $G$. Let $B_H$ denote the block of $H$ containing $e$ and $e_G$, and let $B_G$ denote the block of $G$ containing $e_G$; hence, $B_G = B_H \setminus e$. Since each block in $G$ different from $B_G$ is a block in $H$, it is sufficient to show that $t(B_G) \leq t(B_H)$. Let $e'$ be an edge of $B_H$ such that $t(B_H) = \min\{t(B_H \setminus e'), t(B_H/e')\} + 1$; it follows that $t(B_H \setminus e') < n$ or $t(B_H/e') < n$. If $e'$ is not parallel to $e$, then $B_G \setminus e'$ and $B_G/e'$ are simpler than $B_H \setminus e'$ and $B_H/e'$, respectively. Since $t(B_H \setminus e') < n$ or $t(B_H/e') < n$, it follows from the induction hypothesis that $t(B_G \setminus e') \leq t(B_H \setminus e')$ or $t(B_G/e') \leq t(B_H/e')$. Consequently, $t(B_G) \leq t(B_H)$. So, we may assume that $e' = e$ or $e'$ is parallel to $e$. If $t(B_H) = t(B_H/e') + 1$, then $B_H \setminus e' \cong B_G$; consequently, $t(B_G) < t(B_H)$. If $t(B_H) = t(B_H/e') + 1$, then $B_G/e_G$ is simpler.
than $B_H/e'$; it follows from the induction hypothesis that $t(B_G/e_G) \leq t(B_H/e')$. Hence, $t(B_G) \leq t(B_H)$; thus $t(G) \leq t(H)$. □

(2.2.4) LEMMA. If $G$ is cosimpler than $H$, then $t(G) \leq t(H)$.

Proof. The proof of this lemma is similar to the proof of (2.2.3); so we omit some of the details. As in (2.2.3), we may assume that $|E(G)| = |E(H)| - 1$. It follows that $H$ is obtained by subdividing some edge $e_G$ in $G$ into two edges $e$ and $e_G$ in $H$; thus $H/e = G$. We proceed by induction on $t(H)$.

If $t(H) = 1$, then, by (2.1.2), $H$ is a forest with a (perhaps empty) set of loops. It is clear that $H/f$ is a forest with a set of loops, for each $f \in E(H)$. Thus, $t(G) = t(H/e) \leq 1$.

Now, assume that the result holds for all graphs $G_0$ and $H_0$, where $t(H_0)$ is less than an integer $n$ exceeding 1, the graph $G_0$ is cosimpler than $H_0$, and $|E(G_0)| = |E(H_0)| - 1$. Let $G$ and $H$ be graphs such that $t(H) = n$, the graph $G$ is cosimpler than $H$, and $|E(G)| = |E(H)| - 1$. If $e_G$ is a cut-edge in $G$, then $e_G$ and $e$ are cut-edges in $H$. It follows that each block of $G$ is also a block of $H$; hence, $t(G) \leq t(H)$. So we may assume that $e_G$ is not a cut-edge in $G$; it follows that $e$ and $e_G$ belong to the same block $B_H$ of $H$. Let $B_G$ denote the block of $G$ containing $e_G$; then $B_G = B_H/e$. As in (2.2.3), it is sufficient to show that $t(B_G) \leq t(B_H)$. Let $e'$ be an edge of $B_H$ such that $t(B_H) = \min\{t(B_H/e'), t(B_H/e')\} + 1$. If $e'$ is not in series with $e$, then $B_G/e'$ and $B_G/e'$ are cosimpler than $B_H/e'$ and $B_H/e'$, respectively. Since $t(B_H/e') < n$ or $t(B_H/e') < n$, it follows from the induction hypothesis that $t(B_G/e') \leq t(B_H/e')$ or $t(B_G/e') \leq t(B_H/e')$. Consequently, $t(B_G) \leq t(B_H)$. So, we may assume that $e' = e$ or $e'$ is in series with $e$. Let $S_G$ denote the subgraph of $B_G$ induced by the subset of edges of $B_G$ each element of which is $e_G$ or is in series with $e_G$; similarly, let $S_H$ be the subgraph induced by the subset $E(S_G) \cup \{e\}$ of edges of $B_H$ each element of which is $e$ or is in series
with e. Note that $B_G \setminus S_G = B_H \setminus S_H$, and that, for each $f_G \in E(S_G)$, the graph $B_G \setminus f_G$ is the union of $B_G \setminus S_G$ and a nonnegative number $s_G = |E(S_G)| - 1$ of cut-edges of $B_G \setminus f_G$; similarly, for each $f_H \in E(S_H)$, the graph $B_H \setminus f_H$ is the union of $B_H \setminus S_H$ and a positive number $s_H = s_G + 1$ of cut-edges of $B_H \setminus f_H$.

It follows that $t(B_G \setminus e_G) = t(B_H \setminus e')$. Consequently, if $t(B_H) = t(B_H \setminus e') + 1$, then $t(B_G) \leq t(B_H)$. If, however, $t(B_H) = t(B_H \setminus e') + 1$, then $B_H / e' \cong B_G$; consequently, $t(B_G) < t(B_H)$. Hence, $t(G) \leq t(H)$. □

We conclude this chapter by proving the three propositions below regarding the type of the $n$-fan, the type of the $(n, n - 2)$-multicycle, and the type of the $(n, n - 2)$-comulticycle.

(2.2.5) Proposition. The type of the $n$-fan is $\lceil \log_2 n \rceil + 1$, for each positive integer $n$.

**Proof.** Let us consider the augmented $n$-fan $F'_n$, which is the graph obtained by adding an edge $f'_n$ that is parallel to $f_n$, where $f_n$ is the edge of $F_n$ as illustrated in Figure 2.5 below. Both $F_n$ and $F'_n$ are shown in Figure 2.5.

![Figure 2.5: The graphs $F_n$ and $F'_n$.](image)

Note that, by (2.2.3) and (2.2.4), we have $t(F_n) \leq t(F'_n) \leq t(F_{n+1})$, for each positive integer $n$, since $F_n$ is simpler than $F'_n$, and $F'_n$ is cosimpler than $F_{n+1}$. In particular, it follows that $t(F_m) \leq t(F_n)$ and $t(F'_m) \leq t(F'_n)$ whenever $m$ and $n$ are positive integers and $m \leq n$. 

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We shall prove this proposition by induction on \( n \). In addition to showing that 
\[ t(F_n) = \lceil \log_2 n \rceil + 1, \]
we shall also show that 
\[ t(F'_n) = \lceil \log_2 (n+1) \rceil + 1. \]

If \( n = 1 \), then \( F_1 \) is a link-edge, and \( F'_1 \) is a proper multi-edge; hence, 
\[ t(F_1) = 1 = \lceil \log_2 1 \rceil + 1 \]
and 
\[ t(F'_1) = 2 = \lceil \log_2 (1+1) \rceil + 1, \]
as required. Now, let us assume that \( n \) is an integer exceeding 1 and that 
\[ t(F'_n) = \lceil \log_2 n' \rceil + 1 \]
and 
\[ t(F^n) = \lceil \log_2 (n' + 1) \rceil + 1, \]
for each positive integer \( n' \) less than \( n \).

First, let us show that 
\[ t(F_n) \leq \lceil \log_2 n \rceil + 1 \]
and 
\[ t(F'_n) \leq \lceil \log_2 (n+1) \rceil + 1 \]
when \( n \) exceeds 1. Consider \( F_n \setminus (\frac{n}{2}) \) and \( F'_n \setminus (\frac{n}{2}) \). The graph \( F_n \setminus (\frac{n}{2}) \) consists of a block that is isomorphic to \( F[\frac{n}{2}] \) and another block that is isomorphic to \( F[\frac{n}{2}] \). It follows that 
\[ t(F_n) \leq t(F[\frac{n}{2}]) + 1 = (\lceil \log_2 (\frac{n}{2}) \rceil + 1) + 1 = \lceil \log_2 n \rceil + 1. \]
Similarly, \( F'_n \setminus (\frac{n}{2}) \) consists of a block that is isomorphic to \( F[\frac{n}{2}] \) and another block that is isomorphic to \( F'[\frac{n}{2}] \). If \( n \) is even, then \( F[\frac{n}{2}] = F'_{\frac{n}{2}} \) is simpler than \( F'[\frac{n}{2}] = F'_{\frac{n}{2}} \). It follows that 
\[ t(F'_n) \leq t(F'_{\frac{n}{2}}) + 1 = (\lceil \log_2 (\frac{n}{2} + 1) \rceil + 1) + 1 = \lceil \log_2 (n + 2) \rceil + 1 + 1 = \lceil \log_2 n \rceil + 1, \]
when \( n \) is even. On the other hand, if \( n \) is odd, then \( F[\frac{n}{2}] = F'_{\frac{n-1}{2}} \) is cosimpler than \( F[\frac{n}{2}] = F'_{\frac{n+1}{2}} \). It follows that if \( n \) is odd, then 
\[ t(F'_n) \leq t(F'_{\frac{n+1}{2}}) + 1 = (\lceil \log_2 (\frac{n+1}{2}) \rceil + 1) + 1 = (\lceil \log_2 (n + 1 - 1) \rceil + 1) + 1 = \lceil \log_2 n \rceil + 1. \]

It remains to show that 
\[ t(F_n) \geq \lceil \log_2 n \rceil + 1 \]
and 
\[ t(F'_n) \geq \lceil \log_2 (n+1) \rceil + 1 \]
when \( n \) exceeds 1. Let us start by showing that the induction hypothesis implies that 
\[ t(F_n) = \lceil \log_2 n \rceil + 1, \]
for any integer \( n \) exceeding 1. If \( e_i, f_1, \) or \( f_n \) is deleted from \( F_n \), where \( i \in [n - 1]_+ \), then the resulting graph consists of two blocks, one of which is isomorphic to \( F_n' \), for some integer \( n' \) satisfying \( \lfloor \frac{n}{2} \rfloor \leq n' < n \); so, 
\[ t(F_n') \geq t(F[\frac{n}{2}]) = \lceil \log_2 (\frac{n}{2}) \rceil + 1 = \lceil \log_2 n \rceil. \]

Thus, 
\[ t(F_n \setminus e_i) \geq \lceil \log_2 n \rceil \]
and 
\[ t(F_n \setminus f_{j}) \geq \lceil \log_2 n \rceil \]
for each \( i \in [n - 1]_+ \) and for each \( j \in \{1,n\} \). If \( f_i \) is deleted from \( F_n \), where \( i \) is an integer satisfying \( 1 < i < n \), then \( F_{n-1} \) is cosimpler than the resulting graph \( F_n \setminus f_i \); hence, 
\[ t(F_n \setminus f_i) \geq t(F_{n-1}) = \lceil \log_2 (n - 1) \rceil + 1 \geq \lceil \log_2 n \rceil, \]
for each \( i \) satisfying \( 1 < i < n \). If \( e_i, f_1, \) or \( f_n \) is contracted in \( F_n \),

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where $i \in [n - 1]_+$, then $F_{n-1}$ is simpler than the resulting graph $F_n/e$, where $e \in \{e_i : i \in [n - 1]_+\} \cup \{f_1, f_n\}$; hence, $t(F_n/e) \geq t(F_{n-1}) \geq \lceil \log_2 n \rceil$, for each $e \in \{e_i : i \in [n - 1]_+\} \cup \{f_1, f_n\}$. If $f_i$ is contracted in $F_n$, where $i$ is an integer satisfying $1 < i < n$, then the resulting graph $F_n/f_i$ consists of two blocks, one of which is isomorphic to $F'_{n'}$, for some integer $n'$ satisfying $\lceil \frac{n}{2} \rceil \leq n' < n$. Hence, $t(F_n/f_i) \geq t(F'_{n'}) = \lceil \log_2 (\lceil \frac{n}{2} \rceil + 1) \rceil + 1 \geq \lceil \log_2 \frac{n}{2} \rceil + 1 = \lceil \log_2 n \rceil$, for each integer $i$ satisfying $1 < i < n$. Thus, $t(F_n/e) \geq \lceil \log_2 n \rceil$ and $t(F_n/e) \geq \lceil \log_2 n \rceil$, for each $e \in E(F_n)$; consequently, the induction hypothesis implies that $t(F_n) \geq \lceil \log_2 n \rceil + 1$ for any positive integer $n$.

In order to complete the proof, we still must show that $t(F'_n) \geq \lceil \log_2 (n + 1) \rceil + 1$ when $n$ exceeds 1. The proof of this uses an argument similar to the previous one. If $e_i$ or $f_1$ is deleted from $F'_n$, where $i \in [\lceil \frac{n}{2} \rceil]_+$, then the resulting graph contains a block that is isomorphic to $F'_{n'}$, for some integer $n'$ satisfying $\lceil \frac{n}{2} \rceil \leq n' < n$. It follows that $t(F'_{n'}) \geq t(F'_{\lceil \frac{n}{2} \rceil}) = \lceil \log_2 (\lceil \frac{n}{2} \rceil + 1) \rceil + 1 \geq (\lceil \log_2 (n + 2) - 1 \rceil) + 1 \geq \lceil \log_2 (n + 1) \rceil$ when $\lceil \frac{n}{2} \rceil \leq n' < n$. If $e_i$, $f_n$, or $f'_n$ is deleted from $F'_n$, where $i$ is an integer satisfying $\lceil \frac{n+1}{2} \rceil \leq i < n$, then the resulting graph contains a block that is isomorphic to $F'_{n'}$, for some integer $n'$ satisfying $\lceil \frac{n+1}{2} \rceil \leq n' \leq n$. It follows that $t(F'_{n'}) \geq t(F'_{\lceil \frac{n+1}{2} \rceil}) = \lceil \log_2 \lceil \frac{n+1}{2} \rceil \rceil + 1 \geq \lceil \log_2 (n + 1) - 1 \rceil + 1 \geq \lceil \log_2 (n + 1) \rceil$. Hence, $t(F'_{n'} \setminus e \setminus \{f_1, f_n, f'_n\}) \geq \lceil \log_2 (n + 1) \rceil$ for each $e \in \{e_i : i \in [n-1]_+\} \cup \{f_1, f_n, f'_n\}$. If $f_i$ is deleted from $F'_n$, where $i$ is an integer satisfying $1 < i < n$, then $F'_{n-1}$ is cosimpler than the resulting graph $F'_{n'} \setminus f_i$; hence, $t(F'_{n'} \setminus f_i) \geq t(F'_{n-1}) = \lceil \log_2 n \rceil + 1 \geq \lceil \log_2 (n + 1) \rceil$, for each $i$ satisfying $1 < i < n$. If $e \in \{e_i : i \in [n-1]_+\} \cup \{f_1, f_n, f'_n\}$ is contracted in $F'_n$, then $F'_{n-1}$ is simpler than the resulting graph $F'_{n'}/e$; hence, $t(F'_{n'}/e) \geq t(F'_{n-1}) \geq \lceil \log_2 (n + 1) \rceil$, for each $e \in \{e_i : i \in [n-1]_+\} \cup \{f_1, f_n, f'_n\}$. If $f_i$ is contracted in $F'_n$, where $i$ is an integer satisfying $1 < i \leq \lceil \frac{n}{2} \rceil$, then the resulting graph contains a block $B$ such that $F'_{n'}$ is simpler than $B$, for some integer $n'$ satisfying $\lceil \frac{n}{2} \rceil \leq n' < n$; similarly, if $f_i$ is contracted in $F'_n$, where $i$ is an
integer satisfying \( \left\lfloor \frac{n+1}{2} \right\rfloor \leq i < n \), then the resulting graph contains a block that is isomorphic to \( F'_n \), for some integer \( n' \) satisfying \( \left\lfloor \frac{n}{2} \right\rfloor \leq n' < n \). It follows that 
\[
t(F'_n / f_i) \geq t(F'_n) \geq \left\lceil \log_2 \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \right\rceil + 1 = \left\lceil \log_2 \left( \left\lfloor \frac{n+1}{2} \right\rfloor \right) \right\rceil + 1 = \left\lceil \log_2 (n + 1) \right\rceil,
\]
for each integer \( i \) satisfying \( 1 \leq i < n \). Thus, \( t(F'_n / e) \geq \left\lceil \log_2 (n + 1) \right\rceil \) and \( t(F'_n / e) \geq \left\lceil \log_2 (n + 1) \right\rceil \), for each \( e \in E(F'_n) \); consequently, \( t(F'_n) \geq \left\lceil \log_2 (n + 1) \right\rceil + 1 \) when \( n \) exceeds \( 1 \), as required. \( \square \)

(2.2.6) **Proposition.** The type of the \((n, n - 2)\)-multicycle is \( n \), for each integer \( n \) exceeding \( 3 \).

**Proof.** We prove a more general statement regarding the type of \( C_{n,m} \), where \( m \) and \( n \) are integers exceeding \( 1 \) and \( 3 \), respectively, and \( C_{n,m} \) represents any graph obtained by replacing one edge of \( C_n \) with a multi-edge containing exactly \( m \) edges and by replacing each of the remaining edges of \( C_n \) with a multi-edge containing at least \( m \) edges. The result that we prove here is that \( t(C_{n,m}) = \min\{n, m + 2\} \).

It will follow immediately that \( t(C_{n,n-2}) = n \), for each integer \( n \) exceeding \( 3 \).

First, let us obtain lower and upper bounds for \( t(C_{n,m}) \). Observe that, for each edge \( e \in C_{n,m} \), each of \( C_{n,m} \setminus e \) and \( C_{n,m} / e \) has a block that is neither a cycle, nor a multi-edge, nor an isolated vertex. It follows from (2.1.3) that \( t(C_{n,m} \setminus e) > 2 \) and \( t(C_{n,m} / e) > 2 \), for each \( e \in E(C_{n,m}) \); consequently, \( t(C_{n,m}) > 3 \). Next, we shall obtain an upper bound for \( t(C_{n,m}) \). Let \( E_1 \) be a multi-edge of \( C_{n,m} \) consisting of \( m \) edges. Then each block of \( C_{n,m} \setminus E_1 \) is a multi-edge. It follows from (2.1.3) that \( t(C_{n,m} \setminus E_1) \leq 2 \); hence, \( t(C_{n,m}) \leq |E_1| + t(C_{n,m} \setminus E_1) \leq m + 2 \). Let \( E_2 \) be a set of \( n - 2 \) edges of \( C_{n,m} \) such that if \( f \) and \( f' \) are distinct edges in \( E_2 \), then \( f \) and \( f' \) belong to distinct multi-edges of \( C_{n,m} \). Then \( C_{n,m} / E_2 \) consists of a block that is a multi-edge and blocks that are loops. It follows from (2.1.3) that \( t(C_{n,m} / E_2) \leq 2 \); hence, \( t(C_{n,m}) \leq |E_2| + t(C_{n,m} / E_2) \leq n \). Thus, \( t(C_{n,m}) \leq \min\{n, m + 2\} \). We proceed by induction on \( m \) and \( n \).
Consider a graph $C_{n,\geq 2}$. By the lower bound, $t(C_{n,\geq 2}) > 3$. By the upper bound, $t(C_{n,\geq 2}) \leq 2 + 2 = 4$. So $t(C_{n,\geq 2}) = 4$. Thus, $t(C_{n,\geq m}) = \min\{n, m + 2\}$ when $m = 2$. Now, consider a graph $C_{4,\geq m}$. By the lower bound, $t(C_{4,\geq m}) > 3$. By the upper bound, $t(C_{4,\geq m}) \leq 4$. So $t(C_{4,\geq m}) = 4$. Thus, $t(C_{n,\geq m}) = \min\{n, m + 2\}$ when $n = 4$.

Now, let us assume that if $m'$ and $n'$ are integers satisfying $2 \leq m' < m$ and $4 \leq n' < n$, then $t(C_{n,\geq m'}) = \min\{n, m' + 2\}$, and $t(C_{n',\geq m'}) = \min\{n', m + 2\}$. Consider a graph $C_{n,\geq m}$. For any edge $e \in E(C_{n,\geq m})$, the graph $C_{n,\geq m}/e$ is the union of a block that is a graph $C_{n-1,\geq m}$ and blocks that are loops. It follows from the induction hypothesis that $t(C_{n-1,\geq m}) = \min\{n-1, m + 2\}$. If $e$ belongs to a multi-edge of $C_{n,\geq m}$ that contains exactly $m$ edges, then $C_{n,\geq m}/e$ is a graph $C_{n,\geq m-1}$. It follows from the induction hypothesis that $t(C_{n,\geq m-1}) = \min\{n, m + 1\}$. If $e$ belongs to a multi-edge of $C_{n,\geq m}$ that contains more than $m$ edges and $f$ belongs to a multi-edge of $C_{n,\geq m}$ that contains exactly $m$ edges, then $C_{n,\geq m}\{e, f\}$ is a graph $C_{n,\geq m-1}$ that is simpler than $C_{n,\geq m}/e$. By (2.2.3) and the induction hypothesis, $t(C_{n,\geq m}/e) \geq t(C_{n,\geq m-1}) = \min\{n, m + 1\}$ when $e$ belongs to a multi-edge containing more than $m$ edges. It follows that $t(C_{n,\geq m}) = \min\{\min\{n-1, m + 2\}, \min\{n, m + 1\}\} + 1 = \min\{n-1, m + 1\} + 1 = \min\{n, m + 2\}$, as required. □

(2.2.7) PROPOSITION. The type of the $(n, n-2)$-comulticycle is $n$, for each integer $n$ exceeding 3.

Proof. This result can be proven using a graph theoretic argument that is similar to the argument used to prove (2.2.6); instead, we present a matroid theoretic argument that uses (2.2.6) and the fact that the type of a matroid is the same as the type of its dual.

Note that $C_{n,n-2}^*$ is the graphic dual of $C_{n,n-2}$. By Lemma 2.3.7 in [O] the cycle matroid $M(C_{n,n-2}^*)$ of $C_{n,n-2}^*$ is isomorphic to the dual $M^*(C_{n,n-2})$ of the cycle matroid $M(C_{n,n-2})$ of $C_{n,n-2}$. Since the type of a matroid is the same as
the type of its dual, as noted in (1.1.6), it follows that \( t(C_{n,n-2}^*) = t(M(C_{n,n-2}^*)) = t(M^*(C_{n,n-2})) = t(M(C_{n,n-2})) = t(C_{n,n-2}) = n. \) □
CHAPTER 3

A CHARACTERIZATION OF 3-CONNECTED GRAPHS OF LARGE TYPE

In this chapter we shall show that a 3-connected graph has large type if and only if it has a large fan as a minor. First we shall prove that if a graph has a large fan as a minor, then it has large type; this is the main result of Section 3.1. Then, in Section 3.2, we shall prove that if a 3-connected graph has large type, then it contains a large fan as a minor. We shall see that the concept of what we mean by "large" is somewhat different for these two results.

3.1 Graphs That Have a Large Fan as a Minor

In this section we will show that if a graph \( G \) contains a large fan as a minor, then the type of \( G \) is large. This is stated more precisely as follows.

\[ (3.1.1) \text{ Proposition. If } G \text{ is a graph that contains } F_n \text{ as a minor, then the type } t(G) \text{ of } G \text{ is at least } \lceil \log_2 n \rceil + 1. \]

Note that if this lower bound on type can be improved, then it cannot be improved much, since \( t(F_n) = \lceil \log_2 n \rceil + 1 \), as shown in (2.2.5). Before the proof of (3.1.1) is presented, we need to prove some lemmas.

\[ (3.1.2) \text{ Lemma. If } n \text{ is an integer exceeding 1, and } F_n \leq_m G, \text{ then } G \text{ contains a vertex set } S = \{v_i: i \in [n]+\}, \text{ a } v_1v_n\text{-path } P, \text{ and a tree } T \text{ whose set of leaves is } S, \text{ such that } P \cap T = S \text{ and } F_n \leq_m P \cup T. \]

Proof. If \( F_n \leq_m G \), then there are disjoint subsets \( E_d \) and \( E_c \) in \( E(G) \) such that \( F_n \cong (G\setminus E_d/E_c)_E \), where \( H_E \) denotes the subgraph of a graph \( H \) obtained by deleting all isolated vertices from \( H \); equivalently, \( H_E = H[E(H)] \). Among all pairs \( (E_d, E_c) \) of disjoint sets of edges of \( E(G) \) such that \( F_n \cong (G\setminus E_d/E_c)_E \), fix one for which \( |E_c| \) is minimum; let \( (D, C) \) denote this pair. Let \( G' = (G\setminus D)_E \). A typical \( G' \) is illustrated in Figure 3.1 below; the dashed edges and solid edges of \( G' \)
form a path $P$ and a tree $T$, respectively, that satisfy the conditions of this lemma, and $F_8$ is obtained by contracting the unshaded edges.

![Diagram](image)

**Figure 3.1.** A typical $G'$ that contains an $F_8$-minor.

If $C$ is empty, then $G' \cong F_n$, and $S$, $P$, and $T$ are obvious. So, we may assume that $C = \{e_i : i \in [k]\}$, for some positive integer $k$. Let the sequence $(e_i)_{i=1}^k$ be an arbitrary ordering of the elements of $C$. Let $G_0 = G'$, and, inductively, let $G_i = (G_{i-1}/e_i)E$ for each $i \in [k]$; then $G_k \cong F_n$. Note that since $G_k$ is a block, $E(G_k)$ is contained in a single block $B_i$ of $G_i$, for each $i \in [k]$. This can be seen by observing that if distinct edges $e$ and $f$ are in different blocks of a graph $H$, and $g$ is any edge in $E(H) - \{e, f\}$, then $e$ and $f$ lie in different blocks of $H \setminus g$ and in different blocks of $H/g$. Moreover, $G_i$ is a block for each $i \in [k]$, that is, $G_i = B_i$. This can be seen as follows. If $G_i$ contained a block $B'_i \neq B_i$, then $(D \cup E(B'_i), C - E(B'_i))$ would be a pair of disjoint sets of edges such that $(G \setminus (D \cup E(B'_i))/(C - E(B'_i)))E = F_n$, but, since $G_i$ has no isolated vertices (hence, $E(B'_i)$ is non-empty), $|C - E(B'_i)| < |C|$, a contradiction to the minimality of $C$.

Now, we show that $G' = P \cup T$ by induction on $j$, for $j \in [k]$. We want to prove that, for each $j \in [k]$, the graph $G_{k-j}$ contains a $v_1v_n$-path $P_{k-j}$ and a tree $T_{k-j}$ whose set of leaves is $S$, such that $P_{k-j} \cap T_{k-j} = S$. If $j = 0$, then $k - j = k$, and, since $G_k \cong F_n$, it is obvious what $S$, $P_k$, and $T_k$ are. Assume that $G_{k-j}$ contains
subgraphs $P_{k-j}$ and $T_{k-j}$ that have the required properties, for each nonnegative integer $j < k$, and let $i = k - j$. Let $v$ denote the vertex of $G_i$ obtained by contracting $e_i$ in $G_{i-1}$.

First, if $v \in V(P_i) - S$, then $v$ is incident in $G_i$ with exactly two edges $e$ and $f$, which lie in $P_i$. After expanding $v$ in $G_i$ to $e_i$ to obtain $G_{i-1}$, since $G_{i-1}$ is a block, $e_i$ is neither a loop nor a cut-edge in $G_{i-1}$; hence, $G_{i-1}[E(P_i) \cup e_i]$ is a $v_1v_n$-path that contains the subpath $ee_if$. Let $P_{i-1} = G_{i-1}[E(P_i) \cup e_i]$ and $T_{i-1} = T_i$; it is straightforward that $P_{i-1}$ and $T_{i-1}$ have the required properties.

Now, if $v \in V(T_i) - S$, then $v$ is incident in $G_i$ with only edges in $T_i$. After expanding $v$ in $G_i$ to $e_i$ to obtain $G_{i-1}$, since $G_{i-1}$ is a block, $e_i$ is neither a loop nor a cut-edge; hence, $G_{i-1}[E(T_i) \cup e_i]$ is a tree whose set of leaves is $S$. Let $P_{i-1} = P_i$ and $T_{i-1} = G_{i-1}[E(T_i) \cup e_i]$; it follows that $P_{i-1}$ and $T_{i-1}$ have the required properties.

Finally, assume that $v \in S$; then either $v$ is not an endvertex of $P$, or $v \in \{v_1, v_n\}$. We consider the case in which $v$ is not an endvertex of $P$; the proof when $v \in \{v_1, v_n\}$, which is similar, is only sketched. If $v$ is not an endvertex of $P$, then $v$ is incident in $G_i$ with exactly three edges $e$, $f$, and $g$, where $\{e, f\} \subseteq E(P_i)$ and $g \in E(T_i)$. After expanding $v$ in $G_i$ to $e_i$ to obtain $G_{i-1}$, since $G_{i-1}$ is a block, $e_i$ is neither a loop nor a cut-edge. It follows that one vertex of $e_i$ is trivalent in $G_{i-1}$; call this vertex $v$. Then one of the following holds for $G_{i-1}$.

(i) $G_{i-1}[E(P_i) \cup e_i]$ is a $v_1v_n$-path that contains the subpath $ee_if$, in which case, $P_{i-1} = G_{i-1}[E(P_i) \cup e_i]$ and $T_{i-1} = G_{i-1}[E(T_i)]$ have the required properties.

(ii) $G_{i-1}[E(T_i) \cup e_i]$ is a tree whose set of leaves is $S$, in which case, $P_{i-1} = P_i$ and $T_{i-1} = G_{i-1}[E(T_i) \cup e_i]$ have the required properties.

If $v \in \{v_1, v_n\}$, then $v$ is incident in $G_i$ with exactly two edges $e \in E(P_i)$ and $f \in E(T_i)$. If $v$ is expanded in $G_i$ to $e_i$ to obtain $G_{i-1}$, then $ee_if$ is a subpath
in $G_{i-1}$; let $v$ denote the vertex in $G_{i-1}$ common to $e_i$ and $f$. It follows that the graphs $P_{i-1} = G_{i-1}[E(P_i) \cup e_i]$ and $T_{i-1} = T_i$ have the required properties. \[\Box\]

(3.1.3) **Lemma.** If $n$ is an integer exceeding 1, and $F_n \leq_m G$, then $F_{\left\lceil \frac{n}{2} \right\rceil} \leq_m G \setminus e$ for every $e \in E(G)$.

**Proof.** Assume that $F_n \leq_m G$; then there are subgraphs $P$ and $T$ of $G$ that satisfy the requirements specified in (3.1.2). If $e \notin P \cup T$, then $P \cup T$ is a subgraph of $G \setminus e$; hence $F_{\left\lceil \frac{n}{2} \right\rceil} \leq_m F_n \leq_m P \cup T \leq_m G \setminus e$. If $e \in P$, then one component $P'$ of $P \setminus e$ is a subpath of $P$ containing at least $\left\lceil \frac{n}{2} \right\rceil$ vertices in $S$; hence, $F_{\left\lceil \frac{n}{2} \right\rceil} \leq_m P' \cup T \leq_m G \setminus e$. If $e \in T$, then one component $T'$ of $T \setminus e$ is a subtree of $T$ containing at least $\left\lceil \frac{n}{2} \right\rceil$ vertices in $S$; hence, $F_{\left\lceil \frac{n}{2} \right\rceil} \leq_m P \cup T' \leq_m G \setminus e$. \[\Box\]

(3.1.4) **Lemma.** If $n$ is an integer exceeding 1, and $F_n \leq_m G$, then $F_{\left\lceil \frac{n}{2} \right\rceil} \leq_m G / e$ for every $e \in E(G)$.

**Proof.** Assume that $F_n \leq_m G$; then there are subgraphs $P$ and $T$ of $G$ that satisfy the requirements stated in (3.1.2). If $e$ is a loop, or if some vertex of $e$ does not lie in $P \cup T$, then $P \cup T$ is a subgraph of $G / e$; hence $F_{\left\lceil \frac{n}{2} \right\rceil} \leq_m F_n \leq_m P \cup T \leq_m G / e$. So we may assume that $e || x y$, and $x$ and $y$ are distinct vertices of $P \cup T$.

We shall use the following notation in proving this lemma. If $\{u, v\} \subseteq V(P)$, then let $P_{uv}$ denote the $uv$-subpath of $P$, and, for each $v \in S$, let $e_v$ denote the edge of $T$ incident with $v$.

If $\{x, y\} \subseteq S$, then at least one of $P_{xy}$ or $(P \setminus P_{xy}) \cup e$ contains $m$ vertices of $S$, for some $m \geq \left\lceil \frac{n}{2} \right\rceil + 1$; choose $P' \in \{P_{xy} \cup e, (P \setminus P_{xy}) \cup e\}$ so that $P'$ contains $m$ vertices of $S$. Let $S'$ denote the set of the $m$ vertices in $S \cap V(P')$. Contract $e$ to $x$; it follows that $P' / e$ contains a path $P''$ that contains the $m - 1$ vertices of $S' - y$. Since $(G / e)[E(T)]$ is obtained from $T$ by identifying $x$ and $y$, it follows that $\bigcup_{e \in S' - y} e_v$ is acyclic in $G / e$; hence, there is a tree $T'$ in $(G / e)[E(T)]$ that contains $\bigcup_{e \in S' - y} e_v$. Furthermore, $P'' \cap T' = S' - y$, and the set of leaves of $T'$

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contains $S' - y$. If follows that $F_{m-1} \leq_m G/e$; hence, since $m - 1 \geq \left\lceil \frac{n}{2} \right\rceil$, we have $F_{\left\lceil \frac{n}{2} \right\rceil} \leq_s F_{m-1} \leq_m P'' \cup T' \leq_s G/e$.

The proof of the next case is similar to the proof just presented; so, we only give an outline of it.

If $\{x, y\} \subseteq P$, but $\{x, y\} \cap S \leq 1$, then at least one of $P_{xy}$ or $(P \setminus P_{xy}) \cup e$ contains $m$ vertices of $S$, for some $m \geq \left\lceil \frac{n}{2} \right\rceil$; choose $P' \in \{P_{xy} \cup e, (P \setminus P_{xy}) \cup e\}$ so that $P'$ contains $m$ vertices of $S$, and let $S' = S \cap V(P')$. Contract $e$; then $P'/e$ contains a path $P''$ that contains the $m$ vertices of $S'$, and $T$ is a tree in $G/e$ whose set of leaves is $S$. Since $m \geq \left\lceil \frac{n}{2} \right\rceil$, we have $F_{\left\lceil \frac{n}{2} \right\rceil} \leq_s F_m \leq_m G/e$.

Now, assume that $e \parallel xy$ has both vertices in $T - S$. Then $P$ is a path in $G/e$. It is clear that $\bigcup_{v \in S} e_v$ is acyclic in $G/e$, and, since $(G/e)[V(T) - e]$ is connected, $(G/e)[E(T) - e]$ contains a tree $T'$ whose set of leaves is $S = P \cap T'$. It follows that $F_{\left\lceil \frac{n}{2} \right\rceil} \leq_s F_n \leq_m P \cup T' \leq_s G/e$.

Finally, assume, without loss of generality, that $x \in P$ and $y \in T - S$. Then there are not more than 4 edges of $P \cup T \cup e$ incident with $x$. One of these edges is $e$, and there are two distinct edges $e^x$ and $f^x$ of $P$ incident with $x$. Consider the graph $(P \cup T \cup e)/e \setminus \{e^x, f^x\}$. One component $P'$ of $(P \cup e)/e \setminus \{e^x, f^x\}$ is a path that contains at least $\left\lceil \frac{n}{2} \right\rceil$ vertices of $S$; let $S'$ denote $V(P') \cap S$. It follows that $\bigcup_{v \in S'} e_v$ is acyclic in $G/e$. Since $(T \cup e)/e$ is connected, $(T \cup e)/e$ contains a tree $T'$ whose set of leaves is $S' = P' \cap T'$. Then $F_{\left\lceil \frac{n}{2} \right\rceil} \leq_m P' \cup T' \leq_s (P \cup T \cup e)/e \setminus \{e^x, f^x\} \leq_s G/e$. □

Now, we are ready to prove (3.1.1).

Proof of (3.1.1). Suppose that (3.1.1) is false. Let $\mathcal{G}$ be the collection of counterexamples to (3.1.1); that is, for each $H$ in $\mathcal{G}$, there is a positive integer $n(H)$ such that $F_{n(H)} \leq_m H$, but $t(H) < [\log_2 n(H)] + 1$. Let $\mathcal{G}_0$ be the subcollection of $\mathcal{G}$ each of whose elements contain a minimum number of edges and no isolated vertices, and let $n = \min\{n(H): H \in \mathcal{G}_0\}$. Then $\mathcal{G}_0$ contains a graph $G$ such
that \( n(G) = n \). Any such \( G \) is a minimal counterexample to (3.1.1) in the sense defined above. Note that \( n \) is at least 2 since if \( F_1 \) is a minor of a graph \( H \), then 
\[
t(H) \geq \lfloor \log_2 1 \rfloor + 1 = 1.
\]
It follows that since \( F_n \leq_m G \), there are subgraphs \( P \) and \( T \) of \( G \) that have the properties specified in (3.1.2).

We claim that \( G \) is a block. This can be seen as follows. There is a block \( B \) of \( G \) that contains \( F_n \) as a minor. Clearly, \( t(B) \leq t(G) \); consequently, \( B \) is a counterexample. Since \( G \) is a minimal counterexample and \( |E(B)| \leq |E(G)| \), it follows that \( |E(B)| = |E(G)| \), and, since \( G \) has no isolated vertices, \( B = G \). Thus \( G \) is a block; consequently, there is an edge \( e \) such that \( \min\{t(G \setminus e), t(G/e)\} = t(G) - 1 \).

First, suppose that there is an edge \( e \) such that \( t(G \setminus e) = t(G) - 1 \). By (3.1.3), \( F_{\lfloor \frac{n}{2} \rfloor} \leq_m G \setminus e \). Since \( G \setminus e \) is not a counterexample,

\[
t(G \setminus e) \geq \left\lfloor \log_2 \left\lfloor \frac{n}{2} \right\rfloor \right\rfloor + 1 \geq \left\lfloor \log_2 \frac{n}{2} \right\rfloor + 1 = \lfloor \log_2 n - 1 \rfloor + 1 = \lfloor \log_2 n \rfloor;
\]

hence, \( t(G) \geq \lfloor \log_2 n \rfloor + 1 \), a contradiction. Thus, if \( G \) is a counterexample, then there is an edge \( e \) such that \( t(G/e) = t(G) - 1 \).

Let \( e \) be an edge such that \( t(G/e) = t(G) - 1 \). By (3.1.4), \( F_{\lfloor \frac{n}{2} \rfloor} \leq_m G/e \); that is, \( F_{\lfloor \frac{n}{2} \rfloor} \leq_m G/e \) if \( n \) is even, and \( F_{\lfloor \frac{n-1}{2} \rfloor} \leq_m G/e \) if \( n \) is odd. Since \( G/e \) is not a counterexample,

\[
t(G/e) \geq \left\lfloor \log_2 \frac{n}{2} \right\rfloor + 1 = \left\lfloor \log_2 n \right\rfloor \quad \text{if } n \text{ is even},
\]
\[
t(G/e) \geq \left\lfloor \log_2 \frac{n-1}{2} \right\rfloor + 1 = \lfloor \log_2 (n-1) \rfloor = \lfloor \log_2 n \rfloor \quad \text{if } n \text{ is odd}.
\]

Hence, \( t(G) \geq \lfloor \log_2 n \rfloor + 1 \), a contradiction. It follows that there are no counterexamples, which proves (3.1.1). \( \square \)

### 3.2 3-Connected Graphs That Have Large Type

If \( n \) is a positive integer exceeding 1, then let \( W_n \) denote the \( n \)-wheel, which is the graph whose vertex set is \( \{v_i : i \in [n] \} \) and whose edge set is the union of

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\{v_0v_i : i \in [n]_+\} and the edge set of an n-cycle on \{v_i : i \in [n]_+\}. An edge $v_0v_i$, for $i \in [n]_+$, is called a spoke of $W_n$, and any edge of the cycle on \{v_i : i \in [n]_+\} is called a rim edge of $W_n$. It is evident that deleting any rim edge of $W_n$ results in the n-fan $F_n$. The 4-wheel and a general n-wheel are illustrated in Figure 3.2.

![Figure 3.2. The 4-wheel $W_4$ and the n-wheel $W_n$.](image)

The main result of this section is that if a 3-connected graph has large type, then it contains a large wheel as a minor. We state this more precisely as follows.

(3.2.1) **Theorem.** For each positive integer $n$ exceeding 2, there is an integer $t_n$ such that if $G$ is a 3-connected graph and $t(G) \geq t_n$, then $W_n \leq_m G$.

Before we prove (3.2.1), we need to prove several lemmas.

(3.2.2) **Lemma.** For each cycle $C$ in $G/e$, either $G[E(C)]$ is a cycle of $G$, or $G[E(C) \cup e]$ is a cycle of $G$.

**Proof.** Let $v_e$ be the vertex obtained by contracting $e$ in $G$, and let $C$ be any cycle of $G/e$. If $v_e \notin V(C)$, then $C$ is a cycle of $G$. Now, assume that $v_e \in V(C)$, and $v_e$ is expanded to $e \parallel xy$ in $G$. If the two edges of $C$ incident with $v_e$ in $G/e$ are incident with the same vertex $x$ or $y$ in $G$, then $G[E(C)]$ a cycle of $G$; if, however, one of the two edges of $C$ incident with $v_e$ in $G/e$ is incident with $x$ in $G$ and the other is is incident with $y$ in $G$, then, after expanding $v_e$ to $e$, the subgraph $G[E(C) \cup e]$ of $G$ is a cycle. $\square$
(3.2.3) Lemma. If C and C' are longest cycles in a 2-connected graph, then C and C' intersect in at least two vertices.

Proof. We shall prove the lemma by showing that if C and C' are cycles of equal length in a 2-connected graph G that have at most one vertex in common, then there is a cycle C_1 in G whose length exceeds |E(C)|.

First, assume that C and C' have no vertices in common. Since G is 2-connected, G contains a cycle C_0 that contains e and e', where e \(\in E(C)\) and e' \(\in E(C')\) are arbitrarily chosen. Since C_0 meets both C and C', there are vertices \(v_1 \in V(C) \cap V(C_0)\) and \(v'_1 \in V(C') \cap V(C_0)\) such that all internal vertices of one of the \(v_1v'_1\)-paths \(P_1\) in \(C_0\) lie outside \(V(C) \cup V(C')\). Similarly, in the other \(v_1v'_1\)-path \(P'_1\) of \(C_0\), there are vertices \(v'_2 \in V(C') \cap V(C_0)\) and \(v_2 \in V(C) \cap V(C_0)\) such that all internal vertices of the \(v'_2v_2\)-subpath \(P_2\) of \(P'_1\) lie outside \(V(C) \cup V(C')\). This is illustrated in Figure 3.3; if C and C' are cycles of equal length in a 2-connected graph, and \(C_0\) is the cycle indicated by the bold black edges, then there is a cycle \(C_1\), indicated by the shaded edges, whose length, we shall see, must exceed |\(E(C)\)|.

![Figure 3.3](image-url)

**Figure 3.3.** The cycles C, C', C_0, and C_1 in a 2-connected graph.

C consists of two \(v_1v_2\)-paths, one of which, say \(P_C\), has length at least \(\frac{|E(C)|}{2}\). Similarly, \(C'\) contains a \(v'_1v'_2\)-path \(P_{C'}\) of length at least \(\frac{|E(C')|}{2}\). Then \(C_1 = P_C \cup P_1 \cup P_{C'} \cup P_2\) is a cycle in G of length at least |\(E(C)\)| + 2.
For the remaining case, let us assume that $C$ and $C'$ have only one vertex $v$ in common. The argument for this case is very similar to the one just presented; so we only sketch this argument. Let $e \in E(C)$ and $e' \in E(C')$ be such that each of $e$ and $e'$ is incident with $v$. Let $C_0$ be a cycle of $G$ containing $e$ and $e'$. Then $C_0$ contains distinct vertices $v_0 \in V(C)$ and $v'_0 \in V(C')$ such that the internal vertices of one of the $v_0v'_0$-paths $P_0$ in $C_0$ lie outside $V(C) \cup V(C')$. There are a $vv_0$-path $P$ contained in $C$ and a $vv'_0$-path $P'$ contained in $C'$, each of length at least $\frac{|E(C)|}{2}$. Then $C_1 = P \cup P_0 \cup P'$ is a cycle in $G$ of length at least $|E(C)| + 1$; consequently, the lemma holds. □

We use (3.2.2) and (3.2.3) to prove (in a purely graph-theoretic manner) (3.2.5) below, which is a special case of the following matroid result of Seymour (see [DOO]).

(3.2.4) THEOREM (Seymour). Let $C$ be a largest circuit of a connected matroid $M$. Then the size of each circuit of $M/C$ is less than $|C|$. □

(3.2.5) LEMMA. Let $C$ be a longest cycle of a 2-connected graph $G$. Then the length of each cycle of $G/C$ is less than $|E(C)|$.

Proof. Suppose that $G/C$ contains a cycle $C'$ of length at least $|E(C)|$. It follows from (3.2.2) that $E(C')$ is contained in a cycle of $G$. Since $C$ is a longest cycle of $G$, it follows that $C'$ is a cycle of $G$ and that $|E(C')| = |E(C)|$. It then follows from (3.2.3) that $C$ and $C'$ meet in $G$ in at least two vertices $v_1$ and $v_2$. When $C$ is contracted in $G$, $v_1$ and $v_2$ are identified to a single vertex; hence, $C'$ becomes a non-empty edge-disjoint union of cycles whose lengths are less than $|E(C')|$. Consequently, $C'$ is not a cycle in $G/C$, a contradiction. Thus, the lemma holds. □

The following upper bound for the type of a connected matroid mentioned in [DOO] is an immediate consequence of (3.2.4). For completeness, in (3.2.7),
we shall state and prove the special case of this upper bound for the type of a 2-connected graph.

(3.2.6) **Theorem.** If the type of a connected matroid $M$ exceeds $\frac{N(N+1)}{2}$, for some positive integer $N$, then $M$ contains a circuit with more than $N$ elements. □

(3.2.7) **Lemma.** If the type of a 2-connected graph $G$ exceeds $\frac{N(N+1)}{2}$, for some integer $N$ exceeding 1, then $G$ contains a cycle of length exceeding $N$.

**Proof.** Let $G$ be a 2-connected graph, and assume that a longest cycle $C$ of $G$ has length $N$. We shall show that the type $t(G)$ of $G$ is at most $\frac{N(N+1)}{2}$ by induction on $N$.

If $N = 2$, then it follows that $G$ is multi-edge. By (2.1.3), $t(G) = 2 < \frac{2(2+1)}{2}$; thus, the result holds when $N = 2$.

Now, assume that $N > 2$ and that if the length of a longest cycle in a 2-connected graph $G'$ is $N'$, for some $N' < N$, then $t(G') \leq \frac{N'(N'+1)}{2}$. After contracting $C$ in $G$, every cycle of $G/C$ has length less than $N$, by (3.2.5); in particular, each cycle of each block of $G/C$ has length less than $N$. By hypothesis, the type of each block that is neither a loop nor a cut-edge of $G/C$ does not exceed $\frac{(N-1)N}{2}$, and it is evident that the type of a block that is a loop or a cut-edge does not exceed $\frac{(N-1)N}{2}$. It follows that $t(G) \leq N + \frac{(N-1)N}{2} = \frac{N(N+1)}{2}$, as required, since $G/C$ was obtained by contracting $N$ elements in $G$, and since $t(G/C) \leq \frac{(N-1)N}{2}$. □

Now, we are ready to outline the proof of the main result (3.2.1) of this section. If a 3-connected graph $G$ has sufficiently large type, then $G$ has a large cycle, by (3.2.7). Ding, Oporowski, Oxley, and Vertigan have proved in [DOOV] that if a 3-connected binary matroid $M$ contains a sufficiently large circuit, whose size depends on an integer $n > 2$, then an element of $\{M(W_n), M^*(K_{3,n}), S_{n+2}\}$ is a minor of $M$; the matroid $M^*(K_{3,n})$ is the bond matroid of $K_{3,n}$, and $[I_n, J_n - I_n]$ is a binary representation of $S_n$, where $[I_n]$ is the $n \times n$ identity matrix and $[J_n]$ is
the $n \times n$ matrix each entry of which is 1. Since $M^*(K_{3,n})$ and $S_n$ are graphic for only small values of $n$, it follows that $M(W_n) \leq_m M(G)$; hence, $W_n \leq_m G$.

We state this result of Ding, Oporowski, Oxley, and Vertigan more precisely below.

(3.2.8) **Theorem.** For each positive integer $n$ exceeding 2, there is an integer $N$ such that if $M$ is a 3-connected binary matroid containing a circuit with at least $N$ elements, then $M(W_n)$, $M^*(K_{3,n})$, or $S_{n+2}$ is a minor of $M$.

It is pointed out that (3.2.8) is not stated explicitly as a separate result in [DOOV]; rather, this result can be extracted from there (see the proof of Theorem (1.5) in [DOOV]). A similar unavoidable-minor result for 3-connected graphs that uses only graph-theoretic means can be extracted from [OOT]. In some sense, this similar result, which is not stated explicitly in [OOT], is simpler than (3.2.8) since it does not rely on matroid theory, but the extraction of it requires a deep understanding of [OOT]; whereas, it is relatively easy to extract (3.2.8) from [DOOV].

Note that the conclusion of (3.2.8) states that some matroid $M'$ is a minor of another matroid $M$; in general, if $G$ and $H$ are graphs whose cycle matroids $M$ and $M'$, respectively, satisfy $M' \leq_m M$, then we cannot say that $H \leq_m G$. This is illustrated in Figure 3.4. The cycle matroids of $F_6$ and $G_6$ are isomorphic; it is clear, however, that neither graph is a minor of the other.

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F_6

G_6
```

**Figure 3.4.** $M(F_6) \cong M(G_6)$, but neither $F_6 \leq_m G_6$ nor $G_6 \leq_m F_6$.

We have just seen that if $M$ and $M'$ are the cycle matroids of graphs $G$ and $H$, respectively, and $M \cong M'$, then $G$ need not be isomorphic to $H$; Whitney
has shown, however, that such graphs $G$ and $H$ are $2$-isomorphic, which we define below. Immediately following this definition is the formal statement of Whitney's $2$-Isomorphism Theorem ([O], Theorem 5.3.1).

(3.2.9) **Definition.** A graph $G$ is $2$-isomorphic to a graph $H$, denoted $G \cong_2 H$, if and only if there is a positive integer $n$ and a sequence $(G_i)_{i=1}^n$ of graphs such that $G_1 = G$, the final graph $G_n = H$, and if $i \in [n-1]_+$, then $G_{i+1}$ is obtained by performing one of the three following operations on $G_i$.

(i) **Vertex identification:** If $v_1$ and $v_2$ are vertices in distinct components of $G_i$, then $G_{i+1}$ is obtained by identifying $v_1$ and $v_2$ to a new vertex $v$. If neither component is an isolated vertex, then the vertex identification is proper; otherwise, the vertex identification is trivial.

(ii) **Vertex cleaving:** If $G^1$ and $G^2$ are disjoint graphs such that $G_i$ can be obtained from $G^1$ and $G^2$ by identifying a vertex $v_1$ of $G^1$ and a vertex $v_2$ of $G^2$ to a single vertex $v$, then let $G_{i+1} = G^1 \cup G^2$. If neither $G^1$ nor $G^2$ is an isolated vertex, then the vertex cleaving is proper; otherwise, the vertex cleaving is trivial. This is the reverse of vertex identification, and it is clear that if $G_{i+1}$ is a proper vertex cleaving of $G_i$ at $v$, then $v$ is incident with a loop in $G_i$ or is a cut-vertex of $G_i$.

(iii) **Twisting:** Assume that $G^1$ and $G^2$ are disjoint graphs and that $u_j$ and $v_j$ are distinct vertices for each $j \in [2]_+$. Further assume that $G_i$ is obtained from $G^1$ and $G^2$ by identifying $u_1$ and $u_2$ to a single vertex $u$ and by identifying $v_1$ and $v_2$ to a single vertex $v$. Call $G_{i+1}$ a twisting of $G_i$ about \{u, v\} if $G_{i+1}$ is obtained from $G^1$ and $G^2$ by identifying $u_1$ and $v_2$ to a single vertex $u'$, and by identifying $u_2$ and $v_1$ to a single vertex $v'$. □

(3.2.10) **Whitney's $2$-Isomorphism Theorem.** If $G$ and $H$ are graphs, then the cycle matroids $M(G)$ and $M(H)$ of $G$ and $H$, respectively, are isomorphic if and only if $G$ and $H$ are $2$-isomorphic. □
We are finally ready to present the proof of (3.2.1), the main result of this section.

Proof of (3.2.1). Let $t_n = \frac{N(N-1)}{2} + 1$, where $N$ is the number depending on $n$ from (3.2.8), for each integer $n$ exceeding 2. We are concerned more with the existence of $N$, given $n$, than with its particular value, but this value is given for the sake of completeness. Let $N = 2^k + 1$, where $k = (R_2(m))^m$ and $R_2(i)$ is the smallest integer such that the complete graph on $R_2(i)$ vertices with its edges colored by two colors contains a single-colored complete subgraph on $i$ vertices; let $\tau(i) = 3^R_2(i)+1$; and let

$$m = \max\{11(\tau(2k_0) - 1)(n+2)^2(n+3)^2(n+6), 128r(2r(2n+2))(n+1)^2\},$$

where $k_0$ is the element in the sequence of integers $(k_i)_{i=0}^{4R_2(2n+4)}$ which is defined as follows: $k_{i+4R_2(2n+4)} = n + 3$, and $k_{i+1} = R_2(\tau(k_i)) + 1$ for each $i \in [4R_2(2n+4)]_+$. If $G$ is a 3-connected graph whose type is at least $t_n$, then, by (3.2.7), $G$ contains a cycle of length at least $N$; hence, the simplification $\tilde{G}$ of $G$ contains a cycle $C$ of length at least $N$. The cycle matroid $M(\tilde{G})$ of $\tilde{G}$ is a graphic (hence, binary) 3-connected matroid that contains $C$ as a circuit. Evidently, any minor of a graphic matroid is graphic. By (3.2.8), $M(\tilde{G})$ contains $M(W_n)$, $M^*(K_3,n)$, or $S_{n+2}$ as a minor. $M^*(K_3,n)$ is not graphic if $n > 2$, and the Fano matroid (which is not graphic) is a minor of $S_n$ (hence, $S_n$ is not graphic) if $n > 3$. Since $n$ exceeds 2, neither $M^*(K_3,n)$ nor $S_{n+2}$ is graphic; so $M(W_n) \leq_m M(\tilde{G})$. Since $W_n$ and $\tilde{G}$ are 3-connected graphs, it follows from (3.2.10) that $W_n$ and $\tilde{G}$ are the unique graphs whose respective cycle matroids are $M(W_n)$ and $M(\tilde{G})$. Consequently, $W_n \leq_m \tilde{G} \leq_s G$. □

We conclude this chapter by presenting an informal remark that characterizes, in a weak sense, 3-connected graphs of large type, and by presenting the analogue.
of (3.2.1) for cographic matroids. The remark, (3.2.11) below, is obtained immediately on combining (3.1.1) and (3.2.1), and on observing that $F_n \leq m W_n$. The proof of (3.2.12), the analogue of (3.2.1) for cographic matroids, is essentially the same as that of (3.2.1), and is only sketched.

(3.2.11) **Remark.** A 3-connected graph contains a large fan as a minor if and only if it has large type. □

(3.2.12) **Corollary.** Let $M$ be a 3-connected cographic matroid, and, for each integer $n$ exceeding 2, let $t_n$ be as in (3.2.1). If the type $t(M) \geq t_n$ for some integer $n$ exceeding 2, then $M(W_n) \leq_m M$.

*Proof.* It follows from the matroid definition of type that $t(M^*) = t(M)$, where $M^*$ is the dual of $M$; moreover, $M^*$ is 3-connected since $M$ is 3-connected ([O], Proposition 8.1.5). Also, $M^*$ is graphic since $M$ is cographic. By (3.2.6), $M^*$ contains a circuit of cardinality at least $N$, where $N$ is as in (3.2.8). Thus, an element of the set \{ $M(W_n)$, $M^*(K_{3,n})$, $S_{n+2}$ \} is a minor of $M^*$. Since $M^*$ is graphic, $M(W_n) \leq_m M^*$; hence, $M(W_n) \cong M^*(W_n) \leq_m (M^*)^* \cong M$. □
CHAPTER 4
UNAVOIDABLE MINORS OF
2-CONNECTED GRAPHS OF LARGE TYPE

In this chapter, we shall see that if \( n \) is an integer exceeding 3, and if a
graph has sufficiently large type depending on \( n \), then it contains an element of
\( \{F_n, C_{n,n-2}, C_{n,n-2}^*\} \) as a minor. We can see that it is sufficient to consider only
2-connected graphs of large type as follows. If a graph \( G \) has type \( t(G) \) exceeding
1, then it has a block \( B \) such that \( t(B) = t(G) \), and since \( t(B) > 1 \), it follows
that \( B \) is 2-connected. So, what we shall show in this chapter is that if \( n \) is an
integer exceeding 3, and a 2-connected graph has sufficiently large type depending
on \( n \), then it contains an element of \( \{F_n, C_{n,n-2}, C_{n,n-2}^*\} \) as a minor. The proof
of this will be broken up into several steps. We cannot even state now what these
steps are, because they rely heavily on a canonical decomposition, due to Tutte,
of 2-connected graphs, and on similar decompositions (that are not necessarily
canonical). Several of these decompositions are considered in Section 4.1.

4.1 2-Sums and Tree Structures

In order to prove several of the results in this dissertation, we will rely on a result
of Tutte that states that every 2-connected graph has a canonical decomposition
into simple 3-connected graphs, cycles, and multi-edges. Before stating this more
precisely, some definitions and observations are needed.

If \( G \) is a graph, \( E_0 \) is a subset of \( E(G) \), and \( S \) is a set, then define a function
\( L_G: E_0 \rightarrow S \times (V(G) \times V(G)): e \mapsto (s(e), (u(e), v(e))) \) so that for each \( e \) in \( E_0 \), \( u(e) \)
and \( v(e) \) are the endvertices of \( e \), and if \( s(e) = s(f) \), then \( e = f \). Intuitively, we
may think of \( L_G \) as a function which assigns to each edge \( e \) in \( E_0 \) a label \( s(e) \) and a
direction where \( u(e) \) and \( v(e) \) are the tail and head, respectively, of \( e \); frequently, we
shall describe these functions in this intuitive way. Also, it is convenient to think
of the function \( L_G \) on \( E_0 \) as a partial function \( L_G: E(G) \rightarrow S \times (V(G) \times V(G)) \),
where \( L_G(e) \) is defined if and only if \( e \in E_0 \); often, we shall consider such functions
$L_G$ without specifying the domain of definition. Call $L_G$ a directed labeling of $G$. It is clear that restricting the domain of $L_G$ to a subset $E' \subseteq E_0$ results in a directed labeling $L'_G$ of $G$; call $L'_G$ a restriction of $L_G$. If the domain of $L_G$ is the empty set, then call the directed labeling $L_G$ of $G$ trivial; we may also say that $G$ is unlabeled. It is also clear that if $G'$ is a minor of $G$, then

$L_{G'} : E(G') \cap E_0 \rightarrow S \times (V(G') \times V(G')) : e \mapsto (s(e), (u'(e), v'(e)))$ is a directed labeling of $G'$, where $u'(e)$ and $v'(e)$ are the vertices in $G'$ that correspond to $u(e)$ and $v(e)$, respectively, in $V(G)$; call $L_{G'}$ the directed labeling of $G'$ induced by $L_G$.

Assume that $L_H : E(H) \rightarrow S \times (V(H) \times V(H)) : e \mapsto (s(e), (u_H(e), v_H(e)))$ and $L_K : E(K) \rightarrow S \times (V(K) \times V(K)) : e \mapsto (s(e), (u_K(e), v_K(e)))$ are directed labelings of disjoint graphs $H$ and $K$, respectively, and there is only one pair, $h \in E(H)$ and $k \in E(K)$, of edges such that $s(h) = s(k)$. Then the edge-sum of $H$ and $K$ (with respect to $L_H$ and $L_K$), denoted $(H, L_H) \oplus_2 (K, L_K)$ or, more commonly, $H \oplus_2 K$, is the graph defined as follows. If neither $h$ nor $k$ is a loop, then $H \oplus_2 K$ is obtained by first identifying $h$ and $k$ head-to-head and tail-to-tail, and then deleting the identified edge. If at least one of $h$ and $k$ is a loop, then $H \oplus_2 K$ is obtained by first contracting $h$ to a vertex $v_h$ and $k$ to a vertex $v_k$, and then identifying $v_h$ and $v_k$. We may sometimes refer to $H \oplus_2 K$ as the edge-sum of $H$ and $K$ along $h$ and $k$ when $L_H$ and $L_K$ are understood. The operation of edge-summing is illustrated in Figure 4.1, where $s(h) = s(k)$, and $L_H$ and $L_K$ assign the indicated directions to $h$ and $k$, respectively.

![Figure 4.1](https://example.com/figure4.1.png)

**Figure 4.1.** $H \oplus_2 K$ is the edge-sum of $H$ and $K$.  

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It is clear from the definition that edge-summing is commutative. Evidently, if \( H \) and \( K \) can be edge-summed along \( h \) and \( k \) (with respect to \( L_H \) and \( L_K \)), then the edge set of \( H \oplus_2 K \) is \( (E(H) - h) \cup (E(K) - k) \). It is easy to see that there is a partial function \( L_{H\oplus_2 K} : E(H \oplus_2 K) \to S \times (V(H \oplus_2 K) \times V(H \oplus_2 K)) : e \mapsto (s(e), (u_{H\oplus_2 K}(e), v_{H\oplus_2 K}(e))) \), where \( u_{H\oplus_2 K}(e) \) and \( v_{H\oplus_2 K}(e) \) are the vertices in \( H \oplus_2 K \) that correspond to the tail and head, respectively, of \( e \) determined by \( L_H \) or \( L_K \) (depending on whether \( e \) is in \( E(H) - h \) or in \( E(K) - k \)). Moreover, \( L_{H\oplus_2 K} \) is a directed labeling of \( H \oplus_2 K \) since \( s(e) \neq s(f) \) for any two distinct edges \( e \) and \( f \) in \( (E(H) - h) \cup (E(K) - k) \); we shall call \( L_{H\oplus_2 K} \) the directed labeling inherited from \( L_H \) and \( L_K \). If \( L'_H \) and \( L'_K \) are the directed labelings of \( H \) and \( K \), respectively, obtained by reversing the directions assigned by \( L_H \) and \( L_K \) to the edges \( h \) and \( k \), then it is evident that \( (H, L_H) \oplus_2 (K, L_K) = (H, L'_H) \oplus_2 (K, L'_K) \); call this process of obtaining \( L'_H \) and \( L'_K \) from \( L_H \) and \( L_K \) pair direction reversal. If \( h \) is not a block of \( H \), and \( k \) is not a block of \( K \), then \( H \oplus_2 K \) is called the 2-sum of \( H \) and \( K \).

(4.1.1) PROPOSITION. If \( H \) and \( K \) are 2-connected graphs that can be 2-summed along \( h \in E(H) \) and \( k \in E(K) \), then \( H \oplus_2 K \) is 2-connected.

Proof. To show that \( H \oplus_2 K \) is 2-connected it is necessary and sufficient to show that given any two edges \( e \) and \( f \) in \( H \oplus_2 K \), there is a cycle in \( H \oplus_2 K \) containing them. Since \( H \) and \( K \) can be 2-summed, there is a pair of edges \( h \in E(H) \) and \( k \in E(K) \) such that \( s(h) = s(k) \). Since each of \( H \) and \( K \) is 2-connected, neither \( h \) nor \( k \) is a loop; hence, \( H \setminus h \cong (H \oplus_2 K)[E(H \setminus h)] \) and \( K \setminus k \cong (H \oplus_2 K)[E(K \setminus k)] \). Since \( H \setminus h \) and \( K \setminus k \) are isomorphic to subgraphs of \( H \oplus_2 K \), it follows that if there is a cycle \( C \) in \( H \setminus h \) or in \( K \setminus k \) containing both \( e \) and \( f \), then \( C \) is a cycle in \( H \oplus_2 K \).

To prove the remaining cases, observe that if \( h \) is contained in a cycle \( C_H \) in \( H \) and \( k \) is contained in a cycle \( C_K \) in \( K \), then \( (E(C_H) - h) \cup (E(C_K) - k) \) is the edge set of a cycle in \( H \oplus_2 K \). If both \( e \) and \( f \) are in \( E(H) - h \) or in \( E(K) - k \) (without
loss of generality, \( \{e, f\} \subseteq E(H) - h \), but there is no cycle in \( H \setminus h \) that contains both \( e \) and \( f \), then there is a cycle \( C_H \) in \( H \) that contains \( e, f, \) and \( h \). Since \( K \) is 2-connected, there is a cycle \( C_K \) of length at least 2 that contains \( k \). It follows that \( e \) and \( f \) are contained in \((E(C_H) - h) \cup (E(C_K) - k)\), which is the edge set of a cycle in \( H \oplus_2 K \), as noted above. By symmetry, it remains to examine the case when \( e \in E(H) - h \) and \( f \in E(K) - k \). Then, since \( H \) and \( K \) are 2-connected, there are cycles \( C_H \) in \( H \) containing \( \{e, h\} \) and \( C_K \) in \( K \) containing \( \{f, k\} \). Again, \( e \) and \( f \) are contained in \((E(C_H) - h) \cup (E(C_K) - k)\), which is the edge set of a cycle in \( H \oplus_2 K \). □

We have noted that edge-summing is commutative. In general, however, edge-summing is not associative, but there is "conditional" associativity. The condition that we must impose is that if \( H, J, \) and \( K \) are pairwise disjoint graphs with directed labelings \( L_H, L_J, \) and \( L_K \), respectively, then exactly two elements of \( \{H \oplus_2 J, H \oplus_2 K, J \oplus_2 K\} \) are defined. For example, assume that \( H \oplus_2 J \) and \( J \oplus_2 K \) are defined. Then there are distinct edges \( h \) in \( H \), \( j_1 \) and \( j_2 \) in \( J \), and \( k \) in \( K \) whose labels \( s(h), s(j_1), s(j_2), \) and \( s(k) \), respectively, satisfy \( s(h) = s(j_1) \neq s(j_2) = s(k) \neq s(h) \). It follows that \((H \oplus_2 J) \oplus_2 K \) and \( H \oplus_2 (J \oplus_2 K) \) are defined, and it is straightforward to show that \((H \oplus_2 J) \oplus_2 K = H \oplus_2 (J \oplus_2 K) \).

Additionally, this condition implies that if \((H \oplus_2 J) \oplus_2 K \) is defined, then exactly one of \( H \oplus_2 (J \oplus_2 K) \) or \( J \oplus_2 (H \oplus_2 K) \) is defined and is equal to \((H \oplus_2 J) \oplus_2 K \).

Given a collection of pairwise disjoint graphs \( G \) on which we want to perform edge-sums, it is convenient to use a tree \( T \) whose vertex set corresponds to \( G \) and whose edge set corresponds to a subset of the set of labels used in the directed labelings of the elements of \( G \). To avoid confusion between vertices and edges of elements of \( G \) and those of \( T \), we shall call elements of \( V(T) \) nodes and elements of \( E(T) \) links; moreover, Greek letters will be used to denote nodes and links of \( T \).
and Roman letters will be used to denote vertices and edges of elements of $G$. We describe this correspondence between $G$ and $T$ more precisely as follows.

(4.1.2) **DEFINITION.** Let $G = \{G_i : i \in [n]\}$ be a collection of pairwise disjoint graphs, let $L_G = \{L_{G_i} : i \in [n]\}$ be a collection of directed labelings of the elements of $G$, and let $T$ be a tree on the node set $\{\xi_i : i \in [n]\}$, where $n$ is a nonnegative integer. Then $T = (G, L_G, T)$ is an **edge-sum tree** if and only if the following hold.

(i) If $e = \xi_i \xi_j \in E(T)$, then there are precisely two graphs of $G$, namely $G_i$ and $G_j$, each containing an edge labeled $e$.

(ii) If $G_i \in G$ has an edge labeled $e$, then there is exactly one other graph $G_j \in G$ in that has an edge labeled $e$; moreover, $\xi_i \xi_j \in E(T)$.

It will be useful to look at a more general kind of tree structure (that includes the edge-sum trees) obtained by relaxing condition (ii). Call $T = (G, L_G, T)$ a **labeled edge-sum tree** if and only if $G$, $L_G$, and $T$ are as above, and $T$ satisfies condition (i) above and condition (ii)' below.

(ii)' If $G_i \in G$ has an edge labeled $e$, then there is at most one other graph $G_j \in G$ that has an edge labeled $e$, and if there is such a $G_j$, then $\xi_i \xi_j \in E(T)$.

If $T = (G, L_G, T)$ is a labeled edge-sum tree, then call the elements of $G$ the **node graphs** of $T$, call $L_G$ the directed labeling of $T$, and call $T$ the **tree** of $T$. \qed

Given an edge-sum tree $T = (G, L_G, T)$ and a subtree $T'$ of $T$, we can form the edge-sum tree $T' = (G', L_{G'}, T')$, where $G'$ is the subcollection of $G$ corresponding to $V(T')$, by restricting the directed labeling associated with each element of $G'$ in the appropriate way (that is, for each $G_i \in G'$, there is an edge of $G_i$ labeled $e$ if and only if $e \in E(T')$ and $\xi_i$ is a vertex of $e$). We shall say that $T'$ is a **restriction** of $T$ and that $T'$ is the **restriction** of $T$ induced by the subtree $T'$ of $T$. In particular, if the subtree $T'$ is obtained by deleting a leaf $\xi$ from $T$, then we shall say that $T'$ is obtained by **deleting $\xi$ from $T$** and let $T - \xi$ denote $T'$. 

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A basic operation that we shall perform on a labeled edge-sum tree is forming its composition, which we define as follows. Given a labeled edge-sum tree $T = (G, L_g, T)$, we can obtain a graph $G(T)$ (with a directed labeling, that is, perhaps, trivial) called the composition of $T$, by edge-summing as dictated by the links of $T$ in the following manner. If $T$ has no links, then $T$ consists of a single node, $G$ contains exactly one element, namely $G_0$, and there is nothing to do; hence $G(T) = G_0$, and the edges of $G(T)$ are assigned labels and directions according to $L_{G_0}$. Inductively, if $E(T)$ is non-empty and $e = \xi_i \xi_j$ is a link of $T$, then form $T' = (G', L_{g'}, T')$, where $G'$ is obtained from $G$ by replacing $G_i$ and $G_j$ with their edge-sum, $L_{g'}$ is obtained from $L_g$ by replacing $L_{G_i}$ and $L_{G_j}$ with the directed labeling $L_{G_i \oplus G_j}$ inherited from $L_{G_i}$ and $L_{G_j}$, and $T'$ is obtained from $T$ by contracting $e$ to a node $\xi$ that corresponds to $G_i \oplus G_j$. Let us say that $T'$ is obtained from $T$ by contracting $e$ in $T$, and let $T/e$ denote $T'$. It is clear that $T'$ is a labeled edge-sum tree. In particular, if $T$ is an edge-sum tree, then $T'$ is an edge-sum tree, and it follows that $G(T)$ is unlabeled. In general, when the directed labeling $L_g$ of a labeled edge-sum tree $T = (G, L_g, T)$ is understood, we shall let $(G, T)$ denote $T$. Also, we shall not indicate when edges of node graphs and compositions are assigned labels and directions except as needed.

It follows from the definition of the composition of a labeled edge-sum tree $T = (G, T)$ that there is a sequence $(T_i)_{i=0}^n$ of labeled edge-sum trees where $T$ has $n$ links, $T_0 = T$, and $T_i$ is obtained by contracting a link in $T_{i-1}$, for each $i \in [n]_+$; it follows that $T_n = (G(T), K_1)$. Call each $T_i$ in the above sequence a partial composition of $T$, and if $i \in [n-1]_+$, then the partial composition $T_i$ is proper. Such a sequence of partial compositions determines a natural way to edge-sum the elements of $G$.

Figure 4.2 shows an edge-sum tree $T$ and its composition $G(T)$. The nodes of the tree $T$ of $T$ are indicated by the ovals, and the line segments that connect the
ovals are the links of $T$. Each node graph of $T$ is drawn inside its corresponding oval. The directed labeling of $T$ assigns labels and directions to edges of the node graphs, as indicated. It follows that the line segment that connects the two nodes of $T$ whose node graphs each contain an edge labeled $e_i$ is the link $e_i$. The edges of $G(T)$ are the solid edges. For each $i \in [6]_+$, the dotted line segment labeled $i$ shows where two node graphs were edge-summed along the two edges labeled $e_i$ (but it is not an edge of $G(T)$).

![An edge-sum tree $T$ and its composition $G(T)$.

The terminology has been referring to the composition (rather than a composition) $G(T)$ of a labeled edge-sum tree $T$. So we must show that any composition of $T$ results in a unique graph $G(T)$.

(4.1.3) **Proposition.** If $T = (G, T)$ is a labeled edge-sum tree, and $G$ and $G'$ are compositions of $T$, then $G = G'$. 

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Proof. We may assume that $E(T) = \{\epsilon_i : i \in [n]_+\}$, for some integer $n \geq 2$, for there is nothing to prove if $T$ has fewer than two links. By an appropriate naming of the links of $T$, we may assume that $T[\{\epsilon_j : j \in [i]_+\}]$ is connected for each $i \in [n]_+$; further, we may assume that, for each $i \in [n]_+$, one end of $\epsilon_i$ is $\xi_i$ and the other end lies in $\{\xi_j : j \in [i-1]\}$. It follows that $\xi_n$ is a leaf of $T$ and that $G = ((\cdots ((G_0 \oplus_2 G_1) \oplus_2 G_2) \oplus_2 \cdots) \oplus_2 G_{n-1}) \oplus_2 G_n$ is a composition of $T$.

Assume that the result holds for all labeled edge-sum trees with fewer than $n$ links. Let $G'$ be an arbitrary composition of $T$ where $\mathcal{T}_{n-1} = (\{H, K\}, \epsilon_i)$ is the last proper partial composition in the corresponding sequence of partial compositions. Deleting $\epsilon_i$ from $T$ results in two trees $T_H$ and $T_K$ such that if $\mathcal{G}_H$ and $\mathcal{G}_K$ are the subcollections of $\mathcal{G}$ corresponding to $V(T_H)$ and $V(T_K)$, respectively, then, by the induction hypothesis, the compositions of $\mathcal{T}_H = (\mathcal{G}_H, T_H)$ and $\mathcal{T}_K = (\mathcal{G}_K, T_K)$ are $H$ and $K$, respectively. Since $\mathcal{G}_H$ and $\mathcal{G}_K$ partition $\mathcal{G}$, exactly one of $\mathcal{G}_H$ and $\mathcal{G}_K$ contains $G_n$; without loss of generality, assume that $G_n \in \mathcal{G}_K$.

First assume that $i < n$. By the induction hypothesis, we may compose $\mathcal{T}_K$ in any way; in particular, we may compose $\mathcal{T}_K$ so that the last proper partial composition is $\{\{K', G_n\}, \epsilon_n\}$, where $K' \oplus_2 G_n = K$. Hence, $G' = H \oplus_2 (K' \oplus_2 G_n)$; moreover, condition (ii)' in (4.1.2) implies that $G' = (H \oplus_2 K') \oplus_2 G_n$. Again by the induction hypothesis, since $((\cdots ((G_0 \oplus_2 G_1) \oplus_2 G_2) \oplus_2 \cdots) \oplus_2 G_{n-2}) \oplus_2 G_{n-1}$ and $H \oplus_2 K'$ are compositions of $T - \xi_n$, they are equal; it easily follows that $G = G'$.

Now, assume that $i = n$. Then $K = G_n$ and $H \oplus_2 G_n$ is a composition of $T$. It follows from the induction hypothesis that $H = ((\cdots ((G_0 \oplus_2 G_1) \oplus_2 G_2) \oplus_2 \cdots) \oplus_2 G_{n-2}) \oplus_2 G_{n-1}$; hence, $G = G'$. \[\square\]

Let $\mathcal{T} = (\mathcal{G}, L_G, T)$ and $\mathcal{T}' = (\mathcal{G}, L'_G, T)$ be directed edge-sum trees such that $L'_G$ is obtained from $L_G$ by a sequence of pair direction reversals. Let us say that $\mathcal{T}$ and $\mathcal{T}'$ are equivalent; it is easy to see that $G(\mathcal{T})$ and $G(\mathcal{T}')$ are the same.
In general, we are more interested in edge-sum trees than in partial edge-sum trees. We now look at some special kinds of edge-sum trees.

(4.1.4) Definition. If each element of $\mathcal{G}$ is 2-connected, then we shall call an edge-sum tree $T = (\mathcal{G}, T)$ a block tree. The next important kind of edge-sum tree, namely a 3-block tree, due to Tutte, requires the following terminology. A 3-block is a simple 3-connected graph, a cycle with at least 3 edges, or a multi-edge with at least 3 edges. A 3-block tree is an edge-sum tree $T = (\mathcal{G}, T)$ such that each element of $\mathcal{G}$ is a 3-block and such that if $\xi_i \xi_j \in E(T)$, then $\mathcal{G}_i$ and $\mathcal{G}_j$ are not both cycles and not both multi-edges. □

Obviously, a 3-block tree is a block tree. Let us note that the edge-sum tree $T$ that we saw in Figure 4.2 is a block tree, but it is not a 3-block tree. It follows easily from the above proposition and (4.1.1) that composing a block tree produces a unique (unlabeled) 2-connected graph. A natural question to ask is whether a 2-connected graph has a unique decomposition into some kind of block tree. It is clear that, in general, a 2-connected graph cannot be decomposed into a unique block tree since each partial composition of a block tree is a block tree, but Tutte has proved in [T] the following important relationship, which we state without proof, between 2-connected graphs and 3-block trees.

(4.1.5) Theorem (Tutte). If $G$ is a 2-connected graph containing at least three edges, then it can be decomposed into a 3-block tree; moreover, this decomposition is unique (up to equivalence of 3-block trees). □

For brevity, let us speak of the 3-block tree of a 2-connected graph $G$ rather than the class of equivalent 3-block trees of $G$. Next, we shall prove a useful lemma regarding the composition of a special kind of restriction of an edge-sum tree.
(4.1.6) Lemma. If \( T = (G, T) \) is an edge-sum tree and \( T' = (G', T') \) is a restriction of \( T \) so that, for each node \( \xi_j \) in \( V(T) - V(T') \), the corresponding node graph \( G_j \) is 2-connected, then \( G(T') \preceq_m G(T) \).

Proof. We show that the hypotheses imply a stronger conclusion, namely \( G(T') \preceq_t G(T) \). We may assume that \( T' = T - \xi_j \), where \( \xi_j \) is a leaf of \( T \) whose corresponding node graph \( G_j \) is 2-connected, since any subtree \( T' \) can be obtained from \( T \) by deleting leaves and since the taking of restrictions of edge-sum trees and the \( \preceq_t \) relation on graphs are transitive. Let \( \epsilon = \xi_i \xi_j \) denote the link of \( T \) incident with \( \xi_j \). Then \( (\{H, G_j\}, \epsilon) \) is a partial composition of \( T \), where \( H \), viewed as an unlabeled graph, is isomorphic to \( G(T') \). In this partial composition, each of \( H \) and \( G_j \) has an edge \( h \) and \( g \), respectively, labeled \( \epsilon \). Since \( G_j \) is 2-connected, there is a cycle \( C \) of length at least 2 that contains \( g \). It follows that \( H \oplus_2 C \preceq_t H \oplus_2 G_j = G(T) \). Note that \( H \oplus_2 C \) is isomorphic to the unlabeled graph obtained from \( H \) by subdividing \( h \) with \( |C| - 2 \) new vertices; hence, \( H \preceq_t H \oplus_2 C \). Since \( H \cong G(T') \), it follows that \( G(T') \preceq_t G(T) \). □

Next, we prove two results that indicate sufficient conditions for a 2-connected graph to have a large fan minor. In proving these results, we will make use of the 3-block tree of a 2-connected graph and (4.1.6). The first result, an immediate consequence of (4.1.6) and the proof of (3.2.1), shows that if some 3-connected node graph of the 3-block tree of a 2-connected graph \( G \) contains a long cycle, then a large wheel is a minor of \( G \). The second result, which is the subject of the next section, shows that if the tree of the 3-block tree of a 2-connected graph \( G \) has a long path, then a large fan is a minor of \( G \).

(4.1.7) Proposition. Let \( G \) be a 2-connected graph, and let \( T = (G, T) \) be its 3-block tree. If some element \( H \in G \) is a 3-connected graph that has a cycle of length at least \( N \), where \( N \) is the number from (3.2.8), then \( W_n \preceq_m G \) (hence, \( F_n \preceq_m G \)).
**Proof.** Let \( H \in \mathcal{G} \) satisfy the hypotheses of the proposition, and let \( \xi \) be the node of \( T \) corresponding to \( H \). Then the composition of the restriction of \( T \) induced by the subtree of \( T \) consisting of the single vertex \( \xi \) is \( H \). So \( H \leq_m G \), by (4.1.6). Since \( H \) is a 3-connected graph containing a cycle of length at least \( N \), it follows from the proof of (3.2.1) that \( W_n \leq_m H \); consequently, \( W_n \leq_m G \). \( \square \)

### 4.2 A Long Path in a 3-Block Tree

The following is the main result of this section.

(4.2.1) **Theorem.** Let \( G \) be a 2-connected graph with at least three edges, and let \( T = (\mathcal{G}, T) \) be its 3-block tree. If \( n \) is a positive integer, and \( T \) contains a path of length at least \( 4(n - 1) + 1 \) as a subgraph, then \( F_n \leq_m G \).

Before proving (4.2.1), we state (without proof) a matroid result of Seymour [S] which, when restricted to graphs, yields the corollary immediately following it, that we will use in the proof of (4.2.1).

(4.2.2) **Theorem (Seymour).** If \( M \) is a 3-connected matroid that has a minor in \( \mathcal{F} = \{U_{2,4}, M(K_4)\} \), and \( X \) is any subset of \( E(M) \) that has at most two elements, then \( M \) has a minor in \( \mathcal{F} \) using \( X \). \( \square \)

(4.2.3) **Corollary.** If \( G \) is a simple 3-connected graph, and \( e \) and \( f \) are edges of \( G \), then there is a \( K_4 \)-minor of \( G \) that uses \( e \) and \( f \).

Before proving (4.2.3), we shall show that each 3-connected graph contains a \( K_4 \)-minor. This follows easily from Tutte's Wheels and Whirls Theorem ([O], Theorem 8.4.5). For completeness, we present an elementary proof that \( K_4 \) is a minor of any 3-connected graph.

(4.2.4) **Lemma.** Each 3-connected graph contains a \( K_4 \)-minor.

**Proof.** Clearly, it suffices to show that \( K_4 \) is a minor of the simplification of a 3-connected graph \( G \). So we may assume that \( G \) is simple. Since \( G \) is 3-connected
(hence 2-connected), it contains a cycle $C$ of length at least 3. Next, we examine the bridges of $C$ in $G$.

If some bridge $H$ meets at least three vertices $u$, $v$, and $w$ of $C$, then $V(H) - V(C)$ is non-empty; in particular, $V(H) - V(C)$ contains (not necessarily distinct) vertices $u'$, $v'$, and $w'$ that are adjacent to $u$, $v$, and $w$, respectively. Since $H[V(H) - V(C)]$ is connected, it contains a smallest connected subgraph $T$ that contains $u'$, $v'$, and $w'$; note that $T$ is a tree whose leaves are contained in $\{u', v', w'\}$. It follows that $C \cup T \cup \{uu', vv', ww'\}$ is a subdivision of $K_4$; thus, $K_4 \leq_m G$.

Clearly, no bridge of $C$ in $G$ has only one vertex of attachment, because if some bridge had only one vertex $v$ of attachment, then the bridge would be a loop, or $v$ would be a cut-vertex, which would contradict the fact that $G$ is a simple 3-connected graph. So it remains to consider the case in which each bridge of $C$ in $G$ has exactly two vertices of attachment. We can see that each bridge consists of a single edge as follows. Suppose that some bridge $H$ of $C$ in $G$ contains more than one edge. Then $V(H) - V(C)$ is non-empty. It follows that the two vertices of attachment of $H$ form a vertex-cut, which contradicts the 3-connectedness of $G$. Hence, each bridge consists of a single edge, and, since $G$ is simple and 3-connected, $C$ has at least four vertices and at least two bridges.

Assume that $V(C) = \{v_i : i \in [n]\}$ and $E(C) = \{v_0v_1, v_1v_2, \ldots, v_{n-1}v_n, v_nv_0\}$ for some positive integer $n$ exceeding 2. Among all bridges of $C$ in $G$, choose a bridge $e$ for which the distance in $C$ between its endvertices is minimal. If $i$ denotes this minimum distance, then, by shifting the indices of the vertices of $C$, we may assume that $e = v_0v_i$. Since $G$ is simple, $i > 1$. Since $G$ is 3-connected, there is a bridge $f$ that meets $v_1$. Suppose that the other vertex of attachment of $f$ is $v_j$. Then $j \notin [i]$; otherwise, we would have chosen $f$ over $e$; hence, $j > i$. It follows that the subgraph $C \cup \{e, f\}$ of $G$ is as in Figure 4.3 below. Clearly, $C \cup \{e, f\}$ is a subdivision of $K_4$; hence, $K_4 \leq_m G$. \qed
Now, let us prove (4.2.3).

Proof of (4.2.3). Assume that $G$ is a simple 3-connected graph. Since $U_{2,4}$ is not a graphic matroid, $U_{2,4}$ cannot be a minor of $M(G)$. By (4.2.4), $G$ has a $K_4$-minor. Since $G$ is a simple 3-connected graph, the cycle matroid $M(G)$ of $G$ is a 3-connected matroid. It follows from (4.2.2) that if $\{e, f\} \in E(G)$, then $M(G)$ has a $M(K_4)$-minor that uses $e$ and $f$. Since $M(K_4)$ is 3-connected, it follows from (3.2.10) that $K_4$ is the only graph that has $M(K_4)$ as its cycle matroid; consequently, $G$ contains a $K_4$-minor using $e$ and $f$. □

Now, we state and prove two lemmas that will be useful in proving (4.2.1). The second of these lemmas, (4.2.7), will also be useful in proving many of the results in Section 4.4.

(4.2.5) Lemma. If $T = (G, L_G, T)$ is an edge-sum tree, and $T' = (G, L_G', T')$ is an edge-sum tree obtained from $T$ by reversing the directions of some of the labeled edges of elements of $G$, then $G(T) \cong_2 G(T')$.

Proof. Assume that $L_G$ and $L_G'$ are different, and let $n$ be the number of labeled edges whose directions assigned by $L_G'$ differ from those assigned by $L_G$. First, assume that $n = 1$. Let $k$ be the labeled edge that is assigned different directions by $L_G$ and $L_G'$, and let $K$ be the node graph that contains $k$. If the label of $k$ is $e$, then there is exactly one other edge $h$ in some other node graph $H$ that is labeled $e$, and $L_G$ and $L_G'$ assign the same direction to $h$. Consider the partial compositions $T_1 = T/(E(T) - e)$ and $T'_1 = T'/(E(T) - e)$. The set of node graphs of $T_1$ and

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.3.png}
\caption{The subgraph $C \cup \{e, f\}$ of $G$.}
\end{figure}
$T'_1$ consists of the two graphs $G_h$, which contains $h$, and $G_k$, which contains $k$. Let $L_1$ and $L'_1$ denote the directed labelings of $T_1$ and $T'_1$, respectively. If $h$ is a loop in $G_h$, or if $k$ is a loop in $G_k$, then $T_1/\epsilon = T'_1/\epsilon$, and since $T_1/\epsilon = G(T)$ and $T'_1/\epsilon = G(T')$, it is evident that $G(T) \cong_2 G(T')$. So assume that $h$ is not a loop in $G_h$, and $k$ is not a loop in $G_k$. Let $L_1$ and $L'_1$ denote the directed labelings of $T_1$ and $T'_1$, respectively, of $h$ assigned by $L_1$ and $L'_1$, and let $u_k$ and $v_k$ denote the head and tail, respectively, of $k$ assigned by $L'_1$; it follows that $u_k$ and $v_k$ denote the head and tail, respectively, of $k$ assigned by $L'_1$. Then $T_1/\epsilon$ (that is, $G(T)$) is obtained by identifying $h$ and $k$ so that $u_h$ and $u_k$ are identified to a new vertex $u$, and $v_h$ and $v_k$ are identified to a new vertex $v$, and then deleting the identified edge. Similarly, $T'_1/\epsilon$ (that is, $G(T')$) is obtained by identifying $h$ and $k$ so that $u_h$ and $v_k$ are identified to a new vertex $u'$, and $v_h$ and $u_k$ are identified to a new vertex $v'$, and then deleting the identified edge. It is straightforward that $G(T')$ is a twisting of $G(T)$ about $\{u,v\}$; hence, $G(T) \cong_2 G(T')$.

Now assume that $n$ is an integer exceeding 1 and that the result holds for all edge-sum trees $U = (H, L_H, U)$ and $U' = (H, L'_H, U)$ that have fewer than $n$ labeled edges whose directions assigned by $L_H$ differ from those assigned by $L'_H$. Let $\{e_i : i \in [n]_+\}$ be the set of edges in $\bigcup_{G \in \mathcal{G}} E(G)$ whose directions assigned by $L'_G$ differ from those assigned by $L_G$. Let $T'' = (G, L''_G, T)$ be the edge-sum tree obtained by reversing the direction of $e_i$ assigned by $L''_G$, for each $i \in [n-1]_+$. By the induction hypothesis, $G(T') \cong_2 G(T'')$, and it follows that $e_n$ is the only labeled edge that is assigned different directions by $L''_G$ and $L''_G$. Again, by the induction hypothesis, $G(T) \cong_2 G(T'')$. It is evident that $\cong_2$ is an equivalence relation; hence, $G(T) \cong_2 G(T')$. □

Before stating (4.2.7), let us consider a particular partial composition $T_{\#}$ of an edge-sum tree $T = (G, L_G, T)$ that we shall use in proving (4.2.7) and several results in Section 4.4.
(4.2.6) **DEFINITION.** Let $\mathcal{T} = (\mathcal{G}, L_\mathcal{G}, T)$ be an edge-sum tree for which $|\mathcal{G}| > 1$, let $H$ be a specified node graph in $\mathcal{G}$, and let $\xi$ be the node that corresponds to $H$. Let the positive integer $m$ denote the number of links in $T$ incident with $\xi$, and let $\{\epsilon_i : i \in \{m\}_+\}$ denote the set of links adjacent to $\xi$ in $T$. Then the *star of $\mathcal{T}$ (at $H$)*, denoted $\mathcal{T}_{st}$, is the partial composition $\mathcal{T}/(E(T) - \{\epsilon_i : i \in \{m\}_+\})$ of $\mathcal{T}$. We now define some additional notation regarding $\mathcal{T}$ and $\mathcal{T}_{st}$, for each $i \in \{m\}_+$. Let $h_i$ be the edge of $H$ that is labeled $\epsilon_i$. Let $\xi_i$ be the endnode of $\epsilon_i$ in $T$ that is not $\xi$, let $H_i$ be the node graph of $\mathcal{T}$ corresponding to $\xi_i$, and let $k_i$ be the edge of $H_i$ that is labeled $\epsilon_i$. Let $\mathcal{T}_i$ be the restriction of $\mathcal{T}$ induced by the component $T_i$ of $T\setminus \epsilon_i$ containing $\xi_i$, and let $K^i = G(\mathcal{T}_i)$.

It is straightforward that the set of node graphs of $\mathcal{T}_{st}$ is $\{H\} \cup \{K^i : i \in \{m\}_+\}$, where $H$ is labeled as it is in $\mathcal{T}$, and where $K^i$ has exactly one labeled edge, namely $k_i$, for each $i \in \{m\}_+$. □

(4.2.7) **LEMMA.** Let $\mathcal{T} = (\mathcal{G}, L_\mathcal{G}, T)$ be an edge-sum tree, let $D$ and $C$ be disjoint subsets of $E(G(T))$, and let $\mathcal{T}\setminus D/C$ denote the edge-sum tree obtained by replacing each node graph $H \in \mathcal{G}$ with $H' = H \setminus (E(H) \cap D)/(E(H) \cap C)$, and by replacing each directed labeling $L_H$ in $L_\mathcal{G}$ with the directed labeling $L_{H'}$ of $H'$ induced by $L_H$. Then $G(\mathcal{T}\setminus D/C) = G(\mathcal{T})\setminus D/C$.

**Proof.** It is straightforward that it is sufficient to show that $G(\mathcal{T}\setminus e) = G(\mathcal{T})\setminus e$ and that $G(\mathcal{T}/e) = G(\mathcal{T})/e$, where $e$ is an unlabeled edge in some node graph $H \in \mathcal{G}$. If $\mathcal{G} = \{H\}$, then there is nothing to show since $G(\mathcal{T}\setminus e) = H\setminus e = G(\mathcal{T})\setminus e$ and $G(\mathcal{T}/e) = H/e = G(\mathcal{T})/e$. Now, we may assume that $|\mathcal{G}|$ exceeds 1. Let $\mathcal{T}_{st}$ denote the star of $\mathcal{T}$ at $H$ as described in (4.2.6). Recall that the set of node graphs of $\mathcal{T}_{st}$ is $\{H\} \cup \{K^i : i \in \{m\}_+\}$, where $K^i$ has an edge $k_i$ labeled $\epsilon_i$ for each $i \in \{m\}_+$. Note that $G(\mathcal{T}) = ((\cdots ((H \oplus_2 K^1) \oplus_2 K^2) \oplus_2 \cdots) \oplus_2 K^{m-1}) \oplus_2 K^m$. It follows that $G(\mathcal{T}\setminus e) = ((\cdots ((H\setminus e \oplus_2 K^1) \oplus_2 K^2) \oplus_2 \cdots) \oplus_2 K^{m-1}) \oplus_2 K^m$ and
\[ G(T/e) = ((\cdots ((H/e \oplus_2 K^1) \oplus_2 K^2) \oplus_2 \cdots) \oplus_2 K^{m-1}) \oplus_2 K^m. \]

We proceed by induction on \( m \).

If \( m = 1 \), then \( G(T) = H \oplus_2 K^1 \), and \( h_1 \in E(H) \) and \( k_1 \in E(K^1) \) are the edges along which \( H \) and \( K^1 \) are 2-summed. If \( h_1 \) or \( k_1 \) is a loop, then it is straightforward that \( H \setminus e \oplus_2 K^1 = (H \oplus_2 K^1) \setminus e \) and \( H/e \oplus_2 K^1 = (H \oplus_2 K^1)/e \). Now, assume that neither \( h_1 \) nor \( k_1 \) is a loop. Then it is straightforward that \( H \setminus e \oplus_2 K^1 = (H \oplus_2 K^1) \setminus e \). If \( e \) and \( h_1 \) are not parallel, then it is also straightforward that \( H/e \oplus_2 K^1 = (H \oplus_2 K^1)/e \). So assume that \( e \) has the same set of endvertices \( \{u, v\} \) as \( h_1 \). It follows that \( h_1 \) is a loop in \( H/e \) with endvertex \( v_h \), and \( H/e \oplus_2 K^1 \) is obtained by contracting \( h_1 \) to \( v_h \) and \( k_1 \) to a vertex \( v_k \), and then identifying \( v_h \) and \( v_k \) to a new vertex. \( H \oplus_2 K^1 \) is obtained by identifying \( h_1 \) and \( k_1 \) head-to-head and tail-to-tail, and then deleting the identified edge. (Let the two identified vertices of \( H \oplus_2 K^1 \) be called \( u \) and \( v \), as in \( H \).) Note that \( \{u, v\} \) is the intersection of \( H \setminus h \) and \( K^1 \setminus k \), viewed as subgraphs of \( H \oplus_2 K^1 \). On contracting \( e \) in \( H \oplus_2 K^1 \), \( u \) and \( v \) are identified to a single vertex, and it is straightforward that the resulting graph is identical to \( H/e \oplus_2 K^1 \). Hence, we obtain the desired result when \( m = 1 \).

Now, assume that \( m > 1 \) and that \((\cdots ((H/e \oplus_2 K^1) \oplus_2 K^2) \oplus_2 \cdots) \oplus_2 K^{l-1}) \oplus_2 K^l = ((\cdots ((H \oplus_2 K^1) \oplus_2 K^2) \oplus_2 \cdots) \oplus_2 K^{l-1}) \oplus_2 K^l = ((\cdots ((H/e \oplus_2 K^1) \oplus_2 K^2) \oplus_2 \cdots) \oplus_2 K^l)/e \) if \( l < m \). Then \((\cdots ((H/e \oplus_2 K^1) \oplus_2 K^2) \oplus_2 \cdots) \oplus_2 K^{m-1}) \oplus_2 K^m = ((\cdots ((H \oplus_2 K^1) \oplus_2 K^2) \oplus_2 \cdots) \oplus_2 K^{m-1}) \oplus_2 K^m \) by the induction hypothesis; again, by the induction hypothesis, \((\cdots ((H \oplus_2 K^1) \oplus_2 K^2) \oplus_2 \cdots) \oplus_2 K^{m-1}) \setminus e \oplus_2 K^m = ((\cdots ((H \oplus_2 K^1) \oplus_2 K^2) \oplus_2 \cdots) \oplus_2 K^{m-1}) \setminus e \oplus_2 K^m \) and \((\cdots ((H/e \oplus_2 K^1) \oplus_2 K^2) \oplus_2 \cdots) \oplus_2 K^{m-1}) \setminus e \oplus_2 K^m = ((\cdots ((H/e \oplus_2 K^1) \oplus_2 K^2) \oplus_2 \cdots) \oplus_2 K^{m-1}) \setminus e \oplus_2 K^m \) by the induction hypothesis; thus, \( G(T \setminus e) = G(T) \setminus e \) and \( G(T/e) = G(T)/e \). \( \square \)
Now, we are finally ready to present the proof of (4.2.1), the main result of this section.

*Proof of (4.2.1).* Assume that $T$ contains a subtree $P_0$ that is a path of length at least $4(n - 1) + 1$. If each of the elements of $G$ corresponding to the endnodes of $P_0$ is a multi-edge, then let $P$ be a subpath of $P_0$ obtained by deleting an endnode from $P_0$; it follows from the definition of a 3-block tree in (4.1.4) that one endnode of $P$ corresponds to either a 3-connected graph or a cycle. If one of the elements of $G$ corresponding to the endnodes of $P_0$ is not a multi-edge, then let $P = P_0$. In either case, $T$ contains a subpath $P$ of length $N$, for some integer $N \geq 4(n - 1)$, one endnode of which corresponds to a 3-connected graph or a cycle. Let $T' = (G', P)$ be the 3-block tree that is the restriction of $T$ induced by $P$, and let $G' = G(T')$. By (4.1.6), $G' \leq_m G$.

By renumbering indices we may assume that the node set of $P$ is $\{\xi_i : i \in [N]\}$, the link set of $P$ is $\{e_i = \xi_{i-1}\xi_i : i \in [N]_+\}$, and $G' = \{G_i : i \in [N]\}$, where $G_i$ is the 3-block corresponding to $\xi_i$, and $G_N$ is not a multi-edge. We now want to partition $G'$ into singletons and pairs as follows. Let $i$ be the largest index such that $G_i$ does not belong to a singleton or a pair of elements of $G'$. If $G_{i-1}$ is a multi-edge, then form the pair $\{G_{i-1}, G_i\}$; otherwise, form the singleton $\{G_i\}$. If all elements of $G'$ have not been placed in a singleton or pair, then repeat this process. It is straightforward that this process produces a partition $P(G')$ of $G'$ where each element of $P(G')$ is a singleton consisting of a cycle or a 3-connected graph, or a pair $\{G_{i-1}, G_i\}$ consisting of a multi-edge $G_{i-1}$ and a 3-block $G_i$ that is not a multi-edge. Since each element of $P(G')$ consists of at most two 3-blocks, $|P(G')| = N' + 1$ for some integer $N' \geq \lceil \frac{N+1}{2} \rceil - 1$. Note that it follows from the way that $P(G')$ is defined and from the fact that $G'$ is a 3-block tree that if $i \in [N]_+$, and $G_i$ is a cycle that makes up a singleton in $P(G')$, then $G_{i-1}$ is a 3-connected graph.
Note that \( E(P) \) is partitioned into sets \( E' \) and \( E'' \) of links such that if \( e' \in E' \), then the node graphs that contain an edge labeled \( e' \) are contained in different elements of \( \mathcal{P}(\mathcal{G}') \), and if \( e'' \in E'' \), then the node graphs that contain an edge labeled \( e'' \) form a pair in \( \mathcal{P}(\mathcal{G}') \).

Let \( \mathcal{T}'' \) be the block tree obtained by contracting \( E'' \) in \( \mathcal{T}' \); this amounts to 2-summing the elements of each pair in \( \mathcal{P}(\mathcal{G}') \) and contracting \( E'' \) in \( P \). It follows that \( P' = P/E'' \) is the tree of \( \mathcal{T}'' \), and \( \mathcal{G}'' = \{ G_i : \{ G_i \} \in \mathcal{P}(\mathcal{G}') \} \cup \{ G_i \oplus G_{i+1} : \{ G_i, G_{i+1} \} \in \mathcal{P}(\mathcal{G}') \} \) is the set of node graphs of \( \mathcal{T}'' \); furthermore, \( |\mathcal{G}'| = |\mathcal{P}(\mathcal{G}')| = N' + 1 \). Also, it is evident that \( G(\mathcal{T}''') = G(\mathcal{T}') = G' \) since \( \mathcal{T}''' \) is a partial composition of \( \mathcal{T}' \). Let \( G_0' \) denote the element of \( \mathcal{G}'' \) that is either \( G_N \) or \( G_{N-1} \oplus G_N \), let \( \xi_0' \) denote the node of \( P' \) corresponding to \( G_0' \) (hence, \( \xi_0' \) is an endnode of \( P' \)), and if \( E(P') \) is non-empty, then let \( e_1' \) denote the link of \( P' \) incident with \( \xi_0' \). If \( P' \) contains additional links, then let \( \{ e_i' \}_{i=2}^{N'} \) denote the remaining links of \( P' \) such that \( e_i' \) and \( e_{i+1}' \) are adjacent for each \( i \in [N' - 1]_+ \), and rename the nodes of \( P' \) and elements of \( \mathcal{G}'' \) such that, for each \( i \in [N']_+ \), the endnodes of the link \( e_i' \) are \( \xi_{i-1}' \) and \( \xi_i' \), and \( G_i' \) is the element of \( \mathcal{G}'' \) corresponding to \( \xi_i' \). It follows that if \( G_{i-1}' \) is a cycle, then \( G_i' \) is a 3-connected graph.

Note that each element of \( \mathcal{G}'' \) is a 3-block that is not a multi-edge, or it is the 2-sum of a 3-block that is a multi-edge and a 3-block that is not a multi-edge. It follows that all edges of \( G_0' \) are unlabeled except for one edge \( f_0 \) that is labeled \( e_1' \), and all edges of \( G_{N'}' \) are unlabeled except for one edge \( e_{N'} \) that is labeled \( e_{N'}' \). Furthermore, if \( G_0' \) is not a simple graph, then \( f_0 \) is contained in the proper multi-edge of \( G_0' \); also, if \( G_{N'}' \) is not a simple graph, then \( e_{N'} \) is a trivial multi-edge of \( G_{N'}' \). If \( i \in [N' - 1]_+ \), then all edges of \( G_i' \) are unlabeled except for an edge \( e_i \) that is labeled \( e_i' \) and an edge \( f_i \) that is labeled \( e_{i+1}' \); moreover, if \( G_i' \) is not simple, then \( e_i \) is a trivial multi-edge of \( G_i' \), and \( f_i \) belongs to the proper multi-edge of \( G_i' \). Let \( e_0 \) be an unlabeled trivial multi-edge of \( G_0' \), and let \( f_{N'} \) be an edge that is
contained in a largest unlabeled multi-edge of $G_{N'}$, so that each element $G_i$ of $G''$ has exactly two specified edges $e_i$ and $f_i$.

We shall show that, for each $i \in [N']$, the graph $G_i$ contains a particular minor isomorphic to one the three graphs in Figure 4.4 below. First, we shall show that if $G_i$ is a 3-block that is a cycle, then $D_i \leq_m G_i$. Then, we shall show that if $G_i$ is the 2-sum of a 3-block that is a cycle and a 3-block that is a multi-edge, then $D_i \leq_m G_i$. For the remaining case, in which $G_i$ is a 3-connected graph, we shall show that $D_i'' \leq_m G_i$.

![Figure 4.4](image)

**Figure 4.4.** The graphs $D_i$, $D_i'$, and $D_i''$.

First, assume that $G_i$ is a 3-block that is a cycle. Since $G_i$ has at least three edges, $G_i \setminus \{e_i, f_i\}$ consists of a proper path $P_1$ and a (perhaps trivial) path $P_2$. By contracting, in $G_i$, the paths $P_1$ to a single edge and $P_2$ to a vertex, we obtain a graph $G_i''$ that is isomorphic to $D_i$.

Now, assume that $G_i$ is the 2-sum of a 3-block $C$ that is a cycle and a 3-block $C^*$ that is a multi-edge. Clearly, the simplification of $G_i$ is a cycle with at least three edges. As already mentioned, $e_i$ is a trivial multi-edge of $G_i$, and $f_i$ is contained in the proper multi-edge of $G_i$. As in the case in which $G_i$ is a cycle, the graph obtained by deleting the proper multi-edge and $e_i$ from $G_i$ consists of a proper path $P_1$ and a (perhaps trivial) path $P_2$. If we contract, in $G_i$, the paths $P_1$ to a single edge and $P_2$ to a vertex, then the resulting graph contains a subgraph $G_i''$ that is isomorphic to $D_i'$.

Finally, assume that $G_i$ is a 3-connected graph. If $G_i$ is not simple, then $e_i$ is a trivial multi-edge, and $f_i$ is contained in the proper multi-edge of $G_i$; hence, $e_i$
and \( f_i \) are not parallel. Consequently, if \( G'_i \) is not simple, then we may take the simplification \( \widetilde{G}_i \) of \( G'_i \) so that \( \{e_i, f_i\} \subseteq E(\widetilde{G}_i) \). By (4.2.3), \( \widetilde{G}_i \) has a \( K_4 \)-minor using \( e_i \) and \( f_i \). Thus, one of the two graphs in Figure 4.5 is a minor of \( \widetilde{G}_i \), and, by contracting the shaded edge in either graph, we obtain a graph \( G''_i \) that is isomorphic to \( D''_i \). Since \( \widetilde{G}_i \preceq G'_i \), it is clear that \( G''_i \cong D''_i \preceq G'_i \).

![Figure 4.5](image)

**Figure 4.5.** One of the above graphs is a minor of \( G'_i \).

Let \( T''' = (G''', P') \) be the block tree where \( G''' = \{G''_i : i \in [N']\} \) and \( G''_i \) is the graph that corresponds to the node \( x'_i \) of \( P' \). Since, for each \( i \in [N'] \), the graph \( G''_i \) is a minor of \( G'_i \) in which no labeled edges are contracted, it follows from (4.2.7) that the composition \( G'' \) of \( T''' \) is a minor of \( G' \); moreover, it follows from (4.1.1) that \( G'' \) is 2-connected (hence, loop-free). The next thing that we want to show is that \( G'' \) is a graph that is, in some sense, similar to a fan.

So far, we have been disregarding the directions assigned by the directed labeling of \( T''' \) to the labeled edges in the elements of \( G''' \). We consider these directions now. By performing the appropriate pair direction reversals, we may assume that, for each \( i \in [N' - 1] \), the edge \( f_i \) of \( G''_i \) is directed so that its head is incident with \( e_i \). Let \( T^* \) denote the block tree obtained from \( T''' \) by directing \( e_i \in E(G''_i) \) so that its head is incident with \( f_i \), for each \( i \in [N']_+ \). Since the simplification of each \( G''_i \) is a triangle (that is, a 3-cycle) for each \( i \in [N'], \) call the vertex common to \( e_i \) and \( f_i \) the point of \( G''_i \), and let \( g_i \) denote the edge of \( G''_i \) that is not adjacent to the point of \( G''_i \).

We want to show that the simplification of \( G'' \) is 2-isomorphic to \( F_{N'+2} \). By (4.2.5) it suffices to show that the simplification of \( G(T^*) \) is isomorphic to \( F_{N'+2} \).
Informally, in the composition of $G(T^*)$, the first node graph $G_0'$ contributes 2 to the size of the fan, and each additional node graph $G_i''$ contributes 1 to the size of the fan. Let us recall that if $G_{i-1}'$ is a cycle, then $G_i'$ is a 3-connected graph; it follows that if $G_{i-1}' = D_{i-1}$, then $G_i'' = D_i''$. It is straightforward that $\widetilde{G}(T^*) \cong F_{N'+2}$, given the way that the labeled edges of $T^*$ are directed and the fact that if $G_{i-1}' = D_{i-1}$, then $G_i'' = D_i''$. Hence, $\widetilde{G}'' \cong 2 F_{N'+2}$.

Next, we want to show that the $\left[\frac{N'+2}{2}\right]$-fan is a minor of $\widetilde{G}''$. Since $\widetilde{G}''$ is 2-isomorphic to $F_{N'+2}$ and 2-connected, $\widetilde{G}''$ can be obtained from a finite sequence of twistings of $F_{N'+2}$ about vertex-cuts of size 2. It is straightforward that $\widetilde{G}''$ is similar to a fan, where some of the triangles may point up and some may point down instead of all triangles pointing in the same direction. Figure 4.6 shows a typical graph that is 2-isomorphic to the 8-fan.

![Figure 4.6](image)

**Figure 4.6.** The above graph is 2-isomorphic (but not isomorphic) to $F_8$.

We show that the $\left[\frac{N'+2}{2}\right]$-fan is a minor of $\widetilde{G}''$ as follows. Informally, at least half of the triangles of $\widetilde{G}''$ point in the same direction (up or down). By contracting the appropriate edges, we obtain a graph whose simplification is a fan of size at least $\left[\frac{N'+2}{2}\right]$. For example, in Figure 4.6, where $N' = 6$ (hence, $\left[\frac{6+2}{2}\right] = 4$), if we contract $g_2$, $g_4$, and $g_5$, we obtain a graph that has an $F_4$-minor (and, in fact, an $F_5$-minor). We proceed to describe this more precisely.

Define the function $f : G'''' \to \{-1, 1\}$ as follows. Let $f(G_0'') = 1$, and inductively, for each $i \in [N']^+$, if the directed labeling of $T''''$ directs $e_i$ so that its head is the point of $G_i''$, then $f(G_i'') = f(G_{i-1}'')$; otherwise, $f(G_i'') = -f(G_{i-1}'')$. Informally, we shall say that the triangle of $\widetilde{G}''$ with base $g_i$ points up if $f(G_i'') = 1$ and...
points down if \( f(G'_{i}) = -1 \). It follows that if \( \sum_{i=0}^{N'} f(G'_{i}) \geq 0 \), then at least half of the triangles of \( G'' \) point up; otherwise, more than half of the triangles point down. If \( \sum_{i=0}^{N'} f(G'_{i}) \geq 0 \), then contract \( \{g_{i}: f(G'_{i}) = -1\} \) in \( G'' \); otherwise, contract \( \{g_{i}: f(G'_{i}) = 1\} \). It follows that the simplification of the resulting graph is isomorphic to a fan of size at least \( \left\lceil \frac{N'+2}{2} \right\rceil \). Hence, \( F_{n} \leq_{m} G'' \leq_{m} G' \), and

\[
\left\lceil \frac{N'+2}{2} \right\rceil \geq \left\lceil \frac{N'+1}{2} + 1 \right\rceil \geq \left\lceil \frac{4(n-1)+1}{2} + 1 \right\rceil = \left\lceil \frac{2(n-1)+2}{2} \right\rceil = n.
\]

Thus, \( F_{n} \leq_{m} G \). □

### 4.3 A Sufficient Condition for a \( C_{n,n}^{*} \)-Minor or a \( P_{n,n} \)-Minor

In this section we state and prove a lengthy lemma which states that if a graph \( G \) satisfies certain conditions that depend, in part, on an integer \( n \) exceeding 3, then an element of \( \{C_{n,n}^{*}, P_{n,n}\} \) is a minor of \( G \), where \( P_{n,n} \) is obtained from the path \( P_{n} \) on \( n \) edges by replacing each edge of \( P_{n} \) with a multi-edge of size \( n \). We shall see that the proof of this lemma uses the vertex form ([T], Theorem II.35) and the edge form ([BM], Theorem 11.5) of Menger’s Theorem, and the Pigeonhole Principle [B], which are stated below. The statement and proof of the lemma immediately follows these three results. The last result of this section is a corollary which applies the lemma to block trees so that if a block tree satisfies certain conditions that depend, in part, on an integer \( n \) exceeding 3, then its composition contains an element of \( \{C_{n,n-2}, C_{n,n-2}^{*}\} \) as a minor.

(4.3.1) **Menger’s Theorem (vertex form).** If \( x \) and \( y \) are distinct vertices of a graph \( G \), and a smallest \( xy \)-vertex-cut in \( G \) has size \( k \), then there are \( k \) pairwise internally vertex-disjoint \( xy \)-paths in \( G \). □

(4.3.2) **Menger’s Theorem (edge form).** If \( x \) and \( y \) are distinct vertices of a graph \( G \), and a smallest \( xy \)-edge-cut in \( G \) has size \( k \), then there are \( k \) pairwise edge-disjoint \( xy \)-paths in \( G \). □
(4.3.3) **Pigeonhole Principle.** Let \( \{P_i: i \in [m]\} \) be a partition of a set of cardinality \( m(n-1)+1 \), where \( m \) and \( n \) are positive integers. Then there is an \( i \in [m] \) such that \( |P_i| = n \). □

(4.3.4) **Lemma.** Let \( G \) be a graph with two specified vertices \( x \) and \( y \) such that \( G \cup e \) is 2-connected, where \( e \parallel xy \), and let \( n \) be an integer exceeding 3. If every \( xy \)-path in \( G \) has length at least \( n(n-1) \) and every \( xy \)-edge-cut in \( G \) has size at least \( n2^n \), then at least one of the following holds.

(i) \( C^*_{n,n} \leq_m G \), and the vertices of \( C^*_{n,n} \) that have degree \( n \) are \( x \) and \( y \).

(ii) \( P_{n,n} \leq_m G \), and the endvertices of \( P_{n,n} \) are \( x \) and \( y \).

**Proof.** Label each vertex \( v \) of \( G \) with its distance \( l(v) \) from \( x \). Then \( l(y) = N \) for some \( N \geq n(n-1) \). For \( i \in [N] \), let \( V_i \) be the set of those vertices \( v \) labeled with \( i \), such that there is a \( vy \)-path in \( G \), each of whose vertices, except \( v \), is labeled with an integer exceeding \( i \). It is clear that \( V_i \) is non-empty if \( i \in [N] \), that \( V_0 = \{x\} \) and \( V_N = \{y\} \), and that \( V_i \) is an \( xy \)-vertex-cut if \( i \in [N-1] \). Let \( V_i^x \) be the set of vertices in the component of \( G - V_i \) containing \( x \) for \( i \in [N-1] \), and let \( V_i^y \) be the set of vertices in the component of \( G - V_i \) containing \( y \) for \( i \in [N-1] \). We now establish some properties of these two sets.

(4.3.1) \[ V_i \subseteq V_i^x \quad \text{if } 0 \leq i' < i \leq N. \]

To see this, it suffices to show that, for every \( v \in V_i \), there is an \( xv \)-path in \( G - V_i \). Let \( v \) be an arbitrarily chosen vertex in \( V_i \). Since the label on \( v \) is determined by its distance from \( x \), it follows that \( G \) contains an \( xv \)-path \( P_x \) of length \( l(v) = i' \) and that the label of each vertex of \( P_x \) is at most \( i' \). In particular, \( P_x \) has no vertex of \( V_i \); thus, \( P_x \) is contained in \( G - V_i \), as required.

(4.3.2) \[ V_i \subseteq V_i^y \quad \text{if } 0 \leq i < i' \leq N. \]
The proof of (4.3-2) is very similar to the proof of (4.3-1). Let \( v \) be an element of \( V_i \). It follows that \( G \) contains a \( vy \)-path \( P_v \) and that the label of each vertex of \( P_v \) is at least \( i' \). In particular, \( P_v \) contains no vertex of \( V_i \); thus, \( P_v \) is contained in \( G - V_i \). Consequently, (4.3-2) holds.

For each \( i \) in \([N - n]\), define \( V_i^0 = V_i^V \cap V_{i+n}^x \). Statements (4.3-1) and (4.3-2) immediately imply that for each such \( i \), \( V_{i+1} \subseteq V_i^0 \); hence, \( V_i^0 \) contains an \( xy \)-vertex-cut.

Due to the lengthiness and technicality of the proof, we present an outline of it before filling in the details. The proof is divided into two cases depending on the existence of small \( xy \)-vertex-cuts in each \( V_i^0 \).

First, we will consider the case when there is an \( i \) such that a smallest \( xy \)-vertex-cut \( S_i \) contained in \( V_i^0 \) has at least \( n \) vertices. Then, we will show that \( S_i \) is a smallest \( xy \)-vertex-cut of a specified minor \( G_0 \) of \( G \), which, by (4.3.1), implies the existence of at least \( n \) pairwise internally vertex-disjoint \( xy \)-paths in \( G_0 \). Then, we will show that each of these internally vertex-disjoint \( xy \)-paths has length at least \( n \), which implies that \( C_{n,n}^* \leq m G_0 \); hence, \( C_{n,n}^* \leq m G \). In the remaining case, each \( V_i^0 \) has an \( xy \)-vertex-cut with fewer than \( n \) vertices. Let \( S_i' \) be a smallest \( xy \)-vertex-cut from \( V_i^0 \) for each \( i \) in \([n - 2]\). We recall that every \( xy \)-edge-cut of \( G \) has size at least \( n2^n \). In particular, a smallest \( xy \)-edge-cut of \( G \) has size \( M \), for some \( M \geq n2^n \), which, by (4.3.2), implies that \( G \) has \( M \) pairwise edge-disjoint \( xy \)-paths. Then we will consider a specified minor \( G_2 \) of \( G \) which has at least \( M \) pairwise edge-disjoint \( xy \)-paths, each of which has length at least \( n \). We will then show that there are fewer than \( \frac{M}{n} \) distinct \( xy \)-paths possible in the simplification of \( G_2 \). On applying (4.3.3), we will conclude that some \( n \) pairwise edge-disjoint \( xy \)-paths in \( G_2 \) use exactly the same vertices in the same order. This implies that \( P_{n,n} \leq m G_2 \); hence, \( P_{n,n} \leq m G \). Now, we fill in the details.
Assume first that there is an \( i \in [N - n] \) such that a smallest \( xy \)-vertex-cut \( S_i \) of \( G \) contained in \( V_i^0 \) has at least \( n \) vertices. Since \( G \) is connected, it is clear that there are three kinds of bridges of \( V_i \cup V_{i+n} \) in \( G \): those that meet only \( V_i \), those that meet only \( V_{i+n} \), and those that meet both \( V_i \) and \( V_{i+n} \). Figure 4.7 below illustrates the structure of the bridges of \( V_i \cup V_{i+n} \) in a typical \( G \). In this illustration, the shaded area containing \( x \) represents the union of the bridges of \( V_i \cup V_{i+n} \) in \( G \) meeting only \( V_i \); similarly, the shaded area containing \( y \) represents the union of those bridges meeting only \( V_{i+n} \).

![Figure 4.7](image)

Figure 4.7. The structure of the bridges of \( V_i \cup V_{i+n} \) in \( G \).

Now, we want to contract all of the bridges of \( V_i \cup V_{i+n} \) except those that meet both \( V_i \) and \( V_{i+n} \); that is, we want to contract the shaded areas of \( G \) in Figure 4.7. More precisely, let us contract \( E_0 = E(G) - (E(G[V_i^0]) \cup E(V_i, V_i^0) \cup E(V_i^0, V_{i+n})) \) in \( G \), where \( E(X_1, X_2) \) denotes the set of edges whose elements have one vertex in \( X_1 \) and the other vertex in \( X_2 \) for disjoint sets \( X_1 \) and \( X_2 \) of vertices. Let \( G_0 = G / E_0 \). On contracting \( E_0 \) in \( G \), it is easy to see that \( x \) and the vertices of \( V_i \) are identified, and that \( y \) and the vertices of \( V_{i+n} \) are identified. It is natural to let \( x \) and \( y \), respectively, denote these vertex identifications; it follows that \( V(G_0) = V_i^0 \cup \{x, y\} \).
The next part of the proof uses the following two simple observations. First, no edge of $E_0$ has a vertex in $V_i^0$. Second, two vertices are in the same component of a graph if and only if after contracting any set of edges, those vertices (which may become identified) are in the same component of the resulting graph.

We now show that $S_i$ is a smallest $xy$-vertex-cut of $G_0$ by showing that $S_i$ contains an $xy$-vertex-cut of $G_0$, and then showing that no subset of $V(G_0) - \{x, y\}$ having size less than $|S_i|$ is an $xy$-vertex-cut of $G_0$. Clearly $x$ and $y$ are in different components of $G - S_i$ since $S_i$ is an $xy$-vertex-cut of $G$. Also, in view of the first observation above, $(G - S_i)/E_0$ is well-defined since $S_i \subseteq V_i^0$; thus, $(G - S_i)/E_0 = (G/E_0) - S_i = G_0 - S_i$. Hence, $x$ and $y$ are in different components of $G_0 - S_i$, by the second observation above; so $S_i$ contains an $xy$-vertex-cut in $G_0$. Now let $S$ be any subset of $V_i^0$ that has fewer than $|S_i|$ vertices. Then $x$ and $y$ are in the same component of $G - S$ since $S_i$ is a smallest $xy$-vertex-cut of $G$ contained in $V_i^0$. By the first observation above, $(G - S)/E_0$ is well-defined since $S \subseteq V_i^0$; thus, $(G - S)/E_0 = G_0 - S$. By the second observation above, $x$ and $y$ are in the same component of $G_0 - S$. Hence, no subset of $V_i^0$ that has fewer than $|S_i|$ vertices is an $xy$-vertex-cut of $G_0$. It follows that $S_i$ is a smallest $xy$-vertex-cut of $G_0$. Since $|S_i| \geq n$, (4.3.1) implies that there are $xy$-paths $P_1, P_2, \ldots, P_n$ in $G_0$ that are pairwise internally vertex-disjoint.

Now, we show that each $P_j$ has length at least $n$ for $j \in [n]_+$. It follows from the first observation above that $G[V_i^0] = G_0[V_i^0]$. Let $P$ be any $xy$-path in $G_0$. Then $P' = P - \{x, y\}$ is a path in $G_0[V_i^0] = G[V_i^0]$; hence, $P'$ is a path in $G$ that has one endvertex adjacent to some vertex of $V_i$ and the other endvertex adjacent to some vertex of $V_{i+n}$. It is clear that if two vertices are adjacent in $G$, then their labels differ by 0 or 1. This implies that if the labels of the endvertices of a path in $G$ are $l_1$ and $l_2$, then the length of that path is at least $|l_2 - l_1|$. Furthermore, one endvertex of $P'$ is labeled at most $i + 1$, and the other endvertex is labeled at
least \(i + n - 1\). So the length of \(P'\) is at least \((i + n - 1) - (i + 1) = n - 2\); hence, the length of \(P\) in \(G_0\) is at least \(n\). In particular, \(P_j\) has length at least \(n\), for each \(j \in [n]_+\).

Let \(G'_0\) be the subgraph of \(G_0\) that is the union of \(P_1, P_2, \ldots, P_n\); then \(G'_0\) consists of \(n\) pairwise internally vertex-disjoint \(xy\)-paths, all of length at least \(n\). On contracting an appropriate number of interior edges of \(P_j\) in \(G'_0\), for each \(j \in [n]_+\), we obtain a minor of \(G'_0\) that is isomorphic to \(C^*_{n,n}\), whose vertices of degree \(n\) are \(x\) and \(y\). So \(C^*_{n,n} \leq_m G'_0 \leq_s G_0 \leq_m G\) (hence, \(C^*_{n,n} \leq_m G\)), and the vertices of degree \(n\) of \(C^*_{n,n}\) are \(x\) and \(y\). Thus, the lemma holds if there is an \(i \in [N - n]\) such that \(V_i^0\) lacks an \(xy\)-vertex-cut of size less than \(n\).

Now, for the remaining case, assume that, for each \(i \in [N - n]\), if \(S_i\) is a smallest \(xy\)-vertex-cut in \(V_i^0\), then \(|S_i| < n\). Let \(S'_i\) be a smallest \(xy\)-vertex-cut from \(V_i^0\) for each \(i \in [n - 2]\). As \(S'_i\) is an \(xy\)-vertex-cut, each \(xy\)-path must pass through some vertex \(s_i\) in \(S'_i\), for each \(i \in [n - 2]\).

Let us consider the bridges of \(\bigcup_{i=0}^{n-2} S'_i\) in \(G\). Since \(G\) is connected, potentially, we could have the following kinds of bridges: those that meet exactly one \(S'_i\), those that meet only \(S'_i\) and \(S'_{i+1}\) for some \(i \in [n - 3]\), and those that meet \(S'_i\) and \(S'_j\) (and, perhaps, additional sets \(S'_k\)) for some \(0 \leq i < j - 1 < n - 2\). Next, we show that \(G\) has no bridges of the last kind by showing that any \(s_is_j\)-path in \(G\) contains a vertex \(s_{i+1} \in S'_{i+1}\), when \(0 \leq i < j - 1 < n - 2\), \(s_i \in S'_i\), and \(s_j \in S'_j\).

First, we point out that if the labels of the endvertices of a path in \(G\) are \(l_1\) and \(l_2\), then certainly the path has at least one vertex labeled \(l'\) for each integer \(l'\) between \(l_1\) and \(l_2\), since the labels of adjacent vertices in \(G\) differ by 0 or 1. It follows that if \(P\) is an \(s_is_j\)-path in \(G\), where \(0 \leq i < j - 1 < n - 2\), and \(s_i\) and \(s_j\) are arbitrary elements of \(S'_i\) and \(S'_j\), respectively, then \(P\) contains a vertex whose label is \((i + 1)n\) since \(l(s_i) = in\) and \(l(s_j) = jn\). Let \(s_{i+1}\) be the vertex labeled \((i + 1)n\) that is closest in \(P\) to \(s_j\). Then each vertex of the \(s_{i+1}s_j\)-subpath of \(P\),
except $s_{i+1}$, is labeled greater than $(i + 1)n$. Since $s_j \in S'_j$, there is an $s_jy$-path each of whose vertices is labeled at least $jn$. The union of the $s_{i+1}s_j$-subpath and the $s_jy$-path contains an $s_{i+1}y$-path each of whose vertices is labeled greater than $(i + 1)n$, except $s_{i+1}$ which is labeled $(i + 1)n$. Hence, $s_{i+1} \in S'_{i+1}$, which establishes that $G$ has no bridges that meet $S'_i$ and $S'_j$, where $0 \leq i < j - 1 < n - 2$. Consequently, the structure of the bridges of $\bigcup_{i=0}^{n-2} S'_i$ in $G$ is as in Figure 4.8.

![Figure 4.8](image)

**FIGURE 4.8.** The structure of the bridges of $\bigcup_{i=0}^{n-2} S'_i$ in $G$.

Now, let us consider the minor $G_1$ of $G$ that is obtained by contracting those bridges of $\bigcup_{i=0}^{n-2} S'_i$ in $G$ that contain neither $x$ nor $y$ and that meet only $S'_i$, for each $i \in [n - 2]$; these bridges are represented by the shaded portions of $G$ in Figure 4.8. We note that, for each $i \in [n - 2]$, some vertices of $S'_i$ may become identified on contracting $G$ to $G_1$; let $S'_{i,1}$ denote the subset of $V(G_1)$ that corresponds to $S'_i \subseteq V(G)$. It is clear that $|S'_{i,1}| \leq |S'_i|$.

We now consider the minor $G_2$ of $G_1$ that is obtained by contracting the edge set $E_1$ contained in $G_1$ that is defined as follows. $E_1 = E(G_1) - (E(\{x\}; y) \cup \bigcup_{i=0}^{n-2} E(S'_{i,1}; y))$, where $E(S; y)$ is the set of those edges of $G_1$ each of which has one vertex in $S$ and the other vertex in the component of $G_1 - S$ containing $y$, for $S \subseteq V(G_1) - y$. For each $i \in [n - 2]$, some vertices of $S'_{i,1}$ may become identified on
contracting \(G_1\) to \(G_2\); let \(S'_{i,2}\) denote the set of vertices of \(G_2\) that corresponds to \(S_{i,1}\) in \(G_1\). Then \(|S'_{i,2}| \leq |S_{i,1}|\), and \(G_2 = G_1/E_1\). Figure 4.9 shows a typical \(G_2\).

**Figure 4.9. A typical \(G_2\).**

We now show that \(G_2\) has at least \(n2^{n^3}\) pairwise edge-disjoint \(xy\)-paths. Recall that every \(xy\)-edge-cut of \(G\) has size at least \(n2^{n^3}\). Then by (4.3.2), \(G\) has at least \(n2^{n^3}\) pairwise edge-disjoint \(xy\)-paths. Note that, given any \(xy\)-path \(P\) of \(G\), if we contract (in \(G\)) a set \(S\) of edges that contains no \(xy\)-path, then the subgraph \(P'\) of \(G/S\) induced by \(E(P) - S\) is connected; hence, \(P'\) contains an \(xy\)-path \(P''\). Moreover, \(E(P'') \subseteq E(P') \subseteq E(P)\); this containment and the fact that \(G\) has at least \(n2^{n^3}\) pairwise edge-disjoint \(xy\)-paths imply that \(G_2\) has at least \(n2^{n^3}\) pairwise edge-disjoint \(xy\)-paths. Next, we show that the simplification of \(G_2\) has fewer than \(2^{n^3}\) \(xy\)-paths.

Note that each edge of \(G_2\) is of the form \(xs_0, s_{n-2}y\), or \(s_is_{i+1}\), where \(s_0 \in S'_{0,2}\), \(s_{n-2} \in S'_{n-2,2}\), and \(s_i \in S'_{i,2}\), for \(i \in [n-3]\). It follows that each \(xy\)-path in \(G_2\) has length at least \(n\); moreover, the simplification \(\widetilde{G}_2\) of \(G_2\) has at most \(|S'_{0,2}|\) edges between \(x\) and \(S'_{0,2}\), at most \(|S'_{i,2}|\) \(|S'_{i+1,2}|\) edges between \(S'_{i,2}\) and \(S'_{i+1,2}\) if \(i \in [n-3]\), and at most \(|S'_{n-2,2}|\) edges between \(S'_{n-2,2}\) and \(y\). Since \(|S'_{i,2}| \leq |S_{i}'| < n\), for each \(i\) in \([n-2]\), \(\widetilde{G}_2\) has at most \(n-1 + (n-2)(n-1)^2 + n-1\) edges; hence, \(\widetilde{G}_2\) has fewer than \(n^3\) edges. Clearly, the collection of \(xy\)-paths in \(\widetilde{G}_2\) is contained in the
collection $\mathcal{G}$ of subgraphs of $\widetilde{G}_2$ that lack isolated vertices. Since $|\mathcal{G}| < 2^{n^3}$, there are fewer than $2^{n^3}$ $xy$-paths in $\widetilde{G}_2$.

Since $G_2$ has at least $n2^{n^3}$ pairwise edge-disjoint $xy$-paths and $\widetilde{G}_2$ has fewer than $2^{n^3}$ $xy$-paths, there are at least $n$ pairwise edge-disjoint $xy$-paths, $P'_1, P'_2, \ldots, P'_n$ in $G_2$, each of length at least $n$, that use the same vertices in the same order, by (4.3.3). If the length of $P'_j$ is greater than $n$, for each $j$ in $[n]_+$, then we can contract in $\bigcup_{j=1}^n P'_j$ a parallel class whose edges are incident to neither $x$ nor $y$ repeatedly until we obtain a graph isomorphic to $P_{n,n}$ whose endvertices are $x$ and $y$. So $P_{n,n} \leq_m \bigcup_{j=1}^n P'_j \leq_s G_2 \leq_m G_1 \leq_m G$ (hence, $P_{n,n} \leq_m G$), and the endvertices of $P_{n,n}$ are $x$ and $y$. Thus, the lemma holds if $V^0_i$ contains an $xy$-vertex-cut of size less than $n$, for each $i$ in $[N - n]$. □

Now, we shall describe how (4.3.4) can be applied to 2-connected graphs and block trees. The application to 2-connected graphs, stated in (4.3.5) below, is more intuitive and requires less notation than the application to block trees in (4.3.6) that follows it.

(4.3.5) COROLLARY. Let $B$ be a bridge of $\{x, y\}$ in a 2-connected graph $G$, for distinct vertices $x$ and $y$ in $G$. If each $xy$-path in $B$ has length at least $n(n - 1)$, and if each $xy$-edge-cut in $B$ has size at least $n2^{n^3}$, then an element of $\{C_{n,n}, C^*_n\}$ is a minor of $G$.

We omit the proof of (4.3.5) because it is very similar to the proof of (4.3.6), which is presented below, and to prove (4.3.5) would require the introduction of a large amount of notation, as in the statement of (4.3.6). It will be straightforward, once (4.3.6) is proved, that (4.3.6) is, in some sense, a special case of (4.3.5). Now, we state and prove (4.3.6).

(4.3.6) COROLLARY. Let $\mathcal{T} = (\mathcal{G}, T)$ be a block tree. For each link $\epsilon$ of $T$, consider the partial composition $\mathcal{T}_\epsilon = (\{H^1_\epsilon, H^2_\epsilon\}, T/(E(T) - \epsilon))$ of $\mathcal{T}$, and let $h^\epsilon_\epsilon \parallel u^t_\epsilon v^t_\epsilon$.
denote the edge of $H^i_x$ labeled $e$, for each $i \in [2]^+$. If there are a link $e \in E(T)$, an index $i \in [2]^+$, and an integer $n$ exceeding 3, such that each $u^i_e v^i_e$-path in $H^i_x \backslash h^i_x$ has length at least $n(n-1)$ and each $u^i_e v^i_e$-edge-cut in $H^i_x \backslash h^i_x$ has size at least $n2^{n^2}$, then $C_{n,n-2} \leq m G(T)$ or $C^{*}_{n,n-2} \leq m G(T)$.

Proof. Assume that the link $e$ of $T$ and the integers $i$ and $n$ satisfy the hypotheses. Since $T$ is a block tree, $H^i_x$ is 2-connected. By (4.3.4), either $C^{*}_{n,n} \leq m H^i_x \backslash h^i_x$, or $P_{n,n} \leq m H^i_x \backslash h^i_x$ and the endvertices of $P_{n,n}$ are $u^i_e$ and $v^i_e$. Since $H^i_x$ is the composition of one of the restrictions of $T$ induced by one of the components of $T \backslash e$, it follows from (4.1.6) that $H^i_x \leq m G(T)$. Consequently, $C^{*}_{n,n} \leq m G(T)$ or $P_{n,n} \cup h^i_x \leq m G(T)$. Since $P_{n,n}$ and $h^i_x$ each have $u^i_e$ and $v^i_e$ as endvertices, $(P_{n,n} \cup h^i_x)/h^i_x \cong C_{n,n}$. The result follows. □

In the next section we shall see that we may restrict our attention to block trees that do not satisfy the hypothesis of (4.3.6), given an integer $n$ exceeding 3. If $T$ is a block tree that does not satisfy this hypothesis, given an integer $n$ exceeding 3, then let us call $T$ an $n$-close block tree; an $n$-close 3-block tree is defined similarly.

4.4 Results for $n$-Close Block Trees

Recall, that the goal of this chapter is to show that if $n$ is an integer exceeding 3, and a 2-connected graph has sufficiently large type depending on $n$, then it has an element of $\{F_n, C_{n,n-2}, C^{*}_{n,n-2}\}$ as a minor. We have already seen in (4.1.7) that if the 3-block tree of a 2-connected graph has a 3-connected node graph with a cycle of length at least $N$, where $N$ is the number from (3.2.8) that depends on $n$, then $F_n$ is a minor of $G$. Also, we have seen in (4.2.1) that if the tree of the 3-block tree of $G$ contains a path of length at least $4(n-1)+1$, then $F_n$ is a minor of $G$. Additionally, we have seen in (4.3.6) that if the 3-block tree of $G$ is not $n$-close, for some integer $n$ exceeding 3, then $C_{n,n}$ or $C^{*}_{n,n}$ is a minor of $G$. So, we may restrict our attention to an arbitrary $n$-close 3-block tree $T$ whose tree has no path of length exceeding $4(n-1)$ and whose 3-connected node graphs have
no cycles of length exceeding \( N \), where \( n > 3 \) and \( N \) is the number from (3.2.8) depending on \( n \). In this section, we shall show that if \( T \) is such a 3-block tree, then the type of \( G(T) \) is bounded from above by a function of \( n \), or \( C_{n,n-2} \leq m \ G(T) \), or \( C_{n,n-2}^* \leq m \ G(T) \). The desired result will follow.

Before we can state and prove any results in this section, we need to make some definitions and assumptions, and develop some terminology. By a rooted edge-sum tree, we mean an edge-sum tree \( T = (G, T) \) whose tree \( T \) is a rooted tree (that is, \( T \) contains a distinguished node \( \xi \) called the root of \( T \)); we will frequently abuse the terminology by calling \( \xi \) the root of \( T \). If \( H \) is the node graph in \( G \) that corresponds to \( \xi \), then call \( H \) the root graph of \( T \). The depth of \( T \), denoted \( D(T) \), is \( \max\{d_T(\xi, \eta) : \eta \in V(T)\} \), where \( d_T(\xi, \eta) \) is the distance in \( T \) between the root \( \xi \) of \( T \) and \( \eta \); we will frequently abuse the terminology and notation by referring to the depth \( D(T) \) of \( T \) rather than the depth \( D(T) \) of \( T \).

Note that if \( T \) has no path of length exceeding \( 2M \), where \( M \) is a nonnegative integer, then \( T \) can be viewed as a rooted edge-sum tree of depth at most \( M \). We can see this as follows. Clearly, we may assume that \( T \) contains a nontrivial path; otherwise, \( T \) consists of a single node, and the result is immediate. Pick a longest path \( P \) in \( T \). If the length of \( P \) is odd, then append an edge and a vertex to one of the monovalent vertices of \( P \) so that the longest path \( P' \) in the resulting tree \( T' \) has even length; otherwise, let \( P' = P \) and \( T' = T \). Then there is a positive integer \( M \) such that \( P' \) has length \( 2M \). Let \( v \) denote the central vertex of \( P' \) (that is, the distance from \( v \) to either endvertex of \( P' \) is \( M \)). Let us think of \( T' \) as being rooted at \( v \). If \( u \) is a leaf of \( T' \) that is not an endvertex of \( P' \), then the intersection of \( P' \) and the \( uv \)-path \( P_{uv} \) in \( T' \) is a (perhaps trivial) subpath \( P_{xv} \) of \( P' \) with endvertices \( v \) and \( x \), for some \( x \in V(P') \). Let \( P_{z}^1 \) and \( P_{z}^2 \) denote the two subpaths of \( P' \) whose union is \( P' \) and whose intersection is \( z \). Without loss of generality, \( v \in V(P_{z}^2) \). Then \( P_{z}^2 \) is the union of the subpath \( P'' \) of \( P' \) of length \( M \) and the \( xv \)-path \( P_{xv} \).
in $T'$ such that the intersection of $P''$ and $P_{zu}$ is $v$. Let $l(P_0)$ denote the length of $P_0$, given an arbitrary path $P_0$. Then

\begin{equation}
(4.4-1) \quad l(P') = l(P^1_z) + l(P^2_z) = l(P^1_z) + l(P_{zu}) + l(P'') = 2M.
\end{equation}

Since $P'$ is a longest path in $T'$, and since the $ux$-path $P_{uz}$ in $T'$ meets $P^2_z$ only in $z$, we have $l(P_{uz}) + l(P^2_z) \leq 2M$; it follows from (4.4-1) that the length of $P_{uz}$ is at most $l(P^1_z)$. Thus, the distance between $u$ and $v$, which is $l(P_{uz}) + l(P_{zu})$, is, by (4.4-1), at most $l(P^1_z) + l(P_{zu}) = M$. Since the distance between any leaf $u$ of $T'$ and the root $v$ is at most $M$, we can view $T'$ as having depth $M$.

As noted earlier, we may restrict our attention to an arbitrary $n$-close 3-block tree whose tree has no path of length exceeding $4(n - 1)$ and whose 3-connected node graphs have no cycles of length exceeding $N_n = N$, where $n$ is an integer exceeding 3 and $N$ is the number form (3.2.8) depending on $n$. Clearly, if we think of such a 3-block tree as being rooted, then we may view it as having depth at most $M_n = 2(n - 1)$. We shall see that these values $M_n$ and $N_n$, that depend only on an integer $n$ that exceeds 3, appear in several of the results of this section.

(4.4.1) **Definition.** Let $n$ be an integer exceeding 3. If $T$ is an edge-sum tree with the properties below, then call $T$ a $(d, c; n)$-edge-sum tree; furthermore, if $T$ is a block tree or a 3-block tree, then call $T$ a $(d, c; n)$-block tree or a $(d, c; n)$-3-block tree, respectively.

(i) $T$ can be viewed as a rooted block tree of depth at most $d$, for some nonnegative integer $d$ that does not exceed $M_n$.

(ii) Each block of each node graph of $T$ either has no cycle of length exceeding $N_n$ or is a cycle; moreover, if $D(T) = d$, and $B$ is a block of the root graph of $T$ that is not a cycle, then $B$ has no cycle of length exceeding $c$, for some integer $c$ that exceeds 1 but does not exceed $N_n$. 

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If $T$ is a $(d, c; n)$-edge-sum tree, and each node graph different from the root graph is 2-connected, then call $T$ a $(d, c; n)$-near-block tree. If $T$ is a $(d, c; n)$-block tree or a $(d, c; n)$-3-block tree, and $T$ is $n$-close, then call $T$ a $(d, c; n)$-close block tree or a $(d, c; n)$-close 3-block tree, respectively; in particular, each $(0, c; n)$-block tree is a $(0, c; n)$-close block tree. Note that if $2 \leq d' \leq c \leq N_n$, then each $(d, c'; n)$-edge-sum tree is a $(d, c; n)$-edge-sum tree; also, note that if $0 \leq d' < d \leq M_n$, then each $(d', N_n; n)$-edge-sum tree is a $(d, c; n)$-edge-sum tree. □

Recall that we may restrict our attention to an arbitrary $n$-close 3-block tree $T$, that has depth $d$, which is at most $M_n$, when viewed as rooted, and whose 3-connected node graphs have no cycles of length exceeding $N_n$, where $n$ is an integer exceeding 3. Let $c$ denotes the length of a longest cycle in the root graph $H$ of $T$ if $H$ is not a cycle; otherwise, let $c = 2$. Then, we may assume that $T$ is a $(d, c; n)$-close 3-block tree. Now, we are ready to state the main result (4.4.2) of this section. The statement of (4.4.2) will be followed by several lemmas that will be used in its proof.

(4.4.2) THEOREM. Let $T = (G, T)$ be a $(d, c; n)$-close 3-block tree. Then one of the following holds.

(i) $t(G(T)) < F(n)$, where $F(n) = \frac{n^8(N_n+1)^4}{16} + \frac{2^{n^8} n^2(N_n+1)^3}{3} + \frac{N_n(N_n+1)}{2}$.

(ii) $C_{n,n-2} \leq_{m} G(T)$ or $C_{n,n-2}^{*} \leq_{m} G(T)$.

The first lemma that we shall state and prove is a simple fact regarding edge-cuts in connected graphs. In the lemma that follows it, we shall show that, for each block $B$ of $G(T)$ that contains more than one edge, there is a $(d, c; n)$-block tree $T_B$, called a block-tree reduction of $B$ in $T$, such that $G(T_B) = B$.

(4.4.3) LEMMA. If $G$ is a connected graph and $S$ is an $xy$-edge-cut in $G$, then $G \setminus S$ is made up of two components, $C_x$ containing $x$ and $C_y$ containing $y$. 

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Proof. Since $S$ is an $xy$-edge-cut in $G$, there is no $xy$-path in $G \setminus S$. Suppose that $G \setminus S$ contains a component $C_0$ that is neither $C_x$ nor $C_y$. Let $C$ denote the union of such components $C_0$ of $G \setminus S$; then $G \setminus S = C_x \cup C_y \cup C$. Since $G$ is connected, it follows that there is a shortest path $P$, whose edges are contained in $S$, that has one endvertex in $C$ and the other endvertex in $C_x$ or $C_y$. It follows that $P$ is a proper path, and it meets $C$ and exactly one of $C_x$ and $C_y$; by symmetry, we may assume that $P$ meets $C$ and $C_x$. Let $S' = S - E(P)$. Consider $G \setminus S' = (G \setminus S) \cup P = (C_x \cup C_y \cup C) \cup P$. Since $P$ does not meet $C_y$, the graphs $C_y$ and $C_x \cup C \cup P$ are disjoint. Hence, there is no $xy$-path in $G \setminus S'$. Thus $S'$, which is a proper subset of $S$, contains an $xy$-edge-cut of $G$, a contradiction. Thus, the result holds. □

(4.4.4) Lemma. Let $T_G = (G, T)$ be a $(d, c; n)$-edge-sum tree whose composition is $G$, where $n$ is an integer that exceeds 3. If $B$ is a block of $G$ that contains at least one edge, then there is a $(d, c; n)$-edge-sum tree, namely, $T_B$, whose composition is $B$. In particular, if $B$ is 2-connected, then there is a $(d, c; n)$-block tree $T_B$ whose composition is $B$. Moreover, if $e \in E(B)$ belongs to the root graph of $T_G$, then $e$ belongs to the root graph of $T_B$.

Proof. If $B$ is a link-edge of $G$ or a cycle of $G$, then it follows that $T_B = ({\{B\}, K_1})$ is a $(0, 2; n)$-edge-sum tree. It is trivial to show that $T_B$ satisfies the remaining conditions stated in the lemma. So, for the remainder of the proof, we may assume that $B$ is 2-connected and not a cycle. If $D(T) = 0$, then $T_G = ({G}, K_1; n)$, and $T_G$ is a $(0, c; n)$-edge-sum tree. It follows that $B$ has no cycles of length exceeding $c$. Then $T_B = (\{B\}, K_1; n)$ is a $(0, c; n)$-block tree. The remaining condition of the lemma is satisfied since $D(T_B) = 0$.

Now, we may assume that $d$ is a positive integer, and that the lemma holds for all $(d - 1, N_n; n)$-edge-sum trees. Consider $T = T_G/(E(G) - E(B))$. By (4.2.7), $G(T) = G/(E(G) - E(B))$, which is 1-isomorphic to $B$. Let $\xi$ denote the root of
$T$, and let $H$ denote the root graph of $\mathcal{T}$. Let us consider the star $\mathcal{T}_{st}$ of $\mathcal{T}$ at $H$ as defined in (4.2.6) and the notation used in (4.2.6).

If there is some $j \in [m]_+$ so that the set of node graphs of $\mathcal{T}_{st}/\{\epsilon_i: i \in ([m]_+ - \{j\})\}$ is made up of $K^j$ and a graph $H_0$ whose set of edges consists of a single edge, namely $h_j$, then consider the edge-sum tree $T_j' = \mathcal{T}/(E(\mathcal{T}) - E(T_j))$. The tree of $T_j'$, which is isomorphic to $T_j$, is obtained by contracting $E(\mathcal{T}) - E(T_j)$ in $T$ to the node $\xi_j$; let us view $T_j'$ as being rooted at $\xi_j$. Since $T_j'$ is a partial composition of $\mathcal{T}$, it follows that $G(T_j') = G(\mathcal{T}) \cong_1 B$. The set of node graphs of $T_j'$ is obtained from the set of node graphs of $T_j$ by replacing the root graph $H_j$ of $T_j$ by $H_j/k_j$ if $h_j$ is a loop, and by $H_j\backslash k_j$ if $h_j$ is not a loop. It follows that $T_j'$ is a rooted edge-sum tree of depth less than $d$ whose composition is $1$-isomorphic to $B$, and that each block of each node graph of $T_j'$ either lacks a cycle of length exceeding $N_n$ or is a cycle. Hence, $T_j'$ is a $(d - 1, N_n; n)$-edge-sum tree, and, by hypothesis, there is a $(d, c; n)$-block-tree $T_B$ of $B$, and the result holds.

Finally, we may assume that the node graph $H_j'$ of $\mathcal{T}_{st}/\{\epsilon_i: i \in ([m]_+ - \{j\})\}$ that is not $K^j$ has at least two edges (hence, at least one unlabeled edge), for each $j \in [m]_+$. It follows that all edges of $K_i'$ belong to a single block of $K_i$, for each $i \in [m]_+$; otherwise, $G(\mathcal{T})$ would have more than one proper block. For each $i \in [m]_+$, if $K_i'$ consists of a single edge, then contract the link $\epsilon_i$ in $\mathcal{T}_{st}$. Let $\mathcal{T}' = (\mathcal{G}', T')$ denote the resulting rooted edge-sum tree, and let $H'$ denote the root graph of $\mathcal{T}'$. It follows that each node graph $K_i'$ in $\mathcal{G}' - H'$ is a 2-connected graph with, perhaps, some isolated vertices. It then follows that $H'$ is a 2-connected graph with, perhaps, some isolated vertices; otherwise, $G(\mathcal{T})$ would have more than one proper block. Note that, for each node graph $K_i'$ in $\mathcal{G}' - H'$, the graph $K_i'$ is the composition of the edge-sum tree $\mathcal{T}_i$ rooted at $\xi_i$, for some $i \in [m]_+$, and the edge $k_i \in E(K_i')$ belongs to the root graph of $\mathcal{T}_i$. By hypothesis, for each node graph $K_i'$ in $\mathcal{G}' - H'$, there is a $(d - 1, N_n; n)$-block tree $T_{K_i'}$ whose composition
is \( K' \) and whose root graph contains \( k_i \). Consider the rooted edge-sum tree \( T^* \) defined as follows. Let \( H^* = H'[E(H')] \) be the root graph of \( T^* \), let the directed labeling of \( H^* \) in \( T^* \) agree with the directed labeling of \( H' \) in \( T' \), and let \( \xi^* \) denote the root of the tree \( T^* \) of \( T^* \). We obtain \( T^* \) by connecting \( \xi^* \) to the root of the tree of \( T_{K'} \) with a link, for each \( K' \) in \( G' - H' \). If \( h_i \) is a labeled edge of \( H^* \), then it follows that there is a \( (d - 1, N_n; n) \)-block tree \( T_{K'} \), for some \( K' \) in \( G' - H' \), whose root graph contains \( k_i \). Assign the label \( e^* \) to \( k_i \), and direct \( k_i \) in \( T^* \) so that its direction agrees with its direction in \( T' \). Note that \( H^* \) is obtained by deleting all isolated vertices from \( H' \). Also, note that \( H' \) is obtained from \( H \) by edge-summing \( H \) with graphs \( K^i \) consisting of a single edge; this amounts to deleting or contracting an edge of \( H \), depending on whether such a graph \( K^i \) is a link-edge or a loop. Last, note that \( H \) is obtained by contracting a set of edges in the root graph \( H_G \) of \( T_G \); thus, \( H^* \leq_m H_G \). Since each block of \( H_G \) is a cycle or contains no cycle of length exceeding \( c \), and since \( H' \) is a 2-connected graph with, perhaps, some isolated vertices, it follows that \( H^* \) is a cycle, or \( H^* \) is a block that contains no cycle of length exceeding \( c \). It follows that \( T^* \) is a \( (d, c; n) \)-block tree whose composition is \( B \), as required. \( \square \)

We shall prove (4.4.2) by induction on the indices \( d \) and \( c \). The next few lemmas will handle the details of certain steps of the induction in order to make the proof of (4.4.2) shorter and more readable.

(4.4.5) **LEMMA.** Let \( T = (G, L_G, T) \) be a \( (0, c; n) \)-close block tree, for some integer \( n \) exceeding 3. Then \( t(G(T)) \leq \frac{c(c+1)}{2} \).

**Proof.** Since \( D(T) = 0 \), it follows that \( G \) contains only one node graph \( H \), which is an unlabeled 2-connected graph, and \( G(T) = H \). Recall that \( 2 \leq c \leq N_n \). If \( H \) is a cycle with at least 2 edges, then \( t(H) = 2 < \frac{c(c+1)}{2} \). So, we may assume that \( H \) is a 2-connected graph, each cycle of which has length at most \( c \). By (3.2.7), \( t(H) \leq \frac{c(c+1)}{2} \). \( \square \)
(4.4.6) **Lemma.** Let $\mathcal{T} = (G, L_0, T)$ be a $(d, c; n)$-close block tree whose root graph is a cycle of length at least $n$, for some integers $n$ and $d$ exceeding $3$ and $0$, respectively. Then one of the following holds.

(i) There is a set $S_\chi$ of at most $n - 3$ edges in $G(\mathcal{T})$, so that if $B$ is a 2-connected block of $G(\mathcal{T}) \setminus S_\chi$ for which $t(B) = t(G(\mathcal{T}) \setminus S_\chi)$, then there is a $(d - 1, N_n; n)$-block tree $\mathcal{T}_B$ whose composition is $B$.

(ii) $t(G(\mathcal{T})) \leq n - 2$.

(iii) $C_{n,n-2} \leq_m G(\mathcal{T})$.

**Proof.** Let $H$ denote the root graph of $\mathcal{T}$, and let $\xi$ denote the root of $T$. The cycle $H$ has length $N$, for some integer $N \geq n$. We may assume that $V(H) = \{v_i : i \in [N]_+\}$ and that $E(H) = \{v_1 v_2, v_2 v_3, \ldots, v_{N-1} v_N, v_N v_1\}$; also, it will be convenient to think of $v_1$ as sometimes having the name $v_{N+1}$. Let us consider the star $\mathcal{T}_st$ of $\mathcal{T}$ at $H$ as defined in (4.2.6) and the notation used in (4.2.6).

Since $\mathcal{T}$ is a block tree, it follows that $\mathcal{T}_i$ and $\mathcal{T}_{st}$ are block trees and $K^i$ is 2-connected, for each $i \in [m]_+$. It also follows that the labeled edge $k_i$ in $K^i$ is a link-edge, and we shall let $x_i$ and $y_i$ denote the endvertices of $k_i$, for each $i \in [m]_+$. It follows that there are $N$ distinct vertices in $G(\mathcal{T}_{st}) = G(\mathcal{T})$ corresponding to the $N$ vertices of $V(H)$; for each $i \in [N]_+$, let the vertex in $G(\mathcal{T}_{st})$ corresponding to $v_i$ also be called $v_i$. Since the edges $h_i$ and $k_i$ are identified (and then deleted) when $e_i$ is contracted in $\mathcal{T}_{st}$, the composition $G(\mathcal{T}_{st})$ is obtained from $H$ by replacing $h_i$ with $K^i \setminus k_i$ so that $x_i$ is identified with one endvertex of $h_i$ and $y_i$ is identified with the other endvertex of $h_i$ (as determined by the directed labeling of $\mathcal{T}_{st}$), for each $i \in [m]_+$; let $K^i_k$ denote the subgraph of $G(\mathcal{T}_{st}) = G(\mathcal{T})$ that is isomorphic to $K^i \setminus k_i$ and replaces $h_i$ in $H$, for each $i \in [m]_+$. Note that, for each $i \in [m]_+$, $K^i_k$ and $K^i \setminus k_i$ are identical except that $x_i$ and $y_i$ in $K^i \setminus k_i$ are renamed in $K^i_k$ with the endvertices of $h_i$ in $H$. Figure 4.10 illustrates a typical $\mathcal{T}_{st}$ and its composition. In
this figure, the cycle whose vertex set is \( \{v_i : i \in [6]_+\} \) is the root graph \( H \) of \( \mathcal{T}_{st} \), and, for each \( i \in [3]_+ \), the node graph of \( \mathcal{T}_{st} \) containing \( k_i \) is \( K^i \).

\[ \mathcal{T}_{st} \]

\[ G(T) = G(\mathcal{T}_{st}) \]

**Figure 4.10.** A typical \( \mathcal{T}_{st} \) and its composition \( G(\mathcal{T}_{st}) \).

First, let us assume that some edge \( e \) of \( H \) is not labeled by the directed labeling \( L_G \) of \( \mathcal{T} \); then, \( e \in E(G(T)) \). By shifting the indices of the vertices of \( H \), we may assume that \( e = u_1 u_N \). If \( e \) is deleted from \( G(T) \), then, for each integer \( i \) such that \( 1 < i < N \), the vertex \( v_i \) is a cut-vertex of \( G(T) \setminus e \). It follows that each unlabeled edge in \( E(H) \setminus e \), viewed as a subgraph of \( G(T) \setminus e \), is a block of \( G(T) \setminus e \), and \( K^i_k \) is a union of blocks of \( G(T) \setminus e \), for each \( i \in [m]_+ \). Since \( K^i_k \) has at least one edge, for each \( i \in [m]_+ \), it follows that \( t(G(T) \setminus e) = \max\{t(K^i_k) : i \in [m]_+ \} \).

Let \( l \in [m]_+ \) be an index for which \( t(G(T) \setminus e) = t(K^l_k) \), and let \( B \) be a block of \( K^l_k \) for which \( t(K^l_k) = t(B) \). If \( |E(B)| = 1 \), then it follows that each block of \( G(T) \setminus e \) is a single edge; hence, \( t(G(T)) \leq |\{e\}| + t(B) = 2 \leq n - 2 \). So, we may assume that \( B \) has more than one edge; hence, \( B \) is 2-connected. Note that \( K^l_k \cong K^l \setminus k_l = G(\mathcal{T}_l \setminus k_l) \). Since \( \mathcal{T}_l \setminus k_l \) is a \((d - 1, N_n; n)\)-edge-sum tree, by (4.4.4), there is a \((d - 1, N_n; n)\)-block tree whose composition is \( B \), as required.

For the rest of the proof, we may assume that each edge of \( H \) is labeled by \( L_G \); it follows that \( m = N \). By an appropriate permutation of \([N]_+\) applied to the index \( i \) in \( \{e_i, h_i, k_i, \mathcal{T}_i, T_i, K^i, \mathcal{K}^i\} \), and \( K^l_k \), we may assume that \( h_i = v_i v_{i+1} \) in \( H \).
For the next case, which is similar to the first, let us assume that there is an index \( l \in [N]+ \) for which \( K_l \) has an \( x_l y_l \)-edge-cut \( S^0_l \) containing at most \( n - 2 \) edges. By shifting the indices, we may assume that \( l = N \). Clearly, \( k_N \in S^0_N \). Let \( S_N = S^0_N - k_N \). Then \( S_N \) is made up of unlabeled edges, and \(|S_N| \leq n - 3\).

Note that \( K^N \setminus S_N \cong K^N \setminus S^0_N \). Since \( K^N \) is 2-connected, it follows from \((4.4.3)\) that \( K^N \setminus S^0_N \) is made up of two components, \( C_z \) containing \( x_N \) and \( C_y \) containing \( y_N \). Thus, \( K^N \setminus S_N \) is made up of two components, \( C_1 \) containing \( v_1 \) and \( C_N \) containing \( v_N \), and \( \{C_1, C_N\} \) are identical except for the names of \( v_1 \) and \( v_N \) in \( \{C_z, C_y\} \). It follows that \( G(T) \setminus S_N = G(T_q \setminus k_q) \setminus S_N \) is as in Figure 4.11.

![Figure 4.11. A typical \( G(T) \setminus S_N \).](image)

Note that, for each integer \( i \) such that \( 1 < i < N \), the vertex \( v_i \) is a cut-vertex of \( G(T) \setminus S_N \). Furthermore, if \( C_1 \) is not isomorphic to \( K_1 \), then \( C_1 \) is a union of bridges of \( v_1 \) in \( G(T) \setminus S_N \); similarly, if \( C_N \) is not isomorphic to \( K_1 \), then \( C_N \) is a union of bridges of \( v_N \) in \( G(T) \setminus S_N \). It follows that \( K^i_k \) is a union of blocks of \( G(T) \setminus S_N \) for each \( i \in [N - 1]_+ \), and, \( C \) is a union of blocks of \( G(T) \setminus S_N \), for each element \( C \) of \( \{C_1, C_N\} \) that is not isomorphic to \( K_1 \). If there is an index \( q \in [N - 1]_+ \) such that \( t(K^q_k) = t(G(T) \setminus S_N) \), then let \( B \) be a block of \( K^q_k \) such that \( t(B) = t(K^q_k) \). If \( B \) consists of a single edge, then each block of \( G(T) \setminus S_N \) is a single edge; hence, \( t(G(T)) \leq |S_N| + t(B) \leq n - 2 \); so, as before, we may assume that \( B \) is 2-connected. Recall, \( K^q_k \cong K^q_k \setminus k_q = G(T_q \setminus k_q) \). Since \( T_q \setminus k_q \) is a \((d - 1, N; n)\)-edge-sum tree, by \((4.4.4)\), there is a \((d - 1, N; n)\)-block
tree whose composition is \( B \), as required. If there is no index \( q \in [N - 1]_+ \) for which \( t(K^q_k) = t(G(T) \setminus S_N) \), then \( t(C_1 \cup C_N) = t(G(T) \setminus S_N) \). Let \( B \) be a block of \( C_1 \cup C_N \) for which \( t(B) = t(C_1 \cup C_N) \). As before, \( t(G(T)) \leq |S_N| + t(B) \leq n - 2 \) if \( B \) consists of an edge; so we may assume that \( B \) is 2-connected. Note that \( C_1 \cup C_N \cong K^N \setminus S^0_N = G(T \setminus S^0_N) \). Since \( T \setminus S^0_N \) is a \((d - 1, N_n; n)\)-edge-sum tree, by (4.4.4), there is a \((d - 1, N_n; n)\)-block tree whose composition is \( B \), as required.

For the final case, let us assume, for each \( i \in [N]_+ \), that every \( x_iy_i \)-edge-cut in \( K^i \) has at least \( n - 1 \) edges. The following holds for each \( i \in [N]_+ \). Let \( S^0_i \) be an \( x_iy_i \)-edge-cut in \( K^i \); clearly, \( k_i \in S^0_i \), and the edges in \( S^0_i - k_i \) are unlabeled in \( K^i \). Let \( S_i = S^0_i - k_i \); then \( |S_i| \geq n - 2 \). By (4.4.3), \( K^i \setminus S^0_i \cong K^i_k \setminus S_i \) consists of two components, \( C_{i,1} \) containing \( v_i \) and \( C_{i,2} \) containing \( v_{i+1} \). It is straightforward that \( S_i \) is a \( v_i v_{i+1} \)-edge-cut of \( K^i_k \). From this it follows that \( C_{i,1} \cup C_{i,2} \cup S \) is connected, for each \( s \in S_i \); in particular, each \( s \in S_i \) has one endvertex in \( V(C_{i,1}) \) and the other endvertex in \( V(C_{i,2}) \). Next, we show that a multi-edge of size at least \( n - 2 \) is a minor of \( K^i_k \).

Consider \( K^i_k / E(C_{i,1} \cup C_{i,2}) \), for any \( i \in [N]_+ \); we can see that this is isomorphic to a multi-edge of size at least \( n - 2 \) with endvertices \( v_i \) and \( v_{i+1} \) as follows. When \( E(C_{i,1}) \) is contracted in \( K^i_k \), we may identify all of the vertices of \( C_{i,1} \) to \( v_i \); similarly, when \( E(C_{i,2}) \) is contracted in \( K^i_k \), we may identify all of the vertices of \( C_{i,2} \) to \( v_{i+1} \). Since \( C_{i,1} \) and \( C_{i,2} \) are disjoint, \( v_i \) and \( v_{i+1} \) are distinct vertices in \( K^i_k / E(C_{i,1} \cup C_{i,2}) \). Hence, in \( K^i_k / E(C_{i,1} \cup C_{i,2}) \), one endvertex of \( s \) is \( v_i \) and the other endvertex of \( s \) is \( v_{i+1} \), for each \( s \in S_i \). Consequently, \( K^i_k / E(C_{i,1} \cup C_{i,2}) \) is a multi-edge of size at least \( n - 2 \) with endvertices \( v_i \) and \( v_{i+1} \), for each \( i \in [N]_+ \).

Since a multi-edge of size at least \( n - 2 \) with endvertices \( v_i \) and \( v_{i+1} \) is a minor of \( K^i_k \), for each \( i \in [N]_+ \), it follows that \( C_{N,n-2} \leq m G(T) \). Since \( N \geq n \), it follows that \( C_{n,n-2} \leq m G(T) \), as required. \( \square \)
In (4.2.7) we saw that, given an edge-sum tree $T$ and disjoint sets $C$ and $D$ of edges in $G(T)$, contracting each edge of $C$ in its appropriate node graph and deleting each edge of $D$ in its appropriate node graph is equivalent to first taking the composition of $T$ and then performing contractions on the edges of $C$ and deletions on the edges of $D$. In order to prove the next lemma, namely (4.4.8), we would like, in some sense, to be able to perform a contraction or a deletion on a labeled edge from the root node of a near-block tree $T$ and describe the effect of this on $G(T)$. This is described more precisely in the definition below.

(4.4.7) **Definition.** Let $T$ be a near-block tree of depth at least 1, and let $h$ be a labeled edge in the root graph $H$ of $T$. Let $e$ denote the link of the tree $T$ of $T$ with which $h$ is labeled, and let $k$ denote the other edge that is labeled $e$. Let $T_K$ denote the restriction of $T$ induced by the component $T_K$ of $T \setminus e$ that does not contain the root $\xi$ of $T$, and let $K = G(T_K)$. It follows that $k \in E(K)$, and, since $T_K$ is a block tree, $K$ is 2-connected. Hence, $k$ is a link-edge; so let $x_k$ and $y_k$ denote the endvertices of $k$. Since $K$ is 2-connected, there are in $K$ a cycle $C_k$ containing $k$ and an $x_ky_k$-edge-cut $D_k$ containing $k$. Let $C_0 = E(C_k) - k$ and $D_0 = D_k - k$. Consider the partial composition $T/E(T_K)$ of $T$. The node graph of $T/E(T_K)$ that corresponds to the endnode of $e$ that is not $\xi$ is $K$, and $K$ (viewed as a node graph of $T/E(T_K)$) has exactly one labeled edge $k$; by symmetry, we may assume that the $x_k$ and $y_k$ are the tail and head, respectively, of $k$.

First, let us consider $T_1 = (T/E(T_K))/C_0$. Since the edges of $C_0$ form an $x_ky_k$-path in $K$, the labeled edge $k$ is a loop in the node graph $K/C_0$ of $T_1$. Now consider $T_1/e$. It is straightforward that $T_1$ and $T_1/e$ are the same, except that the link $e$ is contracted to $\xi$ in $T_1/e$, and the node graph in $T_1/e$ corresponding to $\xi$ is $H_1 = H \oplus_2 (K/C_0)$. Since $k$ is a loop in $K/C_0$, it follows from the definition of edge-summing that $H_1$ is 1-isomorphic to the disjoint union of $H/h$ and $K/C_k$. Also note that $K/C_k$, viewed as a subgraph of $H_1$, has no labeled edges.
and is a union of blocks of $H_1$ (provided that $K/C_k$ has at least one edge). It is straightforward that $G(T_1/e)$ is 1-isomorphic to the disjoint union of $K/C_k$ and $G(T/h)$, where $T/h$ is obtained by contracting $h$ in the root graph $H$ of the restriction $T - V(T_K)$ of $T$. (Note that we may contract $h$ in $T - V(T_K)$ since $h$ is unlabeled in $T - V(T_K)$.) Note that $K/C_k$ is the composition of $T_K/C_k$; let us abbreviate $T_K/C_k$ as $T/h$. It follows from the way that $T_1$, $T/h$, and $T_{i/C_k}$ are defined that $G(T/C_0)$ is 1-isomorphic to the disjoint union of the compositions of $T/h$ and $T_{i/C_k}$. Thus, $t(G(T)/C_0) = t(G(T/C_0)) = \max\{t(G(T/h)), t(G(T_{i/C_k}))\}$; hence, $t(G(T)) \leq |C_0| + \max\{t(G(T/h)), t(G(T_{i/C_k}))\}$. Note that $T_{i/C_k}$ is an edge-sum tree of depth less than $D(T)$ and that $T/h$ is a near-block tree. So let us say that we can essentially contract a labeled edge $h$ in the root graph $H$ of a near-block tree $T$ by contracting $C_0$ in $T$, and, after essentially contracting $h$, it is sufficient to consider $\{T/h, T_{i/C_k}\}$, as described above. The process of essentially contracting the labeled edge $h$ in $H$ in $T$ is illustrated in Figure 4.12 below.

![Diagram](image)

**Figure 4.12.** $T/h$ and $G(T/C_k) = K/C_k$ are obtained by essentially contracting $h$.

Now, consider $T_2 = (T/E(T_K)) \setminus D_0$. Since $D_k$ is an $x_k y_k$-edge-cut in $K$, it follows from (4.4.3) that $k$ is a cut-edge of $K \setminus D_0$ whose deletion results in components $K_{x_k}$ and $K_{y_k}$ containing $x_k$ and $y_k$, respectively (hence, $K \setminus D_k = K_{x_k} \cup K_{y_k}$).
It follows that $T_2$ and $T_2/\epsilon$ are the same, except that $\epsilon$ is contracted to $\xi$ in $T_2/\epsilon$, and the node graph in $T_2/\epsilon$ corresponding to $\xi$ is $H_2 = H \oplus_2 (K \setminus D_0)$. Note that, whether $h$ is a loop or a link-edge, $H_2 \cong (H \setminus h) \cup (K \setminus D_k)$. Also note that each component $K'$ of $K \setminus D_k$, viewed as a subgraph of $H_2$, is an unlabeled union of blocks of $H_2$, provided that $E(K') \neq \emptyset$. It follows that $G(T_2/\epsilon) \cong (K \setminus D_k) \cup G(T_h)$, where $T_h$ is obtained by deleting $h$ from $T - V(T_K)$. (Note that we may delete $h$ from $T - V(T_K)$ since $h$ is unlabeled in $T - V(T_K)$.) Note that $K \setminus D_k = G(T_K \setminus D_k)$; let us abbreviate $T_K \setminus D_k$ as $T_{D_k}$. It follows from the way that $T_2$, $T_h$, and $T_{D_k}$ are defined that $G(T/D_0) \cong G(T_h) \cup G(T_{D_k})$. Thus, $t(G(T)) \leq |D_0| + t(G(T/D_0) = |D_0| + t(G(T/D_0)) = |D_0| + \max \{t(G(T_h)), t(G(T_{D_k}))\}$. Note that $T_{D_k}$ is an edge-sum tree of depth less than $D(T)$ and that $T_h$ is a near-block tree. So we can essentially delete a labeled edge $h$ from the root graph $H$ of a near-block tree $T$ by deleting $D_0$ from $T$, and, after essentially deleting $h$, we may consider $\{T_h, T_{D_k}\}$, as described above. The process of essentially deleting a labeled link-edge $h$ from $H$ in $T$ is illustrated in Figure 4.13 below.

![Figure 4.13](image.png)

**Figure 4.13.** $T_h$ and $G(T_{D_k}) = K \setminus D_k$ are obtained by essentially deleting $h$.

Finally, let us extend the definition to disjoint sets $C$ and $D$ of labeled edges in the root graph $H$ of a near-block tree $T$ so that we essentially contract $C$ and essentially delete $D$. Consider the star $T_{st}$ of $T$ at $H$ as defined in (4.2.6).
Recall from (4.2.6) that \( \{h_i: i \in [m]_+\} \) is the set of labeled edges in the root graph \( H \) of \( T \). Since \( T \) is a near-block tree, it follows that \( T_i \) is a block tree and that \( K^i \) is 2-connected, for each \( i \in [m]_+ \). Hence, \( K^i \) contains a cycle \( C_k^i \) and an \( x_k^i, y_k^i \)-edge-cut \( D_k^i \), where \( x_k^i \) and \( y_k^i \) are the tail and head, respectively, of \( k_i \) as determined by the directed labeling of \( T_{st} \), for each \( i \in [m]_+ \).

Note that there are subsets \( I_C \) and \( I_D \) of \( [m]_+ \) so that \( C = \{h_i: i \in I_C\} \) and \( D = \{h_i: i \in I_D\} \). Let \( C_i = C_k^i - k_i \) for each \( i \in I_C \), and let \( D_i = D_k^i - k_i \) for each \( i \in I_D \). Let \( T_{C_k^i} = T_i / C_k^i \) for each \( i \in I_C \), let \( T_{D_k^i} = T_i / D_k^i \) for each \( i \in I_D \), and let \( T_{C_k^i D_k^i} \) be the near-block tree that is obtained from the restriction \( T - \bigcup_{i \in (I_C \cup I_D)} V(T_i) \) of \( T \) by replacing its root graph \( H \) with \( H / C \setminus D \). Let us consider the collection \( \mathcal{I} = \{T_{C_i D_i} \} \cup \{T_{C_k^i} : i \in I_C\} \cup \{T_{D_k^i} : i \in I_D\} \) of edge-sum trees. It is straightforward that the disjoint union of the compositions of the elements of \( \mathcal{I} \) is 1-isomorphic to \( G(T) / \bigcup_{i \in I_C} C_i \setminus \bigcup_{i \in I_D} D_i \). It follows that \( t(G(T)) \leq |\bigcup_{i \in I_C} C_i| + |\bigcup_{i \in I_D} D_i| + \max\{t(G(U)) : U \in \mathcal{I}\} \). So let us say that we can essentially contract \( C \) in and delete \( D \) from the root graph \( H \) of a near-block tree \( T \) by contracting \( \bigcup_{i \in I_C} C_i \) in and by deleting \( \bigcup_{i \in I_D} D_i \) from \( T \). After essentially contracting \( C \) and essentially deleting \( D \), it is sufficient to consider \( \mathcal{I} \), as described above. \( \square \)

(4.4.8) **Lemma.** Let \( T = (G, T) \) be a \((d, c; n)\)-close block tree whose root graph \( H \) contains no cycle of length exceeding \( c \), for some integers \( n, c, \) and \( d \) exceeding 3, 1, and 0, respectively, and assume that each edge of \( H \) is labeled. Then one of the following holds.

(i) There are disjoint subsets \( E_f \) and \( E_\setminus \) of \( E(G(T)) \) containing fewer than \( \frac{c^2 n^2}{8} \) edges and \( 2n^2 - c^2 n^2 \) edges, respectively, so that if \( B \) is a 2-connected block of \( G(T)/E_{\setminus} \) for which \( t(B) = t(G(T)/E_{\setminus}) \), then there is a \((d, c - 1; n)\)-block tree if \( c \geq 3 \) or a \((d - 1, N_n; n)\)-block tree if \( c = 2 \), whose composition is \( B \).
(ii) $t(G(T)) < \frac{c^2n^2}{8} + 2^{n^2-1}c^2n^2 + 1$.

(iii) $C^*_n,n-2 \leq m G(T)$.

Proof. Let $T_i$ be the star of $T$ at $H$, as defined in (4.2.6). For each $i \in [m]_+$, let us assign a weight of $s$ or $l$ to $h_i \in E(H)$ as follows. If every cycle in $K^i$ that contains $k_i$ has length exceeding $n(n - 1)$, then let the weight $w(h_i)$ of $h_i$ be $l$; otherwise, let $w(h_i) = s$.

Let $C$ be a longest cycle of $H$; clearly, $|E(C)| = c$. For each pair $\{u,v\}$ of vertices of $C$, let $P_{uv}$ be a $uv$-path in $H$ made up of edges weighted $s$ such that $V(P_{uv}) \cap V(C) = \{u, v\}$, if such a path exists; otherwise, let $P_{uv}$ be the subgraph of $H$ made up of the vertices $u$ and $v$. Let $P_s = \bigcup_{\{u,v\} \subseteq V(C)} P_{uv}$, and let $F_s$ be a spanning forest of $P_s$; hence, $|E(F_s)| \leq |E(P_s)|$. Note that if $P_{uv}$ is a path, then the length of $P_{uv}$ is at most the distance between $u$ and $v$ in $C$, since $C$ is a longest cycle in $H$. It follows that

$$|E(F_s)| \leq c \sum_{i=1}^{\frac{c^2n^2}{8}} i = c \cdot \frac{c^2_n - 1}{2} \cdot \frac{c+1}{2} = \frac{c(c^2 - 1)}{8}$$

if $n$ is odd, and

$$|E(F_s)| \leq c \sum_{i=1}^{\frac{c^2n^2}{8}} i + \frac{c}{2} \cdot \frac{c}{2} = c \cdot \frac{c}{2}(\frac{c}{2} - 1) + \frac{c^2}{4} = \frac{c^3}{8}$$

if $n$ is even.

Let $I_F$ be the subset of $[m]_+$ such that $i \in I_F$ if and only if $h_i \in F_s$. For each $i \in I_F$, let $C_i$ be a cycle in $K^i$ containing $k_i$ whose length is at most $n(n - 1)$, and let $C_i = E(C_k) - k_i$; thus, $C_i$ consists of unlabeled edges for each $i \in I_F$. Let us essentially contract $F_s$ in $H$ in $T$ by contracting $C_s = \bigcup_{i \in I_F} C_i$ in $T$. Note that $|C_s| \leq \frac{c^2}{8} \cdot n(n-1) < \frac{c^2n^2}{8}$. So, after contracting fewer than $\frac{c^2n^2}{8}$ edges in $T$, we may consider the collection $\mathcal{I} = \{T_{/F_s}\} \cup \{T_{/C_i} : i \in I_F\}$ of edge-sum trees. Given any non-empty collection $\mathcal{U}$ of edge-sum tree, let $G(\mathcal{U})$ denote the disjoint union of the compositions of the elements of $\mathcal{U}$. It follows that $G(T/C_s) = G(T)/C_s \simeq_1 G(\mathcal{I})$. Note that $T_{/F_s}$ is a $(d, c; n)$-near-block tree, and $T_{/C_i}$ is a $(d - 1, N_n; n)$-edge-sum tree, for each $i \in I_F$. 

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Let \( V' \) denote the set of vertices in the root graph \( H' = H/F_s \) of \( T' = T/F_s \), corresponding to \( V(C) \) in \( H \) in \( T \). Clearly \( |V'| \leq |V(C)| = c \). We consider the cases when \( |V'| = 1 \) and when \( 1 < |V'| \leq c \) separately.

First, assume that \( |V'| = 1 \). Then, the length of each cycle in \( H' \) is at most \( c - 1 \); we can see this as follows. If \( C' \) is a longest cycle of \( H \) different from \( C \), then, by (3.2.3), \( C \) and \( C' \) have at least two vertices in common. When \( F_s \) is contracted in \( H \), the vertices \( v_1 \) and \( v_2 \) are identified to a single vertex; thus, the subgraph of \( H' \) corresponding to \( C' \) is an edge-disjoint union of cycles of length less than \( c \). It follows that \( T' \) is a \((d, c - 1; n)\)-near-block tree if \( c > 2 \). If \( c = 2 \), then, since \( H \) is a 2-connected graph, each block of \( H' \) is a loop. It follows that \( G(T') \) is 1-isomorphic to \( \bigcup_{i \in I_{H'}} G(T_i/k_i) \), where \( I_{H'} = \{ i : h_i \in H' \} \). If \( c > 2 \), then let \( \mathcal{I}_0 = \mathcal{I} \); if \( c = 2 \), then let \( \mathcal{I}_0 = (\mathcal{I} - \{ T' \}) \cup \{ T_i/k_i : i \in I_{H'} \} \). Note that \( T_i/k_i \) is a \((d - 1, N_n; n)\)-edge-sum tree, for each \( i \in I_{H'} \). It follows that \( \mathcal{I}_0 \) is made up of a number of \((d - 1, N_n; n)\)-edge-sum trees and, if \( c > 2 \), one \((d, c - 1; n)\)-near-block tree. It is straightforward that \( G(T/C_s) \cong_1 G(\mathcal{I}_0) \).

If \( t(G(T/C_s)) = 1 \), then \( t(G(T)) \leq |C_s| + t(G(T/C_s)) < \frac{s \cdot n^2}{8} + 1 \), in which case we are done. If \( t(G(T/C_s)) = t(G(T)/C_s) > 1 \), then there is a 2-connected block \( B \) for which \( t(B) = t(G(T)/C_s) \). Since \( G(T/C_s) \cong_1 G(\mathcal{I}_0) \), the block \( B \) is isomorphic to some block of \( G(T_0) \), for some \( T_0 \in \mathcal{I}_0 \). By (4.4.4), there is a \((d, c - 1; n)\)-block tree or, if \( c = 2 \), a \((d - 1, N_n; n)\)-block tree whose composition is \( B \). So, if we let \( E_f = C_s \) and \( E_\emptyset = \emptyset \), we are done.

Now, let us consider the case in which \( 1 < |V'| \leq c \). Note that \( H' \) is connected since \( H \) is connected. So, for each pair \( \{ u, v \} \) of vertices in \( V' \), there is a \( uv \)-path in \( H' \), but we can see that there is no \( uv \)-path in \( H' \) consisting only of edges weighted \( s \), as follows. Let \( \{ u, v \} \subseteq V' \) be arbitrary. If there were a \( uv \)-path in \( H' \) consisting only of edges weighted \( s \), then \( H \) would contain, for some \( \{ u_0, v_0 \} \subseteq V(C) \), a \( u_0v_0 \)-path consisting only of edges weighted \( s \). It follows that \( F_s \) would contain a
u_0v_0-path; consequently, \( u_0 \) and \( u_0 \) would be identified to the same vertex when contracting \( F \) in \( H \), a contradiction. So, for each pair \( \{u, v\} \) of vertices in \( V' \), each \( uv \)-path in \( H' \) contains an edge weighted \( l \). Then \( H' \) contains a \( uv \)-edge-cut consisting only of edges weighted \( l \), for each pair \( \{u, v\} \) of vertices in \( V' \). We can see this, for an arbitrary pair \( \{u, v\} \) of vertices in \( V' \), as follows. Since there is no \( uv \)-path made up only of edges weighted \( s \), the vertices \( u \) and \( v \) belong to different components of \( H' \setminus \{h \in E(H') : w(h) = l\} \). Hence, there is a smallest set \( S_{uv} \) of edges weighted \( l \) in \( H' \) so that \( H' \setminus S_{uv} \) is not connected.

If \( |S_{u'v'}| \geq n \), for some pair \( \{u', v'\} \) of vertices of \( V' \), then consider the restriction \( T_\ast \) of the star \( (T')_\ast \) of \( T' \) induced by \( \{e_i : i \in I_{u'v'}\} \), where \( i \in I_{u'v'} \) if and only if \( h_i \in S_{u'v'} \). By (4.1.6), \( G(T_\ast) \leq G((T')_\ast) = G(T') \). Since the weight of \( h_i \) is \( l \), each cycle in \( K^i \) using \( k_i \) has length exceeding \( n(n-1) \), for each \( i \in I_{u'v'} \). Let \( C_k \) be a cycle in \( K^i \) using \( k_i \) for each \( i \in I_{u'v'} \). Consider \( T''_\ast = T_\ast \setminus \bigcup_{i \in I_{u'v'}} (E(K^i) - E(C_{k_i})) \); this is well-defined since \( k_i \) is the only labeled edge in \( K^i \), for each \( i \in I_{u'v'} \). It follows that \( G(T''_\ast) \) is obtained from \( H' \) by subdividing the edge \( h_i \) with \( |E(C_{k_i})| - 2 \) new vertices (that is, at least \( n(n-1) - 1 \) new vertices), for each \( i \in I_{u'v'} \), and adding, perhaps, some isolated vertices. Let \( P_i \) denote the path obtained by subdividing \( h_i \), for each \( i \in I_{u'v'} \), as just described. It follows that \( H' \setminus S_{u'v'} \) and \( G(T''_\ast) \setminus \bigcup_{i \in I_{u'v'}} P_i \) are identical, except, perhaps, for additional isolated vertices in \( G(T''_\ast) \setminus \bigcup_{i \in I_{u'v'}} P_i \). By (4.4.3), \( H' \setminus S_{u'v'} \) consists of two components \( C_{u'} \) containing \( u' \) and \( C_{v'} \) containing \( v' \). Since \( S_{u'v'} \) is a \( u'v' \)-edge-cut in \( H' \), exactly one endvertex \( u_i \) of \( h_i \) lies in \( C_{u'} \), and the other endvertex \( v_i \) of \( h_i \) lies in \( C_{v'} \), for each \( i \in I_{u'v'} \); hence, in \( G(T''_\ast) \setminus \bigcup_{i \in I_{u'v'}} P_i \), for each \( i \in I_{u'v'} \), one endvertex of \( P_i \) lies in \( C_{u'} \), and the other endvertex of \( P_i \) lies in \( C_{v'} \). Let us contract \( C_{u'} \) to a single vertex \( u^* \) and \( C_{v'} \) to a single vertex \( v^* \) in \( G(T''_\ast) \). Since \( P_i \) is obtained by subdividing \( h_i \) in \( H' \), for each \( i \in I_{u'v'} \), it follows that \( G_0 = G(T''_\ast)/E(C_{u'} \cup C_{v'}) \) is made up of \( |S_{u'v'}| \geq n \) pairwise internally
vertex-disjoint \( u^*v^* \)-paths, each having length at least \( n(n-1) \), and, perhaps, some isolated vertices. Hence, \( C_{n,n-2} \leq m \leq G(\mathcal{T}^*) \leq m \leq G(\mathcal{T}^t) \leq m \leq G(\mathcal{T}) \), and we are done.

It remains to consider the case when \(|S_{uv}| < n\), for each pair \( \{u,v\} \subseteq V' \). Let \( S_i = \bigcup_{\{u,v\} \subseteq V'} S_{uv} \), let \( I_S = \{ i \in [m]_+ : h_i \in S_i \} \), and let \( x_i \) and \( y_i \) denote the endvertices of \( k_i \) in \( K^i \). Since \( T \) is \( n \)-close (hence, \( T_{st} \) is \( n \)-close), and since the weight of \( h_i \) is \( l \), there is an \( x_iy_i \)-edge-cut \( D_{k_i} \) of size at most \( n2^{n^2} \) in \( K^i \), for each \( i \in I \). Let \( D_i = D_{k_i} - k_i \) for each \( i \in I \). Now, let us essentially delete \( S_i \) from \( H' \) in \( T' \) by deleting \( D_i = \bigcup_{i \in I_i} D_i \) from \( T' \). Note that \( |D_i| \leq \sum_{i \in I_i} |D_i| < |S_i| \cdot 2^{n^2} < n \cdot \frac{c(c-1)}{2} \cdot n2^{n^2} < 2^{n^2-1}c^2n^2 \). So, after deleting fewer than \( 2^{n^2-1}c^2n^2 \) edges from \( T' \), we may consider the collection \( \mathcal{T}' = \{ \mathcal{T}'_{S_i} \} \cup \{ \mathcal{T}'_{D_{k_i}} : i \in I_S \} \) of edge-sum trees in place of \( T' \). Note that \( G(T') \cong_1 G(T) \). Let \( \mathcal{T}'' = (\mathcal{T} - \{ T' \}) \cup \mathcal{T}' = \{ \mathcal{T}'_{F_s} \} \cup \{ \mathcal{T}'_{C_{k_i}} : i \in I_F \} \cup \{ \mathcal{T}'_{D_{k_i}} : i \in I_S \} \). It follows that \( G(T)/C_s \cdot D_i \cong_1 G(T'') \). Note that each element of \( \mathcal{T}'' - \mathcal{T}'_{F_s} \) is a \((d-1,N_n;n)\)-edge-sum tree. Also, note that \( \mathcal{T}'' = \mathcal{T}'_{F_s} \) is a \((d,c;n)\)-near-block tree; in fact, we show below that the root graph \( H'' = H/F_s \cdot S_i \) of \( T'' \) has only cycles of length less than \( c \).

Let \( C' \) be a longest cycle in \( H \) different from \( C \), if there is such a cycle. Since \( T \) is a block tree, \( H \) is 2-connected; consequently, by (3.2.3), \(|V(C) \cap V(C')| \geq 2 \). Let \( \{v_1,v_2\} \subseteq V(C) \cap V(C') \). If \( v_1 \) and \( v_2 \) are identified to the same vertex when \( F_s \) is contracted in \( H \), then the subgraph of \( H' \) corresponding to \( C' \) is an edge-disjoint union of cycles of length of less than \( c \). If \( v_1 \) and \( v_2 \) are identified to distinct vertices that we shall call \( v_1 \) and \( v_2 \), respectively, in \( H' \), then \( v_1 \) and \( v_2 \) belong to distinct components of \( H'' \) since \( S_{v_1v_2} \) was essentially deleted from \( T' \) in forming \( T'' \) (hence, \( S_{v_1v_2} \) was deleted from \( H' \) in forming \( H'' \)). It follows that \( S_{v_1v_2} \cap E(C') \) is non-empty. So, the subgraph of \( H'' \) corresponding to \( C' \) contains fewer than \( c \) edges; consequently, \( C' \) is not a cycle of length \( c \) in \( H'' \). Hence, \( H'' \) has only cycles of length less than \( c \).
We conclude that $T''$ is a $(d, c-1; n)$-near-block tree if $c > 2$. Hence, $T''$ consists of one $(d, c-1; n)$-near-block tree and a number of $(d-1, N_n; n)$-edge-sum trees if $c > 2$. If $c = 2$, then each block of $H''$ contains at most one edge. Let $I_1$ and $I_2$ denote the subsets of $[m]_+$ so that $i \in I_1$ if and only if $h_i$ is a loop in $H''$, and $i \in I_2$ if and only if $h_i$ is a link-edge in $H''$. It follows that $G(T'')$ is 1-isomorphic to $\bigcup_{i \in I_1} G(T_i/k_i) \cup \bigcup_{i \in I_2} G(T_i/k_i)$. If $c > 2$, then let $T'_0 = T''$; If $c = 2$, then let $T'_0 = (T'' - \{T''\}) \cup \{T_i/k_i: i \in I_1\} \cup \{T_i/k_i: i \in I_2\}$. It follows that $G(T)/C_s \setminus D_1 \cong_1 G(T'_0)$. Note that $T'_0$ consists of a number of $(d-1, N_n; n)$-edge-sum trees and, if $c \geq 2$, one $(d, c-1; n)$-near-block tree.

If $t(G(T)/C_s \setminus D_1) \leq 1$, then we have $t(G(T)) \leq |C_s| + |D_1| + t(G(T)/C_s \setminus D_1) < \frac{c^2 n^2}{8} + 2n^2 - 1 \cdot c^2 n^2 + 1$, in which case we are done. So, we may assume that $t(G(T)/C_s \setminus D_1) = t(G(T)/C_s \setminus D_1) > 1$. Then there is a 2-connected block $B$ for which $t(B) = t(G(T)/C_s \setminus D_1)$. Since $G(T)/C_s \setminus D_1 \cong_1 G(T'_0)$, the block $B$ is isomorphic to some block of $G(T'_0)$, for some $T'_0 \in T'_0$. By (4.4.4), there is a $(d, c-1; n)$-block tree or, if $c = 2$, a $(d-1, N_n; n)$-block tree whose composition is $B$. So, if we let $E_f = C_s$ and $E_\backslash = D_1$, we are done. Thus, the lemma holds. □

The next lemma is an extension of (4.4.8) in which some edges of the root graph of a $(d, c; n)$-close block tree may be unlabeled.

(4.4.9) Lemma. Let $T = (G, T)$ be a $(d, c; n)$-close block tree whose root graph contains no cycle of length exceeding $c$, for some integers $n, c$, and $d$ exceeding 3, 1, and 0, respectively. Then one of the following holds.

(i) There are disjoint subsets $E_f$ and $E_\backslash$ of $E(G(T))$ containing fewer than $\frac{c^2 n^2}{8}$ edges and $2n^2 - 1 \cdot c^2 n^2$ edges, respectively, so that if $B$ is a 2-connected block of $G(T)/E_f \backslash E_\backslash$ for which $t(B) = t(G(T)/E_f \backslash E_\backslash)$, then there is a $(d, c-1; n)$-block tree if $c \geq 3$ or a $(d-1, N_n; n)$-block tree if $c = 2$, whose composition is $B$. 

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(ii) $t(G(T)) < \frac{\varepsilon n^2}{8} + 2n^3 - 1c^2n^2 + 1$.

(iii) $C_{n,n-2} \leq_m G(T)$, or $C_{n,n-2}^* \leq_m G(T)$.

Proof. We may assume that the root graph $H$ of $T$ contains at least one unlabeled edge; otherwise, the desired result is immediate, by (4.4.8). Let $E_0$ denote the set of unlabeled edges in $H$. For each $e \in E_0$, we can assign a direction and a new label $\varepsilon_e$ to $e$, add a pendant link $\varepsilon_e = \xi \eta_e$ at the root $\xi$ of the tree $T$ of $T$, let the node graph corresponding to $\eta_e$ be a 2-cycle $C_e$, and assign a direction and the label $\varepsilon_e$ to one of the edges of $C_e$. Let $\hat{T}$ denote the resulting $(d, c; n)$-block tree, and let $f_e$ denote the unlabeled edge of $C_e$, for each $e \in E_0$. It is evident that $G(\hat{T}) \cong G(T)$. If $\hat{T}$ is not $n$-close, then, by (4.3.6), an element of $\{C_{n,n-2}, C_{n,n-2}^*\}$ is a minor of $G(\hat{T}) \cong G(T)$, and we are done. So, we may assume that $\hat{T}$ is $n$-close.

If $t(G(\hat{T})) < \frac{\varepsilon n^2}{8} + 2n^3 - 1c^2n^2 + 1$, or if $C_{n,n-2} \leq_m G(\hat{T})$, then we are done, since $G(\hat{T}) \cong G(T)$. Otherwise, by (4.4.8), there are disjoint subsets $\hat{E}_f$ and $\hat{E}_\setminus$ of $E(G(\hat{T}))$ containing fewer than $\frac{\varepsilon n^2}{8}$ edges and $2n^3 - 1c^2n^2$ edges, respectively, so that if $B$ is a 2-connected block of $G(\hat{T})/\hat{E}_f \setminus \hat{E}_\setminus$ for which $t(B) = t(G(\hat{T})/\hat{E}_f \setminus \hat{E}_\setminus)$, then there is a $(d, c - 1; n)$-block tree if $c \geq 3$ or a $(d - 1, N_n; n)$-block tree if $c = 2$, whose composition is $B$. Note that when (4.4.8) is applied to $\hat{T}$, each edge in $E_0$ is weighted $s$ in $\hat{T}$, and the edges in $\hat{E}_\setminus$ correspond to edges of $H$ weighted $l$ in $\hat{T}$. Let $E_f = (\hat{E}_f - \{f_e : e \in E_0\}) \cup \{e : f_e \in E_f\}$ and $E_\setminus = \hat{E}_\setminus$. It is straightforward that $|E_f| = |\hat{E}_f|$ and $|E_\setminus| = |\hat{E}_\setminus|$, that $E_f$ and $E_\setminus$ are disjoint subsets of $E(G(T))$, and that $G((T)/E_f \setminus E_\setminus) \cong G(\hat{T})/\hat{E}_f \setminus \hat{E}_\setminus$. The result follows. □

Now, we are nearly ready to prove (4.4.2), whose proof uses several of the above lemmas. Before we prove (4.4.2), let us obtain a lower bound for $N_n$. Recall, that $N_n = N$, where $N$ is the number from (3.2.8), and that $N = 2^k + 1$, where $k = (R_2(m))^m$ and $m$ are the numbers depending on $n > 2$ described in the proof of (3.2.1). It is straightforward to check that $m$ is much greater than $n$, and

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it is easy to see from the definition of $R_2(i)$, given in the proof of (3.2.1), that $R_2(n) > n$, when $n > 2$. It follows that $N_n > 2^{R_2(n)n} > 2n^n$.

Proof of (4.4.2). Let $\mathcal{T} = (G,T)$ be a $(d, c; n)$-close 3-block tree. Recall that $n$ is an integer exceeding 3, $0 \leq d \leq 2(n - 1)$, and $2 \leq c \leq N_n$. What we shall show is that

$$t(G) \leq f(d) = d \sum_{i=1}^{N_n} \left( \frac{i^3n^2}{8} + 2n^3 - 1i^2n^2 \right) + \frac{N_n(N_n + 1)}{2},$$

or $C_{n,n-2} \leq_m G(T)$, or $C_{n,n-2}^* \leq_m G(T)$. Note that

$$f(d) = d \sum_{i=1}^{N_n} \left( \frac{i^3n^2}{8} + 2n^3 - 1i^2n^2 \right) + \frac{N_n(N_n + 1)}{2}
\leq 2(n - 1) \left( \frac{n^2}{8} \sum_{i=1}^{N_n} i^3 + 2n^3 - 1 \sum_{i=1}^{N_n} i^2 \right) + \frac{N_n(N_n + 1)}{2}
< 2n \left( \frac{n^2N_n^2(N_n + 1)^2}{8 \cdot 4} + \frac{2n^3 - 1n^2N_n(N_n + 1)(2N_n + 1)}{6} \right) + \frac{N_n(N_n + 1)}{2}
< \frac{n^3(N_n + 1)^4}{16} + \frac{2n^3(N_n + 1)^3}{3} + \frac{N_n(N_n + 1)}{2} = F(n).$$

We proceed by induction on $d$ which includes within it induction on $c$. If $d = 0$, then, by (4.4.5), $t(G(T)) \leq \frac{c(c+1)}{2} \leq \frac{N_n(N_n + 1)}{2} = 0 \cdot \sum_{i=1}^{N_n} \left( \frac{i^3n^2}{8} + 2n^3 - 1i^2n^2 \right) + \frac{N_n(N_n + 1)}{2}$, as required. For the remainder of the proof, let us assume that $d > 0$, and the result holds for each $d' \in [d - 1]$.

If the root graph of $\mathcal{T}$ is a cycle of length at least $n$, then, by (4.4.6), either $C_{n,n-2} \leq_m G(T)$, in which case we are done, or $t(G(T)) \leq n - 2 < 2^n < N_n < f(d)$, in which case we are done, or there are set $S\setminus$ of at most $n - 3$ edges in $G(T)$ and a $(d - 1, N_n; n)$-block tree $\mathcal{T}_B$ whose composition is a 2-connected block $B$ of $G(T)\setminus S\setminus$ for which $t(B) = t(G(T)\setminus S\setminus)$. It follows that $B \leq_m G(T)$ and that $t(G(T)) \leq |S\setminus| + t(B)$. If $\mathcal{T}_B$ is not $n$-close, then, by (4.3.6), $C_{n,n-2} \leq_m G(T)$ or $C_{n,n-2}^* \leq_m G(T)$, and we are done. So, we may assume that $\mathcal{T}_B$ is $n$-close. By the induction hypothesis, $t(B) \leq f(d - 1)$. It follows that $t(G(T)) \leq n - 3 + f(d - 1) < \sum_{i=1}^{N_n} \left( \frac{i^3n^2}{8} + 2n^3 - 1i^2n^2 \right) + f(d - 1) = f(d)$, as required.
We may assume for the remainder of the proof that the root graph of $\mathcal{T}$ contains no cycles of length exceeding $c$. By (4.4.9), either an element of $\{C_{n,n-2}, C_{n,n-2}^*\}$ is a minor of $G(\mathcal{T})$, in which case we are done, or $t(G(\mathcal{T})) < \frac{c^2n^2}{8} + 2n^3-1c^2n^2 + 1 < \frac{N_n^2}{8} + N_n^2n^2 + \frac{N_n(N_n+1)}{2} < f(d)$, in which case we are done, or conclusion (i) in (4.4.9) holds. So let us assume that conclusion (i) in (4.4.9) holds.

Now, we shall show that $t(G(\mathcal{T})) \leq g(c) = \sum_{i=1}^{c} \left( \frac{i^2n^2}{8} + 2n^3-1i^2n^2 \right) + (d-1) \sum_{i=1}^{N_n} \left( \frac{i^2n^2}{8} + 2n^3-1i^2n^2 \right) + \frac{N_n(N_n+1)}{2} \leq f(d)$. Hence, it will follow that $t(G(\mathcal{T})) \leq f(d)$, or $C_{n,n-2} \leq_m G(\mathcal{T})$, or $C_{n,n-2}^* \leq_m G(\mathcal{T})$. If $c = 2$, then there are disjoint sets $E_f$ and $E_{\setminus}$ containing fewer than $n^2$ and $2n^3+1n^2$ edges, respectively, in $G(\mathcal{T})$ and a $(d-1, N_n; n)$-block tree $\mathcal{T}_B$ whose composition is a 2-connected block $B$ of $G(T)/E_f \setminus E_{\setminus}$ such that $t(B) = t(G(\mathcal{T})/E_f \setminus E_{\setminus})$. If $\mathcal{T}_B$ is not $n$-close, then, by (4.3.6), $C_{n,n-2} \leq_m G(\mathcal{T})$ or $C_{n,n-2}^* \leq_m G(\mathcal{T})$, and we are done. So, we may assume that $\mathcal{T}_B$ is $n$-close. By the induction hypothesis, $t(B) \leq f(d-1)$. It follows that $t(G(\mathcal{T})) \leq |E_f| + |E_{\setminus}| + t(B) < n^2 + 2n^3+1n^2 + f(d-1) < \sum_{i=1}^{2} \left( \frac{i^2n^2}{8} + 2n^3-1i^2n^2 \right) + f(d-1) = g(2) \leq f(d)$, as required. So, let us assume that $3 \leq c \leq N_n$ and that $t(G(\mathcal{U})) \leq g(c')$, or $C_{n,n-2} \leq_m G(\mathcal{U})$, or $C_{n,n-2}^* \leq_m G(\mathcal{U})$ when $\mathcal{U}$ is a $(d, c' ; n)$-close block tree and $c'$ satisfies $2 \leq c' < c$.

There are disjoint subsets $E_f$ and $E_{\setminus}$ of $G(\mathcal{T})$ containing fewer than $\frac{c^2n^2}{8}$ edges and $2n^3-1c^2n^2$ edges, respectively, and a $(d, c-1; n)$-block tree $\mathcal{T}_B$ whose composition is a 2-connected block $B$ of $G(\mathcal{T})/E_f \setminus E_{\setminus}$ such that $t(B) = t(G(\mathcal{T})/E_f \setminus E_{\setminus})$, by (4.4.9). Again, by (4.3.6), if $\mathcal{T}_B$ is not $n$-close as before, $C_{n,n-2} \leq_m G(\mathcal{T})$ or $C_{n,n-2}^* \leq_m G(\mathcal{T})$, and we are done. So, we may assume that $\mathcal{T}_B$ is $n$-close. By the second induction hypothesis, $t(B) \leq g(c-1)$. It follows that $t(G(\mathcal{T})) \leq |E_f| + |E_{\setminus}| + t(B) < \frac{c^2n^2}{8} + 2n^3-1c^2n^2 + g(c-1) = g(c) \leq f(d)$, as required. The theorem follows. □
Since each 2-connected graph with more than two edges can be decomposed into a unique 3-block tree, and since a 2-connected graph with at most 2 edges has very small type, the theorem below follows immediately on combining results (4.1.7), (4.2.1), and (4.4.2).

(4.4.10) **THEOREM.** If $G$ is a 2-connected graph whose type is at least \( \frac{n^3(n_+1)^4 + 2^{n^2}(N_n+1)^2}{3} + \frac{N_n(n_+1)}{2}, \) for some integer $n$ exceeding 3, then an element of \( \{F_n, C_n,n-2, C^*_n,n-2\} \) is a minor of $G$. \( \square \)

In Chapter 3, we saw that a 3-connected graph has large type if and only if it has a large fan as a minor. Note that the converse of (4.4.10) is not true. Recall that the graph in Figure 2.4 can be generalized to a graph $H$ so that $C^*_n,n-2 \leq_m H$, for an arbitrarily large $n$, but $t(H) \leq 4$. 

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CHAPTER 5
ON CONTRACTION-TYPE

In this chapter, we shall see that the upper bound for the type of a connected matroid, described in (3.2.6), that follows from Seymour's result (3.2.4), cannot be significantly improved, even when restricted to the class of 2-connected graphs. The contraction-only nature of this result of Seymour leads to questions about what we shall call the contraction-type of a graph, which we define later.

Let us recall that (3.2.4) states that if $C$ is a largest circuit of a connected matroid $M$, then $M/C$ contains only circuits of size less than $|C| = n$. The bound $t(M) \leq \frac{n(n+1)}{2}$ in (3.2.6) is obtained by repeatedly applying (3.2.4). It is convenient to think of reducing a matroid $M$ (or a graph $G$) in this way to the empty matroid (or an edgeless graph) by performing a number of steps, where the first step consists of contracting an element (or an edge) in each component (or block) of $M$ (or $G$), and the $n$th step consists of contracting an element (or an edge) in each component (or block) of the matroid (or graph) obtained after the $(n - 1)$st step. So, the number of steps needed to contract a connected matroid $M$ to the empty matroid by applying (3.2.4) repeatedly is (at worst) quadratic in the size of a largest circuit in $M$. We ask the following question, which we shall answer negatively.

(5.1.1) QUESTION. In general, is the number of steps required to contract a 2-connected graph $G$ to an edgeless graph by repeatedly applying (3.2.4) better than quadratic in $n$, where $n$ is the length of a longest cycle in $G$?

We can see that this question is answered negatively by considering the generalization $D_{4n}$ of the graph $D_{16}$ in Figure 5.1 below. Let $D_{4n}$, where $n$ is an integer exceeding 1, be obtained as follows. Choose a vertex $x_i$ of the cycle $C_{4i}$ with $4i$ edges, and let $y_i$ be the vertex opposite $x_i$ (that is, $C_{4i}$ is made up of two distinct $x_iy_i$-paths, each of length $2i$), for each $i \in [n]$. For each $i \in [n - 1]$,
identify $y_i$ with $x_{i+1}$; let us refer to this vertex identification as $x_{i+1}$, and let $y_n$ be called $x_{n+1}$. Finally, add an edge $x_i x_{i+2}$, for each $i \in [n-1]_+$. Note that $D_{4n}$ is a block that contains a unique longest cycle whose length is $4n$. It is straightforward to show that, by repeatedly applying (3.2.4), it takes $\sum_{i=1}^{n} 4i + 1$ steps (that is, $2n^2 + 2n + 1$ steps) to obtain an edgeless graph. We describe this repeated application of (3.2.4) to $D_{16}$ immediately following Figure 5.1.

![Figure 5.1. The graph $D_{16}$.](image)

The first time that we apply (3.2.4), we contract the 16 edges of the unique longest cycle of $D_{16}$; thus, we have taken 16 steps. The resulting graph $G_1$ is isomorphic to $D_{12} \cup x_3 x_4$. Also, $G_1$ is a block and has a unique longest cycle $C_{12}$ with 12 edges. In the second application of (3.2.4), we contract $C_{12}$ in $G_1$; thus, we require 12 steps. The resulting graph $G_2$ is isomorphic to $D_8 \cup x_2 x_3$ with a loop $e_2$ attached at $x_3$. The two blocks of $G_2$ are the subgraphs of $G_2$ that are isomorphic to $e_2$ and $D_8 \cup x_2 x_3$. Note that $e_2$ is a 1-cycle and that $D_8 \cup x_2 x_3$ contains a unique longest cycle $C_8$ of length 8. So, the third time that we apply (3.2.4), we simultaneously contract $e_2$ and $C_8$ in $G_2$; thus, we require 8 steps. The resulting graph $G_3$ is isomorphic to $K_{2,3}$ with a loop $e_3$ attached at one of the vertices whose degree in $K_{2,3}$ is 3. Clearly, $G_3$ contains a unique longest cycle $C_4$ of length 4. When we simultaneously contract $e_3$ and $C_4$ in $G_3$ for the fourth application of (3.2.4), the resulting graph consists of a loop, which requires one
last application of (3.2.4) to contract the last loop to an edgeless graph. Thus, we needed \(16 + 12 + 8 + 4 + 1 = \sum_{i=1}^{4} 4i + 1\) steps to obtain an edgeless graph.

We have just seen an example of a 2-connected graph \(G = D_{4n'}\) for which the number of steps needed to contract a \(G\) to an edgeless graph by repeatedly applying (3.2.4) is quadratic in \(n\), where \(n = 4n'\) is the length of a longest cycle of \(G\), and \(n'\) is an integer exceeding 1; here, the number of steps required is \(\sum_{i=1}^{4} 4i + 1 = \frac{n^2}{8} + \frac{n}{2} + 1\).

If we are more selective about the edges that we contract in each block of a graph \(G\), then can we significantly improve the upper bound given in (3.2.7) for contracting a graph to an edgeless graph? This naturally leads to a concept similar to type in which we can only contract edges; we define this concept, namely contraction-type, as follows.

(5.1.2) DEFINITION. Let \(G\) be a graph. If \(G\) is edgeless, then the contraction-type of \(G\), denoted \(ct(G)\), is 0. If \(G\) is a block, then \(ct(G) = \min\{ct(G/e) : e \in E(G)\}\} + 1\). If \(G\) is not a block, then \(ct(G) = \max\{ct(B) : B\) is a block of \(G\}\}. □

Intuitively, the contraction-type of a graph \(G\), is the smallest nonnegative integer \(n\) such that there is a sequence of graphs \(G = G_0, G_1, \ldots, G_n\), where \(G_i\) is obtained by contracting one edge from each block of \(G_{i-1}\), and where \(G_n\) is edgeless. It follows from (3.2.7) that the contraction-type of a 2-connected graph \(G\) is at most \(\frac{n(n+1)}{2}\), where \(n\) is the length of a longest cycle in \(G\). We show that the contraction-type of \(D_{16}\) is at most 13 immediately following Figure 5.2 below.

![Figure 5.2. \(D_{16}\) with some specified edges.](image-url)
Let us start by contracting the five bold edges of $D_{16}$, as indicated in Figure 5.2. The resulting graph $G_1$ consists of a block that is a 2-cycle and blocks that are 4-cycles, 6-cycles, and 8-cycles (two of each, in fact). It will take at most 8 more steps to contract simultaneously each block of $G_1$ to obtain an edgeless graph. Thus, $ct(D_{16}) \leq 13$. It is straightforward to show that this generalizes to $ct(D_{4n}) \leq 3n + 1$. Note that the bound for the contraction-type of $D_{4n}$ that we obtained by applying (3.2.4) repeatedly is quadratic in the length of the longest cycle of $D_{4n}$. We have just seen, however, that $ct(D_{4n})$ is (at worst) linear in $n$ (hence, is at worst linear in the length of the longest cycle of $D_{4n}$). We conclude this chapter by asking the following.

(5.1.3) **Question.** Is there a linear function $f(n)$, for which $ct(G) \leq f(n_G)$, for each graph $G$, where $n_G$ is the length of a longest cycle in $G$ if $G$ contains a cycle, and $n_G = 1$ if $G$ is acyclic?
CHAPTER 6
SUMMARY OF RESULTS

In this dissertation we have proven that if a graph has a large fan as a minor, then it has large type. Then, we proved that if a 3-connected graph has large type, then it has a large wheel as a minor. Let us recall that the proof of this relied on a result in [DOOV] which implies that if a 3-connected graph has a sufficiently large cycle, then it contains a large wheel as a minor. On combining these results we obtained a weak characterization of 3-connected graphs of large type.

For the other main result of this paper, we proved that if a graph $G$ has sufficiently large type depending on an integer $n$ exceeding 3, then an element of $\{F_n, C_{n,n-2}, C^*_{n,n-2}\}$ is a minor of $G$. The proof relied on Tutte's decomposition of 2-connected graphs into 3-block trees. First, we saw that, given the 3-block tree $T$ of a 2-connected graph $G$, if some 3-connected node graph of $T$ has a long cycle, then a large fan is a minor of $G$. Then, we saw that if $T$ contains a long path, then a large fan is a minor of $G$. Next, we saw that if $T$ is not $n$-close, then $C_{n,n-2}$ or $C^*_{n,n-2}$ is a minor of $G$. Finally, we proved that if $T$ is a $(d, c; n)$-close block tree, then either $G$ has small type, or an element of $\{C_{n,n-2}, C^*_{n,n-2}\}$ is a minor of $G$.

Let us note that the techniques used in this dissertation do not appear to be of any use in proving (1.1.4), but they may be useful in extending the main result (4.4.10) of Chapter 4 in some way to regular matroids by using Seymour's decomposition theorem for regular matroids (see [O], Section 13.2). Another direction to extend this work is to further investigate contraction-type.
REFERENCES


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