The Role of the Unit as a Cognitive Bridge Between Additive and Multiplicative Structures.

Nancy Sutton Alexander

Louisiana State University and Agricultural & Mechanical College

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THE ROLE OF THE UNIT AS A COGNITIVE BRIDGE BETWEEN ADDITIVE AND MULTIPLICATIVE STRUCTURES

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy in The Department of Curriculum and Instruction

by

Nancy Sutton Alexander
B.S., Louisiana Tech University 1976
M.S., Louisiana Tech University 1983
May, 1997
DEDICATION

To my grandchildren

During the final weeks of writing this dissertation, I was blessed by the birth of my first grandchild—Miss Jordon Destiny Nalley. I dedicate this work to Jordon and my future grandchildren. I hope that they will experience the types of learning experiences advocated in this study so that their own learning of mathematics will be both meaningful and pleasurable.
ACKNOWLEDGEMENTS

There are many people I wish to thank for their assistance as I was engaged in the researching and writing of this dissertation.

First of all I thank Dr. David Kirshner for directing this study. I am grateful for his invaluable criticism and exacting standards as well as his encouragement and friendship. I also appreciate the time and considerable talents that Dr. Bill Doll, Dr. George Noell, Dr. Miles Richardson, and Dr. Jim Wandersee invested in this project. Each read the manuscript carefully and offered very valuable suggestions.

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Finally, to my relatives and friends, I am also very grateful. I was blessed by the pride of my father, Jesse A. Sutton, who was most eager for me to complete "that little paper." I am thankful for the support I received from my precious children -- Ginger, Rod, and Lisa. I hope their pride in their mother for completing this work makes up for all those evenings and weekends I couldn't spend with them; we'll do that now!

Most of all I am grateful for my wonderful husband, Rodney Alexander, who supported and believed in me even in times when I doubted myself. His
loving partnership has made it possible for me to have the things I want most in life—meaningful and intellectually satisfying work, and a full and very happy personal life.
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ABSTRACT

The rational number domain is known for the great difficulties it presents for students and teachers. As students construct knowledge of fractions, they search for connections to existing knowledge. Procedures from the whole number domain are tried, but many do not work on the new numbers! Recent research has focused on the unit as a way to link whole number and rational number understanding (Behr, Harel, Post, & Lesh, 1992).

Studies have suggested that students intuitively form units (Lamon, 1994), and that this intuitive knowledge can serve as a foundation for rational number understanding (Lamon, 1994; Mack, 1990). Golding (1994) found that the unit concept can link whole number and rational number domains for addition and subtraction. This study examined the role of the unit as a link between whole and rational number domains for multiplication and division. Further it explored whether students' learning could carry over from the group setting to individual performance, and whether their new understandings could be applied to standard school tasks.

This study describes the evolving cognitive processes of four seventh-grade students of varying mathematical ability selected from a seventh-grade class of a rural K-12 school. A fifteen lesson teaching experiment was designed to build on students' existing knowledge of unit and extend this to the rational number domain. Data were collected through videotapes, audiotapes, researcher journal, students' written work, and individual student interviews.
Several conclusions were drawn: (a) students developed a flexible concept of unit; (b) modeling provided continuity between conceptual domains; (c) equipartitioning remained a persistent difficulty; (d) sustained focus on the measuring unit is difficult; (e) faulty selection and use of measurement units handicap development of models; (f) unitizing skills endure and are extendable; and (g) models can inform procedural methods and/or provide alternative solution methods.

The study points to the need for more school practice in partitioning and measurement activities and more extensive use of modeling to facilitate development of unit concepts. Future research should investigate strategies for enabling students to overcome constraints of primitive models of division.
CHAPTER 1

INTRODUCTION

The purpose of this study is to examine students' development of the unit concept of rational number and to examine the role of the unit as a possible link between whole number and rational number domains within multiplication and division. The intent of this chapter is to justify the study by situating it within the larger field of mathematics didactics. The first section will provide a brief look at some of the difficulties students encounter in learning rational number concepts. Some probable causes are presented in the second section, while a possible solution is examined in the third section. The concluding section contains the research questions arising from the first three sections as well as an overview of the organization of the study.

The Problem

Rational number concepts are among the most complex and important mathematical ideas students encounter in the presecondary school years (Behr, Lesh, Post, & Silver, 1983), as evidenced by the time devoted in the middle grades for developing rational number computational skills. Yet student performance with rational numbers on recent national assessments indicates the lack of understanding of rational number concepts (Lindquist, 1989); even those students who have learned how to calculate fractions by applying memorized algorithms seem to have little or no development of the pertinent underlying concepts (Lindquist, 1989). Much of the literature attests to the frustration
students endure as they encounter rational number situations (Behr, Harel, Post, 
& Lesh, 1992; Behr, Lesh, Post, & Silver, 1983; Kieren, 1976). This poor 
performance with rational numbers could be attributable to students’ failure to 
"internalize a workable concept of rational number" (Behr, Wachsmuth, Post, & 

A concept can be defined as a perceived regularity in events or objects 
theory, the meaning of any concept is defined by the network of propositions the 
learner has connected to it (Novak, 1977). The development of knowledge 
about a concept is achieved by the construction of relationships between pieces 
of information (Hiebert and Lefevre, 1986).

Number concepts are foundational to mathematics. Rational numbers, or 
fractions, represent an extension of the set of integers, which are in turn an 
extension of the set of natural numbers. Thus the concept of fraction can be 
considered a more mature concept of number. In similar fashion, rational 
number concepts are foundational to further study of real numbers and algebraic 
generalizations, which follow in the curriculum. Thus students’ lack of 
understanding of rational numbers hinders further mathematical development 
(Hart, 1988; Kieren, 1988; Ohlsson, 1988). "Many student difficulties in algebra 
can be traced back to an incomplete understanding of earlier fraction ideas" 
(Behr & Post, 1988, p. 192).
Some Causes

The low performance of students on rational number items persists despite the emphasis which school programs give to procedural skills and computational algorithms (Behr, Lesh, Post, & Silver, 1983). Indeed Behr et al. (1983) contend that this poor performance could be the result of this curricular emphasis on procedures instead of the conceptual development of rational number ideas. Many students' understanding of rational numbers extends only to a knowledge of rote procedures, often inappropriately applied (Mack, 1990). Students are asked to memorize numerous symbol manipulation rules that have no conceptual meaning for them (Weare & Hiebert, 1988). As a result these memorized rules are often forgotten, distorted, or rigidly applied. When students memorize rules with no conceptual understanding the "procedural skill outstrips conceptual competence" (Weare & Hiebert, 1988, p. 233); that is, students can find an answer by executing a rule but they have no rationale for the validity of the procedure and few resources for judging the reasonableness of the answer.

As students search for meaning in the teacher-taught algorithms, they strive for a connection with the already familiar whole number domain (Fischbein, Deri, Nello & Marino, 1985; Mack, 1990). When a connection is not identified, misconceptions occur as the new knowledge is forced to conform to pre-existing schemes (Skemp, 1987). Students often have difficulty overcoming their whole number ideas while working with fractions (Hiebert & Wearne,
Evidence of a student generated strategy referred to as "whole number dominance" (Behr, Wachsmuth, Post, & Lesh, 1984, p. 328) was seen in numerous student responses on rational number items in the Rational Number Project. When asked to order two fractions, the majority of fourth graders responded that one third is less than one fourth "because three is less than four" (Behr et al., 1984, p. 328). Additionally we see instances of students using simple additive strategies to solve multiplicative problems (Hart, 1981). For example, suppose there are two similar geometric figures, with two corresponding sides of 4 units and 6 units.

If students are asked for the length of a side in the larger figure, corresponding to a side of 7 units in the smaller figure, many will say 9 units. They interpret the problem as one in which the numbers are related additively rather than multiplicatively.

Whole number dominance is also evidenced by the limited models students hold for multiplication and division (Greer, 1992). Apparently resulting from a curricular emphasis on multiplication and division of whole numbers, students form conceptions about multiplication and division such as "multiplication makes bigger; division makes smaller," or MMB:DMS (Graeber & Baker, 1991). Although such conceptions are in accord with the operations of
multiplication and division in the whole number domain, they are incongruent
with these operations in the rational number domain and form a barrier for
solving many problems whose quantities are fractions (Harel, Behr, Post, &
Lesh, 1994). Such misconceptions established by students are not easily
outgrown (Graeber, Tirosh, & Glover, 1989). Many of the same
misunderstandings or misconceptions that have been identified in students are
also prevalent among teachers (Post, Harel, Behr, & Lesh, 1988). There are
even concerns over reasonably educated adults' knowledge of rational numbers
(Boulet, 1995). This raises broad based questions concerning the teaching and
learning of rational numbers.

A Possible Solution

Informal Knowledge

Although many students' knowledge of fractions is initially faulty or
disconnected, recent research shows that many students possess a rich store of
intuitive or informal knowledge related to fractions prior to instruction and that
they are able to draw on this knowledge to solve a variety of problems (Kieren,
1988; Leinhardt, 1988; Mack, 1990). This informal knowledge may be correct or
incorrect (Mack, 1990) and significantly influences what students learn from
instruction (Carpenter & Peterson, 1988). Although some studies assert that
misconceptions can arise from this primitive or informal knowledge (Fischbein et
al., 1985; Kerslake, 1986; Mack, 1990), other studies have shown that informal
knowledge can be used as a foundation upon which meaningful concepts can be built (Hiebert, 1988; Lamon, 1994; Mack, 1993; Pothier & Sawada, 1983).

**Unit Formation**

Numbers arise through counting procedures applied to discrete objects and through measuring procedures applied to continuous quantities (Greer, 1992). Those numbers which can be traced back to counting procedures but that go outside the natural numbers through application of division at some point are known as rational numbers, and are represented in the form $a/b$ where $a$ and $b$ are integers and $b$ is nonzero. For the purpose of the proposed study, consider the positive rational numbers, customarily called fractions (Freudenthal, 1983). In the fraction $a/b$, $a$ is called the numerator and $b$ the denominator. Fractions having the integer 1 as a numerator and any other natural number as denominator (i.e., $1/n$) are called unit fractions.

Children's initial understanding of rational numbers is not derived from $a$ and $b$ or from the mathematical notation $a/b$, but instead, their initial comprehension is derived from their physical embodiments (Post, Wachsmuth, Lesh, & Behr, 1985). Examples of such embodiments might include a picture of an object partitioned into $b$ equal pieces with $a$ of them shaded, or a set of $b$ black chips with $a$ of them covered with red chips. The object being partitioned, or "fractured" (Freudenthal, 1983), is referred to as a unit. This unit is the focus of the present study.
It has been shown that the formation of units develops intuitively prior to schooling (von Glasersfeld, 1981) and that this informal knowledge can be used as a foundation for building rational number understanding (Golding, 1994; Lamon, 1994). A unit is an entity that is treated as a whole (Confrey & Harel, 1994, p. xvii). It may consist of a single object, a collection of objects, or it may be composed of units itself. For example, a case of cola could be considered as a single unit, or one 24-pack. Partitioning of this unit into subgroups would result in the formation of composite units (units of units), for example, two 12-packs or four 6-packs. Further partitioning of these composite units could result in units of units of units; the two 12-packs could be packaged as two groups of three 4-packs. Considering each of the 24 cans as singletons, you have 24 one-units or 24 cans. An understanding of the unit concept involves viewing a whole as a nested system of units (Steffe, 1994).

Mathematics of Quantity

Recent research has focused on the unit concept of rational number because this construct is regarded as critical for establishing links between research on rational numbers and research on whole number arithmetic (Behr, Harel, Post, & Lesh, 1992; Steffe, 1994). Most traditional whole number problems focus on singleton units, or units of one, as illustrated by the usual solution method to the following problem:

Four boxes of golf balls, each containing one dozen balls, are to be repackaged into mini-sets of three balls each. How many three-packs can be formed?
The traditional approach, or *mathematics of number* (Schwartz, 1988) approach, suggests a solution as follows: $4 \times 12 = 48; 48 \div 3 = 16$. In this approach, all units are changed to singletons, or units of size one. If children were to model the solution to this problem, a more natural approach would be to go directly to units of three. Each box of 12 balls (12 [one-unit]s) could be grouped as 4 [three-unit]s. Four boxes would thus contain 4 (4 [three-unit]s), or 16 [three-unit]s. This approach, termed the *mathematics of quantity* (Schwartz, 1988) approach, emphasizes the quantities involved, both the unit of measure and the magnitude of the quantity, rather than just the numbers.

By viewing problems from a mathematics of quantity approach, the focus is on the quantities, the number and the unit, not just on the numbers. Research by Lamon (1994) has shown that the mathematics of number approach is not the naturally occurring approach taken by children on problems in real-world situations or concrete objects. Instead they will intuitively conceptualize the situation in terms of groups, or sets, or bunches, to solve problems. Looking at the mathematics from the perspective of units of quantity provides a link between additive and multiplicative structures (Behr, Harel, Post, & Lesh, 1992; Golding, 1994). Schwartz (1988) contends that this link is essential for mathematical modeling.

**Unitizing and Norming**

In addition to making students more aware of the units or referents in the mathematics, instruction should likewise focus students' attention on the unit
composition and decomposition already a part of their informal knowledge (Lamon, 1994). The whole number problem above involves the reconceptualization of the situation in terms of a fixed unit or standard (the three-packs of golf balls), a process that is called "norming" (Freudenthal, 1983). Students usually encounter this concept in the domain of rational numbers, wherein this ability is critical to the solving of problem situations.

The process of dividing fractions may be interpreted as a norming process. It requires the selection of the divisor as a norming unit and the reinterpretation of the dividend in terms of the divisor. Consider the problem $\frac{3}{4} \div \frac{1}{2}$. This problem can be interpreted as "How many $\frac{1}{2}$'s are in $\frac{3}{4}$?" For this situation, $\frac{1}{2}$ is the unit whole and $\frac{3}{4}$ is reinterpreted in terms of that unit. The solution would be conceptualized as follows:

$$\frac{3}{4} = 1 \left[\frac{1}{2}\right] + \frac{1}{2} \left[\frac{1}{2}\right]$$

Answer: $\frac{3}{4} = 1 \frac{1}{2}$ of the unit whole

The Present Study

Research Questions

The present study will investigate students' development of the unit concept of rational number and examine the unit as a cognitive bridge between
whole number and rational number domains. By building on students' existing informal knowledge of units and extending this to the formation and reformation of units, a natural connection can be established between whole numbers and rational numbers. Such a linkage would serve to aid the conceptual understanding of rational numbers by expanding students' conceptual models beyond those classified as limited (Greer, 1992) or primitive (Fischbein et al., 1985).

The present study does not have a specific testable hypothesis. Instead the focus will be to gain an understanding of students' knowledge of units, their abilities to construct composite units, and whether or not attention to the unit concept will increase their understanding of rational number concepts and operations. The following questions will be considered:

1. How does the concept of unit develop in a multi-representational learning environment designed to use unitizing and norming to link understanding from whole number to rational number domains?

2. What degree of independence of thinking can students achieve through this instructional experience?

3. To what degree can students' concept of unit be used to inform their choice of operations and their algorithmic performance on routine school word problems?

These research questions are addressed through analysis of a variety of qualitative data collected during the teaching experiment.
Organization of the Dissertation

Chapter 2 contains a review of the literature, separated into three sections. The first section presents a general overview of the problems that both students and adults experience with rational number concepts, and then discusses rational number constructs. The second section focuses on the complexities of multiplication and division for whole numbers and rational numbers and the need for students to expand their conceptual models for multiplication and division. The third section situates concepts from the first two sections into the broader multiplicative conceptual field, in which the role of the unit is emphasized.

In Chapter 3 a discussion of the qualitative nature of the study is given, along with justification of the teaching experiment as the selected research methodology. The present study focused on the cognitive processes of a group of four students as they engaged in a three week teaching sequence designed to investigate the development of the unit concept and the role of the unit as a possible link between whole number and rational number domains. Data were collected from a variety of sources in order to better understand the subjects' cognitive development of the unit concept.

Chapter 4 presents entering profiles of the subjects participating in the study, an overview of the 15 lessons in the teaching sequence, and provides an analysis of the qualitative data collected from initial and exit interviews, videotapes, audiotapes, transcripts of tapes, the researcher's journal, and
students' worksheets. The analysis will recapitulate the data, but from the view of the individual participants in order to explore the research questions posed in the earlier chapters.

Chapter 5 provides a summary of the study, limitations of the study, a synopsis of the results, the conclusions reached, and implications of this study for future practice and research.
CHAPTER 2
REVIEW OF THE LITERATURE

The review of the literature presented in this chapter serves to provide a framework for how unit formation and reformation is an intuitive or informal ability that can be expanded to serve as a link between whole number and rational number domains. The review is divided into three sections. The first section presents a general overview of the problems that both students and adults experience with rational number concepts, then discusses rational number construct theories. The second section focuses on the complexities of multiplication and division for whole numbers and rational numbers. The reported studies highlight the conceptual and procedural errors experienced by both students and their teachers, curriculum deficiencies, and the role of multiplication and division as models of situations. The third section situates the concepts from the first two sections into the broader multiplicative conceptual field, in which the concept of unit is emphasized by examining its importance in the mathematics of quantity approach, its pre-instructional presence in children's informal knowledge, and its impact in unitizing and norming.

Rational Numbers

This section will examine a broad view of the research on rational numbers, beginning with the problems experienced by both students and their
teachers. Rational number construct theories will then be discussed, with the importance of the unit fraction as a concluding segment.

**Problems Experienced with Rational Number Concepts**

The consensus of many mathematics educators, as reported in numerous research articles on the topic of children's construction of rational number knowledge (Behr, Harel, Post, & Lesh, 1992; Freudenthal, 1983; Kieren, 1988; Ohlsson, 1988), is that "rational number concepts remain a serious obstacle in the mathematical development of children" (Behr, Harel, Post, and Lesh, 1992, p.296). Results from the Fourth Mathematics Assessment of the National Assessment of Educational Progress (NAEP) highlight the significant difficulties children have learning and applying rational number concepts. Lindquist (1989) reported that only slightly more than half of the grade seven pupils could execute basic arithmetical operations with fractions. Although 80% of the pupils could transform a mixed fraction into an improper one, fewer than half of the seventh- and eleventh-grade students tested knew that 5 1/4 was the same as 5 + 1/4, thus illustrating their weaknesses in underlying rational number concepts. Other reports on this same NAEP data indicated that only about 40% of seventh-graders could identify the point on a number line that represented a simple fraction (Kouba, Brown, Carpenter, Lindquist, Silver, & Swafford, 1988). Furthermore, fewer than 40% were able to identify the largest and smallest of four fractions in a simple problem situation.
One of the more important conclusions drawn from NAEP assessment (Lindquist, 1989) is that many pupils seem to have learned how to calculate fractions by applying memorized algorithms without having developed the pertinent underlying concepts. Mack (1990) contended that these rote procedures, which were often incorrect, characterize the extent of many students' understanding of fractions. NAEP findings are generally consistent with many other studies, indicating low performance on rational number computation and problem solving and weakness of basic rational number concepts.

The "Concepts in Secondary Mathematics and Science" project (CSMS) was designed to investigate the difficulties that children experienced in a number of areas of mathematics, of which rational numbers was one (Hart, 1981; Kerslake, 1986). The aim of this project was to develop a hierarchy of understanding in mathematics in the secondary school curriculum. Four tests on rational numbers were used. Two of these tests contained items presented in problem or diagrammatic form. One was given to 12- and 13-year-olds and contained items on the identification of fractions (the shaded portion of a figure), fraction equivalence, and addition and subtraction of fractions. The other test consisted of similar items but also contained items on multiplication and division and was used with 13- and 14-year-olds. The remaining two tests were a parallel set to the previous tests, and presented items in computational form with no words or diagrams. In all, over 1,000 students were tested. The
CSMS mathematics team reached several conclusions from the results of these tests:

1. **Comparison between problem and computation questions:** There seemed to be no connection in many student's minds between the problem version and the computation version of a question. Students could often solve the problem, but not the computation. For example, children were presented with this problem:

   Three bars of chocolate are to be shared equally between five children. How much should each child get?

   When presented in this problem version, the success rates were 65.9% for 12-year-olds and 63.4% for 13-year-olds. However, when presented with the computation version alone, $3 \div 5$, the proportion of successes were reduced to 35% for the 12-year-olds and 31% for the 13-year-olds. Hart (1981) hypothesized that it was as if two different types of mathematics were involved, one where the children could use common sense, the other where they had to remember a rule. Kerslake (1986) observed that children used other than taught algorithms when given a problem.

2. **Avoidance of fractions:** Rather than giving answers in fractional form, students displayed a tendency to give answers in remainder form, the system acceptable before they had learned of the existence of fractions. Both Hart and Kerslake attributed this tendency to students' reluctance to accept fractions as numbers.
3. **Equivalence and addition of fractions:** A very common error was found to be the adding of numerators and denominators. This was even more prevalent where the denominators were different. Although 70% of the students were able to recognize equivalent fractions when tested as an isolated skill, fewer than half of the children found a common denominator for $\frac{2}{3} + \frac{3}{4}$. Further, it was found that the ability to solve addition and subtraction computations declined with the age of the student, while the ability to solve problems did not decrease with age. Hart (1981) attributed this phenomenon to the fact that students do not connect algorithms with problem solving, but rather use their own methods.

4. **Multiplication and division:** The hardest level in the hierarchy for older children was composed of multiplication and division problems. Results from the tests suggest that the meaning of multiplication is firmly rooted in the student's experience of whole numbers, where the operation can be replaced by repeated addition. For example, the answer to the problem $3 \times 10 \frac{1}{2}$ can be obtained by repeated addition since one of the elements is a whole number. That is, the problem can be interpreted as three groups of $10 \frac{1}{2}$, or $10 \frac{1}{2} + 10 \frac{1}{2} + 10 \frac{1}{2}$. If used consistently, this approach to multiplication would make the meaning of a problem such as $\frac{1}{3} \times \frac{6}{7}$ unclear because the student can no longer count to obtain the answer.

Numerous studies have also investigated adults' knowledge of rational numbers. Boulet (1995) conducted a study to assess university students'
knowledge of fractions, the purposes of which were to determine the long-term effects of the current methods of teaching fractions and to explore how preservice teachers themselves conceive rational numbers. Three tests were administered to two groups: Group A and Group B. Group A consisted of freshmen university students (average age 21 years old) enrolled in various programs in a Quebec university, while Group B consisted of second year Quebec university student-teachers (average age 22 years old) enrolled in an elementary education program. The three tests were designed to evaluate subjects' knowledge of decimal numbers, fraction arithmetic skills, and understanding of fraction arithmetic. In terms of mathematical background of the sample, 27% of Group A subjects and 89% of Group B subjects had taken post-secondary level mathematics courses. Additionally, all subjects in Group B (preservice teachers) had completed a 45-hour refresher course on elementary school mathematics. Results from Test 2 revealed that most subjects could add fractions with a common denominator but had extreme difficulty in other cases such as problems involving unlike denominators or mixed fractions. Over half could not estimate an answer to $\frac{7}{8} + \frac{13}{15}$. About one fourth were unable to multiply ordinary fractions and more than half could not multiply mixed fractions. Almost one third could not carry out divisions involving one or two fractions. Test 3 was designed to highlight the difference between the performance of an algorithm (as in Test 2) and the understanding of underlying fractional arithmetic concepts. Subjects were asked to model various rational
number problems. "Even the simple task of partitioning a whole into thirds eluded some of the subjects" (Boulet, 1995, p. 22). Approximately one fourth of the subjects could not represent fraction addition involving one improper fraction or the addition of fractions without a common denominator.

The results for multiplication and division modeling revealed deeper problems. Only a handful of the subjects could represent the product of two unit fractions; almost none could represent a whole number divided by a fraction. Of the 84 students involved in this study, only one could represent the division of one unit fraction by another. Comparisons of results between the two groups highlighted the problems experienced by the preservice teachers (Group B), who would be asked to teach this same material within a few years. Boulet (1995) emphasized that even a refresher course in elementary school mathematics failed to significantly effect the performance of the student teachers.

Other studies have focused on inservice teachers' experiences with rational number concepts. Post, Harel, Behr, and Lesh (1988) conducted a study called the Rational Number Project, designed to determine a profile of the mathematical understanding of teachers. Instead of trying to assess what teachers did or did not know, the study focused on trying to understand the way teachers understand important concepts about rational numbers. The assessment instrument, given to 218 middle school teachers, was comprised of items from rational number topics such as part-whole relationships, ordering of
fractions, fraction equivalence, the unit concept, and operations with fractions. Ten to 25% of the teachers missed items at the rudimentary level; in some cases almost half the teachers missed fundamental items. For example, $1/3 + 3$ was answered correctly by only 54% of the teachers. Although the mathematical content reflected the conceptual underpinnings of rational number topics for grades 4, 5, and 6 only, assessment results indicate that "many teachers simply do not know enough mathematics" (Post et al., 1988, p. 195). Further, of those teachers who were able to solve the problems correctly, only a minority were able to explain their solutions in a pedagogically acceptable manner.

Kerslake (1986) suggested that some teachers find it difficult to think of a fraction as a number. In a survey of mathematics teachers in secondary schools, Kerslake asked teachers whether they thought of a fraction as one number, two numbers, or not a number at all. Nearly one in eight of these teachers chose the third option. Many of the same misunderstandings of students are also prevalent among their teachers (Post et al., 1988). With teachers themselves experiencing conceptual misunderstandings in rational number concepts, is it any wonder that students have such difficulties with elementary rational number concepts?

The low level of performance of students on rational number items in the NAEP assessments might seem surprising in light of the emphasis on procedural skills and computational algorithms in most school programs.
However, this generally poor performance could be the result of curricular emphasis on procedures rather than the careful development of important conceptual understandings (Behr, Lesh, Post, and Silver, 1983). Many mathematics programs are dominated by textbooks (Post, 1992). Behr, Lesh, Post, and Silver (1983) suggested that the lack of conceptual knowledge of our teachers has resulted in a dependency on the textbook, thus resulting in a delivering of a curriculum which emphasizes procedures rather than understanding.

Post (1992) asserted that this textbook domination of the mathematics program served to create a mismatch between the nature of the learner's needs and the mode in which mathematical content is to be assimilated or learned. Bruner (1966) suggested three levels or modes of representation: (a) iconic, (b) symbolic, and (c) enactive. The iconic mode is based on the use of the visual medium: films, pictures, diagrams, and the like. Symbolic learning is that stage in which one uses abstract symbols to represent reality. At the enactive level, learning involves hands-on or direct experience. Bruner contended that all three modes are important. Post (1992) maintained that the modes used by most commercial textbook series are exclusively iconic and symbolic. He argued that an enactive void is created whenever textbook activities are not supplemented with real-world experiences.

An additional impediment to the development of rational number concepts concerns time. Many researchers (Behr et al., 1983; Boulet, 1995)
attest to the shortage of time in the mathematics curriculum devoted to rational number topics, with the majority of this time focusing on calculations. Behr et al. (1983) contended that this leaves even less time for the development of the unit fraction concept and its impact in developing rational number concepts.

**Rational Number Construct Theory**

The previous studies suggest a lack of conceptual understanding for both teachers and students. Post, Behr, and Lesh (1986) claimed that "many students do not have a workable internal concept of rational number" (p.40). Such an assertion attests to the complexity of the domain of rational numbers.

Kieren (1976) first introduced the idea that rational numbers consist of several constructs and that understanding the concept of rational number means that students understand these constructs, or interpretations of rational number, as well as how these different interpretations interrelate. Likewise, much work in this area of rational number construction has been done by Behr and his colleagues (Behr, Harel, Post, & Lesh, 1994; Behr, Lesh, Post, & Silver, 1983; Behr, Wachsmuth, Post, & Lesh, 1984; Post, Behr, & Lesh, 1986), who suggested that rational number concepts involve the coordination of several variables. Behr referred to these variables as personalities or subconstructs. Although Behr and his colleagues differ slightly from Kieren in the identification of rational number subconstructs, it appears that five subconstructs of rational number suffice to clarify the meaning of rational
number: part-whole, ratio number, measure, quotient, and operator (Behr, Harel, Post, & Lesh, 1992).

Consider the five subconstructs of the fraction 3/4. From the part-whole perspective, this fraction could represent three equal slices out of a pizza cut into four equal pieces:

![Diagram of a circle divided into four equal parts, with three shaded]

The part-whole interpretation of rational numbers depends directly on the ability to partition either a continuous quantity or a set of discrete objects into equal sized subparts or sets. Behr and Post (1992) considered the part-whole notion of rational numbers as being fundamental to the other interpretations.

The ratio interpretation of rational numbers conveys the notion of relative magnitude. The fraction symbol 3/4 should be interpreted in the ratio context as three for every four or as three out of four. For example, the ratio aspect of the fraction 3/4 is exemplified in the statement "three black balls for every four."

![Diagram of black and white balls arranged in a ratio]

The fraction 3/4 may be a measure number represented by a position on the number line or a ruler:
To see a quotient in $\frac{3}{4}$ is to focus on the arithmetic operation of three divided by four. Behr and Post (1992) claimed that the major component of understanding involved in the quotient interpretation is that of partitioning. Thus three-fourths can represent the problem of dividing three pizzas equally among four persons, which can be solved by cutting (partitioning) each of the three pizzas into four equivalent parts and then distributing one part from each pizza to each person.

\[ \text{\begin{figure}[h] 
\centering 
\includegraphics[width=0.5\textwidth]{pie.png} 
\caption{Partitioning pizzas.} 
\end{figure}} \]

Hence each will receive $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$, or $\frac{3}{4}$.

\[ \text{\begin{figure}[h] 
\centering 
\includegraphics[width=0.5\textwidth]{pie_cut.png} 
\caption{Distributing equal parts.} 
\end{figure}} \]

Finally, the fraction $\frac{3}{4}$ could be an operator in that it may be seen as taking three-fourths of something. This suggests that three-fourths can be
interpreted as a function which is applied to some number, object, or set (Behr, Harel, Post, & Lesh, 1993).

Behr, Lesh, Post, and Silver (1983) claimed the part-whole subconstruct as fundamental to the development and understanding of other rational number personalities. Thus the part-whole subconstruct should be considered as the first and most basic definition of fraction in that all other interpretations of fraction (ratio, quotient, measure, and operator) should presuppose an understanding of it. An understanding of the ratio between a part and the whole can best be facilitated by examination of the simplest case—the unit fraction.

**Unit Fractions**

A unit fraction is any fraction of the form $1/n$, where $n$ is any other natural number. From a theoretical perspective, a unit fraction may be defined as the "quantification of the part-whole relationship" (Boulet, 1995, p.11). This quantification is a result of the equipartitioning of the whole and then the focusing on the ratio between the part and the whole.
Although there is no record of the usage or understanding of Stone Age people's concept of rational numbers, the more advanced Bronze Age cultures (c. 2000 B.C.) did see the need for expressing such mathematical concepts in writing. The Egyptian hieroglyphics had a special notation for the unit fraction (Kline, 1972). General fractions such as m/n were considered by the Egyptians as much more complex than unit fractions since they were obtained by combination of unit fractions (e.g., adding 1/5 to 1/3 yields 8/15). The historical primacy of the unit fraction suggests its possible ontogenetic primacy (Boulet, 1995).

Rational numbers were constructed as a result of the lack of closure of the integers under division. By extending the concept of number to fractions and extending the definitions of multiplication and division accordingly, closure is thus achieved for division as well. Boulet (1995) asserted that viewing rational numbers as an extension of the integers can be considered as a more mature concept of number. This extension of integers is paralleled within applications. For example, as seen above, three pizzas cannot be shared equally among four children as long as a pizza is considered an indivisible whole. By a "shift of perspective," however, in which a pizza is considered as something that can be fractured (cut into fractions), a solution becomes possible (Greer, 1992).
Multiplication and Division Concepts

Although the operations of multiplication and division of rational numbers seem relatively simple from a computational point of view, Greer (1992) emphasized the psychological complexity behind this simplicity. He contended that a fundamental restructuring is necessary whenever multiplication and division are extended beyond the domain of positive integers. Much of this complexity arises from a combination of factors, such as students' conceptual and procedural errors in multiplication and division, curriculum deficiencies, and students' limited models for multiplication and division.

Conceptual and Procedural Errors

Schwartz (1988) claimed that a commonly held view among mathematics educators is that children's early number knowledge and number strategies will lead to an understanding of multiplicative structures and rational numbers. He opposed this view, arguing that whole number knowledge has led to both conceptual and procedural errors. Students are usually introduced to multiplication and division through the ideas of repeated addition and sharing activities, respectively. Schwartz challenged this approach, suggesting it leads to both conceptual and procedural flaws in the understanding of multiplication and division.

Conceptual knowledge has been characterized by Hiebert and Wearne (1986) as knowledge that is rich in relationships. It may be viewed as a
connected web of knowledge—a network in which the linking relationships are as prominent as the individual pieces of information. Hiebert and Wearne contend that the development of conceptual knowledge may be achieved by the construction of relationships between pieces of information, either by forming linkages between information already stored in memory or by linking an existing piece of knowledge with one that is newly acquired.

Procedural knowledge embodies two distinct parts (Hiebert & Wearne, 1986). One part is comprised of the formal language of mathematics—the symbol representation system. The other part consists of the algorithms or rules used to complete mathematical tasks.

The importance of linking procedural knowledge to conceptual knowledge has been emphasized by Carpenter (1986), who contended that procedural knowledge has been isolated from related conceptual knowledge because of a failure to develop the necessary conceptual base or because of a failure to link procedures to the conceptual knowledge that has been acquired. Other researchers claim that true mathematical competence is characterized by connections between conceptual and procedural knowledge (Hiebert & Lefevre, 1986; Hiebert & Wearne, 1986).

A conceptual flaw occurring through the repeated addition and sharing models is that these models "lead the student to believe that the resulting computed quantity is of the same sort and has the same referent as one of the quantities that entered into the computation" (Schwartz, 1988, p. 47). Another
conceptual error that can develop from these limited models of multiplication
and division is the notion that multiplication always results in a product larger
than either factor and that division always leads to a quotient which is smaller
than the dividend. This is commonly referred to as "multiplication makes bigger

According to Carpenter (1986), results from the Second NAEP
assessment revealed weaknesses in conceptual knowledge. As an example,
students were asked to estimate the answer to the problem 12/13 + 7/8. Over
half of the 13-year olds chose answers of 19 or 21 as the sum of two fractions
each less than one (Carpenter, Corbitt, Kepner, Lindquist, and Reys, 1981).
Carpenter et al. (1981) asserted that these students apparently simply added
either the numerators or denominators without relating their answer to their
knowledge of fractions or knowledge of addition. Thus a student's chosen
procedure was applied to the numbers given in the problem without relating
that procedure to his or her conceptual knowledge.

The procedures for multiplying common fractions are straightforward
and easy to learn; the numerators and denominators are simply multiplied.
This procedure is so simple that it can be learned without linkages to
conceptual knowledge, and Carpenter (1986) argued that this appears to be the
extent of students' learning of multiplication. Fraction multiplication is more
complex than the straightforward extension of the concept of whole number
multiplication held by most students (Behr & Post, 1992; Greer, 1992).
Carpenter (1986) asserted that students appear to have such little understanding of the meaning of fraction multiplication that, as a consequence, they cannot apply the procedure to solve even simple problems. For example, over 70% of the 13- and 17-year olds in the Second NAEP mathematics assessment could multiply two common fractions, yet only 20% of them could solve the following word problem (Carpenter et al., 1981):

Jane lives 2/3 mile from school. When she has walked 2/5 of the way, how far has she walked?

Greer (1992) asserted that instruction in fractions provides an introduction to simple fraction concepts, but it fails to extend these concepts to develop an adequate knowledge base so that procedures like multiplication can be learned with meaning. Further he contended that procedures are not always clearly connected to the conceptual knowledge that the children have acquired. Hiebert & Wearne (1986) claimed that students are not fully competent in mathematics if either kind of knowledge is deficient or if they both have been acquired but remain separate entities.

Hiebert and Wearne (1986) listed three sites where links between conceptual and procedural knowledge are extremely productive and in which the absence of such links is especially damaging. These sites are important for educators because they provide opportunities to diagnose the source of many learning problems. Site one is the initial point in the problem-solving process when the problem statement is interpreted. It is at this point that the numerical and operational symbols of the problem are given meaning.
Symbolic meaning may come from the syntax (from the store of procedural knowledge), or from the connection of the symbols with their conceptual referents. Consider for example the expression $2/3 + 1/4$. A syntactic meaning of "+" might connect the symbol with the algorithm "invert and multiply." Yet a semantic meaning could connect the symbol with the conceptual notion of "how many fourths are contained in two thirds." Hiebert and Wearne suggested that it is the connections between symbols and conceptual referents that provide the foundation for mathematical competence. Further they argued that connections between conceptual and procedural knowledge at site one are essential for establishing connections at the remaining sites.

Site two involves the execution of the selected procedures in order to solve the problem. One of the procedural rules that students encounter when learning to add and subtract fractions is to "find a common denominator." The rationale for such a rule is based on the notion that an answer will have meaning when things are combined that have been measured with the same unit. The problem at this site is that students may not link their procedural rules to conceptual referents. Hiebert & Wearne (1986) claimed that such an absence will not impede performance, provided that these procedures are recalled correctly and applied appropriately. However, they insisted that linkages here between procedures and their conceptual underpinnings will contribute to genuine competence.
At site three a response is produced that represents the student's answer to the problem. Conceptual knowledge about the meaning of the symbols and about the operation used to solve the problem provide ability to evaluate the reasonableness of the answer. For example, the answer to $8/9 \div 1/3$ should be close to 3 since the fraction $8/9$ is close to 1 and we are partitioning it into units of size $1/3$ each.

**Curriculum Deficiencies**

Behr, Harel, Post, and Lesh (1992) believed that many of the limited conceptions (or misconceptions) of rational numbers held by both children and teachers were a result of deficiencies in the curricular experiences provided in school. Some researchers attributed misconceptions to students' conclusions from instruction that occurred through overgeneralization of rules from the domain of whole numbers to the domain of rational numbers (Behr, Wachsmuth, Post, & Lesh, 1984; Graeber & Tirosh, 1991). Additionally Tirosh and Graeber (1989) contended that teacher-taught procedural rules may not only support, but also could be a source of such misconceptions. From another study, Tirosh and Graeber (1990) reported the heavy reliance of preservice teachers on the domain of whole numbers, "perhaps treating the whole numbers as a paradigmatic model for any set of numbers" (p. 106). For example, the well-reported intuitive models children form about multiplication and division (e.g., MMB:DMS) apparently result from a curricular emphasis on multiplication and division of whole numbers. This phenomenon has resulted
in "whole number dominance" in the conceptions of students (Behr, Wachsmuth, Post, & Lesh, 1984).

Textbooks have also contributed to the reinforcement of such misconceptions with students. They rarely expose students to counterexamples (Nesher, 1987), thereby providing limited opportunity for practice with counterintuitive cases (Graeber and Baker, 1991). The lack of counterexperience resulted in limited conceptions that lasted into adult life (Behr, Harel, Post, and Lesh, 1992). As well as being extremely unbalanced in terms of problem variety, word problems in texts are generally stereotyped (Nesher, 1980). The result of this is that pupils evolve "short cuts," or strategies, to identify the operation required on the basis of the surface structure alone. A major curricular problem is that these strategies can be undeservedly successful. Schoenfeld (1982) reported that in a widely used elementary textbook series, 97% of the problems could be correctly solved by using the key-word strategy. There is evidence that although fourth and fifth grade students can recognize word problems as division situations, their ability to symbolize such divisions are weak. There is confusion in many students surrounding the various notations for division: \( \frac{b}{a} \), \( a \div b \), and \( \frac{a}{b} \). In evaluating results from a study, Graeber and Tirosh (1988) suggested that even though students had "learned" about fractions such as \( \frac{3}{5} \), they often did not recognize this notation as division. In a subsequent examination of curriculum materials available for teaching multiplication and division, Graeber and Baker (1991) noted that the
"a/b" notation for division was usually introduced in the chapter on fractions, but this notation still accounted for less than 10% of the notations in the division chapters. Usage was confined primarily to the fraction chapters alone.

**Multiplication and Division as Models of Situations**

Fischbein and colleagues (Fischbein, Deri, Nello, and Marino, 1985) claimed that the dominant models children use to solve multiplication and division problems have a very limited range of applicability and are responsible for many of the difficulties students encounter in solving arithmetic word problems. Furthermore, they hypothesized that each fundamental operation of arithmetic generally is dominated by an implicit, unconscious, and primitive intuitive model. These primitive, intuitive models are stored in our subconscious and surface whenever we attempt to perform one of the fundamental operations. They serve to impose constraints on students' predictions of the operation needed when solving multiplication and division problems.

Fischbein and his associates (1985) contended that the primitive model associated with multiplication, the repeated addition model, tacitly affects the meaning and use of multiplication, even in persons with considerable training in mathematics. These researchers feel that the misconception, multiplication always makes bigger, develops from operator-operand ideas. In the problem $3 \times 5$, the 3 is considered to be the operator; it indicates the number of equivalent collections. The number 5 is the operand, indicating the magnitude
of each collection. In this interpretation, the operator must be a whole number and, therefore, the multiplication will "make bigger." No restriction applies to the multiplicand.

The partitive model is the intuitive primitive model for division (Fischbein et al., 1985). It may also be termed sharing division since it refers to dividing a collection of objects into a number of equal subcollections. Work by Kouba (1986) and Greer (1987) referred to this also as a "distribution" model for division. Assumptions in this primitive model include (a) the dividend must be larger than the divisor, (b) the divisor must be a whole number, and (c) the quotient must be smaller than the dividend. Fischbein et al. (1985) contended that the quotitive, or measurement, model of division is acquired later through instruction. This view of division involves determining how many subgroups of a given size are contained in a collection or quantity. For the operation to be readily perceivable as quotitive division, the divisor must be smaller than the dividend.

To test their hypotheses of primitive models, Fischbein and his colleagues (1985) conducted a study of 628 students in grades 5, 7, and 9. These students were given a 42-item test which included problems in addition, subtraction, multiplication, and division. On each of the problems the students were asked to indicate the operation used to solve the problem, but they were not required to perform the calculation. The results of the study supported the hypotheses of primitive models. The multiplication problems which included a
decimal operator, thereby violating the repeated addition model, were more difficult for the students. Likewise, when the constraints of the primitive model of division were violated, the percent of incorrect responses increased.

Fischbein and colleagues also addressed the issue as to the source of these primitive models. First, the model is a direct reflection of the way the operation was initially taught in school. Second, these primitive models "correspond to features of human mental behavior that are primary, natural, and basic" (p. 15). The question regarding what can be done about these primitive models is not easily answered. Some research findings have indicated that children do not alter these models (Fischbein et al., 1985; Mack, 1990). Many adolescents and adults continue to face difficulties arising from the conflict of their primitive models and the correct operations. Fischbein (1987) argued that when the concepts of multiplication and division are extended, the intuitive models established earlier continue tacitly to affect thinking. Further, he contended that helping students to become aware of and control these covert influences on their thinking was a major pedagogical challenge.

Fischbein's research has been repeated several times with different populations, both students and teachers, always yielding the same results (Greer, 1987; Tirosh, Graeber, & Glover, 1986).

As a way of alleviating such problems as these, many researchers have sought ways to improve students' experiential bases. Kaput (1985) referred to multiplication and division as having dimensional complexity, a complexity
which becomes particularly evident when attention is paid to the referents of
the numbers in the problem situation. Greer (1994) holds that students should
be exposed to a much wider range of situations modeled by the operations. He
further suggested that the operations and extensions of such should be
introduced within problem-posing contexts in an attempt to broaden conceptual
understandings.

Greer argued that in a problem that can be modeled by a single
operation, even though the numbers carry no information as to the appropriate
operation, these numbers drastically influence the difficulty of the choice-of-
operations task. The early established intuitive models (MMB:DMS) continue
to affect thinking. In order to overcome the limitations of intuition, a strategy
can be employed of replacing the "hard" numbers with "easy" ones, thus
enabling the student to identify the appropriate operation to apply to the
original numbers. However, a difficulty arises in that most students assume
that changing the numbers can change the operation (Bell, Swan, & Taylor,
1981). As an explanatory construct for the tendency of students to alter their
prediction of needed operation based on the type of multiplier or divisor used,
Greer (1994) offered the term "nonconservation of operations" (p. 69). He
considered invariance of the operation over the numbers involved as "the
keystone of extension of meaning for multiplication and division" (p. 70).
The Multiplicative Conceptual Field

Rather than analyzing the cognitive development of rational numbers or
the complexities of multiplication and division as isolated concepts, many
researchers contend that these ideas are interwoven into a field of related
concepts, the "multiplicative conceptual field" (MCF). Vergnaud (1988) defines
this field as consisting of "all situations that can be analyzed as simple and
multiple proportion problems and for which one usually needs to multiply or
divide" (p. 141). He identified broad strands of the multiplicative conceptual
field to include: multiplication, division, fractions, ratio, rational number,
linear and n-linear functions, dimensional analysis, and vector spaces.
Vergnaud (1994) argued that researchers should seek a conceptual analysis of
the domain of multiplication and its related concepts. He proposed that
investigations into these related ideas of mathematics should be made in such a
way as to include (a) the "conceptual operations needed to progressively master
this field" (p. 42), (b) the "situations and problems that offer a sound
experiential reference" (p. 42), (c) a "bulk of concepts" (p. 43) for analysis, and
(d) language and symbols for communicating and thinking. Confrey and Harel
(1994) suggested that the topics included in the MCF seem increasingly critical
in school mathematics because "these topics create the critical juncture in
middle school, separating those students who persist and those who drop out"
(p. xii). It is here, the development of multiplicative reasoning, that the
competence and confidence of many students begin to break down (Hiebert &

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Behr, 1988). Kieren (1994) exhorted that the growth of multiplicative structures "is critical for a person's conceptualizing or bringing forth the world in which he or she lives" (p. 387).

**The Role of the Unit**

The unit concept is emerging as a fundamental concept in the multiplicative conceptual field (Behr, Harel, Post, & Lesh, 1994; Confrey & Harel, 1994; Kieren, 1994). Some researchers advocate that the use of a units-approach to elementary multiplicative and divisional relations "would greatly enhance students' entry and successful acquisition of rational number proficiency" (Behr et al., 1994). A greater understanding of the role of the unit can be gained by examining its importance in the mathematics of quantity approach, its pre-instructional presence in children's informal knowledge, and its impact in unitizing and norming.

**The Mathematics of Quantity**

By viewing problems from a mathematics of quantity approach, the focus is on the quantities, the number and the unit, not just on the numbers. A focus on only the numbers involved, the traditional pedagogical approach to problem solving, is utilizing the mathematics of number approach. Research by Lamon (1994) has shown that this is not the naturally occurring approach taken by children on problems involving real-world situations or concrete objects. Instead they will intuitively conceptualize the situation in terms of groups, or sets, or bunches, thus enabling them to solve the problem.
Schwartz' Role of Intensive Quantities. All quantities arising in the course of counting, measuring, or computing have referents (Schwartz, 1988). Whenever two mathematical quantities are composed, the result is a third quantity which is either like or unlike the original quantities. Schwartz referred to the results of this process as either referent preserving compositions or referent transforming compositions. Examples of referent preserving compositions of quantity include addition and subtraction since the quantity produced is like the original referents (e.g., 13 yds + 6 yds = 19 yds). Problems within the rational number domain also are included in this composition. For example, using the mathematics of quantity approach, the problem 3/5 + 1/5 could be interpreted as 3[1/5-unit]s + 1[1/5-unit]s = 4[1/5-unit]s. For problems with unlike denominators, a conversion of units would be necessary, e.g., 7/8 - 1/4 = 7[1/8-unit]s - 1[1/4-unit]s = 7[1/8-unit]s - 2[1/8-unit]s = 5[1/8-unit]s. This conversion of units is similar to the task involved in combining unlike measurement units, such as 4 yards + 2 feet, in which the solution would proceed as follows: 4 yards + 2 feet = 12 feet + 2 feet = 14 feet. Thus we see that in each case, the resulting quantity is of the same referent (unit) as the original quantities that were combined.

Schwartz (1988) describes multiplication and division as referent transforming compositions of quantity since the quantity produced is unlike either of the two original quantities. Consider these examples: 6 lbs. jelly beans x $3 cost/lb. = $18 (total cost), or 45 candies ÷ 5 bags = 9 candies/bag.
Repetitions of addition or subtraction will not result in the referent that is appropriate for the product in a multiplicative situation or the quotient in a division situation. Consider the referents in the bag of jelly beans: (1) "lbs." (weight of beans), (2) "$" (total cost of beans), and (3) "/lb." (cost per pound). While the first two referents describe the entire bag of beans, the cost per pound referent can describe the entire bag of beans, a single jelly bean, or only a handful of jelly beans. The cost per pound referent is a different sort of descriptor of the jelly beans. It indicates a quality of the beans and not the amount of beans.

Typical measurement problems encountered by middle grade students might include such as these: 3 cm x 9 cm = 27 sq cm, or the rate equals distance/time (e.g., R = 192.5 mi / 3.5 hrs = 55 mph). Cartesian multiplication also illustrates referent transforming compositions of quantity. For example, 4 skirts x 3 blouses = 12 skirt-blouse pairs or 12 outfits. These can be extended to the rational number domain as follows: 1/3 x 2/5 = 1[1/3-unit] x 2[1/5-unit]s = (1 x 2) ([1/3 x 1/5]-unit)s = 2[1/15-unit]s. The resultant referent (1/15-unit) is of a differing type than the composed quantities.

Schwartz (1988) described a referent transforming composition as being composed of two different kinds of quantity—extensive quantity and intensive quantity. An extensive quantity is a quantity that can be counted or measured directly, such as the number of marbles or the length of a rope (Simon &
An intensive quantity is defined as a "generalization of the notion of attribute density" (Schwartz, 1988, p. 43), and is a statement of a relationship between two, usually extensive, quantities. In the example given above involving miles, hours, and miles per hour, the extensive quantities would be miles and hours while the intensive quantity is miles per hour. On the other hand, 27 sq cm and 2[1/15-unit]s (also from the above examples) are not each a relationship between two extensive quantities being composed, but rather an extensive quantity in its own right. Schwartz (1988) contended that an introduction of intensive quantity is essential to understanding the vast majority of situations that call for the arithmetic acts of multiplication and division. The current pedagogic strategy that teaches referent transforming compositions (multiplication and division) as an extension of referent preserving compositions (addition and subtraction) "misses the essential feature of that which is being introduced," that is, "that referent transforming composition gives rise to a quantity of a new kind" (Schwartz, 1988, p.48).

Schwartz (1988) asserted that the idea of referent transforming composition which distinguishes between intensive and extensive quantities will not only improve the future understanding of multiplicative structures and rational numbers, but will also offer an opportunity to repair a substantial amount of poorly taught and poorly learned mathematics. Furthermore, he suggested what he saw as a new approach for the teaching and learning of mathematics--"an approach predicated on the vision of mathematics as a tool
for modeling" (Schwartz, 1988, p. 52). This new approach is based on making a clear distinction between those operations that result in referent preserving compositions of quantity and those operations that result in referent transforming composition of quantities. Like other researchers stressing the importance of the mathematics of quantity approach (Harel, Behr, Post, & Lesh, 1994), Schwartz postulated that this link between numbers and their referents is an essential component of the mathematics used for modeling.

Simon and Blume. Simon and Blume (1992) designed a study to see how learners develop intensive quantities and to determine the extent to which they can relate a repeated addition notion of multiplication to Cartesian products and the area of a rectangle. Twenty-six preservice elementary majors at the junior level were involved in an eight week teaching experiment which examined the development of understanding of the area of a rectangular region as a multiplicative relationship between the lengths of the sides. Results indicated that, although these students had successfully memorized the traditional formula, they had no conceptual understanding of the connection between area and multiplication. Although the majority of students were successful in obtaining the answer by multiplying, none could offer a satisfactory explanation as to why this strategy worked. Most explained that this was the rule or formula which they were taught. Additionally, none of the students could explain what this answer or number signified; there was no attachment of a referent to the answer.
Behr's Units Analysis. Behr and his associates viewed the mathematics of quantity approach as a way to improve teaching of multiplicative concepts, claiming that the units of measure and magnitude of quantities are both significant to understanding number relations and operations (Behr, Harel, Post, & Lesh, 1992; Harel, Behr, Post, & Lesh, 1994). Research by Behr and Harel (1990) showed that both of these variables affect the problem representations made by a solver of proportion problems. Further efforts by Behr and his colleagues involving the multiplicative conceptual field incorporated an emphasis on units analysis and the mathematics of quantity (Behr, Harel, Post, & Lesh, 1994). Through this approach a complex notational language system was created, in both the additive and multiplicative conceptual fields, to represent the mathematics used by children. With this semantic analysis, Behr and his colleagues created a way of symbolizing the construction and reformation of units through the use of parentheses. The notational language consists of two systems: iconic and linguistic. In each, the word "unit" is the basic element, from which composite units (units of units) can be formed. These researchers exhorted that the use of these two systems serves to provide a stable interpretive structure while allowing a view of each world (additive or multiplicative) from the units of quantity perspective. Analysis of these fields from the perspective of the unit emphasizes some of their commonalities. Based on this common structure, Behr and his colleagues argued that using a units-approach to elementary multiplicative and divisional
relations would greatly enhance students' entry and successful acquisition of rational number proficiency. These systems are not designed for use with children in instructional activities, but rather as an aid to researchers in communicating about children's conceptions of specific additive or multiplicative situations, in hypothesizing the cognitive structures that develop (or fail to develop) in acquiring an understanding of the concepts discussed, and in suggesting kinds of learning activities that children should experience in order to develop these structures. These researchers suggested that early attention to working with units, from whole number addition on, can prepare students for later content that has been traditionally difficult (e.g., adding fractions with unlike denominators or arithmetic of intensive quantities).

In Informal Knowledge

Several researchers recently have begun to focus on the knowledge that students bring to formal instruction and its role in students' learning and teachers' instruction (Lamon, 1994; Leinhardt, 1988; Mack, 1990, 1995; Pothier & Sawada, 1983). This type of knowledge has been referred to as informal (Mack, 1990), or intuitive (Leinhardt, 1988), and has generally been characterized as "real-life circumstantial knowledge constructed by the individual student [that] can be drawn upon by the student in response to problems posed in the context of situations" (Mack, 1990, p.16). Much research has shown that many students come to instruction on rational numbers with a rich store of practical knowledge related to fractions and that they are able to
draw on this knowledge to solve a variety of real-world problems (Kieren, 1988; Leinhardt, 1988; Mack, 1990). However, this knowledge appears to be limited initially in three ways (Mack, 1993): (a) Students' informal strategies treat rational number problems as whole number partitioning problems, (b) students' informal conception of rational number influences their ability to reconceptualize the unit, and (c) students' informal knowledge initially is disconnected from their knowledge of formal symbols and procedures associated with rational numbers.

Some research has suggested that children do not use formal, or taught, methods in mathematics, but use instead their own informal methods (Lamon, 1994; Mack, 1990). Kerslake (1986) argued that, in the case of fractions, the position appears to be somewhat different. Here, children are seen to rely on rote memory of previously learned techniques or half-remembered rules that are inappropriately applied. Kerslake attributed the underlying problem to a lack of any attachment of meaning to the notion of fraction. "With the exception of certain simple examples (i.e., 1/2 or 1/4), fractions do not form a normal part of a child's environment, and the operations on them are abstractly defined and not based on natural activity" (Kerslake, 1986, p. 87). Mack (1990) indicated the existence of a conflict between informal knowledge and rote procedures, in which the child's informal knowledge may be pre-empted by attempts to conform to teacher-taught algorithms. In such instances the students rarely stop to consider the reasonableness of their answer in the sense
of its being connected to their informal knowledge. Other studies have also documented that one of the major consequences of students learning rote procedures for operations with rational numbers prior to building on their informal knowledge is that students' rote knowledge of procedures, which is often faulty, tends to dominate their thinking (Hart, 1988; Hiebert & Wearne, 1986).

Mack. One way to help students learn mathematics with understanding is by relating mathematical symbols and procedures to real-world problems that draw on students' informal knowledge. Mack (1990) designed a study to investigate ways of tying instruction to children's informal or intuitive knowledge. The study involved eight average ability sixth graders, who received 11 to 13 individual instructional sessions of 30-minute duration over a six week period. Students were presented problems verbally and asked to think aloud as they solved the problems. The basis of the instructional content was determined by the student's informal knowledge and estimation with fractions was emphasized. Students were allowed to use concrete materials during the first sessions, but this was discouraged after the fifth week. The specific tasks given to the students were based on responses to previous questions. Students were encouraged to draw upon informal knowledge and to relate fraction symbols and procedures to that knowledge. Tasks consistently shifted between real-world problems and symbolic problems in an attempt to examine the students' ability to relate informal knowledge to a symbolic form. The results
of the study supported Mack's contention of the existence of informal knowledge. All students demonstrated a store of informal knowledge which allowed them to solve many real-world problems. The study also indicated that students can build on this informal knowledge to give meaning to formal symbols and procedures.

While this study revealed the presence of informal knowledge, Mack found limits to this knowledge. Students were successful in using informal knowledge to solve problems presented in the context of real-world situations, but could not solve similar problems that were presented symbolically. These results suggested that "initially students' informal knowledge of fractions is disconnected from their knowledge of fraction symbols and procedures" (p. 29). Moreover, results indicated that, when different answers were obtained for problems posed in different contexts, students often resolved the inconsistencies in favor of the faulty procedures. Mack contended that students typically do not overcome the interference of this knowledge on their own; a concerted effort on the part of the instructor is required. Although the interference of rote knowledge of procedures is strong, Mack found that it could be overcome by moving back and forth between problems represented symbolically and those drawing on informal knowledge.

The concept of informal knowledge is also reflected in Fischbein, Deri, Nello, and Marino's (1985) discussion of primitive models. These researchers hypothesized that "each fundamental operation of arithmetic generally remains
linked to an implicit, unconscious, and primitive intuitive model" (p.4). These primitive, intuitive models surface whenever we attempt to perform one of the fundamental operations, and impose constraints on students' predictions of the operation needed to solve the problem. Mack (1993) contends that these constraints can be overcome by instruction which builds on students' informal knowledge.

**Leinhardt.** Leinhardt (1988) assessed fourth-grade students' knowledge of fraction concepts in two ways: (a) by asking students to answer questions about fractions that were represented symbolically and in pictures, and (b) by asking students to answer questions about fractions that were embedded in a real-life story that was read to them. Students' responses to the story questions were frequently correct, which suggested that they possessed a strong understanding of rational number concepts. However, their performance on problems presented symbolically and in pictures was quite weak. Leinhardt concluded that students' informal knowledge of fractions was available only when questions were asked in a way that did not look mathematical.

**Pothier and Sawada.** Pothier and Sawada (1983) conducted a study of 43 children in grades kindergarten through three, designed to reveal their partitioning abilities. The analysis of the study resulted in a five-level theory describing the child's development of the partitioning process. These researchers argued that skill in partitioning was attained by a gradual progression through five levels, each indicating successive attainment of five
subsets of the unit fractions: (a) sharing (the fraction 1/2), (b) algorithmic halving (fractions whose denominators are powers of two), (c) evenness (fractions with even denominators), (d) oddness (fractions with odd denominators), and (e) composition (fractions with composite denominators).

Pothier and Sawada contended that the sharing level is learned in a social setting, a notion consistent with Mack's (1990) theory of informal knowledge. From this entry level, the child must proceed through the other four levels of the hierarchy in order for the partitioning process to be mastered. This notion of progression through levels is consistent with Mack's (1990) assertion that, by building on what students already know, their informal conceptions can be extended in meaningful ways.

**Unitizing and Norming**

Boulet. Boulet (1995) examined the construction of the unit fraction concept in fourth graders and the difficulties encountered by them during this construction. Six children participated in a teaching experiment consisting of 16 fraction teaching activities. Results showed that these children all exhibited understanding of the unit fraction at the conclusion of the teaching experiment and that they were able to construct unit fraction concepts without any direct instruction (e.g., teaching in the form of lecturing). In the exit interviews, some of the children were even able to use this unit fraction knowledge to solve more complex problems (e.g., adding fractions with different denominators) even without having received prior instruction in such areas. Boulet contended
that these results indicate that fourth grade is certainly not too early for introduction of the unit fraction concept since children at that level possess the necessary readiness to construct this concept.

Lamon. Lamon (1994) analyzed students' pre-instructional abilities to handle situations involving ratio and proportion. Twenty-four sixth graders participated in clinical interviews in which five problems were designed to elicit information relative to children's power of unitizing, "the ability to construct a reference unit or whole" (p. 92), and norming, "to reinterpret a situation in terms of that unit" (p. 92). Although none of the children used symbols to represent his or her thinking, those who were successful consistently reasoned using a complex set of units. Three fourths of the students naturally formed ratios and engaged in the process of norming. Even without the use of symbolic representations, student thinking showed that they were quantifying relationships and, more significantly, they were performing arithmetic operations on those quantities. Lamon contends that "students' preinstructional reasoning provides some basis on which to build an understanding of intensive quantities" (p. 113). Just as students need to be made more aware of the referents, or units, in the mathematics, they need to likewise focus their attention on the unit composition and decomposition capabilities already a natural part of their informal knowledge (Lamon, 1994). By making students aware of this existing knowledge they will then be able to tap into it to link this understanding to their developing rational number concepts.
Golding. Golding (1994) studied the cognitive processes of five preservice elementary teachers as they engaged in the formation of units and examined the role of the unit concept as a possible link between the whole number and rational number domains. A six lesson teaching experiment was devised to provide situations in which students could explore the concept of unit. Results supported the existence of an intuitive notion of unit. Students' performances on individual tasks clearly showed the formation of units, whereas their comments regarding unit selection suggested it occurred naturally and without mental processing.

While the intuitive nature of unit formation was evident, it was noticed that unit formation was somewhat hampered by students' previous knowledge of school mathematics. Golding referred to this tendency to suppress an instinct in favor of a memorized approach as "algorithm dominance" (p.161). Another obstacle experienced by students was "mathematical perception," or the notion that school mathematics differs from street mathematics (p. 161). As the problems became more realistic to students, the more flexible their unit structure became. Golding concluded that results from her study suggest that curriculum development should provide experiences with problem situations emphasizing a mathematics of quantity approach.

The issue of the unit concept as a connector between the whole number and rational number domains was examined by Golding through teaching interviews at the end of the lessons. While the format of each interview
varied depending on individual responses, there were two main areas of interest examined in each interview. The first area was whether or not the students would notice and/or use the unit concept in addition of fractions. Students were asked to add two fractions with common denominators (e.g., 4/5 + 3/5) and justify their answer. All of the students used the traditional algorithm in their justification except one student who immediately recognized the unit structure. The others were able eventually to identify the unit structure after a series of probing questions by the researcher.

The second area of interest concerned division of rational numbers. Once the unit concept was examined in the operation of addition, the focus of the interview examined the extension of the unit concept to the operation of division (e.g., 9/12 ÷ 3/12). A new approach to division of fractions was realized by all of the students in the study. This new approach consisted of reunitizing the fractions (converting to a common denominator) and then dividing the numerators. Although all students claimed that using the traditional algorithm (invert and multiply) was faster, this new approach made more sense to them. While the students had worked with fractions for at least 10 years prior to this study, this new conceptualization of division was not reached until the end of the teaching experiment. Golding contended that these results provide strong evidence that knowledge of the unit concept can facilitate understanding of concepts in the domain of rational numbers.
Conclusions

This chapter presented a broad view of research, beginning with the vast array of problems experienced by students and teachers with rational numbers. Rational number construct theories were discussed, along with the complexities of multiplication and division for both whole numbers and rational numbers. These conceptual entities were then situated within the broader context of the multiplicative conceptual field, in which the notion of unit was emphasized as foundational for development of understanding of multiplicative structures. The existence of informal knowledge was discussed, along with students' abilities to build on this informal knowledge as a way to develop conceptual understanding. The review serves to support the assumptions underlying the present research contentions that (a) students need to become aware of their informal knowledge of the unit and their models of multiplication and division, (b) students need to build on this informal knowledge, starting through work with various whole number units and focusing on the measurement aspect of modeling, and (c) students will be able to extend these skills to the rational numbers, revealing a natural connection to the whole number domain.
CHAPTER 3

METHODOLOGY

The purpose of this study was to understand students' development of the unit concept of rational number and to examine the unit as a possible link between whole number and rational number domains of multiplication and division. I attempted to build on students' existing informal knowledge of the unit in order to establish a natural connection between whole numbers and rational numbers (Lamon, 1994; Mack, 1990). My purpose in using this approach was to facilitate students' conceptual understanding of rational numbers by expanding their conceptual models beyond those classified as limited (Greer, 1992) or primitive (Fischbein et al., 1985).

Research Questions

Through my reflections on this area of interest, I have constructed the following explicit questions:

1. How does the concept of unit develop in a multi-representational learning environment designed to use unitizing and norming to link understanding from whole number to rational number domains?

2. What degree of independence of thinking can students achieve through this instructional experience?

3. To what degree can students' concept of unit be used to inform their choice of operations and their algorithmic performance on routine school word problems?
Background

I have taught high school mathematics for 12 years. Regardless of the mathematical abilities of my students, a great majority of them have had difficulties with rational numbers, both conceptual and procedural. In the algebra classes that I taught, many students struggled over fractional numbers that were presented in problem situations. At that time I began to realize the limited nature of their conceptual understanding and the resultant impact on the understanding of advanced algebraic concepts.

I felt that a case study methodology was necessary for this study since developing students' conceptions involves communicating cognitive processes of a personal nature. The teaching experiment was utilized in order to gain an understanding of how students understand the unit concept and to determine whether or not attention to and understanding of the unit increases their overall understanding of multiplication and division of whole numbers and rational numbers. I selected the teaching experiment as the most appropriate research method for this study because it combines the advantages of Piaget's clinical method with those of a flexible teaching unit.

The Teaching Experiment

The teaching experiment is an extension of Piaget's diagnostic interview (Skemp, 1987), but differs from the diagnostic interview in that it involves a teaching component based on the method introduced by Vygotsky in the 1920s (Kantowski, 1978). The purpose of the teaching experiment is to make and
test hypotheses, not only about the nature of a child's thinking at a particular time, but about how this thinking develops from one state to another (Skemp, 1987).

The aim of this research method is to capture students' development of certain processes and to determine how instruction can optimally influence those processes. Analysis of quantitative data in this research is of less concern than the daily subjective analysis of qualitative data. This method combines interview and observation with a flexible teaching component. A sequence of lessons is structured before the experiment but modified continually during the experiment, based on information acquired in the previous lessons. In addition to being unrestricted in the teaching strategies used, the teaching experiment is also flexible in the number of participants. It can be undertaken with small groups or on an individual basis (Kantowski, 1978). In the present study, I used small groups consisting of four students.

Although the teaching experiment is flexible and adaptable, it is somewhat structured. Steffe (1977) summarized the components of a teaching experiment as follows: (a) daily teaching of small groups of children by experimenters, (b) intensive observation of individual children as they engage in mathematical activities, (c) prolonged involvement with the same children over periods ranging from about six weeks to the academic year, (d) clinical interviews with children, and (e) detailed records of observations through audio-video taping and the written work of children.
Prominent researchers have used the teaching experiment in studies involving rational number concepts (Behr, Wachsmuth, Post, & Lesh, 1984; Cobb & Steffe, 1983; Hunting, 1983; Olive, 1993; Thompson, 1982). Additionally, the doctoral dissertations of Boulet (1995) and Golding (1994) used the teaching experiment to study the development of the unit concept in students and preservice teachers respectively. This research method seemed most appropriate for the present study since the objective was to follow students' development of the unit concept and to examine the role of the unit concept as a possible link between whole number and rational number domains.

**Subjects**

Although a teaching experiment can be conducted with an individual, a small group, or a large group (Kantowski, 1978), the present study was best facilitated by means of small group teaching. This allowed a mixed ability setting resembling a modern classroom environment. Additionally, it served to promote the advantages of cooperative learning (Skemp, 1989), thus allowing the researcher to gain valuable data from student interactions.

The researcher selected to conduct the study at a rural school in northern Louisiana which contained about 600 students in grades K-12. Since the researcher had previously been a teacher at this school, it was an easy matter to obtain approvals from the administrators, the classroom teacher, and the parents of students involved in the study. History has shown this
community to be very supportive of the school and its teachers. The community was part of an economically-stable area in which there were numerous business and industrial opportunities for the families. Thus the socio-economic status of students from this school was considered average.

In the standard teaching experiment, data often are gathered only from a sampling of strong, average, or weak students, usually categorized and selected with the aid of the classroom teacher (Kantowski, 1978). In the present study, input from the classroom teacher in regard to students' existing rational number knowledge was used to select the eight students, four from each of two seventh-grade classes, who participated in the clinical case study. The researcher asked the classroom teacher to take into consideration the attendance habits of each student and each student's willingness to verbalize thoughts as additional criteria in the selection of students. Thus observations of two separate student groupings and discussions involving their cognitive processes were employed in this study. Heterogeneous grouping was used to mirror in the study the diversity inherent in classrooms, and its resultant effect on the learning environment. Each of the two groups consisted of four students with varying mathematical abilities; where the teacher considered one to be above average in mathematics, two average, and one below average. This allowed each group to better simulate an ordinary classroom in terms of mathematical aptitude.
All subjects selected for this study were then 12 years of age and had never repeated a grade level. The classroom teacher reported vast differences in ability between the two classes from which the participants were selected. Her second class had no high ability students. Consequently, the high ability student selected from this class to participate in Group B was actually the highest average student from that class.

The original groups each consisted of three females and one male. The only male student from Group A was a chronic behavior problem during the study. This behavior was consistent with that displayed in the regular mathematics classroom and resulted in his dismissal from the study. The remaining male student, a member of Group B, withdrew from the study a few days later. While his explanation for leaving indicated dissatisfaction in being the only male in the study, the researcher suspected that this student also had been pressured into leaving by the other male student. These departures left only three female participants per group.

Chronic absences by Group B participants had adverse effects on the study. One girl dropped out of the study due to excessive absences, only to return for the last lesson of the study. By that time, she had missed most of the unitizing activities. With these circumstances, there were only two students who remained a part of Group B, but they each accumulated several absences as well. Student absences from Group B were attributed to the time period in the school day. This class period, meeting the last hour of the day, was
confronted with numerous interferences. Extra-curricular activities (e.g., club meetings) pulled some students out from the teaching sessions. There were also community events which impacted student attendance. Two of the local churches held weekend youth retreats during the study. On the Fridays of these retreats, the participating students checked out of school early. Due to their inconsistent participation in the study, data from Group B will not be reported or analyzed in this study. The researcher felt the discontinuity of progress attained by these participants would not lend itself to a viable analysis of the effects of this study on the acquisition of unit concepts. Consequently, the results from this group are not included in the analysis of data contained in Chapter 4.

Research Design

This study involved three phases spread over a five-week period, as outlined below.

Initial Interview

During the first week, these eight students participated in an individual interview with the researcher to assess their initial cognitive abilities relative to both unit concept and rational number concepts. Student responses were used to assess students' existing awareness of unit fraction concepts, facility with rational number computations, and ability to model fractional situations. A detailed listing of assessment items is contained in Appendix A. Analysis of
student responses for Group A participants is contained in the student profiles section within Chapter 4.

Teaching Sequence

Each group attended the teaching experiment sessions daily for three weeks for approximately 45 minutes per day. These sessions consisted of three parts: instruction, observation, and discussion. In the first part the previous lessons were summarized and the daily activities were introduced. As opposed to the traditional lecture format, the instruction phase consisted of interactions between the students and the researcher, primarily in the form of questioning and modeling. For the observation part, subjects were observed as they were involved in the planned activities for the lesson. Since most of the activities were of an individual nature, the researcher was able to observe individual reactions to the tasks. The researcher monitored students' engagement in the problem solving situations and recorded student comments and solution procedures. In this phase, students sought to build on their existing knowledge of the unit concept and to extend this to the whole number and rational number operations of multiplication and division. Videotaping of lessons allowed continued observations of participants. The discussion part consisted of whole group discussions of the students' solution strategies. Interactions among the students occurred during this segment. The researcher looked for evidence of conceptual knowledge linkages between whole number and rational number domains. This discussion segment allowed both students and the
researcher to verbalize reactions and questions concerning the problem situations. Although the lessons were pre-structured, they were modified on a daily basis. Observation notes and daily reviews of the videotapes were used to assess needed modifications for the next session. Appendix B contains the problems presented to students in each lesson.

Exit Interview

After the conclusion of the teaching sessions and during the fifth week of the teaching experiment, the researcher conducted individual teaching interviews with each student. These interviews sought to determine the degree of independence of thinking students achieved through this instructional experience and the degree to which the concept of unit could inform students' procedural methods of the usual school curriculum. Rational number tasks were designed to determine if, and to what extent, the students could link knowledge of the unit concept from the whole number to the rational number domain. A variety of manipulatives were available to aid students in their solution process—fraction strips, fraction circles, and chips. Students had the responsibility of deciding which manipulative would best facilitate a solution. Since many of these questions were similar in format to those in the initial interview, results served to provide an index of students' development of rational number concepts during the teaching sequence. Results of this interview are contained in the analysis section of Chapter 4 since these
responses provide much of the data needed for answering the research questions. A detailed listing of assessment items is contained in Appendix C.

**Lesson Content**

The lessons followed the selection of subjects and the initial interview. Each lesson was designed to illustrate various aspects of the unit concept including unitizing, reunitizing, and norming. Unitizing refers to the ability to construct a reference unit or whole (Lamon, 1994). For example, a case of soda could be considered as a single unit, or one 24-pack. Reunitizing involves the composition and/or decomposition of existing units to form new units. The partitioning of this unit of soda into subgroups would result in the formation of composite units (units of units), such as two 12-packs or four 6-packs. Further partitioning, or reunitizing, of these composite units could result in units of units of units; the two 12-packs could be packaged as two groups of three 4-packs, or $2\{3[4\text{-unit}]\text{-unit}\}$s. Considering each of the 24 cans as singletons, you have 24 one-units or 24 cans. An understanding of the unit concept involves viewing a whole as a nested system of units (Steffe, 1994). The task of reunitizing a case of cola, one 24-pack, into 6-packs involves the reconceptualization of the situation in terms of a fixed unit or standard, the 6-packs. This process of reinterpreting a situation in terms of a new reference unit is called norming (Freudenthal, 1983; Lamon, 1994). The ability to apply composition, decomposition, and conversion principles such as these on
quantities is essential in the development of a flexible concept of unit (Behr, Harel, Post, & Lesh, 1994).

The ability to compose units has its origin in modeling and counting activities (Steffe, 1994). Thus the lessons in this study began at this entry level in Lesson One to assess students’ abilities to form composite units in simple counting situations and to help students become more aware of their unit composing abilities. Subsequent lessons extended this awareness of unit composition to the whole number domain. Lesson Two examined students’ formation of units in whole number word problems. Lesson Three focused on students’ abilities in unitizing and norming in problem situations where the reconceptualization of whole numbers as composite units could aid the solution process.

In order to extend the unit concept of number from the whole number to the rational number domain, it was critical that students’ models for multiplication and division be expanded. Lessons Four and Five concentrated on the development of the partitive and measurement (quotitive) models for division and emphasized the presence of division in measurement activities. Lessons Six through Eight centered around development of multiplication models. Problem situations started with the repeated addition model, then progressed to highlight the need for other models such as the array and area models. The issue of change of unit, occurring in some multiplicative situations, is so dynamic that several lessons were taken to emphasize the unit
structure emerging in these referent-transforming compositions and the impact of this unit change on a part-whole relationship. Concrete models used consisted of chips, fraction strips (embedded model of units), and the area model. Once the conception of change in unit was established, students experienced whole number problem situations in which unitizing and norming skills would facilitate the solution process.

Attention was then directed to extending students' existing knowledge of fractions to include awareness of the unit in each fractional situation. Seven lessons served to highlight two major areas of fraction concepts: (1) representational understanding, and (2) unitizing and norming.

Representational understanding entails the semantic representation of a part-whole situation. Lessons Nine and Ten emphasized the necessity of equality in partitioning, verbalization of the meaning of the unit fraction, and a focus on the unit in fractional situations. While a focus on the unit in a fraction situation might seem trivial to some people, this focus is viewed as essential by many rational number researchers in linking concepts within the whole number domain to the rational number domain (Behr, Harel, Post, & Lesh, 1992). For example, one of the most critical understandings in rational numbers is that the numerical symbol, such as 1/3, can represent different amounts, depending on what the unit or the unit whole happens to be. The quantity implied by the symbol 1/3 is ambiguous unless we know to which whole it refers. Lessons Eleven through Fourteen were designed to build on students' existing skills in
unitizing and norming. Students used various manipulatives for concrete exposure to unit composition and decomposition. These activities enabled students to view non-unit fractions as iterations of unit fractions (i.e., $3/5 = 3[1/5\text{-unit}]$s), to view unit fractions as iterations of unit fractions (i.e., $1/4 = 3[1/12\text{-unit}]$s), and to enable students to reconstruct the unit if only the unit fraction is known. Models for multiplication and division of whole numbers were extended to rational numbers in Lessons Thirteen and Fifteen.

Data Collection and Analysis

Data Sources

A variety of qualitative data were obtained from the teaching sequence: videotapes and audiotapes, transcripts of videotapes and audiotapes, students' worksheets, and the researcher's journal.

Videotapes and audiotapes. Each lesson and interview was both videotaped and audiotaped, with the assistance of a graduate student. The researcher is visible throughout the tapes. The researcher viewed the video at the conclusion of each lesson to assess students' unit conceptualization and to determine extensions or modifications needed for the following lesson. Audiotapes served as a backup for areas where the videotape is unclear. These tapes provided the researcher with an opportunity to review each lesson and to assess students' development of rational number concepts. Furthermore, they provided the opportunity for the researcher to catch student actions that were missed during each teaching session.
Transcripts. The videotapes and audiotapes were transcribed to allow for continued analysis throughout the teaching experiment. These descriptive observations became focused observations during analysis and led to selective observations at the conclusion of the data analysis phase of the present study (Spradley, 1980).

Researcher journal. The researcher’s journal was updated daily throughout the teaching experiment with personal observations of students, reflections on each lesson, questions and ideas emerging from students’ progress, and decisions made concerning lesson modifications.

Students' written work. Students’ worksheets provide another source for data analysis. Worksheets consisted of the results of the activities in the initial and exit interviews and in the daily lessons.

Data Analysis

The analysis of data was guided by numerous sources. The researcher consulted sources such as Spradley (1980) and Wolcott (1988). Additional guidance was received by reading other qualitative studies and through conversations with experienced researchers (Behr, personal communication, November 7, 1994; Ginn, personal communication, September 25, 1996). Data analysis was a long-term, continual process in this study and can be categorized into three distinct phases.

The initial analysis occurred during the actual study. Daily examination of the researcher’s observation notes and reflections served to keep the
 researcher focused on the research goals. The daily examination of students' written work provided the researcher with incidences of conceptual changes of students and insights for future analysis.

The second phase involved the transcriptions of both the individual interviews and the group lessons. Since there were approximately 46 hours of videotapes to be viewed by the researcher, the transcription process was a long, tedious endeavor occurring over a three month period. Once the transcriptions were completed, the researcher began a process of continuous readings of these transcripts in order to become familiar with the data and to look for patterns and emerging themes.

The third phase of analysis consisted of the organization of data into categories determined by the research questions. The researcher adopted a color-coding scheme used in a recent study by Golding (1994) in which research questions were assigned a color code. Supporting data found in the transcripts were highlighted with the color corresponding to each research question. Data having pertinence to multiple research questions were highlighted with each appropriate color code. During the writing phase, this color-coding process enabled the researcher to determine the supporting material for the research questions. The researcher also conducted a lesson-by-lesson analysis of the performance of each individual student to assess students' progression in unit concepts over the timeframe of this teaching experiment.
These methods of analysis were applied to all eight students in the study, even though only the data for one group of four students are reported here.
CHAPTER 4
RESULTS AND ANALYSIS

This chapter is divided into three sections. The first portion provides a profile of the subjects involved in the study. The second section reports the progress of the 15 lessons of the teaching experiment. The concluding section recapitulates the data, but from the point of view of the students' individual development, as a means to address the research questions.

The student profiles contain both personal and academic backgrounds on the subjects participating in this study. The classroom teacher's rationale for selecting each student is included, along with the researcher's assessment of individual student abilities emerging from the initial interviews.

The second part of the chapter is divided into 15 sections—one for each lesson of the teaching experiment. This sequencing is designed to show the progression of unit conceptualization skills acquired by the individual students. Each individual lesson analysis begins with a description of the content addressed in that lesson and then provides an analysis of the individual responses.

The final section of this chapter recapitulates the data, but from the point of view of the individual participants in order to explore the research questions posed in the earlier chapters. In order to provide a developmental profile of the participants, each question will be answered for each student.
Student Profiles

An initial interview was conducted with each student before commencing the teaching sequence (Appendix A). This interview was designed to reveal the extent of unit fraction knowledge the students possessed. In addition, the results obtained served for evaluative comparison with the results of the teaching sequence. There was no mention of the word "unit" prior to students' completion of the tasks. Results from the interview enabled the researcher to develop a profile of each student's existing understanding concerning unit concepts.

Task 1 of the initial interview was designed to determine if students' existing models for multiplication and division were limited to the primitive models described by Fischbein and his associates (1985). Students did not have to solve the problems; instead, they only had to identify the operation needed to solve the problem and the problem sentence used to determine the solution. Eight multiplication and division problems were presented, some of which violated constraints on primitive models. A few addition and subtraction problems were interspersed to reduce the likelihood that correct answers would result from guessing, which might be the case if students knew the answers were limited to multiplication and division. Task 2 consisted of 10 rational number problems for students to calculate. These problems enabled the researcher to examine students' procedural skills. Both tasks were completed without assistance from the researcher.
Task 3 required students to explain their interpretation of a fraction. Fraction understanding was explored further in Task 4 in which students were shown seven cards, some modeling various interpretations of the fraction three fourths, and were asked to explain which cards could help someone understand the meaning of the fraction three fourths.

Task 5a explored students' unitizing skills in both composition and decomposition situations. The second problem assessed students' abilities to equipartition a unit whole which contained an already existing equipartitioning used as a distractor. All students had great difficulty with Task 5b. Those who were able to solve it correctly did so only with assistance from the researcher in the form of focused questioning. Students' existing models for multiplication and division of whole numbers and rational numbers were examined in Tasks 6 and 7. Task 8 required students to reconstruct the whole if given a unit fraction. None of the students demonstrated an awareness of the unitizing skills needed to solve this problem.

Four students initially comprised the group in the teaching experiment. The only male participant dropped out early, but this occurrence served to make a stronger working group. Three girls remained in this group; each one being present for the entire teaching experiment. Results from the initial interview, along with the classroom teacher's rationale for selecting each student, are contained in the following profiles. Pseudonyms are used for the participants in the study.
Judy

This student was considered above average by the classroom teacher and generally made straight A's, as did her older siblings. Her older sister is a first-year elementary teacher at the same school. Judy's mother is of foreign descent, but English is the primary language spoken in the home. Judy was extremely shy and soft-spoken. It was difficult to get her to talk freely or to express her thoughts voluntarily. Judy had the highest academic record of all the participants in this study, but her procedural knowledge far exceeded her conceptual knowledge. Judy usually knew the correct procedure to apply when solving a problem, but often was unable to explain why that procedure worked.

Judy scored 90% on her fraction calculations. The single error seemed to be the result of carelessness while rewriting a fraction. Although her procedural skills appeared to be strong, Judy's efforts on Task 1 revealed the presence of limited models for multiplication and division. Judy missed one of the three multiplication problems and all five of the division problems. She correctly identified the operation on three of these five division problems, but exchanged the two numbers when giving the problem statement.

Judy described a fraction as "a part of something." When asked to explain further, Judy remarked that a fraction was part of "a whole number." On Task 4, Judy correctly identified cards displaying both the part-whole and measure interpretations as being representative of three fourths; however, she incorrectly identified the last card as also representing three fourths. There
were three parts out of four shaded on this card, but the partitionings were unequal. Questioning by the researcher revealed Judy’s awareness of the unequal partitioning; however, Judy indicated that this difference did not matter.

In Task 5a, Judy quickly partitioned the 12 jacks into four groups, and revealed her conceptualization of unitizing by stating, "there’s three in each one fourth. That’s 1/4, that, and that." Judy combined these three units to form the composite unit for three fourths. The repartitioning activity in Task 5b proved to be difficult for Judy; however, she was able to find one third of the candy bar once the researcher added the context of sharing a candy bar with two of her friends. Judy left the original partitioning into fourths and repartitioned one of these fourths into thirds. To represent the solution, Judy shaded one of the original partitionings (1/4-unit) and one third of the repartitioned portion (1/3 of 1/4-unit) to represent one third of the candy bar.

For Task 6a, Judy explained that the problem 12 \div 3 meant to "divide [12] into three sections." She used her chips to form three groups with four chips per group. Judy’s understanding of the partitive model was further revealed by her formulation of a story problem: "You have 12 pencils and you want to give them to three people evenly." Judy revealed her awareness of the unit in this story situation by stating the answer meant "four pencils" for each person. This partitive model was the only model Judy was able to envision as a solution for the task. The success Judy experienced with whole number
division did not recur in her solution of the fraction division problem in Task 6b. She was unable to correctly model the problem or create a story problem for \(6 \div 1/2\). In explaining the meaning of the division statement, Judy responded, "It would be like one half of six. It would be three." However, she realized this interpretation was faulty when she performed the calculation and obtained an answer of 12. Judy could not explain the discrepancy between her two answers for the problem, but was confident that her algorithmic solution was indeed the correct answer.

Similar modeling difficulties occurred in Task 7. Judy revealed her understanding of the repeated addition model for multiplication of whole numbers with the creation of a story problem and with a physical model; however, she was unable to interpret the rational number multiplication situations. She solved each problem correctly but still could not determine appropriate story problems or models for Tasks 7b and 7c. The researcher asked Judy what was causing the problem on Task 7b. Judy responded, "the one half. I can't figure out what it means." Her composed story problem for Task 7c, \(1/3 \times 1/2\), was as follows: "If you have like . . . a third of a candy bar and a half of a candy bar, it would equal one sixth of a candy bar." Judy's answer to her calculation of the problem was forced into a story situation no longer conveying the intended operation of multiplication.

Instead of considering the units in Task 8, Judy instead focused on the denominators. She reasoned that if four dots equalled one third, then three
dots must equal one half because "one third is one more than one half." In response to further questioning by the researcher, Judy concluded that the whole unit would consist of five dots since "it's more than one third."

**Larry**

This student was considered below average by the classroom teacher. His grades for the past few years have been low; all previous grade levels were passed marginally. Larry's family moved to the community from out-of-state, and this was his first year at this school. Larry was failing his mathematics class at the time of the study and the classroom teacher expressed doubt that he would pass grade seven for the year. Several of Larry's teachers recently had referred him to the pupil appraisal department for evaluation by the special education team, but no evaluations had been made at the time of this study. This researcher found Larry to be a chronic behavior problem. Not only did he refuse to do the daily tasks, but he also disturbed and distracted other students during the study. He liked to tease and flirt with the girls in Group A. His behavior worsened daily, despite numerous after-class discussions with the researcher and the classroom teacher. Larry was finally asked to leave during Lesson Ten and told not to return until he could behave. He never returned.

Larry was able to identify the correct operation on 40% of the problems in Task 1, but his procedural skills were weaker. He scored 20% on the computation tasks, missing the addition problem, all the multiplication
problems, and four of the six division problems. Analysis of his errors revealed Larry's tendency to exchange the numbers in instances where the divisor was larger than the dividend. Whole number dominance was most evident in Larry's method for multiplying fractions. He disregarded the fraction bar, multiplied the numbers, then brought down the bar as follows:

\[
\begin{array}{c}
\frac{1}{2/3} \\
\times 5 \\
\hline
11/5
\end{array}
\]

Larry's understanding of fraction was limited to a part-to-whole interpretation; however, he did not recognize the importance of equipartitioning. Even with questioning, Larry was unable to find one fourth of the 12 jacks in Task 5. His verbalization of his thinking revealed no evidence of unit conceptualization. Larry was unable to model any of the multiplication or division problems and could not think of story situations for these problems.

**Laura**

The classroom teacher considered Laura to be an average student. Her family recently moved to this area, so Laura was new to the community and to the school this year. Her grades in math had been C's all year. She warmed up quickly to the study, contributing often and eagerly to the discussions. Quite often she was the first one in her group to grasp unit concepts.

Laura scored 20% on the calculations. Her work exhibited numerous instances of misapplied rules and inappropriate strategies. Laura inverted the first number in rational number multiplication problems, regardless of whether
this first number was a whole number or a fraction. She continued the multiplication process by obtaining common denominators, multiplying the numerators, and keeping the same denominator. She seemed to be using her addition rules for part of her multiplication procedure. For division problems, Laura exchanged the numbers whenever the divisor was greater than the dividend. For either multiplication or division problems in which there appeared a whole number and a fraction, Laura converted the whole number, \( n \), to the fraction \( \frac{1}{n} \).

Laura's ability to recognize the needed operation in a problem situation was better than her performance on the calculations. In Task 1 she scored 50%, missing one multiplication statement and four division statements. All four of these missed division statements contained a divisor consisting of a fraction or a whole number larger than the dividend. In such cases, Laura selected multiplication as the needed operation.

Laura had great difficulty verbalizing her understanding of fractions; this problem was compounded by the fact that her fraction terminology was weak. After a long hesitation, she replied that a fraction was "something you could like . . . half something else." She added that a picture of a fraction had "some of it shaded" and indicated that the task facing students consisted of telling "what percent is shaded or not." From such statements as these, it could not be determined if she held a part-whole interpretation of fractions. She correctly identified the part-whole situations for three fourths from the seven cards in
Task 4 as being representations for the fraction $\frac{3}{4}$, but also included the last card which was comprised of four unequal parts. None of the other fraction interpretations on the remaining cards made sense to her.

Laura demonstrated no unitizing ability in Task 5, finding $\frac{3}{4}$ of 12 jacks. Her strategy for partitioning a whole seemed to consist of guessing. Laura decided the answer would be eight "because half is six." With guided questions from the researcher, Laura found one half, but could not determine one fourth. Laura also was unable to find one third of the candy bar which was already partitioned into fourths until the researcher provided the context of sharing with two of her friends. Although Laura was unable to physically partition one of the existing fourths into three pieces, she seemed to realize the necessity of accomplishing this task as indicated by her response: "Give them all one whole piece and then not quite half it . . . like one square halfed [sic] into three . . . equalled into three."

Laura produced models for the whole number multiplication and division statements. She created a story problem for the multiplication problem, but not for the division problem. Laura experienced no successful attempts, however, on the rational number problems—neither in calculations nor in modeling.

**Melanie**

Melanie was referred to this study as an average ability student and had made mostly C's in math during the year. She was extremely talkative. She
was more than willing to share her ideas with the group. This helped the researcher from the standpoint of providing insight into her thinking; however, the problem facing the researcher involved keeping Melanie from dominating the group discussions. She loved to talk, but did not want to listen to others in the group. Instead she would draw pictures or play with the manipulatives. The researcher frequently had to redirect her attention back to the group.

Melanie scored 30% on both the calculations and the operation identification tasks. She missed all of the division problems because of her failure to invert the divisor. She found a common denominator on all the multiplication problems; although her usage of this strategy did not cause the errors on the problems she missed. Melanie described a fraction as "part of a whole," adding further that "it's not a whole." Her part-whole interpretation of rational number also was evident in the fraction card identification in Task 4; however, she denied the importance of equipartitioning in a part-whole situation.

For Task 5, Melanie was unable to partition the 12 jacks so the researcher asked her to find three fourths of 12 chips. Melanie separated the chips into three groups of four and proclaimed that the entire group of chips represented three fourths since "there's four in each group and there's three groups and 3 times 4 equals 12." For Task 5b, finding one third of a candy bar already partitioned into fourths, Melanie decided to "take one of the blocks
out." She crossed out one of the $\frac{1}{4}$-units, shaded one of the remaining three blocks to represent one third.

Although Melanie experienced success in modeling whole number problems in multiplication and division, she was unable to create models for rational number operations or to perform the calculations correctly. She was, however, successful in formulating a story problem for $6 \div \frac{1}{2}$: "Someone had a party and they invited 12 people. There was enough cake for six people and they had to half each piece." Melanie used the same story structure for the problem $4 \times \frac{1}{2}$: "I have eight [people] invited to the party and there were four pieces of cake. You’d have to half each piece and there would be eight pieces." When the researcher asked if there was a difference between these two problems, Melanie replied, "there’s no difference in how to work them. It’s just the problem sign."

Melanie insisted that the problem, four dots equals one third, was written incorrectly in Task 8. She stated: "They don’t equal one third; they equal one fourth. There are four dots!" After a brief hesitation, she added: "No! It’s not one fourth, they are all filled in. It’s a whole." With further questioning by the researcher to determine the basis for her thinking, Melanie concluded that the denominator "tells you how many [dots] there are."

The Lessons

There is no universally accepted definition for the unit concept.

Throughout this study the word unit was used to describe any grouping
conceptualized as an entity or whole. Each lesson in this teaching experiment illustrates various aspects of the unit concept, such as unitizing, reunitizing, and norming. The activities in which the students were asked to participate were designed to promote the learning of unit concepts. Indications of students’ conceptualization of units were ascertained through students’ verbal communication, by the markings on students’ written worksheets, and/or by physical gestures, such as the students’ hand motions observed by the researcher.

This portion of Chapter 4 provides an overview of the content addressed in each lesson of the teaching experiment. It is not intended to represent a detailed script of each entire lesson, but rather a sampling of student responses to the problem situations and typical group interactions. This structure serves to reveal the progression of unit conceptualization skills acquired by the individual students.

**Lesson One**

The first lesson of the teaching experiment was designed to reveal whether or not the students would naturally use grouping or unitizing in everyday counting procedures. No mention of grouping was made prior to the lesson. Students were given three different counting tasks, all designed to examine if and to what extent students formed units other than one. The fourth task required students to examine the counting strategies they had used on the previous tasks, looking for similarities. Students completed all four
tasks individually, then came together as a group to discuss their strategies for completing each task. This discussion focused on helping students become more aware of their unit-composing abilities.

For Task 1, each student was given a package of chips and asked to count them, with results to be recorded on the worksheet. Each packet contained a different number of chips, varying from one hundred to two hundred. Laura, Larry, and Melanie each counted using a units-of-one approach. They emptied the bag of chips into a pile on the table, then removed and counted them one at a time, forming another pile where all chips were placed after being counted. Melanie went one step further, in that she recounted the chips as they were returned to the bag, but she still used the units-of-one method.

Judy counted her chips in groups of size 20, forming little stacks as she counted. She ended with six stacks of 20 chips and one pile of 19. Judy counted the number of stacks, multiplied this number by 20, then added on the remaining odd chips to obtain her final count. Analysis of Judy's procedure shows units corresponding to the stacks, units of 20, and a unit of 19. This units-of-units procedure can be described as $6[20\text{-unit}] + 1[19\text{-unit}]$. Each student was asked to explain the strategy used to count. Judy was the only one to use groups other than ones. When asked why she had counted by 20s, Judy replied that this approach "made it easier."
For Task 2, students were given a large container of multi-colored cubes, which included individual cubes as well as various stacks of snapped cubes ranging in size from two cubes to 10 cubes. As in the previous task, students were asked to count the cubes and record their results on the worksheet. Judy formed stacks of 20 to facilitate her count. She would start each stack by picking up a long stack from her pile of pre-grouped cubes, counting the number of cubes it contained, then adding on other shorter stacks until she obtained a 20-unit. The original units thus became embedded in the larger unit of 20. Then Judy counted the number of stacks lying on her desk, multiplied by 20, and added on the remaining cubes by counting by ones. Analysis of her procedure reveals usage of a similar units of units strategy as she had used on the previous task.

Laura and Larry both used a units-of-one approach for counting the cubes. The cubes were left in their original groupings. A count was obtained by picking up a stack, counting the number of cubes, placing the stack to the side, picking up another stack, counting these by adding onto the previous count, and continuing this process until the entire container of cubes had been exhausted. There was no visible evidence that the variation in pregroupings was even considered.

Melanie counted by ones for a while, but then changed her counting strategy and started over. This time she regrouped the stacks by color, forming long rods. Utilizing this technique, she was not counting at all at this point, but
merely reorganizing the groups. Once all the cubes had been separated, she noticed they were all the same length. Melanie's process of regrouping by color had resulted in the norming of the container of cubes to create units of 25. She then counted the number of cubes in one stack and multiplied by the number of stacks. Melanie explained that this new method was easier than her previous strategy of counting by ones. In multiplying stacks and cubes, Melanie was exhibiting the same units-of-units approach as Judy. When asked to compare her counting method with Judy's, Melanie responded that Judy's way was harder because she had to "count each stack up and then count those up," referring to Judy's odd cubes which did not form a 20-unit. Apparently she did not recognize the similarity in their methods; instead, Melanie saw that she had "less to count."

Task 3 required students to estimate the number of people in a crowded scene and then record this estimate on their worksheet. Students were instructed not to count all the people and not to guess, but to think of a reasonable strategy for estimating. Judging by the responses made by students, estimation of the number of people in this particular picture was an unreasonable task for this age group. No one offered a valid strategy. An explanation offered by Melanie seemed to describe the technique used by most of the students—"I looked at all the people and took a good guess."

Task 4 asked students to rethink the counting strategies used on the preceding tasks and to analyze if similar strategies were used. Responses were
recorded on the worksheets. No mention of groups or units had been made before the lesson. Yet two students acknowledged the use, or lack thereof, of groups when explaining their counting strategies on the worksheets. Judy remarked that she counted by putting items into "groups of 20." Laura said she "just counted them," adding further that she "didn't put them in groups or anything." Although Melanie used a grouping technique to count the cubes in Task 2, she made no mention of groups in describing her counting methods. However, she did specify that she had "color-coordinated—put all in each color together—" the blocks. This act of "putting together" signifies Melanie's focus on units, although she did not recognize this process as grouping or unitizing.

During the group discussion each student explained his or her counting strategy. Whenever a student revealed that a type of grouping strategy had been used, follow-up questioning by the researcher attempted to have students verbalize why this method had been selected. Judy remarked that counting by 20s made it easier. Melanie acknowledged that she had "regrouped by color" and it made it even easier than Judy's method since her method had "come out even." Melanie's use of the word "regrouped" is an indication that she not only recognized existing groups, but also chose to reunitize to form groups which would facilitate a solution.

To discern if students were aware of a focus on grouping or unitizing, the researcher asked the question: "If you had a huge piggy bank full of pennies and you dumped them out on this table, what would be your strategy
to count them?" Laura, who used a units-of-one approach on all previous
tasks, was the first to respond, "I'd probably count by groups of 20s." Judy
previously grouped her worksheet tasks by 20s, but decided in this instance she
would "put them into dollars." Melanie agreed that this dollar grouping would
be her strategy also. The students now seemed to be aware of their abilities to
use grouping and the impact of grouping in facilitating a solution in a counting
situation.

The researcher commented about the strategies used. "What I want you
to see is what each of you are talking about doing. You all used different
amounts. Judy liked counting by 20s, until we got to the piggy bank problem,
and then she counted by dollars. Melanie separated cubes by colors. Now
Laura is using 20s to count those pennies in the piggy bank. In each case, what
is the advantage of doing that?"

Melanie: "It's shorter. If you counted them one by one, it takes forever
and you might make a mistake." Laura and Judy nodded in agreement.

**Lesson Two**

Lesson Two of the teaching experiment was designed to see if students'
awareness of unit composition abilities could be extended to whole number
word problems. Students were given three word problems to solve individually.
On the first task, students were asked to use the manipulatives on the table to
aid their solution process. With these physical objects, students could simulate
the problem. For Task 2, a diagram on the worksheet helped students to
visualize the situation. The concluding task was written as it might appear in their textbook, with no physical or visual aids to provide realistic settings. This lesson was designed to enable the researcher to compare unit formation abilities in this lesson with those exhibited in the previous lesson.

Task 1: Lisa has 5 bags containing 4 candies and 1 bag with 2 candies. Mark has 2 bags with 4 candies and 5 bags with 2 candies. If they combine their candies, putting them in bags with 4 candies each, how many bags will they use?

All students used the bags of candy to help them simulate the problem. It was interesting to watch how the bags were handled. Everyone sorted the bags into two piles, one pile representing Lisa's candy and the other pile for Mark's; but from here, the strategies changed.

Laura immediately started counting bags. "I put 2 two bags with 2 candies in each and counted it as 1 [bag of 4] and the ones with 4, counted those in and I came up with 9 bags" (although her drawing revealed 10 bags of 4 candies). Clearly Laura was thinking in groups or units. By picking up two bags with two candies in each and counting it as one bag, she was reconceptualizing to form composite units, i.e., $2[2\text{-unit}]\text{-units} = 1\{2[2\text{-unit}]\text{-unit}\}\text{-s-unit} = 1[4\text{-unit}]\text{-unit}$. Laura had not used grouping in the tasks of the previous lesson; yet her initial response to this problem consisted of the formation of 4-units.
Melanie pushed all the bags into one pile, then began counting individual candies without regarding the bags. In this units-of-one approach, it had not occurred to her that she had not answered the question of how many bags would be used. In explaining her strategy, Melanie related a "shorter way" to work the problem. "Times 5 times 4 is 20 and 2 times 1 would be 2—that would be 22. Then 2 times 4 would be 8 and that would be 30. Then 5 times 2 is 10, so 40."

Researcher: "It sounds like you are trying to find out how many pieces of candy there are."

Melanie: "But if you found out how many pieces of candy there are, then you could find out how many bags there would be."

Researcher: "That would work, but is it shorter?"

In response to this question, Melanie stated, "To me the multiplying is easier and more simpler." This remark is an indication that the calculation is a more natural approach to Melanie, perhaps because this is the taught procedure.

Judy jumped into the conversation to contrast her counting method with Melanie's. Picking up two bags of two candies, she remarked, "I just put these two as one and counted them as four pieces of candy." This was an indication that Judy physically and mentally regrouped 2 bags of 2 candies into 1 bag with 4 candies. Judy also had a different sorting method when solving the problem. She first separated the bags of two candies from the bags of four, then picked
up bags to count. Her worksheet description of this strategy states, "I counted
the bags with groups of 4 then I put 2 of the bags together that had 2 in them,"
(counting them as one bag of 4).

Task 2: Five boxes of golf balls, each containing one dozen balls, are to be repackaged into mini-sets of four balls each. How many four-packs can be formed?

Task 2 on the worksheet did not offer students the ability to use manipulatives, but instead contained a diagram which could facilitate the solution. All students reunitized each dozen balls into three 4-units, then formed composite units to solve the problem. By multiplying 3 by 5, they transformed the five sets of three 4-units (5{3[4-unit]-unit}s-units) into fifteen 4-units (15[4-unit]-units).

Researcher: "Did anybody count them to see how many balls there were all together?"

Melanie: "I started to, after I separated them into fours. Every top two, then I’d draw a line. Then I started going one, two, three, four, five. Then I said 'hang on.'"

Researcher: "Why did you say that?"

Melanie: "Cause that would take longer. So then I counted and I got 15 little things of them and I said 15 times 4 is 60."
Melanie's mention of "15 little things of them" is an indication that she had indeed constructed composite units. Each of these 15 things was now an entity in its own right. However, she still did not realize that she had counted individual balls with her multiplication of 15 and 4.

The researcher then pursued the question of how they would have worked this problem in class without the presence of the visual aid. "Would you have worked it the same way?"

Judy: "I would just draw it."

Laura: "I think I would have drawn it too. I would have put four in a row."

Melanie: "It's easier to have examples."

Researcher: "So we didn't need to make some math statement that 5 times 12 is 60?"

Melanie: "All you need is to separate all the balls into fours."

Even though Melanie realized that 15 was the answer to the question, she went further and counted the individual balls. It could be that the usual math textbook approach—the units of one approach—is a well-entrenched habit that is hard to override.

Task 3: The home economics class is planning to bake 120 chocolate cakes for a fundraiser. The recipe calls for 3 cups of flour for each cake. Flour comes in bags with 15 cups of flour per bag. How many bags of flour are needed to bake all the cakes?
Melanie, who previously focused on units of one, was the only one to use grouping. She explained her strategy as follows: "I divided 3 into 15 and I got 5."

Researcher: "What does that tell you?"

Melanie: "Five cakes can be made out of each bag of flour. So I divided 120 by 5 and I got 24."

Judy and Laura used a units-of-one approach. They multiplied 120 by 3, an indication that the unit of focus was the cup. Next Judy divided 360 by 15 to obtain the answer of 24 bags, whereas Laura performed several multiplications of 15 by a different number until she obtained a product of 360.

This was the first instance during the study in which Judy chose to use a units-of-one approach. When asked to contrast her method with Melanie's, Judy stated "it seems about the same." The researcher noted that Melanie's method resulted in smaller numbers to work with. There was no indication as to Judy's reason for using a units-of-one approach. Since this was the only problem in the lesson in which students did not utilize concrete materials, it is possible that this task seemed more like a "textbook problem" and, in this event, Judy perhaps resorted to her usual school math approach.

Lesson Three

Lesson Three focused on students' abilities in unitizing and norming. Three tasks offered problem situations where whole numbers could be reconceptualized as composite units. Students worked each problem
independently then came together for a group discussion after each task was completed.

Task 1: If you can buy 3 tapes for $5, how much will you pay for 24 tapes?

This problem was easy for most of the students to solve. Students were encouraged to use chips to model the problem. Everyone except Larry used the chips to regroup the 24 individual tapes into eight groups of three tapes. This norming process can be described as a regrouping of 24[1-unit]-units into 8[3-unit]-units. These students were aware of their usage of group composing abilities in solving the problem.

Judy and Laura counted their groups by fives, realizing that each group of three tapes represented a $5 expenditure. Melanie explained that she "counted how many groups there were and timesed [sic] it by 5." Larry was the only one unable to solve this problem. Instead of grouping into threes, he counted each tape by fives to obtain $120. On his worksheet he designated that each circle, or tape, represented $5.

Task 2: The Movie Center rents all their videos at 3 video movies for $7. In one week there were 672 videos rented. How much money did The Movie Center earn in rentals?

No manipulatives were provided for this task which is similar to the preceding problem. The researcher wanted to examine students' unit formation abilities in problems with numbers large enough to discourage the use of
drawings. This task was difficult for most of the students. Few saw this problem as having a similar structure to the previous one.

Melanie was unable to determine a solution process; she finally resorted to copying Judy's work. Judy was the only student to solve the problem. Her description of her solution process succinctly states that she "divided 3 into 672 and got 224, then timesed [sic] it by 7."

After hearing Judy's strategy, Laura stated, "I may have mine wrong. I just put 672 times 7. I didn't divide or anything."

Researcher: "How much were those videos for rent?"
Laura: "Seven dollars for three videos."
Researcher: "Does that mean $7 for each one?"
Laura: "That's what I did wrong! I just put 7 instead of dividing."
Researcher: "Look at Tasks 1 and 2. The numbers are bigger here, but do you see a similar situation?" (No response.) "The price of these are by?"
Judy: "Groups."
Researcher: "Groups of?"
Laura: "Three."
Researcher: "So you need to find out how many groups there are. How did you determine the number of groups?"
Judy: "Divide."
Researcher: "What was the problem that kept you from seeing this as a grouping situation?"
Laura: "I just didn't. It didn't come to me. I wasn't sure so I just timesed [sic] it by 7."

The researcher again asked students if they saw this problem as a similar situation to the previous task. Melanie made no response, whereas Judy and Laura nodded yes. The following question was posed by the researcher: "If you had 672 chips on the table to use to model this problem, what would you do?" In response, Judy stated, "Group them by threes and then times it by 7." Judy demonstrated her conceptual understanding of this task, but it was unclear whether the others were expanding their thinking in terms of grouping.

Task 3: Mrs. Bryan wants to give Valentine cookies to the students in her math class. She buys one bag with 15 cookies, two bags with 24 cookies each, and one bag with 9 cookies. If Mrs. Bryan allows 3 cookies per student, how many students will receive cookies?

The researcher began the discussion with the question, "Did you use grouping or did you count by ones?" Melanie was the first to volunteer her solution strategy: "I used ones. I divided 3 into 15 and got 5, then I divided 3 by 9 and got 3. Then I divided 24 by 3 and got 8. Then I added and got 16." If Melanie had read the problem more carefully, she possibly would have found the correct solution. Her solution revealed that she had only counted one bag with 24 cookies instead of two bags. Melanie used units other than one, yet she still saw her approach as counting by ones. This was not the first time that
Melanie's comments revealed a lack of distinction between counting by ones or counting units. The researcher probed further into Melanie's thinking about units of one.

Researcher: "You say you counted by ones. Is that what you did? Why did you divide by 3?"

Melanie: "Cause I... I don't know. I thought it would be shorter than adding everything up. I just divided each of them by 3 because 3 would go into them."

This last statement left the researcher still unable to determine if Melanie herself knew why she had divided by 3. This question was used to continue the probe: "What does the 3 represent in this problem?"

Melanie: "How many cookies each student would get."

The researcher asked the entire group, "What do you think about Melanie's method? Did she count by ones or did she use grouping?" Melanie was the first to respond. "I used grouping." The dialogue continues.

Researcher: "Why do you think it's grouping?"

Melanie: "Because I used the 15 and the 3 and they are grouped together."

It still was unclear to the researcher if Melanie truly knew whether she was grouping, so a follow-up question was posed: "What does that 15 divided by 3 tell you?"

Melanie: "That five people got cookies."
Researcher: "So it tells you the number of groups of three in that package."

Larry would not explain his strategy to the group. His worksheet revealed the same units of three approach as Melanie's. He also had the same mistake of only counting one bag with 24 cookies instead of two. Since he would not voice his thoughts, there is no way to determine Larry's conceptualization for this problem. It is doubtful that the group discussion helped his thinking. At this point of the lesson, Larry had withdrawn his attention and was instead preoccupied with distracting the girls.

Judy and Laura each explained their strategy, commenting that they had counted by 1s to determine the total number of cookies contained in all bags and then divided this total by 3. Thinking that perhaps this approach was the one these girls might use in a textbook problem, the researcher tried to bring further context into the task by having students model the solution with chips. Each student was assigned a bag of cookies (chips) and asked to determine how many students would receive cookies from each bag. Instead of counting the individual cookies, all students chose to group them by threes. After allowing students time to model their portion of the problem, the dialogue continued.

Researcher: "I asked you to rethink how many people could get cookies from your bag." (The researcher wrote each student's answer on the board in a grouping or unit notation.)
Melanie: "Five." (5[groups of 3])

Judy: "Eight." (8[groups of 3])

Larry: "Eight." (8[groups of 3])

Laura: "Three." (3[groups of 3])

Researcher: "So how many students get cookies?"

Judy and Laura: (In unison,) "Twenty-four."

Researcher: (Pointing to the coefficient in each grouping statement on the board) "That's 5 groups of three plus 8 groups of three plus 8 groups of three plus 3 groups of three. This gives us 24 groups of three. Is this problem similar at all to the one we had yesterday with the bags of candy?" (The girls all nod yes.)

Melanie: "It's like working with groups of numbers."

Laura: "It's working with groups of threes. But like yesterday you had to do it with twos and fours. It's just the grouping problem is similar."

Researcher: "On the back of your worksheet, respond to this question: Is there any advantage to using grouping? If so, tell me what it is."

All students indicated that the process of grouping made the problem easier or shorter. During the discussion of their responses, Melanie related that "grouping is easier, but sometimes is not easier." When asked to explain this remark, Melanie stated, "Like when you have larger numbers. Like on number 2, grouping would have been pretty hard if you had used things."

Again Melanie's response indicates uncertainty about the concept of unitizing.
Could she think grouping is only a physical reorganization? Further questioning by the researcher ensued.

**Researcher:** (Holding up a chip) "Do you have to always be able to use something like this to be able to group?"

**Melanie:** "Well, if you don't, then grouping is pretty easy because you don't have to go to real large numbers. And you can stay simple with the little numbers."

It was often difficult for the researcher to interpret Melanie's comments. What was she saying this time? The questioning continues.

**Researcher:** "What about that problem number two? What did you do with the 672?"

**Melanie:** "I halfed [sic] it. I divided it by 3 to see how many groups there were.

**Researcher:** "So you did group it?"

**Melanie:** "I grouped on all of them."

**Researcher:** "You didn't have 672 chips, but you grouped it in your head. Did you think, perhaps, something about groups?"

**Melanie:** "Yes ma'am. On the smaller numbers you can work it either way. It is easy."

It seemed that this line of questioning had served to clarify Melanie's thinking about whether grouping is a physical or mental process. She now seemed to view grouping as a process possible under either condition.
With the time remaining in this session, the researcher sought to examine students' thinking about division since subsequent lessons would examine students' models for division. The following question was presented to the students, with the instruction to write the answer on the back of their worksheets: "What does the operation of division mean to you? What are you doing or what are you looking for when you are dividing?"

Laura explained that division "tells how many per person can be used, or group things like on problem 2." This response revealed that she was linking division with the idea of grouping. Laura was asked to give a story problem example for her description of division. After a brief hesitation, she replied, "15 divided by 3. Say three people and how many pieces of candy does each person get." Laura's description of division revealed the presence of the partitive model for division.

Researcher: "So if we had 15 pieces of candy to share among three people, then division would tell us how much each person would get."

Judy described division as "splitting something into parts. The answer tells you how many times something will go into something." For a story problem example, Judy explained division was "like if you had 20 people and you want to put four people in a group." When asked what the answer, 5, implied, she responded, "How many groups you have." Judy's description of division revealed a view of the quotitive model of division.
Melanie gave another confusing response in her description of division: "It tells you the simpler number making it easier to work." When asked to clarify her response, Melanie stated, "Like on number 2, you divided 3 into 672 and it reduced that number down to how many groups there were and it tells you how many groups." Melanie was associating the idea of grouping with division. The fact that she viewed the process of division as resulting in a simpler or smaller number could indicate the presence of whole number dominance in her understanding of division. Her statement indicates that "division makes smaller."

Larry's worksheet description for division consisted of the statement, "How many times will a number go into another number." He did not want to elaborate further on his response.

**Lesson Four**

The first three lessons allowed the researcher to examine students' abilities to form composite units and to use unitizing and norming in whole number problem situations. Before the extension of these concepts to the rational number domain, the researcher saw the need to examine and extend students' existing models of multiplication and division in whole number situations.

Lesson Four allowed the researcher to examine students' existing models of division. Two problems were designed to see if students could model both
the partitive and quotitive models for division. Students worked both problems individually, then explained their procedures to the group. The problems were:

1. Use your chips to show how to share 15 cookies with three children. (Give fair shares.) Draw a picture to show how you solved the problem with chips.

2. You have a bag with 15 cookies. You want to form snack bags having three cookies per bag. How many snack bags can be formed? Use chips to model the solution, then make a drawing to show how you solved it.

The drawings for both tasks were influenced by the students' prior knowledge of the answer. All students drew a model to represent their known solution rather than a model which would help them find the solution. This is best illustrated by Laura's explanation for her solution for the first problem: "I got 15 [chips] out of the stack and I put five into each group. So each child gets five." When asked what would she do if the problem required that 297 cookies be shared with three children, she replied, "I wouldn't know the problem; I would have to divide it."

Researcher: "I want you to figure out how you would solve it without doing the division."

Melanie laughed at this request; Laura frowned as if she was puzzled. The researcher drew 15 circles in a row to represent the cookies and three faces below to represent children. A question was posed for the class: "If you
didn't know what 15 divided by 3 is, how would you figure out how many cookies each child would get if you are going to share equally?"

Laura looked at the drawing on the board and immediately replied, "I would give them by one—like one and one and one" (points to each face). The researcher asked her to go to the board and draw her distribution process. "I'd give one to that person, one to that person, one to that person, and just go all over." Laura had drawn lines from cookie 1 to person 1, from cookie 2 to person 2, and from cookie 3 to person 3. The researcher asked her to complete the task. Once all cookies had been distributed, Laura was asked to determine from this drawing how many cookies each person got. Laura replied, "Not from this. Like if I used the chips I could." Melanie prompted her to "look at the lines around the head." Laura then counted the lines around one head and responded, "five."

Researcher: "Have you done this before in any situation of sharing?"

Laura: "Cards! Like dealing cards."

Researcher: "Look back at the problem, 15 divided by 3. What did the 3 stand for?"

Laura: "Three children."

Researcher: "What were you looking for?"

Laura: "How many cookies each of them would get."

Researcher: "So you knew how many groups you were going to have in the end. You would have three groups since you were sharing with three
children. The question is, how many things are in that group? That's sharing division; you want to know how much is in each part."

The discussion turned to problem 2. The researcher commented, "We said in problem one that the answer, 5, tells us how much is in each group. Is that what the 5 tells us this time?" All students nodded no. Laura explained that the 5 represented "how many snack packs" could be formed. When asked for another word that could be used for snack packs, Judy responded, "Groups."

Researcher: "What does the 3 tell us?"
Melanie: "How many are in each bag."
Researcher: "What does that mean in terms of groups?"
Laura: "Each group will have three in it."

Researcher: "Both problems have the same division statement, 15 divided by 3, but are you looking for the same meaning in terms of the answer? That's what I want you to respond to now. Compare the first two problems and see how they are alike or different in terms of groups. Think about groups. Write your answer at the bottom of your worksheet." The students' written responses are as follows:

Judy: "They both have 15 cookies divided into three things or divided by three people. The first one asked how many cookies would three children get and the second one asked how many bags could you make putting three cookies in one bag."
Laura: "They are alike because they both have 15 cookies divided into three bags or with children. They are different because one is three cookies per bag and one is how many cookies (out of 15) did three children get. They are alike because they both have three groups of something or three in a group. They are different because one is asking how many groups can be formed with three in each group and one is asking how many cookies three children would get."

Melanie: "They both use 15 and involve the numbers 3 and 5. The first one it asks how many cookies each child would get. On Number 2 it asks how many bags."

Although the current discussion had highlighted the answer to this question, the researcher was assessing students' ability to verbalize the role of groups in division models. All students displayed a good understanding of the meaning of the two models for division. Laura's comparison went further to explain the role of groups in the two division types. Whether or not Judy and Melanie possessed this same understanding of groups would be assessed on the four upcoming tasks. Much more time was consumed by this discussion than anticipated by the researcher, but the quality of responses was an indication that students were putting much thought into their remarks.

Once these problems were discussed, students were given four tasks to work individually; each one required students to draw the appropriate division model. Students were encouraged to use the chips to help model each
solution. There were also paper strips models available for Tasks 2 and 4. This type of manipulative had not been used in this teaching experiment and was introduced at this time to examine students' abilities to envision paper strips as a model for measurement division.

Task 1: Five cakes are to be divided between four people. Draw a picture of the five cakes and shade in the amount one person would get. How much cake does each person get?

Judy explained that she had made fourths because "there was four people." When asked if she divided all five cakes into fourths, she replied, "I gave each person one cake and divided the last one into fourths. One and one-fourth." Her concluding phrase revealed her answer to the problem.

Melanie said she had first separated the cakes into three fourths each because she "thought there were four cakes and five people." She admitted that she had looked on Judy's paper and discovered she was wrong. Although Melanie copied Judy's solution, there was some evidence that at least she understood it. Her drawing revealed Laura's partitive strategy of drawing lines from cakes to people to determine how much each person will receive.

Laura also got one and one fourth, but revealed her previous strategy had been to "give each [person] one whole cake and divide the last one into three." The researcher noted that Laura was able to partition this discrete set of five cakes into fourths; however, on her initial interview, she was unable to
partition the continuous case of fourths (the candy bar) into thirds without leading questions from the researcher.

Task 2: Three candy bars are to be shared equally between four children. How much candy will each person get? Draw a picture to show how much each child would get.

Melanie quickly responded that each person would get three fourths. In questioning her as to how she had derived this answer, Melanie admitted that she again had looked at Judy's paper. The researcher attempted to have Melanie think aloud to solve the problem by asking a series of questions.

Researcher: "Look back at the problem. What is the problem statement?"

Melanie: "Three is divided by. Okay, you count up how many pieces of candy bar there would be [counts from her drawing]--12."

Researcher: "There is no 12 in the problem. You are getting ahead; you are looking at your solution. Look at the problem."

Melanie: "Three candy bars is divided by four people."

Researcher: "What is our answer for 3 divided by 4?"

Melanie: "One and one fourth."

This response revealed Melanie's weak understanding of division. She apparently was applying a strategy, displayed on the initial interview, of dividing the smaller number into the larger. Such a strategy is indicative of the
presence of whole number dominance. Again the researcher posed the question, "What is 3 divided by 4?"

Melanie: "One and three. One and . . . ."

Judy: "Three fourths."

Laura: "Each person gets one fourth."

Melanie: "No! They do get three fourths and there's one fourth [of each cake] left over."

The researcher then repeated the question to the group, "So what is 3 divided by 4?" Everyone used decimal division to solve. Judy and Laura correctly solved the problem, getting 0.75, but Melanie exchanged the numbers, making the divisor the smaller number. The researcher asked if there was another way to write their 0.75 answer. Melanie said "75%.

Researcher: "Can you think of another way?"

Melanie: "Three fourths."

The researcher wrote "3 ÷ 4" on the board and asked, "This means what?"

Laura: "Three fourths."

Researcher: "What was the problem statement in Task 1?"

Laura: "Five divided by."

Melanie: "Five fourths. Four under five."

Melanie's uncertainty in naming the fraction revealed a weakness in fraction terminology.
Task 3. Twenty-seven Easter eggs are to be packed into 3-egg packs. How many 3-packs can be formed? Draw a picture to show your 3-egg packs.

The discussion began with a focus on groups and how this problem differed from the previous two. The researcher asked the students what they were looking for in this problem. Judy responded, "How many groups."

Laura was the only student having difficulty with the task. She recognized this as the problem 27 divided by 3; however, after solving it, she began separating the "eggs" into groups of nine to form three groups. She reread the problem, thought about what she had done, then changed her strategy to groups of three.

We had not previously drawn a model for measurement division, but all students were able to draw an appropriate model. In each instance, the "eggs" were grouped by threes on the worksheet, and the answer was obtained by counting the number of groups.

Task 4: It takes 3 yards of material to make one suit. There are 15 yards of material on a bolt of suit fabric. How many suits can be made from this bolt of material?

Judy explained that she "just drew a circle for [each] yard and underlined three of them for three yards." The researcher described Judy's process as a measurement process, then asked for the size of her groups. Laura quickly responded, "three." Laura never worked this problem because of the time she had spent on the previous task; however, her contributions during
the discussion of the solution revealed an understanding of the problem situation. Judy explained further that she wanted groups of size three because each group of three stood for one suit.

Researcher: "So how many suits can be made from this bolt of fabric?"
Melanie: "Five."

Envelopes had been provided for each student in which there were two paper strips. One represented the 15 yard bolt of material and the other strip, labeled 3 yards, represented one suit. Melanie said she had used them after she "had already figured it out." Laura also tried to use the strips, but explained she did not understand them.

**Lesson Five**

As an introduction to the next series of lessons on unitizing, the researcher presented a seemingly nonmathematical situation to the students. Students were given a picture of a large teddy bear and a packet of "rulers." Each student received a different type of ruler: large or small paper clips, buttons, or counters. The task consisted of measuring the height of the bear. Most students gave only a numerical answer. Larry was the only student to specify also the unit or ruler used for measuring. The researcher asked how they all had measured the same bear yet had different heights or measurements. The ensuing discussion emphasized the measurement results of a change in unit and the importance of identifying the unit of measurement. Melanie's response summed up the ideas voiced by the group: "Everything is
going to be a different size if you have different types of measurement." Her remark was somewhat unclear; nonetheless, it highlighted the importance of the unit in measurement situations.

Students used cuisenaire rods to solve five measurement tasks. The goal of this activity was to establish the concept that a measurement depends on the measurement unit (i.e., the unit used as a "ruler"). After solving each task individually, the students created drawings to show how the measures were obtained. Group discussion followed each measurement task. As the tasks progressed, the measure of each rod took on a different value as the ruler or measuring unit was changed. Appendix D illustrates the relationship among the different rods.

Task 1: Use the dark green-unit as a ruler to measure the light green-unit.

1 (light green-unit) = ____ (dark green-unit).

None of the students seemed to have difficulty in solving this task. Melanie's explanation was typical for the strategy used by all students: "Because if you put both [light green] rods on the dark green, it equals it. So if you just put one [light green] rod, it equals half of it" (the dark green rod).

Task 2: Use the light green-unit as a ruler to measure the dark-green unit.

1 (dark green-unit) = ____ (light green-unit).
The same rods were used for this task, except the ruler had changed. Most students' drawings reflected this change by inverting the positions of the object measured and the ruler from the positions used on the previous task. Judy was the only student using the same drawing for both tasks. The researcher acknowledged that the same drawing could be used to model both problems, and then asked why the answer was different for this task. Laura responded, "This time the light green is the ruler."

When asked how long the dark green rod was, most students merely responded, "two." Our beginning discussion about the height of the bear had emphasized the importance of specifying the unit of measure whenever stating a measurement result, yet students failed to verbalize two what.

After discussion of this task, students were able to verbalize the meaning of this measurement process. The researcher inquired as to the operation used to solve these tasks. Melanie stated she had divided because "it's sort of like measurement." The researcher asked if they could now figure out how to use the envelope of paper strips given out yesterday for task 4. Melanie picked up her strips and demonstrated a "sliding method" while explaining, "Put one here, then mark it, then slide it over and mark it."

Researcher: "You are using that as a ruler; you are measuring it. What operation is that?"

Melanie: "Divide."

Task 3: Use the light green-unit as a ruler to measure the white-unit.
1 (white-unit) = ____ (light green-unit).

For this task, the ruler was longer than the object to be measured. It was at this point in the lesson where uncertainty with fractions began to surface. All students had good drawings, but most had difficulty stating the measure. Laura admitted having difficulty with this task. "I wasn't sure, so I just said one fourth." Her answer was the result of a guess rather than a measurement. Melanie and Larry had accurate drawings; they each drew one light green rod with three white rods beneath it. Since three white-units equalled one light green-unit, they presumed the measure of one white to be three instead of one third. This signified their loss of sight of the unit—confusing the unit to be measured with the measuring unit.

Task 4: Use the light green-unit as a ruler to measure the red-unit.

1 (red-unit) = ____ (light green-unit).

Laura acknowledged that her solution strategy had again consisted of guessing the answer to be one fourth. When questioned about the meaning of one fourth as the answer, Laura answered, "There are four little pieces and it's one of the little four." She did not seem to realize that verification of her guess could be achieved by placing four white-units beside one light green-unit.

Larry could not determine a measurement strategy. He seemed confused because the red-unit did not fit beside the green-unit an even number of times. Larry quit working at this point, perhaps discouraged by the difficulty of this task. He spent the remainder of the class period playing with the rods.
and kicking the girls, of which the researcher was unaware until viewing the video.

Melanie introduced the use of a third unit to help her establish the relationship between the red- and light green-units. "I drew the red one, then I drew the white one on top of the red one to see how many the green equalled up to and it was three." In justifying her usage of the white-unit, Melanie revealed that she needed "to see how many little parts of the green the red was." In analyzing this relationship, she conjectured, "Two [whites] equal [one red]. There are three [whites] up there so that tells you the whole number that equals to the green. And two white ones equal one red, so it covers up two spaces of the three. So, two thirds." Even though the measuring unit was longer than the object of measure, Melanie did not lose sight of the unit here as in the previous task.

Task 5: Use the red-unit as a ruler to measure the light green-unit.

1 (light green-unit) = _____ (red-unit).

This task proved to be the hardest for many students. The major difficulty concerned the selection of the appropriate number for the denominator. After the discussion from the previous task, Laura was able to establish a measurement relationship between the two rods without the necessity of guesswork. She explained, "I had the red one to measure the light green. It was bigger than the red, so I just took another red one and put it on."
So [the green-unit is] one red one and half of a red one. [It's] one and one half."

Melanie countered that the answer was one and one third. She had resorted to the use of the white rod to establish a relationship between the red and green rods, but decided that one white rod represented one third. Again Melanie lost sight of the measuring unit. During her explanation, Melanie recognized that she was confusing the unit to be measured with the measuring unit. She explained: "I used the white ones because the green was longer than the red. So it would equal one--a whole. Because the green is longer. And then I used the white and equalled it up to the green. But I think I should have equalled it up to the red. And it should be one and one half." In the process of explaining her reasoning, Melanie had realized her loss of sight of the measuring unit. By refocusing on the red as ruler, she was enabled to correct her thinking.

This was the only task Judy labored with. Her worksheet revealed an answer of one and one fourth, but Judy would not elaborate on her thinking, other than to mention that she had worked it like Melanie. She seemed to be embarrassed about her faulty logic.

**Lesson Six**

This lesson was the first of a series of lessons designed to develop students' models for multiplication. Because of the change in unit structure occurring in some multiplicative situations, it is imperative that students
become more aware of the role of the unit in measurement situations. The only task for this lesson was designed to enable students to contrast the units for perimeter and area. The focus of the lesson involved the appropriate selection of units of measure for area and perimeter. It was hoped that, by comparing the units in these two measurement situations, students would become more sensitive to the role of the unit.

Students were given a sheet of graph paper and challenged to draw as many rectangles as possible having a perimeter of 24 units. Discussion began with a search for the meaning of perimeter. Judy and Laura indicated they did not know. Melanie stated, "It's the measure of something. We had that last year." Seconds later she blurted, "I know. It's like when you add up all the sides!" Melanie seemed excited that she remembered.

Students were observed as they searched for rectangles having a perimeter of 24 units. This proved to be a quite difficult task for all the students. The primary difficulty involved the determination of the measuring unit once students began to focus on the squares within the rectangle. From the start, students were perplexed as to whether they should count segments or boxes. This was a problem in two ways: (a) some students counted all the squares inside, giving them a number for area rather than perimeter, and (b) some students who did use the exterior squares only to calculate length and width did not want to count the corner square twice.
The researcher interrupted the students to draw them back into discussion. She asked, "What are we measuring when we say the length of the side of this rectangle is six units?"

Melanie: "Six blocks."

Laura: "The dots or whatever."

Melanie: "The lines."

The researcher drew a 6 by 6 rectangle on the board and continued, "This rectangle does have a perimeter of 24. It is 6 plus 6 plus 6 plus 6. What are we counting when we say 24? Are we counting these blocks in here?"

Judy: (nods no) "The lines."

Researcher: "What other rectangles have you found besides a 6 by 6?"

Judy: "A 10 and 2."

Researcher: "How many blocks are inside that rectangle?"

Judy: (after counting) "Twenty."

Melanie: "I don't understand. How can a 10 by 2 give 24?"

The researcher drew the 10 by 2 rectangle and asked students to count the length of each side. Then she replied, "So we do have a perimeter of 24 units. How many blocks are inside?" In each rectangle example, the researcher endeavored to highlight the discrepancy between distance around the rectangle and the number of boxes contained inside.

Melanie: "Twenty. Okay, I see."
Laura asserted that she had found a 5 by 7 rectangle, adding further that it contained 35 blocks. Melanie retorted, "How does a 7 by 5 give you 24?"

Laura demonstrated with her rectangle, "It's 7 plus 5 plus 7 plus 5."

Melanie: "That would be 35, wouldn't it?"

Researcher: "Would it? Let's count." (She counted length plus width and wrote on the board: 12 + 12).

Melanie: "So why is there 35 [boxes] in it?"

Judy and Laura seemed to have resolved their confusion over area and perimeter measurement by the end of the lesson. Melanie, however, was unable to distinguish between the two measurement situations.

Lesson Seven

This lesson continued the area concept of the previous lesson, striving to extend students' models of multiplication to include the area model. Reunitizing activities were designed to help students to focus on the change of unit occurring in many multiplicative situations. Students completed the following task, then engaged in discussion about the strategies involved. Throughout the discussions, the researcher continually stressed for students to specify the measuring units in each task, reasoning that this emphasis on units would encourage students to intentionally focus on the unit structure in the problem situations.

Task 1: Cups are packaged in a box so that there are 4 rows with 6 cups in each row.
A. Use the number of cups in each row and the number of cups in each column to write an arithmetic statement to show how many cups are in this box. What are your measuring units?

B. Suppose the manufacturer decided to package these cups into groups of 4, with each package the same shape as shown.

Find the number of packages in each box by looking at rows and columns. How did you determine your answer? What are your measuring units? How can you use rows and columns to determine how many cups are in the box? How does this differ from Question A? How can you explain this difference?

There was no difficulty in finding the number of cups in the box—everyone multiplied 6 and 4. The confusion arose in determining the measuring units. All students said 6 and 4 were the measuring units.

For part B, various counting strategies were used. Laura merely counted to obtain total packages. Larry and Melanie used the strategy of multiplying 2 and 3. When questioned as to why this method was selected, Melanie explained that, in multiplying 2 and 3, she "counted rows and columns." Judy multiplied rows and columns also, but her written description
suggested a deeper understanding of unit structure. By counting "the number of packages in each row and the number of rows of packages," Judy's units-of-units conceptualization can be described as the formation of $2[3\text{-unit}]$-units.

Researcher: "So 2 times 3 gives you 6 what?"

Melanie: "Six packages in each box."

Researcher: "What are our measuring units?"

Melanie: "The packages."

Questioning then focused students' attention to the change in units occurring in a row by column multiplicative situation. Students were asked to use rows and columns of packages to determine the number of cups in each box. Melanie exhorted, "We just did that!" The researcher requested that students use the drawing in Problem B rather than going back to the one in Problem A. Melanie claimed this would take longer "cause you would have to like count [the number of cups per package]; that would be 4. Then 4 and 4 and 4 and 4." Melanie's last remark symbolized the counting of the cups per package in all packages.

Laura remarked that she had "just counted down--one, two, three, four. Then counted over--one, two, three, four, five, six. Then 6 times 4 equals 24." Judy revealed her utilization of the identical strategy. The researcher explained that in doing so, the girls were disregarding packaging (or units) and counting by ones as in the first problem. The dialogue continues:
Researcher: "Remember 6 times 4 gives you cups, not packages. Let's look at size. What are the sizes of these rows? There are two rows, but each is really a double row. We have three columns, looking at our packages, but what are the sizes of these columns?" (She wrote the following unit structure statement on the board: $2\text{[2-unit]s} \times 3\text{[2-unit]s}$.)

Laura: "There are two [columns] in each column."

Melanie: "Oh. I see! It's 2 times 3 is 6 [packages] and 2 times 2 is 4 [cups per package]."

Perhaps the unit structure statement on the board had enabled Melanie to determine the procedure for calculation as:

$$(2\times3) \times (\text{[2-unit]s} \times \text{[2-unit]s}) = 6\text{[4-unit]s}$$

Researcher: "So we have six packages and the size of these packages is four cups."

Melanie: "[Problem] A wants to know how many cups are in the box and [Problem] B asks how many groups of four there are in each box.

Here Melanie demonstrated her understanding of the distinctions between the two grouping situations.

Task 2: Use your grid paper to model the problem $20 \times 30$.

This problem required students to reunitize the existing linear units into a variety of sizes, all leading to a change of unit structure when multiplying. It was hoped by the researcher that awareness of the results of such a change in
units would enable the students to contrast linear unit structure with its associated area unit structure.

An early problem arose in students' measurement processes—many wanted to count the starting point as one unit. For example, consider the line segment marked as follows: X____X____X. Many students would measure a segment such as this and state a length of 3 units rather than 2 units. The use of such a measurement technique on the current task, drawing a 20 by 30 rectangle, would result in the creation of a 19 x 29 rectangle.

Once students drew a 20 x 30 rectangle on grid paper as a model for the multiplication problem, the tasks involved reunitizing this 20 by 30 into units other than one. One goal of this activity was to enable students to see how such a regrouping could facilitate a solution. For example, the 20[1-unit]s could be regrouped into 2[10-unit]s or 4[5-unit]s. Another goal was for students to see the effects of reunitizing on unit structure. In multiplying 20[1-unit]s x 30[1-unit]s, the answer 600[1 square-unit]s, has a change of unit structure which is often not apparent to students. By reunitizing the problem into other units, this change is more evident. Considering the 20 x 30 rectangle example and reunitizing the sides by 10s, the problem becomes 2[10-unit]s x 3[10-unit]s = (2x3) x ([10-unit]s x [10-unit]s) = 6 x [100 square-unit]s.

Students were asked how this rectangle could be a model for multiplication problem 20 x 30. Laura explained that multiplication means "how many little squares are inside the 20 by 30 rectangle." The focus of
discussion centered on the regrouping tasks from the worksheet. Students easily regrouped each side into 10-units and were able to state the length of the sides as 2[10-units] and 3[10-units]. Melanie was the first to comment that this regrouping gave us six "groups." Discussion now focused on these six groups.

Researcher: "What does the six refer to?"
Melanie: "How many groups of 10 there are."

Researcher: "How many blocks do you see within one of those little groups on your paper?"
Melanie: "Ten."
Laura: "Twenty."
Judy: "One hundred."
Melanie: "Oh! There's a hundred."

Researcher: "How did you find the hundred?"
Melanie: "It's 10 times 10. Ten rows and 10 columns."
Laura: "Hundred. I was adding!"

The conversation then contrasted the two rectangles, focusing on the unit structure of each.

Researcher: "We started with a 20 by 30 rectangle. What were the units?"
Laura: "Each one."

Researcher: "We were measuring by ones. When you grouped by 10s, the side that was 20 one-units became what?"
Melanie: "200."

Laura: "Two."

Researcher: "You now have two rows. And the 30 columns? When you grouped them by 10s, how many columns do you have now?"

Judy: "Three."

Researcher: "So your 20 by 30 becomes a 2 by 3 rectangle, but are the units still ones?" (no response) "We have two units of size what?"

Melanie: "Hundred."

Researcher: "Look at your drawing, Melanie. When you grouped them by 10s, what's the size?"

Melanie: "Twenty."

Laura: "Ten."

Researcher: "That's your measuring unit. You have two groups that are 10-units and three columns that are size 10. That gives us six blocks. What is the size of those blocks?"

Melanie: "A hundred! Because there are 10 this way and 10 this way."

Earlier in the lesson, Melanie had told the group of a "short cut" for multiplying 20 by 30. The researcher attempted to connect the current discussion to Melanie's scheme by writing this problem statement on the board:

\[ 2[10\text{-units}] \times 3[10\text{-units}] = 6[10\times10\text{-units}] = 6[100\text{-units}] . \]

Researcher: "Melanie, would this have anything to do with that shortcut multiplication you told us about earlier? What was that method?"
Melanie: "Multiply 2 and 3 and add two zeros."

Researcher: "Look at your model. Do you see any way in which it can explain why that shortcut works?"

Judy: "Three rows and two columns."

Laura: "That is 2 times 3 is 6" (pointing to her six blocks) "and add two zeros."

Researcher: "Why do you add two zeros?" (no response) "You have six groups of what size?"

Laura: "Hundred."

Researcher: "How did you get the hundred?"

Laura: "By 10 times 10."

There was no evidence that this discussion broadened Melanie's understanding of her shortcut. Larry also seemed unaffected by this discussion; however, judging by facial expressions and the tones of voice, both Judy and Laura seemed to understand the concept behind the shortcut.

The students were then asked to regroup the rows and columns of their 20 by 30 rectangle into units of five. Discussion focused on the results of this regrouping on rows and columns. Students were never able to verbalize the difference between units used for the lengths of the sides and units used to represent the answer to the multiplication. With follow-up questioning, many students were able to state that the reunitizing of each side from units of one to units of five resulted in 24 blocks of 25 rather than 600 individual squares;
however, this concept was not firmly rooted in their minds at this point. Evidence for this is seen in the inconsistent answers given by students. A correct answer might be given at one point in the discussion, only to find the same student miss a similar question later in the discussion.

Lesson Eight

This lesson continued model building for multiplication. Students worked in pairs to solve three tasks. Group discussion after the solution of each task focused on the unitizing strategies used by the students. Continuing the previous emphasis on specifying units, the researcher insisted that students specify the unit in each measurement result. This emphasis should not be considered as the researcher’s insistence for preciseness in students’ terminology, but rather as a strategy employed by the researcher to encourage students to focus on the units in each measurement situation. For the first two tasks, students used cubes to help them model the situation and then they drew their solution strategy on the worksheets.

Task 1: It takes 2 yards of material to make one skirt. How many yards of material are needed to make 5 skirts?

Everyone used a similar strategy. Allowing each cube to represent one yard, cubes were grouped by twos. Each two cubes then represented one skirt-unit. Groupings by two were continued until there were five skirts represented in the model.
When asked how the model could help find the solution, most students said to multiply 5 by 2 because there were five groups of two, which could be represented as \(5 \times 2\). Larry responded that he viewed the solution in their model as \(2 + 2 + 2 + 2 + 2\).

Task 2: Donna has 3 pairs of pants and 7 shirts. If all the shirts will match all the pants, how many different outfits can she wear?

Laura and Larry selected seven different colors of cubes and connected them to form one stack. Then they found three different colors of cubes to represent the three pants. In describing their model, Laura held up a stack of "shirts" and said, "You can wear all seven shirts with that [pant] and that [pant] and that [pant]."

Judy and Melanie used a similar modeling strategy, except they formed three stacks of seven shirts and placed one stack with each of the three pants. In addition, they added some social constraints to the task. Melanie explained that they rearranged the order of the colors of the seven shirts in each of the three groups of seven "so she won't be wearing the same shirt the same day of each week. She has a different shirt for each day of the week."

Task 3: Work with your partner to determine how many small rectangles will fit inside the large rectangle. Your rectangles cannot be overlapped or cut, nor can they hang over the edge of the rectangle. Keep them all turned in the same direction to cover the rectangle.
This task proved to be extremely difficult for the students, primarily because of weak measurement skills. Sometimes the students would not butt the edge of the "ruler" (small rectangle) against the mark where it had last been placed. This would cause a little gap between placements of the ruler. In contrast, some students would overlap the edge of the ruler with the markings from the previous measure. Such errors as these caused measurement discrepancies.

Larry explained the measurement he and Laura had found. "It would take 10 [across] and down, 8. So 8 times 10--80." When asked why he had multiplied, Larry replied, "Because there should be 10 going down like that and 8 across. Times--you get 80."

Researcher: "So this multiplying is a way of... ?"

Laura: "I was going to say grouping."

Larry: (looking down the left edge of his rectangle) "If I take all [the rows] and count them straight down and it would be 10. And it would be 10 here and here and all the way" (moving his finger from column to column).

Lesson Nine

This lesson began a series of lessons designed to extend students' existing knowledge of fractions to include awareness of the unit in fractional situations. This lesson emphasized the necessity of equality in partitioning, verbalization of the meaning of the unit fraction, and a focus on the unit. Students used fraction circles to aid their thinking in two tasks. Appendix D
illustrates the relationship among the different pieces. Discussion occurred after the completion of each task.

Task 1: Using your fraction circles, make one orange the unit. What fraction name can you give these pieces?

1 yellow = _________  1 brown = _________
1 pink = ___________  1 red = ___________
1 clear = ____________  1 blue = _________

Students had no difficulty determining the fractional relationship between the orange-unit and the first five units listed on the task. None of them, however, specified the unit with their answer on the worksheet. Most of the discussion focused on the necessity of identifying the unit, or ruler, in a measurement situation. The last problem, however, proved to be more challenging for the students and brought about a richer discussion. Students had to determine the size of one blue-unit, using the orange as the unit whole.

Researcher: "Judy says the blue won't work. Why?"

Judy: "It won't come out evenly."

Melanie: "That means that there is some more left."

Laura: "About two and one half."

Judy: "Two thirds?"

Researcher: "Tell me what you are thinking, Laura."

Laura: "Two blues and one yellow equal the orange. The blue is one third."
Judy: "They all have to be blue for it to be one third."

Melanie: "They don't all have to be blue!"

Researcher: "What does it mean to say that something is one third?"

Melanie: That it's one out of three."

Researcher: "Any three? Suppose this [orange-unit] is Judy's pie. Judy says she will share it equally with you and cuts it into three pieces, in which two are blue-units and one is a yellow-unit. Does it matter which piece you get?"

Melanie: "Un huh, because some are bigger."

Researcher: "So can you say that you all get one third of that pie?"

Melanie: (nods no) "They all have to be even."

Researcher: "Go back to what you were thinking about the blues and yellow. Can you figure out a way we might use them to figure out what one blue is?"

Laura: "Five of the yellow equal the orange."

Melanie: "The yellow is half of the blue."

Laura: "Five of these yellows make up one orange and two of these yellows make up one blue. So the answer is two fifths."

Melanie: "I don't get it! What does two fifths have to do with anything?"

Laura: "Because two yellows make up the blue and five of them make up the whole thing."
Melanie: "But how can you get one third out of that?"

Researcher: "What is the question we are trying to answer?"

Melanie: "We're trying to get blue into the orange and two pieces will go in evenly, but you have a little bit left so we are trying to make it where we can find it evenly."

By "trying to get blue into the orange," Melanie had lost sight of the unit whole. Laura repeated her reasoning to explain why one blue-unit is two fifths of the orange-unit. Melanie replied, "So it would be two and one half though! Because two yellows make up one blue and there are five yellows on the [orange-unit] so two yellows go into one blue and that would make two and that would be a half left. So there is one yellow left and it equals half of the blue, so it would be two and one half."

Researcher: "Are you answering the question: What is the size of one blue compared to the orange? What is the question she is answering?"

Judy: "How many blues does it take to equal the orange piece."

Task 2: Now make green the unit whole. What fraction name can you give these pieces?

1 pink = _______  1 white = _______  1 red = _______

Laura determined that one pink-unit equaled three fourths of the green-unit. In explaining her reasoning, Laura stated, "I put pink on the green. Then I put a white one on [the green-unit] so I saw [1 pink-unit + 1 white-unit = 1 green-unit]. Then I saw that four whites equal a green, so the pink is
three fourths." Laura's modeling actions and reasoning strategies showed great improvement from similar measurement encounters in Lesson Five.

Judy stated that she had discovered that one white is one fourth of the green. Melanie explained that red is one half of the green-unit because "two reds equal one green." This was an improvement in her previous reasoning with blue- and orange-units in the previous task.

The researcher wrote on the board several of the relationships used during the lesson. These statements formed the basis for the next discussion, emphasizing the importance of knowing the unit whole before determining the name of a fractional unit.

Researcher: "When we were looking at the clear as our unit whole, we said orange was one half and yellow is one tenth of the whole. But here, when we changed our unit to the orange piece, we said yellow is one fifth. How can that same piece have two different?"

Laura: (anticipating the question) "Different measuring units."

Researcher: "Look at the statements about the pink unit. We first said it was one fourth of the whole, or it's one half of the orange-unit, or it's three fourths of the green. We came up with three different names for this pink unit. So what do you need to know in order to be able to name the fraction?"

Laura: "What's your measuring unit."

It was evident from the results of these activities that Laura and Judy were developing increased awareness of the unit structure in measurement
tasks. Melanie's measurement skills, however, were still limited. Much of her difficulties stemmed from her continued inability to stay focused on the measuring unit. The researcher's journal remarked that the biggest success realized by Melanie in this lesson was her acknowledgement of the necessity of equality of parts in a part-whole relationship. Whether or not this notion of equality was firmly rooted in her thinking would be re-examined in following lessons.

**Lesson Ten**

This lesson continued the investigation into students' representational understanding of rational numbers. Some confusion was evident in the previous lesson when students stated the value of a certain fractional piece as compared to a specified unit whole. Activities for this lesson sought to alleviate this confusion. All three tasks were solved individually, then group discussion centered around the strategies used to obtain the answers.

**Task 1:** Use your fraction circles to name each fraction piece.

(a) 1 green-unit = _________ red-unit.

(b) 1 green-unit = _________ orange-unit.

How can the same piece have different names?

(c) 1 orange-unit = _________ green-unit.

Parts a and b were devised to focus students' attention on the importance of identifying the unit whole in order to name a specific fractional value. The same fraction piece was compared to two different unit wholes,
resulting in a change in both the value and name of that given fraction piece. One major obstacle still occurring in this lesson was students' loss of sight of the unit whole. When requested to find the name of the green-unit if compared to the red-unit, many students inverted this relationship. The answer given often reflected students' use of the ruler as the object to be measured instead of its use as the measuring unit.

Laura began the group discussion of part a by stating her answer was "two" (red-units). Although she gave only a number, without reference to the measuring unit, Laura did specify the unit when asked to explain her reasoning in obtaining the answer of two. Laura replied, "it takes two red-units to cover the green."

Melanie disagreed with Laura's answer, stating that "the green [was] the ruler." When asked what she was measuring the size of, she commented, "the red to the green." This statement indicated to the researcher a possible uncertainty in Melanie's thinking, which prompted a series of questions by the researcher.

Researcher: "But what is it we are trying to measure?"

Melanie: "The red. No, we're trying to measure the green."

Researcher: "I hear some confusion there. What are we measuring?"

Melanie: "No, we're measuring the red. Trust me!"

Melanie's confusion seemed to be spreading, as evidenced by Judy's response, "the red." The researcher asked Judy what she had measured with.
Her response, "the green," revealed Judy had also lost sight of the unit whole—the measuring unit—even though she had solved the task correctly on her worksheet.

The researcher continued this line of thinking by asking if anyone had found the size of the red. Melanie quickly exclaimed, "You mean one green." Perhaps Melanie understood that the written problem statement indicated a request for the size of one green; however, she seemed unaware that her solution process resulted in a measurement of the red-unit as compared to the green-unit. Students were asked to look back at the problem statement to determine which unit they were to be measuring. Melanie’s continued confusion was evident with her reply: "the red piece—the green piece. The green because you are using the two reds to see how many reds are in the green." Laura and Judy agreed with Melanie’s last response.

All three girls obtained a different answer for part b. Judy stated that "Green is two-thirds of the orange-unit." Melanie responded that her answer was one and one third, but added, "I might have done it backwards." Such awareness could have been illuminated by Melanie’s realization that her answer differed from Judy’s. Throughout the study, Melanie tended to look toward Judy for guidance, perhaps thinking that Judy was usually correct. Laura revealed her answer was one and one half. Students were asked to explain their thinking.
Melanie: "It takes two reds to make up one green. And I filled in the extra space with another red. So that would be—the orange is one whole of the green and there's one extra piece."

Melanie's explanation revealed an understanding that the green- and red-units together equaled the orange-unit, but she could not use this understanding to determine a relationship between the green- and orange-units. Particularly, she seemed unsure of the unit whole in this situation. This uncertainty was also revealed in Laura's response to the researcher's question: "What are we measuring?"

Laura: "the orange—with the green."

Laura demonstrated her reasoning in obtaining her answer of one and one half by placing a green-unit on top of the orange-unit. Next she placed one red-unit beside the green-unit, an action which allowed complete coverage of the orange-unit. Laura explained that since "one red one equals half of the green one, it's one and one half." Laura's reasoning had resulted in a correct answer to the wrong question: Find the value of the orange-unit when compared to the green-unit.

Judy explained her reasoning for the task. She placed one green-unit on top of the orange unit. Next she reasoned: "Two of the reds is one green. And I put [the red-unit] on to complete the orange. So green is two thirds of the orange." Judy's model was identical to Laura's, but their reasoning strategies were opposites because each student had focused on a different measuring unit.
Still there was no agreement by the students that any one of their explanations of strategy was correct. Melanie questioned, "Whose is right?" The researcher decided to ask students to model the meaning of Laura's answer—that one green-unit equalled one and one half orange-units.

Researcher: "What does one and one half mean?"

Laura: "That it's one whole and a half more."

Researcher: "Everybody pull out one orange-unit. (pause) Now show me half of the orange-unit. Put them together, trying to make a circle out of them. (pause) Now is that one and one half of the orange-unit?"

Laura's model consisted of the orange-unit and two pink-units. The result was a whole circle, or two unit wholes. Melanie was the first to point out that Laura's answer was incorrect. Pointing to Laura's model, she stated, "That's two!"

Judy's model contained one orange- and one pink-unit. The researcher held up one orange piece and stated, "this is one orange-unit." She held up the pink piece and commented, "since two of these equal one orange, this is half of an orange-unit. Put them together like this—this is one and one-half orange-unit." The researcher made a reference to Laura's answer by holding up one green-unit and asking, "Is this one green piece equal to one and one half orange-unit?" All students nodded no.

Melanie responded, "See, what I think we were doing is measuring the orange to the green. But I rearranged mine now and I have two thirds like
Judy. The researcher wondered if Melanie was really changing her thinking or merely trying to model Judy's answer. Even in Melanie's response that she now had two thirds, she made no indication as to two thirds what! Did she clearly see the measuring unit in the problem?

The researcher directed attention to Judy's model. "She says one green is two thirds of the orange." After writing Judy's statement on the board, the researcher held up one orange-unit and asked, "What would be two thirds of this orange-unit?"

Melanie: (still demonstrating uncertainty) "One green--three reds. Three reds make up one orange."

Researcher: "So what would be two thirds of this orange?"

Melanie: "Two reds."

Judy & Laura: "One green."

Researcher: "Melanie, tell me what you are thinking. You said you changed your answer. What is the object you are measuring and what is your measuring unit?"

Melanie: "You are using the orange to measure the green. No, the green to measure the orange."

Melanie still was confused with the measurement process. The researcher posed another question to the group: "If I say to measure the green-unit in terms of the orange-unit, what is our measuring unit or ruler?"

Melanie: "Green."
Laura: "Orange is our measuring unit. You are measuring the green."

Melanie: "No. The green is the measuring unit and you are measuring the orange."

Researcher: "Are you trying to find out the size of one orange-unit or one green-unit?"

Laura: "The size of one green."

Melanie: "You are trying to find out how many oranges one green is."

The researcher focused the discussion on the answers from the two tasks: "From the previous problems, we found that one green-unit is two red-units. Then we said that one green-unit is two thirds orange-unit. How can that same green piece have different names?"

Laura: "Different measuring units."

Melanie: "It depends on the way you are measuring it—what you are using it to measure."

Laura: "It depends on the measuring unit."

Task 2: Jenny says this tan piece is one third. John says it is one fourth. Who is correct? Use drawings or models to help you explain your answer.

All students obtained different answers. The researcher acknowledged this fact and asked students to explain the reasoning involved in their solution to the problem. Laura remarked that she was unable to find an answer because "it depends on the measuring unit." She qualified her response by
adding, "if it's pink it's one third, and if it's green it's one fourth." Such awareness by Laura was interpreted by the researcher as an indication that Laura realized the importance of identifying the unit whole in a measurement situation before naming the value of the fractional unit.

While observing the students engaged in solving the task, the researcher noticed that Judy and Melanie used a similar strategy; however, it resulted in different answers. They each traced the tan fraction piece onto their worksheet, but each iterated this piece a different number of times. Judy drew three tan-units then found a fraction unit equal in size to her drawing of three tan-units—the pink unit. In stating her answer, Judy replied, "I said it was the pink and it's one third." Melanie traced her tan-unit four times, then said her drawing revealed the answer was one fourth because she "measured the tan [unit] to it." She admitted that now she thought Laura's answer was correct because "it depends on the measuring unit." Follow-up questioning by the researcher attempted to determine if students were aware of the unit whole in this problem situation. Melanie still demonstrated uncertainty about the measuring unit.

Researcher: "So what were you measuring in that problem?"

Melanie: "The tan."

Researcher: "What was your measuring unit?"

Melanie: "The green and pink."

Task 3: Use your fraction circles to model the following fractions:
The process of iterating a unit fraction results in the formation of composite units. The purpose of this task was to reinforce the notion that non-unit fractions are iterations of unit fractions. Each student was assigned a different fraction to model with their fraction circles. Discussion about the representation of each fraction began once all models had been completed. Rather than using an iteration process, as they did on Task 2, all students created a unit whole, then removed pieces to leave only the designated part-whole relationship.

Judy modeled the fraction three fifths. Her model consisted of three of the blue-units. In explaining her thinking, Judy revealed, "I got five of [the blue-units] and pulled out three." When asked why she started with five, Judy replied, "because there are five pieces in all."

In her explanation of how she modeled four tenths, Laura responded, "I brought 10 of these [yellow-units] together and made a whole, then I pulled four of them back to make four. So four tenths." Her decision to use the yellow-units instead of another color was based on her awareness that "there were 10 of these that made the whole circle." Laura explained that the denominator told her "how many pieces there are in all."

Melanie explained how she modeled the fraction five sixths. "I started with red because it takes six of them to make up a circle. Then I substituted three of the reds for an orange because three reds make up one orange so it
makes it simpler. Then I took out one of the reds." As a result, Melanie's model consisted of one orange-unit and two red-units.

Modeling the last fraction for the group, Melanie pulled out all of the tan pieces. When asked why she had selected this color, Melanie replied, "because that's the color you need and there are 12 of these and your big number is 12." The researcher sought clarification for her reasoning for using "the big number."

Melanie: "The bottom number is 12."

Researcher: "Now I'm confused. Once you said the big number is 12, then you said the bottom number is 12. Which do we need to look at?"

Melanie: "The denominator. You have to find the group that has 12 in it and that is the tan. Put them all together and make one big circle. Then take five of them out and that is seven-twelfths."

Researcher: "What is the measuring unit?"

Melanie: "Seven."

Perhaps Melanie was demonstrating a lack of understanding of unit terminology. By stating that the measuring unit was seven, Melanie displayed an awareness that "something" was being counted out or measured seven times, but she was unaware of what this unit of measure was.

Lesson Eleven

The three tasks for this lesson involved students in a variety of unitizing and norming situations designed to provide opportunities for students to
conceptualize a unit fraction as a composite fraction. In Task 1, students used fraction strips to model non-unit fractions. This task was a continuation of similar activities from the previous lesson in which students used their part-whole interpretation to form composite units.

Task 1: Use your fraction strips to model the following fractions:

(a) 3/5  
(b) 7/12  
(c) 4/10

Each student was given unit fraction strips (1/2, 1/3, 1/4, 1/5, 1/6, 1/8, 1/10, 1/12) and instructed to use these strips to draw a model for each fraction. This exercise highlighted the fact that a non-unit fraction can be determined by focusing on the unit fraction as the measuring unit and then iterating this unit fraction the number of times as specified by the numerator.

Judy immediately proceeded to work. Laura looked puzzled, but eventually started working. Melanie watched Judy, then used the same technique to draw her model. Once all models were finished, the students explained how they had determined their models.

Judy: "I got one fifth and traced it five times, then shaded in three out of five."

Laura: "I was going to take this [one-third strip] and see how many times five went into it. Then I did like Judy and traced one fifth five times and put a line through between three of them and two of them." (She drew a vertical line to separate three fifths and two fifths on her fraction strip.)

Researcher: "So what's the piece everyone decided to use?"
Melanie: "The whole and the one fifth."
Researcher: "You took the one-fifth and?"
Laura: "We put them across until we made up five of them."
Researcher: So what was your measuring unit?"
Laura: "We measured with the big one" (the unit whole).
Melanie: "The big one and the one fifth."
Researcher: "So if you think about it, you have two units in that problem. Melanie said both the whole thing, the unit whole strip, and the one fifth strip. So how did you get three fifths out of the one fifth?"
Melanie: "You count three one fifths."

Not only did Melanie realize the necessity of using the one fifth unit to find three fifths; she also voiced the importance of recognition of the unit whole in establishing the part-whole relationship necessary for determining the one fifth unit.

Task 2: Use your fraction strips to model each fraction.

(a) \( \frac{3}{2} \)  (b) \( 1 \frac{1}{4} \)  (c) \( \frac{6}{5} \)

The problems in Task 2 required students to model mixed fractions, or non-unit fractions having a numerator greater than the denominator. The mixed fractions were easy for all students to model, but the modeling of improper fractions proved to be confusing for the students. Those students who were successful in modeling the improper fractions did so by making the
conversion to mixed fractions in order to model them. Students were reluctant to consider an improper fraction, such as 3/2, as the iteration of the 1/2-unit.

All three girls were stunned by the first problem. Judy stated she did not know what to do with "three over two." She looked on Melanie's paper (this was a change!) and borrowed the idea to use the 1/2-strip, but drew two halves instead of three halves. The discussion of Task 1 problems had highlighted the importance of the numerator as an indicator of the number of iterations of the measuring unit; yet Judy did not seem to consider that strategy as being relevant to the problems in this task.

Laura converted the improper fraction to the mixed fraction, one and one half. Her model consisted of one whole unit and one half unit beneath it. Laura shaded the one half unit, but not the whole unit.

Melanie's strategy and model were both correct, but she still exhibited weak unit terminology. She iterated the 1/2-unit three times to create her model, but Melanie explained that she looked at the problem "as three twos, then drew the twos and a half of a two." Melanie considered her unit of measure to be a two-unit. Judy intervened to add that the measuring unit was a one half-unit. Even though she had converted three halves to a mixed fraction and used the unit whole to model three halves, Judy now realized that she could have iterated the 1/2-unit three times to form the composite fraction of three halves.
Students did not have difficulty with the second problem, 1 1/4.

Everyone drew one unit whole and one fourth of the unit whole. For the last problem, another improper fraction, there was renewed discussion on the units. Laura explained that she had changed six fifths to one and one fifth, but added that now she realized she could have left it as six fifths. The group discussion of Melanie's solution strategy on the first problem served to expand Laura's conceptualization of units to include the model of an improper fraction as the iteration of a unit fraction. The researcher asked a series of questions focusing on the measuring unit in each fraction form of six fifths.

Researcher: If you change it to one and one fifth, what is your unit?"

Melanie: "A whole."

Researcher: "If you look at it as six fifths, what is your unit?"

Melanie: "Draw six fives."

Judy: "Six one fifths."

Researcher: "So you have different ways of looking at fractions, depending on how you look at the units."

From Melanie's two responses, the researcher concluded that she indeed conceptualized six fifths in two distinct unit situations even though she lacked ability to verbalize one fifth as a measuring unit.

The researcher wrote 5/3 on the board and asked students to draw a model for this fraction. Once everyone had completed the task, students were asked to identify the unit. Laura responded, "the three--one third." When
asked how she used this unit to draw five thirds, Laura replied, "I drew five of the one thirds." Clearly Laura was thinking of this fraction as iterations of the unit fraction, one third. Melanie acknowledged usage of the same strategy. Judy converted five thirds into a mixed fraction. She explained that her model was formed by drawing two unit whole strips, partitioning each one into thirds, then shading "five of these thirds." Although her drawing revealed a focus on the unit whole, her shading was based on her recognition of one third, the nested unit within the unit whole, as a measuring unit.

Task 3: Use your chips to help you model the given fractional unit in each problem. Then draw your solution strategy on the worksheet to help you explain your model.

(a) If 16 chips are your unit, find 3/4.

(b) If 28 chips are your unit, find 4/7.

For Task 3, students revealed their abilities in unitizing and norming. Problem situations required students to decompose the existing composite fraction in order to reveal the original fraction as iterations of a unit fraction. The manipulative employed was a discrete model—the chips or counters. Students worked each part individually, then explained their strategies in the group discussion. Students were required to model the fractional portion of the unit whole. They were not asked to state the number of chips in the requested part, yet they all did so on their worksheets.
Judy and Melanie went to work quickly, but Laura had trouble. She attempted to draw several times, but frequently stopped and erased her work. Laura finally drew a 4 x 4 array, then just looked at it. After about five minutes, she looked up and exclaimed, "I don't understand what we're doing."

The researcher asked Judy to explain her solution. Judy stated she put her chips into four groups. When asked why she had formed four groups, Judy replied, "I was looking at the denominator. That's my ruler." Judy continued her explanation, stating she "took three of them—three little group things out of four and then counted them." On her worksheet, Judy drew a 4 x 4 array, then circled three of the four columns.

Laura had been unable to work either problem before the discussion, but was asked to work the last one now since she seemed to understand. She partitioned the chips into seven groups. When asked to explain what Laura was doing in this partitioning process, Judy remarked, "Putting them into groups of seven—seven groups." After Laura obtained her seven groups, she separated four groups away and concluded, "16 is four sevenths of 28 chips."

Lesson Twelve

This lesson sought to build on students' existing skills in unitizing and norming to include the decomposition of unit fractions. Three paperfolding tasks were designed to enable students to view unit fractions as iterations of other unit fractions. The paperfolding exercises also allowed extension of students' models for multiplication from the area model of multiplication,
developed in the domain of whole numbers, to the "embedded units" (Behr et al., 1994) or "nested units" (Steffe, 1994) model for rational numbers.

Task 1: Fold the paper strip and use shading to model the unit fraction 1/3. Now partition this unit fraction (1/3) into two equal parts. One of these new parts is what part of the unit whole strip?

Students acknowledged that they had never used paperfolding to model fractions, so the researcher guided students through the paperfolding requirements on the first task. Folding a strip into three equal parts proved to be problematic for some students. Judy had no difficulty in determining a folding strategy to form thirds. Laura was unable to fold her paper strip into thirds and sought help from Judy. Once everyone had their strip folded properly, the researcher asked students to open the strips and then refold them in half. Students were asked, "What happened to one of your thirds?"

Melanie: "It halfed [sic] it between the other two. It got cut in half."

Laura: "It got folded in the middle."

Questioning focused on the size of one of these newer units as compared to the whole strip. Disregarding the fact that the four sections were not all the same size, Melanie counted the number of sections in her strip and replied, "One fourth." The researcher reviewed the process of paperfolding that had occurred, then focused attention to one of the newer units.
Researcher: "We split one of the thirds into two equal parts. Shade one of your two equal parts. Now look at that one and compare it to the one third."

Melanie: "It's half."

Researcher: "It is one half of the one third. But this new part is what part of the whole strip?"

Laura and Judy both responded that it was one sixth. Laura explained, "because each of the wholes, [each 1/3-unit], is two [1/2-units] and that would be six." Laura was able to conceptualize the embedded units occurring when each 1/3-unit was partitioned into two 1/2-units. After this explanation, Melanie added, "Half each one of them and you would get six."

Students were given a new paper strip and asked to fold it into thirds. With the strip remaining folded into thirds, students folded it in half. This folding technique enabled students to see the halving of each 1/3-unit, with the end result being the formation of six 1/6-units.

Task 2: Fold a strip and use shading to model the unit fraction 1/4. Now partition this unit fraction (1/4) into three equal parts. One of these new parts is what part of the unit whole strip?

Laura also had difficulty folding fourths. She started folding from one end and continued to fold toward the other edge. By the time she completed the foldings, there was a large portion hanging off the edge of her last folding. Laura commented, "It's extra." This precipitated a group discussion about the
necessity of equality in partitioning. Melanie guided Laura in the process of folding, telling Laura to "fold the paper in half, then fold it in half again."

Next students were instructed to fold this folded strip into three equal parts. Laura only performed one fold. Melanie also experienced difficulty in folding. The researcher asked students to go back to the folded fourths and shade in one of these 1/4-units with their pencils. The folded strips were refolded into thirds, with one of the 1/3-units shaded with a different colored pencil than before. The researcher asked, "What did you just shade in?"

Melanie: "One third."

Researcher: "One third of what?"

Melanie: "Of three."

Judy: "Of the whole."

Researcher: "What was the whole?"

Laura: "The fourth. One fourth."

This statement indicated to the researcher that Laura was able to conceptualize the 1/4-unit as the new unit whole.

Researcher: "So you shaded one third of one fourth. Now open it up. Find the last shading you did. What part is that of the whole unit?"

Melanie: "One eighth."

Judy: "One tenth."

Laura and Judy: "One twelfth."

Melanie: (recounts her squares) "Yeah. It's one twelfth."
The researcher asked students to consider how they could draw a model to represent the folding activities just completed. All students drew a fraction strip and separated it into four portions, then partitioned one of these 1/4-units into thirds. The researcher asked, "Can you use that drawing to figure out what one of those new parts is?"

Judy: "Three times four."

Researcher: "Why, Judy?"

Judy: "Because there's three [1/3-units] in one fourth and there's four fourths."

With this explanation, Judy was demonstrating her conceptualization of embedded units in the problem situation. But no one had verbalized a name for these smaller units, so the researcher asked, "What's one of these new parts called?"

Melanie: "One third."

Laura: "One twelfth because there's three in each little box."

Melanie: "It's one third of that little box, [the 1/4-unit], but it's one twelfth of the whole."

Melanie was also able to view the embedded units in this situation. Furthermore, she was able to verbalize the name of the smaller unit in two ways, each dependent on the unit of focus.
Task 3: Draw a unit whole strip and shade the unit fraction $\frac{1}{5}$. Now use partitioning to separate this unit fraction into two equal parts. One of these new parts is what part of the unit whole strip?

Students completed this task individually. They were instructed not to use actual paperfolding, but rather to model the result of a paperfolding process on the paper strip displayed on the worksheet. The question in this task required the determination of the size of one of the smaller units when compared to the unit whole strip. All students were successful on this task.

Melanie stated, "I got a half or one tenth." She was aware that this smaller unit could be named in two ways, depending on the unit whole; however, she was unaware that this question was asking for its name as compared only to the entire strip. Judy created an embedded units model. She partitioned her unit whole strip into fifths, then used vertical partitioning to partition one of these $\frac{1}{5}$-units into halves. Laura also partitioned her unit whole strip into fifths, but used horizontal partitioning to partition the entire strip into halves. This resulted in a $2 \times 5$ array.

Lesson Thirteen

The activities in this lesson were a continuation of activities from the previous lesson, designed to extend students' existing skills in unitizing and norming to include the decomposition of unit fractions. Problem situations in this lesson also provided an opportunity to connect models for multiplication of whole numbers with models for multiplication of rational numbers.
Task 1: Fold a strip and use shading to model the unit fraction $1/6$.

Now partition this unit fraction ($1/6$) into 3 equal parts. One of these new parts is what part of the unit whole strip?

All students were able to obtain the answer of $1/18$-unit. Perhaps the hardest problem encountered was that of folding the strip into sixths. Each student used a different sized strip to complete the folding task. Discussion highlighted that this variation in folding did not change the answer to the question. Even though the sizes of their rectangles differed among the different strips, each one was still one out of 18 or $1/18$-unit whole.

Seeking some indication that the students remembered the drawings of partitioned strips as models for the paperfolding tasks, which students had completed in the previous lesson, the researcher asked students if they could have answered the question in this task without folding the paper. Melanie's strip originally had been folded into a $2 \times 3$ array to obtain sixths. With the paper folded to reveal only a $1/6$-unit, Melanie had folded it vertically into thirds. She examined the rows and columns on her opened strip as she considered the researcher's question. Melanie looked at each of the two rows as three groups of three, as evidenced by her response, "You would say 3 times 3 and then times 2." Melanie's response perhaps was not one that she could have made without the benefit of the results of her paperfolding exercise, but it was an indication to the researcher of Melanie's awareness of the embedded unit structure she had created.
Laura exclaimed, "Oh, I know one. One third of one sixth. And then 3 times 6 is 18." With this response, Laura appeared very close to discovering that this paperfolding exercise served to model the operation of multiplication of rational numbers.

It was apparent to the researcher that both Melanie and Laura were conceptualizing the task in terms of units. The researcher drew a strip on the board and asked Judy to find one third of the 1/6-unit. She correctly partitioned the strip into sixths, then partitioned the shaded one sixth into thirds; however, she did not highlight any of the three new units. Once her model had been completed, Judy wrote the statement: $1/3 \times 1/6 = 1/18$.

Researcher: "How does your model help you figure that out?"

Judy: "There's three thirds in 1/6. I timesed [sic] it by 6 boxes."

Judy had connected her embedded units model to the operation of multiplication of rational numbers. By stating there were "three thirds in 1/6," Judy was revealing her awareness of the underlying unit structure existent in her model. Since Judy had not highlighted the 1/18-unit in her model, the researcher continued the questioning to see if indeed she did understand where the answer, 1/18-unit, was displayed on her model.

Researcher: "So show me with a different colored pen where your answer is."
Judy: (She partitioned each of the other 1/6-units into three parts, but still did not shade the answer.) "You would count all of them and it would be one eighteenth."

Researcher: "What would be one eighteenth?"

Judy: (She shaded one of the three original thirds contained within the shaded 1/6-unit.) "One of these."

Task 2: Create your own problem like the one above. Draw a unit whole strip to model a unit fraction of your choice. Then partition this unit fraction into ____ equal parts (you choose a number). One of these new parts is what part of the unit whole strip? Explain your problem and your solution to the class.

Judy immediately proceeded to model the problem, 1/3 of 1/12. Laura hesitated briefly, then constructed a model for 1/4 of 1/13. Melanie looked to see the type of partitioning being used by the others, then modeled 1/7 of 1/10.

Judy partitioned her strip into 12 sections, using vertical partitionings. She then used horizontal partitioning to form thirds, resulting in the formation of a 3 x 12 array. She explained the next step as follows: "I divided it in three equal parts and one of them is one thirty-sixth." The researcher questioned Judy to see the depth of her understanding about her model.

Researcher: "Where would one twelfth of the whole strip be?"
Judy: "All three of these together" (referring to the original shaded 1/12 unit, now partitioned into three portions).

Researcher: "When you take one twelfth and partition it into three pieces and then look at just one of those pieces, what part is this piece of the one twelfth-unit?"

Judy: "One third--one out of 12."

Researcher: (Pointing to the 1/36-unit) "So this is one third of what?"

Judy: "One twelfth."

Researcher: "What is it in terms of the whole thing?"

Judy: "One thirty-sixth."

Laura explained her problem by stating, "I did 13 across and four equal parts down and so it's one fifty-two. And you can either 13 times 4 or one fourth of one thirteenth equals one fifty-two." Further questioning by the researcher sought to illuminate Laura's conceptualization of the units within the problem situation.

Researcher: "When you broke this strip into 13 pieces, where is one of those pieces?

Laura: (Pointed to the original shaded 1/13-unit.)

Researcher: "What's the size of that piece?"

Laura: "That's one thirteenth."

Researcher: "Of what?"

Laura: "Of the whole."
Researcher: "When you took the 1/13-unit and broke it into four pieces, what were you doing there? What does that give you?"

Laura: "Well, it's one fifty-two of the whole and one fourth of the one thirteenth."

The students were displaying a good understanding of their models. The researcher decided to probe for a connection of this model with their previous area model for whole numbers. She drew a 6 x 5 rectangle on the board, shaded in one of the squares, and asked, "How many squares are inside?"

Laura: "Thirty."

Researcher: "How did you get it?"

Laura: "Multiplied 6 times 5."

Researcher: "Let's use this same model for fractions. Suppose that's a fraction strip and we broke it into six sections. What's the length of this [shaded] section in terms of the whole length?"

Laura: "One sixth of the whole."

Researcher: "If we took the width that is one unit long and broke it up into five parts, what would be the length of one part?"

Judy: "One fifth of the whole side."

Researcher: "How many squares are inside?"

Judy and Laura: "Thirty."

Researcher: "Did you do the same thing as you did for whole numbers to figure it out?"
Laura: "Yeah."

Researcher: "What part is this shaded portion of the whole thing?"

Melanie: "One fifth."

Laura: "One thirtieth."

Researcher: (in response to Melanie's answer) "It's one fifth of what?"

Laura: "One sixth."

Researcher: "And so that's one thirtieth--of what?"

All: "Of the whole."

Task 3: John has a rectangular garden. He is going to plant flowers on half his garden. He wants 1/3 of the flowers to be daisies. Draw John's garden and show the fraction of the garden that will be daisies.

Melanie took great delight in the fact that she was the only one to solve this problem correctly. Laura and Judy disregarded the "half" and drew flowers on one third of the whole garden. Melanie demonstrated her model on the board, then fielded questions from the researcher.

Researcher: "So the question is, what fraction of the garden will be daisies?"

Melanie: "One third. It will be this little part right here" (pointing to 1/3 of 1/2).

Researcher: "What is that? Give it a name."

Melanie: "One third of the whole."

Researcher: "This is one third of the whole?"
Melanie: "One third of the half."

Researcher: "But the question is, what fraction of the garden is daisies?"

Melanie: "One sixth."

Judy and Laura both asserted that they understood this problem, but had failed to read it carefully.

**Lesson Fourteen**

Unitizing and norming were again the focus of this lesson, but with a different emphasis. Problem situations in this lesson required students to reconstruct the unit whole if given only the unit fraction. This was a reverse strategy from preceding lessons in which students were given the unit whole and asked to find a fractional portion of the unit. Two tasks were designed for this lesson.

**Task 1:** This pink piece is half of something. How can we find the value of these other pieces which are part of the same unit?

1 tan = ________ 2 reds = ________

1 brown = ________ 3 greens = ________

Melanie became frustrated when the researcher would not specify the unit whole. This frustration became more apparent throughout the lesson. Melanie was unable to work this task without knowledge of the unit whole. As a result, she looked at Judy's models to see which fraction circle pieces she was using.
Judy and Laura identified the unit whole quickly, then proceeded to find the value of the other colored pieces. In explaining how she determined the unit whole, Laura remarked, "it's half the orange. I put two pinks on the orange." The remainder of this task was a repeat of problem situations students had worked in Lesson Nine, but the pace was much quicker in this lesson. Students themselves remarked at how much easier these problems were now as compared to their previous encounters. The students also displayed a greater understanding of unit structure, as evidenced by Melanie's explanation for the size of the brown: "It takes two browns to make up a pink and it takes two pinks to make up an orange. So you times it, two times two, and you get four--one fourth."

Task 2: This • • • is 3/5 of a unit. Model the unit fraction with your chips, then draw the model you used to find the unit whole.

The researcher tied the solution of this task to the preceding one, emphasizing that students needed first to determine the measuring unit, or unit fraction. This unit could then be iterated to reconstruct the unit whole. All three models were different, so the researcher asked each student to explain the reasoning used to complete the task.

Melanie drew a 3 x 5 array of dots and explained, "I was thinking three rows of five--for the three fifths. So I got 15 and I added in an extra row of dots at the bottom." The researcher drew Melanie's model on the board (see...
below) and asked, "Would this original part that is shaded here be three fifths of the whole thing?" Judy and Laura both replied no.

Laura explained that she "put 10 as [the whole] unit and sectioned off two in each group." Her drawing was a 2 x 5 array, with vertical partitions between the columns. When asked what one column represented, Laura remarked, "one fifth." To qualify her answer, Laura added, "It's one group of the five." Both Laura's model and her explanation revealed that she easily decomposed the 3/5-unit into 3[1/5-units], then iterated the 1/5-unit five times to obtain the unit whole.

Lesson Fifteen

Lesson Fifteen offered students an opportunity to extend their models of division of whole numbers to the rational number domain. Three tasks were designed. Task 1 consisted of a partitive division problem situation having a rational number answer. A correct solution strategy would require the students to model the fracturing of a whole number. The Task 2 situation required students to develop a procedure for measuring an object with a ruler that was longer than the object being measured. In this quotitive measurement situation, no numbers were given for the lengths of the two paper strips and no partitionings were on the strips. Task 3 required students to model quotitive or measurement division problem situations involving rational numbers.
Task 1: Use partitive (sharing) division to model the solution of $3 ÷ 4$.

Draw your solution strategy below.

The researcher read Task 1 and asked if anyone remembered sharing division. Melanie immediately responded, "Laura's line method." Throughout the study, Melanie had referred to partitive division with this name since Laura was the first student to devise a strategy for modeling sharing division. The researcher asked students to add context to the division statement.

Judy: "Three cookies."

Researcher: "And how are we sharing them?"

Laura: "Three cookies and four children."

Melanie: "That won't work! Well, you will have to break up the cookies. Each person would get a fraction of the cookies. Each person would get three fourths of a cookie."

Researcher: "Can you model that?"

Melanie went to the board, partitioned each cookie into fourths, then performed the sharing process of cookie-part to person until all cookie-parts had been distributed. Pointing to the lines at each smiley face, Melanie concluded, "Each person gets three parts, so each person will have three."

Researcher: "Three what?"

Melanie: "Three cookies. Three fourths of cookies."

Researcher: "So what is the answer to the problem three divided by four?"
Laura: "Three fourths."

Researcher: "Melanie, when we first began the problem, you said it
wouldn't work."

Melanie: "I know. That's when I was thinking you had to give them
separate cookies—it wouldn't work."

Researcher: "Okay. So we had to break or fracture the cookies to be
able to get an answer."

Task 2: Use the long strip as your ruler to measure the shorter strip.
Make a drawing to model your measurement process.

Each student spent a few minutes thinking about a strategy before
coming together to discuss their thinking. Judy seemed puzzled with this
measurement task. She looked at the paper strips and did not attempt any
measurement process. During the group’s discussion of ideas, she did not
contribute.

Laura was the first student to develop a plan. She partitioned the
longer strip in half, then partitioned each half into halves, resulting in the
formation of 1/4-units. Laura modeled her measurement process, stating, "I
put [the long strip] up to the smaller strip. It's three fourths of the long strip."

The confusion experienced by Judy and Melanie concerned the act of
measuring a smaller object by a larger one. The researcher used an example
to illustrate a common measurement action in which such a situation occurs:
"Think about it. If you have a 12-inch ruler, what do you look at if you want to
measure something smaller? Do you use the whole ruler?" Melanie shook her head no, picked up the package of fraction strips, and stated, "I would break it down. If I had started with [these] and I got to looking, I would have used the little things to help me" (referring to the partitionings on the fractions strips). Judy nodded her head in agreement.

Researcher: "So the small strip is three fourths of the long ruler. So three divided by four again gives us three fourths. So can we divide a small number by a large number?"

Melanie: "Un huh. Un huh. Un huh!"

Laura: (nods yes)

Researcher: "But what happens, Melanie?"

Melanie: "Add a decimal and divide."

Researcher: "Or?"

Laura: "It gives a fraction."

Melanie: (Referring to the placement of numbers in long division) "It means three goes under and four comes out."

The researcher asked students if they could look at the problem statement and see the answer. All students indicated yes. The researcher wrote the problem statement $5 \div 8$ on the board and asked, "What would this be?"

Laura: (responded quickly) "Five eighths."
Researcher: (Writes the problem 6 ÷ 5.) "Could we do the same thing if the first number is larger?"

Melanie: "You could, but you'd have to change it."

Laura: "One and one fifth."

Researcher: "What about 9 divided by 12?"

Laura: "Nine twelfths—three fourths."

Researcher: "This might give you another way of looking at fractions. A division statement can show you a fraction statement."

Task 3: Use your fraction strips to help you solve each problem. Then draw your solution strategy below. What does the answer mean in terms of units?

a. Measure the 3/5-unit with the 1/5-unit.

b. Measure the 1/4-unit with the 1/6-unit.

c. Measure the 2/3-unit with the 1/6-unit.

d. Measure 2-units with the 2/3-unit.

e. Measure 1-unit with the 3/10-unit.

f. Measure the 1/3-unit with the 2/3-unit.

The entire group worked together to solve the first problem, with guiding questions by the researcher. Students worked the remainder of the exercises individually, then explained their solution strategies to the group.

The researcher wrote the fraction 3/5 on the board and asked, "What is our unit in this number?"
Melanie: "Five."

Laura: "Three out of five."

Researcher: "What is our unit?"

Laura: "One fifth."

Researcher: "You have three of these 1/5-units and you want to measure with what?"

Laura: "One fifth."

Researcher: "What would your answer be?"

Laura: "Three."

Researcher: "What does this answer mean? Three what?"

Laura: "Of your measuring unit."

Researcher: "So we are looking to see how many of these 1/5-units are contained in three fifths. How do you find your answer looking at your fraction strips?"

Laura demonstrated the sliding method used on the previous task with whole numbers. She placed the 1/5-strip beneath the 3/5-strip and counted as she slid the 1/5-unit alongside the 3/5-strip, "one, two, three."

Students worked the remaining tasks individually, with the researcher monitoring their efforts. Occasionally students would lose sight of the unit—they would forget which strip was to be used as the ruler. In instances such as these, the researcher asked the student to specify the ruler in order to help them focus on it.
Melanie was confused because of the different units appearing in the second problem. She pulled out the appropriate fraction strips, knew that the 1/6-unit was the ruler, but was disturbed by the unequal partitionings in the two strips and the fact that the 1/6-unit did not "go into 1/4" a whole number of times. She was unable to determine a measurement process. The researcher asked Melanie to watch as Judy demonstrated the sliding method of measurement. Judy placed the 1/6-unit beneath the 1/4-strip and marked onto the 1/4-strip where the 1/6-unit ended. Judy slid the 1/6-unit down the 1/4-strip, aligning its new starting edge with the mark she had made for the end of the first measure. Noticing that the end of her ruler (the 1/6-unit) exceeded the length of the 1/4-strip and that the end of the 1/4-strip fell in the middle of her ruler, Judy replied, "It's half of it." The researcher asked for the answer to the problem.

Judy: "One and one half."

Laura: "There's one 1/6 and a half of a one."

Melanie had been fidgeting and drawing on her paper during Judy's modeling strategy. She replied, "I don't understand." The researcher reminded Melanie that she had not been watching during the discussion and asked her to concentrate. After this, Melanie perked up and corrected her drawings, which were the best in the group. None of the students had difficulty with problem c. For problem d, two different student models emerged.
Judy drew two whole unit strips, one beneath the other, and shaded all of both strips. She drew another whole unit strip beneath these two, partitioned it into thirds, and shaded two of the 1/3-units. This last strip represented Judy's ruler. She returned to the two whole unit strips and partitioned each into a 2/3-unit and 1/3-unit. Judy explained her reasoning by pointing to each 2/3-unit and stating, "this is two thirds and this here equals two thirds." Next she pointed to the two remaining 1/3-units and stated, "and then both of these equal another two thirds." She concluded the answer was "three two thirds."

Laura was unable to see how Judy determined her third 2/3-unit. The researcher pointed to Judy's partitioning of each whole unit into a 2/3-unit and a 1/3-unit, and asked, "If this is two thirds, what's this?" Laura identified the 1/3-unit in each unit whole. The researcher continued, "Put these two together. What have you got?" Melanie replied, "Two thirds." Laura nodded in agreement and stated, "Yeah. Two thirds."

The researcher questioned students about the possibility of another model for this problem. Judy again went to the board and drew two unit whole strips, but this time she placed them side by side with the ends butted together to form a long strip. The ruler, the 2/3-unit, was drawn beneath her double strip. Next Judy formed 2/3-unit partitionings throughout the length of the double strip, then counted, "one, two, three" to determine her answer. Laura smiled and added, "I like that one." This model seemed to make more sense to
her. The researcher questioned further to see if students were thinking about
the answer in terms of units.

  Researcher: "What does this 3 mean?"
  Melanie: "Three wholes of the unit."
  Judy: "Three rulers."
  Researcher: "What is our ruler?"
  Laura: "Two thirds."
  Researcher: "So what does the answer mean to you?"
  Judy: "Three two thirds."

Laura had difficulty on problem e also. Melanie eagerly explained her
strategy for solving the task: "I got the 10-strip. I put it on a whole [unit],
counted over three and marked it, counted over three more and marked it,
counted over three more and marked. There was one [1/10-unit] left over [on
the unit whole strip] so three and one third." When asked what she had three
and one third of, Melanie pointed to her fraction strip for three tenths and
stated, "this."

All students experienced difficulty on the last problem. Judy drew only
a 1/3-strip for a model, then wrote an answer of one and one third. Melanie
had an accurate drawing but was unable to determine an answer. Laura also
had an accurate drawing but was bothered that her ruler was longer than the
object to be measured.

  Laura: "It's—one?"
Reseacher: "There's one whole measuring unit in there?"

Laura: "No."

Melanie: "No. One third is half of two thirds."

Reseacher: "So how many of these 2/3-units are contained in one third?"

Melanie: "One half."

Judy nodded yes. Laura still seemed confused until the researcher returned her thinking to Task 2, in which Laura herself had determined the modeling strategy for measuring a 3-unit strip with a 4-unit strip. This connection seemed to clarify Laura's thinking on the current problem. She replied, "I got it now!"

Lesson Fifteen marked the conclusion of the teaching sequence. The final phase of the teaching experiment consisted of individual exit interviews with students. The objectives of these interviews were to assess students' autonomous performance on the concepts of unit developed during the teaching sequence and to examine the degree to which students' concept of unit could be used to inform their choice of operations and algorithmic performance on routine school word problems.

Research Questions

Thus far the chapter largely has been devoted to examining the progress of the individual lessons. This section recapitulates the data, but from the point of view of the individual participants in order to explore the research questions.
posed in the earlier chapters. Each question will be answered for each of the three participants. (Recall that Larry withdrew from the study.) In Chapter 5, we will synthesize and attempt to generalize these findings.

**Question 1**

How does the concept of unit develop in a multi-representational learning environment designed to use unitizing and norming to link understanding from whole number to rational number domains?

The sequence of activities in this teaching experiment, and the manner in which they were presented, constituted the learning environment for promoting the acquisition of unit concepts. The students were encouraged to discuss their questions and ideas concerning the tasks at hand. Thus the group interactions served as a tool for students' continued development of the unit concept.

Lessons were designed to focus on four areas of unit concepts: (a) unitizing in whole number situations, (b) model-building for multiplication and division of whole numbers, (c) unitizing in rational number situations, and (d) model-building for multiplication and division of rational numbers. All activities were designed to provide students with a variety of situations in which they could construct units.

This research question addresses the ultimate goal of the teaching sequence: to uncover students' development of the unit concept as well as to identify some of the cognitive obstacles they encountered in this development. By examining students' existing abilities and understandings of the concept of
unit at the start of the teaching experiment (Appendix A), assessing the progress attained by students during the four phases of lessons on specific constructs of unit concept development (Appendix B), and identifying cognitive obstacles that emerged during the teaching experiment, the researcher traces the developmental path of each student through the teaching sequence and then draws conclusions across subjects about the development of the unit concept.

**Judy's Profile**

Judy was the above average student selected by the classroom teacher for participation in the teaching experiment. She was extremely shy and quiet. The researcher found it difficult to encourage Judy to discuss her thoughts freely during early lessons of the teaching sequence; however, she opened up more as the lessons progressed.

**Entering concepts of unit and rational number.** Judy scored 90% on the calculations in her initial interview. Although her procedural skills were strong, her efforts on Task 1 revealed the presence of limited models for multiplication and division. In the choice of operations task, she missed one of three multiplication problems and all five of the division problems. Judy identified the correct operation on three of these division problems, but exchanged the two numbers in the problem statement.

Judy's initial understanding of a fraction consisted of the part-whole interpretation, but she did not recognize the importance of equality in the partitionings. Judy demonstrated unitizing skills by partitioning a discrete set of
12 jacks into 1/4-units, then combining three of the units to form the composite unit for three fourths. Repartitioning, however, proved to be a harder task for Judy. The presence of the existing partitioning of a candy bar into fourths served as a distractor in her efforts to repartition the bar into thirds. Only with guiding questions from the researcher was Judy able to accomplish this task.

Judy formed models for multiplication and division problems involving whole numbers, using the repeated addition and partitive models respectively, and was able to formulate story problems to accompany her models. Both her models and her stories exhibited the formation of units; however, the researcher was unable to determine if Judy was aware of her unit formations. Judy’s abilities did not transfer from whole numbers to rational numbers. She was successful in calculating the rational number problems, but was unable to think of story situations for the tasks. In addition, she could not create appropriate models for these operations. Judy expressed her own doubts about the models she created, although she had confidence in the answers obtained by her procedural solutions.

Unitizing in whole number situations. Even though Judy succeeded on many of the items in the initial interview, her performance in the teaching sequence demonstrated consistent growth in the understanding of unit concepts. She exhibited an intuitive knowledge of units in the counting tasks of Lesson One by utilizing unit formations in all the counting activities. In the group discussion which followed, Judy revealed her awareness of her group
compositions in the tasks and stated that usage of grouping "made it easier to count."

Unit formations were also apparent in Judy's solution strategies for whole number problems in Lesson Two. Task 1 presented students with the task of determining the number of bags, with four candies each, which could be formed from two sets of bags, each having bags of four and bags of two. Students were provided with bags representative of the situation. Judy physically and mentally grouped the bags to obtain her solution. Her conceptualization of this grouping situation is revealed in her explanation: "I counted the bags with groups of four, then I put two of the bags together that had two in them." Judy was the only student to successfully solve the second task in Lesson Three. While the other students used a units-of-one approach, Judy conceptualized the need to reunitize the 672[1-unit] videos into 224[3-unit]s "because the price was for groups."

Model-building for multiplication and division of whole numbers.
Experiences acquired in the group environment enabled Judy to expand her models for multiplication and division. Her models of division were limited initially to the partitive model for whole number division. Judy developed the quotitive model for division without direct instruction in the form of lecturing. In the task of determining how many suits could be made from 15 yards of material, with each suit requiring 3 yards, Judy drew circles for each yard, then underlined each group of three. She explained that she used underlining to form "groups of size three because each group of three yards was for one suit." In
another task, students modeled two interpretations for the problem 15 ÷ 3. After the group discussion about these two types of division, Judy wrote her description to distinguish these two types of division: "They both have 15 cookies divided by three things. One asks how many cookies three children would get and the second asks how many bags could you make putting three cookies in a bag." These explanations reveal that Judy understood the role of grouping depicted in her models.

Judy's extension of models for multiplication and division was not accomplished without problems. The major obstacle encountered by Judy centered around some inconsistencies in thinking about measurement concepts. Unlike the other students, Judy had no difficulty in determining the appropriate unit of measure—segments or blocks—for perimeter or area tasks respectively until, however, she encountered the task of determining the number of small rectangles which would cover the surface of a large rectangle. As were all the students, Judy was aware that she could obtain the solution by multiplying the length and width of the rectangle. In determining the dimensions of the rectangle, Judy, like the other students, demonstrated the propensity not to count the corner space twice, thinking instead that "it had already been counted." As Judy's understanding of multiplication grew to include a units-of-units conceptualization, this measurement obstacle disappeared.

Another measurement inconsistency occurred in Lesson Five. Students were given four rods (see Appendix D) to use as measuring units or rulers and...
were asked to measure each one in terms of the others. As the tasks progressed, the measure of each rod took on a different value as the measuring unit was changed. Judy had little difficulty in the first four tasks, but became confused in the last one. In measuring the light green-unit with the red-unit, she lost sight of the measuring unit. Judy's drawing revealed her recognition that one white-unit added to the red-unit equalled the length of the light green-unit, but she was unable to use this information to quantify the relationship between the light green- and red-units. Instead she focused on the light green-unit as her measuring unit and compared the intended ruler (the red-unit) with the intended object of measurement (the green-unit). Judy established two relationships: (a) the light green-unit equalled three white-units, and (b) the red rod equalled two whites. Not realizing that she had lost sight of the measuring unit, Judy concluded that the green-unit equalled one red-unit and one white-unit. But since a white-unit represented 1/3 green-unit, Judy concluded that one light green-unit equalled one red-unit plus 1/3 light green-unit—or 1 1/3 (red-units). Only with a concerted effort of focusing on the role of the unit in quantifying each part-whole measuring task was Judy able to manage this obstacle.

Judy demonstrated awareness of grouping in multiplicative situations. In Lesson Seven students were asked to count the number of packages of cups contained in a box. Judy conceptualized a units-of-units grouping structure, as evidenced by her explanation of her counting strategy: "count the number of packages in each row and the number of rows of packages." Judy was the only
student to recognize the change in unit structure occurring when the sides of a 20 x 30 rectangle were repartitioned into 10-units, resulting in blocks (area units) of size 100.

**Unitizing in rational number situations.** During Lesson Nine Judy demonstrated a strong representational understanding of units. As the unit whole changed during the tasks, Judy seemed to have a clear focus on the relationship between the units in each measurement situation. It was at this point in the teaching sequence that Judy acknowledged for the first time the importance of equality of parts in the part-whole relationship and, additionally, verbalized that the quantification of a unit was dependent on the measuring unit or ruler. Unlike the other students, Judy had little difficulty in establishing comparative relationships between fraction circles. All students had difficulty in determining the size of the blue-unit as compared to the orange-unit whole. Judy said the blue did not "work" because it didn't "come out evenly." As students explored other colors as possible aids in establishing the relationship between the blue- and orange-units, they began combining different colored-units in order to cover the orange-unit. In doing so, it was discovered that two blues and one yellow equalled the orange-unit. Judy argued that "they all [had] to be blue for it to be one third." Her awareness of the necessity of equality of parts also was evidenced in the paperfolding tasks of Lesson Twelve. Laura had been unsuccessful in determining a strategy for folding the paper strip into four parts. Her folded strip contained unequal parts. Judy criticized Laura's folding
technique, arguing that the result would model one fourth only if "all of the pieces [were] equal."

In Judy's initial construction of composite units for rational numbers, she did not see a composite fraction as an iteration of a unit fraction. She exhibited instead a tendency to refer back to the original unit whole instead of focusing on the unit fraction. For Judy, the whole was a very important referent. For example, in the construction of three fifths, Judy formed a unit whole consisting of five 1/5-units, then removed two of them to reveal a part-whole interpretation. Modeling experiences and group interactions during the teaching sequence served to expand Judy's conceptualization of a composite unit to include iteration of the unit fraction so that, for example, a fraction like 6/5 could be conceptualized as the iteration of six 1/5-units. With a reconceptualization of the part-whole relationship, the unit fraction became the new unit whole to be iterated the number of times as indicated by the numerator.

Model-building for multiplication and division of rational numbers. Judy expanded her models of multiplication to include the embedded units model through the use paperfolding tasks which introduced the notion of decomposition of unit fractions—the determination of a unit fraction of a unit fraction. Such decomposition activities fostered Judy's ability to link the operation of multiplication of rational numbers to her multiplication models. For example, in Lesson Thirteen Judy revealed her understanding of the underlying unit structure
in the task of partitioning a 1/6-unit into thirds. Modeling her fraction strip solution on the board, Judy partitioned a unit whole into sixths, split the shaded 1/6-unit into three parts and explained, "there's three thirds in one sixth." As a conclusion to her demonstration, Judy wrote the problem statement on the board: \( \frac{1}{3} \times \frac{1}{6} = \frac{1}{18} \). Without instruction, Judy had connected her model to the operation of multiplication of rational numbers.

Additional connections occurred in a later group discussion during Lesson Thirteen. In some of the folding tasks, students had folded their paper strips from two directions, resulting in rows and columns of blocks. The group discussion focused on the relationship of this area model for fractions to their previous area model for whole number multiplication. Judy contrasted the models using first whole numbers and then rational numbers for dimensions of the sides. She remarked that "6 times 5 gives how many squares are inside." Judy stated the length of each side of the shaded square as "1/6 of this whole side and 1/5 of this whole side" and concluded that the shaded square was 1/30 of the unit whole.

Models for division of rational numbers were not easily connected to Judy's models for whole number division. Students worked as a group to solve Task 2 of Lesson Fifteen, which required them to measure a short paper strip with a longer one as the ruler. The researcher noticed the frustration on Judy's face as she stared at her strips. While Judy had developed a model of measurement division of whole numbers in a previous lesson (Lesson Four), that
model was for discrete sets. She was unable to adapt that procedure on this continuous set. Perhaps the change in manipulative distracted her. These were blank paper strips—no shadings or markings or numbers. Only when Laura suggested a partitioning of the ruler did Judy begin to see a solution strategy. With this clue she was able to solve the task. The researcher could only surmise what thoughts Judy might have been experiencing on this task. It was rare during the teaching sequence that Judy was ever completely stumped on a task. When such an instance occurred, Judy was reserved and unwilling to discuss her thoughts.

After the group discussion about measurement division, students worked the problem tasks individually. Judy explained that her procedure for measuring the 1/4-unit with the 1/6-unit consisted of a sliding method. She placed the ruler (the 1/6-unit) beneath the 1/4 strip and marked the point where the ruler ended, calling this "one 1/4." Sliding the ruler down so that the starting edge was on the mark where the first measure ended, Judy noticed the ruler exceeded the unit and remarked, "It's half of it." This statement indicated she had mentally partitioned her ruler to measure the remaining portion which was smaller than her ruler. She concluded that her answer was "one and one half." She used a similar strategy on all remaining problems, having no difficulty on any of them until the last problem. The measurement obstacle in this task involved a ruler longer than the unit to be measured. Judy could determine no strategy for accomplishing this measurement, even though it was similar to the
problem with unmarked paper strips that Judy encountered at the start of the lesson.

Laura's Profile

The classroom teacher selected Laura as an average student. She contributed often and eagerly to the group discussions. Often she was the first one in her group to grasp unit concepts.

Entering concepts of unit and rational number. Laura's score of 20% on the calculations in the initial interview was the result of consistently misapplied rules. She inverted the first number in rational number multiplication problems, regardless of whether this first number was a whole number or a fraction. She found common denominators, then multiplied the numerators and retained the same denominator. In the division problems, Laura exchanged the numbers whenever the divisor was greater than the dividend. For either multiplication or division problems in which there appeared a whole number and a fraction, Laura converted the whole number, n, to the fraction 1/n. Laura fared better on the choice of operations tasks. She scored 50%, missing one multiplication and four division statements. All four of these division statements contained a divisor, fraction or whole number, that was larger than the dividend. In these cases, Laura selected multiplication as the needed operation.

Laura identified all the part-whole situations for three fourths, but also included the card containing four unequal parts. She was unable, however, to adequately verbalize her understanding of fractions due to her weak fraction
terminology. Laura's unitizing abilities were limited. In the task of finding \( \frac{3}{4} \) of 12 jacks, she could not find \( \frac{1}{4} \). Her strategy for partitioning seemed to consist of guessing where to draw a line.

Laura successfully produced models for whole number multiplication and division statements and created a story problem for the multiplication situation. She experienced no success, however, on the rational number problems. She was also unable to reconstruct the unit whole when given a discrete set representing a unit fraction. Laura focused on the size of the denominator, as did all the students, and determined that \( \frac{1}{3} \) was greater than \( \frac{1}{2} \) since three was greater than two.

**Unitizing in whole number situations.** Results from the initial interview indicate that Laura began the teaching sequence with limited unitizing skills and extremely weak rational number competencies. Laura used a units-of-one approach on all the counting tasks in the first lesson. Evidence of unit formation was seen, however, in Task 1 of Lesson Two. Laura picked up two bags of two candies and counted them as one bag of four. She was clearly thinking in groups. By picking up two bags she demonstrated her conceptualization of composite groups. Laura alternated between units of one and the formation of composite units on the remaining tasks in Lessons Two and Three. In Lesson Three, she regrouped 24 chips into eight groups of three to solve the first task, but did not unitize in the second task. Although these problems were identical in structure, the second one contained larger numbers.
which apparently served to distract Laura's attention from the similarity of the tasks. Instead of grouping 672 into 7-units, she multiplied by seven. After the group discussion of solution strategies, Laura acknowledged the similarity of the two tasks.

**Model-building for multiplication and division of whole numbers.** Laura began to develop models for division of whole numbers in Lesson Four. She was the first person to determine a model for partitive division, which became known as "Laura's Line Method" throughout the study. After the group discussion about the two interpretations for the problem $15 \div 3$, Laura wrote her distinction between the two divisions: "One is asking how many groups can be formed with three in each group and one is asking how many cookies three children would get." This response indicated her awareness of the grouping situation in each interpretation.

Like Judy, Laura was plagued with some measurement dilemmas throughout the lesson. She successfully used partitioning to share five cakes with four people (Task 1), but had difficulty sharing three candy bars with four children (Task 2). While she successfully partitioned the three circles to model the sharing task, Laura was unable to quantify the result. In the group discussion, Laura was the first student to identify the problems as division tasks. Laura was unable to determine a model for either of the measurement division tasks while working alone, but seemed to understand Judy's model when presented in the group discussion.
Laura did not recognize the role of the unit in measurement. The introductory activity of Lesson Five required students to measure the height of a bear, with each student using a different ruler. During the group discussion, Laura was the only student demonstrating confusion about the fact that everyone got a different answer. Even though the other students indicated this difference was a result of their use of different rulers, Laura insisted this "wouldn't make a difference [because] it's supposed to be the same length." Another obstacle, involving her loss of sight of the measuring unit, occurred in Lesson Five. Laura only had difficulty with the measuring tasks in which the ruler was longer than the rod being measured. For Task 3, Laura admitted to guessing the answer to be one fourth. This perhaps could be interpreted as a weak part-whole interpretation for rational numbers. It never occurred to her that her guess could be verified by placing four white rods beside one light green rod. The group discussion about solution strategies for Task 3 highlighted the fact that additional white rods were used by the other students to fill in the remaining space beside the light green rod, yet Laura still resorted to guessing on Task 4 as well. The researcher suspected that this task possibly represented a concrete example of the physical constraints of Laura's models of division—the divisor must be less than the dividend. Laura was successful on the final task of the lesson. In using the red-unit to measure the light green rod, Laura explained: "It was bigger than the red, so I just took another red one and put it on. So one red one and half of a red one--one and one half." However, Laura never
demonstrated this understanding in instances where the ruler was longer than the rod to be measured.

Laura initially confused concepts of area and perimeter, thinking that perimeter could be determined by a procedure of multiplying the length and width. Although this confusion seemed to be resolved by the end of Lesson Six, another measurement obstacle, concerned with linear measurement, emerged. Laura realized that she needed to add the measures of the sides to obtain perimeter, and she realized additionally that she needed to "count the dots" along each side to determine the length, but she began the measure of each side with a count of one at the corner of each side of the rectangle.

The area model was developed in Lesson Seven. In the second task, students used grid paper to draw a 20 x 30 rectangle. Laura again wanted to count the corner point or dot as "one." Once the rectangle was drawn, Laura explained the connection between her 20 x 30 rectangle and the operation of multiplication. She verbalized the meaning of multiplication as the determination of "how many little squares are inside" the rectangle, and realized that this could be determined by multiplying length and width, but she still began her measurement of each side with a count of one at the corner. Once the lessons turned to the formation of composite groups, however, Laura's difficulty in measuring the lengths of the sides seemed to vanish. It seemed to the researcher that the regrouping of each side into 10-units possibly helped Laura to focus her attention to the segments along each side rather than on the points.
as she measured each side. Laura correctly determined the length of each side using her new composite units, but she was confused by the change in unit structure for area, which was not apparent while using units of one. Laura multiplied length and width of her composite units to determine the number of large blocks inside the rectangle, but was unable to use this strategy to determine the size of these blocks within the rectangle. Instead of multiplying the sides of these interior blocks, she added length and width and concluded the size of the blocks to be 20. For the first time, Laura experienced confusion between linear and area units. Like Melanie, she now considered the squares or boxes when determining length and width of the rectangle. It was unclear to the researcher whether the group discussion served to clarify her thinking in this regard because of the inconsistency of her answers on the next task—reunitizing the sides into 5-units. Although Laura correctly identified the dimensions of each side of the repartitioned rectangle and was the first student to identify the number of boxes inside, she mistakenly concluded that the size of each box (block) was five.

Models of multiplication were further explored in Lesson Eight. Task 3 from the previous lesson had been assigned for homework; however, Laura was the only student to attempt it. It was interesting to hear her explanation, as her family owned a carpet store and her mother helped her solve the task. Together they had modeled the task by creating a rectangle simulating a 9' x 12' room. The interior space was filled with 12 squares of real carpet of varying colors,
each representing one square yard. Laura's explanation of her solution is as follows: "I timesed [sic] 12 by 9 and got 108. My mom helped me and said to divide by 9. I don't know why. So I drew a little box and wrote 9 and 12 on the sides. And I divided each side by 3 and I got 3 and 4. So it would be 3 by 4." Laura disregarded her mom's advice to divide by 9, and instead regrouped each side into 3-units, demonstrating her awareness that three feet equalled one yard. She multiplied the new dimensions of the room to obtain the number of square yards of carpet needed. The other students were impressed, as was the researcher, with the quality of Laura's model and her explanation. As the discussion centered on the unit structure in the task, none of the students could explain the meaning of a square yard. Even after the discussion highlighted that there were three feet in a yard, Laura could not determine the number of square feet in her blocks of carpet and was, therefore, never able to determine why her mother had suggested for her to divide by nine.

Laura successfully modeled the three tasks in Lesson Eight, concluding after the first one that the operation of multiplication was represented by her model. When another student explained multiplication as repeated addition, Laura demonstrated the broadening of her conceptualization of multiplication by adding further that multiplication involved "counting the groups." More measurement problems surfaced in the last task, involving the determination of the number of small rectangles contained in a large rectangle. Laura encountered difficulty because of a lack of preciseness in her measuring.
procedures. In the process of tracing the ruler (the small rectangle) onto the large rectangle, she often overlapped the edge of her ruler with the previous measurement or left a gap between the two measurements.

**Unitizing in rational number situations.** Laura demonstrated an understanding of the part-whole interpretation of rational numbers in Lesson Nine, but only as long as an equal number of parts were contained in the unit whole. When the last problem of Task 1 violated this constraint, Laura's unconcern with inequality of parts in a part-whole relationship returned. In the task of measuring the blue-unit with the orange-unit, Laura realized that the blue-units alone would not cover the orange-unit so she resorted to the use of another unit to fill up the orange-unit. Her discovery that "two blues and one yellow equal the orange" led her to the conclusion that the blue-unit equalled one third of the orange-unit. As a result of this statement, a lengthy group discussion ensured, and concluded with Laura's ability to resolve the task. Her analysis of the task led her to discover that five yellow-units equalled the orange-unit and that two yellow-units equalled the blue-unit. Demonstrating a stronger understanding of a part-whole relationship, Laura concluded that the blue-unit therefore equalled two fifths of the orange-unit. Laura plowed through the remaining tasks in the lesson with seeming ease and clarity of thinking. In summarizing the lesson, the researcher highlighted the fact that, based on measurement results from the tasks, the same fraction circle piece was found to have different values. Laura was the first student to justify that this discrepancy was the result of using
different measuring units. Only a week prior to this lesson, Laura insisted that the same bear should be the same height, regardless of the rulers used.

In Lesson Ten Laura had difficulty remaining focused on the measuring unit, partly because Melanie, who was so verbal about her own confusion, was confusing Laura's thinking. She obtained the correct answer of two for the first problem, but could not explain what this two meant. Laura had lost sight of the unit and was unable to distinguish whether the green- or red-unit was the ruler. For the second problem (1 green = ___ orange-unit), Laura again confused the units, as evidenced by her rationale for her answer of one and one half: "I put one green on top of the orange. And then one red equals half of the green one and so I just put the red on [the orange] to see that it's one and one half." She explained to the researcher that her answer meant "it's one whole and a half more," but she was unaware of her loss of sight of the intended unit whole. After the group discussion, Laura displayed a clearer understanding of the importance of focusing on the measuring unit in each remaining task. She was the only student to correctly solve Task 2, which asked students to determine if the tan fraction circle represented one third or one fourth. In explaining that an answer could not be determined, Laura remarked: "It depends on the measuring unit. If it's pink it's 1/3 and if it's green it's 1/4."

For Task 3, Laura used her part-whole interpretation of rational number to form composite numbers. To obtain the fraction four tenths, she explained that "she brought 10 of these [yellow-units] together and made a whole, then
pulled four of them back to make four, so four tenths." She remarked that she focused on the denominator to determine "how many pieces there are in all." Laura initially focused on the unit whole to model improper fractions by changing each fraction to a mixed number. The group discussion enabled Laura to expand her thinking to view the modeling of an improper fraction as the creation of a composite unit formed by iteration of the unit fraction. When the manipulatives changed from continuous to discrete sets, Laura was initially unable to transfer her understanding of unit structure. When attempting to solve the last task of Lesson Eleven, Laura experienced much frustration and was unable to determine a strategy while working alone. The group discussion, however, seemed to help her connect with her unitizing strategies for the circles or strips. Laura solved the last problem in Task 3 for the group by partitioning the 28 chips into seven groups, then pulling away four groups to represent the composite fraction four sevenths.

**Model-building for multiplication and division of rational numbers.** The paperfolding activities of Lesson Twelve were initially difficult for Laura because of her inability to fold the paper strips into thirds or fourths; however, they enabled her, in the end, to envision the embedded-unit structure involved in the decomposition of unit fractions. For example, after folding a strip into thirds and then folding only one of these thirds into halves, Laura envisioned the embedded units which would occur if each 1/3-unit was partitioned into two 1/2-units. She concluded that the size of this smaller unit was one sixth because
"each of the wholes, [each 1/3-unit], is two [1/2-units] and that would be six."

Laura not only conceptualized the embedded units in these situations, but was, furthermore, able to verbalize the name of the smaller unit in two ways, each dependent on the unit of focus. Perhaps aided by Judy’s verbalization of the same notion, Laura was, nonetheless, able to link this structure to the operation of multiplication of rational numbers. In Task 3 Laura demonstrated the results of folding by drawing a unit whole strip and then partitioning it into five pieces, using vertical cuts. Next she used horizontal partitioning to section the entire strip into halves. This resulted in a 2 x 5 array. Laura linked with her ideas of multiplication from the area model problems with whole numbers to conclude that one of these rectangles would be one tenth of the entire strip. Laura stated that the length of one rectangle within the array was 1/5 of the whole side, while the width was one half of the whole side.

An incident of loss of sight of the unit occurred when Laura drew on the board a model for finding 3/4 of 1/2. She drew a unit whole strip, partitioned it into 1/2-units, then shaded one half; however, she ignored the shaded 1/2-unit and partitioned each 1/2-unit into halves to obtain 1/4-units. When Judy reminded Laura that she had found three fourths of the whole unit, Laura returned to the board and corrected her partitionings to reflect three fourths of only the shaded 1/2-unit. She was able to name this new region in two ways: 3/4 of the 1/2-unit or 3/8 of the whole strip.
Using her decomposition skills, Laura developed a method to reconstruct the unit whole by focusing on the unit fraction embedded in the composite fraction and then iterating it to form the unit whole. For example, Task 2 of Lesson Fourteen presented students with a 2 x 3 array of dots representing three fifths of a unit. A similar task in the initial interview held no meaning for Laura; in this instance, however, she established the need to expand this array into a 2 x 5 model. She then partitioned between each column of dots, stating that each column represented one fifth because "it's one group of the five." Both Laura's model and her explanation revealed that she easily decomposed the 3/5-unit into 3[1/5-units], then iterated the 1/5-unit five times to obtain the unit whole.

For the first task in Lesson Fifteen, Laura provided "three cookies and four children" as context for the problem 3 ÷ 4. She concluded that the answer would be "three fourths or three 1/4-units." In the second task, measuring a small blank paper strip with a longer one, all students had difficulty determining a strategy until Laura suggested folding the longer strip into 1/4-units. Laura demonstrated the folding technique by folding the strip in half, then folding it in half again. This paperfolding was much improved from her initial paperfolding attempts in Lesson 12. She recognized that the smaller strip was "three fourths of the long strip." Yet Laura could not transfer this technique to the last measurement problem of Task 3, measuring the 1/3 strip with the 2/3 strip. Instead, she lost sight of the measuring unit and concluded the answer was two.
Laura's performance in this lesson provided evidence that the development of a model of division of rational numbers is difficult. She had little difficulty with the first three tasks, but was stymied by the last three. Time did not permit the researcher to determine Laura's thinking on all of the problems. She experienced loss of sight of the measuring unit on many of the tasks and was again perplexed by the challenge of measuring a strip with a ruler longer than the unit being measured.

**Melanie's Profile**

Melanie was an average-ability student selected by the classroom teacher. She was extremely talkative and tended to dominate the group discussions. Often the researcher found it difficult to understand what she was saying because Melanie seemed to have a very confusing way of expressing her thoughts.

**Entering concepts of unit and rational number.** Melanie scored 30% on both the calculations and the choice of operations tasks of the initial interview. She seemed to be guessing the operation on many of these tasks. The only recurring pattern detected in her missed calculations involved her failure to invert the divisor in the division tasks. Melanie held a part-whole interpretation of fractions. She identified all the part-whole cards for 3/4, but also included the card containing unequal parts. She denied the importance of equality in parts in modeling a fraction.
She was unable to find three fourths of 12 jacks. Melanie's only attempt consisted of the formation of three groups of four, which she concluded represented three fourths. In finding one third of a candy bar split into 1/4-units, Melanie decided to "take one of the blocks out." With this response, she crossed out one of the 1/4-units and shaded one of the three remaining blocks.

Melanie correctly solved and drew models for multiplication and division tasks for whole numbers. She created good stories for both as well. Although she was unable to solve or model the rational number problems for multiplication and division, she successfully created a measurement interpretation story problem for division, but then used the same story structure for the multiplication problem. Melanie was unable to reconstruct the unit whole from a given unit fraction. In fact, she insisted that the problem was written incorrectly because "four dots don't equal 1/3, they equal 1/4." She concluded that the denominator "tells you how many [dots] there are." It was evident to the researcher that Melanie began the teaching sequence with extremely weak rational number understandings and skills.

**Unitizing in whole number problems.** Melanie used units of one on two of the three counting tasks in Lesson One; however, she did not consider her use of "color coordinating" on the second task as a type of grouping. She used units of one on the first task in Lesson Two, arguing that using the algorithm to obtain the answer was "a shorter way" to work the problem. She added further that "multiplying is easier and more simpler." Melanie formed 4-units on Task 2,
but still insisted she was counting by ones. It appeared to the researcher that perhaps Melanie did not fully understand the concept of what actions constitute grouping. Further evidence for Melanie's lack of understanding occurred in Lesson Three. She divided by three on the last task "because it was shorter than adding everything up," but initially did not see this division as a grouping strategy. Melanie was aware of why she had divided by three—the three represented how many cookies each student would get. She also understood the meaning of her answer, five—"five people got cookies from that bag." Her comments that grouping was "not easier when there were large numbers" suggested that Melanie possibly viewed grouping as a physical process exclusively.

On several of the tasks, Melanie went farther than the requests of the problem task. Even when the task only required determination of the number of groups, Melanie found this answer, but then multiplied by the number per group to obtain the total amount of objects. It seemed to the researcher that perhaps Melanie was following a habit instilled by the usual school math approach—focusing on units of one. By the end of Lesson Three, Melanie acknowledged her use of grouping to solve the tasks, although she continued her solution process until she determined the answer in units of one. It seemed to the researcher that Melanie's usual school math approach—the units-of-one approach—is a difficult habit to overcome.
It was not until the end of the discussion phase of the lesson that Melanie conceded that she had indeed used grouping on all of the tasks in the lesson. She stated that she divided 3 into 672 "and it [told] how many groups" there were. Perhaps Melanie now realized that she had grouped the large number, 672, in her head, thus grouping was also a mental rather than just a physical process.

**Model-building for multiplication and division of whole numbers.**

Melanie demonstrated persistent difficulties in explaining her thinking in an understandable way. In the group discussion about models of division, Melanie explained her interpretation of division as a "simpler way, instead of multiplying, to figure something out." She concluded, "When you multiply you make adding shorter and when you divide you make multiplying shorter." As in the initial interview, Melanie exhibited weak procedural skills. She stated on Task 2 that $3 \div 4$ would be one and one fourth. This answer exhibited two possible misunderstandings: (a) the notion that the divisor must be less than the dividend, and (b) loss of sight of the unit after exchanging the numbers. The researcher suspected that Melanie's weak calculation skills could be the result of her limited understandings of multiplication and division of whole numbers. Her definition of division and her calculation above provide evidence that Melanie viewed the two operations as interchangeable, depending on the numbers involved in the problem situation.
Melanie began to verbalize the role of the unit during the discussion of students' attempts to measure the height of the bear in Lesson Five. Each student used a different ruler, so all had different answers. Laura thought they should have gotten the same answer since they all measured the same bear, but Melanie explained, "They're the same height, but it's a different measurement with each." She added, "The smaller [the ruler] the bigger the number they are going to have [for the measure]."

Students used rods in Lesson Five to establish a relationship between given pairs of rods. Melanie had no difficulty on Task 1. She identified the dark green rod as the ruler and explained that the light green rod measured one half because "it took two of the light green ones to equal the dark green one." On the second task, Melanie concluded that she divided to obtain her answer "because it's sort of like measurement." Task 3, in which the ruler was longer than the rod being measured, presented problems for Melanie. She recognized that three white-units equalled one light green-unit, but nonetheless concluded that one white-unit equalled three light green-units. By using a third unit to establish a relationship between the red- and light green-units in the fourth task, Melanie was successful in measuring the red-unit. She explained that she used the white rod "to see how many little parts of the green the red was." Melanie explained her reasoning as follows: "Two [white-units] equal [the red-unit] and there are three [white-units] . . . that equals to the green. And [since] two white ones equal one red, so [the red-unit] covers up two spaces of the three--so two
thirds. Melanie was unsuccessful in using this same reasoning on the last task because she confused the unit to be measured with the measuring unit. During her explanation of her solution strategy, Melanie caught her mistake—her loss of sight of the measuring unit—and corrected her thinking: "I used the white and equalled it up to the green, but I think I should have equalled it up to the red. And it should be one and one half." Only by refocusing on the red-unit as the ruler was Melanie able to correct her thinking.

Melanie was the only student to recall the meaning of perimeter, but this understanding was limited to a procedural definition—"it's like when you add up all the sides." In Lesson Six students were given a sheet a grid paper and asked to draw rectangles having a perimeter of 24 units. Melanie's confusion began when students started comparing the units obtained in the perimeter with the total number of blocks inside the rectangle. Recognizing that a multiplication of the dimensions would give the number of blocks inside the rectangle, Melanie now considered this number to be the perimeter. She forgot about her opening definition for perimeter. Melanie's confusion was not dispelled at the conclusion of the lesson.

Lesson Seven focused on the area model of multiplication. In Task 2, drawing a 20 x 30 rectangle, Melanie exhibited the same tendency as Laura—wanting to count the starting point (corner) as 1-unit of linear measure. Once the rectangle was drawn, however, Melanie stated that the measuring unit for the length of each side was "the little areas." Again she was confused between
linear and area units. After grouping each side into 10-units, Melanie recognized there were six large blocks inside the rectangle, but stated the size of these blocks as also being 10-units. After the group discussion identified that these blocks were a 100-unit, Melanie then thought that the 20-unit side of the original rectangle was 200-units. At the end of the discussion, Melanie was able to state correctly the number and unit size of the length, the width, and the inside blocks of the rectangle, but this understanding was short-term. In the subsequent task of regrouping the sides into 5-units, Melanie displayed the same confusions and even admitted to her confusion.

Models of multiplication were examined further in Lesson Eight. The third task required students to determine the number of small rectangles contained in a large rectangle. Like Laura, Melanie skipped the corner space when counting the length of the second side because she "had already counted it in that [other] side." Melanie admitted that she was counting the blocks instead of the segments for each side. This focus on the blocks could explain why she did not want to count the corner space twice in obtaining the measure of both sides.

**Unitizing in rational number situations.** Melanie consistently demonstrated weak fraction terminology, but seemed also, at times, to be exhibiting limited fraction understanding in addition to her weak terminology. She verbalized, for example, the fraction 5/4 as "four under five." Melanie's initial interview insistence that the part-whole relationship did not have to
consist of equal parts resurfaced in Lesson Nine. In determining the size of the blue-unit as compared to the orange-unit, Laura had discovered that two blue-units and one yellow-unit equalled the orange-unit. Melanie thought this indicated that the blue-unit was one third of the orange-unit because it was "one out of three other wholes." Her criterion for part-whole relationship was dependent on the number of pieces needed to cover the unit whole, regardless of the size of those pieces. Melanie lost her direction at this point, focusing instead on trying to determine which colored-unit would "give one third out of the orange." She was no longer endeavoring to determine the size of the blue-unit. Once the students redirected her thinking to the blue- and orange-units, Melanie lost sight of the unit and decided the blue-unit would be two and one half. In justifying her answer, Melanie explained, "Two yellows make up one blue and there are five yellows on the half so two yellows go into one blue and that would make two and that would be a half left."

Melanie's loss of sight of the unit continued in Lesson Ten. She obtained the right numerical answer on the first problem in Task 1, but could not determine which of the two units was the ruler. She concluded that she was "measuring the green to measure the orange with the green." Melanie was quite verbal about her confusion, which spread to the other students temporarily. She became so confused that she was rambling. For example, she remarked "the green would be a half of a green." Melanie continued to mix up her units in subsequent tasks.
Melanie used the part-whole interpretation to model the composite fraction five sixths. She explained, "I started with red because it takes six of them to make up a circle." Melanie modeled the fraction, $\frac{7}{12}$, in a similar manner. She stated the measuring unit was seven, then hesitated and added, "Well, really it's the 12 probably because you are using the 12 to see how many seven--no, it's the seven because you are using the seven to find the 12."

Melanie was definitely confused about the measuring unit, but, by stating that the measuring unit was "seven," she was, perhaps, demonstrating an awareness that "something" was being counted out seven times--even though she was unsure what this unit of measure was.

More vagueness about the measuring unit was observed in Lesson Eleven. In modeling the fraction $\frac{3}{5}$, Melanie explained, "I drew the strips with the same number [of parts] as the denominator and shaded in the numerator." When asked what she had three of, Melanie replied, "of five." Perhaps she was linking to her part-whole interpretation. Later in the same discussion, Melanie added that the entire strip was "a whole that's separated into one fifths," adding further that she measured with a "one fifth" to draw three fifths. However, in displaying her drawing before the group, she identified the shaded sections as "one, two, three little shaded fives." In another example she wavered in her decision for the unit in the fraction $\frac{3}{2}$. First she claimed it was "three units," next she decided it was "two units." Melanie was, however, the only student not having difficulty in drawing $\frac{3}{2}$. She selected the $\frac{1}{2}$-unit and iterated it three times to
model $3/2$, but then explained that she "looked at it as three twos and drew the twos and drew a half of a two." Likewise for the fraction $6/5$, Melanie was the only student to draw it initially as an improper fraction. The discussion acknowledged the change in unit structure when $6/5$ was changed to $1 1/5$. Melanie said the unit would be "the whole of $1/5$" for $1 1/5$, but for $6/5$, the unit was "6 fives." Melanie demonstrated an understanding of iteration as a means for forming composite units although her ability to verbalize the unit of iteration was limited.

The first task of Lesson Twelve required students to fold a paper strip into thirds, then fold one of these thirds in half. Melanie initially named one of the newer pieces as one fourth since there were four pieces. The fact that these parts were unequal did not concern Melanie. The group discussion again confirmed the necessity of equality of parts since the other students agreed that equality of partitioning was a prerequisite for a part-whole relationship. After this discussion, Melanie's work on subsequent tasks showed increased awareness of unit structure. Melanie was able, in the second task, to name a unit fraction in two ways, depending on the unit whole. For example, after folding a strip to create $1/4$-units, then folding these $1/4$-units into three equal parts, Melanie named the size of the new part as one third, although she could not identify the unit. When asked to state the name of this new part as compared to the unit whole, she mistakenly replied "one eighth." At this point Melanie had not opened up her folded strip. She was perhaps thinking of the four original parts,
and then creating two new foldings to form three parts. When she unfolded her strip, however, she changed the name of this new part to one twelfth because "there's two rows and six across." With this statement, Melanie demonstrated her linkage of models of multiplication for whole numbers and rational numbers. Melanie summed up the group discussion on this task with her response, "It's one third of that little box, but it's one twelfth of the whole." She was now able to name a unit fraction in two ways, each dependent upon the unit whole. In the third task, Melanie was so focused on the fact that a unit fraction had two names, that she used them both, even if only one name was being requested. The question asked in Task 3 required students to name the new part as compared to the unit whole strip. Melanie replied, "one tenth, but it's half of that [1/5-unit]."

**Model-building for multiplication and division of rational numbers.**

Lesson Eleven continued the study on the creation of embedded units as a way of decomposing unit fractions. The first task began by having students use paperfolding to partition a 1/6-unit into three equal parts. The researcher asked students if they could have answered the question in the task without the aid of paperfolding. Melanie's strip was originally folded into a 2 x 3 array to obtain sixths. With the paper folded to reveal only a 1/6-unit, Melanie had next folded it into thirds. Considering the researcher's question, she examined the rows and columns on her opened strip. Melanie looked at each of the two rows as three groups of three, as evidenced by her response, "You would say 3 times 3 and
then times 2." Although her response was perhaps not one that Melanie could have made without the benefit of the results of her paperfolding exercise, it was, nonetheless, an indication to the researcher of Melanie’s awareness of the embedded unit structure she had created.

Melanie’s insistence that the calculation is a better solution method than modeling was seen again in Lesson 13. After Task 1, the discussion addressed the possibility of drawing a model of paperfolding results so that students would no longer have to fold the strips. This resulted in students’ creation of an embedded units model whereby a unit fraction strip was partitioned once to represent the first folding action, then repartitioned within each initial partitioning to represent the second folding action. Melanie acknowledged that she could multiply rows and columns to obtain the denominator for her part-whole interpretation of the unit fraction. After this Judy discovered that fraction multiplication would also result in the name for a unit fraction as compared to the unit whole. Melanie preferred Judy’s procedure of multiplying to drawing a model. She explained her thinking: "Another way you could do that is $\frac{1}{3}$ of $\frac{1}{6}$ and you get $\frac{1}{18}$. And you can do [this] without having to draw that little chart."

Measurement obstacles continued to plague Melanie. Even though she had connected decomposition of a unit fraction to her multiplication procedures, her previous confusion over area and perimeter persisted and served as a barrier to her ability to think in units. For example, in Task 2, Melanie modeled the
problem 1/4 of 1/10. She explained her model as follows: "I used 10 along the
top and 4 along the sides and I got 40—and that might be 80." Melanie
explained that this multiplication would result in the distance around the outside
of the strip. When Judy confronted Melanie with this inaccuracy, Melanie
corrected her previous statement by adding, "it would tell how many [squares]
there is in the middle. That is 1/40 though, but the 80 is the perimeter." Later
in the discussion she seemed to be remembering perimeter as the addition of all
the sides as evidenced by her new statement. Referring to her 4 x 10 model she
claimed: "This is 10, that's 4, so on the inside there's 40. And on the edges
there's—160."

Melanie viewed the operation of multiplication of rational numbers as not
one operation, but two because "you divide, then multiply." This perhaps
explains why she was confused as to whether a choice of operation task was
multiplication or division. The choice was not a simple one for her because she
envisioned the need for both operations. Likewise, this could account for her
previous definition that division was "a short way of multiplying."

The first task of Lesson Fourteen required students to determine the unit
whole if given a pink fraction circle piece which represented a 1/2-unit. Melanie
was unable to determine the unit whole and resorted to looking at Judy's work
to see which colored-unit she had selected. The remainder of this task consisted
of representational items similar to ones students had encountered in a previous
lesson. Melanie demonstrated a greater understanding of units and a better
ability to verbalize her thinking than on her previous attempts. In determining
the size of the brown-unit, Melanie explained: "It takes two browns to make up a
pink and it takes two pinks to make up an orange. So you times it 2 times 2 and
you get 4—so 1/4." Melanie still experienced loss of sight of unit, but was able to
catch her own mistake. In finding the size of two green-units, she first replied
"one and a half orange." However, when she modeled her solution strategy for
the group, she concludes, "Nope—it's one and one third."

Melanie was unable to perform any of the reconstitution tasks in Lesson
Fourteen and exhibited much frustration throughout the lesson. In the last
reconstitution task, students were given six dots to represent 3/5 and were
required to determine the unit whole. Melanie interpreted this to mean "three
rows of five," but drew a 3 x 5 array in which she embedded the given 2 x 3
array of shaded dots. The remainder of her new array contained unshaded dots.
She refused to listen to correct solutions as they were explained by the other
students.

Lesson Fifteen sought to develop models of division. Students were
required to develop a partitive and measurement model for the problem 3 \div 4.
The researcher asked students to provide context for the problem. It was
decided that this problem would represent the sharing of three cookies among
four children. Melanie immediately replied, "that wouldn't work." Although she
had been unaffected by the problem in its mathematical form, she now viewed
the problem in a realistic setting. After a brief period of thought, she added,
"Well, you will have to break up the cookies. Each person would get a fraction of the cookies. Each person would get three fourths of a cookie." She drew three circles on the board to represent the cookies, then partitioned them with three vertical cuts, forming unequal partitioning. When asked if everyone got a fair share, she replied, "I know, I just drew it real fast." She then properly partitioned the cookies into fourths.

To develop the measurement model of division of rational numbers, Lesson Fifteen presented students with a variety of measurement tasks in which they used fraction strips to model division. Melanie successfully performed all of the measurement problems in Task 3, but had difficulty interpreting the meaning of her answer. One task required students to measure the 3/5 strip with the 1/5-unit. Melanie stated the unit was "five." In the second task, determining the number of 1/6-units contained in the 1/4 strip, Melanie did verbalize the ruler as one sixth. For the fourth task, Melanie obtained the answer of three. When asked what this meant, she replied, "three wholes of the unit" but never explained what the measuring unit was. Melanie demonstrated her measurement model of Task 5 on the board and explained: "I got the 10-strip. I put it on a whole, counted over three and marked it, counted over three more and marked it, counted over three more and marked. There was one left over, so three and one third." When asked to identify the unit, Melanie held up her fraction strip representing three tenths and responded, "this." The final task contained a ruler longer than the strip being measured. None of the students
were successful in their individual attempts on this problem. Although Melanie was the only student to create an accurate drawing, she was unable to use it to determine an answer until the group discussion. At this time she was the first student to establish the relationship between the two fraction strips, which then allowed her to determine the answer. Melanie asserted that "one third is half of two thirds." When asked how many of the 2/3 rulers were contained in one third, she replied, "one half." As with the other students, the presence of a divisor larger than the dividend continued as a major obstacle in Melanie's ability to solve division tasks.

Summary

**Entering concepts of unit and rational number.** Results from the initial interviews revealed that all students began the teaching experiment with limited skills in unitizing. Although all of them recognized a part-whole interpretation, even this skill was limited in that none of the students acknowledged the importance of equipartitioning in a part-whole relationship—essential for an understanding of the unit concept. Except for Judy, none of the students demonstrated proficiency in equipartitioning a unit whole. All students demonstrated limited models for multiplication and division of whole numbers (Fischbein et al., 1985) and no models for these same operations involving rational numbers. Only Judy was successful in solving rational number tasks. None of the students considered units in the task requiring reconstruction of the unit whole from a discrete set of dots which represented a unit fraction. Instead
they focused on the number of dots in the set and the size of the denominator in the given unit fraction to determine a part-whole relationship. These results indicated to the researcher that, at the start of the teaching sequence, none of the students were at an advanced level in comprehension of the unit concept.

**Unitizing in whole number situations.** The activities in Lesson One were designed to allow examination of students' intuitive notions of unitizing in familiar counting situations. No mention of the word "groups" had been made prior to the lesson; students were simply instructed to count various objects. Judy was the only student to form units on all counting exercises. Laura used units of one on all the tasks, whereas Melanie used grouping on one task, but did not recognize her use of color-coding as a type of grouping. The group discussion served to highlight the advantages of grouping in that all students indicated types of counting situations which would be facilitated by the formation of units.

Lesson Two focused on the formation of units in whole number problems. Although Judy and Laura physically and mentally formed groups of bags to solve the first task, Melanie used units of one, considering this a "shorter and simpler way." Even though the problem required determination of the number of bags only, Melanie went further to count the total number of individual candies. This action served as an indication to the researcher that the usual school math approach—the units-of-one approach—is a difficult habit to overcome. All students, however, used units on the second task in which students were given a
Students alternated between using units of one and composite units on Lesson Three. All of them formed composite units on Task 1, but only Judy used grouping on the second task, which, although identical in structure to the first task, contained larger numbers. Melanie's remark, that grouping was not easier when the numbers were large, indicated her uncertainty about the concept of unitizing. Perhaps Melanie thought grouping was a physical process exclusively rather than also a mental process. The group discussion seemed to clarify Melanie's thinking in this regard since she indicated at the conclusion of the lesson that she had grouped on all of the tasks in the lesson.

Model-building for multiplication and division of whole numbers. In Lesson Four, students extended their existing models of division. Laura determined a model for partitive division, while Judy developed the measurement model of division. Both girls were able to explain the role of grouping in these models of division, whereas Melanie displayed confusion about the concept of division. She seemed to view the operations of multiplication and division as interchangeable, depending on the numbers involved. The division tasks in the lesson required equipartitioning of unit wholes to obtain a solution. All students demonstrated weaknesses in basic equipartitioning skills, which served as a handicap in their abilities to develop models of division. Melanie resorted to copying Judy's work in order to obtain a solution on at least two of
the four tasks. Laura was extremely slow in modeling the problems, but eventually succeeded in partitioning; however, she experienced difficulty in quantifying the results. In contrast to the limited understandings of Laura and Melanie, Judy was successful in modeling all of the division tasks.

Lesson Five emphasized the role of the unit in measurement situations. Students were given four rods to use as measuring units or rulers and were asked to measure each one in terms of the others. As the tasks progressed, the measure of each rod took on a different value as the measuring unit was changed. One obstacle emerged during the lesson--all students experienced some instances of loss of sight of the measuring unit. In seeking to establish a relationship between two units in each measurement task, students would sometimes exchange the ruler with the object being measured, resulting in an answer which was the inverse of the requested answer. Analysis of the tasks in which this obstacle occurred reveal a similar situation--these problems all contained rulers longer than the rods being measured. This tendency of the students to invert the relationship illustrates a constraint of primitive models of division (Fischbein et al., 1985): the divisor must be less than the dividend.

Lessons Six through Eight sought to expand students’ models for multiplication of whole numbers. A major recurring obstacle in these lessons involved students’ weak measurement concepts. The first lesson of the series focused on the appropriate selection of units of measure for area and perimeter, the purpose of which was to enable students to become more sensitive to the
role of the unit in measurement situations. From the start, students were perplexed as to whether they should count segments or boxes in order to obtain perimeter. This was problematic in two ways: (a) some students counted all the squares inside, giving them a number for area rather than perimeter, and (b) those students who did use the exterior squares only to calculate length and width did not want to count the corner square twice. In determining the cardinality of a set, students learn that they have to count every object in the set and not to count the same object twice. The latter rule was perhaps at the core of their reluctance to count the corner square twice. Laura and Judy seemed to have resolved their confusion between area and perimeter measurement by the end of the lesson; however, Melanie was still unable to distinguish between these two measurement situations.

Lesson Seven continued to develop students' models for multiplication of whole numbers. Reunitizing activities were designed to help students focus on the change of unit structure occurring in many multiplicative situations. Judy illustrated her units-of-units conceptualization early in the lesson in the task of counting the number of packages of cups in the box. By counting the "number of packages in each row and the number of rows of packages," Judy formed $2\{3\text{-unit}\}s\text{-unit}s$. In the second task, students were required to draw a $20 \times 30$ rectangle on grid paper. An early problem arose in students' measurement processes—many wanted to count the starting point as one unit. This tendency could also be attributed to an interference of students' rules of cardinality. Once
the rectangles were drawn, regrouping activities involved the reunitizing of the sides into 10-units, then 5-units, with an analysis of the change in unit structure occurring within the rectangle as a result of each reunitizing action. None of the students found difficulty in stating the dimensions of the regrouped sides, and all of them realized there were six blocks within the rectangle, but a determination of the size of these blocks was not made instantly by any of the students.

Despite the fact that the group discussions focused on the change in unit structure occurring in each regrouping situation, none of the students were able to generalize this change of structure so that they could apply it to the new grouping situations. That is, students displayed understandings about a grouping structure that were specific to that regrouping situation alone. A correct answer might be given at one point in the discussion, only to find the same student miss a similar question later in the discussion. Based on the inconsistent answers given by the students, it was evident that this concept of change in unit structure in multiplicative situations was not firmly rooted in students' minds.

Weak measurement skills were again the source of problems encountered in Lesson Eight. Students used small rectangles as rulers to determine the number of rulers that would fill the inside of a larger rectangle (Task 3). All students realized that the answer could be determined by multiplying the number of rulers in the length and width of the larger rectangle, but measurement discrepancies made determination of the dimensions of the large rectangle difficult. Sometimes students would not butt the edge of the ruler
against the mark where it had last been placed, causing a gap between placements of the ruler. In contrast, some students would overlap the edge of the ruler with the markings from the previous measure, resulting in an under-measurement. Despite such measurement errors, all the students were able to verbalize the multiplication process as a type of grouping to determine the number of "rulers" or units contained in the large rectangle.

**Unitizing in rational number situations.** The goal of Lesson Nine was to extend students' existing knowledge of fractions to include awareness of the unit in fractional situations. Fraction circles were used in the first task to determine the relationship between the orange-unit as the unit whole and various other fractional units. Judy demonstrated a strong representational understanding of units as she was able to maintain a clear focus on the relationship between the units in each measurement situation. Laura and Melanie had little difficulty in establishing a relationship between units as long as the fractional unit was contained in the unit whole an equal number of times; however, the last problem violated this condition and proved to be more challenging. Judy was the first to recognize that the blue-unit "would not work" because the measurement would not "come out evenly." This discovery led students to consider other fractional units to cover the orange-unit. Laura discovered that two blue-units and one yellow-unit would fill the orange-unit and concluded that "blue is one third." Melanie agreed with this relationship. With this reasoning, both girls exhibited the same weakness of understanding of their part-whole interpretation of
rational numbers as exhibited by them in their initial interviews. Judy, however, revealed her awareness of the necessity of equipartitioning by disputing their claims. Whether or not the group discussion served to clarify Melanie's views of equality in partitioning was not determinable by the researcher. Melanie was confused for the remainder of the lesson and experienced loss of sight of the unit on all remaining tasks. Laura, however, emerged from the discussion with a stronger understanding of a part-whole relationship, as evidenced by the fact that she plowed through the remaining tasks in the lesson with ease. At the conclusion of the lesson, attention focused on the results of the measurement tasks during the lesson and the fact that numerous measurements of the same unit resulted in different values. Laura was the first student to justify the discrepancy, stating this difference was a result of using different measuring units.

All three students initially used their part-whole interpretation to form composite units by first creating the unit whole and then removing parts to model the designated composite unit. As the problem tasks changed to the modeling of improper fractions, complications in modeling these fractions arose. Still depending on the unit whole, Laura converted \( \frac{3}{2} \) to \( 1 \frac{1}{2} \), then modeled the fraction by drawing one whole unit and one \( \frac{1}{2} \)-unit. Judy was unable to determine a modeling strategy for \( \frac{3}{2} \). Melanie, on the other hand, was the first student to use iteration to model a composite fraction. She chose to iterate the \( \frac{1}{2} \)-unit three times to model \( \frac{3}{2} \). Discussion served to expand the thinking of
both Laura and Judy, as evidenced by their ability to model and explain iteration on the subsequent task of modeling $6/5$. As the manipulatives changed from continuous to discrete sets in Task 3, Laura was the only student unable initially to transfer understanding of iteration to the formation of composite units. The group discussion, however, served to support her ability to link the concept of iteration of units to the discrete case. By reconceptualizing the part-whole relationship, all students demonstrated an understanding of the unit fraction as the new unit whole to be iterated the number of times indicated by the numerator.

Model-building for multiplication and division of rational numbers. The paperfolding tasks were successful in enabling students to expand their models of multiplication to include the embedded units model for determining a unit fraction of a unit fraction. Laura and Melanie experienced some initial minor difficulties in folding the strips into thirds or fourths, but soon overcame this obstacle. All students demonstrated an understanding of the embedded-unit structure involved in the decomposition of unit fractions. This is best exemplified by Laura's explanation of her paperfolding efforts to obtain $1/2$ of a $1/3$-unit. She concluded that the answer would be one sixth because "each of the wholes is two and that would be six." By referring to the $1/3$-units as "the wholes," Laura demonstrated her conceptualization that the unit fraction was now the new unit whole. Although her terminology was still weak, Laura's statement revealed further that she not only conceptualized the embedded units
in these situations, she identified the smaller unit in two ways, each dependent on the unit of focus. Early in the lesson, Judy linked the decomposition activities to the operation of multiplication. In finding 1/3 of 1/6-unit, Judy remarked, "there's three thirds in one sixth," then concluded her explanation by writing \( \frac{1}{3} \times \frac{1}{6} = \frac{1}{18} \). Laura and Melanie also verbalized connections of the modeling tasks to multiplication, but there was no way to determine if their realizations were self-made or were the result of Judy's earlier connections. Nonetheless, students' connections to multiplication continued throughout the lesson. When the paperfolding actions were altered to allow folding from two directions, resulting in the formation of a rectangular array, all students connected this model for fraction multiplication to their previous model for whole number multiplication.

Using their decomposition skills, Laura and Judy developed a method to reconstruct the unit whole by focusing on the unit fraction embedded in the composite fraction and then iterating it to form the unit whole. Melanie, however, demonstrated no success in this endeavor. In fact, she became frustrated and refused to listen to the solutions offered by Laura and Judy.

All students experienced difficulty in the development of models of division, primarily because of a change in manipulative. Discrete sets were used to model the majority of measurement situations in whole number division tasks, whereas the manipulative used for rational numbers was the continuous fraction strip model. However, the students developed a sliding method to perform the
measurement tasks and were successful in solving their problems as long as the ruler was less than the unit being measured. When this situation occurred in the last task, students' earlier measurement obstacle for whole number division returned. None of the students were able to determine a strategy for measuring the 1/3-unit with the 2/3-unit during their individual efforts. During the group discussion, however, Melanie established the relationship between the two fraction strips—"one third is half of two thirds"—which then allowed her to model the measurement process. Nevertheless, none of the students were successful in using measurement division to obtain the answer.

**Question 2**

What degree of independence of thinking can students achieve through this instructional experience?

The focus of this question was to determine what unit concept understandings were displayed by students without the support of the peer group environment. Answers for this question were obtained by analyzing students' performance on Tasks 1-5 of the exit interview (Appendix C) in relation to their performance of similar tasks during the teaching experiment.

**Judy**

Judy went through several sequences of partitions, erasures, hesitations, and reflections as she began Task 1. She seemed to realize that each of the existing four segments needed to be reduced, but was unsure as to how this task could be accomplished. On her first attempt, Judy reduced each of the first
three partitions by an equal amount. She paused on the fourth segment, perhaps realizing that this fifth segment, if produced in a similar manner, would be too small. Her second attempt was better, but the five sections were unequal. After a brief pause, she remarked, "I just can't figure out where to put the lines." The researcher suggested that she should look at her fraction strips. Judy pulled out the strips representing four fourths and one fifth, lined them up one beneath the other, then compared the partitionings. Judy ignored the partitionings on the 4/4-strip as she slid the 1/5-unit across the 4/4-strip, marking notches for each 1/5-unit measurement. She refocused attention to the rope on her worksheet. Using both the original (1/4-units) and new (1/5-units) partitionings on her 4/4-strip as a guide, she erased her previous partitionings on the worksheet and successfully repartitioned the rope into 1/5-units. As in the initial interview, Judy demonstrated continued difficulty with the task of partitioning a unit whole into a differing unit from the one inherent in a whole already equipartitioned. Judy's performance with this task indicated that the presence of partitioning was still a powerful distractor to her unitizing abilities.

Judy used fraction circles to measure 4/5 with a 1/5-unit. She formed a circle with the five 1/5-units and traced the circumference on her worksheet. Next she traced four 1/5-units inside the circle, shaded one of these units, and concluded, "there's four of these fifths in four fifths." The researcher asked her to solve the problem 4/5 ÷ 1/5. Judy correctly performed the algorithm, concluding that the answer meant "four wholes." When asked to identify these
wholes, Judy picked up one of the 1/5-units and remarked, "one fifth." Judy recognized the 1/5-unit as the unit whole in this measurement situation.

The second part of this task asked students to consider why the division problem 4/5 + 1/5 is easier than the problem 2/3 ÷ 1/4 and to think about these problems in terms of units. Judy used fraction circles to model 2/3 ÷ 1/4. She traced the 1/3-unit twice to represent 2/3. Next she put the 1/4-unit at one edge of her 2/3 model and traced it. Then she slid the 1/3-unit over and traced another 1/4-unit onto the 2/3-unit. Judy paused briefly, then began erasing her paper. She remarked, "It won't fit." The researcher asked Judy to explain her thinking. Judy responded, "It's not equal." Seeing that Judy seemed to be at an impasse with this model, the researcher suggested that she should look at the fraction strips to help her model the problem. Judy used her sliding method to measure the 2/3-strip with the 1/4-unit, marking notches onto the 2/3-strip for each measure of the 1/4-unit. She stopped, seeming puzzled by the remaining portion which was less than her 1/4-unit. The researcher asked Judy to approximate the size of this remainder. Judy replied, "Maybe two thirds."

The researcher asked Judy to work the problem as she would in math class. Judy quickly performed the algorithm, getting an answer of 2 2/3. Perhaps in recognition that this remainder was identical to her estimate in measuring, Judy smiled. The dialogue continued.

Researcher: "Why is the problem 4/5 ÷ 1/5 easier?"

Judy: "The denominators are the same."
Researcher: "What does this mean in terms of units?"

Judy: "The units are the same."

Researcher: "Why is the problem $2/3 \div 1/4$ not as easy?"

Judy: "The denominators are different, so you have different units."

Researcher: "So what makes $4/5 \div 1/5$ easier?"

Judy: "You only have to look at the top numbers to get your answer."

By performing measurement division on these two different problems, Judy began to view the impact of the unit structure in its facilitation of a solution. She had struggled with the determination of an answer to the measurement task involving differing units, but was then enabled to comprehend that, in problems having the same unit structure, the determination of an answer could be made by focusing only on the numerators and then performing whole number division on these numbers. The researcher asked Judy if she could find two strips to make the fractions in the second problem the same. Judy picked up the $10/12$- and $4/12$-strips and compared them with her original $2/3$- and $1/4$-strips. Deciding they were not equal, she selected instead the $8/12$- and $3/12$-strips, then lined them up beside her original strips.

Researcher: "Can we say that this problem (pointing to the $2/3$- and $1/4$-strips) is the same as this problem (pointing to the $8/12$- and $3/12$-strips)?"

Judy: "Well, you would get the same answer."

Researcher: "What would the answer be?"
Judy: (Used the sliding method to measure the 8/12-strip with the 3/12-strip) "The problem would be 8/12 divided by 3/12. And you would get two--and two thirds."

Researcher: "Why is putting the fractions into twelfths easier?"

Judy: "Because the measuring unit is the same."

Judy was unable to begin Task 3 without assistance, so the researcher asked her to find 1/5 of 15 chips. Judy partitioned off three chips from this set. When asked to find 1/3 of the subset of three, Judy looked puzzled at first, but then pulled out one chip.

Researcher: "How much is this of the whole thing?"

Judy: "One fifteenth."

Researcher: "Can you draw a model that would show 1/3 of 1/5?"

Judy drew a fraction strip and partitioned it into five sections. She pointed to one section and remarked, "This is one fifth." She hesitated, so the researcher added, "Can you take that fraction strip model for one fifth and show me how to find one third of one fifth?" Judy shaded one of the 1/5-units, then partitioned it into three pieces. She concluded by darkly shading one of these three parts. The researcher asked Judy to give two names for this part. Judy responded, "One third of one fifth or one fifteenth of the whole." Since Judy performed this task so well, the researcher extended this task and asked Judy to find 1/3 of 2/5, asking Judy to look at her model and tell how it would be different. Judy remarked immediately, "It would double the shading."
Researcher: "So what would be the answer?"

Judy: "Two fifteenths." (Judy duplicated her partitioning of the first 1/5-unit onto the second 1/5-unit.)

Researcher: "Can you give me two names for the shaded part?"

Judy looked at the six partitions in her two 1/5-units. Since two of these six were now shaded, she concluded, "Two fifteenths [of the whole] and two sixths of two fifths."

These results show that Judy still possessed the decomposition abilities she had demonstrated in the teaching experiment. She even was able to extend this knowledge to decompose a composite fraction. However, in spite of such success, the fact that she initially was unable to determine a solution strategy is evidence of some lingering limitations to this knowledge.

Judy explained the meaning of Task 4 as "trying to get three sevenths from four sevenths" and concluded that the answer was one seventh. She first indicated that the unit in this problem was seven, but changed this to sevenths. The researcher asked Judy to explain the meaning of each fraction in terms of units. Judy indicated that 4/7 was "four out of seven" and 3/7 was "three out of seven." For a similar whole number problem, Judy indicated the problem "four minus three." She stated that the unit in this problem was a whole and distinguished between the units in these two problems by stating that "one is from wholes and one is from parts of wholes."
In solving Task 5, Judy drew 12 dots and circled four of them. She quickly explained: "The whole is 12 and 12 divided by 3 is 4. So there is four in each third." Judy concluded that two thirds would be eight dots. Judy clearly conceptualized the units in this task. Conversely, in the initial interview, Judy considered only the number of dots in determining a unit fraction. For example, in the task of determining 1/2-unit if given that four dots equalled 1/3, Judy responded that three dots would equal 1/2 because "one third is one more than one half," and furthermore she concluded that the unit whole would consist of five dots since "it's more than one third." In this exit interview Judy focused on the existing 1/4-unit, reconstituted the unit whole (12 dots), then unitized into 1/3-units to form the composite unit for 2/3.

Although Judy showed some initial regression in determining a strategy for decomposing a unit fraction, her exit interview results demonstrate an overall endurance of the abilities she had exhibited during the teaching sequence.

Laura

Laura first attempt in solving Task 1 involved the addition of another section to the rope in order to have five parts. Reminded by the researcher that the rope could not be lengthened, Laura thought a moment, then questioned, "Can I draw another one?" She realized that the presence of existing partitions was a factor making the task so difficult. Laura drew another rope, without the partitions, beneath the given one, then correctly separated it into five pieces, adding that she could have changed the original one "if the lines weren’t there."
Laura’s difficulty in this task indicates that the presence of existing partitioning continued to be a strong and persistent distractor in repartitioning tasks.

In Task 2, Laura drew two fraction strips, one with shading to represent four fifths and one for one fifth beneath it. The researcher asked her how this drawing helped her to understand the problem. Laura replied, "there’s four one fifths in that [top drawing]." The researcher showed Laura the problem statement, $4/5 + 1/5$, and asked if her drawing could explain this problem. Laura responded, "How many $4/5$'s, I mean, how many $1/5$'s are in $4/5$'s?"

Researcher: "What would the answer be?"
Laura: "Four."
Researcher: "What does that mean?"
Laura: "Four one fifths."

Laura did not attempt to solve this problem algorithmically to verify her answer of four. She seemed to be confident of her answer. Laura had trouble explaining her thinking, but seemed to realize that, in the problem $2/3 \div 1/4$, the division process would result in a fractional answer. She explained: "Like $2/3$—have to see how many of these [1/4-units] are in there. It like doesn’t come out equal and you have to like half it and it would be--like three I think."

The researcher asked Laura to put the problem in a form to make it easier. Laura looked at the problem on her paper instead of her strips and changed each fraction to twelfths. The researcher asked her to show with the strips what she just did. Laura immediately reached for the twelfth strips, an
indication that she was focusing on the denominator. The researcher asked Laura to compare these new strips, 8/12 and 3/12, with the original ones, 2/3 and 1/4. Laura remarked, "They're equal, but in a different number of parts." The researcher asked Laura to solve the original problem using these new strips. Laura used the sliding method and concluded, "it's two and--two thirds." The conversion of the original fractions into ones with a common unit structure promoted Laura's ability to quantify the relationship necessary for determining a solution.

For Task 3, Laura first picked up her fraction strips for 1/3 and 1/5, tried to use measurement division as on the previous task, but stopped. The researcher asked her to find 1/3 of her 1/5-strip. Laura drew a strip, partitioned it into five equal parts, then partitioned the first part into three equal parts and shaded one of the three parts. She concluded the answer was 1/15. The researcher extended this problem to the problem 1/3 of 2/5. Laura drew another strip, partitioned it into five equal parts, partitioned the first two 1/5-units into three parts each (resulting in six smaller units). Laura shaded two of these six, but was unable to name this new region. In the teaching sequence, the task of finding a fractional value of a composite fraction was confined to discrete sets (chips), so this task involving a continuous set (fraction strip) was a new experience for students. Laura, though unable to quantify the part-whole relationship displayed in her model for the extension task, demonstrated,
nonetheless, her ability to determine a unit fraction of a unit fraction, even without the support of the group environment.

Laura explained the problem in Task 4 in units by stating its meaning as "four 1/7-units take away three 1/7-units." She identified "4 minus 3" as a whole number problem having a similar structure. Laura cited these two problems as being similar in that they "both have one whole unit" for an answer, but she explained further, "Well, seven is not a whole number, but a whole unit."

Laura’s unit terminology was still weak in that she focused on the numeral in the denominator instead of the unit fraction. Nonetheless, it was evident to the researcher that Laura not only was aware of the unit structure existent in each problem, but she distinguished between the unit structure of each.

Laura’s first attempt in Task 5 consisted of a focus on the three dots as the unit whole. She placed three chips on the table, then pulled two chips away from the group to represent two thirds. This indicated to the researcher that Laura still perceived the role of the fraction 2/3 as dividing up a whole instead of specifying an amount. Pointing to the three dots on the worksheet, the researcher asked Laura to show what the whole unit would look like. Laura drew 12 circles arranged to form a 3 x 4 array. The researcher asked her to find 2/3 of the unit, but Laura only looked puzzled. She was also unable to find 1/3 of this unit. The researcher asked Laura for the meaning of one third.

Laura: "One out of three."
Researcher: (Pointing to the unit whole she had drawn) "Can you show me one out of three?"

Laura drew a line to partition the top row away from the other two rows of her drawing. She pointed to the bottom two rows of her drawing to indicate 2/3.

Contrary to her performance in Lesson Fourteen, Laura was unable to perform this task without assistance from the researcher. The precariousness of her reconstitution abilities, as demonstrated in this task, indicated to the researcher that Laura's earlier success on the task was not sufficient proof that she had acquired the ability to reconstitute a unit from a smaller unit.

Melanie

In Task 1, Melanie easily partitioned the rope into five pieces, then paused to look at the sizes of the partitions. Erasing some of the lines, she added, "I'm trying to make sure I'm getting a pretty even estimate. I'm trying to make them all even." She remarked that this task was hard because of "all the other lines and not having a ruler to measure it with to make sure it was right." The researcher showed her the similar task from the initial interview (finding 1/3 of a candy bar split into fourths), in which Melanie obtained one third of the fourths by crossing out one of the fourths and shading one of the three remaining fourths. Melanie remarked, "I could have easily just done this."
immediately repartitioned it into thirds. The presence of the distractors did not hinder Melanie's thinking this time. She not only avoided being distracted by the existing partitioning, she ignored it completely in shading the required fifth.

In Task 2, Melanie drew fraction strips to measure $\frac{4}{5}$ with the $\frac{1}{5}$-unit. She replied, "the answer is four because it takes four one fifths to equal four fifths." The researcher asked her to solve the problem $\frac{4}{5} \div \frac{1}{5}$. Melanie responded: "Hmmm—I don't really know. I'm not good at dividing. I guess—times it—four twenty-fifths?" Melanie looked at her work and frowned. Then she added, "I know it's wrong." She recalculated, getting one fifth. The researcher asked her to explain how she worked it. Melanie added, "No, it's one and one fifth, I guess. I don't know!" The researcher had difficulty understanding how Melanie was obtaining her answers and asked her to explain.

Melanie stated, "I divided [the numerators] together and I got five and I divided [the denominators] together and I got one. But then I thought I would have one left over—I'm not sure. I would probably draw a picture because I do draw pictures normally. I draw circles." Melanie drew five circles and shaded four of them. Beneath them she drew five more circles and shaded one. The researcher remarked that this drawing was similar to Melanie's fraction strip drawing and asked, "What does this drawing tell you?" Melanie responded: "That tells you it takes four of these one fifths. That was my answer."

Melanie remarked that the first problem was easier "because the denominator is the same." She added further, "the parts would be equal."
Melanie did not interpret the researcher's request, to change the form of the fractions so that the problem would be easier to solve, as an indication to change the fractions to equivalent fractions. Instead she demonstrated with the strips how to solve the problem using the measurement division model. Using the 1/4-strip to measure the 2/3 strip by the sliding method, Melanie first concluded the answer would be two and three fourths, but in trying to remeasure while explaining her procedure, she corrected herself and stated the answer was two and "two out of three units." She added, "It takes two and two thirds of these [1/4-units] to equal two thirds of a whole."

Researchers: "Two and two thirds what?"

Melanie "Of fourths. It would take two 1/4-units and two thirds of another 1/4-unit."

Researchers: "Why is the other problem easier?"

Melanie "The little—they have the same little units. So they are easier to measure."

Melanie was unable to perform a symbolic solution for this task, but it was evident to the researcher that Melanie not only understood the concept of unit in measurement division, but now also had an alternative solution method for division of rational numbers.

In Task 3, Melanie selected the strips for 1/3 and 1/5 and tried to use measurement division. The researcher asked if this was the same type of problem as the previous one. Melanie hesitated, then replied: "One third..."
fifth. What am I thinking!" She drew a fraction strip and partitioned it into five sections, then shaded one section to represent 1/5-unit. Next she partitioned this shaded unit into thirds and darkly shaded one of them. She concluded this new region represented "One third of one fifth or one fifteenth of the whole."

As an extension activity, the researcher asked Melanie to find 1/3 of 2/5—an activity performed in the teaching sequence only with discrete sets. This change in problem structure confused Melanie. After a brief hesitation, she wrote these fractions on her worksheet and converted them to common denominators as follows:

\[
\begin{align*}
1/3 & \quad 5/15 \\
2/5 & \quad 6/15 \\
30/15 & = 2
\end{align*}
\]

The researcher asked Melanie if she could draw a picture to model the problem. She replied, "I don't know. This is how I would do it normally if I was in math class." The researcher asked, "What does two fifths look like?" Melanie sketched a fraction strip partitioned into 1/5-units, then shaded two of these units. Melanie's attempts to find 1/3 consisted of the partitioning of only one of these two regions into three parts. The successful modeling of the first task (1/3 of 1/5) did not translate to the new situation—neither in modeling nor in procedural methods.

Researcher: "If this problem was in your math book, how would you find 1/3 of 2/5?"
Melanie "Divide."

Researcher: "Show me what you would do."

Melanie "No! Multiply." (Writes $\frac{1}{3} \times \frac{5}{2} = \frac{5}{6}$.)

Researcher: "Why did you change $\frac{2}{5}$ into $\frac{5}{2}$?"

Melanie (Giggling.) "It needed it."

Researcher: "Is that one of the procedures in multiplication?"

Melanie "I can't remember" (Put her pencil down).

Melanie seemed to have forgotten her previous solution attempt involving multiplication. In this latter solution attempt, she was unable to determine whether she was multiplying or dividing.

For Task 4, Melanie explained the problem in units as "four $\frac{1}{7}$-units minus three $\frac{1}{7}$-units." She indicated that this fractional problem was similar to the whole number problem four minus three because, in each case, she was "subtracting four and three." She identified the units in each problem as $\frac{1}{7}$ for the fractional problem, but as three and four for her whole number problem.

The researcher continued the probe into Melanie's thinking about units on her whole number problem, 4 - 3, by asking what the four meant. Melanie responded, "It's a whole number. It's like four individual wholes."

Melanie began Task 5 by remarking: "First of all, I have to figure it out. It would be . . . ." She placed 12 chips on the table. When asked to explain her thinking, Melanie remarked: "I'm doing this problem so I can separate it in thirds. It would be 12 because 3 times 4 is 12." With this response, Melanie
arranged her chips into a 3 x 4 array. The researcher asked Melanie to explain
the importance of this 12. She replied, "That's my whole." She viewed the task
of finding the unit whole as a necessary first step in determining the answer to
the original problem. This implied her recognition of the importance of
reconstituting the whole. Melanie continued, "Now I can separate them like
this." Next she physically shifted her attention from the four columns each
representing 1/4-unit, to the three rows by widening the space between the rows.
She concluded that this action on her model now "made them into thirds" and
that two thirds would be eight chips.

Melanie repeated her solution process by drawing the chips model on her
worksheet. She explained the arrangements of rows and columns in two
different unit interpretations. Pointing out the columns, Melanie concluded, "this
is how they were in fourths." She shifted attention to the rows and concluded,
"and this is how they were in thirds." Melanie's performance on this
reconstitution task, as well as her explanation of her understanding of the
problem situation, indicate that she perceived the unit as an amount and not
simply as a command for the action of dividing up the whole into parts.

Summary

The presence of existing equipartitioning in Task 1 of the exit interview
continued to be a strong distractor to two students as they endeavored to
repartition the strip into 1/5-units. It is interesting to note that the students who had the best and worst results with a similar task on the initial interviews reversed these roles on the exit interviews. Judy was the only student who successfully performed the repartitioning activity at the outset of the teaching experiment, yet she had the most difficulty of all the students on this task at the conclusion of the teaching experiment. Although she eventually accomplished the task, with help from the researcher, it was evident to the researcher that the presence of the original partitionings served as a strong distractor to Judy's ability to reunitize the unit whole. Melanie, on the other hand, ignored the existing partitionings and easily sectioned the strip into five pieces. The remainder of her concentration involved the chore of trying to "make them all even." Whether or not Melanie realized the necessity of this equality of partitioning or merely acknowledged the directions in the task was not determinable by the researcher. Like Melanie, Laura also was unable to perform the similar task in the initial interview; however, being faced with the distraction of equipartitionings again, she devised a strategy of drawing an identical strip beneath the original one so that she could ignore the existing partitionings. Once the new partitionings were obtained, Laura transferred the partitions to their corresponding positions on the original strip.

Skills in unitizing and norming were examined in Tasks 2 and 3 of the exit interview. All three girls were successful in measuring four fifths with a 1/5-unit, with each one using measurement division in a different way. Judy used
her fraction circles and determined that the answer was "four wholes." She identified the unit whole as the 1/5-unit. Laura, using the fraction strips, indicated her answer signified "four one fifths." Rather than using a manipulative, Melanie chose instead to draw two fraction strips on her worksheet. She concluded that it took four one fifths to equal four fifths. Upon completion of this measurement task, all of the students explained that the division problem 4/5 ÷ 1/5 was easier than the problem 2/3 ÷ 1/4 because the denominators were identical, which, they concluded, indicated that the units were the same. Judy comprehended another way to obtain the answer. With the common unit structure in the first problem, she only had to look at the top numbers to determine the answer: 4 ÷ 1 = 4.

All of the students employed their fraction strips to solve the problem 2/3 ÷ 1/4 using measurement division. Judy easily determined that her ruler "would go into" the 2/3-strip twice and estimated the remainder as 2/3 of the ruler (the 1/4-unit). Melanie's first measurement result ended with the answer of two and three fourths, but she discovered her own error while in the process of explaining her measurement strategy and revised her answer to two and two thirds. Laura measured the 2/3-strip with the 1/4-strip, but concluded the answer would not "come out even." She was unable to quantify the remainder.

The researcher extended this task to a skill not addressed during the teaching sequence by asking students to convert these fractions into a form which would make the problem easier to solve. Laura and Judy used their
common denominator skills to convert the fractions to 8/12 and 3/12 mentally, then modeled measurement division with these new strips and obtained the same answer as with the original strips.

None of the students initially conceptualized a valid solution strategy for finding 1/3 of 1/5-unit (Task 3). Laura and Melanie selected fraction strips for each of the unit fractions in the problem and began a measurement process as in the previous task. Judy made no attempts to solve the problem and was encouraged by the researcher to use 15 chips to aid her thinking. After a successful solution effort with the chips, Judy seemed to recall previous solution methods experienced during the teaching sequence. Without additional guidance from the researcher, all students used the same strategy in their second effort solutions. They partitioned a fraction strip into five parts and shaded a 1/5-unit. Next they partitioned the shaded unit into thirds and highlighted one of these three parts as the solution to the problem. Laura was able to name the new shaded unit only as one fifteenth, whereas both Judy and Melanie named it in two ways: one third of one fifth or one fifteenth of the whole.

As an extension of this task beyond situations experienced during the teaching sequence, the researcher asked students to find 1/3 of 2/5. Decomposition tasks in the teaching sequence were confined to unit fractions. Melanie easily drew a 2/5-unit, but was unable to determine a way of finding 1/3 of 2/5. Rather than considering each 1/5-unit individually, she focused on the 2-block unit and could not envision a strategy for partitioning it into thirds.
This was surprising to the researcher, considering how easily and successfully Melanie performed the repartitioning exercise in Task 1. Laura and Judy drew a 2/5-unit, then partitioned each 1/5-unit into thirds. Both girls shaded two of the six new units to represent the solution to the problem. Laura was unable to name this new unit, but Judy named it in two ways: two fifteenths and as two sixths of two fifths.

All of the students cited 4 - 3 as their example of a whole number problem similar to the rational number problem 4/7 - 3/7. They distinguished between the unit structure of each problem by identifying one as the unit whole for the whole number problem and one seventh or seven as the unit whole for the rational number task. Laura qualified her response of seven by adding, "Well, seven is not a whole number, but a whole unit." Fraction terminology continued as a persistent limitation for Laura.

Reconstitution of the unit whole from a unit fraction was too challenging of a task for any of the students on the initial interview; however, all of them were successful on a similar item in the exit interview. Melanie and Judy quickly conceptualized the unit whole because "3 times 4 is 12." Both girls were aware of the necessity of determining the unit whole before considering the composite unit 2/3. Once the whole was identified, they easily reunitized it into thirds to form the composite unit for 2/3. It was interesting to the researcher to see how easily Melanie solved the problem, considering that she was unable to work similar problems during the teaching sequence only two days prior to this.
interview. Perhaps the group discussions about the similar tasks had aided her
thinking more than Melanie had disclosed during the lessons. For Laura, on the
other hand, solving the task was not an easy matter. She initially focused on the
three dots as the unit whole and took two of the three dots to model two thirds.
Laura reconsidered the situation when asked to identify the whole unit. She
drew 12 circles to form a 3 x 4 array, but was unable to find even 1/3 of this
unit whole without assistance from the researcher.

    Researcher: "What do we mean by one third?"
    Laura: "One out of three."
    Researcher: (Pointed to Laura's unit whole.) "Can you show me one out of three?"

With this assistance, Laura was able to identify one third and then two thirds;
nevertheless, Laura's performance demonstrated that her reconstitution and
reunitizing abilities were limited.

The exit interview was designed not only to measure the effects of the
teaching sequence, but also to account for the social factor inherent in this type
of research. Considering the individual results on the exit interviews, it was
evident to the researcher that participation in the teaching sequence had been
beneficial to the students.
Question 3

To what degree can students’ concept of unit be used to inform their choice of operations and algorithmic performance on routine school word problems?

The 15 lessons of the teaching experiment did not address algorithmic solutions to rational number problem tasks. The researcher felt that students would not provide an intuitive reaction to questions concerning rational numbers if traditional algorithms recently had been reviewed. The discussion of rational number operations was reserved for the exit interview. Tasks 6-8 of this teaching interview presented students with three rational number problem tasks in which they were instructed to model the solution for each task, and then to work the problem as they would in the mathematics class. Questioning by the researcher served to ascertain if students’ understandings of the unit concept could inform their procedural solution methods.

Judy

To solve Task 6 of the exit interview, Judy place 35 chips into seven groups. She explained that she had grouped them in this way "because the denominator is seven and so you get your answer with seven units." At first the researcher was concerned with Judy’s consideration of the unit as seven rather than sevenths; however, further questioning revealed Judy’s knowledge of the unit in her grouping situation.
Researcher: "If you have seven units here, then each one of those units is what part of the unit whole?"

Judy: "One seventh."

The researcher asked Judy to solve the problem as she would in her math class. She wrote $35 + 6/7$, then pause briefly and stated, "that didn't look right." Judy seemed to instinctively believe this problem could be solved by division; however, she also saw that the next step in the division algorithm, inversion of the divisor, would not result in a whole number answer. Judy followed her instinct and began the division process. She stopped after inverting the fraction to $7/6$ and concluded, "that won't work because the answer is 30 people." Judy was confident that her model had produced the correct answer. The researcher asked, "Is it possible that this calculation is right and your model is wrong?" Judy nodded no. This confidence was a contrast with her previous reliance on algorithmic solutions observed during her initial interview!

Researcher: "If it is not division, what else could it be?"

Judy: "Multiply."

Judy then solved the problem using multiplication and obtained the answer of 30 that was dictated by her model. In the initial interview, Judy correctly solve rational number problems but incorrectly created models for the tasks. Looking at the inconsistency of solutions between her procedural solution and her modeling solution during the initial interview, Judy preferred the answer
obtained by her calculation. In this exit interview, however, Judy was confident that her model had indeed produced the correct answer.

Judy selected the fraction strips for $\frac{3}{4}$ and $\frac{1}{8}$ to model the solution for Task 7. Using the $\frac{1}{8}$-unit as a ruler, she measured the $\frac{3}{4}$-strip, then stated her answer was "six." She explained that the $\frac{1}{8}$-unit was the measuring unit and added that the measurement result, six, indicated "that there are six glasses of orange juice." The researcher asked Judy if she knew what operation was represented in her model. She replied, "division." Judy solved the problem using division and obtained the same answer, six, as found in her measurement model.

In solving Task 8, Judy selected the green fraction circles and placed three green-units together to form a whole circle. Searching for the unit in which four fraction pieces equalled the green $\frac{1}{3}$-unit, she selected the tan ($\frac{1}{12}$-unit) and covered the green circle with 12 of these pieces. Judy's need to reconstruct the unit whole was revealed by her process of covering the green circle with the twelve $\frac{1}{12}$-units rather than creating only the $\frac{1}{3}$-unit from an iteration of four $\frac{1}{12}$-units. After covering the unit whole with the $\frac{1}{12}$-units, she removed them all except for one $\frac{1}{12}$-unit. Judy explained that one of the $\frac{1}{3}$-units represented the amount of leftover pizza in the box. She explained that the tan $\frac{1}{12}$-unit represented the amount of pizza Jim ate and added that this represented "one fourth of the one third pizza." Judy sketched a drawing of her pizza model, but only drew the $\frac{1}{3}$ pizza rather than the unit whole pizza.
Next she partitioned this 1/3-unit into four parts, shaded one of them, and concluded that the shaded part represented "one twelfth." Judy correctly calculated the solution to the problem using multiplication. When asked to state two names for the tan piece, using units, Judy stated it was "one fourth of one third and one twelfth of a whole." As Judy named each fractional value, she pointed to the unit being named.

Judy’s procedural skills prior to the teaching sequence were strong; nevertheless, her performance with these rational number tasks demonstrated that her unitizing skills could be used to inform her procedural methods.

Laura

To solve Task 6 of the exit interview, Laura counted out 35 chips and placed them into five groups of size seven. This was accomplished more easily than on her first attempt at Task 5. Laura first attempted to solve the problem by writing the problem statement 35/1 x 7/6, but she stopped and commented, "I thought I was doing the 'flip thing.'" She revealed that she changed her mind because "it didn't look right." Laura’s use of division "did not work" because it did not give the same answer of 30 as in her model. Laura had confidence in her model, but had difficulty with a symbolic solution because of her inability to determine the operation needed to obtain 30. She rewrote the problem statement as 35/1 ÷ 6/7, but then used the multiplication algorithm to obtain her answer of 30. She seemed unaware of whether the needed operation was multiplication or division, perhaps because both operations are involved when
multiplying by a fraction. Nonetheless, Laura demonstrated confidence that her model had produced the correct solution.

Laura used fraction strips for $\frac{3}{4}$ and $\frac{1}{8}$ in Task 7 and counted to see how many $\frac{1}{8}$-units were contained in the $\frac{3}{4}$ strip. She concluded the answer was six. Laura stated that this task required the operation of division. Her symbolic solution, however, began with the problem statement $\frac{3}{4} \times \frac{8}{1}$, which she correctly solved. The propensity of inverting the first number of a multiplication problem, exhibited by Laura on her initial interview, was not exhibited in this task.

Laura quickly modeled Task 8 with her fraction circles. She picked up the green $\frac{1}{3}$-unit, then looked for four smaller units to equal the size of the $\frac{1}{3}$-unit. She placed four tan $\frac{1}{12}$-units on top of the $\frac{1}{3}$-unit to demonstrate the equality. Removing three of them, she concluded that the remaining tan-unit was "$\frac{1}{12}$." There was no hesitation by Laura in solving the problem. She wrote the problem $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$ on her worksheet. In similar problems on the initial interview, the researcher observed Laura's tendencies to first invert the first number in the problem, and then to obtain common denominators before multiplying the fractions. Neither of these strategies were used in Laura's procedural methods to solve this task.

Melanie

Melanie started Task 6 of the exit interview by separating 35 chips into seven groups, then stopped to say: "Ah! I answered the problem. I know what
the answer is. There will be five people that aren't going so 30 people will go. I'll make my little manipulative thing to show you how I answered the problem because I thought about it a minute and said, 'Hey, that's the answer.' Melanie arranged the chips into a 5 x 7 array consisting of six columns of red chips and one column of yellow chips. She explained that she made seven groups "because the denominator is seven and it tells how many groups there should be." When asked to explain the role of the numerator, Melanie replied: "That means how many groups are going. So six groups out of seven are going." She pointed to the last column, the yellow column, and concluded, "these five people are left."

The uncertainty of choice of operation revealed by Laura and Judy was also exhibited by Melanie. In explaining how she would solve the problem in her mathematics class, Melanie replied: "First of all, knowing me, I would probably get it wrong, but I would put 35/1 and then I would divide it—or times it—divide it—by six sevenths I guess. That wouldn't go evenly. Oh, wait! I could cross cancel." She cancelled seven into 35, getting five. Next she multiplied five and six to get her answer of 30. The researcher continued the questioning to see if Melanie knew what operation she had performed.

Researcher: "So is this a multiplication problem or a division problem?

Melanie "It's division, but I multiplied it. See it starts off as a division problem, but when you cross cancel, it turns into a multiplication problem."

Researcher: "Can you relate your model to what you did here?"

Melanie "I divided 7 into 35 and I got 5."
Researcher: "What did that tell you to do with your chips?"

Melanie "Make it in groups of seven--I mean, make it in groups of five. Make five in each group of seven."

Researcher: "And then here in your solution you said multiply 5 times 6 to get 30. Does that relate to anything on your model?"

Melanie (Pointing to her six red columns) "That's 30. My red ones are the people that are going and there are 30 red ones."

Researcher: "So it looks like your model helps to explain what you did here in your solution. The only question remaining is to determine what operation you are doing on this problem. Is there anything about this problem that tells you it must be a division problem?"

Melanie "I don't know. It's the 35. You can look at the 35 and just tell it has to be division. It's just saying, 'Divide, divide, divide.' It's just screaming, 'Divide me!'

Researcher: "Is there any other clue in this problem that indicates division other than the 35?"

Melanie "Just looking at the numbers together. Multiplying it wouldn't look right because you could put 35/1 and 6/7 and then you would times 35 and 6 and--you'd get 210/7. And then--you'd get--(laughs)--the same answer. But it takes longer."

Researcher: "We would get the same answer no matter if we multiply or divide?"
Melanie "Depends on the problem. Depends on if the problem goes in together."

Researcher: "What do you mean by 'goes in together'?"

Melanie "Like the 7 goes into 35 perfectly."

Researcher: "Oh, is that the clue that makes you think it is division?"

Melanie (nods yes.)

Researcher: "So it's the numbers that tell you what operation to use?"

Melanie "Yes."

Melanie was unable to determine a modeling strategy for Task 7 and chose instead to calculate the answer using multiplication. She converted the 3/4 to 6/8 in order to acquire common denominators. Next she multiplied the numerators and brought down the common denominator. Melanie's final answer was obtained by reducing 6/8 to 3/4. She admitted that she used the operation of multiplication because "eight won't go into three." The numbers in the problem directed her choice of operation. Melanie explained that her answer, 3/4, represented "three 1/4-units." She interpreted this to mean "three out of four glasses" and added further, "so you would be able to serve three servings and there would be a little left over." At this point, Melanie seemed to be rambling. Further questioning only resulted in more confused statements.

Melanie "Three servings of four."

Researcher: "Why?"

Melanie "Three 1/4-units--so they would be able to have half of that."
Researcher: "Half of what?"

Melanie: "Half of the 1/8-size glass."

To solve Task 8, Melanie placed a green 1/3-unit on the table and remarked that she now needed to "separate this into four equal parts." She searched for four equal units to fit on top of the 1/3-unit and determined the tan-unit would fit. The researcher asked Melanie how much of the whole pizza was eaten. She counted the number of tan pieces on the 1/3-unit, then counted the remaining tan pieces in her pile of fraction circles. Melanie concluded that Jim had eaten "1/12 of the whole pizza." Although Melanie had counted all the tan-units to determine her answer, she explained that it was better to "times four by three" to determine that 1/12 pizza was eaten. Her procedural solution consisted of the problem statement 1/3 x 1/4 = 1/12. Unlike the previous task, Melanie did not exhibit confusion between the choice of multiplication and division.

Summary

In Task 6, all of the students separated 35 chips into seven groups, removed one of the groups to model 6/7 of 35 students, then concluded that the answer was 30. There was no hesitation by any of the students in the creation of an appropriate model for this problem. However, all of them faltered when deciding on the operation needed to solve the task. All students initially considered division to be the needed operation. Judy followed her initial instinct and began her solution using the division algorithm. Realizing, however, that
this process would not yield the same answer of 30 that was dictated by her model, she abandoned this procedure and chose instead to multiply. This same confidence in models was exhibited by Laura and Melanie. Both students were sure the answer was 30 people, but debated between multiplication and division as the needed operation. Each student wrote the problem statement as a division, but used the multiplication algorithm to obtain the answer of 30. Perhaps the confusion in a choice between these operations involves the fact that both operations are utilized when multiplying by a fraction. An indication of this is Melanie's explanation of her solution process: "It's division, but I multiplied it. See it starts off as a division problem, but when you cross cancel, it turns into a multiplication problem."

Both Judy and Laura used measurement division to successfully solve Task 7. Prior to their procedural attempts, they identified division as the indicated operation. Their subsequent procedural solutions, utilizing division, were successful in producing the same answer of six that was displayed in their models. For Melanie, however, the situation was different. Being unable to produce a model for the task, she resorted instead to her procedural methods. In choosing an operation, Melanie eliminated the possibility of division because "eight won't go into three," and chose instead to multiply. Melanie exhibited faulty procedural strategies. She obtained a common denominator for 3/4 and 1/8, then multiplied the numerators while keeping the same denominator. Although Melanie explained her answer as "three 1/4-units," she seemed to have
forgotten the context of the original problem and of the significance of the 1/8-unit glasses. She interpreted her answer to mean "three out of four glasses," which indicated to her that there would only be "three servings . . . [with] a little left over."

All three students, using a similar procedure of modeling the pizza with their fraction circles, successfully solved Task 8 with the aid of their models. Their procedural solution method consisted of multiplication, which produced the same answer decreed by their models. It was interesting that neither Melanie nor Laura displayed the multiplication errors displayed in their initial interviews.

None of the students could model rational number problems at the outset of the teaching experiment. Without the aid of models, they were forced to rely solely on their procedural skills to determine a solution. The teaching sequence was successful in developing students' concept of unit to enable their construction of models representing both whole number and rational number problem situations. For a student like Judy, who possessed strong procedural skills but limited choice of operations skills, the ability to construct a model to represent a problem situation was the enabling factor that helped her determine the correct operation for solving each problem. In instances where students' procedural skills were faulty, successful modeling of the unit structure of the problem did not enable students to determine the correct operation to solve the problem. Nevertheless, with the ability to model problem situations, both Laura and Melanie, who exhibited numerous instances of misapplied rules for
multiplication and division, now possessed an alternative solution method. Students were able to solve the problems through their modeling efforts by focusing on the unit structure within the problem situation.

Chapter 4 presented the results from the initial interviews, the 15 lessons within the teaching sequence, and the exit interviews, and an analysis of these results in reference to the three research questions. The comprehensive analysis and pedagogical implications will be discussed in Chapter 5.
CHAPTER 5
SUMMARY AND CONCLUSIONS

This chapter presents the conclusions emerging from this study. It begins with a summary of the study, describing the problem, the goals of the research, and the methods used to attain these goals. The second section identifies limitations to this study. A brief overview of the results from the initial interviews, the teaching sequence, and the exit interviews comprises the third section. Then the general conclusions derived from the results of the teaching experiment are discussed. The pedagogical implications and the implications for future research are addressed in the concluding section.

Summary

Many people fail to fully understand mathematics, particularly in the domain of rational numbers. It is thus not surprising to find in the literature that the concept of fraction is problematic for both children and adults. Results from a recent NAEP indicated that the difficulty with learning fractions is caused in large part by the lack of a conceptual base (Lindquist, 1989). Numerous studies have shown that even in instances where the symbolism and algorithms are reasonably mastered, the underlying concepts are often absent (Behr, Harel, Post, & Lesh, 1992; Schwartz, 1988; Wearne & Hiebert, 1988). That is, students learn fraction algorithms without attaching any significance to either the process or the result of this process. To address these problems, many researchers have examined the unit as a means for providing a strong conceptual base for understanding rational numbers (Behr, Harel, Post, & Lesh, 1992, 1993, 1994; Lamon, 1994). An interest in developing a curriculum for rational number development, focusing on the unit concept, has led to this study.
A five week teaching experiment was designed to provide situations to explore seventh grade students' development of the concept of unit and to examine the unit as a connector between whole number and rational number domains. Advice from the classroom teacher was used to select four students, with varying mathematical abilities (one strong, two average, and one weak), from each of two seventh grade classes to participate in this study. Additional criteria taken into consideration in the selection of students were the attendance habits of each student and students' willingness to verbalize thoughts. Heterogeneous grouping allowed each group to better simulate an ordinary classroom in terms of mathematical ability. The original groups each consisted of three females and one male; however, both male students withdrew from the study, leaving only three participants per group. Chronic absences by participants in one group, resulting from conflicts with school and community activities, had adverse effects on the study. Due to their inconsistent participation in the study, data from this group is not reported or analyzed in this report. The researcher felt the discontinuity of progress attained by these participants would not lend itself to a viable analysis of the effects of this study on the acquisition of unit concepts.

Individual initial interviews were conducted during the first week of the teaching experiment in order to ascertain students' existing fraction concepts and abilities. Students participated in fifteen lessons during the experiment, each designed to illustrate various aspects of the unit concept such as unitizing, reunitizing, and norming. The lessons spanned four phases of instruction: (a) unitizing in whole number situations, (b) model-building for multiplication and division of whole numbers, (c) unitizing in rational number situations, and (d) model-building for multiplication and division of rational numbers. Analysis of
the unit as a connector between the whole number and rational number domains involved examination of students' procedures and responses during the fourth phase of lessons, which focused on the development of models for multiplication and division, and students' responses on the exit interview. Additionally, the exit interviews during the last week of the study sought to determine the degree of independence of thinking students achieved through this instructional experience and the degree to which the concept of unit could inform students' procedural methods of the usual school curriculum.

Various means were employed to obtain reliable and accurate data from the teaching experiment. One set of data consisted of results from the initial and exit interviews. The remaining data were obtained during the teaching sequence itself. All lessons were videotaped and audiotaped, providing a source of backup in case of glitches in the videotaping. These videos were viewed after each lesson to assess the teaching sessions and to determine needed modifications in subsequent lessons. Additionally, these videos provided the opportunity to record students' gestures and behaviors that perhaps were unnoticed by the researcher during the teaching sequence. There were also various sources of written data, including students daily worksheets and the researcher's journal, which was comprised of personal observations, reflections, questions, and ideas. The analysis consisted first of highlighting the students' reactions and the procedures they used, both individually and in groups, in solving unitizing and modeling tasks. Next these reactions and procedures were examined more closely through the development of a coding and sorting system, similar to one used by Golding (1994), in order to determine their significance in students' understanding of the unit concept.
Limitations

The adoption of a qualitative methodology in the study of students' construction of unit concepts inevitably raises concerns for reliability and generalizability of results. Some researchers contend that, with the lengthy period of time needed to conduct the teaching sequence, it is difficult to distinguish effects due to social influences from those resulting from the teaching sequence (Kantowski, 1978). Confining the experiment to a five-week period greatly minimizes this environmental factor. As evidenced in the literature, the concept of unit does not develop naturally in a short period time (Behr, Harel, Post, & Lesh, 1992; Boulet, 1995). Since lessons on rational numbers were not taught by the regular classroom teacher during the time of this study, the results obtained most likely ensued from the teaching sequence and not from other factors.

Another troublesome characteristic of the teaching experiment involves the researcher/teacher's liberty to modify the interview plan and to prod the subjects if judged necessary. Although these circumstances would be a cause for the invalidity of the data in other types of research, this flexibility is necessary in this type of research in order to probe students' thought processes and to discover new insights into their learning of unit concepts (Steffe & Thompson, 1996). Given that the goal of this research is not to determine what the student knows about unit concepts but how he or she constructs unit concepts within the social context of small groups, this flexibility is an asset instead of a drawback (Steffe & Thompson, 1996).

A limitation specific to this teaching experiment concerns the deviation from the original experimental design of two groups of four students to one group of three students. This change of plan raises concerns of generalizability.
To the contrary, some researchers argue that it doesn't make sense to demand of teaching experiments that they generalize since the ultimate goal of a this type of study is the development of explanatory models for students' conceptual schemes (Steffe & Thompson, 1996). The critical issue in the teaching experiment is whether or not the results obtained are useful in explaining the thinking of students doing mathematics. The generalizability of results is always tenuous in a qualitative study, but the value of the study usually lies in its ability to raise issues and concerns that can stimulate further research and practice rather than to reveal absolute truths (Steffe & Thompson, 1996).

Results

The following sections identify specific results from the initial interviews, the teaching sequence, and the exit interviews. Assessment questions and lesson tasks, along with a sampling of student responses, are contained in Appendices A, B, and C.

Individual Interviews

The initial interview assessed students' understanding relative to unit fraction concepts such as fraction interpretations; ability to reunitize, equipartition and reconstitute the unit whole; ability to solve and model multiplication and division tasks; procedural skills; and choice of operations tasks. Only one of the students demonstrated proficiency in procedural skills; however, results of the choice of operations task indicate that all of the students had limited models for multiplication and division. Results indicate that the students' concept of fraction involved a part-whole interpretation, yet none of them acknowledged the importance of the equality of the parts in such a relation. Only one student was able to equipartition a discrete set, whereas none of them were able to repartition, without assistance, a continuous whole
with existing equipartitioning. Additionally, none of the students could reconstitute the unit whole when given a unit fraction. All of the students were able to model simple whole number multiplication and division problems, but none of them could model rational number situations. Clearly these results indicate that the students began the teaching experiment with fragmented unit concepts and limited rational number skills.

**Teaching Sequence**

**Unitizing in Whole Number Situations**

Students demonstrated varying awareness of the concept of unit with their formation of units in the early lessons of the teaching sequence. While not all the students used unit formations on the counting tasks in Lesson One, the group discussion served to highlight their awareness of grouping in that all students indicated types of counting situations which would be facilitated by the formation of units. Students alternated between units of one and the formation of composite units in Lessons Two and Three. Although many of the whole number problems were solved using the traditional units-of-one approach, the second effort solutions revealed that students were capable of applying unit formations to the problems. In some instances, however, the traditional approach was hard for students to overcome. For example, Melanie insisted that it was "shorter and simpler" to count by ones. On some tasks which asked for the number of specific groups in a set, Melanie successfully formed groups to aid her count, but then multiplied by the number of items per group to obtain the total individual items.
Lessons Four and Five focused on the development of models of division for whole numbers. The students were successful in determining models for partitive and measurement division tasks without instruction. Additionally, all students verbalized the role of grouping in the two types of division situations. The development of models of division was not without problems. Students exhibited weaknesses in ability to equipartition unit wholes, which served as a handicap in their development of division models. A major obstacle surfaced during these lessons—all students experienced some instances of loss of sight of the measuring unit. In seeking to establish a relationship between two units in each measurement task, students would sometimes exchange the ruler with the object being measured, resulting in an answer which was the inverse of the requested answer.

Models for multiplication of whole numbers were expanded in Lessons Six through Eight. A major recurring obstacle in these lessons involved students' limited measurement concepts. The determination of an appropriate measuring unit was a difficult task for most of the students. For example, in determining the length of the sides of a rectangle, some students could not determine whether to count points, or segments, or squares. This was problematic in two ways: (a) students who counted the points displayed the tendency to begin their count with the number one (instead of zero) at the corner of each side of the rectangle, resulting in measurement discrepancies, and (b) those students who did use only the exterior squares to calculate length and width did not want to count the corner square twice. As the problems focused on area and perimeter tasks, another measurement problem surfaced:
some students counted all the squares inside, giving them a number for area rather than perimeter. Melanie was the only student unable to resolve her confusion between area and perimeter measurement.

Reunitizing activities were designed to help students focus on the change of unit structure occurring in many multiplicative situations. These activities involved the reunitizing of the sides of a rectangle into 10-units, then 5-units, with an analysis of the change in unit structure occurring within the rectangle as a result of each reunitizing action. Although all of the students were able to state the dimensions of the regrouped sides, none of them initially were able to determine the size of the interior blocks (square units) formed by their reunitizing actions. Based on the inconsistency of answers by the students, it was evident that this concept of change in unit structure in multiplicative situations was not firmly rooted in students' minds; however, all students were able to verbalize the multiplication process as a type of grouping to determine the number of "rulers" or units contained in the large rectangles.

Unitizing in Rational Number Situations

Lesson Nine facilitated extension of students' existing knowledge of fractions to include awareness of the unit in fractional situations. Students experienced little difficulty in establishing a relationship between units as long as the fractional unit was contained in the unit whole an equal number of times. When this condition was violated, students' unconcern with equality of parts in an equipartitioning resurfaced. For example, in measuring the blue-unit with the orange-unit as the unit whole, it was discovered by the students that the blue-unit was not contained in the orange unit an equal number of times. This led students to consider using other fractional units to cover the
orange-unit. Discovering that "two blues and one yellow" would fill the orange-unit, Laura concluded that "blue is one third" and was supported in this conclusion by Melanie. Other unitizing activities fostered students' abilities to form composite units by the iteration of the unit fraction. By reconceptualizing the part-whole relationship, all students demonstrated an understanding of the unit fraction as the new unit whole to be iterated the number of times indicated by the numerator.

Model-Building in Rational Number Situations

Students expanded their models of multiplication to include the embedded units model for determining a unit fraction of a unit fraction. Early in these lessons, Judy linked the decomposition activities to the operation of multiplication. This enabled the other students to verbalize connections of the modeling tasks to multiplication of rational numbers. When the paperfolding actions were altered to allow folding from two directions, resulting in the formation of a rectangular array, all students connected this model for fraction multiplication to their previous model for whole number multiplication. Using their decomposition skills, Laura and Judy developed a method to reconstruct the unit whole by focusing on the unit fraction embedded in the composite fraction and then iterating it to form the unit whole. Melanie, however, experienced no success in this endeavor.

All students experienced initial difficulty in the development of models of division, but were successful in developing a sliding method to perform the measurement tasks. They were also successful in solving their measurement tasks as long as the ruler was less than the unit being measured. None of the students were able to determine a strategy for measuring the 1/3-unit with the
2/3-unit, even though Melanie established the relationship that "one third is half of two thirds."

**Exit Interviews**

The exit interview was designed to measure students' abilities to perform unit concept tasks without the support of the peer group, as well as their abilities to use unit concepts to inform their choice of operations and algorithmic performance on routine school word problems. The presence of existing equipartitioning continued to be a strong distractor to two of the students as they endeavored to repartition the strip into 1/5-units. It is interesting that the students who had the best and worst results with a similar task on the initial interview reversed these roles on the exit interview. In contrast to her unsuccessful attempt on the initial interview, Melanie completely ignored the existing partitionings and easily sectioned the strip into five pieces.

Students demonstrated their unitizing and norming skills in modeling tasks. All of the students solved the problem 2/3 + 1/4 using measurement division and all except Laura were able to quantify the result. None of the students initially conceptualized a valid solution strategy for finding 1/3 of 1/5-unit, but they determined a successful second-effort strategy and quantified the result. The researcher extended this task beyond those situations experienced during the teaching sequence by asking them to find 1/3 of 2/5. Laura and Judy were able to use their knowledge from the easier task to solve the harder task and were able to quantify the results.

Students linked the structure of the problem 4/7 - 3/7 to the whole number task 4 - 3. They distinguished between the unit structure of each problem by identifying one (or a whole) as the unit whole for the whole
number problem and one seventh as the unit whole for the rational number task. As in the initial interview, fraction terminology continued as a weakness for two of the students.

Reconstitution of the unit whole from a unit fraction was too challenging of a task for any of the students on the initial interview; however, all of them were successful on a similar item in the exit interview. Such an ability indicates that the student perceives the unit fraction as an amount and not simply as a command for the action of dividing up the whole into parts.

None of the students could model rational number problems at the outset of the teaching experiment. Without the aid of models, they were forced to rely solely on their procedural skills to determine a solution. The teaching sequence was successful in developing students' concept of unit to enable their construction of models representing both whole number and rational number problem situations. Despite the fact that their models were correct, students' initial choice of operations was often incorrect. For a student like Judy, who possessed strong procedural skills but limited choice of operations skills, the ability to construct a model to represent a problem situation was the enabling factor that helped her determine the correct operation for solving each problem. When neither correct procedural skills nor successful selection of operations was possible, having the correct model did not always enable students to critique their own shortcomings. Nevertheless, with the ability to model problem situations, students now possessed an alternative solution method. Students were able to solve the problems through their modeling efforts by focusing on the unit structure within the problem situation. These results indicate that participation in the teaching sequence had been beneficial to the students.
Conclusions

The educational decision about whether to teach for conceptual understanding or for procedural skill is a values question that cannot be determined empirically. Certainly previous data concerning the greater utility of meaningful mathematics for everyday applications, further academic success, and workplace competence argue persuasively for a meaningful mathematics curriculum. But neither can the expedience of procedural drill and practice and its efficacy for short term test performance be denied. Despite the clear preference of the NCTM (1989) for meaningful instruction, the choice of orientations remains problematic for many teachers, students, parents, administrators, and public officials.

Once a choice has been made for conceptual understanding of mathematics, it is necessary to have clear models of conceptual competence, as well as effective strategies for conceptual change, to move students toward competence. In the area of rational number concepts, past research has identified a flexible understanding of the unit as central to conceptual competence (Behr, Harel, Post, & Lesh, 1994). Additionally, a variety of strategies for conceptual change have been advocated, and in some cases tested for students (Boulet, 1995; Lamon, 1994; Mack, 1990, 1995) and for teachers (Post, Harel, Behr, & Lesh, 1988; Golding, 1994).

This study contributes to the conceptual change research on rational numbers by providing a micro-analysis of four students' conceptual development in a five week teaching experiment. For the most part, the strategies employed in teaching rational numbers were derived from the existing literature on conceptual instruction in rational numbers; though elements of the organization and sequencing of instruction were unique to this
study. This research, therefore, contributes to our knowledge of what learning is possible in the relatively ideal circumstances of a small student group, a knowledgeable teacher, and an adequate supply of learning materials. Mediating between these ideal circumstances and the usual circumstances of school mathematics instruction remains for future research initiatives.

But this research project has not been entirely negligent of the debate over a conceptual versus a procedural orientation to instruction. Question 1 seeks to understand students' conceptual development within the microcosm of the learning environment. This means that students' tasks are framed in terms of the material resources (manipulatives) available and accessible to the students in their class. It also means that the conceptual resources of the whole group were available for solving the given problems. To better understand students' development in the terms of usual instructional debates, the exit interview provided an opportunity to investigate question 2, what conceptual competencies can the student display independent of the peer group support system, and question 3, can concepts of unit inform students' performance on traditional word problems. Therefore, this study addressed the development of unit concepts over ideal instructional circumstances and attempted to relate the results to the terms of usual instructional debate, but did not mediate between these ideal circumstances and what is available in more usual teaching circumstances.

Analysis of the students' initial interviews revealed that the students began the teaching sequence with limited concepts of unit and rational numbers. These findings are consistent with those from previous research studies which also identified students' weaknesses in both rational number concepts (Behr, Harel, Post, & Lesh, 1992; Hart, 1981; Lamon, 1994; Lindquist,
1989) and procedural skills (Carpenter, 1986; Wearne & Hiebert, 1988). Furthermore, research indicates that students' rational number understandings are limited because of the traditional curriculum (Behr, Wachsmuth, Post, & Lesh, 1984; Nesher, 1987; Tirosh & Graeber, 1990). Some researchers contend that students' poor performance with rational numbers is attributable to their "failure to internalize a workable concept of rational number" (Behr, Wachsmuth, Post, & Lesh, 1984, p. 323). In an effort to address such concerns, this study focused on the students' development of the concept of unit and their abilities to extend these concepts from whole number to rational number domains.

Through analysis of the results from this teaching experiment, it is evident that students' concept of unit was enriched by participating in the teaching sequence. Several conclusions may be drawn from the research findings in this study.

1. Students developed a flexible concept of unit.

The sequence of tasks involving representational understanding of rational numbers was successful in extending students' existing knowledge of fractions to include awareness of the unit in fractional situations. For example, using the fraction circles, students determined that the green-unit was 1/3 black-unit, but that it also could be represented in other ways such as 2/3 orange-unit or 2 red-units. Such notions have been identified by rational number researchers for their importance in helping students develop a flexible concept of unit (Behr, Harel, Post, & Lesh, 1994). Although the students began the teaching experiment without awareness of the necessity of equality of parts in a part-whole relationship, the tasks in the teaching sequence allowed all but one student to overcome this obstacle. As a result of students'
increased awareness of the unit in fractional situations, their flexibility of units enabled them to extend their part-whole interpretation of rational numbers to envision composite fractions as iterations of unit fractions and to develop a model for unit reconstruction tasks.

2. Modeling activities provided continuity between conceptual domains.

Results from the initial interviews reveal that students had only limited models for multiplication and division involving whole numbers and no models for these same operations involving rational numbers. While there remained unresolved obstacles associated with the development of these models (e.g., the change in unit structure in multiplicative situations, and measurement division involving a divisor greater than the dividend), the teaching sequence was successful, by focusing on the unit, in enabling students to expand models for multiplication and division of whole numbers and, furthermore, to extend these models to the rational number domain. The use of a variety of manipulatives—fraction circles and squares, fraction strips, chips, and paper strips for folding exercises—served to broaden students understandings of unit concepts and to facilitate linkage of concepts from whole number to rational number domains. For example, paperfolding activities served to expand students' models to include the embedded units model for decomposition tasks. By focusing on the embedded unit structure, students were able to recognize the problems modeled as multiplication tasks. Judy was the first to remark that the task of finding 1/3 of 1/4 would result in 1/12 because "there's three [1/3-units] in one fourth and there's four fourths." As the paperfolding tasks progressed to include folding from two directions, students were able to relate this model for fraction multiplication with their area model for whole number multiplication. Results from this study support those from previous studies.
which indicate that it is possible to productively engage students in rational number constructs through the use of manipulatives.

3. Unitizing skills endure and are extendable.

Results from the exit interview provide evidence that individual students do internalize unit concepts and rational number concepts acquired through the group learning experience. Although not all of the participants succeeded on all of the items in both the teaching sequence and exit interview, the results indicate that the knowledge students acquired in unitizing and norming is not only strong, it is extendable or generative in that some of them could even use this knowledge to solve more complex problems than were presented during the instructional sequence (e.g., finding 1/3 of 2/5 instead of just 1/3 of 1/5). Results from the initial interviews indicate that the ability to equipartition an already equipartitioned whole is far from intuitive. However, all of the students were successful on a similar task in the exit interview, even though the lessons during the teaching sequence did not specifically address this skill. This result supports the claims of some researchers (Lamon, 1994; Mack, 1990) as to the power of the unit concept in providing a foundation for rational number understanding. Likewise, the reconstitution of the unit whole from a unit fraction was too challenging of a task for any of the students on the initial interview; yet all of them were successful on a similar item in the exit interview. Such an ability indicates that the student perceives the unit as an amount and not simply as a command for the action of dividing up the whole into parts, and indicates further a maturing of the concept of unit (Behr & Post, 1992).

4. Models can inform procedural methods and/or provide alternative solution method.
Results from the initial interview documented that none of the students could model rational number problems at the outset of the teaching experiment. They relied solely on their procedural skills to determine a solution. The teaching sequence was successful in enabling students to develop models for multiplication and division for both whole number and rational number situations. While the lessons in the teaching sequence did not address computational skills, specific questions in the exit interview were designed to examine students' abilities to use their knowledge of unit concepts to inform their procedural solution efforts. All students were able to correctly model all three tasks, yet their initial choice of operations for a procedural solution was often incorrect. The students consistently displayed confidence in the answer obtained through their modeling efforts and attempted to determine the procedural solution which would yield the same answer.

Judy possessed strong procedural skills but limited choice of operations skills. The ability to construct a model to represent the problem situation enabled her to consistently determine the correct operation for solving each problem. Because of her confidence in her model, she would continually revise her procedural methods until she obtained the answer dictated by her model. This was in contrast to Judy's modeling efforts on the initial interview. At the outset of the teaching experiment, Judy demonstrated strong procedural skills but no models for rational number problem solving. When asked to choose which solution method had produced the correct answer on the initial interview—the procedural solution or the model—Judy relied on her procedural methods. However, in the exit interview, Judy relied on her models to help her determine the correct choice of operations.
For Laura and Melanie, the case was different. Although both girls developed models for multiplication and division of rational numbers on the exit interview, neither possessed strong procedural skills. When neither correct procedural skills nor successful selection of operations was possible, having the correct model did not always enable students to solve the task procedurally. Nevertheless, the ability to model problem situations served to provide students with an alternative solution method. By focusing on the unit structure within the problem situation, students were able to solve the problems through their modeling efforts. These results from the exit interviews indicate that, once acquired, students can use their models successfully either to inform their procedural methods or to solve a problem without the need for a procedural calculation. Such results indicate that the development of models for multiplication and division enhances their problem solving abilities.

Despite these positive features identified by this study, it is clear that not all students' conflicts were successfully negotiated. The following conclusions reflect remaining difficulties that require further attention.

5. Equipartitioning remained a persistent difficulty.

The initial interviews detected students' unconcern with equality in partitioning. Students' limited concepts about equipartitioning in the part-whole relationship were hampered further by their initial weak skills in partitioning. The lack of awareness of the necessity of equality in partitioning continued to impact unit formation abilities of two students during the study. As unitizing activities progressed to the rational number domain, this obstacle resurfaced. Problems in which the given fractional unit was not contained in the unit whole an equal number of times necessitated students' consideration of another measuring unit to aid in establishing a relationship between the two
271

units. For example, in measuring the blue-unit with the orange-unit, students
discovered that two blue-units and one yellow-unit equalled the orange-unit
whole. Students' lack of awareness of the importance of equality of parts in a
part-whole relationship caused them to disregard the size of the parts in the
unit whole and to focus solely on the number of parts, concluding that the
blue-unit equalled 1/3 orange-unit. Even though this problem was overcome by
all but one of the students, it persisted far longer than anticipated.

6. A sustained focus on the measurement unit is difficult to achieve.

A persistently recurring problem emerging in the process of expanding
students' models for multiplication and division concerned students' loss of
sight of the measuring unit. This problem first occurred in the development of
division models for whole numbers as students were seeking to establish a
relationship between two units in a measurement task. Students sometimes
refocused attention from the measuring unit (ruler) in the measurement task to
the object of measurement. Only with a concerted effort of focusing on the
measuring unit were students able to restrict the interference of this obstacle.
Analysis of the situations in which students were unable to overcome loss of
sight of the unit revealed a commonality in problem structure—these problems
all contained rulers longer than the rods being measured.

All students experienced initial difficulty in the development of models
of division of rational numbers; however, they were successful in developing a
sliding method to perform the measurement tasks and were successful in
solving problems as long as the ruler was less than the unit being measured. In
such situations, none of the students were able to determine a successful
strategy, but instead would either exchange the units in the problem situation
or make no efforts to solve the task. This tendency of the students to invert
the relationship between the units illustrates a constraint of primitive models of division (Fischbein et al., 1985): the divisor must be less than the dividend. While the continued emphasis during the teaching sequence on making students aware of the units in each problem task helped to minimize the influence of this obstacle, the teaching sequence was not successful in enabling students to overcome this obstacle in situations where the ruler was longer than the object being measured. More research is needed to determine strategies for helping students overcome this obstacle.

Loss of sight of the measuring unit also occurred in the development of models for multiplication of whole numbers. All students experienced initial confusion about the different units comprising a multiplicative situation—linear units and area units. Perhaps this confusion arose because the change in unit structure in multiplication is not so apparent when considering the situation in terms of units of one. The modeling tasks progressed to the reunitizing of the sides of a rectangle into 10-units and 5-units in order to highlight the change in unit structure, but were unsuccessful in enabling students to resolve their confusion. Students' instinctive response was that the area unit should be the same as the units for the sides. Such a change in unit structure was too challenging of a concept to tackle while these students were still struggling to distinguish between the measurement tasks of perimeter and area.

7. Model-building is hampered by limited measurement concepts.

The development of students' models for multiplication and division was hampered by their limited measurement concepts. This obstacle was problematic in two ways. First, the students experienced difficulty in selecting the appropriate measurement unit for linear or area measurement tasks. In area tasks, the students debated between counting the segments or squares,
whereas in perimeter tasks, students' unit choices included squares, segments, or the points along the sides. A second problem resulted from the selection of an inappropriate unit for linear tasks. When students used the points or dots along the sides to determine the lengths of the sides, many would begin their count with the number one at the first dot. In instances where students chose square units to determine the lengths of the sides of a rectangle, they did not want to count the corner square twice. In determining the cardinality of a set, students learn that they have to count every object in the set, yet cannot count the same object twice. Consequently, we have two examples of how number measurement can be an obstacle in understanding fractions.

In addition to having limited concepts of measure, students also demonstrated weak measurement skills. For example, impreciseness in measurement was obtained by students' failure to butt the edge of the ruler against the mark where it had last been placed, which caused a gap between placements of the ruler. In contrast, students sometimes overlapped the edge of the ruler with the markings from the previous measure, resulting in an under-measurement.

Implications for Teaching and Research

Although the results of the study do not provide answers to all questions concerning the development of the unit concept, they definitely provide clues to the types of experiences students need in order to construct the unit concept, which is foundational to the development of rational number concepts.

Implications for Practice

Equipartitioning Activities

Equipartitioning activities in the curriculum should be expanded. The fact that none of the students in this study recognized the necessity of
equipartitioning in a part-whole relationship provides evidence that more emphasis needs to be placed on the development of the part-whole construct of rational numbers. The ability to equipartition, essential for an understanding of the unit fraction, was a skill identified on the initial interviews as not having been mastered by any of these students. It was evident in this study that these seventh graders had limited prior experiences in partitioning. By their own admission, none of the students had ever engaged in paperfolding tasks, and their exposure to concrete materials in prior years was limited. Yet the limited experiences of these students are typical for students in general. Pothier and Sawada (1990) claimed that partitioning has been virtually ignored in teaching and that students' exposure to partitioning is usually confined to the viewing of prepartitioned figures in textbooks. Their research has shown that students generally complete such textbook exercises without focusing on the geometrical properties of the whole or the parts, frequently causing them to attribute names of fractions to unequal parts of a whole.

Students need more practice with equipartitioning than they presently get. They also need a variety of partitioning activities, ranging from the traditional ones of equipartitioning various continuous shapes and discrete sets to those involving the equipartitioning of already equipartitioned wholes. Another pedagogical consideration is that of emphasizing the result of the equipartitioning process. Without such emphasis on the unit, students are likely to have difficulty viewing the fraction as an amount rather than simply as a command for the action of dividing up the whole into parts.

**Basic Measurement Concepts**

Mastery of basic measurement concepts will facilitate understanding of rational number operations. In addition to increased partitioning activities,
more time in the curriculum should be devoted to mastery of basic measurement concepts prior to the focus on fraction calculations. The fact that many of these seventh graders were unable to determine the appropriate unit for linear measurement tasks indicates that students need more exposure to measurement activities than they presently receive. The concept of measure depends on students’ recognition of the attribute to be measured and the idea that the unit influences number assignment (Wilson & Osborne, 1992). Additional evidence for the need of more measurement activities in the curriculum also is seen in the fact that the students had difficulty determining the length of a rectangle because of their tendency to begin their count with the number one at the endpoint of the segment and their propensity not to count the corner square twice in determining the dimensions of the rectangle. Students need frequent and varied measurement experiences to develop the notion that a single measurement is made by placing congruent units side by side to make a covering of the object being measured, and this covering must be obtained by placing the units without overlaps or vacant spaces.

Use of Representations

Representations promote conceptual understanding of rational numbers. Manipulatives have been used extensively in a variety of domains in mathematics education for years. The results from this study are consistent with results from other studies which indicate the advantages of utilizing concrete aids in the development of mathematical concepts. However, the use of a large variety of manipulatives in this study goes beyond the usual usage recognized in other studies as effective in transferring concepts from one domain to another. For example, paperfolding tasks enabled students to expand their models of multiplication to include the embedded units model.
Such decomposition models are helpful to students in determining a unit fraction of a unit fraction. There were some initial procedural weaknesses in students' efforts to fold the strips into parts, but the paperfolding tasks, when extended to allow folding from two directions, were effective additionally in permitting students to connect this model for fraction multiplication to their area model for whole number multiplication. Furthermore, using these decomposition skills, two of the students were able to develop a method during the teaching sequence to reconstruct the unit whole, consisting of identifying the unit fraction embedded in the composite fraction and then iterating it to form the unit whole. By focusing on the unit in a variety of concrete aids, the teaching sequence was successful in enabling students to expand models for multiplication and division of whole numbers and then to extend these models to the rational number domain. The use of extensive modeling activities serves to promote students' acquisition of rational number concepts.

Implications for Research

Constraints of Primitive Models

Constraints of primitive models of division handicap model-building. Although students were successful in developing models for multiplication and division, there remained unresolved obstacles associated with the development of each model. In the case of division, students were unable to develop successful strategies for modeling division tasks in instances where the ruler was longer than the object being measured. Students either made no attempts to perform the measurement or they would exchange the units in the problem situation. This tendency of the students to invert the relationship between the units illustrates a constraint of primitive models of division (Fischbein et al.,
in which the divisor must be less than the dividend. More research is needed to determine strategies for helping students overcome this obstacle.

Field-Testing

Field-testing of a unit-centered curriculum should be investigated. The success achieved by this teaching sequence warrants further investigation of its effects in a regular classroom setting. It is evident to this researcher that the unit concept can vastly improve students' conceptual understanding of rational number operations.

One of the central issues underlying recent reforms of the mathematics curriculum has been the emphasis that should be placed on developing understanding of basic concepts. There is mounting evidence in the literature which shows that skills cannot be effectively learned in isolation and, additionally, that students must understand the skills they are learning in order to apply them flexibly in a variety of contexts (Lindquist, 1989).

The curricular emphasis on computational skills which has traditionally characterized mathematics instruction has left many students with serious gaps in their knowledge of rational number concepts. As a result, students have experienced great difficulty with many of the more advanced rational number skills and frequently cannot apply the skills which they have learned to solve real world problems. Furthermore, the skills they have learned are in danger of becoming obsolete as increasing technological advances alter the mathematics needed by adults to function productively in society. The development of conceptual understanding in mathematics must become the focus of every mathematics classroom.

Results of this study are consistent with other studies indicating that attention to the unit types in whole number situations provides a more
adequate foundation for understanding whole number arithmetic, while also providing a cognitive bridge to facilitate learning and understanding of rational number concepts and operations. As teachers replace their traditional mathematics curriculum with a unit-centered mathematics curriculum, the conceptual understanding of students will be enhanced, helping to prepare these students for life in the twenty-first century.
REFERENCES


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Steffe, L. (1977, Fall). The teaching experiment. Unpublished manuscript presented at a meeting of the models working group of the Georgia Center for the Study of Learning and Teaching Mathematics, University of New Hampshire.


APPENDIX A

INITIAL INTERVIEW

Task 1: Here are some word problems like you might find in your math book. They might be addition, subtraction, multiplication, or division. We will read the problem together and then you will decide how you would solve it. You only need to tell me what operation you would use and what numbers. You do not have to do the actual calculation.

a. For one cake you need 2 1/4 cups of sugar. How much sugar do you need for 13 cakes?

b. Five pounds of cookies were shared equally by 15 friends. How many pounds did each person get?

c. Tom ate 3/8 of pepperoni pizza. Sam ate 2/5 of the same pizza. How much of the whole pizza was eaten by these boys?

d. Five equal-sized bottles contain a total of 3/4 liter of perfume. How much perfume is in each bottle?

e. A paperhanger needs 3 1/2 rolls of wallpaper to paper a wall. How many walls can be done with 13 rolls of paper?

f. The price of 1 yard of suit fabric is $13. What is the price of 5/8 yard?

g. It takes 5 1/2 yards of ribbon to wrap 3 packages of the same size. How many yards of ribbon are needed to wrap one package?

h. You have 7 liters of punch for a party. If the cups hold 1/5 liter each, how many cups of punch can be served?

i. Donna has 3/4 of a tank of gas in her truck. If she used 1/8 of a tank traveling to the store, how much of the tank of gas is left?

j. A box of sand has a volume of 3/5 cubic meters. What is the volume of 1/4 of a box?
STUDENT | SCORE | ERRORS | ERROR ANALYSIS
--- | --- | --- | ---
Judy | 40% | b,g,h | Exchanged dividend and divisor.
d,e | Said multiply.
f | Said divide.
Larry | 40% | b,h | Said multiply.
e | Exchanged dividend and divisor.
fj | No response.
g | Said subtract.
Laura | 50% | b,d,e,h | Said multiply.
j | Said subtract.
Melanie | 30% | a | Used wrong numbers in problem.
b | Exchanged dividend and divisor.
d,e | Said multiply.
gj | Said subtraction.
i | Exchanged minuend and subtrahend.

Task 2: Perform the following calculations. Show all your work.

a. $5 \times \frac{2}{3}$  
   b. $2 \frac{1}{2} \times 6$  
   c. $\frac{3}{5} \times \frac{2}{7}$

d. $6 \div 3$  
   e. $4 \div \frac{1}{2}$  
   f. $2 \div 6$

g. $\frac{1}{3} \div 6$  
   h. $3 \div 5$  
   i. $\frac{7}{8} \div \frac{1}{4}$

j. $\frac{2}{3} + \frac{1}{4}$

STUDENT | SCORE | ERRORS | ERROR ANALYSIS
--- | --- | --- | ---
Judy | 90% | b | Converted $2 \frac{1}{2}$ to $3/2$.
Larry | 20% | a,b,c,j | Faulty procedures; whole number dominance.
f,h,i | Exchanged dividend and divisor.
g | No response.
Laura | 20% | a,b,c | Inverted whole number, got a common denominator, multiplied numerators, kept same denominator.
e | Inverted whole number, changed $1/2$ to $1/4$,
calculated $1/4 + 1/4 = 1/4$.
f | Exchanged dividend and divisor.
g | Inverted whole number, exchanged dividend and
divisor, calculated $1/6 + 1/3 = 1/2$.
h | Calculated as decimal division, but did not place
decimal in answer: $30 + .5 = 6$.
i | Changed $1/4$ to $1/8$, then calculated as $7/8 + 1/8 = 7/1 = 7$. 

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STUDENT SCORE ERRORS ERROR ANALYSIS
Melanie 30% b Changed 6/1 to 12/1.
d-i Did not invert divisor.

Task 3: How would you explain to someone, who did not know, what a fraction is?

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>RESPONSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>&quot;a part of something ... [part of] a whole number.&quot;</td>
</tr>
<tr>
<td>Larry</td>
<td>No response.</td>
</tr>
<tr>
<td>Laura</td>
<td>&quot;Something you could like half something else.&quot;</td>
</tr>
<tr>
<td>Melanie</td>
<td>&quot;It's like ... it's not a whole. It's part of a whole. It's not exactly whole ... it's part of one ... it tells how many pieces there are.&quot;</td>
</tr>
</tbody>
</table>

Task 4: Which of these cards could help someone to understand what the fraction 3/4 is? Explain how.

![Card Images]
<table>
<thead>
<tr>
<th>STUDENT</th>
<th>ERRORS</th>
<th>ERROR ANALYSIS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>c</td>
<td>&quot;It's 3 + 4, not three fourths.&quot;</td>
</tr>
<tr>
<td></td>
<td>d</td>
<td>&quot;It's not four there ... there's seven.&quot;</td>
</tr>
<tr>
<td></td>
<td>g</td>
<td>Unconcerned about inequality of partitionings.</td>
</tr>
<tr>
<td>Larry</td>
<td>c</td>
<td>&quot;That's a division, not a fraction.&quot;</td>
</tr>
<tr>
<td></td>
<td>d</td>
<td>&quot;That's three out of seven.&quot;</td>
</tr>
<tr>
<td></td>
<td>e</td>
<td>No response.</td>
</tr>
<tr>
<td></td>
<td>g</td>
<td>Unconcerned about inequality of partitionings.</td>
</tr>
<tr>
<td>Laura</td>
<td>c</td>
<td>&quot;No, that's 3 + 4.&quot;</td>
</tr>
<tr>
<td></td>
<td>d</td>
<td>&quot;That's 3/7.&quot;</td>
</tr>
<tr>
<td></td>
<td>e</td>
<td>Unable to explain this card.</td>
</tr>
<tr>
<td></td>
<td>g</td>
<td>Unconcerned about inequality of partitionings.</td>
</tr>
<tr>
<td>Melanie</td>
<td>c</td>
<td>&quot;They would have to work it out.&quot;</td>
</tr>
<tr>
<td></td>
<td>d</td>
<td>&quot;No, there's 4 there and 3 there -- 3/7.&quot;</td>
</tr>
<tr>
<td></td>
<td>e</td>
<td>Unable to explain this card.</td>
</tr>
<tr>
<td></td>
<td>g</td>
<td>Unconcerned about inequality of partitionings.</td>
</tr>
</tbody>
</table>

Task 5a. How would you find 3/4 of these jacks?

![Diagram of jacks]

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>FORMED COMPOSITE UNIT 3/4</th>
<th>UNSUCCESSFUL ATTEMPTS TO SOLVE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Yes</td>
<td>NA</td>
</tr>
<tr>
<td>Larry</td>
<td>No</td>
<td>Could not determine 1/4, even with guided questioning.</td>
</tr>
<tr>
<td>Laura</td>
<td>No</td>
<td>&quot;Six would be half and ... four would be 3/4.&quot; After further thought, added that 3/4 &quot;would be 8 I think, because half is 6.&quot; With help, she determined that 1/4 would be 3 jacks, but could not use this to find 3/4.</td>
</tr>
<tr>
<td>Melanie</td>
<td>No</td>
<td>Separated 12 chips into 3 groups of 4 and said this represented 3/4.</td>
</tr>
</tbody>
</table>

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b. How would you find 1/3 of this candy bar?

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>SOLVED</th>
<th>ATTEMPTS TO SOLVE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ALONE</td>
<td>WITH HELP</td>
</tr>
<tr>
<td>Judy</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td></td>
<td>Repartitioned one of the 1/4-units into thirds. Shaded one of the original partitionings (1/4-unit) and 1/3 of the repartitioned portion (1/3 of 1/4-unit) to represent the answer.</td>
<td></td>
</tr>
<tr>
<td>Larry</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>Shaded one of the four parts.</td>
<td></td>
</tr>
<tr>
<td>Laura</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td></td>
<td>Initially shaded 1/4. With guided questioning, she remarked, &quot;Give them all one whole piece and then like not quite half it. Like one square halfed into three.&quot;</td>
<td></td>
</tr>
<tr>
<td>Melanie</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>Crossed out one of the blocks, leaving three blocks, then shaded one of these three.</td>
<td></td>
</tr>
</tbody>
</table>

Task 6a. Use the chips to show what we mean by 12 ÷ 3. Give me a story problem that this can be used to solve.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>MODEL</th>
<th>STORY PROBLEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Formed 3 groups with 4 chips per group.</td>
<td>&quot;You have 12 pencils and you want to give them to 3 people evenly.&quot;</td>
</tr>
<tr>
<td>Larry</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>Laura</td>
<td>Formed 3 groups with 4 chips per group.</td>
<td>&quot;Three people want ... some have 12 ... give them 12 divided by 3 ~ 4 each.&quot;</td>
</tr>
<tr>
<td>Melanie</td>
<td>Formed 3 groups with 4 chips per group.</td>
<td>&quot;Jamie had a party and she invited two of her friends and she had 12 pieces of candy and wanted to find a way to find an equal sum of candy for each person so ... each person would get four pieces of candy.&quot;</td>
</tr>
</tbody>
</table>

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b. Draw a model (picture) to show what we mean by $6 \div 1/2$. Give me a story problem for this problem. Can you solve the problem?

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>MODEL</th>
<th>STORY PROBLEM</th>
<th>SOLVED</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>None</td>
<td>&quot;If you had 6 pieces of paper and you wanted to divide them in half. It would be 3.&quot;</td>
<td>Yes</td>
</tr>
<tr>
<td>Larry</td>
<td>None</td>
<td>None</td>
<td>No</td>
</tr>
<tr>
<td>Laura</td>
<td>None</td>
<td>None</td>
<td>No</td>
</tr>
<tr>
<td>Melanie</td>
<td>None</td>
<td>&quot;Someone had a party and they invited 12 people. There was enough cake for six people and they had to half each piece.&quot;</td>
<td>No</td>
</tr>
</tbody>
</table>

Task 7a. Draw a picture to show what we mean by $3 \times 4$. Give me a story problem for your model.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>MODEL</th>
<th>STORY PROBLEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>4 x 3 array with chips.</td>
<td>&quot;If 4 people had 3 pencils and you wanted to add them all together.&quot;</td>
</tr>
<tr>
<td>Larry</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>Laura</td>
<td>Formed 3 groups of 4 chips.</td>
<td>&quot;Have 3 boxes and put 4 in each, or you can put 3 in 4 boxes.&quot;</td>
</tr>
<tr>
<td>Melanie</td>
<td>4 x 3 array with chips.</td>
<td>&quot;Johnny was making pants and he wanted to make four pairs of pants and there were three yards for each pair of pants so you would times those ... you'd have to get 12 yards in all.&quot;</td>
</tr>
</tbody>
</table>

b. Draw a picture to show what we mean by $4 \times 1/2$. Give me a story problem for your model. Can you solve the problem?

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>MODEL</th>
<th>STORY PROBLEM</th>
<th>SOLVED</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>None</td>
<td>&quot;If you have 4 halves of a candy bar and you wanted to see how many you had -- 2 wholes.&quot;</td>
<td>Yes</td>
</tr>
<tr>
<td>Larry</td>
<td>None</td>
<td>None</td>
<td>No</td>
</tr>
<tr>
<td>Laura</td>
<td>None</td>
<td>None</td>
<td>No</td>
</tr>
<tr>
<td>Melanie</td>
<td>None</td>
<td>&quot;I have 8 [people] invited to the party and there were 4 pieces of cake. You'd have to half each piece and there would be 8 pieces.&quot;</td>
<td>No</td>
</tr>
</tbody>
</table>
c. Draw a picture to show what we mean by \( \frac{1}{3} \times \frac{1}{2} \). Give me a story problem for your model. Can you solve the problem?

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>MODEL</th>
<th>STORY PROBLEM</th>
<th>SOLVED</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>None</td>
<td>&quot;If you have like a third of a candy bar and a half of a candy bar, it would equal ( \frac{1}{6} ) of a candy bar.&quot;</td>
<td>Yes</td>
</tr>
<tr>
<td>Larry</td>
<td>None</td>
<td>None</td>
<td>No</td>
</tr>
<tr>
<td>Laura</td>
<td>None</td>
<td>None</td>
<td>No</td>
</tr>
<tr>
<td>Melanie</td>
<td>None</td>
<td>None</td>
<td>No</td>
</tr>
</tbody>
</table>

Task 8: If \( \cdot \cdot = \frac{1}{3} \), draw \( \frac{1}{2} \).

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>SOLVED ALONE</th>
<th>WITH ASSISTANCE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Found unit whole</td>
<td>Found ( \frac{1}{2} )-unit</td>
</tr>
<tr>
<td>Judy</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Larry</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Laura</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Melanie</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>

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APPENDIX B

THE LESSONS

Lesson One: Examination of the units used in counting

Task 1: Empty your bag of chips onto the table and count them. How many chips are in your bag?

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Counted by twenties.</td>
</tr>
<tr>
<td>Larry, Laura, Melanie</td>
<td>Counted by ones.</td>
</tr>
</tbody>
</table>

Task 2: Count the number of cubes in your bag. How many do you have?

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Counted by twenties; disregarded color.</td>
</tr>
<tr>
<td>Larry</td>
<td>Counted by ones; disregarded color.</td>
</tr>
<tr>
<td>Laura</td>
<td>Counted by ones; left cubes in original stacks to count.</td>
</tr>
<tr>
<td>Melanie</td>
<td>Counted by ones, then regrouped by colors.</td>
</tr>
</tbody>
</table>

Task 3: Look at the picture and estimate the number of people in the crowd. Do not actually count the people. Think of a reasonable strategy to use that will give you an approximate total. How many people did you estimate? What was the strategy you used to estimate the answer?

All students guessed. No reasonable strategy was offered by anyone.

Task 4: Rethink the strategy used for each counting task above. Did you use a similar strategy for each, or was each strategy different?

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>&quot;I put them all into groups of twenty and then counted them. If there were some left over I just added them after I was finished counting the ones in groups of 20.&quot;</td>
</tr>
<tr>
<td>Larry</td>
<td>&quot;I counted them all one by one.&quot;</td>
</tr>
<tr>
<td>Laura</td>
<td>&quot;I just counted them when I got them out of the bag. I didn't put them in groups or anything.&quot;</td>
</tr>
</tbody>
</table>
Lesson Two: Formation of units in whole number problems

Task 1: Lisa has 5 bags containing 4 candies and one bag with 2 candies. Mark has 2 bags with 4 candies and 5 bags with 2 candies. If they combine their candies, putting then in bags with 4 candies each, how many bags will they use? Use the materials on the table to help you solve the problem. Describe your process for solving this problem. Use drawings to help you explain.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Melanie</td>
<td>&quot;Each was different. For the first task I counted one by one. On the second task, I color coordinated my blocks. On the third task I ... took a good guess.&quot;</td>
</tr>
<tr>
<td>Judy</td>
<td>&quot;I counted the bags with groups of four then I put two of the bags together that had two [candies] in them.&quot;</td>
</tr>
<tr>
<td>Larry</td>
<td>&quot;I put the bags with two in it together to equal one, and counted the ones with four in it.&quot;</td>
</tr>
<tr>
<td>Laura</td>
<td>&quot;I put two bags with two candies in each and counted it as one and the ones with four counted those in and I came up with nine bags.&quot;</td>
</tr>
<tr>
<td>Melanie</td>
<td>&quot;I solved my problem by separating the two peoples bags adding them in my head and then adding them together.&quot;</td>
</tr>
</tbody>
</table>

Task 2: Five boxes of golf balls, each containing one dozen balls, are to be repackaged into mini-sets of four balls each. How many four-packs can be formed?

All students formed three mini-packs from each dozen, then counted the number of mini-packs in five dozen.
Task 3: The home economics class is planning to bake 120 chocolate cakes for a fundraiser. The recipe calls for 3 cups of flour for each cake. Flour comes in bags with 15 cups of flour per bag. How many bags of flour are needed to bake all the cakes?

<table>
<thead>
<tr>
<th>STUDENTS</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy, Larry, Laura</td>
<td>Used units of one.</td>
</tr>
<tr>
<td>Melanie</td>
<td>Used groups: Divided three into fifteen to determine the number of cakes that could be made per bag of flour, then divided this answer into 120.</td>
</tr>
</tbody>
</table>

Lesson Three: Formation of composite units

Task 1: If you can buy 3 tapes for $5, how much will you pay for 24 tapes? Use the counters on the table to help you solve the problem. Draw your solution strategy.

All students formed eight groups of three, then multiplied the number of groups by $5.

Task 2: The Movie Center rents all their videos at 3 video movies for $7. In one week there were 672 videos rented. How much money did The Movie Center earn in rentals?

<table>
<thead>
<tr>
<th>STUDENTS</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy and Melanie</td>
<td>Used groups of three (672 ÷ 3), then multiplied by $7. Saw the similarity to Task 1.</td>
</tr>
<tr>
<td>Larry and Laura</td>
<td>Did not see this problem as similar to Task 1. Resorted to units of one approach: Incorrectly multiplied 672 times $7.</td>
</tr>
</tbody>
</table>

Task 3: Mrs. Bryan wants to give Valentine cookies to the students in her math class. She buys one bag with 15 cookies, two bags with 24 cookies each, and one bag with 9 cookies. If Mrs. Bryan allows 3 cookies per student, how many students will receive cookies?

<table>
<thead>
<tr>
<th>STUDENTS</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy, Larry, Laura</td>
<td>Units of one approach: Counted cookies per bag, added all cookies per bag together, divided total by 3.</td>
</tr>
</tbody>
</table>
Lesson Four: Models of division

Problem A: Use your chips to show how to share 15 cookies with 3 children. (Give fair shares.) Make a drawing to show how you solved the problem with your chips.

Students worked as a group to develop the partitive model of division.

Problem B: You have a bag with 15 cookies. You want to form snack bags having 3 cookies per bag. How many snack bags can be formed? Use chips to model the solution, then make a drawing to show how you solved it.

Students worked as a group to develop the measurement model of division.

Task 1: Five cakes are to be divided between four people. Draw a picture of the five cakes and shade in the amount one person would get. How much cake does each person get?

<table>
<thead>
<tr>
<th>STUDENTS</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Gave each person one cake and divided the last one into fourths. Quantified result as 1 1/4.</td>
</tr>
<tr>
<td>Laura</td>
<td>Gave each person one cake and divided the last one into fourths. Unable to quantify results.</td>
</tr>
<tr>
<td>Melanie</td>
<td>Copied Judy's work.</td>
</tr>
</tbody>
</table>

Task 2: Three candy bars are to be shared equally between four children. How much candy will each person get? Draw a picture of the candy and shade in the amount each child would get.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Partitioned each candy bar into fourths. Gave each person three fourths.</td>
</tr>
</tbody>
</table>
Laura Partitioned two of the bars into halves and the last one into fourths. Gave each person one 1/2-unit and one 1/4-unit, but could not quantify the results.

Melanie Copied Judy's work.

Task 3: 27 Easter eggs are to be packed into 3-egg packs. How many 3-packs can be formed? Draw a picture to show your 3-egg packs.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy and Melanie</td>
<td>Formed nine 3-units.</td>
</tr>
<tr>
<td>Laura</td>
<td>Initially formed three 9-units, then regrouped into nine 3-units.</td>
</tr>
</tbody>
</table>

Task 4: It takes 3 yards of material to make one suit. There are 15 yards of material on a bolt of suit fabric. How many suits can be made from this bolt of material?

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy and Melanie</td>
<td>Formed five 3-units.</td>
</tr>
<tr>
<td>Laura</td>
<td>Unable to solve.</td>
</tr>
</tbody>
</table>

Lesson Five: Measurement division

Task Directions: Use the rods to help you solve each measuring task. Make a drawing to show how you found each answer.

1) Use the dark green-unit as a ruler to measure the light green-unit.

\[ 1(\text{light green-unit}) = \underline{\text{______}}(\text{dark green-unit}) \]

All students used two light green-units to equal the dark green-unit, then concluded that the light green-unit equalled 1/2 the dark green-unit.

2) Use the light green-unit as a ruler to measure the dark green-unit.

\[ 1(\text{dark green-unit}) = \underline{\text{______}}(\text{light green-unit}) \]
All students reversed the previous relationship to conclude that the dark green-unit equalled 2 light green-units.

3) Use the light green-unit as a ruler to measure the white-unit.

\[ 1 \text{ (white-unit)} = \underline{\text{________}} \text{ (light green-unit)} \]

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Recognized that three white-units equalled the light green-unit and concluded that 1 white-unit equalled 1/3 light green-unit.</td>
</tr>
<tr>
<td>Larry and Melanie</td>
<td>Lost sight of the measuring unit: Inverted the relationship and concluded that 1 white-unit equalled three light green-units.</td>
</tr>
<tr>
<td>Laura</td>
<td>Had an accurate drawing but was unable to quantify the result. Admitted guessing the answer as 1/4.</td>
</tr>
</tbody>
</table>

4) Use the light green-unit as a ruler to measure the red-unit.

\[ 1 \text{ (red-unit)} = \underline{\text{________}} \text{ (light green-unit)} \]

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy and Melanie</td>
<td>Correctly established the relationship by using a third unit (white-unit). Concluded that 1 red-unit equalled 2/3 light green-unit.</td>
</tr>
<tr>
<td>Laura</td>
<td>Guessed the answer as 1/4.</td>
</tr>
<tr>
<td>Larry</td>
<td>Unable to determine a strategy. Quit trying.</td>
</tr>
</tbody>
</table>

5) Use the red-unit as a ruler to measure the light green-unit.

\[ 1 \text{ (light green-unit)} = \underline{\text{________}} \text{ (red-unit)} \]

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Incorrectly answered 1 1/4, but would not explain thinking. Seemed embarrassed to be wrong.</td>
</tr>
<tr>
<td>Laura</td>
<td>Correctly established the relationship that 1 light green-unit equalled 2/3 red-unit. (No guessing.)</td>
</tr>
<tr>
<td>Melanie</td>
<td>Lost sight of the unit to conclude the answer was 1 1/3, but caught and corrected her mistake while explaining her reasoning.</td>
</tr>
</tbody>
</table>

Lesson Six: Models of multiplication
Task 1a. How many different rectangles can you draw that have a perimeter of 24 units? Use the grid paper to help you make your rectangles.

b. If these rectangles all have the same perimeter (24 units), will they all be the same size and shape?

c. How can we express their sizes other than stating the perimeter is 24 units?

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>OBSTACLES DISPLAYED</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>None</td>
</tr>
<tr>
<td>Larry</td>
<td>Counted blocks to determine perimeter.</td>
</tr>
<tr>
<td>Laura</td>
<td>Initially counted all interior blocks, but later counted only the sides.</td>
</tr>
<tr>
<td>Melanie</td>
<td>Confused between area and perimeter--wanted to multiply length and width (or count the blocks inside the rectangle) to determine perimeter, even after the group discussion.</td>
</tr>
</tbody>
</table>

Lesson Seven: Models of multiplication

Task 1: Cups are packaged in a box so that there are 4 rows with 6 cups in each row.

```
X X X X X X
X X X X X X
X X X X X X
X X X X X X
```

a. Use the number of cups in each row and the number of cups in each column to write an arithmetic statement to show how many cups are in this box. What are your measuring units?

All students multiplied 6 and 4 to determine the number of cups in the box; however all students identified 6 and 4 as the measuring units.
b. Suppose the manufacturer decided to package these cups into groups of 4, with each package the same shape as shown.

Find the number of **packages** in each box by looking at rows and columns. How did you determine your answer? What are your measuring units?

c. How can you use rows and columns to determine how many **cups** are in the box?

d. How does this differ from question a? How can you explain this difference?

Task 2: Use your grid paper to model the problem 20 x 30.

a. What does the 20 represent?

b. What does the 30 represent?

c. What is the answer for 20 x 30?

d. What does this number mean?

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>OBSTACLES DISPLAYED</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy, Laura</td>
<td>None</td>
</tr>
<tr>
<td>Larry, Melanie</td>
<td>Began the count of each side with the starting point as &quot;one.&quot; Unsure whether to count boxes or segments.</td>
</tr>
</tbody>
</table>
e. Instead of counting all the individual units, can you use groupings to represent the dimensions of each side? What if you grouped them by 10’s? Use your grid paper to show what the new units would look like. Calculate the answer to the multiplication problem using these new units.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>OBSTACLES DISPLAYED</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>None</td>
</tr>
<tr>
<td>Larry and Melanie</td>
<td>Unable to see the change of unit structure in the answer, from linear to square units.</td>
</tr>
<tr>
<td>Laura</td>
<td>Confused area and perimeter: wanted to add the sides to determine the number of blocks inside.</td>
</tr>
</tbody>
</table>

f. Try this problem again using a different grouping of units. Make a drawing to show your grouping, then write a statement that will help you solve the problem.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>OBSTACLES DISPLAYED</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>None</td>
</tr>
<tr>
<td>Larry, Laura, Melanie</td>
<td>Confused about the change in unit structure; Displayed inconsistency in answers.</td>
</tr>
</tbody>
</table>

g. Compare the units used for the dimensions of each side with the unit used to represent the answer. Are they the same or different?

Task 3: You want to put a new carpet in your 9' by 12' bedroom. You plan to use a "patchwork quilt" design, with each different colored piece being one square yard in size. How many colored squares of carpet will you need? Make a drawing to explain how you solved this problem.

Lesson Eight: Models of multiplication

Task 1: It takes 2 yards of material to make one skirt. How many yards of material are needed to make 5 skirts? Use the cubes to help you model the problem. Make a drawing of your model for this problem.

All students formed five 2-units with the cubes. No obstacles were noted.

Task 2: Donna has 4 pairs of pants and 9 shirts. If all the shirts will match all the pants, how many different outfits can she wear? Use the cubes to help you solve the problem. Make a drawing to explain how you solved this problem.
Students correctly modeled the task. No obstacles were noted.

Task 3: Work with your partner to determine how many small rectangles will fit inside your large rectangle. Your small rectangles cannot be overlapped or cut, nor can they hang over the edge of the large rectangle. Keep them all turned in the same direction to cover the rectangle.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>OBSTACLES DISPLAYED</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy and Larry</td>
<td>None</td>
</tr>
<tr>
<td>Laura and Melanie</td>
<td>Weak measurement skills. Sometimes left a gap between placements of the ruler; sometimes overlapped measurements.</td>
</tr>
</tbody>
</table>

Lesson Nine: Representational understanding

Task 1: Use your fraction circles or fraction squares to solve each problem. Make 1 orange the unit whole. What fraction name can you give these pieces?

1 yellow = _________ 1 brown = _________
1 pink = _________ 1 red = _________
1 clear = _________ 1 blue = _________

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>OBSTACLES DISPLAYED</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>None</td>
</tr>
<tr>
<td>Larry and Laura</td>
<td>Unconcern with equality in partitioning.</td>
</tr>
<tr>
<td>Melanie</td>
<td>Unconcern with equality in partitioning. Loss of sight of the unit: confusing the ruler with the object being measured.</td>
</tr>
</tbody>
</table>

Task 2: Now make the green the unit whole. What fraction name can you give these pieces?

1 white = _________ 1 red = _________
1 pink = _________

Lesson Ten: Representational understanding

Task 1: Use your fraction circles to name each fraction piece.

a. 1 green = _________ red-unit.
b. 1 green = _________ orange-unit.
How can the same piece have different names?

c. 1 orange = _________ green-unit.

Make a drawing to help you solve the above problem.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>OBSTACLE DISPLAYED</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy and Laura</td>
<td>Brief episode of loss of sight of unit. Cleared up during lesson.</td>
</tr>
<tr>
<td>Melanie</td>
<td>Continued loss of sight of unit. Exhibited frustration over her confusion about units.</td>
</tr>
</tbody>
</table>

Task 2: Jenny says this white piece is one-third. John says it is one fourth. Who is correct? Use drawings or models to help you explain your answer.

Task 3: Use your fraction strips to model the following fractions:

- a. 3/5
- b. 4/10
- c. 5/6
- d. 7/10

All students used the part-whole interpretation to model the unit whole with fraction circles, using the units dictated by the denominator, then removed the extra pieces to indicate the part-whole relationship indicated by the numerator.

Lesson Eleven: Unitizing and norming

Task 1: Use your fraction strips to model each fraction. Then draw your solution strategy.

- a. 3/5
- b. 7/12
- c. 4/10

Explain the strategy you used to draw each fraction.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Iteration</td>
</tr>
<tr>
<td>Laura</td>
<td>Iteration</td>
</tr>
<tr>
<td>Melanie</td>
<td>Iteration</td>
</tr>
<tr>
<td></td>
<td>(Copied Judy's work on the first task.)</td>
</tr>
</tbody>
</table>
Task 2: Use your fraction strips to model each fraction. Then draw your solution strategy.

a. $\frac{3}{2}$  b. $1 \frac{1}{4}$  c. $\frac{6}{5}$

Explain the strategy you used to draw each fraction.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
<th>OBSTACLES DISPLAYED</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Drew $2/2$ for problem $a$, then copied Melanie's work. Used iteration after Melanie explained it.</td>
<td>Uncertainty about modeling of improper fractions until Melanie explained her process of iteration after everyone attempted the first problem.</td>
</tr>
<tr>
<td>Laura</td>
<td>Initially converted each to mixed fractions. Used iteration after Melanie explained it.</td>
<td>Uncertainty about modeling of improper fractions until Melanie explained her process of iteration after everyone attempted the first problem.</td>
</tr>
<tr>
<td>Melanie</td>
<td>Used iteration.</td>
<td>None</td>
</tr>
</tbody>
</table>

Task 3: Use your chips to help you model the given fractional unit in each problem. Then draw your solution strategy to help you explain your model.

a. If 16 chips are your unit, find $\frac{3}{4}$-unit.

b. If 28 chips are your unit, find $\frac{4}{7}$-unit.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
<th>OBSTACLES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy and Melanie</td>
<td>Used partitive division to separate the chips into the number of groups specified by the denominator. Used the numerator to determine the number of groups needed.</td>
<td>None</td>
</tr>
<tr>
<td>Laura</td>
<td>For the first problem, used partitive division to separate the chips into the number of groups specified by the denominator, but did not know what to do next to obtain the composite fraction until after the group discussion.</td>
<td>Unable to relate this discrete model to the continuous model in the previous task. Tried to use part-whole interpretation to create the composite fraction rather than iteration.</td>
</tr>
</tbody>
</table>
Lesson Twelve: Unitizing and nonning

Task Directions: Start with a unit whole paper strip to model the solution for each problem below.

1. Fold a paper strip and use shading to model the unit fraction 1/3. Now partition this unit fraction (1/3) into three equal parts. One of these new parts is what part of the unit whole strip?

2. Fold a strip and use shading to model the unit fraction 1/4. Now partition this unit fraction (1/4) into three equal parts. One of these new parts is what part of the unit whole strip?

STUDENT OBSTACLES

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>OBSTACLES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>None</td>
</tr>
<tr>
<td>Laura</td>
<td>Difficulty in folding strips into equal parts. Not concerned with equality of parts.</td>
</tr>
<tr>
<td>Melanie</td>
<td>Difficulty in folding strips into equal parts. Not concerned with equality of parts. &quot;They are (one fourths) but they are not equal one fourths.&quot;</td>
</tr>
</tbody>
</table>

3. Draw a unit whole strip and shade the unit fraction 1/5. Now use partitioning to separate this unit fraction into two equal parts. One of these new parts is what part of the unit whole strip?

STUDENT PROCEEDURES

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Used vertical partitioning to partition the 1/5-unit into half. Linked this problem to multiplication -- &quot;you could just multiply numerators and denominators to determine the answer.&quot;</td>
</tr>
<tr>
<td>Laura</td>
<td>Used vertical partitioning to form 1/5-units, then horizontal partitioning to find half of the 1/5-unit. Linked problem to multiplication -- &quot;just multiply the denominators.&quot; Explained the similarity of this model to the area model for whole number multiplication.</td>
</tr>
<tr>
<td>Melanie</td>
<td>Unable to transfer paperfolding techniques to the task of partitioning a strip on the worksheet until after the group discussion. Used vertical partitioning correctly to model the problem, but could not quantify results.</td>
</tr>
</tbody>
</table>
Lesson Thirteen: Unitizing and norming

Task 1: Fold a strip and use shading to model the unit fraction 1/6. Now partition this unit fraction (1/6) into 3 equal parts. One of these new parts is what part of the unit whole strip?

Task 2: Create your own problem like the one above. Draw a unit whole strip to model a unit fraction of your choice. Then partition this unit fraction into ____ equal parts (you choose a number). One of these new parts is what part of the unit whole strip? Explain your problem and your solution to the class.

Task 3: John has a rectangular garden. He is going to plant flowers on half his garden. He wants 1/3 of the flowers to be daisies. Draw John’s garden and show the fraction of the garden that will be daisies.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy and Laura</td>
<td>Misread the problem -- did not see the “half.” Found one third of the whole garden.</td>
</tr>
<tr>
<td>Melanie</td>
<td>Partitioned the rectangle into halves, then partitioned each half into thirds and shaded one block to represent 1/3 of 1/2-garden. Thrilled that she was the only one to solve the problem correctly.</td>
</tr>
</tbody>
</table>

Task 4: Marty ate 3/4 of two-thirds of a candy bar. Make a drawing to show the part of the candy that he ate.

Lesson Fourteen: Reconstructing the Unit

Task 1: This pink piece is half of something. How can I find the value of these other pieces which are part of the same unit?

1 tan = ________  
2 reds = ________  
1 brown = ________  
3 greens = ________

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy and Laura</td>
<td>Easily and quickly determined the unit whole, then used this unit whole to determine the size of each requested unit.</td>
</tr>
<tr>
<td>Melanie</td>
<td>Unable to determine the unit whole. Was irritated that the researcher/teacher would not identify the unit whole for her. Looked at Judy’s modeling actions to determine the unit whole, then was able to complete the task.</td>
</tr>
</tbody>
</table>
Task 2: The chips shown below represent part of a unit whole. Model the unit fraction with your chips, then draw the model you used to find the unit whole.

This is 3/5 of a unit. Draw the unit.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Decomposed 3/5 into three 1/5-units to identify 2 dots as 1/5-unit, then iterated the 2-dot unit five times to construct the unit whole. Underlined the original 6 dots that were embedded in the unit whole.</td>
</tr>
<tr>
<td>Laura</td>
<td>Decomposed 3/5 into three 1/5-units to identify 2 dots as 1/5-unit, then iterated the 2-dot unit five times to construct the unit whole. Used vertical partitioning to separate each two dots into 1/5-units.</td>
</tr>
<tr>
<td>Melanie</td>
<td>Focused on the numbers in the fraction and decided she needed &quot;three rows of five—for the three fifths.&quot;</td>
</tr>
</tbody>
</table>

Lesson Fifteen: Fraction division models

Task 1: Use partitive (sharing) division to model the solution of 3 ÷ 4. Draw your solution strategy.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
<th>OBSTACLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Drew three circles for the cookies, then partitioned them into four equal parts. Used sharing action to divide the cookie parts among four children.</td>
<td>None</td>
</tr>
<tr>
<td>Laura</td>
<td>Drew three circles for the cookies, then partitioned them with three vertical cuts, forming four unequal parts. Used sharing action to divide the cookie parts among four children.</td>
<td>Did not recognize the importance of equipartitioning.</td>
</tr>
<tr>
<td>Melanie</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Task 2: Remove the paper strips from your envelope. Use the long strip as your ruler to measure the shorter strip. Make a drawing to model your measurement process.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
<th>OBSTACLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Unable to determine a strategy.</td>
<td>Unable to link division of rationals to whole number models for division.</td>
</tr>
<tr>
<td>Melanie</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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Laura Used paperfolding to partition the ruler into four equal parts, then compared it to the smaller strip and determined the small strip was 3/4 of the ruler.

Task 3: Use your fraction strips to help you solve each problem. Then draw your solution strategy for each. What does each answer mean in terms of units?

a. Measure 3/5-unit with the 1/5-unit.

b. Measure 1/4-unit with the 1/6-unit.

c. Measure 2/3-unit with the 1/6-unit.

d. Measure 2-units with the 2/3-unit.

e. Measure 1-unit with the 3/10-unit.

f. Measure 1/3-unit with the 2/3-unit.

<table>
<thead>
<tr>
<th>PROBLEM</th>
<th>OBSTACLES</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>None</td>
</tr>
<tr>
<td>b</td>
<td>The units were different on this problem. Laura and Melanie lost sight of the measuring unit—which unit to use as the ruler.</td>
</tr>
<tr>
<td>c</td>
<td>None. (The group discussion after the last problem seemed to alleviate the confusion about the measuring unit.)</td>
</tr>
<tr>
<td>d</td>
<td>Laura was the only student unable to work this problem. She did not know what to do with the whole number, 2.</td>
</tr>
<tr>
<td>e</td>
<td>Laura still was confused by the whole number in the problem.</td>
</tr>
<tr>
<td>f</td>
<td>All students were confused because the ruler was longer than the unit being measured. Judy drew only the 1/3-strip, then stared at her paper. Melanie drew a correct model for the problem, but could not determine an answer. Laura made no drawing. Melanie was able to state the relationship between the units—&quot;1/3 is half of 2/3&quot;—but did not connect her statement with the measuring task.</td>
</tr>
</tbody>
</table>
APPENDIX C

EXIT INTERVIEW

Task 1: A four foot rope needs to be separated into 5 equal pieces. Show how this can be done.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Underwent a series of hesitations, guesses, and erasures, then remarked, &quot;I just can't figure out where to put the lines.&quot; When the researcher suggested that she look at the fraction strips, she used measurement division, with the 1/5-unit as ruler on the 4/4 strip. These markings on the 4/4 strip were then transferred to the &quot;rope&quot; on the worksheet.</td>
</tr>
<tr>
<td>Laura</td>
<td>Initially added another section to the rope to make five sections. When reminded that it could not be lengthened, she drew another rope, without partitions, beneath the original one, then partitioned it into fifths. She transferred the 1/5 partitionings from the new rope to the original one.</td>
</tr>
<tr>
<td>Melanie</td>
<td>Easily partitioned the rope into five parts. Ignored the existing partitionings.</td>
</tr>
</tbody>
</table>

Task 2a. Use your manipulatives to measure 4/5 with the 1/5-unit.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Used fraction circles to create the unit whole with five 1/5-units, then remarked, &quot;that's 1/5 of the whole — (pointed to four of these units) 4/5's.&quot;</td>
</tr>
<tr>
<td>Laura</td>
<td>Drew a fraction strip to represent 4/5 and a strip beneath it to represent 1/5. She pointed to the 1/5-unit and remarked, &quot;there's four 1/5's in that [top drawing].&quot;</td>
</tr>
<tr>
<td>Melanie</td>
<td>Drew fraction strips for 4/5-unit and 1/5-unit. She replied, &quot;the answer is four because it takes four 1/5's to equal 4/5.&quot;</td>
</tr>
</tbody>
</table>
b. As measurement division, think about why the division problem $\frac{4}{5} \div \frac{1}{5}$ is easier than the problem $\frac{2}{3} \div \frac{1}{4}$. Think about this in terms of units.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>RESPONSE</th>
</tr>
</thead>
</table>
| Judy      | "It's easier because the denominators are the same. The units are the same ... they can be compared easier. (Points to the other problem) They're different. You would have to draw two different things to compare them."
| Laura     | Considered the first problem easier "because it's equal." In contrast, the other problem was harder because "you have to see how many of these [1/4-units] are in [the 2/3-unit]. It doesn't come out equal."
| Melanie   | She considered the first problem easier because "the denominator's the same ... the parts would be equal. They have the same little units. So they are easier to measure."

C. Can you put $\frac{2}{3} \div \frac{1}{4}$ in a form to make it easier and then explain the division. Draw pictures to help you explain your thinking.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Remarked that this problem was not as easy &quot;because the denominators are different so you have different units.&quot; Used measurement division with the 8/12 and 3/12 strips, obtaining the correct answer of 2 2/3. Also solved the problem procedurally.</td>
</tr>
<tr>
<td>Laura</td>
<td>Found equivalent fraction strips and used measurement division to calculate the correct answer.</td>
</tr>
<tr>
<td>Melanie</td>
<td>Did not convert to equivalent fractions. Left them as 2/3 and 1/4 and used measurement division to correctly calculate the answer. Tried unsuccessfully to solve the problem procedurally.</td>
</tr>
</tbody>
</table>

Task 3: Draw a picture to show what we mean by $\frac{1}{3}$ of $\frac{1}{5}$-unit.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Was initially unable to recall a procedure so the researcher asked her to find 1/5 of 15 chips, then 1/3 of her 3-chip answer. This enabled her to remember partitioning exercises. She partitioned a strip into five sections, shaded a 1/5-unit, then partitioned it into three sections and darkly shaded 1/3 of 1/5-unit.</td>
</tr>
<tr>
<td>Laura</td>
<td>Initially tried to use measurement division on fraction strips, but the researcher suggested finding 1/3 of the 1/5 strip. Partitioned a strip into five parts, then partitioned the first 1/5-unit into three parts and shaded one of these thirds.</td>
</tr>
</tbody>
</table>

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Extension Task: Draw a picture to show what we mean by 1/3 of 2/5-unit.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Immediately responded, &quot;it would double the shading ... two fifteenths.&quot;</td>
</tr>
<tr>
<td></td>
<td>Duplicated the previous partitioning of 1/3 of 1/5 onto the second 1/5-unit.</td>
</tr>
<tr>
<td>Laura</td>
<td>Drew another strip partitioned into fifths, then partitioned the first two 1/5-units into three parts each (resulting in six smaller units). Shaded two of these six parts, but was unable to quantify this new region.</td>
</tr>
<tr>
<td>Melanie</td>
<td>Drew another strip partitioned into fifths and shaded two of these 1/5-units. Partitioned only one of these 1/5-units into thirds.</td>
</tr>
</tbody>
</table>

Task 4: Consider the subtraction problem 4/7 - 3/7. Think of this problem in terms of units. Can you think of a whole number problem for which the basic idea of subtracting 4/7 and 3/7 is very similar? How are they alike? How are they different? Explain your thinking by drawing pictures to help.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>WHOLE NUMBER PROBLEM</th>
<th>UNITS</th>
<th>CONTRASTS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Whole Number Problem</td>
<td>Rational Number Problem</td>
</tr>
<tr>
<td>Judy</td>
<td>4 - 3</td>
<td>&quot;a whole.&quot;</td>
<td>&quot;Seven ... no, sevenths.&quot;</td>
</tr>
<tr>
<td>Laura</td>
<td>4 - 3</td>
<td>&quot;a whole&quot;</td>
<td>&quot;seven&quot;</td>
</tr>
<tr>
<td>Melanie</td>
<td>4 - 3</td>
<td>4 and 3</td>
<td>1/7</td>
</tr>
</tbody>
</table>

Task 5: If ■ ■ = 1/4, show what 2/3 would look like.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>SOLVED ALONE</th>
<th>SOLVED WITH ASSISTANCE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Found unit whole</td>
<td>Found 2/3-unit</td>
</tr>
<tr>
<td>Judy</td>
<td>Yes</td>
<td>NA</td>
</tr>
<tr>
<td>Laura</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Melanie</td>
<td>Yes</td>
<td>NA</td>
</tr>
</tbody>
</table>
Use manipulatives or drawings to help you model the solution for each problem. Then show how you would work the problem in your math class.

Task 6: There are 35 students in the science class. 6/7 of them went on the field trip. How many students went on the trip?

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>CORRECT MODEL</th>
<th>CORRECT PROCEDURE INITIALLY</th>
<th>ADAPTED PROCEDURE TO FIT MODEL</th>
<th>USED CORRECT PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Laura</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Melanie</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
</tbody>
</table>

Task 7: There is 3/4 of a liter of orange juice in the refrigerator. If juice glasses hold 1/8 liter, how many glasses of juice can be served?

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>CORRECT MODEL</th>
<th>CORRECT PROCEDURE INITIALLY</th>
<th>ADAPTED PROCEDURE TO FIT MODEL</th>
<th>USED CORRECT PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Yes</td>
<td>Yes</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>Laura</td>
<td>Yes</td>
<td>Yes</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>Melanie</td>
<td>No</td>
<td>No</td>
<td>NA</td>
<td>NA</td>
</tr>
</tbody>
</table>

Task 8: Jim found 1/3 pizza in the refrigerator. If he ate 1/4 of the leftover pizza, how much of a whole pizza did he eat?

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>CORRECT MODEL</th>
<th>CORRECT PROCEDURE INITIALLY</th>
<th>ADAPTED PROCEDURE TO FIT MODEL</th>
<th>USED CORRECT PROCEDURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judy</td>
<td>Yes</td>
<td>Yes</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>Laura</td>
<td>Yes</td>
<td>Yes</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>Melanie</td>
<td>Yes</td>
<td>Yes</td>
<td>NA</td>
<td>NA</td>
</tr>
</tbody>
</table>
APPENDIX D

Rods and Circles

A. Size and Color of Fraction Rods

- Dark Green 1 Unit

- Light Green 1/2 Unit

- Red 1/3 Unit

- White 1/6 Unit

B. Size and Color of Fraction Circles

- 1 Unit Black
- 1/2 Unit Orange
- 1/3 Unit Green

- 1/4 Unit Pink
- 1/5 Unit Blue
- 1/6 Unit Red

- 1/8 Unit Brown
- 1/10 Unit Yellow
- 1/12 Unit Tan
Nancy Sutton Alexander, the daughter of Jesse A. Sutton, Jr. and Evelyn Ponder Sutton, was born in Shreveport, Louisiana on June 23, 1948. She later moved to Jonesboro, Louisiana, and graduated from Jonesboro-Hodge High School in 1966 as Salutatorian of her graduating class.

Nancy attended Louisiana Tech University for one year, then married Rodney M. Alexander on October 19, 1967. After starting a family, Nancy returned to Louisiana Tech University and received her Bachelor of Science degree in Mathematics Education in 1976, graduating Magna Cum Laude. The following year, Nancy attended graduate school at Louisiana Tech and held an assistantship in the Mathematics Department. In 1977 Nancy began teaching high school mathematics in her hometown of Quitman, Louisiana. She taught there for twelve years while pursuing graduate studies on a part-time basis. Nancy received her Master of Science degree from Louisiana Tech University in 1983.

In 1989 Nancy took a year of leave from teaching to begin doctoral studies at Louisiana State University in Baton Rouge, Louisiana. She held an assistantship in the Department of Curriculum and Instruction. In 1990 she was hired as an instructor at Louisiana Tech University, working full-time while continuing her doctoral studies. Nancy is currently an assistant professor in the College of Education at Louisiana Tech. She and Rodney live in Quitman, Louisiana. They have three children -- Ginger, Rod, and Lisa -- and one granddaughter, Jordyn.
DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Nancy Sutton Alexander

Major Field: Curriculum and Instruction

Title of Dissertation: The Role of the Unit as a Cognitive Bridge Between Additive and Multiplicative Structures

Approved:

[Signature]

Major Professor and Chairman

[Signature]

Dean of the Graduate School

EXAMINING COMMITTEE:

[Signatures]

Date of Examination:

November 25, 1996