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Applying Significance Testing to the Taguchi Methods of Quality Control.

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TO THE
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ABSTRACT

During recent years, the Taguchi Methods of quality control have been adopted in many industries because the goal of the methods is to produce a product or process that is robust to environmental conditions and component variation. The methodology involves using statistical experimental designs to achieve this goal through analyzing a performance measure, called a signal-to-noise ratio, that takes into account both variation and target value. However, many statisticians have criticized the Taguchi Methods for the lack of use of sound statistical practices. This research will focus on one specific aspect of the Taguchi Methods: the signal-to-noise ratio. Sampling distributions and moment estimates for three of the most popular ratios will be approximated. Using the derived formulas for the variance of the performance measure, statistical significance testing can be applied to designed experiments to determine significant factor effects. A thorough numerical analysis of the derived formulas is included.

CHAPTER 1. INTRODUCTION

1.1 Forward

Quality engineering encompasses two major aspects in the production of goods: evaluation of quality and increased productivity and reliability through the efficient improvement of quality. In recent years, a Japanese quality consultant by the name of Genichi Taguchi has introduced to the United States new ideas and methods for quality engineering. His philosophies and techniques which are based on cost-effective measures for product and process design are known as Taguchi Methods.

The emphasis of this research will be on one particular aspect of the Taguchi Methods, called signal-to-noise ratios. The use of statistical significance testing when selecting optimal parameter values using signal-to-noise ratios in quality engineering studies will be studied.

1.2 Taguchi's Definition of Quality

In order to better understand Taguchi Methods, it is important to first consider Taguchi's definition of product quality. "Quality is the loss a product causes to society after being shipped, other than any losses caused by its intrinsic functions (Taguchi, 1986)." Thus, quality is not expressed in subjective terms such as value or marginal utility. The loss referred to has two possible causes: variability of function and harmful side effects. The purpose of quality engineering is to build quality into the product by designing and producing products that minimize these losses without increasing costs.

1.3 Development Stages

Dr. Taguchi classifies three stages in the development of a product or process: system design, parameter design, and tolerance design. System design involves the innovation of an idea and uses scientific or engineering knowledge to configure the initial system. This includes selecting tentative values for system parameters. During parameter design, the tentative values of the system parameters from the first stage are tested over specified ranges in order to determine the optimal combination of settings. In this situation optimal means those settings that minimize performance variation in the system. The tolerance design stage follows parameter design. Once the optimal factor settings are discovered, bounds for these values are determined by considering the system's sensitivity to each factor.

During the parameter design stage, Taguchi uses statistical experimental designs. Traditionally, designed experiments have been used to analyze the mean response. But Taguchi indicates the importance of also analyzing the variation of response in order to achieve three objectives in the parameter design stage. These objectives are:

1.) Design products and processes to be robust to environmental conditions.

2.) Design and develop products that are robust to component variation.

3.) Minimize variation around a target value.

The goal is to design a product or process that consistently performs at a target level, regardless of surrounding conditions.

1.4 Loss Function

Taguchi (1986) relates the quality of a product to the "loss it imparts upon society." He typically models that loss with a quadratic loss function of the form,

$$L(y) = k(y - T)^2$$

where L = the loss in dollars, k = a cost coefficient, y = the value of the quality characteristic, and T = the target value for that characteristic. See Figure 1.1.

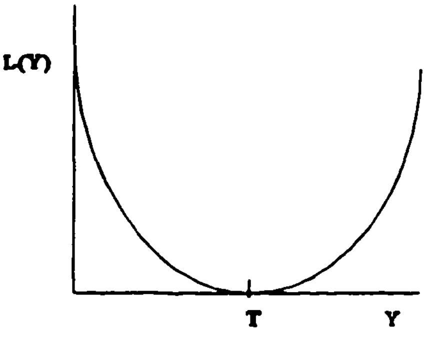

Figure 1.1
Quadratic Loss Function

To justify using this quadratic function to model loss, Taguchi considers a Taylor expansion of L(y) about T.

$$L(y) = L(T) + \frac{L'(T)}{1!}(y-T) + \frac{L''(T)}{2!}(y-T)^2 + \frac{L'''(T)}{3!}(y-T)^3 + \cdots$$

Since it is obvious that loss is at a minimum when y is equal to T, L(T) may be set to zero for the sake of simplicity. Further, since loss increases as y varies from T in any direction, the first derivative of L at T, L'(T), is also equal to zero. Substituting the conditions $L(T) = 0$ and $L'(T) = 0$ into the previous Taylor expansion for L reduces it to the following equation.

$$L(y) = \frac{L''(T)}{2!}(y-T)^2 + \frac{L'''(T)}{3!}(y-T)^3 + \cdots$$

The term involving $(y-T)^3$ should be omitted since it implies that the loss L would become smaller as y deviates lower than target T. By ignoring that term, as well as all other higher order terms, L(y) is then approximated by a quadratic term of the form $k(y-T)^2$, which is Taguchi's quadratic loss function.

1.5 Factor Types and Arrays

Again, in order to minimize this loss, the product must be produced at levels that ensure minimal variation around its target value. In order to satisfy this objective, it is necessary to identify the two types of factors that affect the product's functional characteristics. They are controllable factors, sometimes called design parameters, and uncontrollable factors, often called noise factors. Controllable factors are easy to control. But noise factors are hard to control, impossible to control, or too expensive to control. Thus, Taguchi proposes that the way to achieve robustness is to select the levels of the control factors that minimize the effects of the noise factors. Instead of finding and

eliminating the uncontrollable special causes of variation, the impact of the causes is reduced or removed.

Taguchi's application of experimental designs involves separating the control factors from the noise factors. The control factors and the noise factors are assigned separate, individual experimental designs. The control factors are varied according to an orthogonal array where the columns represent the control factors and the rows represent runs of specific sets of factor levels to be tested. The noise factors are varied according to their own orthogonal array. Then the two arrays are overlaid into a product array where the control array is the inner array and the noise array is the outer array. This means that the outer array is tested across every row of the inner array.

1.6 Signal-to-Noise Ratios

After the experiments are performed, the data must be analyzed. The mean response may be studied by considering each run of the inner (control) array. In the Taguchi approach, if the product array format is used so that the noise factors are varied systematically, using sliding levels, specific interactions between the control and noise factors do not have to be recognized. The optimal parameter values can be discovered by studying the variation of the response by a suitable performance measure, called a signal-to-noise ratio. There are many formulas for this signal-to-noise ratio (SN), depending on the applied situation. Taguchi classifies different situations under study into specific categories and defines an appropriate SN ratio. All are derived from the loss function. In its simplest form, the SN represents the ratio of the mean response (i.e. signal) for each run

in the inner array to the standard deviation (i.e. noise). The purpose of the SN ratio is to separate location (mean) effects and dispersion (variance) effects. Factor levels that maximize this performance measure are generally considered optimal.

Regarding the use of SN ratios, Taguchi says that a statistical analysis of variance with F tests can be used to find the factors that affect the mean and the factors that influence the SN ratios. However, he suggests graphing the SN ratios and mean responses for each control factor level and "picking the winner."

1.7 Confirmation

The final step in Taguchi's approach to parameter design is confirmation. Once the optimal settings are determined, a small number of confirmation runs using the selected settings are made to verify the optimality conclusion.

1.8 An Example

To more clearly understand Taguchi's method of parameter design, an example will be illustrated. The case study selected (Byrne and Taguchi, 1987 and Montgomery, 1991) for this illustration involves an application for automotive engines. The problem is to choose a method for economically assembling an elastomeric connector to a nylon tube that is to be fit to an engine component while delivering the required pull-off performance. There were actually two simultaneous objectives for the experiment: 1) minimize the assembly effort and 2) maximize the pull-off force. The analysis that follows will actually concentrate on the second objective of maximizing the pull-off force. Factors that might affect the connector's pull-off force were identified and separated into four control factors

and three noise factors related to conditioning. Each control factor was tested at three levels and each noise factor at two levels. The following table summarizes the factors and their test levels.

Table 1.1
Factors and Levels

FACTOR	TYPE	LEVELS
A. Interference	Control	Low, Medium, High
B. Wall Thickness	Control	Thin, Medium, Thick
C. Insertion Depth	Control	Shallow, Medium, Deep
D. Percent Adhesive in Pre-dip	Control	Low, Medium, High
E. Conditioning Time	Noise	24h, 120h
F. Conditioning Temp	Noise	72°F, 150°F
G. Conditioning Relative Humidity	Noise	25%, 75%

The method that follows searches for the levels of the control factors that are least influenced by changes in the noise factors and yet yield the maximum pull-off force for the connector. In accordance with the Taguchi Methods, separate experimental designs are selected for the control and noise factors. For this example, an L_9 orthogonal array was chosen. It is considered to be the most efficient design for testing four factors at three levels with only 9 runs. However, an L_8 orthogonal array was selected for the three noise factors. An L_8 array will accommodate up to seven factors at two levels each. Since we have only three, the remaining columns could be used to estimate interaction effects and

experimental error. But this is rarely deemed important in Taguchi's analysis. Recall, the purpose of the noise array is to create variation in order to find the control factor levels that are the least influenced by the noise. The L_9 and L_8 arrays used are given in Tables 1.2 and 1.3. In Table 1.2, the ones, twos, and threes under the factors represent the lowest, middle, and highest factor levels, respectively. In Table 1.3, the ones represent the lowest factor levels while the twos represent the highest factor levels.

The next step in the Taguchi Method is to combine the two separate arrays in order to form the complete parameter design layout. The L_9 array of control factors is the outer array. This design yields a total sample size of 8(9)=72 experimental runs because each of the nine runs from the inner array is tested across the eight runs of the outer array. This combined array is sometimes called the product array. The resulting combined matrix for this experiment filled in with observed pull-off forces for the 72 runs is summarized below.

Table 1.2
L_9 Array for Control Factors

Run		Factor A	Factor B	Factor C	Factor D
1		1	1	1	1
2		1	2	2	2
3		1	3	3	3
4		2	1	2	3
5		2	2	3	1
6		2	3	1	2
7		3	1	3	2
8		3	2	1	3
9		3	3	2	1

Table 1.3
L_8 Array for Noise Factors

Run		Factor E	Factor F	Factor EXF	Factor G	Factor EXG	Factor FXG	error
1		1	1	1	1	1	1	1
2		1	1	1	2	2	2	2
3		1	2	2	1	1	2	2
4		1	2	2	2	2	1	1
5		2	1	2	1	2	1	2
6		2	1	2	2	1	2	1
7		2	2	1	1	2	2	1
8		2	2	1	2	1	1	2

In order to analyze this data, the mean response for each run in the L_9 array was calculated. This is called $\bar{y}$ in Table 1.4 But in order to analyze the variation, a signal-to-noise ratio for each of those runs is also computed. Taguchi has defined several different signal-to-noise ratios depending on the situation and objective of the experiment. However, all are tied to his loss function and are basically the ratio of the mean response to the standard deviation. This aspect of the Taguchi Methods will be discussed in greater detail in Chapter 3.

Since the objective of the experiment illustrated here is to maximize pull-off force, the appropriate signal-to-noise ratio is

$$SN_L = -10\log\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{y_i^2}\right).$$

This signal-to-noise formula is called "larger-the-better" (Montgomery, 1991) and is generally applied when the desired response is to be maximized. The factor levels that maximize the signal-to-noise ratio are considered to be the optimal settings. The computed SN_Ls are also included in Table 1.4.

Table 1.4
Product Array

Outer Array L_8										
E	1	1	1	1	2	2	2	2		
F	1	1	2	2	1	1	2	2		
EXF	1	1	2	2	2	2	1	1		
G	1	2	1	2	1	2	1	2		
EXG	1	2	1	2	2	1	2	1		
FXG	1	2	2	1	1	2	2	1		
error	1	2	2	1	2	1	1	2		
Inner Array L_9 Run A B C D									$\bar{y}$	SN_L
1 1 1 1 1	15.6	9.5	16.9	19.9	19.6	19.6	20.0	19.1	17.53	24.03
2 1 2 2 2	15.0	16.2	19.4	19.2	19.7	19.8	24.2	21.9	19.48	25.52
3 1 3 3 3	16.3	16.7	19.1	15.6	22.6	18.2	23.3	20.4	19.03	25.34
4 2 1 2 3	18.3	17.4	18.9	18.6	21.0	18.9	23.2	24.7	20.13	25.90
5 2 2 3 1	19.7	18.6	19.4	25.1	25.6	21.4	27.5	25.3	22.83	26.91
6 2 3 1 2	16.2	16.3	20.0	19.8	14.7	19.6	22.5	24.7	19.23	25.33
7 3 1 3 2	16.4	19.1	18.4	23.6	16.8	18.6	24.3	21.6	19.85	25.71
8 3 2 1 3	14.2	15.6	15.1	16.8	17.8	19.6	23.2	24.2	18.34	24.83
9 3 3 2 1	16.1	19.9	19.3	17.3	23.1	22.7	22.6	28.6	21.20	26.15

In order to select the factor levels that give maximum results, Taguchi suggests graphing both the average means and the average signal-to-noise ratios for each level of each factor and visually identifying the factors that seem to be significant and at what level. For this example, this has been done and the graphs are represented in Figures 1.2 and 1.3.

In order to "pick the winner" in maximizing the average pull-off force, the following factor levels would be selected: Medium Interference, Medium Wall Thickness,

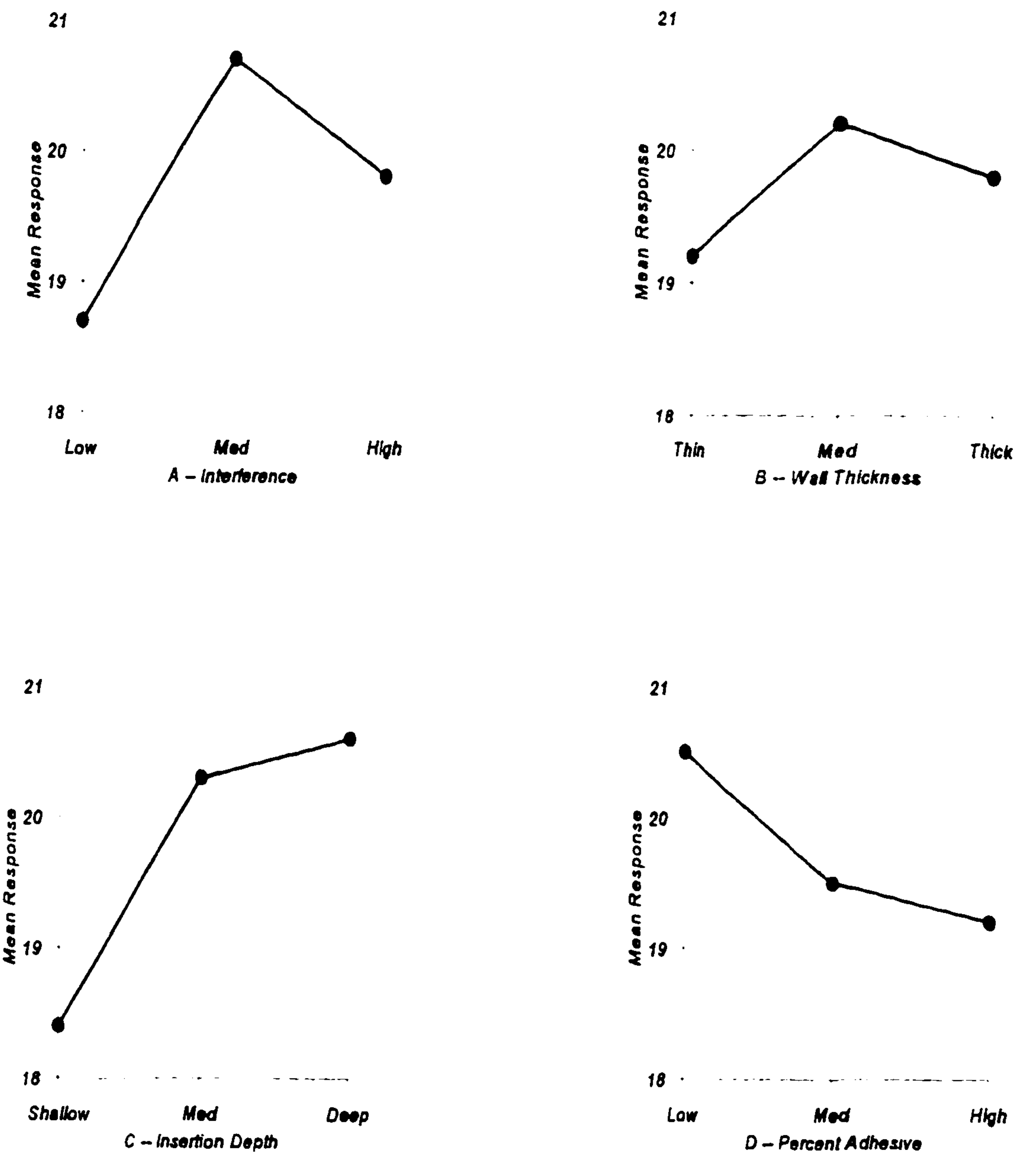

Figure 1.2
Control Factors versus Mean Response

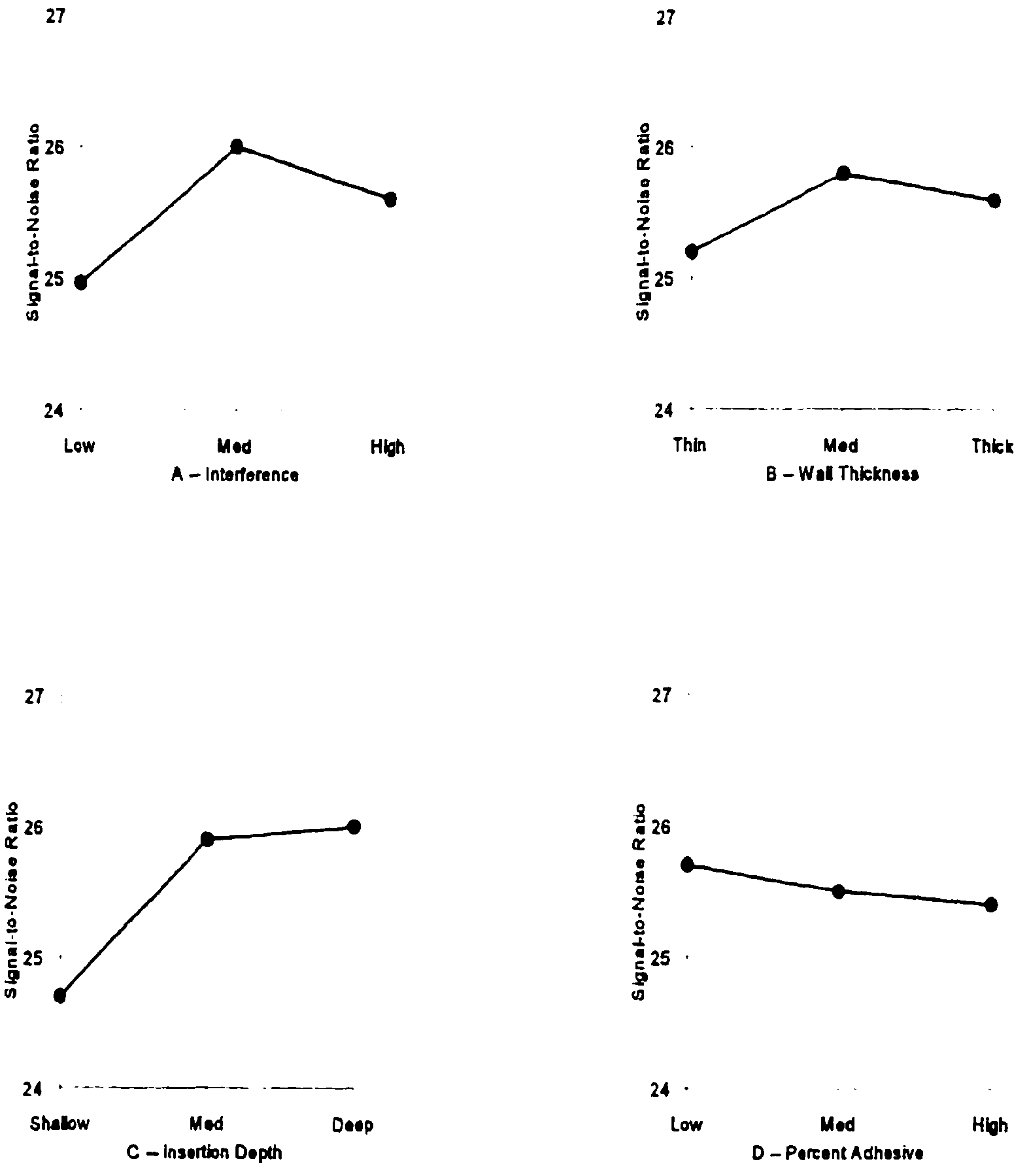

Figure 1.3
Control Factors versus Signal-to-Noise Ratio

Deep Insertion Depth, and Low Percent Adhesive. Even though Medium Wall Thickness was "optimal" on both graphs, cost considerations determined that Thin Wall Thickness should be selected. Since the effects of Wall Thickness did not appear to be as large as the effects of Interference and Insertion Depth, the researchers did not feel this to be a major deviation. The final factor settings selected were: Medium Interference, Thin Wall Thickness, Deep Insertion Depth, and Low Percent Adhesive.

The final step in the experiment was to confirm the results. The combination of factor levels deemed optimal was not run during the initial experiment so five additional tests were performed using these settings. The noise factors were applied at their lowest levels. The average of those five samples was within a predetermined 90% confidence band, thus confirming the experiment results.

1.9 Objectives of the Research

The question addressed in this research is whether the factor settings that were determined to be optimal are statistically significant. This research will focus on determining the sampling distributions for functions of popular choices of SN ratios. Using the sampling distributions to derive the mean and variance for the SN ratios will enable significance testing to be performed on the effects of the experiment.

1.10 Justification of the Research

Many statisticians have criticized the Taguchi Methods for the lack of use of sound statistical tools. However, the Taguchi Methods are being employed by large industries.

This research will tie sound statistical tools, such as significance testing using appropriate distributions for the response variable, to the methods of Taguchi.

In addition, the methodology proposed by this research could be used to prevent unnecessary, costly parameter settings for processes. The Taguchi Methods determine settings for parameters in the process, but if the settings determined are not significantly different from other settings, it could be costly to incorporate the Taguchi settings when they are not necessary.

1.11 Organization of the Research

The research is presented in six chapters. In Chapter 2, a thorough literature review dealing with Taguchi Methods will be discussed. Chapter 3 focuses on one specific aspect of the Taguchi Methods -- signal-to-noise ratios. That chapter will justify their use in specific applications. Chapter 4 derives the sampling distributions and moment estimates for the most popular SN ratios. In Chapter 5, a numerical analysis justifying those formulas is presented. The final chapter summarizes the research and conclusions.

CHAPTER 2. LITERATURE REVIEW

2.1 Introduction of Taguchi Methods to the Western World

While in Japan, Taguchi began developing his ideas and methodologies in the area of quality improvement during the 1950s. But his methods and ideas were practically unknown outside of Japan until he received a grant in 1980 to visit the United States and lecture on his ideas. His audiences' initial reaction was skepticism, although a few individuals associated with some very large U.S. corporations and organizations such as AT&T, Ford, Xerox, and the American Supplier Institute became very interested and started to apply his philosophy and techniques. The Quality Assurance Center of AT&T Bell Labs organized two conferences in 1984 and 1985. These conferences exposed the Taguchi Methods to the statistical community as a whole and thus stimulated research on this new topic. Therefore, the last decade has included a great deal of discussion on Taguchi's ideas, as well as numerous applications in industry. Many articles and even books have been published that explain, review, apply, or criticize the Taguchi Methods.

One of the earliest papers published was by Kackar (1985) who was affiliated with AT&T Bell Labs at the time. His paper summarized the Taguchi Methods, applying terminology that was more widely understood by the statistical community. There had been some difficulty with the translations from Japanese. His paper made it much easier for many interested parties to better understand the Taguchi Methods and see the similarities with existing statistical methods. The article was followed by remarks from several discussants proclaiming that Taguchi was to be applauded for bringing awareness

to the use of statistics in designing quality into products and processes but questioning his practices with regard to choice of experimental design, treatment of interactions, and use of signal-to-noise ratios. In a separate article in the same journal where the paper by Kackar and discussants appeared, Hunter (1985) also made important contributions to the understanding of Taguchi's methods and supported the philosophy behind those methods by explaining some of the existing statistical tools involved, such as experimental design. He also suggested that the choice of experimental design, the role of interactions, and the effects of data transformations should be examined further.

2.2 Resolution of Experimental Designs

Several researchers, including Box and Meyer (1986) and Ryan (1988) have claimed that Taguchi's choice for experimental designs are not optimal in that maximum resolution for the number of given main effects and array size is not achieved. Bullington, Hool, and Maghsoodloo (1990) provided a technique to obtain Resolution IV designs and still use Taguchi's orthogonal inner and outer array approach. Further, Kacker, Lagergren, and Filliben (1991) showed that Taguchi's orthogonal arrays are equivalent to fractional factorials. They claimed this gave credence to the use of Taguchi's published list of orthogonal arrays that are used in many applied applications.

2.3 Inner and Outer Array Approach

Other researchers have questioned the purpose of the inner and outer array approach of the Taguchi Methods. Welch, Yu, Kang, and Sacks (1990) proposed placing both control and noise factors in a single array. But Shoemaker, Tsui, and Wu (1991)

showed how Taguchi's combined array approach can allow for a smaller number of experiments to be run, making the method more economical. Their method "sacrifices" some interactions which they have deemed to be insignificant or unimportant based on some type of previous information.

2.4 Signal-to-Noise Ratios

Another aspect of the Taguchi Methods that has drawn a great deal of concern is the purpose and use of the signal-to-noise ratios. Taguchi does not show how minimizing loss is necessarily connected to maximizing a particular signal-to-noise ratio. León, Shoemaker, and Kackar (1987) explained Taguchi's signal-to-noise ratios in detail and discussed their role in parameter design. They justified the use of a signal-to-noise ratio when special models for the response are assumed. Under those conditions, the signal-to-noise ratios take into account the existence of a special type of design parameter they called an adjustment parameter. They showed that when adjustment parameters exist, using a signal-to-noise ratio permits the optimization problem to be done in two steps. But they went on to show that when these special parameters do not exist, the signal-to-noise ratio does not appear to optimize the situation. Because of those restrictions, Box (1988) proposed the use of transformations as a more appropriate method. Then, Maghsoodloo (1990) derived the precise relation of two widely used signal-to-noise ratios to Taguchi's quality loss function. The two ratios determined were the "smaller-the-better" and the "larger-the-better" formulas. The "nominal-the-better" ratio was also considered, but he was unable to derive an exact relation. His results for all three cases were

tabulated. Miller and Wu (1994) give a more rigorous justification of the use of the signal-to-noise ratios by considering Fieller intervals. Because it is pertinent to the problem to be considered in this research, their work will be discussed in more detail in the next chapter.

Other researchers have been concerned over whether to use a single performance statistic, such as the signal-to-noise ratio, to minimize variation or to analyze the mean response and variation separately. Ullman (1989) proposed a method called Analysis of Ranges (ANOR). He showed it is an extension of an existing statistical method called Analysis of Means (ANOM). The hybrid method involves separately analyzing location and dispersion effects and then evaluating them together.

Steinberg and Burztyn (1994) suggest explicit direct modeling of the noise factor effects. They encourage avoiding summary measures, such as SN ratios. Instead they analyze the full array to determine important dispersion effects.

2.5 Overall Criticisms of the Taguchi Methods

Due to the controversies regarding the Taguchi Methods, Lochner (1991) summarized the areas that are well-respected, such as his overall philosophy, and those that are questioned, such as signal-to-noise ratios and experimental design selection. A more technical and extensive summary of that type was presented by Nair (1992). He compiled the comments of a panel discussion on the Taguchi Methods. The panel was composed of 16 leading practitioners and researchers, many of whom have written previously mentioned papers. Nair's goal was "to provide readers with a balanced and up-

to-date overview of (a) the importance and usefulness of the principles underlying parameter design, (b) Taguchi's methodology for implementing them, and (c) the various research efforts aimed at developing alternative methods."

2.6 Applications of Taguchi Methods

Today, Taguchi Methods are being employed in many U.S. companies, including AT&T Bell Labs, Ford, Xerox, and Exxon. Several papers have been published that demonstrate the use of these methods for some actual applied situation. For example, Crossfield and Dale (1991) used Taguchi Methods for the design improvement process of turbochargers. Cao and Zhou (1993) presented the application of parameter design in the quality control of the lift gate assembly for a minivan. Eaton, Lakers, Prybyla, and Shina (1993) applied Taguchi Methods in the optimization of circuit design.

Organizations, such as the American Supplier Institute (ASI) and the Center for Quality and Productivity Improvement at the University of Wisconsin, offer courses on these methods. ASI even has an annual symposium for the presentation of case studies applying Taguchi Methods. Dr. Taguchi, himself, has published numerous textbooks regarding quality engineering and design of experiments. They are available through ASI.

CHAPTER 3. SIGNAL-TO-NOISE RATIOS

3.1 Validating the Use of Signal-to-Noise Ratios

The validity of the use of Taguchi's signal-to-noise ratio as a performance measure has been researched by several statisticians. Leon, Shoemaker, and Kacker (1987) explained the role of SN ratios in parameter design and gave a specific situation where they found a SN ratio to be an appropriate criteria to consider for quality improvement. Box (1988) supported their presentation and went on to suggest the use of transformations. Maghsoodloo (1990) provided derivations for some of the most commonly used SN ratios based on Taguchi's Quality Loss Function. Most recently, Miller and Wu (1994) provided a rigorous justification for the use of Taguchi's signal-to-noise ratios by considering a signal-response application. [In Taguchi's terminology, this would be a dynamic system.]

Leon, Shoemaker, and Kacker (1987) show that "if certain models for the product or process response are assumed, then maximization of the SN ratio leads to minimization of average squared-error loss." In these situations, the authors show that Taguchi's SN ratios exploit the existence of special design adjustment parameters thus enabling the parameter design optimization process to be broken down into two steps: 1) optimizing the SN ratio and 2) then optimizing the mean.

3.2 A Specific Situation

One specific problem considered was a basic static parameter design problem. For this type of problem, the output y is determined by the noise, N, through a transfer

function, f. The transfer function depends on the design parameters, $\theta=(\underline{d},\underline{a})$. The parameters $\underline{a}$ are considered fine-tuning adjustments that can be optimized after the main design parameters $\underline{d}$ are determined and fixed.

Consider a specific transfer function model:

$$y = f(\underline{d},\underline{a},\underline{N}) = \mu(\underline{d},\underline{a})\xi(\underline{N},\underline{d}),$$

where $\mu(\underline{d},\underline{a})$ is a strictly monotone function of each component of $\underline{a}$ for each $\underline{d}$. [Note that $E\{\xi(\underline{N},\underline{d})\} = 1$ since $\mu_y = \mu(\underline{d},\underline{a})$.] This model implies that $\left(\frac{\sigma_y}{\mu_y}\right)^2$ does not depend on $\underline{a}$ and that σ_y is proportional to μ_y.

Because of the independence of $\underline{d}$ and $\underline{a}$, $(\underline{d}^*,\underline{a}^*)$ can be found that minimize the expected loss $R(\underline{d},\underline{a}) = E\{L(y,T)\} = E\{k(y-T)^2\}$ by the following two-step procedure.

Step 1. Find $\underline{d}^*$ that minimizes $R(\underline{d},\underline{a})$ over all $\underline{a}$.

Let $P(\underline{d}) = \min_a R(\underline{d},\underline{a})$.

Step 2. Find $\underline{a}^*$ that minimizes $R(\underline{d}^*,\underline{a})$.

For the defined loss function and transfer function, it is shown that $P(\underline{d})$ is a decreasing function of SN_T. For convenience, let k=1 in the expected loss function. Then,

$$\begin{aligned}
R(\underline{d},\underline{a}) &= E\{(y - T)^2\} \\
&= E\{[\mu(\underline{d},\underline{a})\xi(\underline{N},\underline{d}) - T]^2\} \\
&= E\{\mu^2(\underline{d},\underline{a})\xi^2(\underline{N},\underline{d}) - 2T\mu_y + T^2\} \\
&= \mu^2(\underline{d},\underline{a})[\sigma(\underline{d})+1]^2 - 2T\mu_y + T^2
\end{aligned}$$

$$= \mu^2(\underline{d},\underline{a})\sigma^2(\underline{d}) + \mu^2(\underline{d},\underline{a}) - 2T\mu_y + T^2$$

$$= \mu^2(\underline{d},\underline{a})\sigma^2(\underline{d}) + [\mu(\underline{d},\underline{a}) - T]^2$$

where $\sigma^2(\underline{d}) = Var[\xi(\underline{N},\underline{d}) = E[\xi^2(\underline{N},\underline{d})] - 1$.

To optimize, consider the first derivative set equal to zero. Then,

$$\mu(\underline{d},a^*(\underline{d})) = \frac{T}{1+\sigma^2(\underline{d})}$$

and

$$P(\underline{d}) = \min_a R(\underline{d},\underline{a}) = \frac{T^2\sigma^2(\underline{d})}{1+\sigma^2(\underline{d})}$$

It can be seen that $P(\underline{d})$ is an increasing function of $\sigma^2(\underline{d})$. Now, consider SN_T.

$$SN_T = -10\log\left(\frac{\sigma_y^2}{\mu_y^2}\right)$$

$$SN_T = -10\log\left[\frac{\mu^2(\underline{d},\underline{a})\sigma^2(\underline{d})}{\mu^2(\underline{d},\underline{a})}\right]$$

$$SN_T = -10\log[\sigma^2(\underline{d})]$$

Therefore, if f applies, maximizing SN_T leads to minimizing quadratic loss.

The authors go on to discuss situations where the SN ratio is not independent of the adjustment parameters and propose a different performance measure that will enable the two-step optimization procedure. They call the measures PerMIA's -- performance measures independent of adjustment.

3.3 Conditions Which Make SN Ratios Appropriate

Based on the results of Leon, Shoemaker, and Kacker (1987) that show SN_T is an appropriate performance measure for the special case in which σ_y is proportional to μ_y, Box (1988) argues that analyzing $\ln y$ is more efficient than analyzing SN_T.

Consider $z = \ln y$ with mean μ_z and constant variance σ_z^2. Then, according to Johnson and Kotz (1970),

$$\mu_y = \exp\left(\mu_z + \frac{1}{2}\sigma_z^2\right)$$

and

$$\sigma_y = \mu_y\sqrt{\exp(\sigma_z^2) - 1}.$$

These relationships for μ_y and σ_y are exact when z is normally distributed, but only approximate otherwise. Notice σ_y is clearly proportional to μ_y. Further, the coefficient of variation,

$$\frac{\sigma_y}{\mu_y} = \sqrt{\exp(\sigma_z^2) - 1},$$

is independent of μ_y. This suggests that analyzing $\frac{\sigma_y}{\mu_y}$, or more precisely, SN_T, is basically the same as analyzing $\sigma_z = \sigma_{\ln y}$. Box suggests that using the variance stabilizing transformation of taking logarithms of y when σ_y is proportional to μ_y is a more efficient criteria than SN_T.

3.4 Contributions of Miller and Wu

The work performed by Miller and Wu (1994) is the most pertinent to the issue to be addressed in this research. The authors considered the situation where the system response takes on different values due to changes in the signal factor. They modeled the relationship as a measurement system with a linear model of the form

$$Y = \alpha + \beta M + \epsilon, \ \epsilon \sim N(0, \sigma^2),$$

where M is the signal factor (i.e. the quantity of interest) and Y is the response whose measurement is used to estimate M. Suppose the true relationship between M and Y is known so that α, β, and σ^2 are known. Then the estimator of M for an observed response $Y = Y_{obs} = \alpha + \beta\hat{M}$ is given by,

$$\hat{M} = \frac{Y_{obs} - \alpha}{\beta}.$$

Calculating the variance of the estimation yields

$$Var(\hat{M}) = Var\left[\frac{Y_{obs} - \alpha}{\beta}\right] = \frac{1}{\beta^2} Var[Y_{obs} - \alpha] = \frac{\sigma^2}{\beta^2}$$

The reciprocal of V(M) would then be $\omega = \frac{\beta^2}{\sigma^2}$, the quantity Taguchi uses for his signal-to-noise ratio.

However, the above derivation assumed α, β, and σ^2 were known. In reality, those values need to be estimated. So Miller and Wu considered the expected length of the Fieller interval for M. But that length is contingent on the type of calibration procedure employed. Two common types were considered in their paper. What is known as the standard calibration type will be considered here. The standard type is where the relationship between Y and M is modeled by observations of Y taken for a series of standards having known values of M. Define the variables y_i for the measured values of Y and m_i for the known values of M associated with the n standards, $(i = 1,\ldots,n)$. The usual estimators for α, β, and σ^2 are given by the following formulas.

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{m}$$

$$\hat{\beta} = \frac{\sum_{i=1}^{n}(y_i-\bar{y})(m_i-\bar{m})}{\sum_{i=1}^{n}(m_i-\bar{m})^2}$$

$$s^2 = \frac{\sum_{i=1}^{n}(y_i-\bar{y})^2 - \hat{\beta}\sum_{i=1}^{n}(y_i-\bar{y})(m_i-\bar{m})}{n-2}$$

The $100(1-\gamma)\%$ prediction interval for Y with a given value of $M = m_0$ can then be given by

$$\hat{\alpha} + \hat{\beta}m_0 \pm t_{\frac{\gamma}{2},n-2}\, s\sqrt{1 + \frac{1}{n} + \frac{(m_0-\bar{m})^2}{\sum_{i=1}^{n}(m_i-\bar{m})^2}}.$$

But now consider the reverse situation where y_i is the observed value of Y for a sample with an unknown value of m. A $100(1-\gamma)\%$ confidence interval for m_0 can be obtained by using the values of M for which y_0 is a member of that $100(1-\gamma)\%$ prediction interval of Y. This is called a Fieller interval. The Fieller interval will contain all values of m that make the following inequality true:

$$(y_0 - \hat{\alpha} + \hat{\beta}m)^2 \leq t^2 s^2\left(1 + \frac{1}{n} + \frac{(m_0-\bar{m})^2}{\sum_{i=1}^{n}(m_i-\bar{m})^2}\right).$$

where $t = t_{\frac{\gamma}{2}, n-2}$. Miller and Wu went on to show that when the set of values of M form a finite interval, (m_L, m_U), the length of the interval is determined by the following formula.

$$m_U - m_L = 2t\left[\left(1 + \frac{1}{n}\right)\left(\hat{\omega} - \frac{t^2}{\sum_{i=1}^{n}(m_i - \bar{m})^2}\right) + \frac{\hat{\omega}(m_0 - \bar{m})^2}{\sum_{i=1}^{n}(m_i - \bar{m})^2}\right]^{\frac{1}{2}} \left(\hat{\omega} - \frac{t^2}{\sum_{i=1}^{n}(m_i - \bar{m})^2)}\right)^{-1}$$

The length of the Fieller interval is dependent on the values of $\hat{\omega} = \frac{\hat{\beta}^2}{s^2}$ and $\sum_{i=1}^{n}(m_i - \bar{m})^2$. The F statistic for testing the hypothesis, $\beta = 0$, is given by $f = \sum_{i=1}^{n}(m_i - \bar{m})^2\hat{\omega}$. Thus, the width of the Fieller interval depends on this F statistic.

Notice as $\hat{\omega} > \frac{t^2}{\sum_{i=1}^{n}(m_i - \bar{m})^2}$ increases, $m_U - m_L$ decreases. The observed length of the

Fieller interval is a random variable itself, and further, the variable quantity f has a noncentral F distribution with 1 and n-2 degrees of freedom. Thus, the expected length of this Fieller interval is given by:

$$E(\hat{\omega}) = \frac{n-2}{n-4}\left(\omega + \frac{1}{\sum_{i=1}^{n}(m_i-\bar{m})^2}\right).$$

So as ω decreases, the expected length of the Fieller interval will also decrease, thereby ensuring that ω (Taguchi's SN ratio) is a valid performance measure for the given situation. A similar result was obtained when Miller and Wu considered the second type of calibration where fixed measurements are taken on a sample with fixed responses instead of standards.

Now that it has been determined that SN ratios are appropriate response variables for specific situations, the distributions related to some of the most popular SN ratios will be developed. This will allow significance tests and/or confidence intervals to be employed to determine if the optimal SN ratio is significantly different from those not deemed optimal.

However, a student once asked Dr. Taguchi to comment on the following statement regarding quality evaluation based on the signal-to-noise ratio: "I think that the form of distribution is important in statistical methods, because the significance and the confidence limits are considered." Dr. Taguchi responded, "Attempts have been made to give sense to significant tests and confidence limits in quality engineering as well. Now

I believe they are nearly worthless. They may be used as references but are useless for practical purposes. In technological fields, improvement is what should be considered. Understanding the present status is not the aim. It must be remembered that improvement of quality is almost always associated with improvement of variability or the mean square error." He went on further to say, "To think about a distribution may be important for scientists or mathematicians who are interested in investigating a fact but have nothing to do with engineering methods regarding the reduction of variability." (Taguchi, 1992) Dr. Taguchi's comments are contradictory to those of most statisticians. In numerous fields where statistics are employed, including medical research, significance testing is deemed to be a crucial element. Why would quality engineering be any different?

CHAPTER 4. SAMPLING DISTRIBUTIONS OF SN RATIOS

4.1 Purpose

In Chapter 3 it was shown that there are situations where signal-to-noise ratios are appropriate response variables. However, when utilizing Taguchi's methodology, the largest SN ratio is found and is supposed to indicate the optimal combination of settings of factors. But, are these "optimal" settings significantly different from other combinations? In order to answer this question, the distribution of the response variable needs to be known so that statistical significance testing can be performed. Therefore, the sampling distributions of Taguchi's SN ratios need to be determined. Further, if the responses are significantly different, looking at only the response and choosing the settings associated with the response that is different can be misleading. A significant difference indicates that the null hypothesis is void. At that stage, the factor effects, rather than just the response, should be analyzed.

In this chapter, the three Taguchi SN ratios predominately used in applications will be researched further. Distributions associated with two of these SN ratios, SN_T and SN_S will be derived. Difficulties with the third, SN_L, will be discussed.

4.2 Nominal-the-better

First consider the nominal-the-better SN ratio:

$$SN_T = -10\log\left[\frac{s^2}{\bar{y}^2}\right]$$

In order to find a known distribution related to this SN ratio, the equation will have to be manipulated. This equation can be rewritten as

$$10^{\frac{SN_T}{10}} = \frac{\bar{y}^2}{s^2}$$

Multiplying through by n results in the following equation which will be used to find a sampling distribution.

$$n10^{\frac{SN_T}{10}} = \frac{n\bar{y}^2}{s^2}$$

By finding the distribution of $\frac{n\bar{y}^2}{s^2}$, the moments of SN_T can then be approximated. The variance of SN_T will be enough information to perform significance testing for the factor effects in the experiment.

It can be proved that $\frac{n\bar{y}^2}{s^2}$ has a noncentral F distribution with 1 and n-1 degrees of freedom and noncentrality parameter $\frac{n\mu^2}{\sigma^2}$ if the y_i are normally distributed with mean μ and variance σ^2 for $i = 1,2,\ldots,n$.

4.3 Proof

Let $y_i \sim N(\mu,\sigma^2)$ for $i = 1,2,\ldots,n$. Then $\bar{y} \sim N(\mu, \frac{\sigma^2}{n})$. It can be written that

$y_i = \mu + \sigma z_i$, $i = 1,2,\ldots,n$ where each z_i is an independent unit normal random variable.

Consider $\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n}$. These results follow.

$$\bar{y} = \frac{\sum_{i=1}^{n} (\mu + \sigma z_i)}{n}$$

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} \mu + \frac{1}{n}\sum_{i=1}^{n} \sigma z_i$$

$$\bar{y} = \mu + \sigma\bar{z}$$

Then, $\bar{y}^2 = \mu^2 + 2\sigma\mu\bar{z} + \sigma^2\bar{z}^2$. Let $U = \sum_{i=1}^{n} z_i$ so that $U \sim N(0,n)$ and $\frac{U}{\sqrt{n}} \sim N(0,1)$.

Substituting $\frac{U}{n}$ for $\bar{z}$ into the equation for $\bar{y}^2$ yields the following equivalent equation.

$$\bar{y}^2 = \mu^2 + 2\sigma\mu\frac{U}{n} + \sigma^2\frac{U^2}{n^2}$$

Factoring the last equation gives the next two equations.

$$\bar{y}^2 = \frac{\sigma^2}{n}\left(\frac{n\mu^2}{\sigma^2} + \frac{2\mu U}{\sigma} + \frac{U^2}{n}\right)$$

$$\bar{y}^2 = \frac{\sigma^2}{n}\left(\frac{\sqrt{n}\mu}{\sigma} + \frac{U}{\sqrt{n}}\right)^2$$

Rearranging leads to the following result.

$$\frac{n\bar{y}^2}{\sigma^2} = \left(\frac{U}{\sqrt{n}} + \frac{\sqrt{n}\mu}{\sigma}\right)^2$$

This equation can be written as

$$\frac{n\bar{y}^2}{\sigma^2} = \sum_{i=1}^{1}\left(\frac{\mu_i}{\sqrt{n}} + \frac{\sqrt{n}\mu}{\sigma}\right)^2.$$

Since the term $\frac{\sqrt{n}\mu}{\sigma}$ is a constant and $\frac{\mu_i}{\sqrt{n}}$ is unit normal, $\frac{n\bar{y}^2}{\sigma^2}$ is distributed as noncentral chi-squared with 1 degree of freedom and noncentrality parameter $\frac{n\mu^2}{\sigma^2}$, (Johnson and Kotz, 1970).

To find the distribution of $\frac{n\bar{y}^2}{s^2}$, note the following equivalent equation.

$$\frac{n\bar{y}^2}{s^2} = \frac{\frac{n\bar{y}^2}{\sigma^2}}{\frac{s^2}{\sigma^2}} = \frac{\frac{\left[\frac{n\bar{y}^2}{\sigma^2}\right]}{1}}{\frac{\left[\frac{(n-1)s^2}{\sigma^2}\right]}{(n-1)}}$$

The expression $\frac{n\bar{y}^2}{\sigma^2}$ has been shown to be noncentral chi-squared with 1 degree of freedom. The expression $\frac{(n-1)s^2}{\sigma^2}$ is well known to be distributed as chi-squared with n-1 degrees of freedom. Thus the ratio $\frac{n\bar{y}^2}{s^2}$ is distributed as noncentral F with 1 and n-1 degrees of freedom and noncentrality parameter $\frac{n\mu^2}{\sigma^2}$. Since $n \cdot 10^{\frac{SN_T}{10}} = \frac{n\bar{y}^2}{s^2}$, $n \cdot 10^{\frac{SN_T}{10}}$ has a noncentral F distribution with 1 and n-1 degrees of freedom and noncentrality parameter $\frac{n\mu^2}{\sigma^2}$ when $y_i \sim N(\mu,\sigma^2)$, $i = 1,2,\ldots,n$.

With this known distribution, the mean and variance of $n \cdot 10^{\frac{SN_T}{10}}$ can be determined using formulas for the noncentral F given in Kotz and Johnson, (1981).

$$E\left(n \cdot 10^{\frac{SN_T}{10}}\right) = \frac{n-1}{n-3}\left[1 + \frac{n\mu^2}{\sigma^2}\right]$$

$$Var\left(n \cdot 10^{\frac{SN_T}{10}}\right) = \frac{2(n-1)^2(n-2)}{(n-3)^2(n-5)}\left[1 + \frac{2n\mu^2}{\sigma^2} + \frac{n^2\mu^4}{\sigma^4(n-2)}\right]$$

Based on this mean and variance of $n \cdot 10^{\frac{SN_T}{10}}$, an approximation for the mean and variance of SN_T can be derived. Let $X = \frac{n\bar{y}^2}{s^2}$, *then* $SN_T = 10 \cdot \log\frac{x}{n}$. Define $H(x) = SN_T = 10 \cdot \log\frac{x}{n}$. Then $H'(x) = 10x^{-1}$ *and* $H''(x) = -10x^{-2}$. According to results in Bain and Engelhardt, (1992),

$$E(H(x)) \doteq H(\mu_x) + \frac{1}{2}H''(\mu_x)\sigma_x^2$$

and

$$Var(H(x)) \doteq \left[H'(\mu_x)\right]^2\sigma_x^2.$$

where $\mu_x = E(x)$ and $\sigma_x^2 = Var(x)$. Thus,

$$E(SN_T) \doteq 10\log\left[\frac{\frac{n-1}{n-3}\left(1+\frac{n\mu^2}{\sigma^2}\right)}{n}\right] - \frac{10(n-2)}{(n-5)\left[1+\frac{n\mu^2}{\sigma^2}\right]^2}\left[1+\frac{2n\mu^2}{\sigma^2}+\frac{n^2\mu^4}{\sigma^4(n-2)}\right]$$

and,

$$Var(SN_T) \doteq \left(\frac{10}{1+\frac{n\mu^2}{\sigma^2}}\right)^2 \cdot \frac{2(n-2)}{(n-5)}\left[1+\frac{2n\mu^2}{\sigma^2}+\frac{n^2\mu^4}{\sigma^4(n-2)}\right].$$

In the next chapter, these approximations will be used to estimate the variance for the factor effect estimates in the designed experiment. That variance will be used to perform significance tests for the factor effect estimates.

4.4 Smaller-the-better

The moments of the distribution for the smaller-the-better signal-to-noise ratio, $SN_S = -10\log\left(\frac{1}{n}\sum_{i=1}^{n} y_i^2\right)$ will be considered next. This performance measure is used in applications when the target value is zero. The formula for the ratio can be rearranged as $n \cdot 10^{\frac{-SN_S}{10}} = \sum_{i=1}^{n} y_i^2$. In order to eventually estimate the moments of SN_S, the distribution

of $\sum_{i=1}^{n} y_i^2$ will be determined when the y_i are normally distributed with mean μ and

variance σ^2 *for* $i = 1,2,\ldots,n$. Under this condition it can be shown that $\sum_{i=1}^{n} y_i^2$ is

distributed as σ^2 times a noncentral chi-squared with n degrees of freedom and a

noncentrality parameter $\frac{n\mu^2}{\sigma^2}$. The proof for this claim follows.

4.5 Proof

Let $y_i \sim N(\mu, \sigma^2)$ *for* $i = 1,2,\ldots,n$. *Then* $\bar{y} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$. It can be written that

$y_i = \mu + \sigma z_i$, $i = 1,2,\ldots,n$ where each z_i is an independent unit normal random variable.

Then, $y_i^2 = (\mu + \sigma z_i)^2 = \mu^2 + 2\mu\sigma z_i + \sigma^2 z_i^2$.

Now consider $\sum_{i=1}^{n} y_i^2$.

$$\sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} (\mu + \sigma z_i)^2 = \sum_{i=1}^{n} (\mu^2 + 2\mu\sigma z_i + \sigma^2 z_i^2)$$

$$\sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} \sigma^2\left[\frac{\mu^2}{\sigma^2} + \frac{2\mu}{\sigma} z_i + z_i^2\right] = \sum_{i=1}^{n} \sigma^2\left[\frac{\mu}{\sigma} + z_i\right]^2$$

Since $\frac{\mu}{\sigma}$ is a constant and z_i is unit normal, $\sum_{i=1}^{n} y_i^2$ is distributed as σ^2 times a noncentral chi-squared with n degrees of freedom and a noncentrality parameter $\frac{n\mu^2}{\sigma^2}$.

The mean and variance of $\sum_{i=1}^{n} y_i^2 = n \cdot 10^{\frac{SN_S}{10}}$ can be found based on the noncentral chi-squared moment formulas found in Johnson and Kotz, (1970). These formulas applied to this specific situation follow.

$$E\left(n \cdot 10^{\frac{SN_S}{10}}\right) = E\left(\sum_{i=1}^{n} y_i^2\right) = \sigma^2\left[n + \frac{n\mu^2}{\sigma^2}\right] = n(\sigma^2 + \mu^2)$$

$$Var\left(n \cdot 10^{\frac{SN_S}{10}}\right) = \sigma^4\left[2n + 4n\frac{\mu^2}{\sigma^2}\right] = 2n\sigma^2(\sigma^2 + 2\mu^2)$$

Based on this mean and variance, an approximation for the mean and variance of SN_S can be derived. Let $x = \sum_{i=1}^{n} y_i^2$, *then* $SN_S = -10\log\frac{x}{n}$. Define $H(x) = SN_S = -10\log\frac{x}{n}$.

Then $H'(x) = -10x^{-1}$ *and* $H''(x) = 10x^{-2}$. Once again, using the result from Bain and Engelhardt, (1992), approximate formulas for the expected value and variance of SN_S can be determined. The resulting formulas follow.

$$E(SN_S) \doteq -10\log\left[\sigma^2 + \mu^2\right] + \frac{5}{n^2\left[\sigma^2 + \mu^2\right]^2}, \textit{ and}$$

$$Var(SN_S) \doteq \frac{200\left[1 + \frac{2\mu^2}{\sigma^2}\right]}{n\left[1 + \frac{\mu^2}{\sigma^2}\right]^2}.$$

As with the results for SN_T, the last approximations will be used to estimate the variance for the factor effect estimates. That is the variance that will be used to perform significance tests for the factor effect estimates.

4.6 Larger-the-better

Finally, the larger-the-better signal-to-noise ratio, $SN_L = -10\log\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{y_i^2}\right)$, will be considered. Applications that apply this SN ratio have a target value of infinity. No

known distribution could be determined to associate with this particular SN ratio. Therefore, no existing formulas for the mean and variance could be used. However, approximations for these two statistics can be derived using formulas for functions of random variables previously cited from Bain and Engelhardt, (1992). When $y \sim (\mu,\sigma^2)$, $\mu \neq 0$,, the two formulas that will be applied are

$$E(g(y)) \doteq g(\mu) + \frac{1}{2} g''(\mu)\sigma^2, \text{ and}$$

$$Var(g(y)) \doteq \left(\frac{dg}{dy}\right)_{\mu}^{2} Var(y).$$

In order to apply these formulas, it is necessary to rearrange SN_L as $n \cdot 10^{\frac{SN_L}{10}} = \sum_{i=1}^{n} \frac{1}{y_i^2}$. To start, let $g(y) = \frac{1}{y^2} = y^{-2}$, $y \neq 0$, where y represents the response

variable for an individual experiment run and each $y \sim (\mu,\sigma^2)$. Then the following derivatives would apply.

$$g'(y) = -2y^{-3}$$

$$g''(y) = 6y^{-4}$$

Applying the formula for the first moment would yield the following.

$$
\begin{aligned}
E(g(y)) = E\left(\frac{1}{y^2}\right) &\doteq \frac{1}{\mu^2} + \frac{1}{2}(6\mu^{-4})\sigma^2 \\
&\doteq \frac{1}{\mu^2} + \frac{3}{\mu^4}\sigma^2
\end{aligned}
$$

Further, the expected value of a summation of these random variables can be found.

$$
\begin{aligned}
E\left(\sum_{i=1}^{n}\frac{1}{y_i^2}\right) &= \sum_{i=1}^{n} E\left(\frac{1}{y_i^2}\right) \\
&\doteq n\cdot\left[\frac{1}{\mu^2} + \frac{3}{\mu^4}\sigma^2\right]
\end{aligned}
$$

Therefore,

$$
E\left(n\cdot 10^{-\frac{SN_L}{10}}\right) \doteq n\cdot\left[\frac{1}{\mu^2} + \frac{3}{\mu^4}\sigma^2\right]
$$

Similar steps can be taken to find a formula for the variance of this function of SN_T.

Substituting into the Bain and Englehart equation for variance gives

$$
Var\left(\frac{1}{y^2}\right) = (-2\mu^{-3})^2\sigma^2 = \frac{4}{\mu^6}\sigma^2.
$$

Because each individual y is independent.

$$Var\left(\sum_{i=1}^{n}\frac{1}{y_i^2}\right)=\sum_{i=1}^{n}Var\left(\frac{1}{y_i^2}\right)=n\cdot\left[\frac{4}{\mu^6}\sigma^2\right].$$

Thus,

$$Var\left(n\cdot 10^{\frac{SN_L}{10}}\right)=n\cdot\left[\frac{4}{\mu^6}\sigma^2\right]$$

By again applying the formulas from Bain and Engelhardt, approximations for the expected value and variance of the larger-the-better SN ratio can be derived. Let $x=n\cdot 10^{\frac{SN_L}{10}}$, then $SN_L=-10\log\frac{x}{n}$ Define $f(x)=SN_L=-10\log\frac{x}{n}$, then

$f'(x)=\frac{-10}{x}=-10x^{-1}$ and $f''(x)=10x^{-2}$ Substituting and simplifying yields the following final approximations.

$$E(SN_L)\doteq-10\log\left(\frac{1}{\mu^2}+\frac{3\sigma^2}{\mu^4}\right)+\frac{20\sigma^2}{n\mu^6\left(\frac{1}{\mu^2}+\frac{3\sigma^2}{\mu^4}\right)^2}$$

$$Var(SN_L) \doteq \frac{400\sigma^2}{n\mu^6\left(\frac{1}{\mu^2} + \frac{3\sigma^2}{\mu^4}\right)^2}$$

As with the previous signal-to-noise ratios discussed, these approximations may be used to estimate the variance of the factor effects so that significance testing may be performed on the effect estimates.

4.7 Variance of Effect Estimates

It is the practice in design of experiments to study factor effects to see if there are any significant factors. It is not meaningful to look at the response variable alone when the null hypothesis that all response variables are equal is actually false. If one row's response variable is affected by some factor, other rows will also be affected because of the use of that factor in determining the response variable value. This is why each row's effect estimate should be computed. These values can be associated with specific factors and are not confounded with other factors.

In order to test the significance of effect estimates, the effect for each row must surely be computed. For two level factors, Yate's method can easily be applied to compute the necessary estimates. Then taking each effect estimate and dividing by the standard error of these estimates yields a value that can be compared to a standard normal distribution for significance. To determine the standard error, a formula for the variance of the effects must be derived.

4.8 Derivation of Variance for Effect Estimates

Based on factors with two levels, the effect estimate, ϵ_i, associated with the ith row can be written as

$$\epsilon_i = \frac{\sum_{j=1}^{N} a_{ji} SN_i}{\frac{N}{2}} = \frac{2}{N} \sum_{j=1}^{N} a_{ji} SN_i$$

where N = the number of runs in the experiment and a_{ji} = ± 1 depending on the row's jth column factor setting associated with the standard form of the design. Then,

$$Var(\epsilon_i) = \frac{4}{N^2} Var(\sum_{i=1}^{N} SN_i) = \frac{4}{N} Var(SN).$$

This formula is dependent on the variance of the signal-to-noise ratio that is used as the response variable. The formulas derived earlier in this chapter for the variances of the most popularly used signal-to-noise ratios can be used to perform significance testing on the effect estimates for a designed experiment using one of these three SN ratios that is based on the Taguchi Methods.

The next chapter will describe the numerical analysis of the validity of the variance formulas derived in this chapter. Results of simulation studies will be given.

CHAPTER 5. NUMERICAL ANALYSIS

5.1 Testing the Distribution Associated with SN_S

In order to numerically test the hypothesis that $n \cdot 10^{\frac{SN_S}{10}}$ is distributed as σ^2 times a noncentral chi-squared with n degrees of freedom and noncentrality parameter $\frac{n\mu^2}{\sigma^2}$, a simulation study, written in FORTRAN, was performed. A designed experiment of 8 runs with 10 replicates (n=10) was used.

The true mean and standard deviation for each run were set at $\mu = 20$ and $\sigma = 4$. The number of trials tested was 10,000. For each trial a row of 10 standard normal variates was generated using the subroutine DRNNOA, a random number generator from the FORTRAN IMSL STAT/LIBRARY. Each variate was non-standardized by multiplying by σ and adding μ. This will be called y_i. For each row the mean and standard deviation were computed, as well as the value for $SN_S = -10\log\left(\frac{1}{n}\sum_{i=1}^{n} y_i^2\right)$.

For each experiment of 8 runs, the overall mean of the row means, $\bar{\bar{y}}$, and the mean of the row standard deviations, $\bar{y}_s$, were computed. These values were used to estimate μ and σ in future formulas since, in realistic situations, the true parameter values

are unknown. If, as proposed, $n \cdot 10^{\frac{SN_S}{10}}$ is distributed as $\sigma^2 \cdot \chi^2_n\left(\frac{n\mu^2}{\sigma^2}\right)$, then

$\frac{1}{\sigma^2} \cdot n \cdot 10^{\frac{SN_S}{10}}$ would be distributed as $\chi^2_n\left(\frac{n\mu^2}{\sigma^2}\right)$. So for each row, $\frac{1}{(\bar{y}_s)^2} \cdot n \cdot 10^{\frac{SN_S}{10}}$ was

calculated and sent to the IMSL subroutine DCSNDF. This subroutine returns the probability of a noncentral chi-squared distribution taking on a value less than or equal to the value sent. If this probability was less than 0.025 or greater than 0.975, a flag was set to indicate rejection of the null hypothesis. For the 10,000 trials, a total of 414 flags were set indicating an alpha of approximately 4%.

To determine if the test was sensitive to changes in the mean and standard deviation, the set mean of 20 and standard deviation of 4 were increased and decreased, both separately and together. Additionally, other values of μ and σ were used to generate more simulations. All provided similar results of $\alpha \doteq 4\%$. Also, the same type of study was performed for a 16 run experiment. For these simulations, $\alpha \doteq 5\%$. These results are summarized in Table 5.1. The standard errors for the alphas are reported in parentheses below each alpha estimate. The following expression was used to calculate the standard errors:

$$\sqrt[2]{\frac{\alpha(1-\alpha)}{10000}}$$

Table 5.1

Simulation Results for $\frac{1}{(\bar{y}_s)^2}\cdot n\cdot 10^{\frac{SN_S}{10}}$ Using SN_S

μ	σ	8 run experiment (10,000 trials)	16 run experiment (10,000 trials)
40	8	α = .0457 (.0042)	α = .0495 (.0043)
	4	α = .0408 (.0040)	α = .0535 (.0045)
	2	α = .0388 (.0039)	α = .0513 (.0044)
20	8	α = .0441 (.0041)	α = .0518 (.0044)
	4	α = .0414 (.0040)	α = .0487 (.0043)
	2	α = .0472 (.0042)	α = .0513 (.0044)
10	8	α = .0437 (.0041)	α = .0476 (.0043)
	4	α = .0427 (.0040)	α = .0544 (.0045)
	2	α = .0454 (.0042)	α = .0494 (.0043)
4	2	α = .0458 (.0042)	α = .0528 (.0045)
	1	α = .0443 (.0041)	α = .0498 (.0044)
	0.5	α = .0416 (.0040)	α = .0539 (.0045)
2	1	α = .0442 (.0041)	α = .0498 (.0044)
	0.5	α = .0465 (.0042)	α = .0547 (.0045)
1	0.5	α = .0435 (.0041)	α = .0491 (.0043)

To determine whether using $\bar{y}_s$ was a good estimate of σ, $\frac{1}{\sigma^2} \cdot n \cdot 10^{\frac{SN_S}{10}}$ was used

in the DCSNDF subroutine call for several simulations. The results were almost identical to those obtained using $\bar{y}_s$.

5.2 Testing Factor Effect Estimates Associated with SN_S

During the testing for the validity of the distribution associated with SN_S, calculations were also performed in order to determine the significance of any effect estimates. The null hypothesis for this type of test would be that all effects are equal to zero. A significant effect would result in rejection of this null hypothesis. To reiterate this in model form, let

$$SN_{S_i} = \mu_{SN_S} + \frac{\epsilon_1}{2}x_1 + \frac{\epsilon_2}{2}x_2 + \ldots + \frac{\epsilon_n}{2}x_n + e_i,\ e_i \sim N(0,1),\ x_i = \pm 1.$$

Then the null hypothesis could be written as:

$$H_0:\ SN_{S_i} = \mu_{SN_S} + e_i.$$

In addition to the steps described in the last section, Yate's formula was used to calculate the effect associated with each row in the experiment. To determine if any of these effects were significant, each effect estimate was divided by the square root of the variance of the factor effects. This is where the equations derived in Chapter 4 came in

to use. If any of these resulting values in absolute value were greater than $z_{.975} = 1.96$, a flag was set to indicate a significant effect. Results of some of these simulations are given in Table 5.2. Reasonable values for α were obtained, thus giving evidence for the validity of using this type of significance testing in conjunction with the Taguchi Methods.

5.3 Testing the Distribution Associated with SN_T

A very similar analysis that was performed for SN_S was performed for SN_T. The major differences were in the calculation of the SN ratio and the subroutine used to return the probability. Recall from Chapter 4 that $n \cdot 10^{\frac{SN_T}{10}}$ is distributed as noncentral F with 1 and n-1 degrees of freedom and noncentrality parameter $\frac{n\mu^2}{\sigma^2}$. Since there was not a subroutine available in the IMSL STAT/LIBRARY that returned a probability associated with a noncentral F, subroutine DTNDF was used instead. That subroutine returns the probability that a noncentral T takes on a value less than or equal to the value sent. By using the square root of the value that is supposed to be noncentral F as the value sent to DTNDF the probabilities returned are equivalent to those of a noncentral F. (The square root of the noncentrality parameter also was sent to DTNDF.) The results obtained from simulations of 8 run and 16 run designs are given in Table 5.3. The results confirm the conclusion that $n \cdot 10^{\frac{SN_T}{10}}$ is distributed as noncentral F.

Table 5.2
Simulation Results for Effect Estimates Using SN_S

μ	σ	8 run experiment (10,000 trials)	16 run experiment (10,000 trials)
40	8	α = .0591 (.0047)	α = .0594 (.0047)
	4	α = .0582 (.0047)	α = .0633 (.0049)
	2	α = .0596 (.0047)	α = .0572 (.0046)
20	8	α = .0633 (.0049)	α = .0611 (.0048)
	4	α = .0613 (.0048)	α = .0590 (.0047)
	2	α = .0611 (.0048)	α = .0615 (.0048)
10	8	α = .0668 (.0050)	α = .0701 (.0051)
	4	α = .0645 (.0049)	α = .0662 (.0050)
	2	α = .0628 (.0049)	α = .0575 (.0047)
4	2	α = .0677 (.0050)	α = .0662 (.0050)
	1	α = .0609 (.0048)	α = .0632 (.0049)
	0.5	α = .0584 (.0047)	α = .0622 (.0048)
2	1	α = .0713 (.0051)	α = .0659 (.0050)
	0.5	α = .0608 (.0048)	α = .0541 (.0045)
1	0.5	α = .0658 (.0050)	α = .0635 (.0049)

Table 5.3

Simulation Results for $n \cdot 10^{\frac{SN_T}{10}}$ Using SN_T

μ	σ	8 run experiment (10,000 trials)	16 run experiment (10,000 trials)
40	8	α = .0428 (.0040)	α = .0450 (.0041)
	4	α = .0394 (.0039)	α = .0494 (.0043)
	2	α = .0427 (.0040)	α = .0500 (.0044)
20	8	α = .0409 (.0040)	α = .0457 (.0042)
	4	α = .0450 (.0041)	α = .0475 (.0043)
	2	α = .0399 (.0039)	α = .0522 (.0044)
10	8	α = .0445 (.0041)	α = .0470 (.0042)
	4	α = .0426 (.0040)	α = .0523 (.0045)
	2	α = .0396 (.0039)	α = .0467 (.0042)
4	2	α = .0413 (.0040)	α = .0439 (.0041)
	1	α = .0425 (.0040)	α = .0480 (.0043)
	0.5	α = .0431 (.0041)	α = .0505 (.0044)
2	1	α = .0383 (.0038)	α = .0492 (.0043)
	0.5	α = .0374 (.0044)	α = .0548 (.0046)
1	0.5	α = .0414 (.0040)	α = .0448 (.0041)

5.4 Testing Factor Effect Estimates Associated with SN_T

This analysis was identical to that described in section 5.2 except that the formulas associated with SN_T were used. The results of some of the simulations appear in Table 5.4. The average Type I error estimate was only about 2%.

5.5 Testing for an Empirical Distribution for SN_L

Since no analytical method to get the distribution associated with the larger-the-better SN ratio was determined, exploratory analysis using Microsoft EXCEL was performed in order to discover an empirical distribution that could be used to determine the validity of the formulas derived in the last chapter. For fixed values of μ and σ, 1000 samples with 5 replicates were generated for which the SN_Ls were computed. Histograms of these samples were examined. The distributions resembled normal distributions in that they were mound-shaped. Normal probability plots were constructed for the samples. The plots were very close to straight lines indicating the data was approximately normally distributed. These graphs appear in the Appendix. Also, the empirical rule was used to determine that the data were approximately normally distributed. Summary statistics for the mean and standard deviation of the samples' SN_Ls, along with the percentage of the 1000 SN_Ls that fell within the empirical rule calculations, appear in Table 5.5.

Also, the samples were analyzed to determine percentiles for use as critical values for testing whether the SN_L ratios came from the assumed distribution. The values were determined by ranking the 1000 SN ratios and using the 25th and 975th largest as bounds. For the four combinations of μ and σ, the results are shown in Table 5.6.

Table 5.4
Simulation Results for Effect Estimates Using SN_T

μ	σ	8 run experiment (10,000 trials)	16 run experiment (10,000 trials)
40	8	α = .0135 (.0023)	α = .0130 (.0023)
	4	α = .0136 (.0023)	α = .0149 (.0024)
	2	α = .0140 (.0023)	α = .0129 (.0023)
20	8	α = .0133 (.0023)	α = .0151 (.0024)
	4	α = .0158 (.0025)	α = .0155 (.0025)
	2	α = .0137 (.0023)	α = .0142 (.0024)
10	8	α = .0307 (.0035)	α = .0280 (.0033)
	4	α = .0153 (.0025)	α = .0135 (.0023)
	2	α = .0136 (.0023)	α = .0135 (.0023)
4	2	α = .0174 (.0026)	α = .0185 (.0027)
	1	α = .0135 (.0023)	α = .0138 (.0023)
	0.5	α = .0115 (.0021)	α = .0125 (.0022)
2	1	α = .0197 (.0028)	α = .0153 (.0025)
	0.5	α = .0143 (.0024)	α = .0140 (.0023)
1	0.5	α = .0172 (.0026)	α = .0199 (.0028)

Table 5.5
Summary Statistics for SN_L Simulations
1000 samples, 5 replicates

μ	σ	$\bar{x}$	s	$\bar{x} \pm s$	$\bar{x} \pm 2s$	$\bar{x} \pm 3s$
1000	**50**	59.96634	.194848	70.7%	94.4%	99.6%
	25	59.99085	.098263	69.5%	94.8%	99.9%
500	**50**	53.86369	.403132	68.1%	95.3%	99.7%
	25	53.95490	.040958	67.1%	95.9%	99.3%

Table 5.6
Percentiles for Empirical Distribution of SN_L

μ	σ	$P_{.025}$	$P_{.975}$
1000	**50**	59.55958	60.33852
	25	59.79677	60.19124
500	**50**	53.05210	54.62763
	25	53.57087	54.33900

Using the critical values for the empirical distribution, the null hypothesis that all SN_Ls are equal for an experiment was tested by comparing each calculated SN ratio with the critical values. The number of rejections of this null hypothesis were tallied to determine an estimate for the true alpha of this test. This analysis was performed for both 8 run and 16 run experiments. The results of 10,000 trials with 5 replicates are given in Table 5.7. An average alpha of approximately 5% was achieved as expected.

Table 5.7
Simulation Results for Empirical Distribution Using SN_L

μ	σ	8 run experiment (10,000 trials)	16 run experiment (10,000 trials)
1000	50	α = .0482 (.0043)	α = .0489 (.0043)
	25	α = .0402 (.0039)	α = .0450 (.0041)
500	50	α = .0512 (.0044)	α = .0541 (.0045)
	25	α = .0467 (.0042)	α = .0520 (.0044)

5.6 Testing Factor Effect Estimates Associated with SN_L

To test the null hypothesis with regard to effect estimates that they are all equal to zero, an analysis very similar to those used for SN_S and SN_T was used. The difference was that the formula for Var(SN_L) derived in Chapter 4 was used in the formula for the variance of the effect estimates. However, before using that formula, an analysis was performed to determine if the formulas that were derived for the mean and variance were similar to the actual mean and variance computed for 1000 samples with 5 replicates. The results of this analysis appear in Table 5.8.

The formula from Chapter 4 very closely approximates the mean. However, the formula estimate for the variance overestimates the actual variance of the SN_L ratios. Therefore, when the effect estimates were tested, fewer rejections were tallied than would have normally been expected. The results of those simulations appear in Table 5.9.

Table 5.8
Comparison of Actual Statistics with Formula Estimates for SN_L from 1000 samples with n=5

μ	σ	Actual $\bar{x}$	Actual s^2	Formula Estimate for $\bar{x}$	Formula Estimate for s^2
1000	**50**	59.96634	.037966	59.97463	.177590
	25	59.99085	.009656	59.99292	.043782
500	**50**	53.86369	.162515	53.89990	.688111
	25	53.95490	040958	53.96336	170204

Table 5.9
Simulation Results for Effect Estimates Using SN_L

μ	σ	8 run experiment (10,000 trials)	16 run experiment (10,000 trials)
1000	50	α = .0008 (.0006)	α = .0011 (.0007)
	25	α = .0017 (.0002)	α = .0022 (.0009)
500	50	α = .0005 (.0004)	α = .0005 (.0004)
	25	α = .0008 (.0006)	α = .0011 (.0007)

5.7 Testing the Power of the Tests

The power of the tests, both for the sampling distribution of the SN ratio and the effect estimates, were analyzed for SN_T and SN_S only. To analyze the power of the test that the signal-to-noise ratios did indeed come from the assumed distribution with a stated mean and standard deviation, the null hypothesis was made false. To do that, each observation in an odd row in the designed experiment was decreased by a constant, while each observation in an even row in the designed experiment was increased by the same constant. This meant that the overall mean stayed the same, but the variation was much different. This analysis was repeated for both 8 run and 16 run experiments with constants of 1σ, 2σ, and then 3σ. The same FORTRAN program that was used to test the distributions associated with the SN ratios was used, but with the observations altered. For a powerful test, the number of rejections of the null hypothesis should be very high. For both SN ratios tested, as the variation from the null hypothesis increased, so did the number of rejections of the null hypothesis. However, the test associated with the smaller-the-better SN ratio was much more powerful. The results of these simulations appear in Tables 5.10 and 5.11. The standard errors of the estimates are reported in the parentheses below the power estimates. The power estimates are based on 10,000 trials with 10 replicates.

To test the power of the test for significant effects, the null hypothesis that all effect estimates are zero was made false. In order to do that, recall that the formula for the SN ratio is a function of the y values during the simulations when each row is

generated. Each of the k replicates for the ith row can be represented by the following model.

$$y_{i,k} = \mu + \frac{\epsilon_1}{2}x_1 + \frac{\epsilon_2}{2}x_2 + \ldots + e_i$$

In order for the alternative hypothesis to be true, at least one of the epsilons must be different from zero. To make this occur for the simulations, the value of ϵ_1 was made significant. In standard form, the odd number rows of the design would have $x_1 = -1$, while each even number row would have $x_1 = +1$. Thus, for each odd number row, the original standard deviation was increased by a multiple, while each even number row had the original standard deviation decreased. In other words, the observations in each odd row were generated by taking μ and adding a constant times σ times a random standard normal variate. The observations in each even row were generated by taking μ and adding the inverse of the constant times σ times a random standard normal variate. Simulations were performed for constants of 2, 4, and 6. Since only the first epsilon was made to be significant, only the absolute value of the first effect estimate will be tested against the critical value of 1.96. The results of the simulations appear in Tables 5.12 and 5.13, indicating that the power of the test for significant factor effects when using the nominal-the-better SN ratio is very large. Testing for significant effects with SN_T is much more powerful than for the smaller-the-better SN ratio. These results for the two different ratios are opposite from those for the power of the distribution test.

Table 5.10
Simulation Results for Testing the Power of the Test of the Distribution of SN_S

μ	σ	1 - β 8 run experiment (10,000 trials) ±1σ	±2σ	±3σ	1 - β 16 run experiment (10,000 trials) ±1σ	±2σ	±3σ
40	8	.9076 (.0058)	1.0000 (.0000)	1.0000 (.0000)	.8955 (.0061)	.9999 (.0002)	1.0000 (.0000)
	4	.9068 (.0058)	1.0000 (.0000)	1.0000 (.0000)	.8955 (.0061)	1.0000 (.0000)	1.0000 (.0000)
	2	.9106 (.0057)	1.0000 (.0000)	1.0000 (.0000)	.8994 (.0060)	1.0000 (.0000)	1.0000 (.0000)
20	4	.9009 (.0060)	1.0000 (.0000)	1.0000 (.0000)	.9005 (.0060)	1.0000 (.0000)	1.0000 (.0000)
	2	.9109 (.0057)	1.0000 (.0000)	1.0000 (.0000)	.8968 (.0061)	1.0000 (.0000)	1.0000 (.0000)
10	2	.8998 (.0060)	1.0000 (.0000)	1.0000 (.0000)	.8999 (.0060)	.9998 (.0003)	1.0000 (.0000)
4	1	.9010 (.0060)	1.0000 (.0000)	1.0000 (.0000)	.8921 (.0062)	1.0000 (.0000)	1.0000 (.0000)
	0.5	.9048 (.0059)	1.0000 (.0000)	1.0000 (.0000)	.8976 (.0061)	1.0000 (.0000)	1.0000 (.0000)
2	0.5	.9028 (.0059)	1.0000 (.0000)	1.0000 (.0000)	.8909 (.0062)	1.0000 (.0000)	1.0000 (.0000)

Table 5.11
Simulation Results for Testing the Power of the Test of the Distribution of SN_T

μ	σ	$1-\beta$ 8 run experiment (10,000 trials)			$1-\beta$ 16 run experiment (10,000 trials)		
		$\pm 1\sigma$	$\pm 2\sigma$	$\pm 3\sigma$	$\pm 1\sigma$	$\pm 2\sigma$	$\pm 3\sigma$
40	8	.1419 (.0070)	.4081 (.0098)	.6261 (.0097)	.1484 (.0071)	.4079 (.0098)	.6242 (.0097)
	4	.0664 (.0050)	.1445 (.0070)	.2781 (.0090)	.0736 (.0052)	.1511 (.0072)	.2786 (.0090)
	2	.0494 (.0043)	.0658 (.0050)	.0957 (.0059)	.0546 (.0045)	.0747 (.0053)	.1035 (.0061)
20	4	.1321 (.0068)	.4020 (.0098)	.6256 (.0097)	.1421 (.0070)	.3950 (.0098)	.6234 (.0097)
	2	.0621 (.0048)	.1449 (.0070)	.2722 (.0089)	.0681 (.0050)	.1498 (.0071)	.2772 (.0090)
10	2	.1355 (.0068)	.4041 (.0098)	.6219 (.0097)	.1439 (.0070)	.4095 (.0098)	.6253 (.0097)
4	1	.1770 (.0076)	.5258 (.0100)	.7092 (.0091)	.1933 (.0079)	.5254 (.0100)	.7068 (.0091)
	0.5	.0787 (.0054)	.1963 (.0079)	.3805 (.0097)	.0859 (.0056)	.2036 (.0081)	.3790 (.0097)
2	0.5	.1867 (.0078)	.5284 (.0100)	.7055 (.0091)	.1938 (.0079)	.5219 (.0100)	.7103 (.0091)

Table 5.12
Simulation Results for Testing the Power of the Test of the Effect Estimates Using SN_S

μ	σ	1 - β 8 run experiment (10,000 trials)			1 - β 16 run experiment (10,000 trials)		
		$2\sigma, \frac{\sigma}{2}$	$4\sigma, \frac{\sigma}{4}$	$6\sigma, \frac{\sigma}{6}$	$2\sigma, \frac{\sigma}{2}$	$4\sigma, \frac{\sigma}{4}$	$6\sigma, \frac{\sigma}{6}$
40	8	.2413 (.0085)	.7163 (.0090)	.9509 (.0043)	.3740 (.0096)	.9301 (.0051)	.9986 (.0007)
	4	.1429 (.0067)	.3387 (.0045)	.5547 (.0010)	.1741 (.0067)	.4969 (.0100)	.7818 (.0083)
	2	.1150 (.0064)	.2079 (.0081)	.2812 (.0090)	.1203 (.0065)	.2612 (.0088)	.3765 (.0097)
20	4	.2440 (.0086)	.7213 (.0087)	.9605 (.0039)	.3694 (.0096)	.9304 (.0051)	.9989 (.0007)
	2	.1393 (.0069)	.3465 (.0095)	.5548 (.0099)	.1716 (.0075)	.4972 (.0097)	.7711 (.0084)
10	2	.2421 (.0086)	.7198 (.0090)	.9542 (.0042)	.3645 (.0096)	.9251 (.0053)	.9991 (.0006)
4	1	.3164 (.0093)	.8655 (.0068)	.9937 (.0016)	.5061 (.0100)	.9878 (.0022)	1.0000 (.0000)
	0.5	.1610 (.0074)	.4293 (.0099)	.6902 (.0092)	.2119 (.0082)	.6342 (.0096)	.9040 (.0059)
2	0.5	.3113 (.0093)	.8649 (.0068)	.9936 (.0016)	.4934 (.0100)	.9880 (.0022)	1.0000 (.0000)

Table 5.13
Simulation Results for Testing the Power of the Test of the Effect Estimates Using SN_T

μ	σ	$1-\beta$ 8 run experiment (10,000 trials)			$1-\beta$ 16 run experiment (10,000 trials)		
		$2\sigma, \frac{\sigma}{2}$	$4\sigma, \frac{\sigma}{4}$	$6\sigma, \frac{\sigma}{6}$	$2\sigma, \frac{\sigma}{2}$	$4\sigma, \frac{\sigma}{4}$	$6\sigma, \frac{\sigma}{6}$
40	8	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)
	4	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)
	2	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)
20	4	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)
	2	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)
10	2	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)
4	1	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)
	0.5	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)
2	0.5	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)	1.0000 (.0000)

5.8 Summary of Numerical Analysis

Having determined the probabilities of Type I and Type II errors associated with applying the formulas from Chapter 4 of this dissertation, the numerical analysis of the

previous tests support the validity of those formulas. Therefore, this analysis measures the effectiveness of using statistical significance tests for factor effect estimates resulting from the use of signal-to-noise ratios as the performance measure in a designed experiment that applies other aspects of the Taguchi Methods.

CHAPTER 6. CONCLUSION

During the parameter design stage of the Taguchi Methods, the goal is to determine optimal parameter values that enable a product or process to be robust to surrounding conditions. The current practice is to graph the mean response and SN ratio from a designed experiment and "pick the winner." But are the values picked to be optimal actually statistically significant? Are there significant factor effects? It was the purpose of this dissertation to provide motivation and justification for applying more advanced statistical methods, such as statistical significance testing of effect estimates, to the Taguchi Methods of quality control. Sampling distributions and moment estimates for three of the most commonly used SN ratios were derived. These formulas will enable statistical significance testing to be applied to designed experiments that incorporate the Taguchi Methods.

An overview of current Taguchi Methods methodology was presented in the first chapter, along with an example. The second chapter reviewed recent literature dealing with various concerns of the Taguchi Methods. Chapter 3 focused on one particular aspect of the methods, the signal-to-noise ratio. The ratios were explained in detail, and justification for the use of SN ratios as a response variable to achieve the goals of the Taguchi Methods was presented. In order to apply significance testing to the SN ratios, the sampling distribution of the ratio is necessary. In Chapter 4, sampling distributions and moment estimates associated with three of the most widely used SN ratios were derived.

The last chapter provided a rigorous numerical analysis of the formulas that were derived in Chapter 4.

The three most commonly used SN ratios that were analyzed in this research were the nominal-the-better (SN_T), the smaller-the-better (SN_S), and the larger-the-better (SN_L). It was determined that the nominal-the-better SN ratio is related to a noncentral F distribution. The smaller-the-better SN ratio was found to be related to a noncentral chi-squared distribution. Approximating formulas for the means and variances of these SN ratios were derived based on their associated sampling distributions. No known distribution could be determined algebraically to associate with the larger-the-better SN ratio. However, approximations for the mean and variance of SN_L were derived based on formulas for functions of random variables. Finally, formulas for the variance of effect estimates were derived in general. Those formulas depend on the variance of the chosen SN ratio. That variance would be used to find significant effect estimates.

The simulations that were performed to test the distributions associated with the smaller-the-better and nominal-the-better SN ratios gave results that confirmed the sampling distributions derived for each. For 8 run experiments with 10,000 trials and 10 replicates the probability of a Type I error occurring was approximately 4%, while for 16 run experiments α was about 5%. The results of the FORTRAN simulations for testing the formulas for factor effect estimates associated with SN_S gave an average Type I error estimate of 6%. All of these tests were based on 95% confidence levels so the achieved results were very reasonable. Testing the formulas for factor effect estimates associated

with SN_T gave an average Type I error estimate of 2%. When testing for effect estimates, a normal approximation was used for the estimates because of the additive method used to arrive at the estimates. This approximation may explain the small alpha that resulted.

The test of the distribution for SN_S was found to be very powerful, $1-\beta \doteq 1$. For the nominal-the-better SN ratio, as the random sample simulations were forced farther away from the null hypothesis, the number of rejections of the null hypothesis did increase. When testing the power of the effect estimates, SN_T had very powerful results. For the smaller-the-better SN ratio, the test became more powerful as the samples varied farther from the null hypothesis.

Exploratory analysis was performed using the larger-the-better SN ratio to determine that it is approximately normally distributed. Based on the normal distribution assumption, simulations were run to test the null hypothesis that all SN ratios were the same and an average Type I error probability of 5% resulted for both 8 run and 16 run experiments. However, a very small alpha was found for tests of effect estimates associated with SN_L. The analysis showed that the approximating formula for the variance of this SN ratio overestimates the actual variance. That would cause fewer rejections of the null hypothesis because the effect estimate is found by dividing with the variance.

The research presented in this dissertation provides formulas for moment estimates of sampling distributions associated with the three most commonly used SN ratios of the Taguchi Methods. These formulas can be used to apply significance testing to those methods, thus lending some statistical credibility to the Taguchi Methods.

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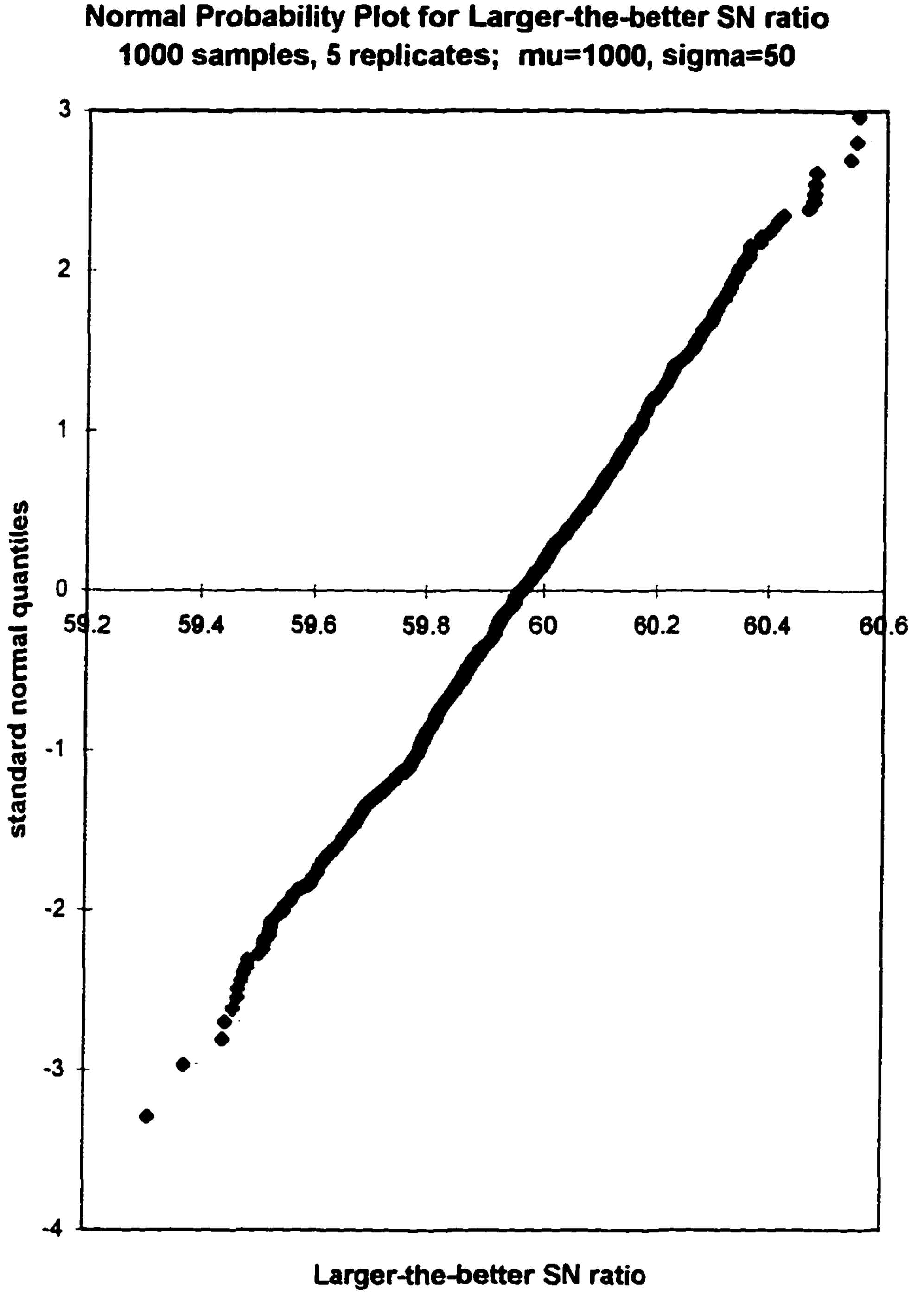
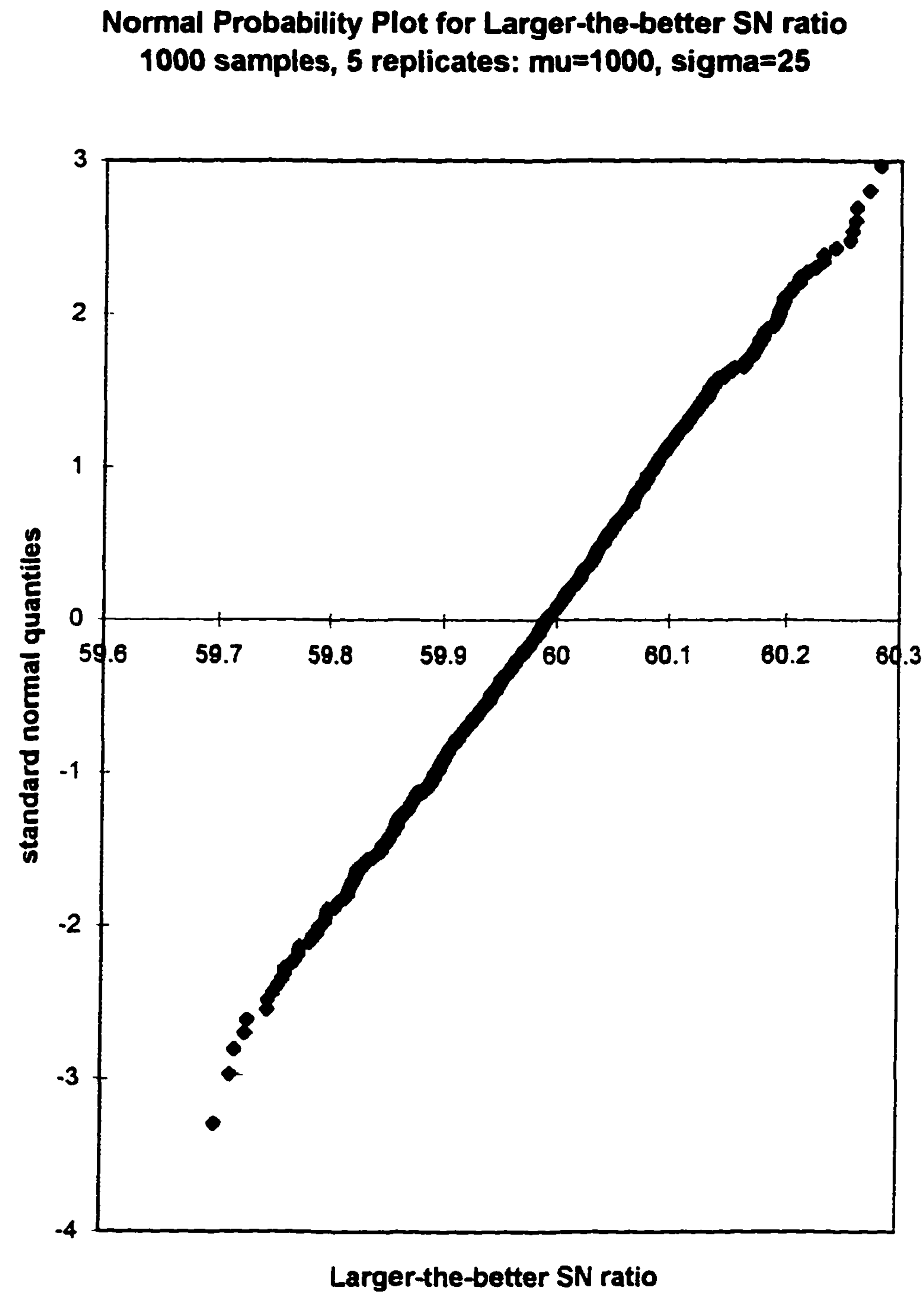
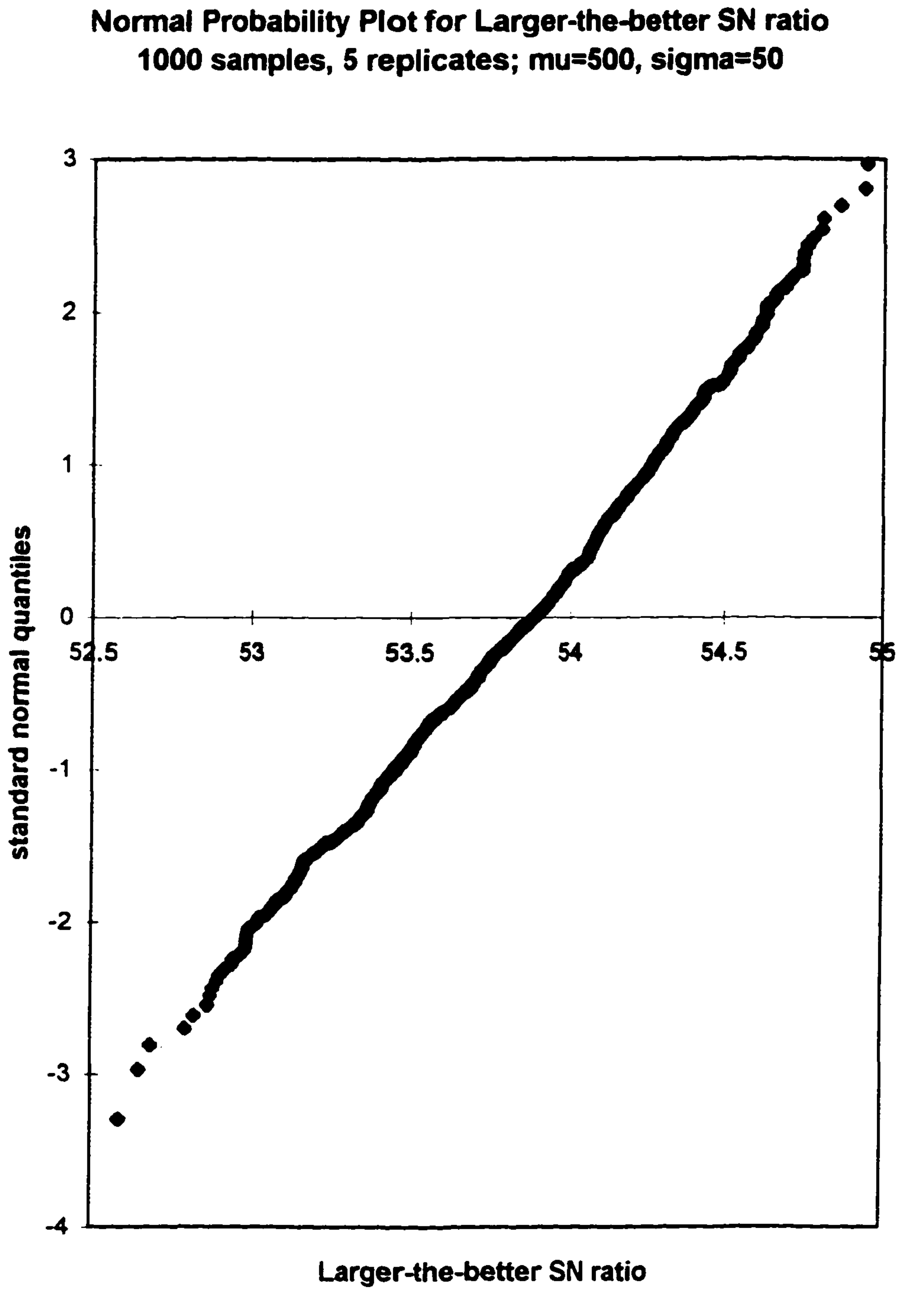
APPENDIX

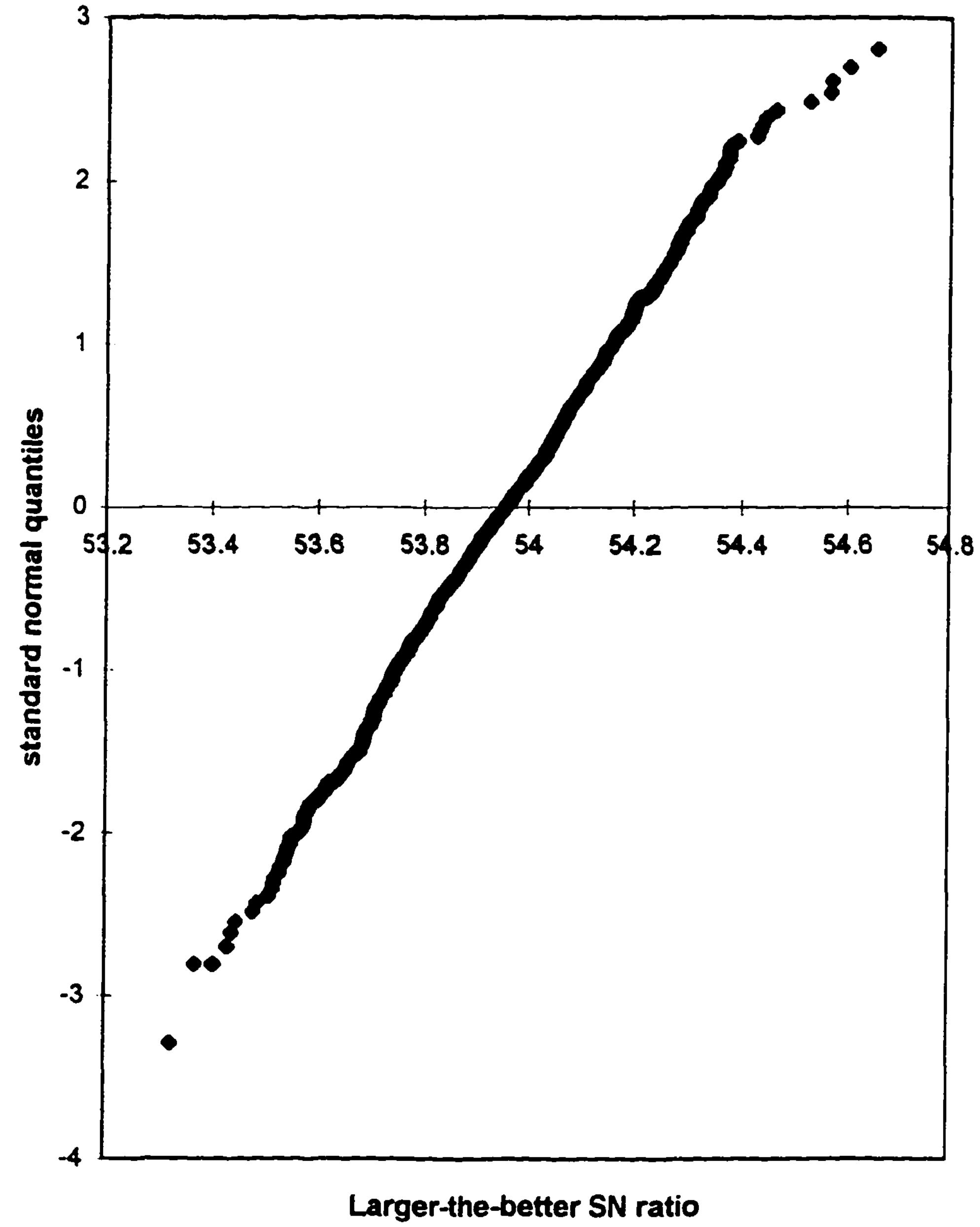


Normal Probability Plot for Larger-the-better SN ratio
1000 samples, 5 replicates; mu=1000, sigma=50
standard normal quantiles
3
2
1
0
-1
-2
-3
-4
59.2
59.4
59.6
59.8
60
60.2
60.4
60.6
Larger-the-better SN ratio


Normal Probability Plot for Larger-the-better SN ratio
1000 samples, 5 replicates: mu=1000, sigma=25
3
2
1
0
-1
-2
-3
-4
59.6
59.7
59.8
59.9
60
60.1
60.2
60.3
standard normal quantiles
Larger-the-better SN ratio


Normal Probability Plot for Larger-the-better SN ratio
1000 samples, 5 replicates; mu=500, sigma=50
3
2
1
0
-1
-2
-3
-4
52.5
53
53.5
54
54.5
55
standard normal quantiles
Larger-the-better SN ratio

Normal Probability Plot for Larger-the-better SN ratio
1000 samples, 5 replicates; mu=500, sigma=25



VITA

Shannon Jo Duchenois Kast was born at Tell City, Indiana. She graduated from Western Kentucky University in 1981 with a Bachelor of Arts in mathematics and minor in business administration. In 1983, she earned a Master of Applied Mathematical Sciences at The University of Georgia. Her main field of interest is applied statistics. She is currently an instructor at Southeastern Louisiana University, where she has taught mathematics courses since 1991. Prior to that, she was an assistant professor in the Mathematics Department at Northwestern State University of Louisiana for three years. She also has previous work experience as a computer programmer. She lives in Loranger, Louisiana, a predominately dairy community, ten miles north of Hammond with her husband of fourteen years, Kevin L. Kast. She is the mother of three children -- Todd, Tori, and Taylor.

DOCTORAL EXAMINATION AND DISSERTATION REPORT

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Major Field: Business Administration (ISDS)

Title of Dissertation: Applying Significance Testing to the Taguchi Methods of Quality Control

Approved:

Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

Date of Examination:

March 31, 1997