Matroid Connectivity.

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MATROID CONNECTIVITY

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy in

The Department of Mathematics

by

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# TABLE OF CONTENTS

Acknowledgements ........................................................................................................ ii

Abstract ........................................................................................................................ iv

Preliminaries .................................................................................................................. 1

Chapter

1 On Coefficients of the Tutte Polynomial ................................................................. 2
  1.1 Introduction ........................................................................................................ 2
  1.2 A Characterization of $M_{ij}$ ........................................................................ 5
  1.3 A Bound on $|E(M)|$ of $M \in M_{ij}$ ............................................................. 9
  1.4 Coefficients that Count Cyclic Flats .............................................................. 16

2 Triads and Triangles in 3-Connected Matroids ...................................................... 27
  2.1 Introduction ..................................................................................................... 27
  2.2 Preliminary Results ...................................................................................... 31
  2.3 The Proof of Theorem 2.1.1 ....................................................................... 33
  2.4 Further Results ........................................................................................... 54

3 A Unique Decomposition for 3-Connected Binary Matroids ............................. 60
  3.1 Introduction .................................................................................................. 60
  3.2 The Proof of the Main Result .................................................................. 61
  3.3 On a Decomposition of 3-Connected Graphs ............................................ 81

4 The Components of the Decomposition ................................................................. 94
  4.1 Introduction .................................................................................................. 94
  4.2 The Basics of Spikes, Wheels, and the Dual of Twirls ............................. 98
  4.3 Multiple Connections and Minimal Splits ................................................. 108
  4.4 On the Size of $\delta(X)$ ........................................................................... 120
  4.5 A Characterization of $\mathcal{M}$ ................................................................. 140

5 Contraction-Minimally 3-Connected Binary Matroids .......................................... 153
  5.1 Introduction .................................................................................................. 153
  5.2 The Components ......................................................................................... 154

References .................................................................................................................. 164

Vita .............................................................................................................................. 166
This dissertation has three parts. The first part, Chapter 1, considers the coefficient $b_{ij}(M)$ of $x^iy^j$ in the Tutte polynomial of a connected matroid $M$. The main result characterizes, for each $i$ and $j$, the minor-minimal such matroids for which $b_{ij}(M) > 0$. One consequence of this characterization is that $b_{11}(M) > 0$ if and only if the two-wheel is a minor of $M$. Similar results are obtained for other values of $i$ and $j$. These results imply that if $M$ is simple and representable over $GF(q)$, then there are coefficients of its Tutte polynomial which count the flats of $M$ that are projective spaces of specified rank. Similarly, for a simple graphic matroid $M(G)$, there are coefficients that count the number of cliques of each size contained in $G$.

The second part, Chapter 2, generalizes a graph result of Mader by proving that if $f$ is an element of a circuit $C$ of a 3-connected matroid $M$ and $M \setminus e$ is not 3-connected for each $e \in C - \{f\}$, then $C$ meets a triad of $M$. Several consequences of this result are proved. One of these generalizes a graph result of Wu. The others provide 3-connected analogues of 2-connected matroid results of Oxley.

The third part, Chapters 3–5, involves a decomposition of 3-connected binary matroids based on 3-separations and three-sums. The dual of this decomposition is a direct generalization of a decomposition due to Coullard, Gardner, and Wagner for 3-connected graphs. In Chapter 3, we define the decomposition and prove that
minimal such decompositions are unique. In Chapter 4, the components of this decomposition are characterized. In Chapter 5, it is shown that, when restricted to contraction-minimally 3-connected binary matroids, the components that are not vertically 4-connected are wheels, duals of twirls, or binary spikes.
PRELIMINARIES

We assume that the reader is familiar with the basic concepts in matroid and graph theory. In particular, the matroid notation and terminology will, in general, follow Oxley (1992). The notation and terminology specific to graph theory is from Bondy and Murty (1976).

The deletion of a set $X$ of elements from a matroid $M$ is denoted by $M \setminus X$. In order to distinguish between the operation of deletion in a matroid (or graph) and the set-theoretic difference operation, we shall denote the difference of the sets $X$ and $Y$ by $X - Y$. The notation $X \subset Y$ will be used to denote that $X$ is a proper subset of $Y$, that is, $X \subseteq Y$ but $X \neq Y$. 

1
CHAPTER 1

ON COEFFICIENTS OF THE TUTTE POLYNOMIAL

1.1 Introduction

The Tutte polynomial of a matroid was introduced by Crapo (1969). It generalizes a polynomial for graphs introduced by Tutte (1947) (now also called the Tutte polynomial). Some evaluations of the Tutte polynomial correspond to important invariants (for example, the characteristic polynomial, Möbius function, number of bases, number of independent sets, number of spanning sets, etcetera).

For a matroid $M$, the Tutte polynomial is a two-variable polynomial and, hence, can be expressed in the form $t(M) = \sum b_{ij}(M)x^iy^j$. Important structural information about $M$ can be obtained from the coefficients. For example, it is well known that a matroid $M$ with at least two elements is connected if and only if $b_{10}(M) > 0$. Also, a connected matroid $M$ is non-uniform if and only if $b_{11}(M) > 0$ (Brylawski 1982). This chapter explores the structural information that can be obtained from the coefficients.

For a matroid $M$, if $X \subseteq E(M)$, the corank and nullity of $X$ are given by $\text{cor}(X) = r(M) - r(X)$ and $\text{null}(X) = |X| - r(X)$, respectively. The main result of Section 1.2 characterizes the set $\mathcal{M}_{ij}$ of connected matroids $M$ such that $b_{ij}(M) > 0$ but $b_{ij}(N) = 0$ for each connected proper minor $N$ of $M$. This characterization, given in Theorem 1.2.8, depends on the existence of a cyclic flat of corank $i$ and nullity $j$, where a flat is cyclic if it is the union of circuits.
In Section 1.3, it is proved that the sets $\mathcal{M}_{ij}$ are finite by providing an explicit bound on the size of the matroids in $\mathcal{M}_{ij}$. This bound and Theorem 1.2.8 also provide the means for creating an explicit list of matroids in $\mathcal{M}_{ij}$ for small values of $i$ and $j$.

Section 1.4 generalizes the dependence of the coefficient $b_{ij}$ on the existence of a cyclic flat of corank $i$ and nullity $j$. In particular, the coefficients $b_{ij}$ that count the cyclic flats of corank $i$ and nullity $j$ are specified. The above result also has interesting applications for certain special subclasses of matroids. For example, if a simple matroid $M$ is representable over $GF(q)$, then there are coefficients of its Tutte polynomial which count the projective subspaces of $M$ for each dimension. These coefficients are identified in Section 1.4. The coefficients that count the cliques of each size in a simple graphic matroid are also specified.

A detailed summary of the basic theory of the Tutte polynomial is given by Brylawski (1982) and some of our terminology will follow his. In particular, note that in Oxley (1992) and elsewhere, the 'corank' of a set $X$ means the rank of $X$ in the dual. For that particular meaning of 'corank', the notation $r_{M^*}(X)$ will be used here, but it will not be referred to as 'corank'. If $X$ is a union of circuits of $M$, that is, $M|X$ has no coloops, then $X$ is called cyclic. Matroid duality will be used to shorten many of the proofs. In this chapter, one of the more frequently used aspects of duality relates cyclic sets and flats. Specifically, $X$ is a cyclic set of a matroid $M$ if and only if $E(M) - X$ is a flat of the dual matroid $M^*$. For matroids $M_1$ and $M_2$ such that $E(M_1) \cap E(M_2) = \{e\}$, the parallel and series connection with respect to
the basepoint \( e \) will be denoted by \( P_e(M_1, M_2) \) and \( S_e(M_1, M_2) \), respectively. The basic results on parallel and series connections used here were proved by Brylawski (1971), and are summarized in Brylawski (1986).

Let \( B \) be the set of bases of \( M \). For a basis \( B \) and element \( e \in E - B \), the fundamental circuit \( C_M(e, B) \) of \( e \) with respect to \( B \) is the unique circuit contained in \( B \cup e \). Suppose that \( E(M) \) is totally ordered. For a basis \( B \), if \( e \in E - B \) and \( e \) is the least element of \( C_M(e, B) \), then \( e \) is called an externally active element of \( B \). Dually, if \( e \in B \) and \( e \) is the least element of \( C_M^*(e, E - B) \), then \( e \) is called an internally active element of \( B \). Let

\[
IA(B) = \{ e \in B : e \text{ is an internally active element of } B \},
\]

\[
EA(B) = \{ e \in E - B : e \text{ is an externally active element of } B \},
\]

\[
IP(B) = B - IA(B), \text{ and}
\]

\[
EP(B) = (E - B) - EA(B).
\]

The elements in \( IP(B) \) and \( EP(B) \) are called internally and externally passive, respectively. The number of elements in \( IA(B) \) is called the internal activity of \( B \) and is denoted \( \iota(B) \). Likewise, \( |EA(B)| \) is called the external activity of \( B \) and is denoted \( \varepsilon(B) \).

For a matroid \( M \), the Tutte polynomial \( t(M) \) is defined by

\[
t(M) = \sum_{X \subseteq \mathcal{E}(M)} (x - 1)^{\text{cor}(X)}(y - 1)^{\text{null}(X)}. \tag{1.1}
\]

Crapo (1969, Theorem 1) showed that this definition is equivalent to the equation

\[
t(M) = \sum_{B \in \mathcal{B}} x^{\iota(B)}y^{\varepsilon(B)}. \tag{1.2}
\]
Evidently this two-variable polynomial can be written in the form \( t(M) = \sum b_{ij} x^i y^j \), where \( b_{ij} \geq 0 \). For the statement of many of the results that follow, it will be convenient to introduce a partial order on the indices of the coefficients of \( t(M) \).

Define \((i, j) \leq (i', j')\) if \( i \leq i'\) and \( j \leq j'\).

### 1.2 A Characterization of \( \mathcal{M}_{ij} \)

The main result of this section is Theorem 1.2.8, which characterizes the matroids which are minor-minimal among connected matroids with \( b_{ij} > 0 \). The proof will require the following results. The first result is due to Tutte (1966a, 6.5; see also Oxley 1992, Theorem 4.3.1).

**(1.2.1)** For a connected matroid \( M \) and an element \( e \in E(M) \), either \( M/e \) or \( M \setminus e \) is connected.

The next two results are due to Brylawski (1972, Proposition 6.8 and Corollary 6.9, respectively; for the former, see also Oxley 1992, Corollary 4.3.7).

**(1.2.2)** Let \( N \) be a connected minor of a connected matroid \( M \). Then there is a sequence \( M_0, M_1, M_2, \ldots, M_n \) of connected matroids such that \( M_0 = N, M_n = M \), and, for each \( i \) in \( \{0, 1, \ldots, n-1\} \), \( M_i \) is a single-element deletion or a single-element contraction of \( M_{i+1} \).

**(1.2.3)** If \( N \) is a non-empty minor of a connected matroid \( M \), then \( b_{ij}(N) \leq b_{ij}(M) \), for all \((i, j)\).

Let \( a_{ij}(M) = |\{X \subseteq E(M) : \text{cor}(X) = i \text{ and } \text{null}(X) = j\}| \). Thus \( t(M) = \sum a_{ij}(x-1)^i(y-1)^j \). The next two results are taken from Brylawski (1982). The
first is contained in the proof of Proposition 5.8; and the second is an immediate corollary to Proposition 6.5.

(1.2.4) For a matroid $M$, $a_{ij}(M) \geq b_{ij}(M)$.

(1.2.5) Suppose that $M$ is a matroid and $(i, j) \leq (h, k) \leq (i', j')$. If $b_{ij} > 0$ and $b_{i'j'} > 0$, then $b_{hk} > 0$.

(1.2.6) Suppose that $M$ is a connected matroid and $(i, j) \neq (0, 0)$. If there is a set $X \subseteq E$ such that $\text{cor}(X) \geq i$ and $\text{nul}(X) \geq j$, then $b_{ij}(M) > 0$.

Proof. Let $S$ be the collection of subsets $X$ of $E$ such that $\text{cor}(X) \geq i$ and $\text{nul}(X) \geq j$. Let $A_{i'j'} = \{X \subseteq E : \text{cor}(X) = i' \text{ and } \text{nul}(X) = j'\}$. Suppose that $S$ is non-empty. Let $(i'', j'')$ be a maximal member of the finite set $\{(i', j') : A_{i'j'} \subseteq S\}$. By (1.1), the members of $A_{i''j''}$ make positive contributions to $b_{i''j''}$. Since $(i'', j'')$ is maximal, these are the only sets making contributions to $b_{i''j''}$. Therefore $b_{i''j''} > 0$. Because $(i, j) \neq (0, 0)$ either $(i, j) \geq (1, 0)$ or $(i, j) \geq (0, 1)$, and since $M$ is connected, $b_{10} = b_{01} > 0$. Because $(i'', j'') \geq (i, j)$, (1.2.5) implies that $b_{ij} > 0$. □

The last preliminary to the main result follows by duality from the definition of $M_{ij}$.

(1.2.7) $M \in M_{ij}$ if and only if $M^* \in M_{ji}$.

A matroid is minimally connected if it is connected but every single-element deletion is disconnected. A matroid $M$ is minor-minimally-connected if $M$ is connected but, for every element $e$, precisely one of $M\setminus e$ and $M/e$ is disconnected. Let $\text{Con}(M)$, or simply $\text{Con}$, be $\{e \in E(M) : M/e$ is disconnected$\}$. 

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(1.2.8) Theorem. Let \( i + j \geq 2 \). The matroid \( M \) is minor-minimal among connected matroids with \( b_{ij}(M) > 0 \) if and only if \( M \) satisfies the following three conditions:

(i) \( M \) is minor-minimally-connected;

(ii) \( \text{Con} \) is the unique set of corank \( i \) and nullity \( j \); and

(iii) \( \text{Con} \) is a cyclic flat.

Proof. Suppose that \( M \) is minor-minimal among connected matroids with \( b_{ij}(M) > 0 \). Note that, since \( i + j \geq 2 \), it follows that \( M \) has at least two elements. By (1.2.4) there is an \( X \subseteq E \) such that \( r(X) = r(M) - i \) and \( |X| = r(X) + j \). Since \( M \) is connected with at least two elements, if \( e \in E - X \), then \( r_{M \setminus e}(X) = r_{M}(X) = r(M) - i = r(M \setminus e) - i \) and \( |X| = r_{M}(X) + j = r_{M \setminus e}(X) + j \). Thus \( \text{cor}_{M \setminus e}(X) = i \) and \( \text{nul}_{M \setminus e}(X) = j \), so by (1.2.6), \( b_{ij}(M \setminus e) > 0 \). Because \( M \) is minor-minimal among connected matroids with \( b_{ij} > 0 \), it follows that, if \( e \in E - X \), then \( M \setminus e \) is disconnected. By duality, if \( e \in X \), then \( M/e \) is disconnected. Therefore, \( M \) is minor-minimally-connected and \( X = \text{Con} \). Hence \( X \) is the unique set of corank \( i \) and nullity \( j \) and it follows that (i) and (ii) hold.

If \( \text{Con} \) is not cyclic, then there is an element \( e \in \text{Con} \) such that \( \text{cor}(\text{Con} - \{e\}) = i + 1 \) and \( \text{nul}(\text{Con} - \{e\}) = j \). Then \( \text{cor}_{M \setminus e}(\text{Con} - \{e\}) = i + 1 \) and \( \text{nul}_{M \setminus e}(\text{Con} - \{e\}) = j \). It follows by (1.2.6) that \( b_{ij}(M \setminus e) > 0 \) and hence, \( M \setminus e \) is disconnected. This contradicts (1.2.1). Therefore \( \text{Con} \) is cyclic. By duality and (1.2.7), \( E - \text{Con} \) is cyclic in \( M^* \). Therefore \( \text{Con} \) is a flat of \( M \). Thus (iii) holds.
Now suppose $M$ satisfies (i)-(iii). Because $\text{cor}(\text{Con}) = i$ and $\text{nul}(\text{Con}) = j$, it follows by (1.2.6) that $b_{ij} > 0$. Assume that $M$ has a proper connected minor $N$ such that $b_{ij}(N) > 0$. Since $i + j \geq 2$, $N$ must be non-empty. Since $M$ is connected, it follows from (1.2.2) that, for some $e \in E$, either $M \setminus e$ is connected and $N$ is a minor of $M \setminus e$, or $M/e$ is connected and $N$ is a minor of $M/e$. Since $b_{ij}(N) > 0$ and $N$ is non-empty, it follows from (1.2.3) that either (a) $e \in \text{Con}$ and $b_{ij}(M \setminus e) > 0$, or (b) $e \in E - \text{Con}$ and $b_{ij}(M/e) > 0$. Suppose that (a) holds. Then, by (1.2.4), there is a set $Y \subseteq E - \{e\}$ such that $\text{cor}_{M \setminus e}(Y) = i$ and $\text{nul}_{M \setminus e}(Y) = j$. Now $\text{Con} \neq Y$, yet $\text{cor}_M(Y) = \text{cor}_{M \setminus e}(Y) = i$ and $\text{nul}_M(Y) = \text{nul}_{M \setminus e}(Y) = j$, contradicting (ii). Hence (a) is impossible. By duality, (b) cannot occur either. Therefore $M$ has no connected minor $N$ such that $b_{ij}(N) > 0$. □

Though Theorem 1.2.8 goes a long way toward characterizing exactly which matroids are in $\mathcal{M}_{ij}$ for particular values of $i$ and $j$, an explicit list of such matroids will be delayed until the end of the next section. A bound provided in Section 1.3 will make giving such a listing almost trivial for small values of $i$ and $j$. The fundamental relationship between the pairs $\mathcal{M}_{ij}$ and $\mathcal{M}_{ji}$ provided by (1.2.7), despite being an immediate consequence of the definitions, is a valuable tool in constructing the list of matroids in $\mathcal{M}_{ij}$. The following corollary to Theorem 1.2.8 appears to be even more basic than (1.2.7), and yet it does not immediately follow from the definitions.

(1.2.9) Corollary. $\mathcal{M}_{ij} \cap \mathcal{M}_{ij'} = \emptyset$ unless $i = i'$ and $j = j'$. 
1.3 A Bound on $|E(M)|$ for $M \in \mathcal{M}_{ij}$

In Theorem 1.2.8, the set $\text{Con}(M)$ played a key role in determining whether or not the matroid $M \in \mathcal{M}_{ij}$. The main result of this section, Theorem 1.3.4, bounds the sizes of $\text{Con}(M)$ and $\text{Del}(M)$ in terms of $i$ and $j$. These bounds imply that the sets $\mathcal{M}_{ij}$ are finite for all $i$ and $j$. The section concludes with an explicit determination of the sets $\mathcal{M}_{11}, \mathcal{M}_{21}$, and $\mathcal{M}_{12}$.

By Theorem 1.2.8, if $i + j \geq 2$ and $M \in \mathcal{M}_{ij}$, then $M$ is minor-minimally-connected. Therefore, results for minor-minimally-connected matroids also hold for matroids in $\mathcal{M}_{ij}$ where $i + j \geq 2$. The converse is not true. For example, the matroid $M$ obtained by extending a triangle by adding an element in parallel to each of two of the three edges is minor-minimally-connected; but, because $\text{Con}(M)$ is not a flat, $M$ is not in $\mathcal{M}_{ij}$ for any $(i,j)$. A characterization of minor-minimally-connected matroids given by Oxley (1984) will be used to find a bound on $|E(M)|$ for $M \in \mathcal{M}_{ij}$. The following is a straightforward consequence of the characterization given in Oxley (1984).

(1.3.1) Suppose that a matroid $M$ is minor-minimally-connected. Then either $\text{Con}(M)$ is a union of 2-circuits; or, given an element $e \in \text{Con}(M)$ not in a 2-circuit, $M = P_e(M_1 \setminus f_1, M_2 \setminus f_2)$, where both $M_1$ and $M_2$ are minor-minimally-connected having at least four elements and $\{e, f_1\}$ and $\{e, f_2\}$ are circuits of $M_1$ and $M_2$, respectively.

Theorem 1.2.8 showed not only that a member $M$ of $\mathcal{M}_{ij}$ is minor-minimally-connected, but also that, for such a matroid, $\text{Con}(M)$ must be a cyclic flat. This
additional condition will be used to tailor (1.3.1) to the present context. For the
relevance of the terms \( \text{cor}(\text{Con}(M)) \) and \( \text{null}(\text{Con}(M)) \) in what follows, note that if
\( M \in M_i \), then \( i = \text{cor}(\text{Con}(M)) \) and \( j = \text{null}(\text{Con}(M)) \).

(1.3.2) Suppose that \( M \) is minor-minimally-connected, that \( \text{Con}(M) \) is a cyclic
flat, and that \( \text{cor}(\text{Con}(M)) + \text{null}(\text{Con}(M)) \geq 2 \). Then either \( \text{Con}(M) \) is a union of
2-circuits; or, given an element \( e \in \text{Con}(M) \) not in a 2-circuit of \( M \), we have that
\( M = P_e(M_1 \setminus f_1, M_2 \setminus f_2) \) where

(i) both \( M_1 \) and \( M_2 \) are minor-minimally-connected;

(ii) \( \{e, f_1\} \) and \( \{e, f_2\} \) are circuits of \( M_1 \) and \( M_2 \), respectively;

(iii) \( \text{Con}(M_1) \) and \( \text{Con}(M_2) \) are cyclic flats of \( M_1 \) and \( M_2 \), respectively;

(iv) \( \text{cor}(\text{Con}(M_1)) + \text{null}(\text{Con}(M_1)) \geq 2 \), and
\[ \text{cor}(\text{Con}(M_2)) + \text{null}(\text{Con}(M_2)) \geq 2; \] and

(v) \( \text{cor}(\text{Con}(M_1)) + \text{cor}(\text{Con}(M_2)) = \text{cor}(\text{Con}(M)), \) and
\[ \text{null}(\text{Con}(M_1)) + \text{null}(\text{Con}(M_2)) = \text{null}(\text{Con}(M)) + 2. \]

Proof. Suppose \( \text{Con}(M) \) is not a union of 2-circuits and \( e \) is in \( \text{Con}(M) \) but is
not in any 2-circuits of \( M \). By (1.3.1), \( M = P_e(M_1 \setminus f_1, M_2 \setminus f_2) \) where both \( M_1 \) and
\( M_2 \) are minor-minimally-connected, \( \{e, f_1\} \) and \( \{e, f_2\} \) are circuits of \( M_1 \) and \( M_2 \)
respectively, and \( M_1 \) and \( M_2 \) have at least four elements. In particular, (i) and (ii)
follow from (1.3.1) directly.
The first step in the proofs of (iii)-(v) will be to show that

\[ \text{Con}(M_i) - \{f_i\} = \text{Con}(M) \cap E(M_i). \]  \hspace{1cm} (1.3)

Since \( \text{Con}(M_i \setminus f_i) \) equals either \( \text{Con}(M) \cap E(M_i \setminus f_i) \) or \( (\text{Con}(M) \cap E(M_i \setminus f_i)) - \{e\} \), in order to prove (1.3), it is sufficient to show that \( \text{Con}(M_i) = \text{Con}(M_i \setminus f_i) \cup \{e, f_i\} \).

Since \( |E(M_i)| \geq 4 \), it follows that \( \{e, f_i\} \subseteq \text{Con}(M_i) \). Suppose \( x \in \text{Con}(M_i) - \{e, f_i\} \) but \( x \notin \text{Con}(M_i \setminus f_i) \). Then \( M_i \setminus f_i / x = M_i / x \setminus f_i \) is not disconnected but \( M_i / x \) is disconnected. Therefore \( f_i \) must be in a parallel class with \( e \) and \( x \). This contradicts the assumption that \( \{e\} \) is a trivial parallel class of \( M \). Now suppose \( x \in \text{Con}(M_i \setminus f_i) \) but \( x \notin \text{Con}(M_i) \). Then \( M_i / x \) is connected but \( M_i \setminus f_i / x = M_i / x \setminus f_i \) is disconnected. However, \( M_i / x \setminus f_i \) must be connected since \( M_i / x \) is connected and \( f_i \) is parallel to \( e \) in \( M_i \). Hence \( \text{Con}(M_i) = \text{Con}(M_i \setminus f_i) \cup \{e, f_i\} \), and (1.3) is proved.

Since \( \text{Con}(M) \) is a flat of \( M \) it follows that \( \text{Con}(M) \cap E(M_i) \) is a flat of \( M_i \setminus f_i \). Thus, by (1.3), \( \text{Con}(M_i) - \{f_i\} \) is a flat of \( M_i \setminus f_i \). Since \( e \in \text{Con}(M_i) \) and \( \{e, f_i\} \) is a circuit of \( M_i \), it follows that \( \text{Con}(M_i) \) is a flat of \( M_i \).

By (1.3), if \( x \in \text{Con}(M_1) - \{e, f_1\} \), then \( x \in \text{Con}(M) \). Since \( \text{Con}(M) \) is cyclic, \( x \in C \subseteq \text{Con}(M) \) for some circuit \( C \) of \( M \). Either \( C \) is a circuit of \( M_1 \setminus f_1 \), or \( C = (C_1 - \{e, f_1\}) \cup (C_2 - \{e, f_2\}) \) where \( C_1 \) and \( C_2 \) are circuits of \( M_1 \setminus f_1 \) and \( M_2 \setminus f_2 \), respectively. In each case, \( x \) belongs to some circuit contained in \( \text{Con}(M_1) \). Recall that \( e \) is contained in the two-circuit \( \{e, f_1\} \). Hence \( \text{Con}(M_1) \) is cyclic. Likewise, \( \text{Con}(M_2) \) is cyclic. Therefore, \( \text{Con}(M_1) \) and \( \text{Con}(M_2) \) are cyclic flats of \( M_1 \) and \( M_2 \), respectively. Hence (iii) holds.
Suppose $\text{cor}(\text{Con}(M_1)) = 0$. Because $\text{Con}(M_1)$ is a flat, $\text{Con}(M_1) = E(M_1)$.

Since $M_1$ is connected, $M_1$ has at least $r(M_1) + 1$ rank-1 flats. Moreover, because $e$ and $f_1$ are in the same rank-1 flat, $|E(M_1)| \geq r(M_1) + 2$, and hence, $\text{null}(\text{Con}(M_1)) \geq 2$. Therefore if $\text{cor}(\text{Con}(M_1)) = 0$, then $\text{cor}(\text{Con}(M_1)) + \text{null}(\text{Con}(M_1)) \geq 2$. Since \{e, f_1\} is a parallel class of $\text{Con}(M_1)$, it follows that $\text{null}(\text{Con}(M_1)) \geq 1$. Therefore, if $\text{cor}(\text{Con}(M_1)) \geq 1$, then $\text{cor}(\text{Con}(M_1)) + \text{null}(\text{Con}(M_1)) \geq 2$. Hence $\text{cor}(\text{Con}(M_1)) + \text{null}(\text{Con}(M_1)) \geq 2$ and, likewise, $\text{cor}(\text{Con}(M_2)) + \text{null}(\text{Con}(M_2)) \geq 2$. Hence (iv) holds.

Since $e \in \text{Con}(M)$, it follows that $\text{Con}(M) \cong P_e(\text{Con}(M_1 \setminus f_1), \text{Con}(M_2 \setminus f_2))$.

Therefore,

\[
\text{cor}(\text{Con}(M)) = r(M) - r(\text{Con}(M)) = [r(M_1 \setminus f_1) + r(M_2 \setminus f_2) - 1] - [r(\text{Con}(M_1) - \{f_1\}) + r(\text{Con}(M_2) - \{f_2\}) - 1] = [r(M_1) + r(M_2) - 1] - [r(\text{Con}(M_1)) + r(\text{Con}(M_2)) - 1] = [r(M_1) - r(\text{Con}(M_1))] + [r(M_2) - r(\text{Con}(M_2))] = \text{cor}(\text{Con}(M_1)) + \text{cor}(\text{Con}(M_2)).
\]

Also,

\[
\text{null}(\text{Con}(M)) = |\text{Con}(M)| - r(\text{Con}(M)) = [|\text{Con}(M_1)| + |\text{Con}(M_2)| - 3] - [r(\text{Con}(M_1)) + r(\text{Con}(M_2)) - 1]
\]
This completes the proof of (v) and the result. □

(1.3.3) Suppose that \( M \) is minor-minimally-connected with \( \text{Con}(M) \) a cyclic flat and that \( \text{cor}(\text{Con}(M)) + \text{null}(\text{Con}(M)) \geq 2 \). Then

\[
\begin{align*}
(i) \quad |\text{Con}(M)| &\leq \text{cor}(\text{Con}(M)) + 2\text{null}(\text{Con}(M)) - 1; \\
(ii) \quad |\text{Del}(M)| &\leq \text{null}(\text{Con}(M)) + 2\text{cor}(\text{Con}(M)) - 1; \text{ and} \\
(iii) \quad |E(M)| &\leq 3\text{cor}(\text{Con}(M)) + 3\text{null}(\text{Con}(M)) - 2.
\end{align*}
\]

Proof. It is sufficient to prove (i) because (ii) follows from (i) by duality and (iii) is just the combination of (i) and (ii).

Suppose that every element of \( \text{Con}(M) \) is in a non-trivial parallel class. Then \( \text{null}(\text{Con}(M)) \geq \frac{1}{2} |\text{Con}(M)| \), and hence, \( |\text{Con}(M)| \leq 2\text{null}(\text{Con}(M)) \) with equality if and only if the simplification of \( \text{Con}(M) \) is independent and each parallel class of \( \text{Con}(M) \) contains two elements. Since \( M \) is connected, if \( |\text{Con}(M)| = 2\text{null}(\text{Con}(M)) \), then \( \text{cor}(\text{Con}(M)) \geq 1 \). Therefore \( |\text{Con}(M)| \leq \text{cor}(\text{Con}(M)) + 2\text{null}(\text{Con}(M)) - 1 \).

Now suppose that \( \text{Con}(M) \) has an element \( e \) that is not in a two-circuit. The bound given by (i) will be proved by induction on the number of elements of \( \text{Con}(M) \) that are not in two-circuits. Note that \( M \) satisfies (v) of (1.3.2) and \( M = \)
\( P_e(M_1 \setminus f_1, M_2 \setminus f_2). \) By the induction assumption, \( |\text{Con}(M_k)| \leq \text{cor}(\text{Con}(M_k)) + 2nul(\text{Con}(M_k)) - 1, \) for \( k = 1, 2. \) Hence,

\[
|\text{Con}(M)| = |\text{Con}(M_1)| + |\text{Con}(M_2)| - 3 \\
\leq [\text{cor}(\text{Con}(M_1)) + 2nul(\text{Con}(M_1)) - 1] \\
+ [\text{cor}(\text{Con}(M_2)) + 2nul(\text{Con}(M_2)) - 1] - 3 \\
= [\text{cor}(\text{Con}(M_1)) + \text{cor}(\text{Con}(M_2))] \\
+ 2[nul(\text{Con}(M_1)) + nul(\text{Con}(M_2))] - 5 \\
= \text{cor}(\text{Con}(M)) + 2(nul(\text{Con}(M)) + 2) - 5 \\
= \text{cor}(\text{Con}(M)) + 2nul(\text{Con}(M)) - 1. \square
\]

Applying (1.3.3) to the matroids in \( \mathcal{M}_{ij} \) provides the following results.

\textbf{(1.3.4) Theorem.} Suppose that \( i + j \geq 2 \) and \( M \in \mathcal{M}_{ij}. \) Then

(i) \( |\text{Con}(M)| \leq i + 2j - 1; \)

(ii) \( |\text{Del}(M)| \leq j + 2i - 1; \) and

(iii) \( |E(M)| \leq 3i + 3j - 2. \)

\textbf{(1.3.5) Corollary.} The sets \( \mathcal{M}_{ij} \) are finite for each \( i \) and \( j. \)

The bounds given by Theorem 1.3.4 and the characterization given by Theorem 1.2.8 provide the means for creating the list of matroids in \( \mathcal{M}_{ij} \) for small values of \( i \) and \( j. \) The following results illustrate this. Recall that \( \mathcal{W}_2 \) denotes the two-spoked wheel.
(1.3.6) Proposition. \( \mathcal{M}_{11} = \{W_2\} \).

Proof. Suppose that \( M \in \mathcal{M}_{11} \). By Theorem 1.2.8, \( \text{Con}(M) \) is a circuit. Moreover, Theorem 1.3.4 (i) implies that \( \text{Con}(M) \) is a 2-circuit. Therefore \( r(M) = 2 \). Clearly, the two-wheel is a minor of every rank-two connected matroid containing a 2-circuit. Since the two-wheel satisfies the conditions of Theorem 1.2.8, the result holds. \( \square \)

![](image)

Figure 1.1: (a) \( G_1 \) (b) \( G_2 \)

(1.3.7) Proposition. Let \( G_1 \) and \( G_2 \) be the graphs in Figure 1.1. Then

\[ \mathcal{M}_{21} = \{M(G_1), M(G_2)\} \]

Proof. Suppose that \( M \in \mathcal{M}_{21} \). By Theorem 1.2.8, \( \text{Con}(M) \) is a circuit. Moreover, Theorem 1.3.4 (i) implies that \( \text{Con}(M) \) is either a 2-circuit or a 3-circuit. Suppose that \( \text{Con}(M) \) is the 3-circuit \( \{e, f, g\} \). By (1.3.1), \( M \cong P_3(N_1, N_2) \) where it may be assumed that \( \{e, f, g\} \subseteq E(N_1) \) and \( |E(N_2) - \{e\}| \geq 2 \). The matroid \( N_1 \) can be similarly decomposed relative to the element \( f \). This procedure can be repeated again for the element \( g \). The result is that there are at least six elements in \( \text{Del}(M) \), contradicting Theorem 1.3.4 (ii). Therefore \( \text{Con}(M) \) is a 2-circuit and \( r(M) = 3 \). Since \( M \) is connected and contains a 2-circuit, it must have at least five elements. Since \( |E(M)| = |\text{Del}(M)| + 2 \), Theorem 1.3.4 (ii) implies that \( |E(M)| \leq 6 \). If \( |E(M)| = 5 \), then \( M = M(G_2) \).
Suppose that $|E(M)| = 6$. Note that $M^* \in \mathcal{M}_{12}$, and $\text{Con}(M^*) = \text{Del}(M)$ and $\text{Del}(M^*) = \text{Con}(M)$. Moreover, $r(M^*) = 3$ and $\gamma_{M^*}(\text{Con}(M^*)) = 2$; and $\text{Con}(M^*)$ is cyclic and without loops in $M^*$. Therefore, in $M^*$, the restriction to $\text{Con}(M^*)$ is either isomorphic to the two-wheel; or isomorphic to the rank-two uniform matroid on four elements; or $\text{Con}(M^*)$ consists of two sets $A$ and $B$ of parallel elements with two elements in each set. In the first two cases, $M^*/x \cong M^*(G_2)$ for some $x \in \text{Con}(M^*)$. Therefore, the last case holds. Each of the two four-element sets $\text{Con}(M) \cup A$ and $\text{Con}(M) \cup B$ is hyperplane of $M$. Each of these hyperplanes is isomorphic to the two-wheel. Therefore, $M = M(G_1)$. Clearly $M(G_1)$ and $M(G_2)$ satisfy Theorem 1.2.8. Hence the result is proved. □

The next result follows immediately from its predecessor by duality.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.2.png}
\caption{(a) $H_1$ (b) $H_2$}
\end{figure}

\textbf{(1.3.8) Proposition.} Let $H_1$ and $H_2$ be the graphs in Figure 1.2. Then

\[ \mathcal{M}_{1,2} = \{ M(H_1), M(H_2) \}. \]

\section{1.4 Coefficients that Count Cyclic Flats}

The main result of this section, Theorem 1.4.11, determines precisely when a coefficient $b_{ij}$ of $t(M)$ counts the subsets of $M$ having corank $i$ and nullity $j$. In particular, it will follow from this that if $M \in \mathcal{M}_{ij}$, then $b_{ij}(M) = 1$. 

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Arranging the positive coefficients of $t(M)$ in an array, the coefficients of Theorem 1.4.11 will be identified by their relative location in this array. Consider the matroid $M = M(G)$ where $G$ is the graph in Figure 1.3.

![Figure 1.3: G](image)

The Tutte polynomial for $M$ is

$$t(M) = x^5 + 4x^4 + 6x^3 + 5x^2 + 2x + x^4y + 8x^3y + 12x^2y + 9xy + 2y + 8x^2y^2 + 12xy^2 + 6y^2 + x^2y^2 + 6xy^3 + 7y^3 + xy^4 + 4y^4 + y^5$$

The boxed terms $x^5$, $x^4y$, $x^2y^2$, $xy^4$, and $y^5$ are called the corners of $t(M)$. In general, the term $b_{ij}x^iy^j$, or more usually, the coefficient $b_{ij}$ is a corner of $t(M)$ if $b_{ij} > 0$ and there is no other positive coefficient $b_{i'j'}$, with $(i', j') \geq (i, j)$.

By (1.2), $b_{ij}$ counts the bases of internal activity $i$ and external activity $j$.

The proof of Theorem 1.4.11 will rely on the existence of a collection of sets $\mathcal{A}$ in one-to-one correspondence with the bases $\mathcal{B}$, such that if $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $\text{cor}(A) = \iota(B)$ and $\text{nul}(A) = \varepsilon(B)$ whenever $A$ corresponds to $B$. Dawson (1981) constructed such a collection $\mathcal{A}$ as follows. Suppose that $M$ is a totally ordered matroid. Let $\mathcal{A} = \{A : A = \text{IP}(B) \cup \text{EA}(B) \text{ for } B \in \mathcal{B}\}$. The definition of $\mathcal{A}$ induces a bijection from $\mathcal{B}$ to $\mathcal{A}$. The proof of the main result will require an
extension of this map to the entire power set of $E$. To create such an extension consider the collection $\mathcal{P}(M)$ of intervals of the power set $2^E$ of $E$ where

$$\mathcal{P}(M) = \{[[IP(B), E - EP(B)] : B \in B\}.$$

Dawson (1981, Lemma 1.4) proved that $\mathcal{P}(M)$ partitions $2^E$ and each interval of $\mathcal{P}(M)$ contains precisely one basis and one member of $\mathcal{A}$. Hence, there are well-defined functions $\alpha$ and $\beta$ from $2^E$ to $\mathcal{A}$ and $B$, respectively. These functions are defined by letting $\alpha(X)$ be the element of $\mathcal{A}$ such that $X$ and $\alpha(X)$ are in the same interval of $\mathcal{P}(M)$. Similarly, $\beta(X)$ is the element of $B$ such that $X$ and $\beta(X)$ are in the same interval of $\mathcal{P}(M)$. Note that $\mathcal{P}(M)$, $\mathcal{A}$, $\alpha$, and $\beta$ depend on the ordering of $E(M)$.

In order to prove the main result of this section, the following additional results will be needed. For a subset $X$ of a totally ordered matroid, let $\min(X)$ denote the least element of $X$ with respect to that order.

(1.4.1) *(Dawson 1981, Corollary 3.3)* $A \in \mathcal{A}$ if and only if

(i) $C \subseteq A$ for all circuits of $M$ such that $C - \{\min(C)\} \subseteq A$; and

(ii) $D \subseteq E - A$ for all cocircuits of $M$ such that $D - \{\min(D)\} \subseteq E - A$.

The next two results are also from Dawson (1981). They appear in the proofs of more general results, Proposition 3.5 and Theorem 3.6, respectively.

(1.4.2) *If $A \in \mathcal{A}$ and $A \in [IP(B), E - EP(B)]$ then $A$ is the unique set in $[IP(B), E - EP(B)]$ such that $\text{cor}(A) = \iota(B)$ and $\text{null}(A) = \varepsilon(B)$. Moreover, there are $\left(\begin{array}{c} \text{cor}(A) \\ i \end{array}\right) \left(\begin{array}{c} \text{null}(A) \\ j \end{array}\right)$ sets of corank $i$ and nullity $j$ in $[IP(B), E - EP(B)]$.***
(1.4.3) If $X$ is a cyclic flat of $M$, then $X \in A$.

Suppose that $E$ is totally ordered and $X$ and $Y$ are subsets of $E$. Then define $X < Y$ if $x < y$ for all $x \in X$ and $y \in Y$.

(1.4.4) Suppose that $M$ is a non-empty connected matroid and $X \subseteq E$. Then $X \not\in B$ if and only if there is a total order on $E$ such that $X \in A$.

Proof. Suppose that $A \in A$ and $A \in B$. By (1.4.2), $cor(A) = \iota(A)$ and $null(A) = \varepsilon(A)$. Since $A \in B$, the corank and the nullity of $A$ are zero. Therefore, $\iota(A) = \varepsilon(A) = 0$. This contradicts the observation that the least element of any ordering (which exists, since $M$ is non-empty) will be active relative to any basis. Hence if $A \in A$, then $A \not\in B$.

Suppose that $A \not\in B$. Let $B \in B$ such that $B \cap A$ is a basis for $A$. Suppose that $|E(M)| = n$ and let $l = n - |B \cap A| - |(E - B) \cap (E - A)|$. We construct an order on $E$ by first arbitrarily assigning $\{1, 2, \ldots, l\}$ to the elements in $(B - A) \cup (A - B)$. Then use the following algorithm to assign $\{l + 1, l + 2, \ldots, n\}$ to the elements in $(B \cap A) \cup [(E - B) \cap (E - A)]$.

Let $N = M$, let $X = B \cap A$, let $X' = (E - B) \cap (E - A)$, and let $k = n$.

1. If $X \cup X'$ is empty, then stop.

2. If there is a two-circuit $\{x, y\}$ of $N$ such that $x \in X$ and $y \in X'$, go to step 6.

3. If there is a two-cocircuit $\{x, y\}$ of $N$ so that $x \in X'$ and $y \in X$, go to step 5.

4. Arbitrarily choose an element $y \in X \cup X'$. If $y \in X$, go to step 5. If $y \in X'$, go to step 6.
5. Assign the number \( k \) to the element \( y \). Replace \( N \) by \( N/y \); replace \( X \) by \( X - \{y\} \); and replace \( k \) by \( k - 1 \). Go to step 1.

6. Assign the number \( k \) to the element \( y \). Replace \( N \) by \( N\setminus y \); replace \( X' \) by \( X' - \{y\} \); and replace \( k \) by \( k - 1 \). Go to step 1.

In order to show that \( A \in \mathcal{A} \) for the above order, it is sufficient to show that \( A \) satisfies (i) and (ii) of (1.4.1). First we prove that the above algorithm satisfies the following two conditions.

(a) No element of \( B \cap A \) is contracted as a coloop.

(b) No element of \( (E - B) \cap (E - A) \) is deleted as a loop.

Suppose that an element \( x \) of \( B \cap A \) was contracted as a coloop. Since \( M \) is connected, there must be an element \( y \) such that, at some step, \( \{x, y\} \) was a two-cocircuit and \( y \) was deleted. Therefore, \( y > x \) and \( \{x, y\} \) was a two-cocircuit when \( y \) was deleted in step 6. Since step 3 applies in this situation but was evidently not reached, \( y \) must have been deleted as a result of step 2. Therefore \( y \) was in a two-circuit at the time. By orthogonality, \( \{x, y\} \) was both a two-circuit and a two-cocircuit at this step. This implies that \( E(N) = \{x, y\} \). Since \( A \not\subset B \), either \( B - A \) or \( A - B \) is non-empty. Since \( (B - A) \cup (A - B) \subseteq E(N) \) at each step, \( E(N) - \{x, y\} \) was non-empty, a contradiction. Therefore, no element of \( B \cap A \) is contracted as a coloop. Hence (a) is proved, and (b) follows by duality.

Suppose that \( C \) is a circuit of \( M \) and \( C - \{\min(C)\} \subseteq A \). Let \( y = \min(C) \) and suppose that \( y \in (E - B) \cap (E - A) \). Since \( A - B < (E - B) \cap (E - A) \), the set
$C - \{y\} \subseteq A \cap B$. Therefore, in the algorithm producing the order, $y$ was deleted as a loop. This contradicts (b) and hence $y \not\in (E - B) \cap (E - A)$. Now suppose that $y \in B - A$. Since $y \in cl(A)$ and $B \cap A$ is a basis for $A$, the element $y \in cl(B \cap A)$. This contradicts the assumptions that $B$ is a basis of $M$ and $y \in B - A$. Therefore $y \not\in B - A$. Hence $y \in A$ and the set $A$ satisfies (i) of (1.4.1). The set $A$ satisfies (ii) of (1.4.1) by duality. Therefore, $A \in \mathcal{A}$. □

(1.4.5) For a matroid $M$, suppose that $A \subseteq E(M)$. Then $A \in \mathcal{A}$ for all total orderings of $E$ if and only if $A$ is a cyclic flat of $M$.

Proof. Suppose that $A$ is not a flat. Then there is a circuit $C$ of $M$ and an element $y \in C$ such that $C \cap A = C - \{y\}$. Construct an order on $E$ such that $y$ is the minimum element. Then the set $A$ does not satisfy (i) of (1.4.1). Therefore, $A \not\in \mathcal{A}$. By duality, if $A$ is not cyclic, then there is an order of $E$ such that $A \not\in \mathcal{A}$. Therefore, if $A$ is not a cyclic flat, then there is an order on $E$ such that $A \not\in \mathcal{A}$. An application of (1.4.3) completes the proof. □

The following is a direct consequence of (1.4.5). Note that it strengthens a result (stated earlier as (1.2.4)) of Brylawski.

(1.4.6) Proposition. For a matroid $M$, suppose that $a_{ij}$ counts the sets of corank $i$ and nullity $j$. Then $b_{ij} < a_{ij}$ if and only if there is a set of corank $i$ and nullity $j$ which is not a cyclic flat.

The following result will be used to generalize (1.4.4) to all matroids. It also provides a combinatorial proof of the well-known result that if $M = M_1 \oplus M_2$, then $t(M) = t(M_1)t(M_2)$. 

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(1.4.7) Proposition. Suppose that $M = M_1 \oplus M_2 \oplus \cdots \oplus M_t$. Further, suppose that $E = E(M)$ is totally ordered and that $E_s = E(M_s)$ has the total ordering induced by the ordering on $E$, for $s = 1, 2, \ldots, t$. Then $[X, Y] \in \mathcal{P}(M)$ if and only if $[X|E_s, Y|E_s] \in \mathcal{P}(M_s)$ for $s = 1, 2, \ldots, t$.

Proof. We may assume that $M = M_1 \oplus M_2$; the general case follows immediately by induction. Since $M = M_1 \oplus M_2$, a subset $B$ of $E$ is a basis of $M$ if and only if $B_s = B|E_s$ is a basis for $M_s$ for each $s \in \{1, 2\}$. Suppose that $B, B_1$, and $B_2$ are bases of $M, M_1$, and $M_2$, respectively, where $B = B_1 \cup B_2$. The proof of the result relies on the claim that $IP(B_s) = IP(B)|E_s$ for each $s \in \{1, 2\}$. Since $M = M_1 \oplus M_2$, for $s \in \{1, 2\}$, a set $D$ meeting $E_s$ is a cocircuit of $M$ if and only if $D$ is a cocircuit of $M_s$. Therefore, for $s \in \{1, 2\}$, if $x$ is an element of $B_s$, then $C_{M^*}(x, E_s - B_s) = C_{M^*}(x, E - B)$. Hence, for $s \in \{1, 2\}$, if $x \in E_s$, then $x \in IP(B_s)$ if and only if $x \in IP(B)$; that is $IP(B_s) = IP(B)|E_s$. By duality, $E_s - EP(B_s) = [E - EP(B)]|E_s$. Since, for $s \in \{1, 2\}$, each interval of $\mathcal{P}(M)$ and $\mathcal{P}(M_s)$ is of the form $[IP(B), E - EP(B)]$ and $[IP(B_s), E_s - EP(B_s)]$, respectively, the result is proved. □

(1.4.8) Corollary. Suppose the matroid $M = M_1 \oplus M_2 \oplus \cdots \oplus M_t$. Then $t(M) = t(M_1)t(M_2) \cdots t(M_t)$.

(1.4.9) Corollary. Suppose that $M = M_1 \oplus M_2 \oplus \cdots \oplus M_t$. Then $b_{ij}(M)$ is a corner of $t(M)$ if and only if $b_{i_1j_1}(M_s)$ is a corner of $t(M_s)$ for every sequence $(i_1, j_1), (i_2, j_2), \ldots, (i_t, j_t)$ such that $\sum_{s=1}^t i_s = i$, and $\sum_{s=1}^t j_s = j$, and $b_{i_1j_1}(M_s) > 0$. 

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The next result follows from (1.4.4) and Proposition 1.4.7.

\[(1.4.10)\] Suppose that \(M\) is a non-empty matroid and \(M = M_1 \oplus M_2 \oplus \cdots \oplus M_t\), where \(M_s\) is a non-empty connected matroid, for \(s = 1, 2, \ldots, t\). Then there is a total order on \(E\) such that \(A \cap E(M_s) \not\subseteq B(M_s)\) if and only if \(A \subseteq A_s\), for each \(s \in \{1, 2, \ldots, t\}\).

The following theorem is the main result of this section. Brylawski (1982) showed that if \(b_{ij}\) is a corner of \(t(M)\), then \(b_{ij}\) counts the sets of corank \(i\) and nullity \(j\) and each such set is a cyclic flat. The proof given by Brylawski was based on an examination of the Tutte polynomial given by (1.1). A new combinatorial proof will be given here. This theorem also strengthens Brylawski’s result.

\[(1.4.11)\] Theorem. Suppose that \(b_{ij} > 0\) for a matroid \(M\). Then the following are equivalent.

(i) \(b_{ij}\) is a corner of \(t(M)\).

(ii) Every set of corank \(i\) and nullity \(j\) is a cyclic flat.

(iii) \(b_{ij}\) counts the sets of corank \(i\) and nullity \(j\).

Proof. The equivalence of (ii) and (iii) follows immediately from Proposition 1.4.6. Statements (i) and (ii) are clearly equivalent in the case of the empty matroid, so we shall assume that \(M\) is not empty.

To prove that (i) implies (ii), suppose that there is a set \(X\) of corank \(i\) and nullity \(j\) which is not a cyclic flat. By (1.4.5), there is an order on \(E\) such that

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Suppose that $\alpha(X) = A$ and $\beta(X) = B$. By (1.4.2),

$$(\varepsilon(B), \varepsilon(B)) = (\text{cor}(A), \text{null}(A)) > (i, j).$$

Therefore, $b_{ij}$ is not a corner of $t(M)$. Hence if (ii) fails, so does (i).

Finally, to prove that (ii) implies (i), note that since a cyclic flat of a matroid is the disjoint union of cyclic flats of each connected component, Corollary 1.4.9 implies that we need only prove this implication in the connected case. Therefore, suppose that $M$ is connected and that $b_{ij}$ is not a corner of $t(M)$. Then $b_{i'j'} > 0$ for some $(i', j') > (i, j)$. Further, suppose that $A \in A$ such that $\text{cor}(A) = i'$ and $\text{null}(A) = j'$. By (1.4.2), there is a set $X$ such that $\alpha(X) = A$ and $\text{cor}(X) = i$ and $\text{null}(X) = j$. Moreover, $X$ is not a cyclic flat, since $X \not\subseteq A$. Hence (ii) implies (i). □

Before applying Theorem 1.4.11, we note that the intervals of $P(M)$ have been used before, in connection with the Tutte polynomial. Crapo (1969) used essentially the same intervals to prove that the definitions of the Tutte polynomial given by (1.1) and (1.2) are equivalent. Also, a recursive definition of these intervals, given by Gordon and Traldi (1990), motivated the algorithm in the proof of (1.4.4).

The next result applies Theorem 1.4.11 to the matroids $M_{ij}$ characterized by Theorem 1.2.8.

(1.4.12) Corollary. A connected matroid $M$ has a connected proper minor $N$ such that $b_{ij}(N) > 0$ if either of the following holds.

(i) $b_{ij}(M) > 1$.

(ii) $b_{i'j'}(M) > 0$ for some $(i', j') > (i, j)$. 

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In the following, we use well-known facts about graphic and representable matroids to apply Theorem 1.4.11 to matroids in those subclasses. Though these results are not difficult to prove directly without Theorem 1.4.11, they appear to be new results. In essence, Theorem 1.4.11 has done more to facilitate the discovery of these results than it has done to facilitate their proofs.

Suppose that $G$ is a simple graph and $M = M(G)$. Then the largest possible sets of rank $k - 1$ in $M$ are $k$-cliques of $G$. If $k \geq 3$, then each $k$-clique of $G$ is cyclic in $M$. Every $k$-clique has nullity $n(k) = \binom{k}{2} - (k - 1) = \binom{k-1}{2}$. Applying Theorem 1.4.11 yields the following.

(1.4.13) Proposition. Suppose that $G$ is a simple graph, $M = M(G)$, and $k \geq 3$. Then

(i) $b_{r(M) - (k-1), n(k)}$ counts the $k$-cliques of $G$;

(ii) $b_{r(M) - (k-1), n(k)} \leq \binom{r(M) + 1}{k}$; and

(iii) if $(i,j) > (r(M) - (k - 1), n(k))$, then $b_{ij}(M) = 0$.

Now suppose that $M$ is a simple matroid representable over $GF(q)$. The largest possible sets of rank $k$ in $M$ are flats isomorphic to the projective space $PG(k - 1, q)$. If $k \geq 2$, then $PG(k - 1, q)$ is cyclic and has nullity $n(k, q) = (q^k - 1)/(q - 1) - k$. Recall that the projective space $PG(r - 1, q)$ contains $\binom{r}{k}_q$ restrictions isomorphic to the projective space $PG(k - 1, q)$, where $\binom{r}{k}_q$ is the Gaussian coefficient

$$\left[ \begin{array}{c} r \\ k \end{array} \right]_q = \frac{(q^r - 1)(q^r - q) \cdots (q^r - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})}.$$
Applying Theorem 1.4.11 yields the following.

(1.4.14) Proposition. Suppose $M$ is a simple matroid representable over $GF(q)$ and $k \geq 2$. Then

(i) $b_{r(M) - k, n(k, q)}$ counts the projective spaces $PG(k-1, q)$ contained in $M$;

(ii) $b_{r(M) - k, n(k, q)} \leq \left[ \begin{array}{c} r(M) \\ k \end{array} \right]_q$; and

(iii) if $(i, j) > (r(M) - k, n(k, q))$, then $b_{ij}(M) = 0$.

In particular, if $M$ is a simple binary matroid, then $b_{r(M) - 2, 1}$ counts the 3-circuits in $M$ and $b_{r(M) - 3, 4}$ counts the Fano submatroids contained in $M$. 

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CHAPTER 2

TRIADS AND TRIANGLES IN 3-CONNECTED MATROIDS

2.1 Introduction

Mader (1996, Proposition 2) proved that if $C$ is a circuit of a 3-connected graph $G$ with the property that $\min_{v \in C} d(v) > 3$, then $C$ has at least two superfluous edges. Here, $d(v)$ denotes the degree of a vertex $v$ and a superfluous edge $e$ is one for which $G\setminus e$ is 3-connected. This chapter shows that the above result is a special case of a more general theorem for matroids. That theorem, the main result of this chapter, follows.

(2.1.1) Theorem. Suppose that $M$ is a 3-connected matroid with at least four elements. Let $C$ be a circuit of $M$ and $f$ be an element of $C$. If $M\setminus e$ is not 3-connected for every $e \in C \setminus \{f\}$, then $C$ meets a triad of $M$.

Oxley (1981b, Lemma 2.3) proved that if a 2-connected matroid $M$, with at least two elements, has a circuit $\{x_1, x_2, \ldots, x_m\}$ such that $M\setminus x_i$ is not 2-connected for all $i$ in $\{1, 2, \ldots, m-1\}$, then $\{x_1, x_2, \ldots, x_{m-1}\}$ contains a 2-cocircuit. By orthogonality, a circuit contains a 2-cocircuit if and only if it meets a 2-cocircuit. Hence, our main result is at once the 3-connected analogue of Oxley's 2-connected result and the matroid generalization of Mader's graph result.

While extremal connectivity results like Theorem 2.1.1 are of interest in their own right, they also tend to be valuable tools in other areas of matroid theory. To
illustrate this, we state several corollaries of Theorem 2.1.1, the proofs of which will be delayed until Section 2.4. The first is a generalization of the following graph result of Mader, which was proved independently by Haidong Wu (see Mader 1996, p. 430).

\((2.1.2)\) If \(C\) is a circuit in a 3-connected graph with at most one superfluous edge \([v_0, v_1]\), then \(d(v_0) = d(v_1) = 3\), or there is a \(v \in V(C) - \{v_0, v_1\}\) with \(d(v) = 3\).

\((2.1.3)\) Corollary. Suppose that \(M\) is a 3-connected matroid with at least four elements. Let \(C\) be a circuit of \(M\) and \(f\) be an element of \(C\). If \(M \setminus e\) is not 3-connected for every \(e \in C - \{f\}\), then either

(i) \(M\) has two triads meeting \(f\), or

(ii) \(M\) has a triad meeting \(C\) which does not contain \(f\).

Theorem 2.1.1 also leads to new results on 3-connected matroids which extend 2-connected results. Oxley (1984) proved several results using the 2-connected analogue of our main theorem. These results bound the number of 2-circuits and 2-cocircuits in a 2-connected matroid. After proving Theorem 2.1.1, we will be able to see these results in the context of what similar results can be proved in the 3-connected case. In view of this more general context, it appears that the original 2-connected results should be able to be strengthened. By examining the proofs of Theorem 3.1 and Proposition 3.5 of Oxley (1984), it is not difficult to see that these proofs actually establish the stronger results (2.1.4) and (2.1.5), which are stated below.
In order to state and prove these results it will be convenient to establish the following notation. If $M$ is an $n$-connected matroid that is not $(n + 1)$-connected, define the set $\text{Del}$ to be those elements $e$ of $M$ for which $M \setminus e$ is not $n$-connected. Let $\text{Con}(M) = \text{Del}(M^*)$. An $n$-connected matroid $M$ is minor-minimally $n$-connected if either $M \setminus e$ or $M/e$ is not $n$-connected for each $e \in E(M)$; that is, $E(M) = \text{Del} \cup \text{Con}$. Let $f_n(M)$ denote the number of dependent rank-$n$ flats of $M$ and let $f_n^*(M) = f_n(M^*)$.

(2.1.4) For all minor-minimally $2$-connected matroids $M$ with rank and corank at least two,

$$f_1(M) + f_1^*(M) \geq r(M/\text{Del}) + r(M^*/\text{Con}) + 1.$$  

(2.1.5) For all minor-minimally $2$-connected matroids $M$ with rank and corank at least two,

$$f_1(M) + f_1^*(M) \geq 2$$

We extend these results to the $3$-connected case as follows.

(2.1.6) Proposition. For all minor-minimally $3$-connected matroids $M$ with rank and corank at least three,

$$f_2(M) + f_2^*(M) \geq \frac{1}{2} [r(M/\text{Del}) + r(M^*/\text{Con})] + 1.$$  

(2.1.7) Proposition. For all minor-minimally $3$-connected matroids $M$ with rank and corank at least three,

$$f_2(M) + f_2^*(M) \geq 2.$$
The basic property of matroids that a circuit and a cocircuit cannot have exactly one common element will be referred to as orthogonality. A partition \( \{X, Y\} \) of the ground set of a matroid \( M \) is called a \( k \)-separation of \( M \) if \( \min\{|X|, |Y|\} \geq k \), and

\[
r(X) + r(Y) - r(M) \leq k - 1.
\] (2.1)

A matroid \( M \) is called \( n \)-connected, if for all \( k < n \), there is no \( k \)-separation of \( M \). The inequality given by (2.1) is equivalent to the inequality

\[
r(X) + r^*(X) - |X| \leq k - 1.
\] (2.2)

In some circumstances, (2.2) will be of more immediate use than (2.1). We will depart slightly from the notation for parallel connection given by Oxley (1992). There, if the basepoint is understood, then the parallel connection of the matroids \( M \) and \( N \) is denoted \( P(M, N) \). For clarity, we shall explicitly specify the basepoint \( x \) of a parallel connection with the notation \( P_x(M, N) \). We will use nothing more than the basic properties of parallel connections, which can be found in Oxley (1992).

Finally, note that Theorem 2.1.1 is similar to the following result of Lemos (1989, Theorem 1).\

\[\text{(2.1.8) Theorem.} \quad \text{Suppose that} \ M \ \text{is a} \ 3 \text{-connected matroid with at least four elements and let} \ C \ \text{be a circuit of} \ M. \ \text{If} \ M\setminus e \ \text{is not} \ 3 \text{-connected for every} \ e \in C, \ \text{then} \ C \ \text{meets at least two distinct triads of} \ M. \]

Though the proof of Theorem 2.1.1 will follow the general outline used by Lemos, our weaker hypothesis will require significant detours from that outline.

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2.2 Preliminary Results

In this section, we provide a list of results that will be used to prove Theorem 2.1.1. Throughout this section it is assumed that $M$ is a 3-connected matroid with at least four elements. The first two results are from Tutte (1966a, 7.1 and 7.2, respectively). Recall that an element $e$ of a 3-connected matroid $M$ is essential if neither $M\setminus e$ nor $M/e$ is 3-connected.

(2.2.1) If $e$ is an essential element of $M$ then $e$ belongs to a triangle or a triad of $M$.

(2.2.2) If $M\setminus e$ and $M\setminus f$ are not 3-connected and $\{e, f\}$ is contained in a triangle $T$ of $M$, then there is a triad $T'$ of $M$ such that $e \in T'$ and $|T \cap T'| = 2$.

The following two results of Oxley (1981c, Lemma 2.6 and 1981a, Lemma 2.1, respectively) were proved for arbitrary connectivity. We restate them for the 3-connected case, which is all that will be needed here.

(2.2.3) If $M\setminus e$ is not 3-connected and $(M/f)\setminus e$ is 3-connected, then there is a triad of $M$ which contains $e$ and $f$.

(2.2.4) Suppose that $N/e$ is 3-connected, but $N$ is not. Then either $e$ is a loop, a coloop, or in a 2-cocircuit of $N$.

Instead of making direct use of (2.2.4), we shall use the following straightforward consequence of it.
(2.2.5) Suppose that $M \setminus f$ is $3$-connected but $M \setminus e$ is not $3$-connected. Then the matroid $(M \setminus f) \setminus e$ is not $3$-connected.

The next five results (2.2.6)-(2.2.10) follow directly from the definition of $k$-connectivity. The last two arise more clearly as consequences of (2.2).

(2.2.6) Suppose that $\{A, B\}$ is a 2-separation of $M \setminus e$. Then neither $A$ nor $B$ spans $e$.

(2.2.7) Let $e$ be an element that does not belong to any triad of $M$. If $\{A, B\}$ is a 2-separation for $M \setminus e$ and some element $f$ of $B$ is spanned by $A$ in $M$ or $M^*$, then $\{A \cup \{f\}, B - \{f\}\}$ is a 2-separation for $M \setminus e$.

(2.2.8) Suppose that $\{A, B\}$ is a 3-separation of $M$. If some element $f$ of $B$ is spanned by $A$ in $M$ or $M^*$, then $\{A \cup \{f\}, B - \{f\}\}$ is a 3-separation of $M$.

(2.2.9) Suppose that $\{A, B\}$ is a partition of $E(M) - \{e\}$ and $|A| = 2$.

(i) If $\{A, B\}$ is a 2-separation of $M \setminus e$, then $A \cup \{e\}$ is a triad of $M$.

(ii) If $\{A, B\}$ is a 2-separation of $M/e$, then $A \cup \{e\}$ is a triangle of $M$.

(2.2.10) If $|A| = 3$ and $A$ is both a triangle and a triad of $M$, then $|E(M)| = 4$.

The following result is due to Seymour (1980, 2.3).

(2.2.11) Suppose that $\{A, B\}$ is a 2-separation of a matroid $N$. If $C$ and $C'$ are circuits of $N$ which intersect both $A$ and $B$, then $C \cap A$ is not a proper subset of $C' \cap A$. 

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The following three results were verified and then used by Lemos to prove Theorem 2.1.8. The proof of Theorem 2.1.1 will also use them.

(2.2.12) (Lemos 1989, 2.5) Suppose that \( e \) is an essential element of \( M \). If \( e \) does not belong to any triad of \( M \), then the simplification of \( M/e \) is 3-connected and every parallel class of \( M/e \) has at most two elements.

(2.2.13) (Lemos 1989, 2.12) Suppose that \( e \) and \( f \) are distinct elements of \( M \) and that \( M \setminus e \) is not 3-connected. If there are triangles \( T \) and \( T' \) of \( M \) such that \( e \in T' \), \( f \in T \), \( |T \cap T'| = 1 \), and \( T' \cup f \) is a cocircuit of \( M \), then \( e \) belongs to a triad of \( M \).

(2.2.14) (Lemos 1989, 2.13) Suppose that \( U \subseteq E(M) \) and \( M \setminus U \cong M(K_4) \) and that \( b \) and \( d \) are distinct elements of \( U \). If there is no triangle of \( M \setminus U \) which contains \( \{b, d\} \) and neither \( M \setminus b \) nor \( M \setminus d \) is 3-connected, then \( U = E(M) \).

2.3 The Proof of Theorem 2.1.1

Suppose \( (M, C, f) \) is a triple for which Theorem 2.1.1 does not hold. We will call such a triple a bad triple. Consider the bad triples in which \( |C| \) is a minimum and, from among these, choose one in which \( |E(M)| \) is as small as possible. Such a triple will be called a minimal triple. By (2.2.2), \( |C| > 3 \).

(2.3.1) If \( e \in C - \{f\} \), then \( e \) is an essential element of \( M \).

Proof. Suppose that \( M/e \) is 3-connected and \( e' \in C - \{e,f\} \). Since \( M \setminus e' \) is not 3-connected, (2.2.3) implies that \( (M/e) \setminus e' \) is not 3-connected. The set \( C - \{e\} \) is a circuit of \( M/e \). If there is no triad of \( M/e \) meeting \( C - \{e\} \), then \( (M/e, C - \{e\}, f) \)
is a bad triple. This contradicts the minimality of \((M, C, f)\). Therefore, there is a triad of \(M/e\) meeting the circuit \(C - \{e\}\). But any triad of \(M/e\) is also a triad of \(M\), contradicting the assumption that \((M, C, f)\) is bad. Therefore \(M/e\) is not 3-connected. By assumption, \(M\setminus e\) is not 3-connected. Therefore \(e\) is essential. □

Combining (2.3.1) and (2.2.1), it follows that if \(e \in C - \{f\}\), then \(e\) is in a triangle or a triad of \(M\). By assumption, \(C\) does not meet a triad. Hence, we get the following result.

(2.3.2) If \(e \in C - \{f\}\), then \(e\) is in a triangle of \(M\).

(2.3.3) Suppose that \(e \in C - \{f\}\) and that \(T\) is a triangle of \(M\) containing \(e\).

(i) If \(f \not\in T\), then \(T \cap C = \{e\}\).

(ii) If \(T'\) is a triangle of \(M\) distinct from \(T\), then \(|T \cap T'| \leq 1\).

Proof. Part (i) follows by (2.2.2). In part (ii), we may assume that if \(|T \cap T'| > 1\), then \(|T \cap T'| = 2\) and hence, \(M|(T \cup T') \cong U_{2,4}\). Therefore, \(M/e\) contains a parallel class with at least three elements. By (2.2.12), \(e\) is in a triad of \(M\); a contradiction. Hence (ii) follows. □

(2.3.4) If \(\{e, f, a\}\) is a triangle of \(M\) and \(e \in C\), then \(C' = (C - \{f\}) \cup \{a\}\) is a circuit of \(M\) and \((M, C', a)\) is a minimal triple.

Proof. By orthogonality, if \(a\) meets a triad, then \(C\) meets a triad. Therefore \(C' = (C - \{f\}) \cup \{a\}\) does not meet a triad. By circuit elimination, \(C'\) contains a circuit of \(M\) and, by the minimality of \(|C|\), the set \(C'\) is a circuit of \(M\). Therefore, \((M, C', a)\) is a minimal triple. □
(2.3.5) If \( \{e, f, a\} \) is a triangle of \( M \) and \( e \in C \), then \( f \) is in a 4-cocircuit of \( M \).

**Proof.** By Theorem 2.1.8, \( M \setminus f \) is 3-connected and, by (2.3.4), \( (C - \{f\}) \cup \{a\} \) is a circuit of \( M \setminus f \). By (2.2.5), for all \( x \in (C - \{f\}) \cup \{a\} \), the matroid \( (M \setminus f) \setminus x \) is not 3-connected. Since \( (M, (C - \{f\}) \cup \{a\}, a) \) is a minimal triple, \( (C - \{f\}) \cup \{a\} \) must meet a triad \( T \) of \( M \setminus f \). By orthogonality, \( T \cap C \) is non-empty. Hence \( T \) is not a triad of \( M \). Therefore, \( T \cup \{f\} \) is a 4-cocircuit of \( M \). \( \Box \)

(2.3.6) There is an element \( e \in C - \{f\} \) that is not in a triangle with \( f \).

**Proof.** Suppose every element of \( C \) shares a triangle with \( f \). Further, suppose \( |C| \geq 5 \) and \( \{e_1, e_2, e_3, e_4, f\} \subseteq C \). By (2.3.3)(ii), if two distinct triangles meet \( f \), and one of these meets some other element of \( C \), then their intersection is \( \{f\} \). Therefore, we may assume that \( \{e_i, f, a_i\} \) is a triangle for all \( i \) in \( \{1, 2, 3, 4\} \), and that \( e_1, e_2, e_3, f, a_1, a_2, a_3 \), and \( a_4 \) are distinct. By (2.3.5), \( f \) is in a 4-cocircuit \( F \) of \( M \) and, by orthogonality, at least one of \( e_i \) and \( a_i \) also belongs to \( F \) for all \( i \) in \( \{1, 2, 3, 4\} \). This is impossible since \( |F| = 4 \). Therefore \( |C| = 4 \).

We may assume that \( C = \{e_1, e_2, e_3, f\} \) and that \( \{e_i, f, a_i\} \) is a triangle for all \( i \) in \( \{1, 2, 3\} \). By orthogonality, \( |F \cap \{e_1, e_2, e_3\}| \geq 1 \). Suppose \( F \cap \{e_1, e_2, e_3\} = \{e_1\} \).

Since \( \{f, e_2, a_2\} \) and \( \{f, e_3, a_3\} \) are triangles, by orthogonality, \( F = \{f, e_1, a_2, a_3\} \). By (2.3.4), \( \{e_1, e_2, e_3, a_1\} \) is a circuit of \( M \). But \( |F \cap \{e_1, e_2, e_3, a_1\}| = 1 \), contradicting orthogonality. Therefore, \( |F \cap \{e_1, e_2, e_3\}| \geq 2 \).

Suppose \( F \cap \{e_1, e_2, e_3\} = \{e_1, e_2\} \). Since \( \{f, e_3, a_3\} \) is a triangle, by orthogonality, \( F = \{f, e_1, e_2, a_3\} \). As \( r(C) = 3 \) and \( a_i \in cl(C) \) for all \( i \) in \( \{1, 2, 3\} \), it follows that
\( r(\{e_1, e_2, e_3, f, a_1, a_2, a_3\}) = 3 \). Hence \( r(\{f, e_3, a_1, a_2\}) \leq 3 \). So \( \{f, e_3, a_1, a_2\} \) contains a circuit. But \( F \cap \{f, e_3, a_1, a_2\} = \{f\} \). Therefore, by orthogonality, \( \{e_3, a_1, a_2\} \) is a triangle. Let \( \{A, B\} \) be a 2-separation of \( M \setminus e_3 \). Neither \( A \) nor \( B \) spans \( e \) by (2.2.6). Therefore, we may assume that \( f \in A \) and \( a_3 \in B \). Since \( \{e_3, a_1, a_2\} \) is a triangle, (2.2.6) implies that if \( a_2 \in A \), then \( a_1 \in B \). Suppose that \( a_2 \in A \). Since \( f \in A \) and \( \{f, e_2, a_2\} \) is a triangle of \( M \), (2.2.7) implies that we may assume that \( e_2 \in A \). If \( e_1 \in A \), then the existence of \( F \), along with our prior assumptions that \( f \) and \( a_2 \) are in \( A \), and (2.2.7) imply that \( \{A \cup \{a_3\}, B - \{a_3\}\} \) is also a 2-separation of \( M \setminus e_3 \). But \( f \) and \( a_3 \) must belong to different sets of the partition. So assume that \( e_1 \in B \). Then the existence of the triangle \( \{f, e_1, a_1\} \), along with our assumption that \( a_1 \) is in \( B \), and (2.2.7) imply that \( \{A - \{f\}, B \cup \{f\}\} \) is a 2-separation of \( M \setminus e_3 \). This leads to the same contradiction as before. Therefore, \( a_2 \in B \) and \( a_1 \in A \). Since \( f \in A \), the existence of the triangle \( \{f, e_1, a_1\} \) and (2.2.7) imply that we may assume that \( e_1 \in A \). Now we have that \( \{f, e_1, a_1\} \subseteq A \) and \( \{a_2, a_3\} \subseteq B \); and (2.2.6) implies that neither \( \{A \cup \{a_3\}, B - \{a_3\}\} \) nor \( \{A - \{f\}, B \cup \{f\}\} \) is a 2-separation of \( M \setminus e_3 \). If \( e_2 \in A \), then (2.2.7) and the existence of \( F \) imply that \( \{A \cup \{a_3\}, B - \{a_3\}\} \) is also a 2-separation of \( M \setminus e_3 \); a contradiction. But, if \( e_2 \in B \), then (2.2.7) and the existence of the triangle \( \{f, e_2, a_2\} \) imply that \( \{A - \{f\}, B \cup \{f\}\} \) is a 2-separation of \( M \setminus e_3 \); a contradiction. Therefore \( F = C \). Hence, \( C \) is both a 4-circuit and a cocircuit of \( M \).

By (2.3.4) and the preceding argument using \( \{a_1, e_1, e_2, e_3\} \) in the place of \( C \), we deduce that \( \{a_1, e_1, e_2, e_3\} \) is also a 4-cocircuit. But the intersection of this cocr-
cuit with the triangle \( \{f, e_2, a_2\} \) is \( \{e_2\} \); contradicting orthogonality. This completes the proof of (2.3.6). \( \square \)

(2.3.7) Suppose that \( \{e', e\} \subseteq C - \{f\} \) and let \( M' \) be the simplification of \( M/e \) and \( N = M'/e' \). Then \( N \) is not 3-connected.

Proof. Suppose that \( N \) is 3-connected. Let \( \{A, B\} \) be a 2-separation of \( M/e' \) for which \( e \in A \); that is, \( r_M(A) + r_M(B) - r(M) \leq 1 \); and

\[
|A| \geq 2 \text{ and } |B| \geq 2. \tag{2.3}
\]

Since \( M \) is 3-connected, \( r_M(A \cup \{e'\}) + r_M(B) - r(M) \geq 2 \). Therefore,

\[
r_M(A) + r_M(B) - r(M) = 1. \tag{2.4}
\]

By (2.2.7) and (2.2.6), we may assume that \( A \) is closed in \( M \). Let \( A' = A \cap E(M') \) and \( B' = B \cap E(M') \). Hence, \( r_N(A') = r_M(A) - 1 \) and \( r_N(B') = r_M(B \cup \{e\}) - 1 \). Moreover, \( r(N) = r(M) - 1 \). Therefore,

\[
r_N(A') + r_N(B') - r(N) = r_M(A) + r_M(B \cup \{e\}) - r(M) - 1. \tag{2.5}
\]

Since \( A \) is closed in \( M \) and \( e \in A \), if \( A \) meets a non-trivial parallel class of \( M/e \), then \( A \) contains that parallel class. Hence, if \( |A'| = 0 \), then \( A = \{e\} \); and if \( |A'| = 1 \), then either \( A = A' \cup \{e\} \) or \( A \) is a triangle containing \( e \). By (2.3.3)(i), \( e \) and \( e' \) do not belong to a common triangle of \( M \). Therefore, since \( A \) is closed, if \( B \) meets a non-trivial parallel class of \( M/e \), then \( B \) contains that parallel class. Hence, if \( |B'| = 0 \), then \( |B'| = 0 \); and if \( |B'| = 1 \), then either \( B = B' \), or \( B \) spans \( e \) in \( M \).
Suppose that $B$ spans $e$ in $M$. Then (2.4) and (2.5) imply that
\[ r_N(A') + r_N(B') - r(N) = 0. \]
Since $N$ is 3-connected, either $A'$ or $B'$ is empty. But this implies that either $|A| = 1$ or $|B| = 0$, contradicting (2.3).

We may now assume that $B$ does not span $e$. Equations (2.4) and (2.5) imply that
\[ r_N(A') + r_N(B') - r(N) = 1. \]
Since $N$ is 3-connected, either $|A'| \leq 1$ or $|B'| \leq 1$. Since $B$ does not span $e$, we have $B = B'$, contradicting (2.3). Therefore, we may assume that $|A'| = 1$ and $|A| \geq 2$. Hence, $A$ is a triangle of $M$ containing $e$. Since $B = E(M) - (A \cup \{e\})$, equation (2.4) is equivalent to the equation:
\[ r_M(A) + r_M(A \cup \{e\}) - |A \cup \{e\}| = 1. \]
Hence $r_M(A \cup \{e\}) = 3$. By (2.2.10), $A$ is not a triad of $M$, and, since $e'$ is not in a triad of $M$, the set $A \cup \{e\}$ is a cocircuit of $M$. By (2.3.2), $e'$ is in a triangle $T$ of $M$. Hence, by orthogonality, $|T \cap A| \geq 1$. Moreover, (2.3.3)(ii) implies that $|T \cap A| = 1$. But (2.2.13) implies that $e'$ is in a triad; a contradiction. Hence $N$ is not 3-connected. □

(2.3.8) Suppose that $e \in C - \{f\}$ and that $e$ is not in a triangle with $f$. Then there is a cocircuit $D$ of $M$ such that $|D \cap C| = 2$, $e \notin D$, and $(D - C) \cup \{e\}$ is a triangle of $M$. 

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**Proof.** Since $C$ is a circuit of $M$, the set $C - \{e\}$ is a circuit of $M/e$. Let $M'$ be the simplification of $M/e$. By (2.3.3), $C - \{e\}$ is also a circuit of $M'$. If there is no triad of $M'$ meeting $C - \{e\}$, (2.3.7) implies that $(M', C - \{e\}, f)$ is a bad triple. This contradicts the minimality of $(M, C, f)$. Therefore $C - \{e\}$ meets a triad $D'$ of $M'$. Thus, there is a cocircuit $D$ of $M$ such that if $M' = M/e \setminus A$, then $D' = D - A$ and $e \not\in D$. Since $C$ meets no triad of $M$, the set $D'$ is properly contained in $D$. Therefore, $e$ is in a triangle $T$ such that $T \cap D$ is non-empty. Since $e \notin D$, the two other elements, $a$ and $b$, of $T$ must be in $D$ by orthogonality. By (2.3.3)(i) and the assumption that $e$ is not in a triangle with $f$, neither $a$ nor $b$ is in $C$. We may assume that $a \in A$, and, hence $b \in D'$. Therefore, $|D' \cap (C - \{e\})| \leq 2$. By assumption, $D' \cap (C - \{e\})$ is non-empty. Therefore, by orthogonality, $|D' \cap (C - \{e\})| = 2$. Hence, $D' = \{b, x, y\}$ where $\{x, y\} \subseteq C - \{e\}$. Suppose that $T'$ is another triangle meeting $D$ and containing $e$. By orthogonality, $T' \cap D = T' - \{e\}$. Hence $T'$ contains either $x$ or $y$. This contradicts (2.3.3)(i). Therefore $T$ is the only triangle meeting both $D$ and $e$. Hence $D = D' \cup \{a\}$. Therefore, $|D \cap C| = |\{x, y\}| = 2$ and $(D - C) \cup \{e\} = \{e, a, b\} = T$. This completes the proof (2.3.8). □

(2.3.9) Suppose that $D$ is as in (2.3.8). Then $D \cap C$ contains an element different from $f$ that is not in a triangle with $f$.

**Proof.** Suppose that $D \cap C = \{f, y\}$, $D - C = \{a, b\}$, and that $\{f, y, z\}$ is a triangle of $M$. Note that $z$ may or may not belong to $\{a, b\}$. Let $\{A, B\}$ be a 2-separation of $M \setminus e$. As $\{a, b, e\}$ is a triangle of $M$, (2.2.6) implies that $\{a, b\}$ cannot be contained in the same set of the partition $\{A, B\}$. By (2.2.7), we may assume that the triangle
\{f, y, z\} is contained in either \(A\) or \(B\). Hence, we may assume that \(\{f, y, z, a\} \subseteq A\) and \(b\) must be in \(B\). By the existence of \(D\), the element \(b\) is in \(cl_M(A)\). Hence, \(b\) is in the closure of \(A\) with respect to the matroid \((M \setminus e)^*\). Therefore, by (2.2.7), we may assume that \(b \in A\); a contradiction. Therefore, \(D\) satisfies (2.3.9) if \(f \in D \cap C\).

Suppose that \(D \cap C = \{x, y\}, D - C = \{a, b\}\), and that \(\{f, x, x'\}\) and \(\{f, y, y'\}\) are triangles of \(M\). By orthogonality, \(|D \cap \{f, x, x'\}|\) and \(|D \cap \{f, y, y'\}|\) are greater than one. Therefore, we may assume that \(x' = a\) and \(y' = b\). Hence, \(\{f, x, y\}\) and \(\{f, y, b\}\) are triangles of \(M\). By (2.3.8), \(\{e, a, b\}\) is also a triangle of \(M\). Since \(\{x, y, e\} \subseteq C\), the set \(\{x, y, e\}\) is independent in \(M\). Hence, \(M[X] \cong \mathcal{W}^3\), where \(X = \{x, y, a, b, e, f\}\) and \(\mathcal{W}^3\) denotes the rank-three whirl. Since \(\{x, y, e, f\}\) is a dependent subset of \(C\), it follows that \(C = \{x, y, e, f\}\). By (2.3.4), \((M, C', a)\) is also a minimal triple, where \(C' = \{x, y, e\}\). Applying (2.3.6) to \(C'\) implies that \(y\) is not in a triangle with \(a\). Let \(D'\) be a cocircuit satisfying the conditions of (2.3.8) relative to the element \(y\) and the circuit \(C'\). Applying the result of the first paragraph, relative to the triple \((M, C', a)\), the cocircuit \(D'\), and the element \(y \in C'\), we deduce that either \(D'\) satisfies (2.3.9) or \(a \notin D' \cap C'\). The former case is impossible since every element of \(C' - \{y\}\) is in a triangle with \(a\). In the latter case, (2.3.8) implies that \(y \notin D'\) either. Therefore \(D' \cap C' = \{x, e\}\). Since \(\{f, x, y\}\) and \(\{f, y, b\}\) are triangles, orthogonality implies that \(D' = \{x, e, f, b\}\). Since \(\{x, y, a, b\}\) is a cocircuit, \(r^*(\{x, y, b, e, f\}) = r^*(X)\). But \(r^*(\{x, y, b, e, f\}) = r^*(D' \cup \{y\}) \leq 4\). Therefore \(r^*(X) \leq 4\). Hence \(r(X) + r^*(X) - |X| \leq 1\). Since \(M\) is 3-connected, \(|E - X| \leq 1\). Since every element of \(\mathcal{W}^3\) is in a triad, \(\mathcal{W}^3\) is not a counterexample.
Suppose $M$ is a 3-connected single-element extension of $W^3$ by an element $z$. Since neither $x$, $y$, nor $e$ can be in a triad of $M$, the element $z$ cannot be added to one of the three-point lines of $W^3$; nor can $z$ be added freely to $W^3$ since $M \setminus e$ cannot be 3-connected. Therefore, $z$ must be added to one of the two-point lines of $W^3$. But any such choice results in a matroid for which the deletion of at least one of $x$, $y$, or $e$ is still 3-connected. Therefore, $M$ cannot be an extension of $W^3$ and so the proof of (2.3.9) is complete. □

If $e \in C \setminus \{f\}$ and $e$ is not in a triangle with $f$, then let $D_e$ be a cocircuit of $M$ which satisfies the properties of (2.3.8) and let $T_e$ be the triangle $(D_e - C) \cup \{e\}$.

(2.3.10) Suppose that $e \in C \setminus \{f\}$, and $e$ is in no triangle with $f$. Then $T_e$ is the unique triangle containing $e$.

Proof. By (2.3.9), there is an element $c$ of $M$ that is in $D_e \cap C$ and not in a triangle with $f$. By (2.3.3)(i), $c$ is the only element of $T_e$ in $C$. Suppose that $T_e = \{a, b, e\}$. By (2.3.8), $D_e - C = \{a, b\}$. By orthogonality, $\{a, b\} \cap T_e$ is non-empty. But by (2.3.3)(ii), $|T_e \cap T_c| \leq 1$, hence, we may assume that $b \in T_e$ and that $a \notin T_e$. Therefore, $T_e = \{b, c, d\}$ for some element $d$ belonging to neither $T_e$ nor $C$. Observe that $\{b, d\} \subseteq D_e$. By (2.3.8), the other two elements of $D_e$ must come from $C$. By orthogonality, one of those other elements must be $e$. Hence $\{b, d, e\} \subseteq D_e$. By the submodularity of the rank function, $r(\{a, b, c, d, e\}) = r(T_e \cup T_c) \leq 3$. If $\{a, e, d, c\}$ contains a triangle, then that triangle would contradict (2.3.3)(ii) with respect to either $T_e$ or $T_c$. Hence, $\{a, e, d, c\}$ has rank at least three. Therefore $r(T_e \cup T_c) = 3$. 

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Suppose that there is another triangle $T'_e$ which contains $e$. By (2.3.3)(ii), neither $a$ nor $b$ is in $T'_e$. By (2.3.3)(i), the other two elements of $T'_e$ are not in $C$. By orthogonality with respect to $D_e$, the intersection of $\{b, d\}$ and $T'_e$ is non-empty. Hence $d \in T'_e$. Let $g$ be the remaining element of $T'_e$; that is, $T'_e = \{d, e, g\}$. Since $\{d, e\} \subseteq T_e \cup T_c$, the element $g \in cl(T_e \cup T_c)$. Hence $r(T_e \cup T_c \cup T'_e) = 3$. Therefore, $\{g, d, e\}$ contains a circuit $C'$ of $M$. As $g, c, a \notin D_e$ and $d \in D_c$, by orthogonality, $C' = \{g, c, a\}$. Hence, $\{a, b, e\}, \{b, c, d\}, \{d, e, g\}$, and now $\{a, c, g\}$ are all triangles of the rank-three set $U = \{a, b, c, d, e, g\}$. Since the existence of any other triangle would violate (2.3.3)(ii), these are the only triangles of $U$. Hence $M|U \cong M(K_4)$. Since neither $M \setminus e$ nor $M \setminus c$ is 3-connected, (2.2.14) implies that $U = E(M)$. Since $f \in E(M) - U$, we have a contradiction. Hence, the triangle $T'_e$ does not exist. □

(2.3.11) Suppose that $\{e_1, e_2, e_3\} \subseteq C - \{f\}$, that no triangle meeting $\{e_1, e_2, e_3\}$ contains $f$, and that $D_{e_1} \cap C = \{e_1, e_2, e_3\}$. Then $T_{e_1} \cap T_{e_2} \cap T_{e_3} = \emptyset$.

Proof. Suppose that $T_{e_1} - e_1 = \{a_1, a_2\}$ and $T_{e_2} - e_2 = \{a_2, a_3\}$. If the result does not hold, then $T_{e_1} - e_3 = \{a_2, a_4\}$. By orthogonality and the characterization given in (2.3.8), $D_{e_1} = \{a_1, a_2, e_2, e_3\}, D_{e_2} = \{a_2, a_3, e_1, e_3\}$, and $D_{e_3} = \{a_2, a_4, e_1, e_2\}$.

Put $U = \{a_1, a_2, a_3, a_4, e_1, e_2, e_3\}$. The existence of the cocircuits $D_{e_1}, D_{e_2},$ and $D_{e_3}$ implies that $\{e_1, e_2, e_3, a_2\}$ spans $U$ in $M^*$. Similarly, the existence of the triangles $T_{e_1}, T_{e_2},$ and $T_{e_3}$ implies that $\{a_1, a_2, a_3, a_4\}$ spans $U$ in $M$. Hence, it follows that $r(U) + r^*(U) - |U| \leq 1$. As $M$ is 3-connected and $|C| > 3$, it follows that $|E(M) - U| = 1$. Thus $E(M) - U = \{f\}$ and $r(\{a_1, a_2, a_3, a_4\}) = 4$. Therefore, $r(\{a_2, a_3, a_4, e_2, e_3\}) = 3$ and $r(\{a_2, a_3, a_4, e_2, e_3\}) + r(\{a_1, e_1\}) - r(M) = 5 - r(M)$. 

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Since $M \setminus f$ is 3-connected, $r(M) \leq 3$. Therefore the rank of $M$ must be three. But if $r(M) = 3$, then $\{e_3, a_1, a_2, a_3\}$ contains a circuit of $M$. By (2.3.3)(ii), $\{a_1, a_2, a_3\}$ is not a triangle and, by (2.3.10), $e_3$ is no other triangle of $M$. Therefore, $\{e_3, a_1, a_2, a_3\}$ is a circuit of $M$. But this circuit intersects the cocircuit $D_{e_3}$ in only one element, $a_2$; a contradiction. Hence the result holds. □

The remainder of the proof will use a sequence $(e_0, e_1, \ldots, e_n, e_{n+1})$ of distinct elements of $C$ and a set $\{a_1, a_2, \ldots, a_{n+1}\}$ disjoint from $C$, where $n \geq 1$. We assume that these elements satisfy the following conditions.

(S1) For all $i$ in $\{1, 2, \ldots, n+1\}$, the element $e_i \in C \setminus \{f\}$ and $e_i$ is not in a triangle with $f$.

(S2) All $a_i$ are distinct and the $a_i$ are defined by:

for all $i$ in $\{1, 2, \ldots, n-1\}$, the singleton $\{a_{i+1}\} = T_{e_i} \cap T_{e_{i+1}}$;

$\{a_1\} = T_{e_1} \setminus \{a_2, e_1\}$ and $\{a_{n+1}\} = T_{e_n} \setminus \{a_n, e_n\}$; and

for $n = 1$, the set $T_{e_1} \setminus \{e_1\} = \{a_1, a_2\}$.

(S3) For all $i$ in $\{1, 2, \ldots, n\}$, the set $\{e_{i-1}, e_{i+1}\} = D_{e_i} \cap C$.

(2.3.12) Such a sequence can be defined for $n = 1$.

Proof. By (2.3.6), there is an element $e_1 \in C \setminus \{f\}$ that is not in a triangle with $f$. Therefore, by (2.3.8) and (2.3.9), there is a cocircuit $D_{e_1} = \{a_1, a_2, e_0, e_2\}$ with triangle $T_{e_1} = \{e_1, a_1, a_2\}$. Moreover, $D_{e_1} \cap C = \{e_0, e_2\}$ where it may be assumed
that \( e_2 \) is not in a triangle with \( f \). The sequence \( e_0, e_1, e_2 \) and elements \( a_1 \) and \( a_2 \) satisfy (S1), (S2), and (S3) for \( n = 1 \). □

(2.3.13) \textit{The element} \( a_{n+1} \in T_{e_{n+1}} \) \textit{(or, if} \( n = 1 \), \textit{this may be assumed)} \textit{and, if} \( a_{n+2} \in T_{e_{n+1}} - \{e_{n+1}, a_{n+1}\} \), \textit{then} \( a_{n+2} \notin \{a_1, a_2, \ldots, a_{n+1}\} \).

\textbf{Proof.} For \( n = 1 \), consider the sequence defined in (2.3.12). By (2.3.3), \( e_1 \notin T_{e_2} \).

Hence \( T_{e_2} \cap D_{e_1} \subseteq \{e_2, a_1, a_2\} \). By orthogonality, \( |T_{e_2} \cap \{a_1, a_2\}| \geq 1 \). But by (2.3.3), \( |T_{e_2} \cap T_{e_1}| \leq 1 \). Therefore, \( |T_{e_2} \cap \{a_1, a_2\}| = 1 \). Hence, we may assume that \( a_2 \in T_{e_2} \) and \( a_1 \notin T_{e_2} \). Therefore, if \( a_3 \in T_{e_3} - \{e_2, a_2\} \), then \( a_3 \notin \{a_1, a_2\} \). Suppose that \( n \geq 2 \). By (S1) and (S3), assertion (2.3.11) holds with \( \{e_1, e_2, e_3\} \) replaced by \( \{e_{n-1}, e_n, e_{n+1}\} \). Also, by (S2), \( T_{e_n} = \{e_{n-1}, a_{n-1}, a_n\} \) and \( T_e = \{e_n, a_n, a_{n+1}\} \).

Since \( e_{n+1} \in D_{e_n} \cap T_{e_{n+1}} \) and \( T_{e_{n+1}} \cap C = e_{n+1} \), by orthogonality, either \( a_n \) or \( a_{n+1} \) belongs to \( T_{e_{n+1}} \). By (2.3.11) we have that \( a_{n+1} \in T_{e_{n+1}} \). If \( a_{n+2} \in T_{e_{n+1}} - \{e_{n+1}, a_{n+1}\} \), clearly \( a_{n+2} \neq a_{n+1} \), and (2.3.3) implies that \( a_{n+2} \neq a_n \). Moreover, \( a_i \in D_{e_i} \) for all \( i \) in \( \{1, 2, \ldots, n - 1\} \). Since the elements \( e_1, e_2, \ldots, e_{n+1}, a_1, a_2, \ldots, a_{n+1} \) are distinct and \( D_{e_i} = \{e_{i-1}, e_{i+1}, a_i, a_{i+1}\} \) for all \( i \) in \( \{1, 2, \ldots, n\} \), neither \( e_{n+1} \) nor \( a_{n+1} \) is in \( D_{e_1} \cup D_{e_2} \cup \cdots \cup D_{e_{n-1}} \). Hence, by orthogonality, \( a_{n+2} \notin \{a_1, a_2, \ldots, a_{n-1}\} \).

Thus, \( a_{n+2} \notin \{a_1, a_2, \ldots, a_{n+1}\} \). □

From now on suppose that \( \{a_{n+2}\} = T_{e_{n+1}} - \{e_{n+1}, a_{n+1}\} \).

(2.3.14) \textit{The element} \( e_n \in D_{e_{n+1}} \). \textit{If} \( e_{n+2} \) \textit{is the element of} \( D_{e_{n+1}} \cap C \) \textit{different from} \( e_n \), \textit{then} \( e_{n+2} \notin \{e_1, e_2, \ldots, e_{n+1}\} \).

\textbf{Proof.} By (2.3.13), \( T_{e_{n+1}} = \{e_{n+1}, a_{n+1}, a_{n+2}\} \). Therefore, (2.3.8) and (2.3.10) imply that \( a_{n+1} \) and \( a_{n+2} \in D_{e_{n+1}} \). Since \( T_{e_n} = \{e_n, a_n, a_{n+1}\} \) and (2.3.8) says that
$|D_{e_{n+1}} \cap C| = 2$, orthogonality implies that $e_n \in D_{e_{n+1}}$; that is, $\{e_n, a_{n+1}, a_{n+2}\} \subseteq D_{e_{n+1}}$. Let $e_{n+2}$ be the element of $D_{e_{n+1}} \cap C$ different from $e_n$. Since $e_{n+1} \notin D_{e_{n+1}}$, the element $e_{n+2}$ equals neither $e_n$ nor $e_{n+1}$. By (2.3.13), $e_1, e_2, \ldots, e_{n+1}, a_1, a_2, \ldots, a_{n+2}$ are distinct, and by (S2), $T_{e_i} = \{e_i, a_i, a_{i+1}\}$ for all $i$ in $\{1, 2, \ldots, n\}$. Hence, the intersection $\{e_n, a_{n+1}, a_{n+2}\} \cap T_{e_i}$ is empty for all $i$ in $\{1, 2, \ldots, n-1\}$. By orthogonality, $e_{n+2} \notin \{e_1, e_2, \ldots, e_{n-1}\}$. Hence, $e_{n+2} \notin \{e_1, e_2, \ldots, e_{n+1}\}$. □

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.1}
\caption{(a) the triangles $T_{e_i} = \{a_i, e_i, a_{i+1}\}$, (b) the cocircuits $D_{e_i} = \{e_{i-1}, a_i, a_{i+1}, e_{i+1}\}$}
\end{figure}

From now on, suppose that $e_{n+2}$ is the element of $D_{e_{n+1}} \cap C$ different from $e_n$. Parts (a) and (b) of Figure 2.1 illustrate the relative positions of the triangles $T_{e_i}$ and the cocircuits $D_{e_i}$, respectively. Moreover, we shall assume that the sequence is maximal and let $\mathcal{F} = f \cup \{x \in C : x$ is in a triangle with $f\}$. Observe that each such sequence satisfies one of the following:
(i) $e_{n+2} \neq e_0$ and either $e_{n+2}$ or $e_0$ is not in $\mathcal{F}$;

(ii) $\{e_0, e_{n+2}\} \subseteq \mathcal{F}$ and either $f = e_{n+2} \neq e_0$ or $f = e_0 \neq e_{n+2}$;

(iii) $e_{n+2} = e_0 \notin \mathcal{F}$;

(iv) $e_{n+2} = e_0 \in \mathcal{F}$;

(v) $\{e_{n+2}, e_0\} \subseteq \mathcal{F}$, and $e_{n+2}, e_0$, and $f$ are distinct.

(2.3.15) In case (i), the sequence is not maximal.

Proof. Suppose that case (i) holds and $e_{n+2} \notin \mathcal{F}$. If we replace $n$ by $n + 1$, then (S1) follows by assumption, (2.3.13) implies (S2) and, (2.3.14) implies (S3). Hence, the original sequence can be lengthened. If $e_0 \notin \mathcal{F}$, then consider the sequence $e'_0, e'_1, \ldots, e'_{n+1}$, where $e'_i = e_{n+2-i}$. Making the appropriate changes in the $a_i$'s, this sequence satisfies (S1)–(S3) and $e_0$ satisfies the conditions on $e_{n+2}$ given by (2.3.14). Hence, as before, this sequence can be lengthened. □

(2.3.16) Case (ii) is impossible.

Proof. If $f = e_0 \neq e_{n+2}$, the sequence can be reordered as in the proof of (2.3.15) so that $f = e'_{n+2} \neq e'_0$. Therefore, we may assume that $f = e_{n+2} \neq e_0$.

Since $\{a_1, a_2, e_0, e_2\}$ is a cocircuit and the element $e_0 \in \mathcal{F}$, by orthogonality, either $\{e_{n+2}, e_0, a_2\}$, $\{e_{n+2}, e_0, a_1\}$, or $\{e_{n+2}, e_0, e_2\}$ is a triangle of $M$. The last set is ruled out by (2.3.3)(i).

Suppose that $\{e_{n+2}, e_0, a_2\}$ is a triangle. Consider the cocircuit $D_{e_{n+1}} = \{a_{n+1}, a_{n+2}, e_n, e_{n+2}\}$ of (2.3.14). By orthogonality and since $n$ is at least one,
\( a_2 = a_{n+1} \). Hence \( n = 1 \) and \( e_{n+2} = e_3 \). Let \( \{A, B\} \) be a 2-separation of \( M \backslash e_1 \). By (2.2.6), neither \( A \) nor \( B \) can span \( e_1 \) in \( M \). Therefore, \( a_1 \) and \( a_2 \) cannot be in the same set of the separation. Suppose that \( a_1 \in B \) and \( a_2 \in A \). Since \( \{a_1, a_2, e_0, e_2\} \) is a cocircuit, (2.2.7) implies that \( e_2 \) and \( e_0 \) must be in different sets of the partition. Suppose that \( e_2 \in A \) and \( e_0 \in B \). If \( e_3 \in B \), then, since \( \{e_3, e_0, a_2\} \) is a triangle and \( e_0 \in B \), (2.2.7) implies that we may assume that \( a_2 \in B \); a contradiction. But, if \( e_3 \in A \) then, by (2.2.7), we may assume that \( e_0 \in A \); also a contradiction. A similar argument resulting in a similar contradiction occurs when assuming that \( e_0 \in A \).

Suppose that \( \{e_{n+2}, e_0, a_1\} \) is a triangle. Since \( \{a_{n+1}, a_{n+2}, e_n, e_{n+2}\} \) is a cocircuit, by orthogonality, \( n = 0 \); a contradiction. □

(2.3.17) Suppose that case (iii) holds. Then \( T_{e_0} = \{e_0, a_1, a_{n+2}\} \), and there is a circuit \( C' \) of \( M \) such that \( \{a_1, a_2, \ldots, a_{n+2}\} \subseteq C' \).

Proof. By (2.3.11) and (2.3.3)(i), \( a_{n+2} \in T_{e_{n+2}} = T_{e_0} \) and \( a_1 \in T_{e_0} \). By (2.3.13), \( a_1 \neq a_{n+2} \). By orthogonality, the intersection of the triangle \( T_{e_{n+2}} \) with the cocircuit \( \{e_n, e_{n+2}, a_{n+1}, a_{n+2}\} \) contains \( e_{n+2} \) and at least one other element. By (2.3.3)(i), that other element cannot be \( e_n \). Since \( a_{n+1} \in T_{e_n} \cap T_{e_{n+1}} \), (2.3.11) implies that \( a_{n+2} \in T_{e_{n+2}} \). Similarly, \( a_1 \in T_{e_0} \). Hence \( T_{e_0} = T_{e_{n+2}} = \{e_0, a_1, a_{n+2}\} \).

As \( a_2 \) and \( e_2 \) are not in series in \( M \backslash e_1 \), there is a circuit \( C'' \) of \( M \) such that \( a_2 \in C'' \) and \( e_1, e_2 \not\in C'' \). Let \( C^{(i-1)} \) be a circuit such that \( e_i \in C^{(i-1)} \) and \( e_j \not\in C^{(i-1)} \) for all \( j \leq i-1 \leq n+1 \). Note that \( C'' \) is such a circuit for some \( i \). By strong circuit elimination, there is a circuit \( C^{(i)} \subseteq (C^{(i-1)} \cup T_{e_i}) - \{e_i\} \) such that \( a_2 \in C^{(i)} \). Hence, there is a circuit \( C^{(i)} \) containing \( a_2 \) for which \( e_j \not\in C^{(i)} \) for all \( j \leq i \). By induction,
there is a circuit \( C^{(\ell+2)} \) such that \( e_j \not\in C^{(\ell+2)} \) for all \( j \leq \ell + 2 \). Let \( C' = C^{(\ell+2)} \). By orthogonality and the existence of the cocircuits \( D_{e_i} \), the set \( \{a_1, a_2, \ldots, a_{\ell+2}\} \subseteq C' \).

\( \square \)

(2.3.18) Case (iii) is impossible.

Proof. By (2.3.17), there is a circuit \( C' \) of \( M \) such that \( \{a_1, a_2, \ldots, a_{\ell+2}\} \subseteq C' \). Let \( \{A, B\} \) be a 2-separation for \( M \setminus e_1 \). We may assume that \( a_1 \in A \). By (2.2.6), \( a_2 \in B \). Recall that \( T_{e_0} = T_{e_{\ell+2}} = \{e_0, a_1, a_{\ell+2}\} \). Suppose that \( T_{e_i} \) is a subset of either \( A \) or \( B \) for each \( i \neq 1 \). Then \( T_{e_{\ell+2}} \subseteq A \), and \( T_{e_2} \subseteq B \). Also, for some \( i \in \{2, 3, \ldots, \ell+1\} \), \( T_{e_i} \subseteq B \) and \( T_{e_{i+1}} \subseteq A \). By combining (S2) along with the results (2.3.13) and (2.3.17), we have that \( a_{i+1} \in T_{e_i} \cap T_{e_{i+1}} \) for all \( i \in \{2, 3, \ldots, \ell+1\} \); a contradiction.

Therefore, for some \( i \neq 1 \), the set \( T_{e_i} \) is contained in neither \( A \) nor \( B \). If \( |T_{e_i} \cap A| \geq 2 \), by (2.2.7), we may assume that \( T_{e_i} \subseteq A \). Therefore, we may assume that, for each \( i \neq 1 \), either \( T_{e_i} \subseteq A \) or \( |T_{e_i} \cap B| \geq 2 \), with the latter case holding for at least one \( i \neq 1 \). If \( T_{e_i} - \{e_i\} \subseteq B \), by (2.2.7), we may assume that \( e_i \in B \). Hence, for some \( i \neq 1 \), we have \( |T_{e_i} \cap A| = 1 \) and \( e_i \in B \). Therefore, \( T_{e_i} \cap A \subseteq \{a_i, a_{i+1}\} \subseteq C' \). Hence \( T_{e_i} \cap A \subseteq C' \cap A \). By (2.2.11), it follows that \( T_{e_i} \cap A = C' \cap A \). Since \( a_1 \in A \cap C' \) and \( i \neq 1 \), it follows that \( i = 0 \). Since \( T_{e_i} \cap A = \{a_1\} \), we may assume that \( a_1 \in B \); a contradiction. \( \square \)

The next three observations apply to case (v).

(2.3.19) Suppose that case (v) holds. Then \( \{f, e_0, a_1\} \) and \( \{f, e_{n+2}, a_{n+2}\} \) are triangles.
Proof. By assumption, \{f, e_0, x\} and \{f, e_{n+2}, y\} are triangles for some \(x\) and \(y\).
Consider the cocircuits \(D_{e_1} = \{e_0, a_1, a_2, e_2\}\) and \(D_{e_{n+1}} = \{e_n, a_{n+1}, a_{n+2}, e_{n+2}\}\).
Since neither \(x\) nor \(y\) is in \(C\), it follows by orthogonality that \(x \in \{a_1, a_2\}\) and \(y \in \{a_{n+1}, a_{n+2}\}\). If \(x = a_2\), then \(\{f, e_0, x\} \cap D_{e_2} = \{x\}\); a contradiction. Therefore, \(\{f, e_0, a_1\}\) is a triangle. Similarly, if \(y = a_{n+1}\), then \(\{f, e_{n+2}, y\} \cap D_{e_n} = \{y\}\); a contradiction. Therefore, \(\{f, e_{n+2}, a_{n+2}\}\) is also a triangle. □

(2.3.20) Suppose that case (v) holds. Then, either \(\{f, e_{n+2}, a_{n+2}\}\) is the only triangle meeting \(e_{n+2}\), or \(n = 1\) and \(\{e_0, e_3, a_2\}\) is also a triangle of \(M\).

Proof. Suppose there is another triangle \(\{e_{n+2}, x, y\}\) meeting \(e_{n+2}\). Because of the existence of the cocircuit \(D_{e_{n+1}}\), orthogonality implies that \(\{x, y\} \cap \{e_n, a_{n+1}, a_{n+2}\}\) is non-empty. Since \(\{f, e_{n+2}, a_{n+2}\}\) is a triangle, the result (2.3.3) implies that \(\{x, y\} \cap \{e_n, a_{n+1}, a_{n+2}\} = \{a_{n+1}\}\). We may assume that \(x = a_{n+1}\). If \(y \in D_{e_n}\), since \(x = a_{n+1}\) and \(\{e_{n+1}, a_{n+1}, a_{n+2}\}\) and \(\{e_n, a_n, a_{n+1}\}\) are triangles, (2.3.3) implies that \(y = e_{n-1}\). By (2.3.3)(ii), this is possible only if \(e_{n-1} \in \mathcal{F}\). By (S1), this is only possible if \(n = 1\) and \(y = e_0\). Hence \(\{e_0, e_3, a_2\}\) is a triangle of \(M\). □

(2.3.21) Suppose that case (v) holds. If \(\{f, e_{n+2}, a_{n+2}\}\) is the only triangle meeting \(e_{n+2}\), then there is a 4-cocircuit \(F\) such that \(\{a_{n+2}, f\} \subseteq F\) and \(e_{n+2} \not\in F\).

Proof. Let \(M'\) be the simplification of \(M/e_{n+2}\). Then we may assume that \(f \in M'\). By (2.3.3), \(\{f, e_{n+2}, a_{n+2}\}\) is the only triangle containing \(e_{n+2}\) that meets \(C\) in more than one element. Therefore, \(C - \{e_{n+2}\}\) is a circuit of \(M'\). By (2.3.7) and the minimality of \(|C|\), the circuit \(C - \{e_{n+2}\}\) meets a triad \(F'\) of \(M'\). Therefore, there
is a cocircuit $F$ of $M$ such that, if $M' = M / e_{n+2} \setminus A$, then $F' = F - A$. Since $F$ properly contains $F'$, the element $e_{n+2}$ is in a triangle $T$ such that $T \cap F$ is non-empty. By assumption, $T = \{f, e_{n+2}, a_{n+2}\}$ and $f \in F$. Since $T$ is the only triangle meeting $e_{n+2}$, we deduce that $A = \{a_{n+2}\}$. Since $F'$ contains only three elements the result follows. □

From now on we may assume that either case (iv) or (v) holds. Let

$$A_0 = \{a_2, a_3, \ldots, a_{n+2}, e_2, \ldots, e_{n+1}\}.$$  

\textbf{(2.3.22)} Suppose that $C'$ is a circuit of $M \setminus a_1$ and $C' \cap A_0 \neq \emptyset$. Then $C' \cap A_0 \subseteq \{a_2, a_3, \ldots, a_{n+2}\}$.

\textbf{Proof.} Suppose that $C' \cap A_0 \subseteq \{a_2, a_3, \ldots, a_{n+2}\}$. By assumption, this intersection is non-empty and $a_1 \notin C'$. Therefore, there is a $k \in \{1, 2, \ldots, n + 1\}$ such that $a_k \notin C'$ and $a_{k+1} \in C'$. Hence, the cocircuit $D_{e_k} = \{a_k, a_{k+1}, e_{k-1}, e_{k+1}\}$ intersects $C'$ at $a_{k+1}$; a contradiction. □

\textbf{(2.3.23)} Suppose that $L_{i,j} = \bigcup_{k=i}^{j} T_{e_k}$, for $2 \leq i < j \leq n + 1$. Then $M|L_{i,j} = P_{a_{i+1}}(L_{i,l}, L_{i,j})$ for $i \leq l < j$.

\textbf{Proof.} Suppose that $j - i = 1$. Then we need to show that $M|(T_{a_{i}} \cup T_{e_{i+1}}) = P_{a_{i+1}}(T_{a_{i}}, T_{e_{i+1}})$. By (2.3.3)(ii), $T_{a_{i}}$ and $T_{e_{i+1}}$ are the only triangles in $L_{i,j}$. Therefore $r(L_{i,j}) = 3$, and hence $r(L_{i,j}/a_{i+1}) = 2$. Since $T_{a_{i}}$ and $T_{e_{i+1}}$ are triangles containing $a_{i+1}$, if $R = T_{a_{i}} - \{a_{i+1}\}$ and $S = T_{e_{i+1}} - \{a_{i+1}\}$, then $R$ and $S$ are 2-circuits of $L_{i,j}/a_{i+1}$. Since $r(L_{i,j}/a_{i+1}) = 2$, it follows that $L_{i,j}/a_{i+1} = M|R \oplus M|S$. Hence, $L_{i,j} = M|(T_{a_{i}} \cup T_{e_{i+1}}) = P_{a_{i+1}}(T_{a_{i}}, T_{e_{i+1}})$.  

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Now suppose that $j - i > 1$. In general, if $x$ belongs to $E(M_2)$, then

$$P_x(P_y(M_1, M_2), M_3) = P_y(M_1, P_z(M_2, M_3)).$$

Therefore, if we assume that the result holds for $L_{i,j-1}$, it is sufficient to show that $L_{i,j} = P_{a_j}(L_{i,j-1}, T_{e_j}).$

Since $L_{i,j-1}$ is the parallel connection of $j - i$ triangles, $r(L_{i,j-1}) = j - i + 1$. Therefore, $r(L_{i,j-1}/a_j) = j - i$. Clearly, $r(T_{e_j}/a_j) = 1$. Hence, it is sufficient to show that $r(L_{i,j}/a_j) = j - i + 1$. This is equivalent to showing that $r(L_{i,j}) = j - i + 2$. By submodularity, $r(L_{i,j}) \leq j - i + 2$. Suppose that $r(L_{i,j}) \leq j - i + 1$. Consider the subset $\{a_i, a_{i+1}, \ldots, a_{j+1}\}$ of $L_{i,j}$. Since $|\{a_i, a_{i+1}, \ldots, a_{j+1}\}| = j - i + 2$, this set must contain a circuit. This contradicts (2.3.22). Hence, $r(L_{i,j}) = j - i + 2$, which completes the proof. □

(2.3.24) There is a 2-separation $\{A, B\}$ of $M \setminus e_1$ such that $\{e_0, e_{n+2}\} \subseteq B$ and $A_0 \subseteq A$.

**Proof.** Let $\{A, B\}$ be a 2-separation of $M \setminus e_1$ such that $a_2 \in A$ and $A$ is closed in $M$ and $M^*$. Since $e_1$ cannot be in the closure of either $B$ or $A$ in $M$, it follows that $a_1 \in B$. Since $B$ cannot span $a_2$, at least one other element of $T_{e_2} = \{e_2, a_2, a_3\}$ must be in $A$. Therefore, $T_{e_2} \subseteq A$. Hence, if $n = 1$, then $A_0 \subseteq A$.

Suppose $n > 1$ and $T_{e_2} = \{e_3, a_3, a_4\} \subseteq A$. Then, $\{a_2, a_3, a_4, e_2, e_3\} \subseteq A$. Further, suppose that $\{a_2, \ldots, a_i, e_2, \ldots, e_{i-1}\} \subseteq A$. Then $D_{e_{i-1}} \cap A \subseteq \{a_{i-1}, a_i, e_{i-2}\}$. Since $A$ is closed in $M^*$, the remaining element $e_i$ of $D_{e_{i-1}}$ must also be in $A$. Therefore, $T_{e_i} \cap A \subseteq \{a_i, e_i\}$. Since $A$ is closed in $M$, the remaining element $a_{i+1}$
of \( T_e \) must also be in \( A \). By induction, \( A_0 \subseteq A \). Therefore, we may assume that \( \{ a_4, e_3 \} \subseteq B \). Hence, \( a_4 \in B \) and \( \{ e_2, a_3 \} \subseteq A \). Since \( A \) is closed in \( M^* \), the remaining element \( e_4 \) of the cocircuit \( D_{e_4} \) must belong to \( B \). Therefore, \( \{ a_4, e_3, e_4 \} \subseteq B \).

Since \( \{ a_3, a_4, e_3 \} \) is a circuit of \( M \), (2.2.7) implies that \( B \cup \{ a_3 \}, A - \{ a_3 \} \) is a 2-separation of \( M \setminus e_1 \). Also, since \( \{ e_2, e_4, a_3, a_4 \} \) is a cocircuit of \( M \), (2.2.7) implies that \( B \cup \{ a_3, e_2 \}, A - \{ a_3, e_2 \} \) is a 2-separation of \( M \setminus e_1 \). But then the set \( B \cup \{ a_3, e_2 \} \) spans \( a_2 \), and hence it spans \( e_1 \), contradicting (2.2.6). Hence, \( A_0 \subseteq A \).

Recall that \( A \) is closed in \( M^* \). Therefore, \( e_0 \in B \), otherwise, \( a_1 \in A \) by the existence of \( D_{e_1} \). In case (iv), the result is proved because \( e_0 = e_{n+2} \). Suppose that case (v) holds; that is, \( \{ f, e_0, a_1 \} \) and \( \{ f, e_{n+2}, a_{n+2} \} \) are triangles. Suppose that \( e_{n+2} \in A \). By the existence of the triangle \( \{ f, e_{n+2}, a_{n+2} \} \), and the fact that \( A \) is closed in \( M \), it follows that \( f \in A \). By (2.3.20), \( \{ f, e_{n+2}, a_{n+2} \} \) is the only triangle meeting \( e_{n+2} \). By (2.3.21), there is a 4-cocircuit \( F \) of \( M \) such that \( \{ a_{n+2}, f \} \subseteq F \) and \( e_{n+2} \not\in F \). By the existence of the triangle \( \{ f, e_0, a_1 \} \) and orthogonality, \( |F \cap \{ e_0, a_1 \}| \geq 1 \). Suppose that \( \{ e_0, a_{n+2}, f \} \subseteq F \). By the existence of \( T_{e_{n+1}} \), either \( a_{n+1} \) or \( e_{n+1} \) also belongs to \( F \). In either case, \( e_0 \) is in the closure of \( A \) in \( M^* \); a contradiction. Suppose that \( \{ a_1, a_{n+2}, f \} \subseteq F \). The existence of \( T_{e_1} \) and \( T_{e_{n+1}} \) implies that \( n \) must equal 1 and the remaining element of \( F \) must be \( a_2 \). But, in this case, \( F \cap C = \{ f \} \); a contradiction. Therefore, we may assume that \( e_{n+2} \) belongs to \( B \). □

(2.3.25) \( C_{ij} = \{ a_i, e_i, e_{i+1}, \ldots, e_{j-1}, a_j \} \) is a circuit of \( M|A_0 \) for \( 2 \leq i < j \leq n+1 \).
Proof. This follows from (2.3.23) and the fact that \( C' \) is a circuit of a parallel connection \( P_x(M_1, M_2) \) if and only if \( C' \) is a circuit of \( M_1 \) or \( M_2 \), or \( C' = (C'_1 \cup C'_2) \setminus \{x\} \) where \( C'_i \) is a circuit of \( M_i \) containing the basepoint \( x \), for \( i = 1, 2 \). □

(2.3.26) If \( C' \) is a circuit of \( M \setminus \{e_0, e_{n+2}, e_1, a_1\} \) which intersects \( A_0 \), then \( C' \) is equal to some \( C_{ij} \).

Proof. Suppose that \( C' \cap A_0 \cap C = \emptyset \). Then \( C' \cap A_0 \subseteq \{a_2, a_3, \ldots, a_{n+2}\} \), contradicting (2.3.22). Therefore, \( C' \cap A_0 \cap C \) is a non-empty subset of \( \{e_2, e_3, \ldots, e_{n+1}\} \).

Hence, \( e_i \in C' \) for some \( i \in \{2, 3, \ldots, n+1\} \). Choose \( C' \) such that \( |C' \cap A_0 \cap C| \) is as small as possible. We may assume that \( C' \neq C_{ii} = \{a_i, e_i, a_{i+1}\} \). Therefore, by circuit elimination, there is a circuit \( C'' \subseteq (C' \cup \{a_i, a_{i+1}\}) \setminus \{e_i\} \). This contradicts the minimality of \( |C' \cap A_0 \cap C| \). □

(2.3.27) The partition \( \{A_0, E(M \setminus e_1) - A_0\} \) is a 2-separation of \( M \setminus e_1 \).

Proof. Since both sets have at least two elements, it is sufficient to show that \( r(A_0) + r(E(M \setminus e_1) - A_0) - r(M \setminus e_1) \leq 1 \). By (2.3.24) there is a 2-separation \( \{A, B\} \) of \( M \setminus e_1 \) such that \( A_0 \subseteq A \) and \( \{e_0, e_{n+2}\} \subseteq B \). Since \( \{e_0, e_{n+2}, e_1, a_1\} \cap A = \emptyset \), we have, by (2.3.26) that if \( C' \) is a circuit of \( M \setminus A \), then \( C' \cap A_0 = \emptyset \) or \( C' - A_0 = \emptyset \). Hence \( r(A) = r(A_0) + r(A \setminus A_0) \). By submodularity, \( r(A - A_0) + r(B) \geq r((A - A_0) \cup B) = r(E(M \setminus e_1) - A_0) \). Therefore,

\[
1 = r(A) + r(B) - r(M \setminus e_1)
= r(A_0) + r(A - A_0) + r(B) - r(M \setminus e_1)
\geq r(A_0) + r(E(M \setminus e_1) - A_0) - r(M \setminus e_1).
\]

□
(2.3.28) The partition \( \{A, B\} \) of \( E(M) \) is a 3-separation of \( M \), where the set \( A = \{e_n, e_{n+1}, a_{n+1}, a_{n+2}\} \).

**Proof.** By (2.3.27), \( \{A_0, E(M \setminus e_1) - A_0\} \) is a 2-separation of \( M \setminus e_1 \). Therefore, the partitions \( \{A_0, E(M) - A_0\} \) and \( \{A_0 \cup \{e_1\}, E(M) - (A_0 \cup \{e_1\})\} \) are 3-separations of \( M \). If \( n = 1 \), then \( A = A_0 \cup \{e_1\} \). If \( n = 2 \), then \( A = A_0 \). Suppose that \( n > 2 \) and consider the 3-separation \( \{A_0, E(M) - A_0\} \). Since \( \{e_1, a_1\} \subseteq E(M) - A_0 \) and \( T_{e_1} = \{e_1, a_1, a_2\} \) is a triangle of \( M \), (2.2.8) implies that \( \{A_0 - \{a_2\}, \{E(M) - A_0\} \cup \{a_2\}\} \) is also a 3-separation of \( M \). Since \( \{e_0, a_1, a_2, e_2\} \) is the cocircuit \( D_{e_1} \) of \( M \), we may add the element \( e_2 \) to \( \{E(M) - A_0\} \cup \{a_2\} \), creating another 3-separation of \( M \). Clearly, this procedure can be repeated, removing first \( a_i \) and then \( e_i \) from \( A_0 - \{a_2, e_2, \ldots, a_{i-1}, e_{i-1}\} \) until \( a_n \) is the last element removed. Thus we obtain the desired partition. □

Assertion (2.3.28) implies that

\[
\text{r(\{e_n, e_{n+1}, a_{n+1}, a_{n+2}\}) + r^*(\{e_n, e_{n+1}, a_{n+1}, a_{n+2}\}) = 6.}
\]

Since \( r(\{e_n, e_{n+1}, a_{n+1}, a_{n+2}\}) = 3 \), it follows that \( \{e_n, e_{n+1}, a_{n+1}, a_{n+2}\} \) contains a cocircuit \( C' \) of \( M \). Since \( e_{n+1} \) and \( e_{n+2} \) do not belong to any triad of \( M \), the set \( C' = \{e_n, e_{n+1}, a_{n+1}, a_{n+2}\} \). But this contradicts (2.2.13) and with this we finish the proof of Theorem 2.1.1. □

### 2.4 Further Results

In this final section, we prove several corollaries of Theorem 2.1.1. We begin by showing how Corollary 2.1.3 follows from the main theorem.
Proof of Corollary 2.1.3. Suppose that (i) is not satisfied. By Theorem 2.1.1, either (ii) holds, or $M$ has only one triad $T$ meeting $C$ and it contains $f$. Suppose the latter case holds. Since $f \in T$, the matroid $M \setminus f$ is not 3-connected. Theorem 2.1.8 contradicts our assumption. Hence (ii) holds. □

Recall that Theorem 2.1.8 counts triads. Next, we shall establish a stronger result which counts rank-2 dependent flats in the dual. This stronger result will be needed to prove Propositions 2.1.6 and 2.1.7. First we prove the following easy but useful lemma.

(2.4.1) Suppose that $M$ is 3-connected and $A \subseteq E(M)$ such that $|A| \geq 2$. Then $r(A) + r^*(A) - |A| \leq 1$ if and only if $r(A) = r(M)$ and $r^*(A) = r(M^*)$.

Proof. If $r(A) + r^*(A) - |A| = 0$, then $|E - A| = 0$ and the result holds. Therefore, we may assume that $r(A) + r^*(A) - |A| = 1$ and $|E - A| = 1$. Thus $1 \leq r(M) + r(M^*) - |E| + 1 = 1$. Hence, $r(M) = r(A)$ and $r(M^*) = r^*(A)$. □

(2.4.2) Proposition. Suppose that $M$ is a 3-connected matroid having rank and corank at least three and let $C$ be a circuit of $M$. If $M \setminus e$ is not 3-connected for every $e \in C$, then $C$ meets at least two dependent rank-2 flats of $M^*$.

Proof. Suppose that, for every element $e \in C$, the matroid $M \setminus e$ is not 3-connected. If $C$ is a triangle, then (2.2.2) implies that either the result holds for $C$, or $C$ is contained in a dependent rank-2 flat of $M^*$. But this implies that $C$ is a triad of $M$; a contradiction. Therefore, the result holds if $|C| = 3$.

By Theorem 2.1.8, $C$ meets at least one dependent rank-2 flat of $M^*$. Let $M$ and $C$ be such that $C$ meets only one dependent rank-2 flat $D$ of $M^*$ and $|C|$ is
minimal among circuits which do not satisfy the result. Hence $|C| \geq 4$. Since $C$ meets at least two triads, $|D| \geq 4$. By orthogonality, $|D - C| \leq 1$.

Let $e \in D \cap C$ and suppose that $M' = M/e$ is 3-connected. Then $D - \{e\}$ is a dependent rank-2 flat $D'$ of $(M')^*$ and $C - \{e\}$ is a circuit $C'$ of $M'$ which meets only one dependent rank-2 flat of $(M')^*$. If $x \in C' - D$, then (2.2.3) implies that $M' \setminus x$ is not 3-connected. Suppose that $x \in C' \cap D$. Then $x$ is in a triad of $M'$ and, hence, $M' \setminus x$ is not 3-connected. By the minimality of $|C|$, either $r(M') = 2$ or $r((M')^*) = 2$. Since $r((M')^*) = r(M^*)$, we may assume that $r(M') = 2$. This implies that $r(M) = 3$. Therefore, $r(D) \leq 3$ and hence, $r(D) + r^*(D) - |D| \leq 1$. By (2.4.1), $r^*(M) = r^*(D) = 2$; a contradiction. Hence, if $e \in D \cap C$, then $M/e$ is not 3-connected. Since $D \cap C$ contains a triad of $M$, Theorem 2.1.8 implies that $D \cap C$ meets a triangle $T$ of $M$. By orthogonality, $|T \cap D| \geq 2$. Hence, $r(T \cap D) + r^*(T \cap D) - |T \cap D| = 4 - |T \cap D|$. By the 3-connectivity of $M$ and our assumptions about the rank and corank of $M$, we have that $|T \cap D| = 2$. But since $|D| \geq 4$, there is a triad contained in $D$ which intersects $T$ in precisely one element, contradicting orthogonality. This completes the proof. □

Next we turn to Proposition 2.1.7. Recall that $f_2$ is the number of dependent rank-two flats of $M$ and $f_2^*$ is $f_2(M^*)$. Proposition 2.1.7 says that, for a minor-minimally 3-connected matroid $M$ with rank and corank at least three, $f_2(M) + f_2^*(M) \geq 2$.

Proof of Proposition 2.1.7. If $Del$ is dependent, then Proposition 2.4.2 implies that $f_2^* \geq 2$, and the result holds. By duality, $f_2 \geq 2$ if $Con$ is codependent.
So suppose that \( \text{Del} \) is independent and that \( \text{Con} \) is coindependent. Hence, \( \text{Del} \) is contained in a basis of \( M \) and \( \text{Con} \) is contained in a cobasis of \( M \). Since \( \text{Del} \cup \text{Con} = E(M) \), the set \( \text{Del} \) must be a basis \( B \) of \( M \) and \( \text{Con} \) must be the cobasis \( E(M) - B \) of \( M \). Since \( M \) is 3-connected, there is an element \( e \in E(M) - B \) and a circuit \( C \subseteq B \cup \{e\} \). Since \( C \) satisfies the hypothesis of Theorem 2.1.1, it meets a triad of \( M \). The same argument applied to the dual proves that \( M \) also contains a triangle. Hence, \( f_2(M) + f_2^*(M) \geq 2 \). □

We now turn to the proof of Proposition 2.1.6. Recall that this result is the 3-connected analogue of (2.1.4). The proof of (2.1.4) used the following result of Oxley (1981b, Corollary 2.6).

(2.4.3) Let \( M \) be a 2-connected matroid having corank at least two. Suppose that \( A \subseteq E(M) \) such that for all \( a \) in \( A \), \( M \setminus a \) is not 2-connected. Then either \( A \) is independent, or \( A \) meets at least \( |A| - r(A) + 1 \) dependent rank-1 flats of \( M^* \).

Likewise, the proof of Proposition 2.1.6 will use the following analogue of (2.4.3).

(2.4.4) Proposition. Let \( M \) be a 3-connected matroid having corank at least three. Suppose that \( A \subseteq E(M) \) such that for all \( a \) in \( A \), \( M \setminus a \) is not 3-connected. Then either \( A \) is independent, or \( A \) meets at least \( \frac{1}{2}(|A| - r(A)) + 1 \) dependent rank-2 flats of \( M^* \).

Proof. Suppose that \( A \) is dependent and let \( A' \) be the union of all circuits of \( M \) contained in \( A \). Let \( X \) be the set of elements of \( A' \) which are contained in rank-2
dependent flats of $M^*$. Then, by Theorem 2.1.1, $X$ meets every circuit of $M|A'$. Since the cobases of a matroid are the minimal sets meeting every circuit, it follows that $X$ contains a cobasis $B^*$ of $M|A'$. Choose an element $a$ of $B^*$. The element $a$ is in a rank-2 dependent flat $F_1$ of $M^*$. Let $C_a$ be the fundamental circuit of $a$ in $M|A'$ with respect to $A' - B^*$. By Proposition 2.4.2, $C_a$ meets a rank-2 dependent flat $F_2$ of $M^*$ different from $F_1$. By orthogonality, $|F_i - C_a| \leq 1$ for $i = 1$ and 2. Thus $|B^* \cap (F_1 \cup F_2)| \leq 3$, and so $|B^* - (F_1 \cup F_2)| \geq |B^*| - 3$. Since $B^*$ contains at most two elements of any rank-2 dependent flat of $M^*$ meeting $A'$, the number of flats meeting $A'$ is at least $|B^*| - 3 + 2 = |B^*| + 1$. Equality holds only if $|B^* \cap F_1| = 2$ and $|B^* \cap (F_1 \cup F_2)| = 3$. Let $b \in (B^* \cap F_2) - F_1$ and let $C_b$ be the fundamental circuit of $b$ in $M|A'$ with respect to $A' - B^*$. By Proposition 2.4.2, $C_b$ meets a rank-2 dependent flat $F_3$ of $M^*$ different from $F_2$. Since $|B^* \cap F_1| = 2$ and $b \not\in F_1$, we have that $|F_1 - C_b| \geq 2$. By orthogonality, $C_b \cap F_1 = \emptyset$. Hence $F_3 \neq F_1$. Now $|B^* - (F_1 \cup F_2 \cup F_3)| \geq |B^*| - 4$, so the number of rank-2 dependent flats meeting $A'$ is at least $|A'| - 4 + 3 = |B^*| + 1$. The result follows from the observation that $A$ meets at least as many rank-2 dependent flats as $A'$ and that $|B^*| = |A'| - r(A') = |A| - r(A)$. $\square$

In order to apply Proposition 2.4.4 in the proof of Proposition 2.1.6, we note that, for a set $A \subseteq E(M)$,

$$|A| - r(A) = r(M^*) - r(M^*|(E - A)) = r(M^*/(E - A)).$$

Hence, in the statement of Proposition 2.4.4, $\frac{1}{2}(|A| - r(A)) + 1$ can be replaced by
Recall that Proposition 2.1.6 states that for a minor-minimally 3-connected matroid \( M \) with rank and corank at least three,

\[
f_2(M) + f_2^*(M) \geq \frac{1}{2} [r(M/\text{Del}) + r(M^*/\text{Con})] + 1.
\]

**Proof of Proposition 2.1.6.** Suppose that \( \text{Del} \) is independent. Then either

- \( \text{Con} \) is a spanning dependent set of \( M^* \), or \( \text{Del} \) is a basis of \( M \).

Suppose that \( \text{Con} \) is a spanning dependent set of \( M^* \). Then \( r(M^*/\text{Con}) = 0 \) and the dual of Proposition 2.4.4 implies that

\[
f_2(M) \geq \frac{1}{2} r(M/(E - \text{Con})) + 1
\]

\[
\geq \frac{1}{2} r(M/\text{Del}) + 1.
\]

Hence the result follows.

Suppose that \( \text{Del} \) is a basis of \( M \) and \( \text{Con} \) is a basis of \( M^* \). Then \( r(M/\text{Del}) = r(M^*/\text{Con}) = 0 \). Hence, the result follows by Proposition 2.1.7. Therefore, the result holds if \( \text{Del} \) is independent. By duality, the result holds if \( \text{Con} \) is independent in \( M^* \). Hence, we may assume that \( \text{Del} \) is dependent in \( M \) and \( \text{Con} \) is dependent in \( M^* \). In this case, Proposition 2.4.4 and its dual imply that

\[
f_2(M) + f_2^*(M) \geq \left[ \frac{1}{2} r(M/(E - \text{Con})) + 1 \right] + \left[ \frac{1}{2} r(M^*/(E - \text{Del})) + 1 \right]
\]

\[
= \frac{1}{2} [r(M/(E - \text{Con})) + r(M^*/(E - \text{Del}))] + 2
\]

\[
\geq \frac{1}{2} [r(M/\text{Del}) + r(M^*/\text{Con})] + 2. \square
\]
CHAPTER 3
A UNIQUE DECOMPOSITION FOR 3-CONNECTED BINARY MATROIDS

3.1 Introduction

In this chapter we introduce a decomposition for 3-connected binary matroids. The main result proves that every 3-connected binary matroid has a unique minimal such decomposition. This is the latest in a natural progression of graph and matroid results in this area. For example, Whitney (1932, Theorem 12), MacLane (1937, Theorem 2), Tutte (1966b, Chapter 11), and Cunningham and Edmonds (1980, Theorems 1 and 18) have each produced uniqueness results for the decomposition of \(k\)-connected graphs and matroids, where \(k \in \{1, 2\}\). Coullard, Gardner, and Wagner (1993, Theorem 1.2) defined a unique decomposition for 3-connected graphs.

Recall that, for a positive integer \(k\), a partition \(\{X, Y\}\) of the ground set of a matroid \(M\) is called a \(k\)-\textit{separation} of \(M\) if \(\min\{|X|, |Y|\} \geq k\), and \(r(X) + r(Y) \leq r(M) + k - 1\). A matroid \(M\) is called \(n\)-\textit{connected}, if for all \(k < n\), there is no \(k\)-separation of \(M\). A partition \(\{X, Y\}\) of the ground set of \(M\) is called a \textit{vertical} \(k\)-\textit{separation} of \(M\) if \(\min\{r(X), r(Y)\} \geq k\), and \(r(X) + r(Y) \leq r(M) + k - 1\). A matroid \(M\) is \textit{vertically n-connected} if, for all \(k < n\), there is no vertical \(k\)-separation of \(M\). A matroid \(M\) is \textit{cyclically n-connected} if \(M^*\) is vertically \(n\)-connected.

In Section 3.2, the terminology and notation needed to define the decomposition are established, as are the basic results needed to prove the main result. These results will also be useful in subsequent development of the theory. In Section 3.3,
we prove that the decomposition of Coullard, Gardner, and Wagner is a special case of the decomposition introduced in this chapter. In the process, we show how the decomposition is related to the generalized parallel connection of Brylawski (1975) and the three-sum of Seymour (1980).

The decomposition to be defined in the next section is based on vertical 3-separations of 3-connected binary matroids. This is similar to previous work. For example, Coullard, Gardner, and Wagner based their decomposition of 3-connected graphs on cyclical 3-separations; and Cunningham and Edmonds based their decomposition of 2-connected matroids on vertical 2-separations. Because of this similarity, many of the terms to be defined in the next section (for example, split, crossing split, compatible split, and good split) are shared by all three works, though the precise definitions vary from work to work.

3.2 The Proof of the Main Result

Let \( M \) be a simple 3-connected binary matroid with a vertical 3-separation \( \{E_1, E_2\} \). Let \( X_i = \text{cl}(E_i) \) for \( i \in \{1, 2\} \). The pair \( X = \{X_1, X_2\} \) is called a split of \( M \). The set \( X_1 \cap X_2 \) is denoted by \( \delta(X) \). Splits \( X = \{X_1, X_2\} \) and \( Y = \{Y_1, Y_2\} \) are distinct if \( X_1 \neq Y_1 \) and \( X_2 \neq Y_2 \). Two splits \( X = \{X_1, X_2\} \) and \( Y = \{Y_1, Y_2\} \) are said to cross if \( Y_i \cap (X_j - \delta(X)) \) and \( X_i \cap (Y_j - \delta(Y)) \) are non-empty for all \( i, j \in \{1, 2\} \). Note that \( Y_1 \cap (X_1 - \delta(X)) = Y_1 - X_2 \). Therefore, \( X \) and \( Y \) cross if and only if \( Y_i - X_j \) and \( X_i - Y_j \) are non-empty for all \( i, j \in \{1, 2\} \). Two distinct splits that do not cross are compatible. A good split is one that is not crossed by any other split. Let \( P(r) = PG(r-1, 2) \). Since \( M \) is simple and binary, \( M \) is isomorphic
to a restriction of $P(r)$ if $r \geq r(M)$. Identify $M$ with such a restriction. For a split $X = \{X_1, X_2\}$, let $\gamma(X) = cl_{P(r)}(X_1) \cap cl_{P(r)}(X_2)$ and let $M_i = P(r)|(X_i \cup \gamma(X))$ for each $i \in \{1, 2\}$. The set $\gamma(X)$ is called the connection of $X$ and $\{M_1, M_2\}$ is called the simple decomposition of $M$ generated by $X$. Observe that the decomposition depends on the particular injective map used to identify $M$ with a restriction of $P(r)$. Assuming that the identification is fixed, we have that a vertical 3-separation and its associated split generate a unique simple decomposition.

We have defined both a split and a decomposition generated by a split to be sets rather than ordered pairs. Yet, given a split $\{X_1, X_2\}$ generating a decomposition $\{M_1, M_2\}$, we will consistently make the assumption that $X_1$ is associated with the first component $M_1$, and that $X_2$ is associated with the second component $M_2$.

Before moving on to general decompositions, we prove some basic results about vertical 3-separations and splits. The first result can be found in Oxley (1992, Lemma 8.2.10).

(3.2.1) Let $X_1, X_2, Y_1,$ and $Y_2$ be subsets of the ground set of a matroid $M$. If $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$, then

$$r(Y_1) + r(Y_2) - r(Y_1 \cup Y_2) \leq r(X_1) + r(X_2) - r(X_1 \cup X_2).$$

(3.2.2) Suppose that $\{X, Y\}$ is a partition of the ground set of a 3-connected matroid $M$ and that $r(X) + r(Y) - r(M) \leq 2$. If $3 \leq r(X) < r(M)$, then $\{X, Y\}$ is a vertical 3-separation of $M$.

**Proof.** Since $r(X) < r(M)$, the set $Y$ is non-empty. Since $r(X) \geq 3$, it suffices to show that $r(Y) \geq 3$. Suppose that $r(Y) \leq 2$. Since $r(X) \leq r(M) - 1$, we have
that $r(X) + r(Y) - r(M) \leq 1$. Hence, $\{X, Y\}$ is either a separation of $M$ or a 2-separation of $M$. This contradicts the assumption that $M$ is 3-connected. □

(3.2.3) Let $X_1$ and $X_2$ be subsets of a 3-connected binary matroid $M$ such that $X_1 \cup X_2 = E(M)$. Suppose that $r(X_1) + r(X_2) \leq r(M) + 2$ and $3 \leq r(X_1) < r(M)$. Then $\{cl_M(X_1), cl_M(X_2)\}$ is a split of $M$.

Proof. By (3.2.2), $\{X_1, X_2 - X_1\}$ is a vertical 3-separation of $M$. Hence,

$$
\begin{align*}
\text{r}(M) + 2 & \geq \text{r}(X_1) + \text{r}(X_2) \\
& \geq \text{r}(X_1) + \text{r}(X_2 - X_1) \\
& = \text{r}(M) + 2.
\end{align*}
$$

Therefore $\text{r}(X_2) = \text{r}(X_2 - X_1)$. By the definition of a split, $\{cl_M(X_1), cl_M(X_2 - X_1)\}$ is a split of $M$. Since $\text{r}(X_2) = \text{r}(X_2 - X_1)$, we have that $cl_M(X_2) = cl_M(X_2 - X_1)$. This completes the proof. □

For the remainder of the chapter we shall assume that $M$ is a 3-connected binary matroid and that $P$ is a binary projective space of which $M$ is a restriction.

(3.2.4) If $X$ is a split of $M$, then $|\gamma(X)| = 3$ and $r(\gamma(X)) = 2$.

Proof. Suppose that $X = \{X_1, X_2\}$. Since $\gamma(X) = cl_P(X_1) \cap cl_P(X_2)$, the set $\gamma(X)$ is a flat of $P$. Also, since $P$ is modular,

$$
\begin{align*}
\text{r}(\gamma(X)) &= \text{r}[cl_P(X_1) \cap cl_P(X_2)] \\
& = \text{r}(X_1) + \text{r}(X_2) - \text{r}[cl_P(X_1) \cup cl_P(X_2)] \\
& = \text{r}(X_1) + \text{r}(X_2) - \text{r}(M) \\
& = 2.
\end{align*}
$$

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The result follows since every rank-2 flat of a binary projective space has precisely three elements. □

(3.2.5) Suppose that $X = \{X_1, X_2\}$ is a split of $M$. Then $\text{cl}_M(X_i - \delta(X)) = X_i$, for each $i \in \{1, 2\}$.

Proof. Let $\{E_1, E_2\}$ be a vertical 3-separation of $M$ that induces the split $X$. Suppose that $\text{cl}_M(X_1 - \delta(X)) \subseteq X_1$. Then $r(X_1 - \delta(X)) \leq r(X_1) - 1$. Therefore,

$$r(X_1 - \delta(X)) + r(X_2) - r(M) \leq r(X_1) + r(X_2) - r(M) - 1$$

$$= 1. \quad (3.1)$$

Since $r(X_2) = r(E_2)$ and $\{E_1, E_2\}$ is a vertical 3-separation of $M$, we have that $r(X_2) < r(M)$ and that $r(X_2) \geq 3$. By (3.2.2), $\{X_1 - \delta(X), X_2\}$ is a vertical 3-separation of $M$; contradicting (3.1). Hence $\text{cl}_M(X_1 - \delta(X)) = X_1$. □

The next result is a straightforward application of (3.2.2) and (3.2.5).

(3.2.6) Suppose that $X = \{X_1, X_2\}$ is a split of a 3-connected binary matroid $M$ and $\{\delta_1, \delta_2\}$ is a partition of $\delta(X)$. Then $\{X_1 - \delta_1, X_2 - \delta_2\}$ is a vertical 3-separation of $M$.

The above result shows that there may be several vertical 3-separations that give rise to the same simple decomposition of $M$ and the same split $X$. Therefore, it is more convenient to associate a simple decomposition with the split that generates it rather than with the set of associated 3-separations. The next result allows us to define general decompositions inductively.
Suppose that \( X = \{X_1, X_2\} \) is a split of \( M \) generating the simple decomposition \( \{M_1, M_2\} \). Then \( M_1 \) and \( M_2 \) are 3-connected.

**Proof.** Suppose that \( \{S, T\} \) is a \( k \)-separation of \( M_1 \) where \( k \in \{1, 2\} \). We may assume that \( |S \cap \gamma(X)| \geq 2 \). Observe that if \( |S \cap \gamma(X)| = 2 \), then

\[
\begin{align*}
    r(S \cup \gamma(X)) + r(T - \gamma(X)) & \leq r(S) + r(T) \\
    & \leq r(M_1) + k - 1.
\end{align*}
\]

If \( |T| > k \), then \( \{S \cup \gamma(X), T - \gamma(X)\} \) is a \( k \)-separation of \( M_1 \). If \( |T| = k \), then

\[
    r(S \cup \gamma(X)) + r(T - \gamma(X)) = r(M_1) + k - 2.
\]

Therefore, if \( \{S, T\} \) is a 1- or 2-separation of \( M_1 \), then \( \{S \cup \gamma(X), T - \gamma(X)\} \) is a 1- or 2-separation of \( M_1 \). Hence, we may assume that \( \gamma(X) \subseteq S \). Thus,

\[
\begin{align*}
    r(S \cup X_2) + r(T) & \leq r(S) + r(X_2) - r(S \cap X_2) + r(T) \\
    & = r(S) + r(T) + r(X_2) - 2 \\
    & = r(X_1) + (k - 1) + r(X_2) - 2 \\
    & = r(X_1) + r(X_2) + k - 3 \\
    & = r(M) + k - 1.
\end{align*}
\]

Since \( \{X_2 - \gamma(X), X_1\} \) is a 3-separation of \( M \), \( |(S \cup X_2) - \gamma(X)| \geq 3 \). Hence, \( \{(S \cup X_2) \cap E(M), T\} \) is either a 1- or 2-separation of \( M \); a contradiction. Therefore, \( M_1 \) is 3-connected, and, likewise, so is \( M_2 \). \( \Box \)

Once again, suppose that \( P(r) \) is the binary projective space of rank \( r \), where \( r \geq r(M) \), and that \( M \) is identified with a particular restriction of \( P(r) \); that is, \( M \)
has some given coordinatization. A **decomposition** $D$ of $M$ is defined inductively. Either $D = \{M\}$ or $D$ is obtained from a decomposition $D'$ of $M$ by replacing a member $N$ of $D'$ by the members of a simple decomposition of $N$ with respect to $P(r)$. For this representation of $M$, two decompositions are *equivalent* if and only if they are equal. Suppose two different coordinatizations of $M$ are used for the decompositions $D_1$ and $D_2$. Since binary matroids are uniquely coordinatizable, there is an automorphism $\phi$ of $P(r)$ which maps the coordinatization of $M$ used in $D_1$ to the coordinatization of $M$ used in $D_2$. In that case, $D_1$ and $D_2$ are called *equivalent* if and only if $D_2 = \{\phi(N) : N \in D_1\}$. Our first main theorem concerning decompositions establishes uniqueness up to this equivalence. By the definition of equivalence, we may assume that the decompositions of $M$ are done with respect to a fixed coordinatization and a fixed projective space $P$. In that case, "equivalence" is the same as "equality". A decomposition $D$ is a *refinement* of a decomposition $D'$ if $D$ is constructed from $D'$ by some sequence of simple decompositions. The refinement is *strict* if that sequence is non-empty; it is *simple* if that sequence has one entry. We have defined decompositions in terms of splits without regard to whether the splits are good or not. But, in order to obtain the desired uniqueness result, we shall define a minimal decomposition in terms of good splits only. A decomposition $D$ is *minimal* if it has no strict refinement and if every simple refinement used to produce $D$ was the result of a good split. We are now able to state the main result.

**3.2.8 Theorem.** *Every 3-connected binary matroid has a unique minimal decomposition.*
The next two results establish basic properties of crossing and compatible splits.

(3.2.9) *The splits* $X = \{X_1, X_2\}$ *and* $Y = \{Y_1, Y_2\}$ *are compatible if and only if* $X_j \supset Y_i$ *and* $Y_k \supset X_l$, *where* $\{i, k\} = \{j, l\} = \{1, 2\}$.

**Proof.** Suppose that two distinct splits $X = \{X_1, X_2\}$ and $Y = \{Y_1, Y_2\}$ are compatible. By definition, either $X_j \supset Y_i$ or $Y_j \supset X_i$ for some $i, j \in \{1, 2\}$. We may assume that $Y_2 \supset X_2$. Since $X \neq Y$, the containment must be proper; that is, $Y_2 \supset X_2$. Hence $Y_1 - \delta(Y) \subset X_1 - \delta(X)$. Therefore, either $Y_1 \subset X_1$ or $\delta(Y) - X_1$ is non-empty. Suppose $y \in \delta(Y) - X_1$. Since $y \in \delta(Y)$, it is in $cl_M(\delta(Y))$. Hence, $y \in cl_M(X_1 - \delta(X))$. By (3.2.5), $y \in X_1$; contradicting our assumption. Therefore, $Y_1 \subset X_1$. This proves that if $X$ and $Y$ are compatible, then $X_j \supset Y_i$ and $Y_k \supset X_l$, where $\{i, k\} = \{j, l\} = \{1, 2\}$. The converse of this statement follows from the definition of compatible splits. \qed

Since two splits either cross or are compatible, but never both, every new characterization of compatible splits implies a new characterization of crossing splits. The next result combines the observations made in the opening paragraph of this section about crossing splits with the information obtained as a result of (3.2.9).

(3.2.10) *Given two splits* $X = \{X_1, X_2\}$ *and* $Y = \{Y_1, Y_2\}$ *of* $M$, *the following are equivalent.*

(i) $X$ and $Y$ cross.

(ii) $Y_i \cap (X_j - \delta(X))$ and $X_i \cap (Y_j - \delta(Y))$ are non-empty for all $i, j \in \{1, 2\}$. 

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(iii) \( Y_i - X_j \) and \( X_i - Y_j \) are non-empty for all \( i, j \in \{1, 2\} \).

(iv) \( Y_i - X_j \) is non-empty for all \( i, j \in \{1, 2\} \).

(v) \( Y_i \cap (X_j - \delta(X)) \) is non-empty for all \( i, j \in \{1, 2\} \).

The next several results establish hereditary properties for splits, good splits, and their connections. Most of the important properties of the decomposition, including Theorem 3.2.8, depend on these results.

\textbf{(3.2.11)} Let \( X = \{X_1, X_2\} \) be a split of \( M \), and let \( \{M_1, M_2\} \) be the simple decomposition of \( M \) generated by \( X \). If \( Y = \{Y_1, Y_2\} \) is a split of \( M \) such that \( Y_2 \supset X_2 \), then \( Y' = \{Y'_1, Y'_2\} \) is a split of \( M_1 \), where \( Y'_1 = \text{cl}_{M_1}(Y_1) \) and \( Y'_2 = \text{cl}_{M_1}(E(M_1) - Y'_1) \).

\textbf{Proof.} Suppose that \( Y = \{Y_1, Y_2\} \) is a split of \( M \) and that \( X_2 \subset Y_2 \). Then \( \gamma(X) \subseteq \text{cl}_P(Y_2) \). Since \( Y_2 \) is closed in \( M \), it follows that \( [\gamma(X) - Y_2] \cap E(M) = \emptyset \). Hence, \( \gamma(X) - Y_2 \subseteq \gamma(X) - Y_1 \) and

\[
\gamma(X) = [\gamma(X) \cap Y_2] \cup [\gamma(X) - Y_2] \subseteq [\gamma(X) \cap Y_2] \cup [\gamma(X) - Y_1].
\]

Since \( E(M) - Y_1 \subseteq Y_2 \), we have that

\[
X_1 - Y_1 = X_1 \cap (E(M) - Y_1) \subseteq X_1 \cap Y_2.
\]

By definition, the set \( E(M_1) = X_1 \cup \gamma(X) \). Therefore, combining (3.2) and (3.3), we get

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\[ E(M_1) - (Y_1 - Y_2) = [E(M_1) - Y_1] \cup [E(M_1) \cap Y_2] \]
\[ = [(X_1 - Y_1) \cup (\gamma(X) - Y_1)] \cup [(X_1 \cap Y_2) \cup (\gamma(X) \cap Y_2)] \]
\[ \subseteq (X_1 \cap Y_2) \cup \gamma(X). \]

Therefore,
\[ r(Y_1 - Y_2) + r[E(M_1) - (Y_1 - Y_2)] - r(M_1) \leq r(Y_1 - Y_2) + r[(X_1 \cap Y_2) \cup \gamma(X)] - r(X_1). \quad (3.4) \]

Since \( \gamma(X) \subseteq \text{cl}_P(X_1) \cap \text{cl}_P(Y_2) \) and since \( X_1 \cap Y_2 \subseteq \text{cl}_P(X_1) \cap \text{cl}_P(Y_2) \), we have that
\[ r[(X_1 \cap Y_2) \cup \gamma(X)] \leq r[\text{cl}_P(X_1) \cap \text{cl}_P(Y_2)]. \quad (3.5) \]

Observe that \( E(M) = X_1 \cup Y_2 \). Hence \( r(M) = r[\text{cl}_P(X_1) \cup \text{cl}_P(Y_2)] \). By submodularity,
\[ r[\text{cl}_P(X_1) \cap \text{cl}_P(Y_2)] \leq r(X_1) + r(Y_2) - r(M). \quad (3.6) \]

Combining (3.4), (3.5), and (3.6), we get that
\[ r(Y_1 - Y_2) + r[E(M_1) - (Y_1 - Y_2)] - r(M_1) \leq r(Y_1 - Y_2) + r(Y_2) - r(M) \]
\[ \leq r(Y_1) + r(Y_2) - r(M). \quad (3.7) \]

Since \( Y \) is a split of \( M \), (3.7) implies that
\[ r(Y_1 - Y_2) + r[E(M_1) - (Y_1 - Y_2)] - r(M_1) \leq 2. \]

By (3.2.9), \( Y_1 \subset X_1 \). Since both \( Y_1 \) and \( X_1 \) are closed relative to \( M \), we have that
\[ r(Y_1) < r(X_1) = r(M_1). \] Hence \( r(Y_1 - Y_2) < r(M_1) \). By (3.2.6), \( \{Y_1 - Y_2, Y_2\} \)
is a vertical 3-separation of $M$. Therefore $r(Y_1 - Y_2) \geq 3$. Hence, (3.2.2) implies that $\{Y_1 - Y_2, E(M_1) - (Y_1 - Y_2)\}$ is a vertical 3-separation of $M_1$. By (3.2.5), $Y_1 = cl_M(Y_1 - Y_2)$. Hence $cl_{M_1}(Y_1 - Y_2) = cl_{M_1}(Y_1)$. Therefore $Y'$ is a split of $M_1$.

(3.2.12) Let $X = \{X_1, X_2\}$ be a split of $M$, and let $\{M_1, M_2\}$ be the simple decomposition of $M$ generated by the split $X$. If $Y' = \{Y'_1, Y'_2\}$ is a split of $M_1$ such that $\gamma(X) \subseteq Y'_2$, then $Y = \{Y_1, Y_2\}$ is a split of $M$, where the set $Y_1 = Y'_1 \cap E(M)$ and $Y_2 = cl_M(E(M) - Y_1)$.

Proof. Suppose that $Y' = \{Y'_1, Y'_2\}$ is a split of $M_1$ and $\gamma(X) \subseteq Y'_2$. Let $Y_1 = Y'_1 \cap E(M)$ and $Y_2 = cl_M(E(M) - Y_1)$. Observe that

$$\gamma(X) \subseteq cl_P[(X_1 - Y_1) \cup \gamma(X)] \cap cl_P(X_2)$$

$$\subseteq cl_P(X_1) \cap cl_P(X_2)$$

$$= \gamma(X).$$

By the modularity of $P$,

$$r[(X_1 - Y_1) \cup X_2] = r[(X_1 - Y_1) \cup \gamma(X)] + r(X_2) - 2. \hspace{1cm} (3.8)$$

Since $\gamma(X) \subseteq Y'_2$ and $Y'_2 = cl_{M_1}[(X_1 \cup \gamma(X)) - Y_1]$, we have that $\gamma(X) \subseteq cl_{M_1}[(X_1 \cup \gamma(X)) - Y_1]$. Therefore,

$$r[(X_1 \cup \gamma(X)) - Y_1] = r[(X_1 - Y_1) \cup \gamma(X)]. \hspace{1cm} (3.9)$$

Combining (3.8) and (3.9), we get
\[ r(Y_1) + r[(X_1 - Y_1) \cup X_2] - r(M) \]
\[ = r(Y'_1) + r[(X_1 - Y_1) \cup \gamma(X)] + r(X_2) - 2 - r(M) \]
\[ = r(Y'_1) + r[(X_1 \cup \gamma(X)) - Y_1] + r(X_2) - 2 - r(M) \]
\[ = r(Y'_1) + r(Y'_2) - 2 + r(X_2) - r(M) \]
\[ = r(X_1) + r(X_2) - r(M) \]
\[ = 2. \]

Since \( \{Y_1, E(M_1) - Y_1\} \) is a vertical 3-separation of \( M_1 \), we have that \( 3 \leq r(Y_1) < r(M_1) \leq r(M) \). By (3.2.2), \( \{Y_1, E(M) - Y_1\} \) is a vertical 3-separation of \( M \). Hence \( Y \) is a split of \( M \). \( \Box \)

(3.2.13) **Suppose that** \( X = \{X_1, X_2\} \) **and** \( Y = \{Y_1, Y_2\} \) **are compatible splits of** \( M \), **where** \( Y_2 \supset X_2 \). **Further, let** \( \{M_1, M_2\} \) **be the simple decomposition of** \( M \) **generated by** \( X \) **and let** \( Y' \) **be the split of** \( M_1 \) **induced by** \( Y \). **Then** \( \gamma(Y) = \gamma(Y') \).

**Proof.** By definition, \( \gamma(Y') = cl_p(Y_1) \cap cl_p[E(M_1) - cl_{M_1}(Y_1)] \). Suppose that \( e \in E(M_1) - cl_{M_1}(Y_1) \). Then \( e \in X_1 - Y_1 \) or \( e \in \gamma(X) - cl_{M_1}(Y_1) \). In the former case, \( e \in Y_2 \). In the latter case, since \( \gamma(X) \subseteq cl_p(X_2) \subseteq cl_p(Y_2) \), the element \( e \in cl_p(Y_2) \). Hence, \( cl_p(Y_2) \supseteq cl_p[E(M_1) - cl_{M_1}(Y_1)] \). By definition, \( \gamma(Y') = cl_p(Y_1) \cap cl_p(Y_2) \). Therefore \( \gamma(Y) \supseteq \gamma(Y') \). By (3.2.4), \( \gamma(Y') \) and \( \gamma(Y) \) are both rank-two flats of \( P \). Hence \( \gamma(Y) = \gamma(Y') \). \( \Box \)

In the next two results, we adjust the statements of (3.2.11) and (3.2.12) by redefining \( Y'_2 \) and \( Y_2 \), respectively. These changes will make several future results easier to prove while providing more precise information about induced splits.
(3.2.14) Let $X = \{X_1, X_2\}$ be a split of $M$, and let $\{M_1, M_2\}$ be the simple decomposition of $M$ generated by $X$. If $Y = \{Y_1, Y_2\}$ is a split of $M$ such that $Y_2 \supset X_2$, then $Y' = \{Y_1', Y_2'\}$ is a split of $M_1$, where $Y_1' = cl_{M_1}(Y_1)$ and $Y_2' = (X_1 \cap Y_2) \cup \gamma(X)$.

Proof. By (3.2.11), we have that $Y' = \{cl_{M_1}(Y_1), Y_2'\}$ is a split of $M_1$ for $Y_2' = cl_{M_1}(E(M_1) - Y_1')$. Since $Y_2' \supset X_2$, we have that $cl_{P}(Y_2) \supset \gamma(X)$. Hence, $(X_1 \cap Y_2) \cup \gamma(X)$ is a closed set of $M_1$. Suppose that $x \in Y_1' \cap \gamma(X)$. Since $\gamma(X) \subseteq cl_{P}(Y_2)$ and $Y_1' \subseteq cl_{P}(Y_1)$, we have that $x \in \gamma(Y)$. By (3.2.13), $\gamma(Y) = \gamma(Y')$. Hence $Y_1' \cap \gamma(X) \subseteq \delta(Y') \subseteq Y_2'$. Clearly, $\gamma(X) - Y_1' \subseteq Y_2'$. Therefore $\gamma(X) \subseteq Y_2'$. Suppose that $x \in X_1 \cap Y_2$. Then, either $x \in \gamma(Y)$ or $x \in X_1 - Y_1$. In either case, $x \in Y_2'$. Therefore $X_1 \cap Y_2 \subseteq Y_2'$. Hence $(X_1 \cap Y_2) \cup \gamma(X) \subseteq Y_2'$. Now, suppose that $x \in Y_2' - [(X_1 \cap Y_2) \cup \gamma(X)]$. Since $(X_1 \cap Y_2) \cup \gamma(X)$ is closed, if $Y_2' - [(X_1 \cap Y_2) \cup \gamma(X)] \subseteq \gamma(Y')$, then (3.2.5) implies that $(X_1 \cap Y_2) \cup \gamma(X) = Y_2'$. Therefore, we may assume that $x \notin Y_1'$. Hence, $x \in E(M_1) - cl_{M_1}(Y_1) \subseteq (X_1 \cap Y_2) \cup \gamma(X)$; a contradiction. This completes the proof. □

(3.2.15) Let $X = \{X_1, X_2\}$ be a split of $M$, and let $\{M_1, M_2\}$ be the simple decomposition of $M$ generated by $X$. If $Y' = \{Y_1', Y_2'\}$ is a split of $M_1$ such that $\gamma(X) \subseteq Y_2'$, then $Y = \{Y_1, Y_2\}$ is a split of $M$, where $Y_1 = Y_1' \cap E(M)$ and $Y_2 = (Y_2' \cup X_2) \cap E(M) = (Y_2' \cap X_1) \cup X_2$.

Proof. By (3.2.12), we have that $Y = \{Y_1' \cap X_1, Y_2\}$ is a split of $M$ for $Y_2 = cl_{M}(E(M) - Y_1)$. Observe that $(Y_2' \cap X_2) \cap E(M) = (Y_2' \cap X_1) \cup X_2$. We will complete the proof by showing that $Y_2 = X_2 \cup (Y_2' \cap X_1)$. Clearly $Y_2 = (Y_2 \cap X_1) \cup X_2$. Suppose
\( x \in (Y'_2 \cap X_1) - \gamma(Y') \). Then \( x \not\in Y'_1 \). Since \( Y_1 = Y'_1 \cap X_1 \), we have that \( x \not\in Y_1 \).

Therefore \( x \in Y_2 \cap X_1 \). Hence \((Y_2 \cap X_1) \supset (Y'_2 \cap X_1) - \gamma(Y')\). Since \( \gamma(Y) = \gamma(Y') \) and \( Y_2 \supset \gamma(Y) \cap X_1 \), we have that \( Y_2 \cap X_1 \supset Y'_2 \cap X_1 \). Therefore \( Y_2 \supset X_2 \cup (Y'_2 \cap X_1) \).

Suppose that \( y \in Y_2 - [X_2 \cup (Y'_2 \cap X_1)] \). Then \( y \not\in Y'_2 \). Hence \( y \not\in \gamma(Y') \). Since \( \gamma(Y) = \gamma(Y') \), we have that \( y \in Y_2 - Y_1 \). Therefore \( y \in Y_2 - Y'_1 \). This implies that \( y \in Y'_2 \cap X_1 \); a contradiction. Therefore \( Y_2 = X_2 \cup (Y'_2 \cap X_1) \). □

(3.2.16) Let \( X = \{X_1, X_2\} \) be a split of \( M \), and let \( \{M_1, M_2\} \) be the simple decomposition of \( M \) generated by \( X \). If \( Y = \{Y_1, Y_2\} \) is a good split of \( M \) such that \( Y_2 \supset X_2 \), then \( Y' = \{Y'_1, Y'_2\} \) is a good split of \( M_1 \), where \( Y'_1 = \text{cl}_{M_1}(Y_1) \) and \( Y'_2 = (X_1 \cap Y_2) \cup \gamma(X) \).

Proof. Suppose that \( Y = \{Y_1, Y_2\} \) is a good split of \( M \), where \( Y_2 \supset X_2 \). By (3.2.14), \( Y' = \{Y'_1, Y'_2\} \) is a split of \( M_1 \), where \( Y'_1 = \text{cl}_{M_1}(Y_1) \) and \( Y'_2 = (X_1 \cap Y_2) \cup \gamma(X) \).

Suppose that \( Y' \) is crossed by the split \( Z' = \{Z'_1, Z'_2\} \) of \( M_1 \). Since \( Z'_1 \) and \( Z'_2 \) are closed in \( M_1 \), we may assume that \( \gamma(X) \subseteq Z'_2 \). By (3.2.15), \( Z = \{Z_1, Z_2\} \) is a split of \( M \) where \( Z_1 = Z'_1 \cap E(M) \) and \( Z_2 = (Z'_2 \cup X_2) \cap E(M) \). Therefore, either \( r(Z'_1) > r(Z_1) \) or \( Z'_1 = \text{cl}_{M_1}(Z_1) \). Suppose that \( x \in Z'_1 - \text{cl}_{M_1}(Z_1) \). Then \( x \in \gamma(X) - \text{cl}_{M_1}(Z_1) \). Therefore \( x \in Z'_2 \). This implies that \( x \in \gamma(Z') \). But this contradicts (3.2.13). Hence \( Z'_1 = \text{cl}_{M_1}(Z_1) \). Since \( Y \) is a good split of \( M \), and \( Z \) and \( Y \) are distinct splits of \( M \), the splits \( Z \) and \( Y \) are compatible. Therefore, \( Z_1 \subset Y_1 \), or \( Y_1 \subset Z_1 \), or \( Z_1 \subset Y_2 \), or \( Y_2 \subset Z_1 \). Since \( Z'_1 = \text{cl}_{M_1}(Z_1) \) and \( Y'_1 = \text{cl}_{M_1}(Y_1) \), the first two containments imply that \( Z' \) and \( Y' \) are compatible; a contradiction. Hence,
either $Z_1 \subset Y_2$ or $Y_2 \subset Z_1$. If $Z_1 \subset Y_2$, then

$$Z'_1 = \text{cl}_{M_i}(Z_1)$$

$$\subseteq \text{cl}_{M_i}(X_1 \cap Y_2)$$

$$\subseteq Y'_2.$$ 

This contradicts the assumption that $Z'$ and $Y''$ cross. Since $Y_2 \supset X_2 \supset \delta(X)$ and $Z_1 \cap X_2 \subset \delta(X)$, the final containment cannot hold. Therefore, $Y'$ is not crossed by $Z'$; contradicting our assumption. □

For a good split $Y$ of $M$ and a decomposition $D$ of $M$, repeated application of (3.2.16) will either produce a good split of one of the members of $D$, or $Y$ will be redundant; that is, $Y$ will not produce a strict refinement of $D$. If $Y$ produces a good split $Y'$ of one of the members of $D$, then $Y$ is said to induce $Y'$. A good split $Y$ may be redundant relative to a decomposition $D$ if it has already been used in the production of $D$. The next result shows that two distinct splits may induce the same split. Hence, it provides another source of redundancy.

(3.2.17) Let $X = \{X_1, X_2\}$ be a split of $M$, and let $\{M_1, M_2\}$ be the simple decomposition of $M$ generated by $X$. Suppose that $X$ is a good split of $M$ and $Y' = \{Y'_1, Y'_2\}$ is a good split of $M_1$ such that $\gamma(X) \subseteq Y'_2$. Either $Y = \{Y_1, Y_2\}$ is a good split of $M$, where $Y_1 = Y'_1 \cap E(M)$ and $Y_2 = (Y'_2 \cup X_2) \cap E(M)$; or $\gamma(X) \subseteq Y'_1$ and $W = \{W_1, W_2\}$ is a good split of $M$, where $W_1 = Y'_2 \cap E(M)$ and $W_2 = (Y'_1 \cup X_2) \cap E(M)$.

Proof. Suppose that $Y = \{Y_1, Y_2\}$ is not a good split of $M$, where $Y_1 = Y'_1 \cap E(M)$ and $Y_2 = (Y'_2 \cup X_2) \cap E(M)$. By (3.2.15), $Y$ is a split of $M$. Let $Z = \{Z_1, Z_2\}$ be

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a split of $M$ which crosses $Y$. Since $X$ is a good split, we may assume that either $X_1 \subset Z_1$ or $X_2 \subset Z_2$. Since $Y_1 \subset X_2$, the first containment implies that $Y$ and $Z$ are compatible. Therefore, we may assume that $X_2 \subset Z_2$. By (3.2.14), there is a split $Z' = \{Z_1', Z_2'\}$ of $M_1$ such that $Z_1' = cl_{M_1}(Z_1)$ and $Z_2' = (X_1 \cap Z_2) \cup \gamma(X)$. Since $Y'$ is a good split, either $Y_1' \subset Z_1'$, or $Y_1' \subset Y_1'$, or $Y_1' \subset Z_2'$, or $Z_2' \subset Y_1'$. The containments $Y_1' \subset Z_1'$, and $Z_1' \subset Y_1'$, and $Y_1' \subset Z_2'$, imply, respectively, that $Y_1 \subset Z_1$, and $Z_1 \subset Y_1$, and $Y_1 \subset Z_2$. In any of these cases, $Z$ and $Y$ are compatible. Therefore, we may assume that $Z_2' \subset Y_1'$, and hence, $\gamma(X) \subset Y_1'$. By (3.2.15), there is a split $W = \{W_1, W_2\}$ of $M$, where $W_1 = Y_2' \cap E(M)$ and $W_2 = (Y_1' \cup X_2) \cap E(M)$. Suppose $W$ is crossed by a split $U = \{U_1, U_2\}$ of $M$. Since $X$ is a good split, we may assume that either $X_1 \subset U_1$ or $X_2 \subset U_2$. Since $W_1 \subset X_1$, the first containment implies that $W$ and $U$ are compatible. Therefore, we may assume that $X_2 \subset U_2$. By (3.2.14), there is a split $U' = \{U_1', U_2'\}$ of $M_1$ such that $U_1' = cl_{M_1}(U_1)$ and $U_2' = (X_1 \cap U_2) \cap \gamma(X)$. Since $Y'$ is a good split, either $Y_2' \subset U_1'$, or $U_1' \subset Y_2'$, or $Y_2' \subset U_2'$, or $U_2' \subset Y_2'$. The first three cases imply that $W_1 \subset U_1$, or $U_1 \subset W_1$, or $W_1 \subset U_2$, respectively. These contradict the assumption that $U$ and $W$ cross. Therefore, $U_2' \subset Y_2'$. Let $S_1' = U_1' \cap Z_1'$. Observe that, $r(S_1') \leq r(U_1') + r(Z_1') - r(U_1' \cup Z_1')$. Since $U_1' \supset Y_1'$ and $Z_1' \supset Y_2'$, we have that $r(U_1' \cup Z_1') = r(M_1)$. Hence,

$$r(S_1') \leq r(U_1') + r(Z_1') - r(M_1).$$

Let $S_2' = cl_{M_1}[E(M_1) - (U_1' \cap Z_1')]$. Observe that $r(S_2') \leq r(U_2' \cup Z_2')$ and that
\[ U'_2 \cap Z'_2 = \gamma(X) \]. Hence,

\[ r(S'_2) \leq r(U'_2) + r(Z'_2) - 2. \]

Therefore,

\[
r(S'_1) + r(S'_2) - r(M_1) \\
\leq [r(U'_1) + r(Z'_1) - r(M_1)] + [r(U'_2) + r(Z'_2) - 2] - r(M_1) \\
= [r(U'_1) + r(U'_2)] + [r(Z'_1) + r(Z'_2)] - 2r(M_1) - 2 \\
= 2[r(M_1) + 2] - 2r(M_1) - 2 \\
= 2.
\]

Since \( U'_1 \supset Y'_1 \), there is an element \( e \in U'_1 - Y'_1 \). Note that \( e \in Y'_2 - \gamma(X) \). Hence \( e \in Z'_1 - \gamma(X) \). Therefore \((U'_1 \cap Z'_1) - \gamma(X) \neq \emptyset \). Since \( \gamma(X) \) is a rank-two flat contained in \( U'_1 \cap Z'_1 \), we have that \( r(S'_1) = r(U'_1 \cap Z'_1) \geq 3 \). Moreover, since \( r(U'_1) < r(M_1) \), we have that \( r(S'_1) = r(U'_1 \cap Z'_1) < r(M_1) \). By (3.2.3), \( S' = \{S'_1, S'_2\} \) is a split of \( M_1 \). Observe that \( S' \) crosses \( Y' \); a contradiction. Therefore, either \( Y \) or \( W \) is a good split of \( M \). □

A decomposition sequence \( \mathcal{D} = (D_0, D_1, \ldots, D_n) \), \( n \geq 0 \), is a sequence of decompositions where \( D_0 = \{M\} \), and \( D_i \) is a simple refinement of \( D_{i-1} \), for \( 1 \leq i \leq n \). By (3.2.17), to each simple refinement of a decomposition, there corresponds at least one good split of \( M \). We shall say that a decomposition sequence is generated by a corresponding sequence of good splits of \( M \). But, in general, a decomposition sequence may be generated by more than one sequence of good splits. However, the
following result shows that a sequence of good splits generates only one decomposition sequence.

\[ (3.2.18) \quad \text{A sequence of good splits of } M \text{ generates a unique decomposition sequence.} \]

**Proof.** Suppose that the decomposition sequences \( \mathcal{D} \) and \( \mathcal{D}' \) of \( M \) are distinct, where \( \mathcal{D} = (D_0, D_1, \ldots, D_n) \) and \( \mathcal{D}' = (D_0', D_1', \ldots, D_m') \). Let \( k = \min\{i : D_i \neq D_i'\} \). Note that \( k \geq 1 \). Hence, \( D_k \) and \( D_k' \) are different simple refinements of the same decomposition \( D_{k-1} \). Suppose that \( D_k \) and \( D_k' \) result from good splits of different members \( G \) and \( H \), respectively, of the decomposition \( D_{k-1} \). Let \( D_k \) result from a good split \( X = \{X_1, X_2\} \) of \( G \) and \( D_k' \) result from a good split \( Y = \{Y_1, Y_2\} \) of \( H \). We may assume that \( \{X_1 \cap E(M), cl_M(E(M) - X_1)\} \) and \( \{Y_1 \cap E(M), cl_M(E(M) - Y_1)\} \) are the corresponding good splits of \( M \), where \( X_1 \subseteq E(G) \) and \( Y_1 \subseteq E(H) \). Since the intersection of \( E(H) \) and \( E(G) \) has rank at most two in \( M \), and the ranks of \( X_1 \cap E(M) \) and \( Y_1 \cap E(M) \) are at least three, \( X_1 \cap E(M) \neq Y_1 \cap E(M) \). It remains to show that \( cl_M(E(M) - X_1) \) does not equal \( Y_1 \cap E(M) \). This is clear if \( D_{k-1} \) has a third member. So suppose that \( k - 1 = 1 \) and \( D_1 = \{G, H\} \). The set \( cl_M(E(M) - X_1) \) intersects \( E(G) \) in a set having rank at least three. Therefore, \( cl_M(E(M) - X_1) - E(H) \) is non-empty. Hence, \( cl_M(E(M) - X_1) \) does not equal \( Y_1 \cap E(M) \). Thus, \( D_k \) and \( D_k' \) are generated by different splits. Suppose that \( D_k \) and \( D_k' \) result from different splits of the same member of the decomposition \( D_{k-1} \).

By (3.2.17), these good splits are induced by different good splits of \( M \). \( \square \)

Before we state and prove a generalization of (3.2.18), which is needed to prove Theorem 3.2.8, we prove the following lemma.
Suppose that $X = \{X_1, X_2\}$ and $Y = \{Y_1, Y_2\}$ are compatible splits of $M$, where $Y_2 \supset X_2$. Let $\{M_1, M_2\}$ be the simple decomposition of $M$ generated by $X$ and let $Y'$ be the split of $M_1$ induced by $Y$. Further, suppose that $\{N_1, N_2\}$ is the simple decomposition of $M_1$ generated by the split $Y'$. Then $E(N_2) = (X_1 \cap Y_2) \cup \gamma(Y) \cup \gamma(X)$.

Proof. By definition, $E(N_2) = Y'_2 \cup \gamma(Y')$. By (3.2.13), $\gamma(Y') = \gamma(Y)$. By (3.2.14), $Y'_2 = (X_1 \cap Y_2) \cup \gamma(X)$. This completes the proof. □

We say that a set of good splits $S$ generates a decomposition $D$ if there is a sequence involving every member of $S$, and only members of $S$, which generates a decomposition sequence for which $D$ is the last member. Just as a sequence of good splits generates a unique decomposition sequence, we have the following generalization.

A set of good splits of $M$ generates a unique decomposition.

The following lemma is the key to the proof of (3.2.20).

Suppose that $X = \{X_1, X_2\}$ and $Y = \{Y_1, Y_2\}$ are good splits of $M$, where $Y_2 \supset X_2$. Then the decomposition generated by $(X,Y)$ equals the decomposition generated by $(Y,X)$.

Rather than prove (3.2.21) directly, we prove a slightly more general result that will be useful in the next chapter. The above result (3.2.21) follows from (3.2.22) by a direct application of (3.2.16).
(3.2.22) Suppose that $X = \{X_1, X_2\}$ is a split of $M$ which generates the decomposition $\{M_1, M_2\}$ of $M$. Further, suppose that $Y' = \{Y'_1, Y'_2\}$ is a split of $M_1$ where $Y'_2 \supset \delta(X)$, and $Y'$ generates the decomposition $\{N_1, N_2\}$ of $M_1$. Finally, suppose that the split $Y = \{Y_1, Y_2\}$ of $M$, where $Y_1 = Y'_1 \cap E(M)$ and $Y_2 = (Y'_2 \cap E(M)) \cup X_2$, generates the decomposition $\{G_1, G_2\}$ of $M$. Then, $G_1 = N_1$ and $G_2$ has the decomposition $\{N_2, M_2\}$.

Proof. Let $\{M_1, M_2\}$ be the decomposition of $M$ generated by $X$ and let $\{G_1, G_2\}$ be the decomposition of $M$ generated by $Y$. By (3.2.16), $Y' = \{c_{M_1}(Y'_1), (X_1 \cap Y_2) \cup \gamma(X)\}$ is a good split of $M_1$. Suppose that $\{N_1, N_2\}$ is the decomposition of $M_1$ generated by $Y'$. By (3.2.14), $X' = \{c_{G_2}(X_2), (Y_2 \cap X_1) \cup \gamma(Y)\}$ is a split of $G_2$. Suppose that $\{H_1, H_2\}$ is the decomposition of $G_2$ generated by $X'$. Suppose that $x \in c_{M_1}(Y_1) - Y_1$. Then $x \in \gamma(X) - Y_1$. Since $c_{P}(Y_2) \supset \gamma(X)$, we have that $x \in \gamma(Y)$. Therefore, $y_1 \cup \gamma(Y) \supset c_{M_1}(Y_1)$. Hence,

\[
E(N_1) = c_{M_1}(Y_1) \cup \gamma(Y') \\
= c_{M_1}(Y) \cup \gamma(Y) \\
= Y_1 \cup \gamma(Y) \\
= E(G_1).
\]

Similarly, $E(H_2) = E(M_2)$. Moreover, by applying (3.2.19), we get that

\[
E(N_2) = (X_1 \cap Y_2) \cup \gamma(Y) \cup \gamma(X) \\
= E(H_1). \square
\]
Proof of (3.2.20). Suppose that \( S = (Z^1, Z^2, \ldots, Z^i, X, Y) \) is a sequence of good splits of \( M \) and that \( D \) is the decomposition generated by \( S \). Suppose that \( (\{M\}, \ldots, \{M^1, M^2, \ldots, M^i\}) \) is the decomposition sequence associated with the sequence \( (Z^1, Z^2, \ldots, Z^i) \). Observe that if \( X \) and \( Y \) induce splits on different members of \( \{M^1, M^2, \ldots, M^i\} \), or if \( X \) or \( Y \) is redundant relative to \( \{M^1, M^2, \ldots, M^i\} \), then \( (Z^1, Z^2, \ldots, Z^i, Y, X) \) also generates \( D \). Therefore, we may assume that \( X \) induces a split \( X' \) of some member \( M' \) of \( \{M^1, M^2, \ldots, M^i\} \) and that \( Y \) also induces a split \( Y' \) of \( M' \). If \( Y' = X' \), then, again, \( (Z^1, Z^2, \ldots, Z^i, Y, X) \) generates \( D \). Therefore, (3.2.21) applies to the splits \( X' \) and \( Y' \) of \( M' \). That result implies that the decomposition generated by \( (Z^1, Z^2, \ldots, Z^i, X, Y) \) is the same as that generated by \( (Z^1, Z^2, \ldots, Z^i, Y, X) \). Therefore, a transposition in any sequence of good splits does not change the final entry in the decomposition sequence. Since any permutation results from a sequence of transpositions, the final entry in a decomposition sequence is independent of the order of the sequence of good splits. This completes the proof. \( \Box \)

Proof of Theorem 3.2.8. Let \( M \) be a 3-connected binary matroid, let \( D \) be a minimal decomposition of \( M \), and let \( \Sigma = \{X : X \in S \text{ and } S \text{ generates } D\} \); that is, a good split \( X \) of \( M \) belongs to \( \Sigma \) if \( X \) belongs to some set of good splits \( S \) which generates \( D \). Suppose that \( Y \) is a good split of \( M \) and that \( (X^1, X^2, \ldots, X^k) \) is a sequence of good splits generating \( D \). By repeatedly applying (3.2.16), we get that either \( Y \) induces a good split of some member of \( D \) or, at some stage, \( Y \) is redundant. In the latter case, \( Y \) may be added to the sequence without changing
the decomposition sequence. Hence, either $Y \in \Sigma$ or $D$ is not minimal. Therefore, $\Sigma = \{X : X$ is a good split of $M\}$ and every minimal decomposition is generated by the set of all good splits. The theorem now follows from (3.2.20). $\square$

3.3 On a Decomposition for 3-Connected Graphs

In this section we prove that the decomposition of 3-connected graphs presented by Coullard, Gardner, and Wagner (1993) is a special case of our decomposition for 3-connected binary matroids. This is stated more precisely in Theorem 3.3.1, where we assume that the decomposition of the graph $G$ is a result of the theory of Coullard, Gardner, and Wagner and that the decomposition of the binary matroid $M^*(G)$ is given by the theory developed in the previous sections. Note that, for a graph $G$, the matroid $M^*(G)$ is the dual of the cycle matroid $M(G)$.

(3.3.1) Theorem. Suppose that the 3-connected graph $G$ has the minimal decomposition $\{G_1, G_2, \ldots, G_n\}$. Then the binary matroid $M^*(G)$ has the decomposition $\{M^*(G_1), M^*(G_2), \ldots, M^*(G_n)\}$.

Of course, before we can understand this result, much less prove it, we need to define the decomposition of Coullard, Gardner, and Wagner. Recall that for a subset $A$ of the edge set $E$ of a graph $G$, the graph $G[A]$ is the subgraph of $G$ induced by $A$. For a subset $U$ of the vertex set $V$ of the graph $G$, the neighborhood of $U$ in $G$ is denoted $N_G(U)$. If $U = \{v\}$, we denote $N_G(U)$ by $N_G(v)$. For an edge $e$ with ends $u$ and $v$, we will sometimes denote $e$ by $uv$.

For a positive integer $k$, a partition $\{E_1, E_2\}$ of $E(G)$ is a $k$-separation if $\min\{|E_1|, |E_2|\} \geq k$ and $|V(G[E_1]) \cap V(G[E_2])| \leq k$. A graph $G$ is $n$-connected if
it has no \( k \)-separation for any \( k < n \). Menger's Theorem states that if \( G \) has at least \( n + 1 \) vertices, then \( G \) is \( n \)-connected if and only if every pair of vertices are joined by at least \( n \) internally disjoint paths. A \( k \)-separation is \textit{cyclic} if neither \( G[E_1] \) nor \( G[E_2] \) is a triad; that is, neither \( G[E_1] \) nor \( G[E_2] \) is the graph consisting of three edges having precisely one common vertex. Suppose that a vertex \( v \) of \( G \) has neighborhood set \( N(v) \) and \( \{N_1, N_2\} \) is a partition of \( N(v) \). We construct a new graph \( G' \) from \( G \) by \textit{splitting} the vertex \( v \) into vertices \( v_1 \) and \( v_2 \). The graph \( G' \) is constructed by letting \( V(G') = (V(G) - \{v\}) \cup \{v_1, v_2\} \) and \( E(G') = (E(G) - \{uv : u \in N(v)\}) \cup \{uv_1 : u \in N_1\} \cup \{uv_2 : u \in N_2\} \cup \{v_1v_2\} \).

The next result relates the 3-connectivity for graphs and their cycle matroids. It is a standard result that can be proved using Menger's Theorem (see, for example, Oxley 1992, Corollary 8.2.8).

\textbf{(3.3.2)} \textit{If \( G \) is a simple graph without isolated vertices and \( |V(G)| \geq 3 \) and \( G \neq K_3 \), then \( M(G) \) is 3-connected if and only if \( G \) is 3-connected.}

The proof of (3.3.2) given in Oxley contains the following result.

\textbf{(3.3.3)} \textit{A partition \( \{E_1, E_2\} \) of \( E(G) \) is a 3-separation of a 3-connected graph \( G \) if and only if \( \{E_1, E_2\} \) is a 3-separation of \( M^*(G) \).}

By the definition of a cyclic 3-separation, a graph \( G \) has a cyclic 3-separation if and only if \( M^*(G) \) has a vertical 3-separation. Suppose that \( G \) is a 3-connected graph with a cyclic 3-separation \( \{E_1, E_2\} \). Let \( A_3 \) be the set of edges that are incident, in \( G[E_1] \) or \( G[E_2] \), to a degree-one vertex of either \( G[E_1] \) or \( G[E_2] \). Let \( A_i = E_i - A_3 \).
for each $i \in \{1, 2\}$. If $\{C_1, C_2\}$ is a partition of $A_3$, then $\{A_1 \cup C_1, A_2 \cup C_2\}$ is a 3-separation for $G$. Indeed, the 3-separation $\{E_1, E_2\}$ is of this form. The ordered triple $A = \{A_1, A_2; A_3\}$ is called a split of $G$. By a straightforward argument with ranks, it is not difficult to see that $A_3 \subseteq \text{cl}_{M^*(G)}(A_i)$ for each $i \in \{1, 2\}$. Therefore, we have the following result.

(3.3.4) The graph $G$ has a split $\{A_1, A_2; A_3\}$ if and only if $M^*(G)$ has a split $\{X_1, X_2\}$, where $X_1 = A_1 \cup A_3$ and $X_2 = A_2 \cup A_3$.

The vertices in the set $V(G[A_1]) \cap V(G[A_2])$ together with the edges of $A_3$ are called the connections of $A$ (or the $A$-connections). The set of $A$-connections is denoted $C(A)$. A split $A$ has exactly three $A$-connections. Moreover, each edge in $A_3$ has one end in $V(G[A_1]) - V(G[A_2])$ and the other in $V(G[A_2]) - V(G[A_1])$.

Coullard, Gardner, and Wagner, defined a decomposition relative to a cyclic 3-separation, and they observed that a split $A = \{A_1, A_2; A_3\}$ of a graph $G$ generates a simple decomposition $\{G_1, G_2\}$ as follows. The graph $G[A_1 \cup A_3]$ has three vertices $x, y$, and $z$ in $V(G[A_1] \cup A_3) \cap V(G[A_2])$. To the graph $G[A_1 \cup A_3]$, add the vertex $t$ and the three edges $xt$, $yt$, and $zt$. Label these edges $e, f,$ and $g$, respectively. Let $H_1$ be the resulting graph. If $a \in \{e, f, g\}$ and $a$ is adjacent to an edge in $A_3$, then contract $a$. The resulting graph is the component $G_1$; that is,

$$G_1 = H_1/\{a \in \{e, f, g\} : a \text{ is adjacent to an edge in } A_3\}.$$  

The component $G_2$ is similarly defined. Note that if $p \in A_3$ and $G/p$ is 3-connected, then $G/p$ has the split $\{A_1, A_2; A_3 - \{p\}\}$ that generates the simple decomposition.
\{G_1, G_2\}. For the proof of Theorem 3.2.8, we will use a construction that is different but equivalent to the above construction. First we will describe a construction having three steps (A1)–(A3) and prove that the resulting graph is \(G_1\). Then we will define the desired construction with two steps (B1) and (B2), and prove that this two-step process results in the same graph as does (A1)–(A3).

Our intermediate construction is defined as follows.

(A1) Construct a new graph \(H_1^{(1)}\) from \(G\) by splitting each of the vertices \(\{x, y, z\}\) of \(G[A_1 \cup A_3]\) determined by the partition \(\{V(G[A_1 \cup A_3]) \cap N(x), V(G[A_2]) \cap N(x)\}\) of \(N(x)\) so that, for example, \(x\) is split into vertices labeled \(x_1\) and \(x_2\). Label the edges \(xx_2, yy_2, \) and \(zz_2\) by \(e, f, \) and \(g,\) respectively.

(A2) Contract from \(H_1^{(1)}\) the edges in \(A_2\). Label this graph \(H_1^{(2)}\).

(A3) Contract from \(H_1^{(2)}\) the edges created by splitting those vertices of \(\{x, y, z\} - C(A)\).

In order to show that this construction is the same as the original, we need to show that \(H_1^{(2)}/(\{x, y, z\} - C(A)) = G_1\). The first step in this process is to show that \(H_1[A_1 \cup A_3] = G[A_1 \cup A_3] = H_1^{(1)}[A_1 \cup A_3] = H_1^{(2)}[A_1 \cup A_3]\). By the construction, it is clear that \(H_1[A_1 \cup A_3] = G[A_1 \cup A_3]\). Since \(H_1^{(1)}\) results by splitting vertices that have degree at most one in \(G[A_1 \cup A_3]\), the graphs \(H_1^{(1)}[A_1 \cup A_3]\) and \(G[A_1 \cup A_3]\) are equal. Since \(H_1^{(2)}[A_1 \cup A_3]\) results from \(H_1^{(1)}[A_1 \cup A_3]\) by contracting edges that are not joined to any vertex in \(H_1^{(1)}[A_1 \cup A_3]\), we have that \(H_1^{(2)}[A_1 \cup A_3] = H_1^{(1)}[A_1 \cup A_3]\). Besides the edges \(A_1 \cup A_2\), the graph \(H_1\) has three edges \(e, f, \) and \(g;\) and these
edges are adjacent to \( x, y, \) and \( z, \) respectively. This is also the case for the graph \( H_1^{(2)} \). Other than \( x, y, \) and \( z, \) each of the edges \( e, f, \) and \( g, \) is adjacent to only one other vertex, say \( t. \) To see this, note that Menger's Theorem implies that \( G[A_2] \) is connected. Therefore \( H_1 = H_1^{(2)} \). Observe that the effect of (A3) on \( H_1^{(2)} \) is the same as the procedure that creates \( G_1 \) from \( H_1 \). Therefore, (A1)--(A3) creates the graph \( G_1. \)

Now we define a new, two-step construction for creating the component \( G_1. \)

(B1) Split each of the vertices of \( C(A) \) as in (A1).

(B2) Contract the edges of \( A_2. \)

Observe that the difference between (A1) and (B1) is that, in (A1) the vertices in \( \{x, y, z\} - C(A) \) are also split. But the edges created by splitting \( \{x, y, z\} - C(A) \) in (A1) are then contracted in (A3). Note that (A2) and (B2) are identical. In order to show that (A1)--(A3) and (B1)--(B2) create the same graph, it is sufficient to show that, in general, the three-step process of

(i) splitting a vertex, thus creating an edge \( e; \) then

(ii) contracting a set of edges \( A \) disjoint from \( e; \) and then

(iii) contracting \( e, \)

results in the same graph as would be produced by simply contracting the set of edges \( A. \) Since order of contraction does not matter, steps (ii) and (iii) can be interchanged. Since splitting a vertex and then contracting the resulting edge amounts
to no change at all, performing steps (i)–(iii) achieves the same result as performing step (ii) alone. Hence, applying (B1) and (B2) produces the component $G_1$. To construct the component $G_2$, we perform (B1) and then contract the edges $A_i$ from the resulting graph.

Before we use the above construction to prove Theorem 3.2.8, we need a few more results about the decomposition of 3-connected binary matroids. In addition, we show how the decomposition of 3-connected binary matroids relates to generalized parallel connections (Brylawski 1975) and three-sums (Seymour 1980). In addition to the insight gained by relating the present work to these other constructions, it will allow us to use results associated with these constructions.

(3.3.5) Suppose that $X = \{X_1, X_2\}$ is a split of $M$ and that $Y_i \subseteq cl_P(X_i) - X_i$ for each $i \in \{1, 2\}$. Further, suppose that $Y = Y_1 \cup Y_2$. Let $M' = P|(E(M) \cup Y)$ and $X_i = X_i \cup Y_i$ for each $i \in \{1, 2\}$. Then $M'$ is a 3-connected binary matroid and $X' = \{X_1', X_2'\}$ is a split of $M'$, where $\gamma(X') = \gamma(X)$.

Proof. Since $M'$ is a restriction of $P$, it is binary. Suppose that $M'$ is not 2-connected. Let $\{Z_1', Z_2'\}$ be a 1-separation of $M'$ and let $Z_1 = Z_1' - Y$ and $Z_2 = Z_2' - Y$. Since $r(Z_1) + r(Z_2) \leq r(Z_1') + r(Z_2') = r(M') = r(M)$, we may assume that $Z_1$ is empty. Therefore $Z_1' \subseteq Y$. Since $r(M' \setminus Y) = r(M')$, we have that $r(Z_2') = r(M')$. Since $M'$ is simple, this contradicts the assumption that $\{Z_1', Z_2'\}$ is a 1-separation of $M'$. Therefore $M'$ is 2-connected.
Suppose that $M'$ is not 3-connected. Let $\{Z_1', Z_2'\}$ be a 2-separation of $M'$ and let $Z_1 = Z_1' - Y$ and $Z_2 = Z_2' - Y$. Observe that

\[ r(Z_1) + r(Z_2) \leq r(Z_1') + r(Z_2') = r(M') + 1 = r(M) + 1. \]

Suppose that $r(Z_1) + r(Z_2) = r(M)$. Since $M$ is connected, we may assume that $Z_1$ is empty. Hence, $r(Z_2') = r(M')$ and $r(Z_1') = 1$. Since $M'$ is simple, $\{Z_1', Z_2'\}$ is not a 2-separation of $M'$. Therefore, we may assume that $r(Z_1') + r(Z_2') = r(M') + 1$. Since $M$ is 2-connected, we may assume that $|Z_1| = 1$. Therefore, $Z_1' \neq Z_1$ and, hence, $r(Z_1') > r(Z_1)$. This contradicts the assumption that $r(Z_1') + r(Z_2') = r(M') + 1$.

Therefore $M'$ is 3-connected.

Since $r(X_i') = r(X_i)$ for each $i \in \{1, 2\}$, we have that $r(X_1') + r(X_2') = r(M') + 2$. Since $X$ is a split of $M$, $r(X_i') = r(X_i) \geq 3$, for each $i \in \{1, 2\}$. Observe that $X_1'$ and $X_2'$ are closed in $M'$. Hence, (3.2.3) implies that $\{X_1', X_2'\}$ is a split of $M'$. Moreover,

\[ \gamma(X') = \text{cl}_p(X_1') \cap \text{cl}_p(X_2') = \text{cl}_p(X_1) \cap \text{cl}_p(X_2) = \gamma(X). \]

This completes the proof. □

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(3.3.6) Suppose that $X = \{X_1, X_2\}$ is a split of $M$ and that $N = M \setminus \gamma'$ is a 3-connected matroid where $\gamma' \subseteq \gamma(X)$. Then $X' = \{X'_1, X'_2\}$ is a split of $N$, where $X'_1 = X_1 - \gamma'$ and $X'_2 = X_2 - \gamma'$. Moreover, $\gamma(X') = \gamma(X)$.

Proof. Observe that $X'_1$ and $X'_2$ are closed sets in $N$. By (3.2.5), we have that $r(X'_i) = r(X_i)$ for each $i \in \{1, 2\}$. Therefore,

\[
\begin{align*}
r(X'_1) + r(X'_2) &= r(X_1) + r(X_2) \\
&= r(M) + 2 \\
&= r(N) + 2.
\end{align*}
\]

By (3.2.3), $X' = \{X'_1, X'_2\}$ is a split of $N$. Since $cl_p(X'_i) = cl_p(X_i)$ for each $i \in \{1, 2\}$, it follows that $\gamma(X') = \gamma(X)$. □ The following result is an immediate corollary to (3.3.6).

(3.3.7) Suppose that $X = \{X_1, X_2\}$ is a split of $M$ generating the decomposition $\{M_1, M_2\}$ and that $N = M \setminus \gamma'$ is a 3-connected matroid where $\gamma' \subseteq \gamma(X)$. Then $\{X_1 - \gamma', X_2 - \gamma'\}$ generates the decomposition $\{M_1, M_2\}$ of $N$.

Suppose that $M_1$ and $M_2$ are matroids and $X_1$ and $X_2$ are subsets of $E(M_1)$ and $E(M_2)$, respectively, where $M_1|X_1 \cong M_2|X_2$. Identify $X_1$ and $X_2$ by an isomorphism and label this set $X$. If $X$ is a modular flat of $M_1$, then the generalized parallel connection $P_X(M_1, M_2)$ exists and is defined as the matroid with ground set $E(M_1) \cup E(M_2)$ (where $X = E(M_1) \cap E(M_2)$), whose closed sets are all sets of the form $C_1 \cup C_2$, where $C_1$ is closed in $M_1$ and $C_2$ is closed in $M_2$. In subsequent chapters we will want to specify the sets $X_1$ and $X_2$ that are identified, and will,
on occasion, denote the generalized parallel connection by \( P_X((M_1, X_1), (M_2, X_2)) \).

Note that the resulting matroid depends on the isomorphism used to identify \( X_1 \) and \( X_2 \).

It is not difficult to show, arguing directly, that if \( M \) has the decomposition \( \{M_1, M_2\} \) generated by the split \( X \) and \( \delta(X) = \gamma(X) \), then \( M = P_{\gamma(X)}(M_1, M_2) \).

Therefore, in general, if \( M \) has the decomposition \( \{M_1, M_2\} \) generated by the split \( X \), then \( M = P_{\gamma(X)}(M_1, M_2) \setminus (\gamma(X) - \delta(X)) \). Given two binary matroids \( N_1 \) and \( N_2 \), each having at least seven elements and a common triangle \( T \) which does not contain a cocircuit in either matroid, the three-sum \( N_1 \oplus_3 N_2 \) of Seymour is defined as \( P_T(N_1, N_2) \setminus T \). Therefore, if \( M \) has the decomposition \( \{M_1, M_2\} \) generated by the split \( X \), and \( \delta(X) = \emptyset \) and both \( M_1 \) and \( M_2 \) have at least seven elements, then \( M = M_1 \oplus_3 M_2 \).

The next result is a special case of a more general result on generalized parallel connections (Brylawski 1975, Proposition 5.11; see also Oxley 1992, Proposition 12.4.14).

(3.3.8) Suppose that \( M = P_T(M_1, M_2) \), where \( M, M_1, \) and \( M_2 \) are binary matroids and \( T \) is a triangle. Then

(i) \( M \setminus e = P_T(M_1 \setminus e, M_2) \) for all \( e \in E(M_1) - T \);

(ii) \( M \setminus e = P_T(M_1, M_2 \setminus e) \) for all \( e \in E(M_2) - T \);

(iii) \( M / e = P_T(M_1 / e, M_2) \) for all \( e \in E(M_1) - T \);

(iv) \( M / e = P_T(M_1, M_2 / e) \) for all \( e \in E(M_2) - T \); and
(v) \( M/e = P_{T/e}(M_1/e, M_2/e) \) for all \( e \in T \).

Moreover, in case (v), the simplification of \( M/e \) is a minor of the parallel connection \( P(M_1/e, M_2/e) \).

The next result is from Tutte (1959; see also Oxley 1992, Theorem 6.6.5).

(3.3.9) \( \) A binary matroid is cographic if and only if it has no minor isomorphic to any of the matroids \( F_7, F_7', M(K_5), \) and \( M(K_{3,3}) \).

We will use the properties of generalized parallel connections given by (3.3.8) and the above result of Tutte to establish our next result.

(3.3.10) \( \) Suppose that \( M_1 \) and \( M_2 \) are cographic matroids and that there is a triangle \( T \) such that \( P_T(M_1, M_2) \) exists. If \( P_T(M_1, M_2) \backslash T \) is cographic, then \( P_T(M_1, M_2) \) is cographic.

Proof. Suppose that \( P_T(M_1, M_2) \) is a smallest counterexample. By (3.3.9), the matroid \( P_T(M_1, M_2) \) has a minor isomorphic to one of \( F_7, F_7', M(K_5), \) or \( M(K_{3,3}) \), but none of the matroids \( M_1, M_2, \) and \( P_T(M_1, M_2) \backslash T \) has such a minor. Suppose that \( e \in E(M_1) - T \). Then \( P_T(M_1, M_2) \backslash e = P_T(M_1/e, M_2) \) and \( P_T(M_1, M_2) / e = P_T(M_1/e, M_2) \). The minimality of \( P_T(M_1, M_2) \) implies that both \( P_T(M_1/e, M_2) \) and \( P_T(M_1/e, M_2/e) \) are cographic. The same conclusion can be drawn if \( e \in E(M_2) - T \). Suppose that \( e \in T \). Then \( P_T(M_1, M_2) / e = P_{T/e}(M_1/e, M_2/e) \). Observe that the matroids \( F_7, F_7', M(K_5), \) and \( M(K_{3,3}) \) are simple and the simplification of \( P_T(M_1, M_2) / e \) is a minor of the parallel connection \( P(M_1/e, M_2/e) \). Since
the parallel connection of cographic matroids is certainly cographic, we have that
$P_T(M_1, M_2)/e$ is cographic.

We conclude that the only non-cographic minors of $P_T(M_1, M_2)$ are of the
form $P_T(M_1, M_2)\setminus S$ where $S \subset T$. Therefore, we may assume that $P_T(M_1, M_2)\setminus S$ is
isomorphic to either $F_7$, $F_7^*$, $M(K_5)$, or $M(K_{3,3})$ for some proper subset $S$ of $T$. Since
the matroids $F_7$, $F_7^*$, $M(K_5)$, and $M(K_{3,3})$ are 3-connected and, by assumption,
neither $M_1$ nor $M_2$ is isomorphic to any of these matroids, one of these matroids
must have a decomposition $\{M_1, M_2\}$ generated by a split $X$, where $\delta(X)$ is non-
empty. The matroids $F_7$ and $M(K_5)$ have no splits and the only splits of $F_7^*$ and
$M(K_{3,3})$ have empty connections. This completes the proof. □

The next two results are the key lemmas in the proof of Theorem 3.2.8.

(3.3.11) Lemma. Suppose that the graph $G$ has a split $A = \{A_1, A_2; A_3\}$ and that
the matroid $M^*(G)$ has the corresponding split $X = \{X_1, X_2\}$. The split $A$ generates
the decomposition $\{G_1, G_2\}$ of $G$ if and only if the split $X$ generates the decomposi-
tion $\{M^*(G_1), M^*(G_2)\}$ of $M^*(G)$.

Proof. First we show that if $G$ has the decomposition $\{G_1, G_2\}$ generated by $A$,
then $M^*(G)$ has the decomposition $\{M^*(G_1), M^*(G_2)\}$ generated by $X$. Suppose
that $G$ is a counterexample such that $3 - |A_3|$ is minimum. By (3.3.4), $X_1 = A_1 \cup A_3$ and $X_2 = A_2 \cup A_3$. Suppose that $C(A)$ consists of edges only. Then
$|A_3| = 3$ and $G_1 = G/A_2$ and $G_2 = G/A_1$. Since $|A_3| = 3$, $A_3 = \gamma(X)$. Therefore,
$X$ generates the decomposition $\{M^*(G)\setminus A_2, M^*(G)\setminus A_1\}$. The result holds since
$M^*(G_1) = M^*(G/A_2) = M^*(G)\setminus A_2$, and, likewise, $M^*(G_2) = M^*(G)\setminus A_1$. Now
assume that $C(A)$ has at least one vertex, say $x$. Then $|A_3| \leq 2$. Split the vertex $x$ according to (B1), creating the edge $e$. Denote the resulting graph by $G'$. Since $G'$ has the same decomposition $\{G_1, G_2\}$ as does $G$, the minimality of $3 - |A_3|$ implies that $M^*(G')$ has the decomposition $\{M^*(G_1), M^*(G_2)\}$. By (3.3.7), $M^*(G)$ also has the decomposition $\{M^*(G_1), M^*(G_2)\}$ generated by the split $X$.

Next, we prove the converse, that if $M = M^*(G)$ has the decomposition $\{M_1, M_2\}$ generated by $X$, then $G$ has the decomposition $\{G_1, G_2\}$ generated by $A$, where $M^*(G_i) = M_i$ for each $i \in \{1, 2\}$. Suppose that $M$ is a counterexample such that $|\gamma(X) - \delta(X)|$ is minimum. We shall begin by assuming that $\delta(X) = \gamma(X)$. Then $M_1 = M^*(G) \setminus A_2$ and $M_2 = M^*(G) \setminus A_1$. Since $|\delta(X)| = 3$, we have that $A_3 = \gamma(X)$. Therefore, $A$ generates the decomposition $\{G/A_2, G/A_1\}$ and, hence, the result holds. Now assume that $|\delta(X)| \leq 2$. Therefore $|A_3|$ is also less than three. Let $e$ be an element of $\gamma(X) - \delta(X)$. By (3.3.5), $\{X_1 \cup \{e\}, X_2 \cup \{e\}\}$ is a split of $M' = P(E(M) \cup \{e\})$ generating the same decomposition as does the split $X$ of $M^*(G)$. By (3.3.10), there is a graph $G'$ for which $M' = M^*(G')$. The minimality of $|\gamma(X) - \delta(X)|$ implies that $G'$ has the decomposition $\{G_1, G_2\}$. It remains to show that $G$ also has the decomposition $\{G_1, G_2\}$. The split of $G'$ generating this decomposition is $A' = \{A_1, A_2; A_3'\}$, where $A_3' = A_3 \cup \{e\}$. Therefore, $G = G'/e$ has a split $A = \{A_1, A_2; A_3\}$, This split generates the same split of $G$ as does the split $A'$ of $G'$. This completes the proof. □

(3.3.12) Lemma. The decomposition $\{G_1, G_2\}$ is generated by a good split of $G$ if and only if $\{M^*(G_1), M^*(G_2)\}$ is generated by a good split of $M^*(G)$.
Proof. In light of Lemma 3.3.11, we need to show that the splits \( A = \{A_1, A_2; A_3\} \) and \( B = \{B_1, B_2; B_3\} \) cross in \( G \) if and only if \( x = \{A_1 \cup A_3, A_2 \cup A_3\} \) and \( y = \{B_1 \cup B_3, B_2 \cup B_3\} \) cross in \( M^*(G) \). Recall that \( \delta(X) = A_3 \) and \( \delta(Y) = B_3 \). By the definition of crossing splits for graphs, \( A \) and \( B \) cross if and only if \( A_i - B_j \) and \( B_k - A_l \) are non-empty for all \( i, j, k, l \in \{1, 2\} \). Suppose that \( \{j, j'\} = \{l, l'\} = \{1, 2\} \). Then \( E(M) - B_j = Y_{j'} \) and \( E(M) - A_l = X_{l'} \). Therefore, \( A_i - B_j \) and \( B_k - A_l \) are non-empty for all \( i, j, k, l \in \{1, 2\} \) if and only if \( (X_i - \delta(X)) \cap Y_{j'} \) and \( (Y_j - \delta(Y)) \cap X_{l'} \) are non-empty for all \( i, j', k, l' \in \{1, 2\} \). This is the definition of crossing splits for binary matroids. \( \square \)

In the theory developed by Coullard, Gardner, and Wagner, decompositions are generated by splits, and a minimal decomposition is produced by a sequence of good splits, where each of the final components has no good split. Therefore, Theorem 3.3.1 is an immediate consequence of Lemmas 3.3.11 and 3.3.12.
CHAPTER 4

THE COMPONENTS OF THE DECOMPOSITION

4.1 Introduction

In the previous chapter, we defined a decomposition for 3-connected binary matroids and proved that a minimal such decomposition is unique. The decomposition process is carried out using good splits until none of the components has a good split. Therefore, if a component has no good splits, then it will be one of the components in the minimal decomposition. Since the vertically 4-connected binary matroids are precisely the 3-connected matroids having no splits, they form an obvious class of possible components in a minimal decomposition. But there could be other components. In particular, a wheel having rank at least four has no good splits, but is not vertically 4-connected. The purpose of this chapter is to characterize those 3-connected binary matroids that have no good splits and are not vertically 4-connected. We shall denote this class of matroids by $\mathcal{M}$.

Recall that our decomposition is the extension to binary matroids of the dual of the decomposition for 3-connected graphs of Coullard, Gardner, and Wagner (1993). They showed that, besides the cyclically 4-connected graphs, the set of 3-connected graphs having no good splits properly contains the sets of wheels of rank at least four, and twirls of rank at least six. Recall that a twirl is a bipartite graph $K_{3,n}$, where $n \geq 3$. They left as an open question what the other members of this set must be. In order to achieve a characterization for the set of possible components in
the decomposition of Coullard, Gardner, and Wagner, one simply takes the duals of
the matroids in $M$ and looks at those that are graphic. Therefore, the main result
of this chapter provides a characterization in the graph case. We have chosen to
pursue a different application of these results in the final chapter.

Recall that the matroids in $M$ arise as components in a minimal decompo-
sition; that is, these matroids cannot be decomposed by good splits. But these
matroids have splits, and therefore, can be decomposed. We have chosen to charac-
terize the matroids in $M$ using this decomposition. The main results of this chapter
are given in terms of wheels, the dual of twirls, and spikes. These basic building
blocks are defined and examined in Section 4.2. There is another special class of
matroids $N$ which is used in our characterization and is defined in Section 4.4.

Finally, before presenting the main results, we define a component $M_1$ in a
decomposition $\{M_1, M_2\}$ of $M$ generated by a split $\{X_1, X_2\}$ to be a maximal wheel
if it is a wheel and there is no decomposition $\{N_1, N_2\}$ of $M$ generated by a split
$\{Y_1, Y_2\}$ where $N_1$ is a wheel and $Y_1 \supset X_1$. A maximal spike is defined similarly.

We are now prepared to present the three main results of this chapter.

(4.1.1) Theorem. Suppose that $M \in \mathcal{M}$ and $M$ has no decomposition with a
component isomorphic to either $M(K_4)$ or $F_7$. Then $M \cong M^*(K_{3,n})$, for some
$n \geq 4$.

(4.1.2) Theorem. Suppose that $M \in \mathcal{M}$ and that $X = \{X_1, X_2\}$ is a split of
$M$ with decomposition $\{M_1, M_2\}$ such that $M_1$ is a maximal wheel of $M$. Then
$|\delta(X)| \leq 2$ and
(i) if $|\delta(X)| = 2$, then $M_2 \in \mathcal{M}$; and

(ii) if $|\delta(X)| \leq 1$, then $M_2 \in \mathcal{M} \cup \mathcal{N} \cup \{F_7, M(K_4)\}$.

(4.1.3) Theorem. Suppose that $M \in \mathcal{M}$ and that $X = \{X_1, X_2\}$ is a split of $M$ with decomposition $\{M_1, M_2\}$ such that $M_1$ is a maximal spike of $M$. Then $|\delta(X)| \leq 1$ and

(i) if $|\delta(X)| = 1$, then $M_2 \in \mathcal{M} \cup \{M(K_4)\}$; and

(ii) if $|\delta(X)| = 0$, then $M_2 \in \mathcal{M} \cup \{F_7, M(K_4)\}$.

Moreover, if $M_2 \cong F_7$, then $M$ is a tipless spike.

In Section 4.3, we define a multiple connection and investigate the splits associated with such a connection. Since this chapter is a characterization of those matroids having no good splits, it is valuable to know when good splits must exist. In Section 4.3, we prove that there is always a good split associated with a multiple connection. In Section 4.4, we analyze a split according to the number of elements in its connection. In Section 4.5, the work of the previous sections is put together to prove the desired characterization of $\mathcal{M}$ that is contained in Theorems 4.1.1, 4.1.2, and 4.1.3.

Most of the notation and terminology specific to the decomposition was developed in Chapter 3. As there, we shall assume throughout this chapter that $M$ is a 3-connected binary matroid and that $P$ is a projective space of sufficiently high rank so that $M$ is isomorphic to a restriction of $P$. Moreover, we shall identify $M$ with such a restriction.
Before proceeding to Section 4.2, we prove two fundamental results that will be used throughout the chapter.

(4.1.4) Suppose that \(X_1\) and \(X_2\) are subsets of \(M\), that \(X = \{\text{cl}_M(X_1), \text{cl}_M(X_2)\}\) is a split of \(M\), that \(X_1 \cup X_2 = E(M)\), and that \(\delta(X) \subseteq X_1 \cap X_2\). Then \(X = \{X_1, X_2\}\) is a split of \(M\).

**Proof.** By definition, \(\delta(X) = \text{cl}_M(X_1) \cap \text{cl}_M(X_2)\). Suppose that \(x \in \text{cl}_M(X_1) - X_1\). Then \(x \in X_2 - X_1\) and \(x \in \text{cl}_M(X_1)\). Hence \(x \in \text{cl}_M(X_2) \cap \text{cl}_M(X_1)\). Therefore \(x \in \delta(X)\). But this contradicts the assumption that \(X_1 \supseteq \delta(X)\). Hence \(X_1 = \text{cl}_M(X_1)\).

Likewise, \(X_2 = \text{cl}_M(X_2)\). □

(4.1.5) Suppose that \(X = \{X_1, X_2\}\) is a split of \(M\) generating the decomposition \(\{M_1, M_2\}\). Further, suppose that \(e \in X_2 - \gamma(X)\), and that \(N'\) is 3-connected, where \(N' = M \setminus e\). Then \(X' = \{X'_1, X'_2\}\) is a split of \(N'\), where \(X'_1 = X_1\) and \(X'_2 = X_2 - \{e\}\). Moreover, \(\gamma(X') = \gamma(X)\) and \(\{M_1, M_2 \setminus e\}\) is the decomposition of \(N'\) generated by \(X'\).

**Proof.** Since \(M_2\) is 3-connected, \(e \in \text{cl}_{M_2}[E(M_2) - \{e\}]\). Hence \(r(M_2 \setminus e) = r(M_2)\). Therefore

\[
r(M_1) + r(M_2 \setminus e) = r(M) + 2. \tag{4.1}
\]

By (3.3.5), \(Y' = \{Y'_1, Y'_2\}\) is a split of \(M' = P|(E(M) \cup \gamma(X))\), where \(Y'_1 = X_1 \cup \gamma(X)\) and \(Y'_2 = X_2 \cup \gamma(X)\). Moreover, \(\gamma(Y') = \gamma(X)\). By (3.2.3) and (4.1) we have that \(Z = \{\text{cl}_{M' \setminus e}(Y'_1), \text{cl}_{M' \setminus e}(Y'_2 - \{e\})\}\) is a split of \(M' \setminus e\). Clearly, \(\text{cl}_P(Y'_1) = \text{cl}_P(X_1)\). Since \(M_2\) is 3-connected, \(r(Y'_2 - \{e\}) = r(Y_2)\) and, hence, \(\text{cl}_P(Y'_2 - \{e\}) = \text{cl}_P(X_1)\) and, hence, \(\text{cl}_P(Y'_2 - \{e\}) = \text{cl}_P(X_1)\).
Clearly, $\text{cl}_P(Y'_{2}) = \text{cl}_P(X_{2})$. Thus, $\text{cl}_P(Y'_{2} - \{e\}) = \text{cl}_P(X_{2})$. Therefore, $\gamma(X) = \gamma(Y') = \gamma(Z)$. Let $\gamma = \gamma(X)$. Observe that $\text{cl}_{M \setminus \{e\}}(Y'_{1}) = Y'_{1}$ and that $\text{cl}_{M \setminus \{e\}}(Y'_{2} - \{e\}) = Y'_{2} - \{e\}$. Therefore $Z = \{Y'_{1}, Y'_{2} - \{e\}\}$. By (3.3.6), $\{Y'_{1} \cap E(N'), (Y'_{2} - \{e\}) \cap E(N')\}$ is a split of $N'$ with connection $\gamma$. Since $Y'_{1} \cap E(N') = X'_{1}$ and $(Y'_{2} - \{e\}) \cap E(N') = X'_{2}$, it remains only to show that this split generates $\{M_{1}, M_{2} \setminus \{e\}\}$.

Let $\{N_{1}, N_{2}\}$ be the decomposition generated by $X'$. We have that $E(M_{1}) = X_{1} \cup \gamma(X) = X'_{1} \cup \gamma(X') = E(N_{1})$. Hence $M_{1} = N_{1}$. Also, $E(M_{2} \setminus \{e\}) = (X_{2} - \{e\}) \cup \gamma(X) = X'_{2} \cup \gamma(X') = E(N_{2})$. Hence $N_{2} = M_{2} \setminus \{e\}$. □

### 4.2 The Basics of Spikes, Wheels, and the Dual of Twirls

Along with vertically 4-connected binary matroids, the fundamental building blocks of our decomposition are spikes, wheels, and the duals of twirls. In this section we introduce these matroids and make some basic observations about them. In particular, we show that the members in each of these classes can only be decomposed into components that are themselves smaller members of the respective class. We shall also present a new characterization for twirls and their duals.

We start with the wheels. Since we are concerned with 3-connected matroids, we will only consider 3-connected wheels; that is, those wheels $W_{r}$, whose rank $r$ is at least three. The wheel $W_{r}$ is defined as the cycle matroid of the graph composed of $2r$ edges and $r + 1$ vertices, where $r$ of the edges and $r$ of the vertices form a cycle, and the other vertex, the hub, is joined to each of these $r$ vertices. The edges adjacent to the hub are the spokes of the wheel. The edges of the $r$-cycle are the
rim edges. Note that if \( r = 3 \), then any vertex could be the hub and any triangle could be the rim. We shall denote both the graph and its cycle matroid by \( \mathcal{W}_r \). The rank-three wheel \( \mathcal{W}_3 \) is the complete graph on four vertices. We will often use the notation \( M(K_4) \) instead of \( \mathcal{W}_3 \) for the cycle matroid of this graph. The following characterization for wheels can be found in Seymour (1980, 6.1; see also Oxley 1992, Lemma 11.1.5).

\[(4.2.1)\]

Suppose that \( M \) is a binary matroid and \( n \geq 2 \). Then \( M \) is the wheel \( \mathcal{W}_n \) if and only if there are disjoint subsets \( \{r_1, r_2, \ldots, r_n\} \) and \( \{s_1, s_2, \ldots, s_n\} \) of \( E(M) \) such that, for all \( i \in \{1, 2, \ldots, n\} \), the set \( \{s_i, r_i, s_{i+1}\} \) is a triangle and the set \( \{r_i, s_{i+1}, r_{i+1}\} \) is a triad, where all subscripts are taken modulo \( n \).

Next, we investigate how 3-connected wheels decompose. First we need the following result.

\[(4.2.2)\]

Suppose that \( X = \{X_1, X_2\} \) is a split of \( M \) generating the decomposition \( \{M_1, M_2\} \). If \( e \in E(M_1) \) and \( e \) is not in a triangle of \( M_1 \), then \( e \in X_1 - \delta(X) \) and \( e \) is not in a triangle of \( M \).

**Proof.** Suppose that \( e \in E(M_1) \) and \( e \) is not in a triangle of \( M_1 \). Since \( \gamma(X) \) is a triangle of \( M_1 \) and \( \delta(X) \subseteq \gamma(X) \), we have that \( e \in E(M_1) - \gamma(X) = X_1 - \delta(X) \).

Since \( X_1 \) and \( X_2 \) are closed sets that cover \( E(M) \), each triangle of \( M \) is contained in either \( X_1 \) or \( X_2 \). Since \( e \notin X_2 \), if \( e \) is in a triangle of \( M \), that triangle must be contained in \( X_1 \). This contradicts the assumption that \( e \) is not in a triangle of \( M_1 \).

Hence, \( e \) is not in a triangle of \( M \). \( \square \)
(4.2.3) Suppose that $M$ is a wheel and that $X = \{X_1, X_2\}$ is a split of $M$ generating the decomposition $\{M_1, M_2\}$. Then $M_i$ is a wheel for each $i \in \{1, 2\}$ and $|\delta(X)| = 2$.

**Proof.** Since $M$ has a split, $r(M) \geq 4$. Let $r = r(M)$. By (4.2.1), $E(M) = R \cup S$, where

(i) $R = \{r_1, r_2, \ldots, r_r\}$ and $S = \{s_1, s_2, \ldots, s_r\}$ are disjoint sets; and

(ii) each set $\{s_i, r_i, s_{i+1}\}$ is a triangle and each set $\{r_i, s_{i+1}, r_{i+1}\}$ is a triad, where the indices are taken modulo $r$.

Each triangle of $M$ is contained in either $X_1$ or $X_2$. Since $S$ is a basis of $M$, neither $X_1$ nor $X_2$ can contain $S$. We may assume that $s_1 \in X_1 - \delta(X)$. Since the sets $\{s_i, r_i, s_{i+1}\}$ cover $E(M)$, where the indices are taken modulo $r$, there must be an element $s_j$ such that $\{s_{j-1}, r_{j-1}, s_j\} \subseteq X_1$ and $\{s_j, r_j, s_{j+1}\} \subseteq X_2$. Similarly, there is an $s_k$ such that $\{s_{k-1}, r_{k-1}, s_k\} \subseteq X_2$ and $\{s_k, r_k, s_{k+1}\} \subseteq X_1$, where the indices are taken modulo $r$. Then $\delta(X) \supseteq \{s_j, s_k\}$. Since $X_2 - \delta(X)$ is non-empty, we have that $k - j \geq 2$. Since $r \geq 4$, we have that $s_j$ and $s_k$ are not contained in a common triangle of $M$. Therefore $\delta(X) = \{s_j, s_k\}$. Hence,

$$X_1 = \{s_k, s_{k+1}, \ldots, s_r, s_1, s_2, \ldots, s_j\} \cup \{r_k, r_{k+1}, \ldots, r_r, r_1, r_2, \ldots, r_{j-1}\};$$

and

$$X_2 = \{s_j, s_{j+1}, \ldots, s_k\} \cup \{r_j, r_{j+1}, \ldots, r_{k-1}\}.$$  

We may relabel the elements so that $X_2 = \{s'_l, \ldots, s'_1\} \cup \{r'_l, \ldots, r'_{l-1}\}$, where $l = k - j + 1$. Then $\delta(X) = \{s'_l, s'_1\}$. Let $r'_l$ be such that $\gamma(X) = \{s'_l, r'_l, s'_1\}$. Since
every set \( \{s'_i, r'_i, s'_{i+1}\} \) is a triangle, where the indices are taken modulo \( l \), in order to show that \( M_2 \) is a wheel, it remains to show that every set \( \{r'_i, s'_{i+1}, r'_{i+1}\} \) is a triad of \( M_2 \), where the indices are taken modulo \( l \). Let \( H \) be the union of the triangles \( \{s'_{i+2}, r'_{i+2}, s'_{i+3}\}, \{s'_{i+3}, r'_{i+3}, s'_{i+4}\}, \ldots, \{s'_{i-1}, r'_{i-1}, s'_i\} \) and \( n \) be the number of triangles in the union. Since each one of these triangles meets the preceding one, we have that \( r(H) \leq n + 1 \). But \( n = l - 2 \). Hence \( r(H) \leq r(M_2) - 1 \). Therefore, \( \{r_i, s_{i+1}, r_{i+1}\} \) contains a cocircuit of \( M_2 \). Since \( M_2 \) is 3-connected, \( \{r_i, s_{i+1}, r_{i+1}\} \) is a triad. Therefore \( M_2 \) is a wheel. By symmetry, \( M_1 \) is a wheel. □

In the proof of (4.2.3), it was shown that if \( X \) is a split of \( \mathcal{W}_n \), then we may assume that \( X = \{\{s_1, r_1, s_2, \ldots, r_{l-1}, s_l\}, \{s_l, r_l, s_{l+1}, \ldots, s_n, r_n, s_1\}\} \), where \( l \in \{3, 4, \ldots, n - 1\} \). But if we let \( Y_1 = \{s_{l-1}, r_{l-1}, s_l, r_l, s_{l+1}\} \) and \( Y_2 = \{E(M) - Y_1\} \cup \{s_{l-1}, s_{l+1}\} \), then \( Y = \{Y_1, Y_2\} \) is a split of \( M \) which crosses \( X \). This shows that if \( r \geq 4 \), then \( \mathcal{W}_r \) has no good splits.

We now turn our attention to spikes.

(4.2.4) Definition. A matroid \( M \) satisfying the following three conditions is called an \( n \)-spike with tip \( p \).

(i) \( E(M) = \bigcup_{i=1}^{n} L_i \), where each \( L_i \) is a three-point line passing through a common point \( p \).

(ii) For all \( k \in \{1, 2, \ldots, n - 1\} \), the union of any \( k \) of \( L_1, L_2, \ldots, L_n \) has rank \( k + 1 \).

(iii) \( r(L_1 \cup L_2 \cup \cdots \cup L_n) = n \).
We note that the above definition and the following observations are from Oxley (1996). Though there are spikes that are not binary, for each \( n \) there is a unique binary matroid satisfying Definition 4.2.4. Such a matroid is called the \textit{binary} \( n \)-\textit{spike}. Because the only spikes we shall consider are binary, they will simply be referred to as \textit{spikes}, or \textit{n}\textit{-spikes}. Such matroids are denoted \( S_n \). The matroid \( S_n \setminus p \) is called the \textit{tipless} \( n \)-\textit{spike}. Since we are concerned only with 3-connected matroids, we shall assume that a spike has rank at least three. Observe that \( S_3 \) is the Fano plane.

In the following result, we state several properties of spikes that will be useful in later proofs.

(4.2.5) \textit{Suppose that} \( M = S_n \) \textit{and the element} \( p \) \textit{and the sets} \( \{L_i\}_{i=1}^{n} \) \textit{are as in Definition 4.2.4. Then,}

(i) \( (L_i \cup L_j) - \{p\} \) \textit{is a circuit of} \( M \) \textit{for all distinct} \( i \) \textit{and} \( j \);

(ii) \textit{if} \( C \) \textit{is a non-spanning circuit of} \( M \), \textit{then either} \( C \) \textit{is as in (i), or} \( C = L_i \) \textit{for some} \( i \in \{1,2,\ldots,n\} \), \textit{or} \( C \) \textit{avoids} \( p \) \textit{and contains a unique element from each of the sets} \( L_i - \{p\} \), \textit{for all} \( i \in \{1,2,\ldots,n\} \);

(iii) \( M/p \) \textit{can be obtained from an} \( n \)-\textit{element circuit by replacing each element by two elements in parallel; and}

(iv) \textit{if} \( L_i = \{p,x_i,y_i\} \), \textit{then each of} \( M \setminus p \setminus x_i \) \textit{and} \( (M \setminus p \setminus x_i)^* \) \textit{is an} \( (n-1) \)-\textit{spike with tip} \( y_i \).
The next result is the analogue for spikes of (4.2.3).

(4.2.6) Suppose that \( \{M_1, M_2\} \) is a decomposition of \( M \) generated by a split \( X = \{X_1, X_2\} \). Then \( M \) is a spike if and only if \( M_i \) is a spike for each \( i \in \{1, 2\} \) and \( \delta(X) = \{p\} \), where \( p \) is a tip for \( M_1 \) and \( M_2 \).

Proof. By (4.2.5), \( E(M) = \bigcup_{i=1}^{r} L_i \), where each \( L_i \) is a triangle containing the point \( p \). Each \( L_i \) is in either \( X_1 \) or \( X_2 \). Therefore, \( \delta(X) \) is either a triangle or \( \{p\} \).

Let \( r_1 = r(M_1) \) and \( r_2 = r(M_2) \). Then there are \( r_1 - 1 \) triangles in \( X_1 \) and \( r_2 - 1 \) triangles in \( X_2 \). Since \( r_1 + r_2 = r + 2 \) and there are \( r \) triangles in all, there can be no triangles common to both \( X_1 \) and \( X_2 \). Hence \( \delta(X) = \{p\} \).

We may assume that \( X_1 = \bigcup_{i=1}^{r_1} L_i \). Let \( L'_r = \gamma(X) = \{p, a, b\} \). In order to conclude that \( M_1 \) is an \( r_1 \)-spike, it remains to show that the union of any proper subset of triangles from \( \{L_1, L_2, \ldots, L_{r_1}, L'_r\} \) has rank one greater than the number of such triangles. Since it is true for \( \{L_1, L_2, \ldots, L_r\} \), it is true for any subset of \( \{L_1, L_2, \ldots, L_{r_1}\} \). If a proper subset \( Y \) of \( \{L_1, L_2, \ldots, L_{r_1}, L'_r\} \) contains \( k \) triangles, one of which is \( L'_r \), this subset must contain a proper \((k-1)\)-member subset of \( \{L_1, L_2, \ldots, L_{r_1}\} \). Hence \( r(\bigcup_{L_i \in Y \setminus \{L'_r\}} L_i) = k \). Suppose that \( r(\bigcup_{L_i \in Y} L_i) \neq k + 1 \). Then \( r(\bigcup_{L_i \in Y} L_i) < k + 1 \). The addition of all subsequent triangles to the set contained in \( Y \) will increase the rank by no more than the number of such triangles added. This implies that if all but one of the remaining triangles is added to \( Y \), then the rank of this set will be less than or equal to \( r_1 - 1 \). But this contradicts the 3-connectivity of \( M_1 \). Therefore, \( M_1 \) is an \((r_1)\)-spike. The result follows by symmetry. \( \square \)
By the proof of (4.2.6), if $X = \{X_1, X_2\}$ is a split of $S_n$, we may assume that $X_1 = \bigcup_{i=1}^{l} L_i$ and $X_2 = \bigcup_{i=l+1}^{n} L_i$, where $l \in \{2, 3, \ldots, n-2\}$. But, if we let $Y_1 = L_i \cup L_{i+1}$ and $Y_2 = (E(M) - Y_1) \cup \{p\}$, then $Y = \{Y_1, Y_2\}$ is a split of $M$ which crosses $X$. This shows that if $n \geq 4$, then $S_n$ has no good split.

Before turning our attention to the twirls and their duals, we examine the smallest 3-connected wheel and spike, $M(K_4)$ and $F_7$, respectively. These two matroids are the only rank-three binary matroids that are 3-connected. They both have the property that each triangle intersects every other triangle. The spike $F_7$ is the unique in which any element can be viewed as the tip. Every spike of higher rank has a uniquely determined tip.

Finally, we discuss twirls and their duals. A complete bipartite graph $K_{3,n}$ ($n \geq 3$) is called a twirl. We shall also call the cycle matroid of such a graph a twirl. Every twirl is 3-connected. In the present context, it is not a twirl $M(K_{3,n})$ that is key, but rather its dual, $M^*(K_{3,n})$.

As in the case of wheels and spikes, for our purposes, it is most useful to characterize the dual of a twirl by how the triangles are arranged. Such a characterization is given by (4.2.7). The proof involves the use of the generalized parallel connection of Brylawski (1975) introduced in Section 3.3.

Let $D_n$ be the matroid obtained by taking a set of $n$ matroids $\{N_1, N_2, \ldots, N_n\}$, each isomorphic to $W_3$ and identifying a triangle from each $N_i$. To be more precise, we can define $D_n$ recursively. Let $D_1 \cong W_3$ and let $\Delta_1$ be an arbitrary triangle of $D_1$. Let $D_n = P_{\Delta}(D_{n-1}, \Delta_{n-1}, (N_n, \Delta))$ for $n \geq 2$, where $N_n$ is isomorphic to
$W_3$ and $\Delta$ is an arbitrary triangle of $N_3$. Observe that $\Delta_n = N_i \cap N_j$ for distinct $i, j \in \{1, 2, \ldots, n\}$. It is easy to verify that $M(K_{3,n}) = D_n \setminus \Delta_n$ for $n \geq 2$.

The above characterization of $M(K_{3,n})$ will help us provide a useful characterization of $M^*(K_{3,n})$, which is given in the next result.

\textbf{(4.2.7)} \textit{Let $n$ be a positive integer exceeding one and $M$ be a binary matroid of rank $r = 2n - 2$, such that}

\begin{enumerate}[(i)]
  \item $E(M)$ is the disjoint union of $n$ triangles $\{T_1, T_2, \ldots, T_n\}$; and
  \item $r(T_1' \cup T_2' \cup \ldots \cup T_k') = 2k$ for every proper subset $\{T_1', T_2', \ldots, T_k'\}$ of $\{T_1, T_2, \ldots, T_n\}$.
\end{enumerate}

Then $M \cong M^*(K_{3,n})$.

Rather than prove (4.2.7) directly, we shall use the characterization of $M(K_{3,n})$ given prior to (4.2.7) to prove the next result, which implies (4.2.7).

\textbf{(4.2.8)} \textit{Let $n$ be a positive integer exceeding one and $M$ be a binary matroid of rank $r = n + 2$, such that}

\begin{enumerate}[(i)]
  \item $E(M)$ is the disjoint union of $n$ triads $\{T_1, T_2, \ldots, T_n\}$; and
  \item $r(T_1' \cup T_2' \cup \ldots \cup T_k') = k + 2$ for every non-empty subset $\{T_1', T_2', \ldots, T_k'\}$ of $\{T_1, T_2, \ldots, T_n\}$.
\end{enumerate}

Then $M \cong M(K_{3,n})$. 

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Proof. The proof consists of using the recursive characterization of $M(K_{3,n})$ to prove by induction that $M \cong M(K_{3,n})$. Suppose that $n = 2$. Then $r(T_1) + r(T_2) = r(M) + 2$. Since $T_1$ is a triad, the set $T_2$ is a hyperplane. Likewise, $T_1$ is closed. Hence $T_1$ and $T_2$ are closed non-intersecting sets. Therefore, there is a triangle $\gamma$ of $P$, where $\gamma = cl_P(T_1) \cap cl_P(T_2)$ and $\gamma \cap (T_1 \cup T_2) = \emptyset$. Observe that $|T_1 \cup \gamma| = 6$ and $r_P(T_i \cup \gamma) = 3$ for each $i \in \{1, 2\}$. Hence, $P|(T_1 \cup T_2 \cup \gamma) = D_2$ and $\gamma = \Delta_2$. Therefore $M = D_2 \setminus \Delta_2 \cong M(K_{3,2})$.

Suppose that $n \geq 3$ and that for $n' < n$ the result holds. Let $M_n = M|(T_1 \cup T_2 \cup \cdots \cup T_{n-1})$. In order to show that $M_n$ satisfies (i) and (ii), we need only show that each $T_i$ is a triad of $M_n$ for $i \in \{1, 2, \ldots, n-1\}$. By symmetry, we need only show that $T_{n-1}$ is a triad. Since $r(T_1 \cup T_2 \cup \cdots \cup T_{n-2}) = n$ and $r(M_n) = n + 1$, the set $T_{n-1}$ contains a cocircuit of $M_n$. Therefore, we need to show that $T_{n-1}$ does not contain a coloop or 2-element cocircuit of $M_n$. Observe that $(M_n)^* = M^*/T_n$. If $T_{n-1}$ contains a coloop $\{a\}$ of $M_n$, then $a \in cl_M(T_n)$. But this contradicts our assumption that $r_{M^*}(T_{n-1} \cup T_n) = 4$. If $T_{n-1}$ contains a 2-element cocircuit $\{a, b\}$ of $M_n$, then $r_{M^*}(T_{n-1} \cup \{a, b\}) = 3$. Again, this contradicts our assumption that $r_{M^*}(T_{n-1} \cup T_n) = 4$. Hence, there is a triangle $\Delta_{n-1} = cl_P(T_i) \cap cl_P(T_j)$ for distinct $i, j \in \{1, 2, \ldots, n-1\}$, and $M_n \cong D_{n-1} \setminus \Delta_{n-1}$. In order to prove the result, we need only show that $\Delta_{n-1} = cl_P(E(M_n)) \cap cl_P(T_n)$. Since $r(M_n) + r(T_n) = r(M) + 2$, there is a triangle $\gamma = cl_P(E(M_n)) \cap cl_P(T_n)$. To complete the proof, we need to show that $\gamma = \Delta_{n-1}$. Let $M_{n-1} = M|(T_1 \cup T_2 \cup \cdots \cup T_{n-2} \cup T_n)$. By symmetry, there are triangles $\Delta'_n$ and $\Delta''_{n-1}$ such that $\Delta'_n = cl_P(T_i) \cap cl_P(T_j)$ for distinct $i, j \in \{1, 2, \ldots, n-2, n\}$.
and \( \Delta_{n-1}' = \text{clp}(T_i) \cap \text{clp}(T_j) \) for distinct \( i, j \in \{1, 2, \ldots, n-3, n-1, n\} \). Therefore, 
\( \Delta_{n-1}' = \text{clp}(T_1) \cap \text{clp}(T_n) \supset \text{clp}(E(M_n)) \cap \text{clp}(T_n) = \gamma \). Since both \( \Delta_{n-1}' \) and \( \gamma \) have rank two, we have that \( \Delta_{n-1}' = \gamma \). Likewise, \( \Delta_{n-1}'' = \text{clp}(T_{n-1}) \cap \text{clp}(T_1) \supset \gamma \).

Since \( n \geq 3 \), we have that \( n-1 \neq 1 \). Therefore, \( \Delta_{n-1} = \text{clp}(T_{n-1}) \cap \text{clp}(T_1) \supset \gamma \).

Since both \( \Delta_{n-1} \) and \( \gamma \) have rank two and are closed, we have that \( \Delta_{n-1} = \gamma \). This completes the proof. \( \square \)

The next result shows how the dual of a twirl will decompose. It is the analogue of (4.2.3) and (4.2.6).

**Theorem 4.2.9.** Suppose that \( \{M_1, M_2\} \) is a decomposition of \( M \) generated by a split \( X = \{X_1, X_2\} \). Then \( M \) is the dual of a twirl if and only if \( M_i \) is the dual of a twirl for each \( i \in \{1, 2\} \) and \( \delta(X) = \emptyset \).

**Proof.** By (4.2.7), \( E(M) = \bigcup_{i=1}^{n} T_i \), where \( \{T_1, T_2, \ldots, T_n\} \) is a set of disjoint triangles and \( r(M) = 2n - 2 \). The rank of the union of any proper subset of these triangles is \( 2k \), where \( k \) is the number of such triangles in the union. Moreover, each \( T_i \) is in either \( X_1 \) or \( X_2 \). Let \( r_1 = r(M_1) \) and \( r_2 = r(M_2) \). Then there are \( n_1 = r_1/2 \) triangles in \( X_1 \) and \( n_2 = r_2/2 \) triangles in \( X_2 \). Since \( r_1 + r_2 = r + 2 \) and there are \( n = (r + 2)/2 \) triangles in all, we have that \( n_1 + n_2 = (r_1 + r_2)/2 = (r + 2)/2 = n \).

Hence, there are no triangles in \( \{T_1, T_2, \ldots, T_n\} \) common to both \( X_1 \) and \( X_2 \). Since no element is in more than one triangle, \( \delta(X) = \emptyset \).

We may assume that \( X_1 = \bigcup_{i=1}^{n_1} T_i \). Let \( T_{n_1+1} = \gamma(X) \). Observe that \( r(M_1) = r(X_1) = 2n_1 = 2(n_1 + 1) - 2 \). Therefore, in order to show that \( M_1 \cong M^*(K_{3,(n_1+1)}) \) it remains to show that, for all \( k < n_1 + 1 \), the union of any \( k \) tri-
angles from \(\{T_1, T_2, \ldots, T_n, T_{n+1}'\}\) has rank \(2k\). This is clearly true for subsets of \(\{T_1, T_2, \ldots, T_n\}\). If a proper subset \(Y\) of \(\{T_1, T_2, \ldots, T_n, T_{n+1}'\}\) contains \(k\) members including \(T_{n+1}'\), it must contain a \((k-1)\)-member subset of \(\{T_1, T_2, \ldots, T_n\}\). Hence, \(r(\bigcup_{T_i \in Y - \{T_{n+1}'\}} T_i) = 2k - 2\). Suppose that \(r(\bigcup_{T_i \in Y} T_i) \neq 2k\). Then \(r(\bigcup_{T_i \in Y} T_i) < 2k\).

The addition of all subsequent triangles to the set \(Y\) will cause the rank to go up no more than twice the number of such triangles added. This implies that if all but one of the remaining triangles of \(M\) is added to \(Y\), then the rank of this set will be less than \(2n_1\). Since \(r(M_1) = 2n_1\), the remaining triangle must contain a cocircuit. But this contradicts the 3-connectivity of \(M_1\). Therefore, \(M_1\) is isomorphic to \(M^*(K_3,(n_1+1))\). The result follows by symmetry. □

As in the case of wheels and spikes, it is not difficult to see that if \(M = M^*(K_3,n)\) and \(n \geq 4\), then \(M\) is 3-connected and has splits, but has no good split.

Let \(X = \{X_1, X_2\}\) be a split of \(M = M^*(K_3,n)\) where \(n \geq 4\). We may assume that \(X_1 = \bigcup_{i=1}^l T_i\) and \(X_2 = \bigcup_{i=l+1}^n T_i\) where \(l \in \{2, 3, \ldots, n-2\}\). If we let \(Y_1 = T_l \cup T_{l+1}\) and \(Y_1 = E(M) - Y_1\), then \(Y = \{Y_1, Y_2\}\) is a split of \(M\) which crosses \(X\).

### 4.3 Multiple Connections and Minimal Splits

Recall that the connection of a split \(X = \{X_1, X_2\}\) is the rank-two flat \(\gamma(X) = cl_p(X_1) \cap cl_p(X_2)\). A connection \(\gamma\) is called a multiple connection if there are at least two distinct splits \(X\) and \(Y\) such that \(\gamma = \gamma(X) = \gamma(Y)\). A split \(X\) is called minimal with respect to its connection \(\gamma(X)\) if, for any other split \(Y\) such that \(\gamma(Y) = \gamma(X)\), the splits \(X\) and \(Y\) are compatible. In this section we have two main results. One of them, Proposition 4.3.12, states that a minimal split of a multiple connection is
a good split. This is important because when analyzing those matroids having no good splits, we can immediately rule out matroids based on Proposition 4.3.12.

In Section 4.4, we analyze a split $X$ based on the number of elements in $\delta(X)$. The other main result of this section, Proposition 4.3.8, begins that analysis. That result states that if $\delta(X) = \gamma(X)$, then there is a good split with connection $\gamma(X)$. We present Proposition 4.3.8 in this section because it is needed to prove the other main result of this section, Proposition 4.3.12.

We now present a sequence of results culminating in the proof of Proposition 4.3.8. Several of these results, including (4.3.6), (4.3.7), and Proposition 4.3.8 itself, will be useful in the proof of later results.

(4.3.1) Suppose that $X = \{X_1, X_2\}$ and $Y = \{Y_1, Y_2\}$ are distinct splits of $M$ with a common connection $\gamma = \gamma(X) = \gamma(Y)$. If $(X_1 \cap Y_1) - \gamma$ is non-empty, then $\{X_1 \cap Y_1, X_2 \cup Y_2\}$ is a split of $M$ with connection $\gamma$.

Proof. Let $M' = P((E(M) \cup \gamma)$, and let $X'_i = X_i \cup \gamma$ and $Y'_i = Y_i \cup \gamma$ for each $i \in \{1, 2\}$. Observe that $r(X'_1) + r(X'_2) + r(Y'_1) + r(Y'_2) = 2r(M') + 4$. By the modularity of $P$, we have that $r(X'_1) + r(Y'_1) = r(cl_P(X'_1) \cap cl_P(Y'_1)) + r(cl_P(X'_1) \cup cl_P(Y'_1))$ and $r(X'_2) + r(Y'_2) = r(cl_P(X'_2) \cap cl_P(Y'_2)) + r(cl_P(X'_2) \cup cl_P(Y'_2))$. Therefore, $r(cl_P(X'_1) \cap cl_P(Y'_1)) + r(cl_P(X'_2) \cap cl_P(Y'_2)) + r(cl_P(X'_1) \cup cl_P(Y'_1)) + r(cl_P(X'_2) \cup cl_P(Y'_2)) = 2r(M') + 4$. Since $\gamma = (X'_1 \cap Y'_1) \cap (X'_2 \cup Y'_2) = (X'_2 \cap Y'_2) \cap (X'_1 \cup Y'_1)$, we have that $r(X'_1 \cap Y'_1) + r(X'_2 \cup Y'_2) = r(M') + 2$. Since $(X_1 \cap Y_1) - \gamma$ is non-empty, we have that $r(X'_1 \cap Y'_1) \geq 3$. By (3.2.3), $\{cl_{M'}(X'_1 \cap Y'_1), cl_{M'}(X'_2 \cup Y'_2)\}$ is a split of $M'$. Moreover, $\gamma$ is the connection of this split. Observe that $X'_i$ and $Y'_i$
are closed subsets of $M'$ for $i \in \{1, 2\}$. Hence $cl_{M'}(X'_1 \cap Y'_1) = X'_1 \cap Y'_1$. Suppose that $x \in cl_{M'}(X'_1 \cup Y'_2) - (X'_2 \cup Y'_2)$. Then $x \in (X'_1 \cap Y'_1) - (X'_2 \cup X'_2)$. Since $\gamma \subseteq X'_2 \cup Y'_2$, this implies that $r[cl_{M'}(X'_2 \cup Y'_2)] \cap (X'_1 \cap Y'_1)] \geq 3$; a contradiction. Hence $X'_2 \cup Y'_2 = cl_{M'}(X'_2 \cup Y'_2)$. Therefore, $\{X'_1 \cap Y'_1, X'_2 \cup Y'_2\}$ is a split of $M'$ with connection $\gamma$. The result follows by applying (3.3.6). □

(4.3.2) Suppose that $X = \{X_1, X_2\}$ is a minimal split of $M$ with respect to the connection $\gamma$ and that $Y = \{Y_1, Y_2\}$ is a different split of $M$ with connection $\gamma$ such that $X_1 \subset Y_1$. Then, for any split $Z = \{Z_1, Z_2\}$ with connection $\gamma$, either $X_1 \subset Z_1$ or $X_1 \subset Z_2$.

Proof. Suppose that $Z = \{Z_1, Z_2\}$ has connection $\gamma$, that $X_1 \not\subset Z_1$, and that $X_1 \not\subset Z_2$. Since $X$ is minimal, we have that either $Z_1 \subset X_1$ or $Z_2 \subset X_1$. We may assume that $Z_2 \subset X_1$ and, hence, $X_2 \subset Z_1$. By (4.3.1), $\{Z_1 \cap Y_1, Z_2 \cup Y_2\}$ is a split of $M$ with connection $\gamma$. Since $X_1 \subset Y_1$, the set $(Z_1 \cap Y_1) \cap (X_1 - \gamma(X))$ is non-empty unless $Z_2 \supset X_1$. Therefore, $(Z_1 \cap Y_1) \cap (X_1 - \gamma(X))$ is non-empty. Likewise, since $X_2 \subset Z_1$, the set $(Z_1 \cap Y_1) \cap (X_2 - \gamma(X))$ is non-empty unless $Y_2 \supset X_2$. So $(Z_1 \cap Y_1) \cap (X_2 - \gamma(X))$ is non-empty. Since $Z_2 \subset X_1$ and $Y_2 \subset X_2$, we have that $Z_2 \cup Y_2$ meets both $X_1 - \gamma(X)$ and $X_2 - \gamma(X)$. Therefore, the split $\{Z_1 \cap Y_1, Z_2 \cup Y_2\}$ crosses $X$; a contradiction. □

For a split $X = \{X_1, X_2\}$ of $M$ and a subset $\gamma$ of $E(M)$ as in (4.3.2), we shall call the set $X_1$ a minimal component of the minimal split $X$. Observe that if both $X_1$ and $X_2$ are minimal components of $X$, then $X$ must be the only split with connection $\gamma$. Therefore, (4.3.2) implies that if $\gamma$ is a multiple connection with
minimal split $X = \{X_1, X_2\}$, then precisely one of the sets $X_1$ or $X_2$ is the minimal component.

The next result shows that the minimal components of a connection cover the ground set of a matroid.

(4.3.3) For any connection $\gamma$, there is a minimal split with respect to $\gamma$ and, for any element $x \in E(M) - \gamma$, there is a minimal component of a minimal split containing $x$.

Proof. The result clearly follows if $\gamma$ has only one split. Therefore, assume that $X = \{X_1, X_2\}$ and $Y = \{Y_1, Y_2\}$ are two distinct crossing splits having connection $\gamma$. We may assume that $x \in X_1 \cap Y_1$. By (4.3.1), $\{X_1 \cap Y_1, X_2 \cup Y_2\}$ is a split of $M$. If this is a minimal split, then (4.3.2) implies that $X_1 \cap Y_1$ is the minimal component and the result follows. If this is not a minimal split, then there is a crossing split $Z = \{Z_1, Z_2\}$ where we may assume that $x \in Z_1$. By (4.3.1), $\{X_1 \cap Y_1 \cap Z_1, X_2 \cup Y_2 \cup Z_2\}$ is a split of $M$. Moreover, if this split is minimal, (4.3.2) implies that $X_1 \cap Y_1 \cap Z_1$ is the minimal component. Clearly this process may be repeated, with the first components of each split containing the element $x$ and being properly contained in the first component of the previous split. Hence, this process must stop with a minimal split and the minimal component containing $x$. □

While the previous result shows that minimal components cover the ground set, the next result shows that they partition the set $E(M) - \gamma$.

(4.3.4) Suppose that $\{\{U_i, V_i\}\}_{i=1}^k$ is the collection of minimal splits with respect to a multiple connection $\gamma$ and that for each split $\{U_i, V_i\}$ the set $U_i$ is the minimal
component of that split. Then \( \{U_i - \gamma\}_{i=1}^k \) is a partition of \( E(M) - \gamma \). Moreover, \( k \geq 3 \), and \( V_i = \bigcup_{j \neq i} U_j \) for each \( i \in \{1, 2, \ldots, k\} \).

**Proof.** Clearly, if \( \{U_i - \gamma\}_{i=1}^k \) partitions \( E(M) - \gamma \), then \( V_i = \bigcup_{j \neq i} U_j \) for each \( i \in \{1, 2, \ldots, k\} \). By (4.3.3), \( \bigcup_{i=1}^k (U_i - \gamma) = E(M) - \gamma \). Since \( \{U_1, V_1\} \) and \( \{U_2, V_2\} \) are compatible, and both \( U_1 \) and \( U_2 \) are minimal components, we have that \( V_2 \supset U_1 \).

This containment implies that \( U_1 \cap U_2 \subseteq V_2 \cap U_2 \subseteq \gamma \). Therefore, \( (U_1 - \gamma) \cap (U_2 - \gamma) = \emptyset \). In general, \( (U_i - \gamma) \cap (U_j - \gamma) = \emptyset \) for distinct \( i, j \in \{1, 2, \ldots, k\} \). Hence, \( \{U_i - \gamma\}_{i=1}^k \) is a partition of \( E(M) - \gamma \). It only remains to show that \( k \geq 3 \). Since \( \gamma \) is a multiple connection, \( k \geq 2 \). If \( k = 2 \), then \( V_1 = U_2 \), contradicting the observation made prior to (4.3.3), that a minimal split of a multiple connection has only one minimal component. Hence \( k \geq 3 \). \( \square \)

(4.3.5) *Suppose that \( X = \{X_1, X_2\} \) is a minimal split with respect to a multiple connection and that \( X_1 \) is the minimal component of this split. Then \( r(X_2) \geq 4 \).*

**Proof.** By (4.3.4), there is a split \( Y = \{Y_1, Y_2\} \) of \( M \) such that \( Y_1 \subset X_2 \). By (3.2.14), the matroid \( M_2 \) has a split. Hence \( r(M_2) \geq 4 \). The result follows from the observation that \( r(X_2) = r(M_2) \). \( \square \)

The next two results are used to prove the first of our main results, Proposition 4.3.8. These two results will also be used in many subsequent results.

(4.3.6) *Suppose that \( X = \{X_1, X_2\} \) and \( Y = \{Y_1, Y_2\} \) are splits of \( M \). Then*

\[
r(X_1 \cap Y_1) + r(X_1 \cap Y_2) \leq r(X_1) + r[cl_{\gamma}(X_1) \cap \gamma(Y)].
\]
Proof. By the modularity of $P$, we have that

$$r[cl_P(X_1) \cap cl_P(Y_1)] + r[cl_P(X_1) \cap cl_P(Y_2)]$$

$$= r(cl_P(X_1)) + r[cl_P(X_1) \cap cl_P(Y_1) \cap cl_P(Y_2)]$$

$$= r(cl_P(X_1)) + [cl_P(X_1) \cap \gamma(Y)].$$

The result follows by applying (3.2.1). $\square$

(4.3.7) Suppose that $X = \{X_1, X_2\}$ and $Y = \{Y_1, Y_2\}$ are crossing splits of $M$ and that $\delta(X) = \gamma(X)$. Then $\gamma(Y) = \gamma(X)$.

Proof. Since $r(Y_1) + r(Y_2) = r(M) + 2$, (3.2.1) implies that $r(Y_1 \cap X_1) + r(Y_2 \cap X_1) \leq r(X_1) + 2$. Since both $Y_1$ and $Y_2$ are closed, we may assume that $\gamma(X) \subseteq Y_1$. Since $X$ and $Y$ cross, $(Y_1 \cap X_1) - \gamma(X)$ is non-empty. Therefore $r(Y_1 \cap X_1) \geq 3$. Again, since $X$ and $Y$ cross, $r(Y_1 \cap X_1) < r(X_1)$. Since $M|X_1$ is 3-connected, (3.2.3) implies that $\{Y_1 \cap X_1, Y_2 \cap X_1\}$ is a split of $M|X_1$. Hence, $cl_P(Y_1 \cap X_1) \cap cl_P(Y_2 \cap X_1)$ is a rank-two flat of $P$. Hence $\gamma(Y) \subseteq X_1$. By a similar argument, $\gamma(Y) \subseteq X_2$.

Therefore $\gamma(Y) = \gamma(X)$. $\square$

The next result follows from (4.3.3), (4.3.7), and the definition of a minimal split. It is the first of two main results in this section. It says that if all three elements of a connection are contained in the matroid, then there is a good split associated with that connection.

(4.3.8) Proposition. Suppose that $\gamma$ is a connection of $M$ such that $\gamma \subseteq E(M)$. Then $\gamma$ is a connection of a good split of $M$. Moreover, a split $X$ with connection $\gamma$ is a good split of $M$ if and only if $X$ is a minimal split with respect to $\gamma$. 

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The next three results will be used to prove the other main result in this section. In addition, (4.3.10) and (4.3.11) will find use in subsequent developments.

(4.3.9) Suppose that $X = \{X_1, X_2\}$ and $W = \{U, V\}$ are distinct splits with the same connection $\gamma$. Further, suppose that $W$ is a minimal split with respect to $\gamma$, that $X_1 \supset U$, and that $X$ generates the decomposition $\{M_1, M_2\}$. Then $\{U \cup \gamma, (V \cap X_1) \cup \gamma\}$ is a good split of $M_1$ with connection $\gamma$.

Proof. Let $M' = P((E(M) \cup \gamma))$. By (3.3.5), both $X' = \{X_1 \cup \gamma, X_2 \cup \gamma\}$ and $W' = \{U \cup \gamma, V \cup \gamma\}$ are splits of $M'$ with the same connection $\gamma$. Observe that $X'$ generates the decomposition $\{M_1, M_2\}$. Suppose that $Z' = \{Z'_1, Z'_2\}$ is a split of $M'$ which crosses $W'$ and has connection $\gamma$. Then (3.3.6) implies that $Z = \{Z_1, Z_2\}$ is a split of $M$ with connection $\gamma$, where $Z_i = Z'_i - (\gamma - \delta(X))$ for $i \in \{1, 2\}$. Since $W$ is a minimal split of $M$, the splits $W$ and $Z$ are compatible and we may assume that $Z_1 \supset U$. Then $Z'_1 = Z_1 \cup \gamma \supset U \cup \gamma$. This contradicts the assumption that $W'$ and $Z'$ cross. Hence, $W'$ is a minimal split with respect to $\gamma$. By Proposition 4.3.8, $W'$ is a good split of $M'$. By (3.2.16), the split $\{U \cup \gamma, (V \cap X_1) \cup \gamma\}$ is a good split of $M_1$ with connection $\gamma$. □

(4.3.10) Suppose that $X = \{X_1, X_2\}$ and $Y = \{Y_1, Y_2\}$ are splits of $M$ such that $\gamma(X) \neq \gamma(Y)$ and $\text{cl}_p(Y_1) \supset \gamma(X)$. Then $X$ and $Y$ are compatible.

Proof. Let $M' = P((E(M) \cup \gamma(X)))$. By (3.3.5), $M'$ is a 3-connected binary matroid and $X' = \{X'_1, X'_2\}$ is a split of $M'$ where $X'_1$ and $X'_2$ are $X_1 \cup \gamma(X)$ and $X_2 \cup \gamma(X)$, respectively, and $\gamma(X') = \gamma(X)$. Let $Y'_i = \text{cl}_{M'}(Y_i)$ for each $i \in \{1, 2\}$. Observe that
\( Y'_1 = Y_1 \cup \gamma(X) \) and \( Y'_2 = Y_2 \cup (\gamma(X) \cap \gamma(Y)) \). By (3.3.5), \( Y' = \{Y'_1, Y'_2\} \) is a split of \( M' \) and \( \gamma(Y') = \gamma(Y) \). Hence \( \gamma(X') \neq \gamma(Y') \). By (4.3.7), \( X' \) and \( Y' \) are compatible.

Therefore, we may assume that either \( Y'_1 \supset X'_1 \) or \( X'_1 \supset Y'_1 \). In the former case,

\[
Y_1 = Y'_1 - (\gamma(X) - \delta(X)) \\
\supseteq X'_1 - (\gamma(X) - \delta(X)) \\
= X_1.
\]

Hence, \( X \) and \( Y \) are compatible. In the latter case,

\[
X_1 = X'_1 - (\gamma(X) - \delta(X)) \\
\supseteq Y'_1 - (\gamma(X) - \delta(X)) \\
= Y_1.
\]

Hence, \( X \) and \( Y \) are compatible. \( \Box \)

\((4.3.11)\) Suppose that \( X = \{X_1, X_2\} \) and \( Y = \{Y_1, Y_2\} \) are crossing splits of \( M \) and \( r(X_1 \cap Y_1) + r(X_1 \cap Y_2) = r(X_1) + 2 \). Then \( \gamma(X) = \gamma(Y) \).

**Proof.** By (4.3.6), \( clp(X_1) \supseteq \gamma(Y) \). The result follows by applying (4.3.10). \( \Box \)

We conclude this section with the second of our two main results.

\((4.3.12)\) **Proposition.** Suppose that \( \gamma \) is a multiple connection of \( M \) and that \( X \) is a minimal split with respect to \( \gamma \). Then \( X \) is a good split of \( M \).

**Proof.** Suppose that \( Z = \{Z_1, Z_2\} \) is a split of \( M \) that crosses \( X = \{X_1, X_2\} \). Since \( Z \) crosses \( X \), and \( X \) is minimal with respect to \( \gamma \), the connections \( \gamma(Z) \) and \( \gamma \) are distinct. By (4.3.10), we have that \( |clp(Z_i) \cap \gamma| \leq 1 \) for each \( i \in \{1, 2\} \).
Suppose that $Z$ crosses every split having $\gamma$ as its connection. Further, suppose that \{\{U, V\}\} is a minimal split with respect to $\gamma$ and that $U$ is the minimal component of this split. By (4.3.11), we have that $r(U \cap Z_1) + r(U \cap Z_2) \leq r(U) + 1$. Let \{\{U_1, V_1\}, \{U_2, V_2\}, \ldots, \{U_k, V_k\}\} be the minimal splits with respect to $\gamma$ with minimal components $U_1, U_2, \ldots, U_k$. By (4.3.4), $k \geq 3$. Suppose that $r(U_i \cap Z_1) + r(U_i \cap Z_2) = r(U_i) + 1$ for each $i \in \{1, 2, \ldots, k\}$. Since $\gamma \subseteq \text{cl}_{\rho}(U_i)$ for each $i \in \{1, 2, \ldots, k\}$, we have that $\gamma \subseteq \text{cl}_{\rho}(U_i^{l-1}_{i=1}) \cap \text{cl}_{\rho}(U_i)$ for all $l \in \{2, 3, \ldots, k\}$. Moreover, $U_i^{l-1}_{i=1}$ is a subset of $V_i$ and $\gamma = \text{cl}_{\rho}(V_i) \cap \text{cl}_{\rho}(U_i)$. Hence $\gamma = \text{cl}_{\rho}(U_i^{l-1}_{i=1}) \cap \text{cl}_{\rho}(U_i)$. Therefore, by the modularity of $P$,

$$r\left(\bigcup_{i=1}^{k-1} U_i\right) + r(U_k) = r(M) + 2,$$

and

$$r\left(\bigcup_{i=1}^{k-2} U_i\right) + r(U_{k-1}) = r\left(\bigcup_{i=1}^{k-1} U_i\right) + 2.$$

Hence,

$$r\left(\bigcup_{i=1}^{k-2} U_i\right) + r(U_{k-1}) + r(U_k) = r(M) + 4.$$

Proceeding in this manner, we get that

$$\sum_{i=1}^{k} r(U_i) = r(M) + 2k - 2.$$

Since $|\text{cl}_{\rho}(Z_i) \cap \gamma| \leq 1$ for each $i \in \{1, 2\}$, we have that $|\text{cl}_{\rho}(U_i^{l-1}_{j=1}) \cap \text{cl}_{\rho}(U_i) \cap \text{cl}_{\rho}(Z_i)| \leq 1$ for each $i \in \{1, 2\}$. By the modularity of $P$, we have that

$$r(Z_i \cap (\bigcup_{j=1}^{l-1} U_j)) + r(Z_i \cap U_i) \leq r(Z_i \cap (\bigcup_{j=1}^{l} U_j)) + 1, \text{ for each } i \in \{1, 2\}. \quad (4.2)$$
Therefore, \( r(Z_1 \cap U_1) + r(Z_1 \cap U_2) \leq r(Z_1 \cap (U_1 \cup U_2)) + 1 \). Applying (4.2) again, we get that \( r(Z_1 \cap (U_1 \cup U_2)) + r(Z_1 \cap U_3) \leq r(Z_1 \cap (U_1 \cup U_2 \cup U_3)) + 1 \). Hence, \( r(Z_1 \cap U_1) + r(Z_1 \cap U_2) + r(Z_1 \cap U_3) \leq r(Z_1 \cap (U_1 \cup U_2 \cup U_3)) + 2 \); and, in general, we get that \( \sum_{i=1}^{k} r(Z_1 \cap U_i) \leq r(Z_1) + k - 1 \). Similarly, \( \sum_{i=1}^{k} r(Z_2 \cap U_i) \leq r(Z_2) + k - 1 \).

Hence,

\[
\sum_{i=1}^{k} (r(Z_1 \cap U_i) + r(Z_2 \cap U_i)) \leq r(Z_1) + r(Z_2) + 2k - 2
\]

\[
= r(M) + 2k.
\]

By assumption, \( r(Z_1 \cap U_i) + r(Z_2 \cap U_i) = r(U_i) + 1 \) for all \( i \in \{1, 2, \ldots, k\} \). Therefore,

\[
\sum_{i=1}^{k} (r(Z_1 \cap U_i) + r(Z_2 \cap U_i)) = \sum_{i=1}^{k} r(U_i) + k.
\]

Hence,

\[
r(M) + 2k \geq \sum_{i=1}^{k} r(U_i) + k
\]

\[
= [r(M) + 2k - 2] + k
\]

\[
= r(M) + 3k - 2.
\]

But this implies that \( k \leq 2 \); a contradiction. Therefore, if \( Z \) crosses every split having \( \gamma \) as its connection, then \( r(U \cap Z_1) + r(U \cap Z_2) = r(U) \) for some minimal component \( U \) in a minimal split \( \{U, V\} \) of \( M \).

Suppose that \( Z \) crosses every split having \( \gamma \) as its connection, and that \( \{U, V\} \) is a minimal split with respect to \( \gamma \) with minimal component \( U \) such that \( r(U \cap Z_1) + r(U \cap Z_2) = r(U) \). Since \( Z \) crosses \( \{U, V\} \), both \( (U \cap Z_1) - \gamma \) and \( (U \cap Z_2) - \gamma \)
are non-empty. Since $P[(U \cup \gamma)$ is 3-connected, we have that

$$r[(U \cap Z_1) \cup \gamma] + r(U \cap Z_2) = r(U \cap Z_1) + r[(U \cap Z_2) \cup \gamma]$$

$$= r(U) + 2.$$  \hfill (4.3)

To see this, first note that, since $r(U \cap Z_1) + r(U \cap Z_2) = r(U)$, the sets $(U \cap Z_i) \cup \gamma$ and $(U \cap Z_j) - \gamma$ partition $U \cup \gamma$ for $\{i, j\} = \{1, 2\}$. Therefore, $r[(U \cap Z_i) \cup \gamma] + r[(U \cap Z_j) - \gamma] \geq r(U) + 1$ with equality only if $Z_i \cup \gamma \supseteq U \cup \gamma$. By (3.2.5), $Z_i \supseteq U$. This contradicts the assumption that $\{U, V\}$ and $Z$ cross. Hence, (4.3) holds. Therefore, by submodularity, $cl_p(U \cap Z_i) \cap \gamma$ is empty for $i \in \{1, 2\}$. Hence, $cl_p(V) \cap cl_p(U \cap Z_i)$ is empty. Therefore, $cl_p(V \cap Z_i) \cap cl_p(U \cap Z_i)$ is empty. Hence $r(V \cap Z_i) + r(U \cap Z_i) = r(Z_i)$ for $i \in \{1, 2\}$. Therefore,

$$r(U) + r(V) = r(Z_1) + r(Z_2)$$

$$= r(V \cap Z_1) + r(U \cap Z_1) + r(V \cap Z_2) + r(U \cap Z_2)$$

$$= r(U) + r(V \cap Z_1) + r(V \cap Z_2).$$

Hence, $r(V) = r(V \cap Z_1) + r(V \cap Z_2)$. By an argument similar to that used for (4.3), we have that

$$r[(V \cap Z_1) \cup \gamma] + r(V \cap Z_2) = r(V \cap Z_1) + r[(V \cap Z_2) \cup \gamma]$$

$$= r(V) + 2.$$  

Suppose that $r(V \cap Z_1) \geq 3$. Then (3.2.3) implies that $\{(V \cap Z_1), (V \cap Z_2) \cup \gamma\}$ is a split of $P[(V \cup \gamma)$. By (4.3.9), $\{U_j \cup \gamma, (V - U_j) \cup \gamma\}$ is a good split of $P[(V \cup \gamma)$ for
each $U_j$ such that $U_j \neq U$. Therefore, either $V \cap Z_1$ or $(V \cap Z_2) \cup \gamma$ must contain either $U_j \cup \gamma$, or $(V - U_j) \cup \gamma$. Suppose that $U_j \subseteq (V \cap Z_2) \cup \gamma$. Since

$$U_j = \text{cl}_M(U_j - \gamma)$$

$$\subseteq \text{cl}_M(V \cap Z_2)$$

$$= V \cap Z_2,$$

we have that $U_j$ is contained by $V \cap Z_2$. Hence, if either $V \cap Z_1$ or $(V \cap Z_2) \cup \gamma$ contains $U_j \cup \gamma$, then $V \cap Z_1$ or $V \cap Z_2$ must contain $U_j$. Therefore, either $Z_1$ or $Z_2$ contains $U_j$. This implies that $\{Z_1, Z_2\}$ is compatible with $\{U_j, V_j\}$; a contradiction.

Since there is at least one minimal split $\{U_i, V_i\}$ other than $\{U, V\}$ and $\{U_j, V_j\}$, (4.3.4) implies that $(V - U_j) \cup \gamma \supseteq U_i \cup \gamma$. Therefore, if either $V \cap Z_1$ or $(U \cap Z_2) \cup \gamma$ contains $(V - U_j) \cup \gamma$, then either $V \cap Z_1$ or $(V \cap Z_2) \cup \gamma$ contains $U_i \cup \gamma$. By the same argument as was used in the case of $U_j \cup \gamma$, this contradicts the assumption that $\{Z_1, Z_2\}$ and $\{U_i, V_i\}$ cross. Hence, we may assume that $|V \cap Z_1| = |V \cap Z_2| = 2$.

Hence, $r(V \cup \gamma) = 4$ and $|V \cup \gamma| \leq 7$. Since $P|(V \cup \gamma)$ is 3-connected and binary, $P|(V \cup \gamma) \cong F_7^*$. But $F_7^*$ contains no triangles, while $P|(V \cup \gamma)$ contains the triangle $\gamma$. Hence, there is at least one split having $\gamma$ as its connection which $Z$ does not cross.

Suppose that $Y = \{Y_1, Y_2\}$ is a split having connection $\gamma$ and that $Y$ and $Z$ are compatible. Since $X$ is minimal, the splits $X$ and $Y$ are compatible. Therefore, we may assume that $X_1 \subseteq Y_1$. Since $Y$ and $Z$ are compatible, we may assume that either $Z_1 \supset Y_1$ or $Y_1 \supset Z_1$. If $Z_1 \supset Y_1$, then $Z_1 \supset X_1$, contradicting the assumption
that $X$ and $Z$ cross. Hence $Y_1 \supset Z_1$. Let $\{M_1, M_2\}$ be the decomposition generated by the split $\{Y_1, Y_2\}$. Let $Z' = \{Z'_1, Z'_2\}$ be the split of $M_1$ induced by $Z$, where $Z'_1 = \text{cl}_{M_1}(Z_1)$ and $Z'_2 = (Z_2 \cap Y_1) \cup \gamma$. Likewise, let $X' = \{X'_1, X'_2\}$ be the split of $M_1$ induced by $X$, where $X'_1 = X_1 \cup \gamma$ and $X'_2 = (X_2 \cap Y_1) \cup \gamma$. By (3.2.13), $\gamma(Z') = \gamma(Z)$ and $\gamma(X') = \gamma$. Therefore, $\gamma(X') \neq \gamma(Z')$. Hence, $\gamma$ is contained in $Z'_1$ or $Z'_2$, but not both. Since $Z_2 \supset Y_2$, we have that $\text{cl}_p(Z_2) \supset \gamma$. Since $\gamma$ does not belong to both $\text{cl}_p(Z_1)$ and $\text{cl}_p(Z_2)$, the set $\gamma \not\in \text{cl}_p(Z_1)$. Therefore, $\gamma \not\in Z'_1$. Hence, $\gamma \subseteq Z'_2$. By (4.3.7), the splits $X'$ and $Z'$ are compatible. Therefore $X'_1 \subset Z'_1$, or $X'_2 \subset Z'_1$, or $X'_1 \supset Z'_1$, or $X'_2 \supset Z'_1$. Since $\gamma \subseteq X'_1$ but $\gamma \not\in Z'_1$, the containment $X'_1 \subset Z'_1$ does not hold. Likewise, $X'_2 \subset Z'_1$ does not hold. If $X'_1 \supset Z'_1$, then $X_1 \supset Z_1$, contradicting the assumption that $X$ and $Z$ cross. Therefore, $X'_2 \supset Z'_1$. Since $X_2 \supset (\gamma \cap E(M))$, we have that $X_2 \supset Z_1$; a contradiction. Therefore, $X$ is a good split. □

4.4 On the Size of $\delta(X)$

Recall that $\mathcal{M}$ is the set of 3-connected binary matroids, each member of which has at least one split but no good split. These are the matroids that can appear as components in minimal decompositions but are not vertically 4-connected. In this section, we proceed with the characterization of $\mathcal{M}$ by analyzing a matroid $M$ based on the number of elements in $\delta(X)$ for a split $X$ of $M$. That analysis was actually begun in the previous section with Proposition 4.3.8. That result shows that if $M$ has a split $X$ such that $|\delta(X)| = 3$, then $M \not\in \mathcal{M}$. Therefore, in this section, we begin with the case $|\delta(X)| = 2$, and conclude with the case $\delta(X) = \emptyset$. The following technical result will be used in all three of the cases we will analyze.
Suppose that \( X = \{X_1, X_2\} \), \( Y = \{Y_1, Y_2\} \), and \( Z = \{Z_1, Z_2\} \) are splits of \( M \) such that \( \gamma(X) \), \( \gamma(Y) \), and \( \gamma(Z) \) are distinct and that \( \{X, Y\} \) and \( \{X, Z\} \) are compatible pairs of splits. Further, suppose that \( \{M_1, M_2\} \) is the decomposition generated by \( X \), and that \( Y' \) and \( Z' \) are the splits of \( M_1 \) induced by \( Y \) and \( Z \), respectively. If \( Y \) and \( Z \) cross, then \( Y' \) and \( Z' \) cross.

**Proof.** Suppose that \( Y' = \{Y'_1, Y'_2\} \) and \( Z' = \{Z'_1, Z'_2\} \). We may assume that \( Y_1 \subseteq X_1 \) and \( Z_1 \subseteq X_1 \). By (3.2.11), \( Y'_1 = \text{cl}_{M_1}(Y_1) \) and \( Y'_2 = (Y_2 \cap X_1) \cup \gamma(X) \). Also, \( Z'_1 = \text{cl}_{M_1}(Z_1) \) and \( Z'_2 = (Z_2 \cap X_1) \cup \gamma(X) \). Suppose that \( Y \) and \( Z \) cross but \( Y' \) and \( Z' \) are compatible. Since \( Y \) and \( Z \) cross and \( \gamma(Y) \neq \gamma(Z) \), (4.3.10) implies that \( \gamma(Y) \not\subseteq \text{cl}_{P}(Z_i) \) for each \( i \in \{1, 2\} \). But, because \( Y' \) and \( Z' \) are compatible, either \( \gamma(Y') \subseteq \text{cl}_{P}(Z'_1) \), or \( \gamma(Y') \subseteq \text{cl}_{P}(Z'_2) \). By (3.2.13), \( \gamma(Y') = \gamma(Y) \). By definition, \( \text{cl}_{P}(Z'_i) \subseteq \text{cl}_{P}(Z_i) \) for each \( i \in \{1, 2\} \). Hence, if \( Y' \) and \( Z' \) are compatible, we run into a contradiction. Therefore, \( Y' \) and \( Z' \) cross. \( \Box \)

The next several results, culminating in (4.4.5), deal with the case \( |\delta(X)| = 2 \). The results (4.4.2) and (4.4.4) are the lemmas used to prove (4.4.5), while (4.4.3) is stated in such a way as to be useful in later cases. The analysis of the cases \( |\delta(X)| = 1 \) and \( |\delta(X)| = 0 \) are summarized in (4.4.8) and (4.4.12), respectively. The sequence of lemmas used to prove (4.4.5) is repeated, with the appropriate changes, in the two subsequent cases. We have stated (4.4.3) and (4.4.8) in such a way that those arguments that do apply in subsequent cases need not be repeated. Even so, the process and proofs may appear unnecessarily repetitive. But a close examination exposes difficulties unique to each case.
Suppose that $X = \{X_1, X_2\}$ is a split of $M$ generating the decomposition $\{M_1, M_2\}$ and that $|\delta(X)| = 2$. Further suppose that $X$ is crossed by a split $Y = \{Y_1, Y_2\}$ such that $\gamma(X) \neq \gamma(Y)$, and that $M_1$ has no splits. Then $M_1 \cong M(K_4)$.

**Proof.** Since $X$ and $Y$ cross and $\gamma(X) \neq \gamma(Y)$, (4.3.10) implies that $|\text{cl}_P(Y_i) \cap \gamma(X)| \leq 1$ for each $i \in \{1, 2\}$. In particular, $|Y_i \cap \gamma(X)| = 1$ for each $i \in \{1, 2\}$ and $[\text{cl}_P(Y_1) \cap \gamma(X)] \cap [\text{cl}_P(Y_2) \cap \gamma(X)] = \emptyset$. Hence $\gamma(Y) \cap \gamma(X) = \emptyset$. Moreover, by the modularity of $P$, $r[(Y \cap X_1) \cup \gamma(X)] = r(Y \cap X_1) + 1$ for each $i \in \{1, 2\}$. By (4.3.11), $r(Y \cap X_1) + r(Y \cap X_1) \leq r(X_1) + 1$. Therefore,

$$r[(Y \cap X_1) \cup \gamma(X)] + r(Y \cap X_1) = r(Y \cap X_1) + r[(Y \cap X_1) \cup \gamma(X)]$$

$$= r(Y \cap X_1) + r(Y \cap X_1) + 1$$

$$\leq r(M_1) + 2.$$

Since $(Y \cap X_1) - \gamma(X)$ is non-empty and $(Y \cap X_1) \cap \gamma(X)$ is non-empty, we have that $|Y_i \cap X_1| \geq 2$. Hence, $r(Y_i \cap X_1) \geq 2$ for each $i \in \{1, 2\}$. Therefore, $r[(Y_i \cap X_1) \cup \gamma(X)] \geq 3$ for each $i \in \{1, 2\}$. Since $M_1$ is 3-connected and has no splits, (3.2.3) implies that $r[(Y_i \cap X_1) \cup \gamma(X)] = r(M_1)$ for each $i \in \{1, 2\}$. Hence, $r(Y_1 \cap X_1) = r(Y_2 \cap X_1) = 2$ and $r(M_1) = 3$. Since $r(Y_i \cap X_1) = 2$ for each $i \in \{1, 2\}$, we have that $|Y_i \cap X_1| \leq 3$ for each such $i \in \{1, 2\}$. Moreover, if $|Y_i \cap X_1| = 3$ for each $i \in \{1, 2\}$, then $(Y \cap X_1) \cap (Y \cap X_1)$ is non-empty. Hence $|X_1| \leq 5$. Since $|\delta(X)| = 2$, we have that $|E(M_1)| = |X_1| + 1 \leq 6$. Since $M_1$ is 3-connected, $M_1 \cong M(K_4)$.

**4.4.3** The following conditions cannot all be satisfied simultaneously.

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(i) $X = \{X_1, X_2\}$ is a split of $M$ generating the decomposition $\{M_1, M_2\}$.

(ii) $Y = \{Y_1, Y_2\}$ is a split of $M$ such that $|\text{cl}_P(Y_i) \cap \gamma(X)| = 1$ for each $i \in \{1, 2\}$.

(iii) $Z'$ is a good split of $M_1$.

(iv) The connections $\gamma(X)$, $\gamma(Y)$, and $\gamma(Z')$ are distinct.

(v) $Y$ crosses $X$ and $W$, where $W = \{W_1, W_2\}$ is a split of $M$ inducing $Z'$.

**Proof.** Suppose that conditions (i)-(v) are all satisfied. Let $\{y_i\} = \text{cl}_P(Y_i) \cap \gamma(X)$ for each $i \in \{1, 2\}$. Note that $y_1$ and $y_2$ may be equal. Let $M' = P|(E(M) \cup \{y_1, y_2\})$.

Let $X' = \{X'_1, X'_2\}$, where $X'_i = X_i \cup \{y_1, y_2\}$ for each $i \in \{1, 2\}$. Let $Y' = \{Y'_1, Y'_2\}$, where $Y'_i = Y_i \cup \{y_i\}$ for each $i \in \{1, 2\}$. Then (i) and (ii) are satisfied, where $X'$ replaces $X$ and $Y'$ replaces $Y$. Observe that $\gamma(X') = \gamma(X)$ and $\gamma(Y') = \gamma(Y)$.

Hence, (iv) is satisfied where $X'$ replaces $X$ and $Y'$ replaces $Y$. Let $W'$ be the split of $M'$ inducing the split $Z'$ of $M_1$. Observe that (v) holds when $Y$, $X$, and $W$ are replaced by $Y'$, $X'$, and $W'$, respectively. Therefore, we may assume that both $y_1$ and $y_2$ are in $E(M)$. We may also assume that $\gamma(X) \subseteq Z'_2$ and that $W_1 \subseteq X_1$. By the modularity of $P$, we have that $r[(Y_i \cap X_1) \cup \gamma(X)] = r(Y_i \cap X_1) + 1$ for each $i \in \{1, 2\}$. By (4.3.11) we have that $r(Y_1 \cap X_1) + r(Y_2 \cap X_1) \leq r(X_1) + 1$. Therefore,

$$r[(Y_1 \cap X_1) \cup \gamma(X)] + r(Y_2 \cap X_1) = r(Y_1 \cap X_1) + r[(Y_2 \cap X_1) \cup \gamma(X)]$$

$$= r(Y_1 \cap X_1) + r(Y_2 \cap X_1) + 1$$

$$\leq r(M_1) + 2.$$
Since both \((Y_i \cap X_1) - \gamma(X)\) and \((Y_i \cap X_1) \cap \gamma(X)\) are non-empty, we have that \(|Y_i \cap X_1| \geq 2\). Hence, \(r(Y_i \cap X_1) \geq 2\) and \(r([Y_i \cap X_1] \cup \gamma(X)) \geq 3\) for each \(i \in \{1, 2\}\).

Either,

(a) neither of the sets \(\{cl_{M_1}([Y_1 \cap X_1] \cup \gamma(X)), cl_{M_1}(Y_2 \cap X_1)\}\) and \(\{cl_{M_1}([Y_2 \cap X_1] \cup \gamma(X)), cl_{M_1}(Y_1 \cap X_1)\}\) is a split of \(M_1\); or

(b) precisely one of \(\{cl_{M_1}([Y_1 \cap X_1] \cup \gamma(X)), cl_{M_1}(Y_2 \cap X_1)\}\) and \(\{cl_{M_1}([Y_2 \cap X_1] \cup \gamma(X)), cl_{M_1}(Y_1 \cap X_1)\}\) is a split of \(M_1\); or

(c) both of the sets \(\{cl_{M_1}([Y_1 \cap X_1] \cup \gamma(X)), cl_{M_1}(Y_2 \cap X_1)\}\) and \(\{cl_{M_1}([Y_2 \cap X_1] \cup \gamma(X)), cl_{M_1}(Y_1 \cap X_1)\}\) are splits of \(M_1\).

In case (a), (3.2.3) implies that \(r([Y_1 \cap X_1] \cup \gamma(X)] = r([Y_2 \cap X_1] \cup \gamma(X)] = r(M_1)\) and, hence, \(r(Y_1 \cap X_1) = r(Y_2 \cap X_1) = 2\). Therefore \(r(M_1) = 3\). This contradicts the assumption that \(M_1\) has a good split. In case (b), we may assume that \(\{cl_{M_1}([Y_1 \cap X_1] \cup \gamma(X)), cl_{M_1}(Y_2 \cap X_1)\}\) is a split and that \(\{cl_{M_1}([Y_2 \cap X_1] \cup \gamma(X)), cl_{M_1}(Y_1 \cap X_1)\}\) is not a split. Then \(r(Y_2 \cap X_1) \geq 3\). By (3.2.3), \(r([Y_2 \cap X_1] \cup \gamma(X)] = r(M_1)\) and, hence, \(r(Y_1 \cap X_1) = 2\). Since \(Z'\) is a good split, either

(1) \(Z_1 \supset cl_{M_1}([Y_1 \cap X_1] \cup \gamma(X)]\);

(2) \(Z_2 \supset cl_{M_1}(Y_2 \cap X_1)\);

(3) \(Z_2 \subset cl_{M_1}([Y_1 \cap X_1] \cup \gamma(X)]\); or

(4) \(Z_1 \subset cl_{M_1}(Y_2 \cap X_1)\).
The first containment contradicts the assumption that $|cl_p(Z_1') \cap \gamma(X)| \leq 1$. Since $r[(Y_1 \cap X_1) \cup \gamma(X)] = 3$, containment (3) implies that $r(Z_2') \leq 2$. This contradicts the assumption that $Z'$ is a split. We will show that if containment (2) holds then $W_2 \supset Y_2$, contradicting the assumption that $Y$ and $W$ cross. Suppose that $Z_2' \supset cl_{M_1}(Y_2 \cap X_1)$ and that $x \in Y_2$. Then either $x \in Y_2 \cap X_1$ or $x \in X_2$. Since $W_2$ contains $X_2$, we may assume that $x \in Y_2 \cap X_1$. Hence $x \in Z_2'$. Observe that $Z_2' \cap X_1 = W_2 \cap X_1$. Hence $x \in W_2$. Therefore, $W_2 \supset Y_2$. Finally, we will show that if containment (4) holds, then $W_1 \subseteq Y_2$, contradicting the assumption that $Y$ and $W$ cross. Suppose that $Z_1' \subset cl_{M_1}(Y_2 \cap X_1)$ and that $x \in W_1$. Clearly, $x \in Z_1'$. Hence $x \in [cl_{M_1}(Y_2 \cap X_1)] \cap E(M)$. Observe that $[cl_{M_1}(Y_2 \cap X_1)] \cap E(M)$ is a closed set of $M$. In particular, $[cl_{M_1}(Y_2 \cap X_1)] \cap E(M) = cl_{M_1}(Y_2 \cap X_1) = Y_2 \cap X_1$. Therefore $x \in Y_2$. Hence $W_1 \subseteq Y_2$. Therefore, case (b) cannot occur. Thus, we may assume that case (c) holds. Since $Z'$ is a good split, $Z'$ and $\{cl_{M_1}[(Y_1 \cap X_1) \cup \gamma(X)], cl_{M_1}(Y_2 \cap X_1)\}$ are compatible. Hence, one of the containments (1)–(4) must hold. By using the same arguments as for case (b), it follows that the first, second, and last containments cannot hold. Hence $Z_2' \subset cl_{M_1}[(Y_1 \cap X_1) \cup \gamma(X)]$. Using an analogous argument when analyzing the compatible splits $Z'$ and $\{cl_{M_1}[(Y_2 \cap X_1) \cup \gamma(X)], cl_{M_1}(Y_1 \cap X_1)\}$, we can conclude that $Z_1' \subset cl_{M_1}[(Y_2 \cap X_1) \cup \gamma(X)]$. Therefore, $Z_1' \supset cl_{M_1}(Y_2 \cap X_1)$ and $Z_1' \supset cl_{M_1}(Y_1 \cap X_1)$. Since $cl_{M_1}(Y_2 \cap X_1) \cup cl_{M_1}(Y_1 \cap X_1) \supset X_1$ we have that $Z_1' \supset X_1$. Hence, $W_1 \supset X_1$, a contradiction. Thus, all three cases (a), (b), and (c) lead to contradictions. $\square$
(4.4.4) Suppose that \( X = \{X_1, X_2\} \) is a split of \( M \) generating the decomposition \( \{M_1, M_2\} \) and that \( |\delta(X)| = 2 \). Further suppose that \( M \in \mathcal{M} \) and \( M_1 \not\in \mathcal{M} \). Then \( M_1 \) has no splits.

Proof. Suppose that \( M_1 \) has a good split \( Z' = \{Z'_1, Z'_2\} \). We may assume that \( \gamma(X) \subseteq Z'_2 \). By Proposition 4.3.12, \( M \) has no multiple connection. Therefore, \( |cl_p(Z'_1) \cap \gamma(X)| \leq 1 \). Let \( Z = \{Z_1, Z_2\} \) be the split of \( M \) inducing \( Z' \). Since \( M \in \mathcal{M} \), there is a split \( Y = \{Y_1, Y_2\} \) of \( M \) which crosses \( Z \). Since \( M \) has no multiple connections, \( \gamma(X) \), \( \gamma(Y) \), and \( \gamma(Z) \) are distinct. Since \( Z' \) is a good split of \( M_1 \) and \( Z \) and \( X \) are compatible, (4.4.1) implies that \( Y \) crosses \( X \). Since \( X \) and \( Y \) cross and \( \gamma(X) \neq \gamma(Y) \), (4.3.10) implies that \( |cl_p(Y_i) \cap \gamma(X)| \leq 1 \) for each \( i \in \{1, 2\} \). But this situation contradicts (4.4.3). Hence, if \( M_1 \not\in \mathcal{M} \), then \( M_1 \) has no splits. \( \square \)

The next result follows from (4.4.2) and (4.4.4).

(4.4.5) Suppose that \( X = \{X_1, X_2\} \) is a split of \( M \) generating the decomposition \( \{M_1, M_2\} \) and that \( |\delta(X)| = 2 \). Further, suppose that \( M \in \mathcal{M} \) and \( M_1 \not\in \mathcal{M} \). Then \( M_1 \cong M(K_4) \).

The next several results, culminating in (4.4.10), deal with the case \( |\delta(X)| = 1 \).

Before proceeding with these results, we define a class \( \mathcal{N} \) of matroids that arise in the characterization. One of the main goals of this chapter is to set the groundwork for the results of Chapter 5. Therefore, we will describe \( \mathcal{N} \) in no more detail than is necessary to establish those results. We have also addressed the issue of characterizing those matroids in \( \mathcal{M} \) so as to characterize the analogous set for the
decomposition of 3-connected graphs. It appears that this characterization can be improved in so far as the set $\mathcal{N}$ can be characterized in more detail. This is not done in this dissertation.

**(4.4.6) Definition.** The set $\mathcal{N}$ is the set of ordered pairs $(N,T)$ where $N$ is a 3-connected binary matroid and $T$ is a triangle of $N$, where $(N,T)$ satisfies one of the following conditions:

(i) $N$ is vertically 4-connected and there is a closed subset $K$ of rank two, such that $K \cap T = \emptyset$; or

(ii) $N$ is not vertically 4-connected, and $N$ has a good split. Moreover, for every good split $Z$ of $N$, the set $\gamma(Z)$ and $T$ are disjoint.

**(4.4.7)** Suppose that $X = \{X_1,X_2\}$ is a split of $M$ generating the decomposition $\{M_1,M_2\}$ and that $|\delta(X)| = 1$. Further, suppose that $X$ is crossed by a split $Y = \{Y_1,Y_2\}$ such that $\gamma(X) \neq \gamma(Y)$, and that $M_1$ has no splits. Then $M$ has a decomposition such that at least one of the components is isomorphic to $M(K_4)$ or $F_7$. Moreover, if $M_1$ is not isomorphic to $M(K_4)$ or $F_7$, then $M$ is vertically 4-connected, and

(i) if $N = M_1$ and $T = \gamma(X)$, then $(N,T) \in \mathcal{N}$;

(ii) there is a subset $K$ of $X_1$ satisfying (4.4.6) (i) and an element $z \in X_2 - \gamma(X)$, such that $K \cup \{z\}$ is a component of a split of $M$, with the corresponding component in the decomposition isomorphic to $M(K_4)$; and

(iii) $M_2$ has a triad.
Proof. Let \( x = \delta(X) \). Suppose that \( y \in \gamma(X) - \{x\} \) and that \( y \in cl_p(Y_i) \) for some \( i \in \{1, 2\} \). By (3.3.5), \( M' = P'(E(M) \cup \{y\}) \) is 3-connected, both \( \{X_1 \cup \{y\}, X_2 \cup \{y\}\} \) and \( \{cl_{M'}(Y_1), cl_{M'}(Y_2)\} \) are splits of \( M' \), and \( \{X_1 \cup \{y\}, X_2 \cup \{y\}\} \) generates the decomposition \( \{M_1, M_2\} \) of \( M' \). Observe that this situation satisfies the hypothesis of (4.4.2). Hence \( M_1 \cong M(K_4) \). Therefore, we may assume that \( cl_p(Y_i) \cap \gamma(X) \subseteq \{x\} \) for each \( i \in \{1, 2\} \), and that \( x \in Y_1 \). Hence, \( cl_p(Y_1) \cap \gamma(X) = \{x\} \). Therefore, either \( x \in Y_2 \) and, hence, \( cl_p(Y_2) \cap \gamma(X) = \{x\} \), or \( x \not\in Y_2 \). Then \( \gamma(X) \cap \gamma(Y) = \emptyset \).

Suppose that \( x \in Y_2 \). By (4.3.11), \( r(X_i \cap Y_i) + r(X_i \cap Y_2) \leq r(X_i) + 1 \). By the modularity of \( P \), we have that \( r[(X_i \cap Y_i) \cup \gamma(X)] = r(X_i \cap Y_i) + 1 \) for each \( i \in \{1, 2\} \). Therefore,

\[
r[(Y_1 \cap X_1) \cup \gamma(X)] + r(Y_2 \cap X_1) = r(Y_1 \cap X_1) + r[(Y_2 \cap X_1) \cup \gamma(X)]
\]

\[
= r(Y_1 \cap X_1) + r(Y_2 \cap X_1) + 1
\]

\[
\leq r(M_1) + 2.
\]

Since \( X \) and \( Y \) cross, the sets \( (Y_1 \cap X_1) - \gamma(X) \) and \( (Y_2 \cap X_1) - \gamma(X) \) are non-empty; and, since \( x \in Y_1 \cap Y_2 \), we have that \( r(Y_i \cap X_1) \geq 2 \) for each \( i \in \{1, 2\} \). Since \( M_1 \) is 3-connected and has no splits, \( r(Y_1 \cap X_1) = r(Y_2 \cap X_1) = 2 \) and \( r(Y_1 \cap X_1) + r(Y_2 \cap X_1) + 1 = r(M_1) + 2 \). Therefore \( r(M_1) = 3 \). Hence, \( M_1 \) is isomorphic to either \( M(K_4) \) or \( F_7 \).

Suppose that \( x \not\in Y_2 \). By (4.3.11), \( r(X_1 \cap Y_1) + r(X_2 \cap Y_1) \leq r(Y_1) + 1 \) and, by (4.3.6), \( r(X_1 \cap Y_2) + r(X_2 \cap Y_2) = r(Y_2) \). Therefore, either \( r(Y_1 \cap X_1) + \)

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r(Y_2 \cap X_1) = r(X_1), or \( r(Y_1 \cap X_2) + r(Y_2 \cap X_2) = r(X_2) \). Suppose that \( r(Y_1 \cap X_1) + r(Y_2 \cap X_1) = r(X_1) \). Since \( r([X_1 \cap Y_1] \cup \gamma(X)) = r(X_1 \cap Y_1) + 1 \), we have that \( r([X_1 \cap Y_1] \cup \gamma(X)) + r(X_1 \cap Y_2) = r(M_1) + 1 \). Since \( M_1 \) is 3-connected and \((X_1 \cap Y_2) - \gamma(X)\) is non-empty, we have that \(|X_1 \cap Y_2| = 1\). Since \( \gamma(X) \) does not meet \( cl_p(Y_2) \), the modularity of \( P \) implies that \( r([X_1 \cap Y_2] \cup \gamma(X)) = r(X_1 \cap Y_2) + 2 \). Hence, \( r(X_1 \cap Y_1) + r([X_1 \cap Y_2] \cup \gamma(X)) = r(M_1) + 2 \). Since \( M_1 \) has no split, we have that \( r(X_1 \cap Y_1) = 2 \). Hence \( r(M_1) = 3 \). Therefore, \( M_1 \) is isomorphic to either \( M(K_4) \) or \( F_7 \). Therefore, we may assume that \( r(Y_1 \cap X_2) + r(Y_2 \cap X_2) = r(X_2) \) and \( r(Y_1 \cap X_1) + r(Y_2 \cap X_1) = r(X_1) + 1 \). Since \( x \in Y_1 \cap X_2 \), the modularity of \( P \) implies that \( r([Y_1 \cap X_2] \cup \gamma(X)) = r(Y_1 \cap X_2) + 1 \). Hence, \( r([Y_1 \cap X_2] \cup \gamma(X)) + r(Y_2 \cap X_2) = r(M_2) + 1 \). Since \( M_2 \) is 3-connected and \((X_2 \cap Y_2) - \delta(X)\) is non-empty, we have that \(|X_2 \cap Y_2| = 1\). Also, \( r([Y_1 \cap X_1] \cup \gamma(X)) + r(Y_2 \cap X_1) = r(M_1) + 2 \). Since \( M_1 \) has no splits, \( r(X_1 \cap Y_2) = 2 \). Let \( K = X_1 \cap Y_2 \) and let \( T = \gamma(X) \). Since \( \gamma(X) \) does not meet \( cl_p(Y_2) \), we have that \( K \cap T = \emptyset \). Let \( \{z\} = X_2 \cap Y_2 \). Then \( r(Y_2) = r(K \cup \{z\}) = 3 \). Hence, \( K \cup \{z\} \) is a component of a split of \( M \). Therefore, this split generates a decomposition with a component isomorphic to \( M(K_4) \). This completes the proof that \( M \) has a component isomorphic to either \( M(K_4) \) or \( F_7 \). In addition, we have shown that if \( M_1 \) is not isomorphic to either \( M(K_4) \) or \( F_7 \), then (i) and (ii) are satisfied. Since \( r(Y_1 \cap X_2) = r(X_2) - r(Y_2 \cap X_2) \), we have that \( r(Y_1 \cap X_2) = r(X_2) - 1 \). This implies that \( (\gamma(X) - \{x\}) \cup \{z\} \) contains a cocircuit of \( M_2 \). Since this set has three elements and \( M_2 \) is 3-connected, the set \((\gamma(X) - \{x\}) \cup \{z\} \) is a triad of \( M_2 \). This completes the proof. \( \square \)
Suppose that the following conditions are satisfied:

(i) \( X = \{X_1, X_2\} \) is a split of \( M \) generating the decomposition \( \{M_1, M_2\} \);

(ii) \( Y = \{Y_1, Y_2\} \) is a split of \( M \) such that \( |cl_P(Y_1) \cap \gamma(X)| = 1 \) and \( cl_P(Y_2) \cap \gamma(X) = \emptyset \);

(iii) \( Z' \) is a good split of \( M_1 \);

(iv) the connections \( \gamma(X), \gamma(Y), \) and \( \gamma(Z') \) are distinct; and

(v) \( Y \) crosses both \( X \) and \( W \), where \( W = \{W_1, W_2\} \) is the split of \( M \) induced by \( Z' \).

Then,

(a) there is an element \( z \in X_2 - \gamma(X) \) such that

\[
U = \{cl_M([Z'_1 \cap X_1) \cup \{z\}], cl_M([Z'_2 \cap (X_1 \cap Y_1)] \cup (X_2 \cap Y_1))\}
\]

is a split of \( M \);

(b) \( \gamma(X) \cap \gamma(W) \) is empty; and

(c) \( M_2 \) has a triad.

Proof. Let \( \{y_1\} = cl_P(Y_1) \cap \gamma(X) \). Using an argument similar to that used at the beginning of the proof of (4.4.3), we may assume that \( y_1 \) is in \( E(M) \). We may also assume that \( \gamma(X) \subseteq Z'_2 \) and that \( W_1 \subset X_1 \). By the modularity of \( P \), we have that

\[
r[(X_i \cap Y_1) \cup \gamma(X)] = r(X_i \cap Y_1) + 1 \text{ and } r[(X_i \cap Y_2) \cup \gamma(X)] = r(X_i \cap Y_2) + 2 \]
for each
i \in \{1,2\}. By (4.3.6), \( r(X_1 \cap Y_2) + r(X_2 \cap Y_2) = r(Y_2) \). By the modularity of \( P \),
\[ r(X_1 \cap Y_1) + r(X_2 \cap Y_2) = r(Y_2) + 1. \]
Hence, either \( r(X_1 \cap Y_1) + r(X_1 \cap Y_2) = r(X_1) \),
and \( r(X_2 \cap Y_1) + r(X_2 \cap Y_2) = r(X_2) + 1 \); or \( r(X_1 \cap Y_1) + r(X_1 \cap Y_2) = r(X_1) + 1 \),
and \( r(X_2 \cap Y_1) + r(X_2 \cap Y_2) = r(X_2) \).

Suppose that \( r(X_1 \cap Y_1) + r(X_1 \cap Y_2) = r(X_1) \), and \( r(X_2 \cap Y_1) + r(X_2 \cap Y_2) = r(X_2) + 1 \). Then \( r((X_1 \cap Y_1) \cup \gamma(X)) + r(X_1 \cap Y_2) = r(M_1) + 1 \). Since \( M_1 \) is 3-connected and \( (X_1 \cap Y_2) - Y_1 \) is non-empty, we have that \( |X_1 \cap Y_2| = 1 \) and \( r(X_1 \cap Y_1) = r(M_1) - 1 \). Therefore, \( r(X_1 \cap Y_1) + r((X_1 \cap Y_2) \cup \gamma(X)) = r(M_1) + 2 \).

Suppose that \( \{cl_{M_1}(X_1 \cap Y_1), cl_{M_1}((X_1 \cap Y_2) \cup \gamma(X))\} \) is a split of \( M_1 \). Since \( Z' \) is a
good split, either

1. \( Z_1' \supset cl_{M_1}((Y_2 \cap X_1) \cup \gamma(X)) \);
2. \( Z_2' \supset cl_{M_1}(Y_1 \cap X_1) \);
3. \( Z_2' \subset cl_{M_1}((Y_2 \cap X_1) \cup \gamma(X)) \); or
4. \( Z_1' \subset cl_{M_1}(Y_1 \cap X_1) \).

By using the same arguments as in the proof of (4.4.3) case (b), except interchanging the roles of \( Y_1 \) and \( Y_2 \), we have that none of these containments hold. Therefore, \( \{cl_{M_1}(X_1 \cap Y_1), cl_{M_1}((X_1 \cap Y_2) \cup \gamma(X))\} \) is not a split of \( M_1 \). Hence, (3.2.3) implies that \( r((X_1 \cap Y_2) \cup \gamma(X)) = r(M_1) \). Thus, \( r(M_1) = 3 \); contradicting the assumption that \( M_1 \) has a split.

Suppose that \( r(X_1 \cap Y_1) + r(X_1 \cap Y_2) = r(X_1) + 1 \), and \( r(X_2 \cap Y_1) + r(X_2 \cap Y_2) = r(X_2) \). Then \( r((X_2 \cap Y_1) \cup \gamma(X)) + r(X_2 \cap Y_2) = r(M_2) + 1 \). Since \( M_2 \) is 3-connected and
\((X_2 \cap Y_2) - Y_1\) is non-empty, we have that \(|X_2 \cap Y_2| = 1\) and \(r(X_2 \cap Y_1) = r(M_2) - 1\).

Suppose that \(\{cl_{M_1}([X_1 \cap Y_1] \cup \gamma(X)], cl_{M_1}(X_1 \cap Y_2)\}\) is a split of \(M_1\). Since \(Z'\) is a good split, either

\((1)'\) \(Z'_1 \supset cl_{M_1}([Y_1 \cap X_1] \cup \gamma(X)]\);

\((2)'\) \(Z'_2 \supset cl_{M_1}(Y_2 \cap X_1)\);

\((3)'\) \(Z'_2 \subset cl_{M_1}([Y_1 \cap X_1] \cup \gamma(X)]\); or

\((4)'\) \(Z'_1 \subset cl_{M_1}(Y_2 \cap X_1)\).

Containments \((1)'\), \((2)'\), and \((4)'\) lead to the same contradictions in this case as did the corresponding containments in the proof of case (b) of (4.4.3). Moreover, the arguments leading to these contradictions are the same.

Suppose that containment \((3)'\) holds; that is, \(Z'_2 \subset cl_{M_1}([Y_1 \cap X_1] \cup \gamma(X)]\) and \(Z'_1 \supset cl_{M_1}(X_1 \cap Y_2)\). Let \(\{z\} = X_2 \cap Y_2\). Let \(U'_1 = Z'_1 \cap X_1\) and \(U'_2 = Z'_2 \cap (X_1 \cap Y_1)\).

Observe that \(r(U'_2 \cup \gamma(X)) = r(Z'_2)\). By the modularity of \(P\), we have that \(r(U'_2) = r(Z'_2) - 1\). Therefore,

\[
\begin{align*}
r(U'_1) + r(U'_2) & = r(Z'_1) + r(Z'_2) - 1 \\
& = r(X_1) + 1.
\end{align*}
\]

Let \(U_1 = U'_1 \cup \{z\}\) and \(U_2 = U'_2 \cup (Y_1 \cap X_2)\). By the modularity of \(P\), we have that

\[
\begin{align*}
r(U_2) & = r(U'_2) + r(Y_1 \cap X_2) - 1 \\
& = r(U'_2) + r(X_2) - 2.
\end{align*}
\]
Since \( r(U_1) = r(U'_1) + 1 \), it follows that

\[
\begin{align*}
    r(U_1) + r(U_2) &= r(U'_1) + r(U'_2) - 1 + r(X_2) - 1 \\
                     &= r(X_1) + r(X_2) \\
                     &= r(M) + 2.
\end{align*}
\]

By (3.2.2), \( \{cl_M(U_1), cl_M(U_2)\} \) is a split of \( M \). Observe that \( z \in U_1 - X_1 \) and \( U'_2 - \gamma(X) \) is non-empty. Therefore, \( U_1 - X_1 \) and \( U_2 - X_2 \) are non-empty. Observe that \( U'_1 - \gamma(X) \) is non-empty. Hence, in order to show that \( \{cl_M(U_1), cl_M(U_2)\} \) and \( X \) cross, it remains to show that \( U_2 - X_1 \) is non-empty. But \( Y_1 - X_1 \) being non-empty implies that \( U_2 - X_1 \) is non-empty. This proves part (a).

Suppose that \( \gamma(Z') \) and \( \gamma(X) \) meet. Then \( cl_P(U_1 \cap \gamma(X)) \) is non-empty. Since \( Z'_2 \supseteq \gamma(X) \) and \( U'_2 = Z'_2 \cap (X_1 \cap Y_1) \), we have that \( cl_P(U_2) \cap \gamma(X) \supseteq \{y_1\} \). This situation contradicts (4.4.3). Hence, \( \gamma(Z') \cap \gamma(X) = \emptyset \). Thus, part (b) is proved.

To complete the proof we must show that \( M_2 \) has a triad. Since \( r(Y_1 \cap X_2) = r(X_2) - 1 \), we have that \( (\gamma(X) - \{y_1\}) \cup \{z\} \) contains a cocircuit of \( M_2 \). Since this set has three elements and \( M_2 \) is 3-connected, the set \( (\gamma(X) - \{y_1\}) \cup \{z\} \) is a triad of \( M_2 \). \( \square \)

(4.4.9) Suppose that \( X = \{X_1, X_2\} \) is a split of \( M \) generating the decomposition \( \{M_1, M_2\} \) and assume \( |\delta(X)| = 1 \). Further, suppose that \( M \in \mathcal{M} \) and \( M_1 \notin \mathcal{M} \).

Then either \( M_1 \) has no splits, or

(i) if \( N = M_1 \) and \( T = \gamma(X) \), then \( (N, T) \in \mathcal{N} \);
(ii) For each good split $Z' = \{Z'_1, Z'_2\}$ of $M_1$, there is an element $z \in X_2 - \gamma(X)$ such that

$$U = \{cl_M((Z'_1 \cap X_1) \cup \{z\}), cl_M((Z'_2 \cap (X_1 \cap Y_1)) \cup (X_2 \cap Y_1))\}$$

is a split of $M$ for $\{i, j\} = \{1, 2\}$; and

(iii) $M_2$ has a triad.

**Proof.** Suppose that $M_1$ has a good split $Z' = \{Z'_1, Z'_2\}$. We may assume that $\gamma(X) \subseteq Z'_2$. Since $\gamma(X)$ is not a multiple connection, $|cl_{p}(Z'_i) \cap \gamma(X)| \leq 1$. Let $Z = \{Z_1, Z_2\}$ be the split of $M$ inducing $Z'$. Since $M \in \mathcal{M}$, there is a split $Y = \{Y_1, Y_2\}$ of $M$ which crosses $Z$. By Proposition 4.3.12, $M$ has no multiple connection. Therefore, $\gamma(X)$, $\gamma(Y)$, and $\gamma(Z)$ are distinct. Since $Z'$ is a good split of $M_1$ and $Z$ and $X$ are compatible, (4.4.1) implies that $Y$ crosses $X$. Since $X$ and $Y$ cross and $\gamma(X) \neq \gamma(Y)$, (4.3.10) implies that $|cl_{p}(Y_i) \cap \gamma(X)| \leq 1$ for each $i \in \{1, 2\}$. If $|cl_{p}(Y_i) \cap \gamma(X)| = 1$ for each $i \in \{1, 2\}$, then this situation contradicts (4.4.3). Therefore, we may assume that $|cl_{p}(Y_1) \cap \gamma(X)| = 1$ and that $|cl_{p}(Y_2) \cap \gamma(X)| = 0$. Then (4.4.8) implies that (i), (ii), and (iii) hold. This completes the proof. ◼

The next result follows from (4.4.7) and (4.4.9).

(4.4.10) Suppose that $X = \{X_1, X_2\}$ is a split of $M$ generating the decomposition $\{M_1, M_2\}$ and assume $|\delta(X)| = 1$. Further, suppose that $M \in \mathcal{M}$ and $M_1 \notin \mathcal{M}$. Then

(i) $M_1$ is isomorphic to $M(K_4)$ or $F_7$;
(ii) $M_1$ is vertically 4-connected and $M$, $M_1$, and $M_2$ satisfy (4.4.7) (i)-(iii); or

(iii) $M_1$ has a good split and $M$, $M_1$, and $M_2$ satisfy (4.4.9) (i)-(iii).

The next four results complete our analysis by treating the case $\delta(X) = \emptyset$.

(4.4.11) Suppose that $X = \{X_1, X_2\}$ is a split of $M$ generating the decomposition $\{M_1, M_2\}$ and assume $\delta(X) = \emptyset$. Further, suppose that $X$ is crossed by a split $Y = \{Y_1, Y_2\}$ such that $\gamma(X) \neq \gamma(Y)$, and that $M_1$ has no splits. Then either

(i) $M$ has a decomposition such that at least one of the components is isomorphic to $M(K_4)$ or $F_7$; or

(ii) $M_1 \cong M^*(K_{3,3})$.

Moreover, if case (i) holds and $M_1$ is not isomorphic to $M(K_4)$ or $F_7$, then $M_1$ is vertically 4-connected and $M$, $M_1$, and $M_2$ satisfy (4.4.7) (i)-(iii).

Proof. If $(\text{cl}_P(Y_1) \cup \text{cl}_P(Y_2)) \cap \gamma(X)$ is non-empty, then let $y \in (\text{cl}_P(Y_1) \cup \text{cl}_P(Y_2)) \cap \gamma(X)$. By (3.3.5), $M' = P\{E(M) \cup \{y\}\}$ is 3-connected, both $\{X_1 \cup \{y\}, X_2 \cup \{y\}\}$ and $\{\text{cl}_{M'}(Y_1), \text{cl}_{M'}(Y_2)\}$ are splits of $M'$, and $\{X_1 \cup \{y\}, X_2 \cup \{y\}\}$ generates the decomposition $\{M_1, M_2\}$ of $M'$. Observe that this situation satisfies the hypothesis of (4.4.7). Hence, $M$ satisfies (i) and, if $M_1$ is not isomorphic to $M(K_4)$ or $F_7$, then $M_1$ is vertically 4-connected and $M$, $M_1$, and $M_2$ satisfy (4.4.7) (i)-(iii). Therefore, we may assume that $(\text{cl}_P(Y_1) \cup \text{cl}_P(Y_2)) \cap \gamma(X)$ is empty.
By the modularity of $P$, we have that $r[(X_i \cap Y_i) \cup \gamma(X)] = r(X_i \cap Y_i) + 2$ for each $i \in \{1, 2\}$. By (4.3.6), $r(X_1 \cap Y_1) + r(X_1 \cap Y_2) = r(X_1)$. Therefore,

$$r[(X_i \cap Y_i) \cup \gamma(X)] + r(X_1 \cap Y_2) = r(X_i \cap Y_1) + r[(X_1 \cap Y_2) \cup \gamma(X)]$$

$$= r(M_i) + 2.$$

Since $X$ and $Y$ cross, $(X_i \cap Y_i) - Y_j$ is non-empty, where $\{i, j\} = \{1, 2\}$. Hence, $r[(X_i \cap Y_i) \cup \gamma(X)] \geq 3$ for each $i \in \{1, 2\}$. Since $M_1$ has no splits, (3.2.3) implies that $r[(X_i \cap Y_i) \cup \gamma(X)] = r(M_i)$ for each $i \in \{1, 2\}$. Therefore, $r(M_i) = 4$ and each of the three sets $X_1 \cap Y_1$, $X_1 \cap Y_2$, and $\gamma(X)$ are rank-two flats of $M_1$. Since $r(X_1 \cap Y_1) + r(X_1 \cap Y_2) = r(M_i)$, the modularity of $P$ implies that $cl_P(X_1 \cap Y_1)$ and $cl_P(X_1 \cap Y_2)$ are disjoint triangles of $P$. Therefore, by (4.2.7),

$$M' = P([cl_P(X_1 \cap Y_1) \cup cl_P(X_1 \cap Y_2) \cup \gamma(X)])$$

$$= M^*(K_{3,3}).$$

Observe that $M_1$ is either $M'$ or a one- or two-element deletion of $M'$. But any one- or two-element deletion of $M^*(K_{3,3})$ results in a matroid that is no longer vertically 4-connected. Therefore, $M_1 \cong M^*(K_{3,3})$. □

(4.4.12) The following conditions cannot all be satisfied simultaneously.

(i) $X = \{X_1, X_2\}$ is a split of $M$ generating the decomposition $\{M_1, M_2\}$.

(ii) $Y = \{Y_1, Y_2\}$ is a split of $M$ such that $cl_P(Y_i) \cap \gamma(X) = \emptyset$ for each $i \in \{1, 2\}$.

(iii) $Z'$ is a good split of $M_1$.
(iv) The connections $\gamma(X)$, $\gamma(Y)$, and $\gamma(Z')$ are distinct.

(v) $Y$ crosses both $X$ and $W$, where $W = \{W_1, W_2\}$ is the split of $M$ induced by $Z'$.

Proof. We may assume that $\gamma(X) \subseteq Z_2'$. By the modularity of $P$, $r[[X_i \cap Y_j] \cup \gamma(X)] = r(X_i \cap Y_j) + 2$ for each $i, j \in \{1, 2\}$. By (4.3.6), $r(X_i \cap Y_j) + r(X_j \cap Y_j) = r(Y_j)$ for each $i, j \in \{1, 2\}$. Therefore, $r[(X_1 \cap Y_1) \cup \gamma(X)] + r(X_1 \cap Y_2) = r(X_1 \cap Y_1) + r[(X_1 \cap Y_2) \cup \gamma(X)] = r(M_1) + 2$. Either,

(a) neither of the sets $\{\text{cl}_{M_1}[(Y_1 \cap X_1) \cup \gamma(X)], \text{cl}_{M_1}(Y_2 \cap X_1)\}$ and $\{\text{cl}_{M_1}[(Y_2 \cap X_1) \cup \gamma(X)], \text{cl}_{M_1}(Y_1 \cap X_1)\}$ is a split of $M_1$; or

(b) precisely one of $\{\text{cl}_{M_1}[(Y_1 \cap X_1) \cup \gamma(X)], \text{cl}_{M_1}(Y_2 \cap X_1)\}$ and $\{\text{cl}_{M_1}[(Y_2 \cap X_1) \cup \gamma(X)], \text{cl}_{M_1}(Y_1 \cap X_1)\}$ is a split of $M_1$; or

(c) both of the sets $\{\text{cl}_{M_1}[(Y_1 \cap X_1) \cup \gamma(X)], \text{cl}_{M_1}(Y_2 \cap X_1)\}$ and $\{\text{cl}_{M_1}[(Y_2 \cap X_1) \cup \gamma(X)], \text{cl}_{M_1}(Y_1 \cap X_1)\}$ are splits of $M_1$.

In case (a), (3.2.3) implies that $r(Y_1 \cap X_1) = r(Y_2 \cap X_1) = 2$ and hence, $r(M_1) = 4$. Therefore,

\[
M' = P[[\text{cl}_P(X_1 \cap Y_1) \cup \text{cl}_P(X_1 \cap Y_2) \cup \gamma(X)]
= M^*(K_{3,3}),
\]

and $M_1$ is either $M'$, or a one- or two-element deletion of $M'$. But each one of these matroids either has no good splits or is not 3-connected.
In case (b), we may assume that \( \{ \text{cl}_{\mathcal{M}_1}(Y_1 \cap X_1) \cup \gamma(X_1), \text{cl}_{\mathcal{M}_1}(Y_2 \cap X_1) \} \) is a split and that \( \{ \text{cl}_{\mathcal{M}_1}(Y_2 \cap X_1) \cup \gamma(X_1), \text{cl}_{\mathcal{M}_1}(Y_1 \cap X_1) \} \) is not a split. By (3.2.3) and the fact that \( r(Y_2 \cap X_1) = r(M_1) - 2 \), we have that \( r(Y_1 \cap X_1) = 2 \) and \( r(Y_2 \cap X_1) \geq 3 \).

Since \( Z' \) is a good split, either

1. \( Z'_1 \supset \text{cl}_{\mathcal{M}_1}(Y_1 \cap X_1) \cup \gamma(X_1) \);
2. \( Z'_2 \supset \text{cl}_{\mathcal{M}_1}(Y_2 \cap X_1) \);
3. \( Z'_2 \subset \text{cl}_{\mathcal{M}_1}(Y_1 \cap X_1) \cup \gamma(X_1) \); or
4. \( Z'_1 \subset \text{cl}_{\mathcal{M}_1}(Y_2 \cap X_1) \).

The same arguments used to prove that containments (1), (2), and (4) result in contradictions in the proof of case (b) of (4.4.3) can be used here as well. Suppose that containment (3) holds. Since \( r([Y_1 \cap X_1 \cup \gamma(X)]) = 4 \), we have that \( r(Z'_2) = 3 \).

Since \( \text{cl}_{\mathcal{P}}(Y_1 \cap X_1) \cap \gamma(X) = \emptyset \), it follows that \( Z'_2 = \gamma(X) \cup \{z_2\} \) for some element \( z_2 \) of \( Y_1 \cap X_1 \). Since \( Z'_1 \) does not contain \( X_1 - \gamma(X) \) and \( Z'_1 \) is a closed subset of \( X_1 \cup \gamma(X) \), we have that \( Y_1 \cap X_1 = \{z_1, z_2\} \) for some element \( z_1 \), and \( Z'_1 \supseteq (Y_2 \cap X_1) \cup \{z_1\} \). Consider the sets \( (Y_2 \cap X_1) \cup \{z_2\} \) and \( \{z_1\} \cup \gamma(X) \). Observe that \( r([Y_2 \cap X_1 \cup \{z_2\}] = r(M_1) - 1 \) and \( r(\{z_1\} \cup \gamma(X)) = 3 \). Let \( W' = \{W'_1, W'_2\} \) be the split of \( M \), where \( W'_1 = \text{cl}_{\mathcal{M}_1}([Y_2 \cap X_1] \cup \{z_2\}) \) and \( W'_2 = \text{cl}_{\mathcal{M}_1}(\{z_2\} \cup \gamma(X)) \).

Observe that this split crosses \( Z' \). Hence, \( Z' \) is not a good split of \( M_1 \). Therefore, containment (3) cannot hold. Therefore, case (b) cannot occur, and we may assume that case (c) holds.
Since $Z'$ is a good split, $Z'$ and $\{c_{M_1}[(Y_1 \cap X_1) \cup \gamma(X)], c_{M_1}(Y_2 \cap X_1)\}$ are compatible. Hence, one of the components (1)–(4) must hold. By using the same arguments as for case (b), it follows that the first, second, and last containments cannot hold. Hence $Z'_2 \subseteq c_{M_1}[(Y_1 \cap X_1) \cup \gamma(X)]$. Using an analogous argument to that given when analyzing the compatible splits $Z'$ and $\{c_{M_1}[(Y_2 \cap X_1) \cup \gamma(X)], c_{M_1}(Y_1 \cap X_1)\}$, we can conclude that $Z'_2 \subseteq c_{M_1}[(Y_2 \cap X_1) \cup \gamma(X)]$. Therefore, $Z'_1 \supset c_{M_1}(Y_2 \cap X_1)$ and $Z'_1 \supset c_{M_1}(Y_1 \cap X_1)$. Since $c_{M_1}(Y_2 \cap X_1) \cup c_{M_1}(Y_1 \cap X_1) \supset X_1$, we have that $W_1 \supset X_1$, a contradiction. Hence, all three cases (a), (b), and (c) lead to contradictions. □

(4.4.13) Suppose that $X = \{X_1, X_2\}$ is a split of $M$ generating the decomposition $\{M_1, M_2\}$ and assume $\delta(X) = \emptyset$. Further, suppose that $M \in \mathcal{M}$ and $M_1 \not\in \mathcal{M}$. Then either $M_1$ has no splits, or $M_1$ has a good split and $M, M_1$, and $M_2$ satisfy (4.4.9) (i)–(iii).

Proof. Suppose that $M_1$ has a good split $Z' = \{Z'_1, Z'_2\}$. We may assume that $\gamma(X) \subseteq Z'_2$. Since $\gamma(X)$ is not a multiple connection, $|c_p(Z'_1) \cap \gamma(X)| \leq 1$. Let $Z = \{Z_1, Z_2\}$ be the split of $M$ inducing $Z'$. Since $M \in \mathcal{M}$, there is a split $Y = \{Y_1, Y_2\}$ of $M$ which crosses $Z$. By Proposition 4.3.12, $M$ has no multiple connection. Therefore, $\gamma(X), \gamma(Y)$, and $\gamma(Z)$ are distinct. Since $Z'$ is a good split of $M_1$ and $Z$ and $X$ are compatible, (4.4.1) implies that $Y$ crosses $X$. Since $X$ and $Y$ cross and $\gamma(X) \neq \gamma(Y)$, (4.3.10) implies that $|c_p(Y_i) \cap \gamma(X)| \leq 1$ for each $i \in \{1, 2\}$. If $|c_p(Y_i) \cap \gamma(X)| = 1$ for each $i \in \{1, 2\}$, then this situation contradicts (4.4.3). If $|c_p(Y_i) \cap \gamma(X)| = 1$ for some $i \in \{1, 2\}$, then (4.4.9) implies that (i)–(iii) of (4.4.9) hold. But $c_p(Y_i) \cap \gamma(X) = \emptyset$ for each $i \in \{1, 2\}$, contradicts (4.4.12). □
The next result follows from (4.4.11) and (4.4.13).

(4.4.14) Suppose that \( X = \{X_1, X_2\} \) is a split of \( M \) generating the decomposition \( \{M_1, M_2\} \) and assume \( \delta(X) = \emptyset \). Further, suppose that \( M \in \mathcal{M} \) and \( M_1 \not\in \mathcal{M} \). Then either

- (i) \( M \) has a decomposition such that at least one of the components is isomorphic to \( M(K_4) \) or \( F_7 \); or
- (ii) \( M_1 \cong M^*(K_{3,3}) \).

Moreover, if case (i) holds and \( M_1 \) is not isomorphic to \( M(K_4) \) or \( F_7 \), then \( M_1 \) is vertically 4-connected and \( M, M_1, \) and \( M_2 \) satisfy (4.4.7) (i)-(iii).

### 4.5 A Characterization of \( \mathcal{M} \)

In this section we prove the main results of this chapter, Theorems 4.1.1, 4.1.2, and 4.1.3. Taken together, these results say that if \( M \in \mathcal{M} \), then either

- (i) \( M \cong M^*(K_{3,n}) \), where \( n \geq 4 \);
- (ii) \( M \) has a decomposition \( \{M_1, M_2\} \), where \( M_1 \) is a wheel, and either \( M_2 \) is \( F_7 \) or \( M(K_4) \), or \( M_2 \in \mathcal{M} \cup \mathcal{N} \); or
- (iii) \( M \) has a decomposition \( \{M_1, M_2\} \), where \( M_1 \) is a spike, and either \( M_2 \) is \( F_7 \) or \( M(K_4) \), or \( M_2 \in \mathcal{M} \).

The first two results (4.5.1) and (4.5.2) of this section deal with the situation when one of the components is a spike.
(4.5.1) Suppose that \( X = \{X_1, X_2\} \) is a split of \( M \) and that \( M_1 \) is a spike.

(i) If \( |\delta(X)| = 2 \), then \( X \) is a good split of \( M \).

(ii) If \( \delta(X) = \{x\} \) and \( x \) is not a tip of \( M_1 \), then \( X \) is a good split of \( M \).

**Proof.** Suppose that \( X \) is crossed by the split \( Y = \{Y_1, Y_2\} \) and that \( \gamma = \gamma(X) \) is a multiple connection. By Proposition 4.3.12, there is a good split \( Z = \{Z_1, Z_2\} \) of \( M \), where \( Z \) is a minimal split with respect to \( \gamma \). By (4.3.3), we may assume that \( Z_1 \subset X_1 \). By (3.2.16), \( M_1 \) has a good split, contradicting (4.2.5). Therefore, \( \gamma \) is not a multiple connection.

Suppose that the rank of \( M_1 \) is \( r \), where \( r \geq 4 \). Let \( L_i = \{a_i, b_i, p\} \) be the triangles of \( M_1 \) for \( i \in \{1, 2, \ldots, r\} \), where \( p \) is the tip of \( M_1 \). Let \( L_r = \gamma(X) \). By (4.3.10), \( |\text{cl}_p(Y_i) \cap \gamma(X)| \leq 1 \) for each \( i \in \{1, 2\} \). Suppose that \( \text{cl}_p(Y_1) \) contains the non-tip element \( a_r \) of \( L_r \). Then \( \text{cl}_p(Y_1) \cap \gamma(X) = \{a_r\} \). Since \( p \notin \text{cl}_p(Y_1) \), the set \( Y_1 \) contains at most one element from each \( L_i \) for \( i \in \{1, 2, \ldots, r - 1\} \). Hence, \( Y_2 \) contains at least one element from each \( L_i \) for \( i \in \{1, 2, \ldots, r - 1\} \). We may assume that \( \{b_1, b_2, \ldots, b_{r-1}\} \subseteq Y_2 \). By (4.2.5), \( \{b_1, b_2, \ldots, b_{r-1}, a_r\} \) is either a circuit or a basis of \( M_1 \). If \( \{b_1, b_2, \ldots, b_{r-1}, a_r\} \) is a circuit, then \( a_r \in \text{cl}_p(Y_2) \) and \( r(\text{cl}_p(Y_2 \cap X_1)) \geq r - 1 \). If \( \{b_1, b_2, \ldots, b_{r-1}, a_r\} \) is a basis, then \( \{b_1, b_2, \ldots, b_{r-1}, b_r\} \) is a circuit. Hence, \( b_r \in \text{cl}_p(Y_2) \) and \( r(\text{cl}_p(Y_2 \cap X_1)) \geq r - 1 \). Therefore, either \( b_r \in \text{cl}_p(Y_2) \) or \( a_r \in \text{cl}_p(Y_2) \), and \( r(\text{cl}_p(Y_2 \cap X_1)) \geq r - 1 \). Hence, \( |\text{cl}_p(Y_i) \cap \gamma(X)| = 1 \) for each \( i \in \{1, 2\} \) and \( p \notin \text{cl}_p(Y_2) \). Hence \( \{a_1, a_2, \ldots, a_r\} \subseteq Y_1 \). Therefore,
\[r(Y_1 \cap X_1) \geq r - 1. \text{ Let } cl_P(Y_1 \cap X_1) \cap \gamma(X) = \{y_i\}. \text{ Then}
\]

\[r(Y_1 \cap X_1) + r[(Y_1 \cap X_2) \cup \{y_i\}] = r(cl_P(Y_1) \cap X_1) + r(cl_P[(Y_1 \cap X_2) \cup \{y_i\}])
\]

\[= r(cl_P(Y_i)) + 1
\]

\[= r(Y_i) + 1.
\]

Hence, \(r(Y_i) = r(Y_i \cap X_1) + r[(Y_i \cap X_2) \cup \{y_i\}] - 1\). Moreover, \(r(Y_1) + r(Y_2) = r(X_1) + r(X_2) = r + r(M_2)\). Since \(X\) and \(Y\) cross, \((X_1 \cap X_2) - \gamma(X)\) is non-empty for each \(i \in \{1, 2\}\). Since \(M_2\) is 3-connected, the matroid \(M_2 \setminus (\gamma(X) - \{y_1, y_2\})\) is certainly 2-connected. Hence,

\[r[(Y_1 \cap X_2) \cup \{y_1\}] + r[(Y_2 \cap X_2) \cup \{y_2\}] \geq r(M_2) + 1.
\]

Therefore,

\[r + r(M_2) = r(Y_1 \cap X_1) + r(Y_2 \cap X_1)
\]

\[+ r[(Y_1 \cap X_2) \cup \{y_1\}] + r[(Y_2 \cap X_2) \cup \{y_2\}] - 2
\]

\[\geq (2r - 2) + (r(M_2) + 1) - 2
\]

\[= r(M_2) + 2r - 3.
\]

Hence, \(r \leq 3\); a contradiction. Since every element of a rank-three spike is a tip, we have shown that (ii) holds in general, and, that (i) holds if \(r(M_1) \geq 4\).

Suppose that \(|\delta(X)| = 2\) and that \(M_1 \cong F_7\). Further, suppose that \(X\) is crossed by the split \(Y = \{Y_1, Y_2\}\). By (4.3.10), \(|cl_P(Y_i) \cap \gamma(X)| \leq 1\) for each \(i \in \{1, 2\}\). Therefore, if we let \(\delta(X) = \{y_1, y_2\}\), we may assume that \(Y_i \cap \delta(X) = \{y_i\}\) for each...
i ∈ {1, 2}. Let \( \{a, b, c, d\} = X_1 - \gamma(X) \). If \(|Y_i \cap \{a, b, c, d\}| \geq 3\), then \( Y_i \supseteq X_1 \), contradicting the assumption that \( X \) and \( Y \) cross. Hence, we may assume that \( \{a, b\} = Y_1 \cap (X_1 - \gamma(X)) \) and \( \{c, d\} = Y_2 \cap (X_1 - \gamma(X)) \). By (4.2.5), \( \text{cl}_{M_i}(\{a, b\}) \) and \( \text{cl}_{M_1}(\{c, d\}) \) meet in a unique point of \( \gamma(X) \). Since \(|\delta(X)| = 2\), we may assume that \( y_1 \not\in \text{cl}_{M_1}(\{a, b\}) \). Hence, \( Y_1 \supseteq X_1 \); a contradiction. Therefore, \( X \) must be a good split of \( M \). This completes the proof. □

(4.5.2) Suppose that \( X = \{X_1, X_2\} \) is a split of \( M \) generating the decomposition \( \{M_1, M_2\} \) where \( M_1 \) is an \( r \)-spike, for some \( r \geq 3 \). Further, suppose that \( Y = \{Y_1, Y_2\} \) is a split of \( M \) which crosses \( X \). Then \( r(Y_1 \cap X_1) + r(Y_2 \cap X_1) \geq r(M_1) + 1 \) and \( \text{cl}_p(Y_i) \cap \gamma(X) \subseteq \{p\} \), where \( p \) is a tip of \( M_1 \).

Proof. By (4.5.1), we may assume that \(|\delta(X)| \leq 1\), and, if \(|\delta(X)| = 1\), then \( \delta(X) = \{p\} \), where \( p \) is a tip of \( M_1 \). The proof of (4.5.1) contains the result that \( \text{cl}_p(Y_i) \cap \delta(X) \subseteq \{p\} \) for each \( i \in \{1, 2\} \). It remains to show that \( r(Y_1 \cap X_1) + r(Y_2 \cap X_1) \geq r(M_1) + 1 \). As was shown in the proof of (4.5.1), neither \( Y_1 \cap X_1 \) nor \( Y_2 \cap X_1 \) can meet each triangle of \( X_1 \cup \{p\} \). Let \( \{L_i\}_{i=1}^{r-1} \) be the set of triangles of \( X_1 \cup \{p\} \) and let \( L_r = \gamma(X) \). Let \( n_j = |\{L_i : L_i \cap Y_j \text{ is non-empty for all } i \in \{1, 2, \ldots, r-1\}\}| \) for each \( j \in \{1, 2\} \). Then \( n_1 + n_2 \geq r - 1 \). Since \( \text{cl}_{M_1}(Y_i) \) contains at least one triangle of \( X_1 \cup \{p\} \) for each \( i \in \{1, 2\} \), we have that \( r(Y_i \cap X_1) = n_i + 1 \). Hence,

\[
\begin{align*}
    r(Y_1 \cap X_1) + r(Y_2 \cap X_1) &= n_1 + n_2 + 2 \\
    &\geq r + 1.
\end{align*}
\]

This completes the proof. □
In the next two results (4.5.3) and (4.5.4) we deal with the situation when one of the components is \( M^*(K_{3,3}) \).

(4.5.3) Suppose that \( X = \{X_1, X_2\} \) is a split of \( M \) generating the decomposition \( \{M_1, M_2\} \) where \( M_1 \cong M^*(K_{3,3}) \). Further, suppose that \( Y = \{Y_1, Y_2\} \) is a split of \( M \) which crosses \( X \). Then \(|\delta(X)| \leq 1\) and either

(i) \( r(Y_i \cap X_1) = 3 \) and \( r(Y_j \cap X_1) = 2 \), if \(|\text{cl}_p(Y_i \cap X_1) \cap \gamma(X)| = 1\) and \( \text{cl}_p(Y_j \cap X_1) \cap \gamma(X) = \emptyset\), where \( \{i, j\} = \{1, 2\} \); or

(ii) \( r(Y_i \cap X_1) = 2 \) and \( \text{cl}_p(Y_i \cap X_1) \cap \gamma(X) = \emptyset \) for each \( i \in \{1, 2\} \).

Proof. The conclusion that \(|\delta(X)| \leq 1\) is a direct consequence of (i) and (ii). Since \( X \) and \( Y \) cross, (4.3.10) implies that \(|\text{cl}_p(Y_i \cap X_1) \cap \gamma(X)| \leq 1\) for each \( i \in \{1, 2\} \).

Suppose that \(|\text{cl}_p(Y_i \cap X_1) \cap \gamma(X)| = 1\) for each \( i \in \{1, 2\} \). Let \( \gamma(X) = \{a, b, c\} \).

Suppose that \( \text{cl}_p(Y_1 \cap X_1) \cap \gamma(X) = \text{cl}_p(Y_2 \cap X_1) \cap \gamma(X) = \{a\} \). Let \( M' = M_1 \setminus b, c \).

Since \( M' \) is connected, we have that

\[
r[(Y_1 \cap X_1) \cup \{a\}] + r[(Y_2 \cap X_1) \cup \{a\}] \geq 5
\]

Since \( Y \) crosses \( X \), the ranks in the above inequality must be less than four. Therefore, we may assume that \( r[(Y_i \cap X_1) \cup \{a\}] = 3 \) for each \( i \in \{1, 2\} \); or that \( r[(Y_1 \cap X_1) \cup \{a\}] = 3 \) and \( r[(Y_2 \cap X_1) \cup \{a\}] = 2 \). If the former case holds, then \( \gamma(Y) \subseteq \text{cl}_p(X_1) \), contradicting (4.3.10). If the latter case holds, then \( (Y_2 \cap X_1) \cup \{a\} \) is a triangle of \( M_1 \). Therefore, \( Y_1 \cap X_1 \) must contain the four points of \( M_1 \) disjoint from the intersecting triangles, \( \gamma(X) \) and \( (Y_2 \cap X_1) \cup \{a\} \). Hence, \( r(Y_2 \cap X_1) = 4 \), a contradiction.
Suppose that $\text{cl}_p(Y_1 \cap X_1) \cap \gamma(X) = \{a\}$ and $\text{cl}_p(Y_2 \cap X_1) \cap \gamma(X) = \{b\}$.

Let $M' = M_1 \setminus c$. Then $M' \cong \mathcal{W}_4$. In particular, $M'$ is 3-connected. Therefore, 
$r((Y_1 \cap X_1) \cup \{a\}) + r((Y_2 \cap X_1) \cup \{b\}) \geq 6$. This implies that $\gamma(Y)$ is contained in $X_1$, contradicting (4.3.10).

We may assume that $|\text{cl}_p(Y_1 \cap X_1) \cap \gamma(X)| \leq 1$ and $\text{cl}_p(Y_2 \cap X_1) \cap \gamma(X) = \emptyset$.

Suppose that $\text{cl}_p(Y_1 \cap X_1) \cap \gamma(X) = \{a\}$. Therefore, $r((Y_1 \cap X_1) \cup \{a\}) + r(Y_2 \cap X_1) = 5$. If $r((Y_1 \cap X_1) \cup \{a\}) = 2$, then $r(Y_2 \cap X_1) = 4$; a contradiction. Hence, 
$r((Y_1 \cap X_1) \cup \{a\}) = 3$ and $r(Y_2 \cap X_1) = 2$.

Suppose that $\text{cl}_p(Y_1 \cap X_1) \cap \gamma(X) = \emptyset$. We have that $M_1 \setminus a, b, c \cong L_1 \oplus L_2$, where $L_i$ is a three-point line for each $i \in \{1, 2\}$. Moreover, if a set meets both lines, then its closure in $M_1$ meets the line $\gamma(X)$. Hence, we may assume that $Y_i \cap X_1 = L_i$ for each $i \in \{1, 2\}$. This completes the proof. $\square$

(4.5.4) Suppose that $X = \{X_1, X_2\}$ is a split of $M$ generating the decomposition

$\{M_1, M_2\}$, where $M_1 \cong M^*(K_{3,3})$. Then $X$ is a good split if either

(i) $M_2 \cong S_r$ for some $r \geq 3$; or

(ii) $M_2 \cong M(K_4)$ and $|\delta(X)| = 2$.

Proof. Suppose that $M_2$ is an $r$-spike and $Y = \{Y_1, Y_2\}$ is a split of $M$ crossing $X$.

By (4.5.2), $r(Y_1 \cap X_2) + r(Y_2 \cap X_2) \geq r + 1$. By (4.5.3), either

(a) case (i) of (4.5.3) holds and we may assume that

$$r(Y_1 \cap X_1) = 3,$$

$$r(Y_2 \cap X_1) = 2,$$
\[ r(Y_1) \geq r(Y_1 \cap X_1) + r(Y_1 \cap X_2) - 1, \text{ and} \]

\[ r(Y_2) = r(Y_2 \cap X_1) + r(Y_2 \cap X_2); \text{ or} \]

(b) case (ii) of (4.5.3) holds and we may assume that \( r(Y_i \cap X_i) = 2 \) and \( r(Y_i) = r(Y_i \cap X_1) + r(Y_i \cap X_2) \) for each \( i \in \{1, 2\} \).

In case (a),

\[
\begin{align*}
  r(Y_1) + r(Y_2) &= [3 + r(Y_1 \cap X_2) - 1] + [2 + r(Y_2 \cap X_2)] \\
  &= (Y_1 \cap X_2) + r(Y_2 \cap X_2) + 4 \\
  &\geq r + 5 \\
  &= r(M) + 3.
\end{align*}
\]

This contradicts the assumption that \( Y \) is a split. We get the same contradiction in case (b). Hence, \( X \) is a good split when \( M_2 \) is a spike. Observe that (ii) is a consequence of (4.4.2). \( \square \)

The next technical result is used to prove (4.5.6).

(4.5.5) Suppose that \( X = \{X_1, X_2\} \) is a split of \( M \) generating the decomposition \( \{M_1, M_2\} \), where \( M_1 \cong M^*(K_{3,3}) \). Further, suppose that \( |\delta(X)| \leq 1 \), and \( R_1 \) and \( R_2 \) are non-empty subsets of \( X_2 \), where \( c_{\mathcal{M}_4}(R_1) \supseteq \delta(X) \). Then there are subsets \( S \) and \( T \) of \( X_1 \) such that

(i) \( r(S) = 3 \) and \( r(T) = 2 \);

(ii) \( r(S \cup R_1) = r(R_1) + 2 \); and
(iii) \( r(T \cup R_2) = r(R_2) + 2. \)

Moreover, \( T \) is a triangle of \( M \) disjoint from \( \gamma(X) \).

Proof. Let \( E(M_1) = \{x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3\} \), where the sets \( \{a_1, a_2, a_3\} \) and \( \{x_i, y_i, z_i\} \) are the triangles of \( M_1 \) for each \( a \in \{x, y, z\} \) and each \( i \in \{1, 2, 3\} \). By symmetry, we may assume that \( \gamma(X) = \{z_1, z_2, z_3\} \). Suppose that \( |\delta(X)| = 1 \). We may assume that \( \delta(X) = \{z_1\} \). Let \( S = \{x_1, x_2, x_3, y_1, z_1\} \) and \( T = \{y_1, y_2, y_3\} \).

Observe that \( S \) and \( T \) satisfy (i) and \( T \) satisfies the last condition of (4.5.5). We claim that \( \operatorname{clp}(S) \cap \gamma(X) = \{z_1\} \). Suppose that \( z_2 \in \operatorname{clp}(S) \cap \gamma(X) \). Since \( z_1 \in S \), we have that \( z_3 \in \operatorname{clp}(S) \). Hence, \( y_2 \) and \( y_3 \) also belong to \( \operatorname{clp}(S) \). But this implies that \( E(M_1) \subseteq \operatorname{clp}(S) \), contradicting the observation that \( r[\operatorname{clp}(S)] = r(S) = 3 \). The claim follows by symmetry. By the modularity of \( P \), we have that

\[
\begin{align*}
  r(S \cup R_1) &= r[\operatorname{clp}(S) \cup \operatorname{clp}(R_1)] \\
  &= r[\operatorname{clp}(S)] + r[\operatorname{clp}(R_1)] - r[\operatorname{clp}(S) \cap \operatorname{clp}(R_1)] \\
  &= r(R_1) + 2.
\end{align*}
\]

This completes the proof of (ii) in the case when \( |\delta(X)| = 1 \). If \( \delta(X) = \emptyset \), let \( S = \{x_1, x_2, x_3, y_1\} \). Then \( z_1 \in \operatorname{clp}(S) \) and the proof of (ii) in the case when \( \delta(X) = \{z_1\} \) applies in this situation as well. Hence, (ii) holds.

Suppose that \( \delta(X) \subseteq \{z_1\} \). Since \( T = \{y_1, y_2, y_3\} \) is a triangle, it is closed in \( P \). Hence, \( \operatorname{clp}(T) \cap \operatorname{clp}(R_2) = \emptyset \), for every subset \( R_2 \) of \( X_2 \). By the modularity of \( P \), we have that \( r(T \cup R_2) = r(R_2) + 2 \). Hence, (iii) holds. This completes the proof. \( \square \)
Suppose that $X = \{X_1, X_2\}$ is a split of $M$, generating the decomposition $\{M_1, M_2\}$. Further, suppose that $M \in \mathcal{M}$, that $M_1 \cong M^*(K_{3,3})$, that $M_2 \in \mathcal{M}$, and that $\delta(X) = \emptyset$. If $M_2$ has a component isomorphic to either $M(K_4)$ or $F_7$, then so does $M$.

**Proof.** Suppose that $M_2$ has a split $Y = \{Y_1, Y_2\}$ generating a decomposition $\{N_1, N_2\}$, where $N_1 \cong F_7$. Observe that $\gamma(Y) \subseteq N_1$. We may assume that the closure in $M_2$ of either $Y_1$ or $Y_2$ contains $\gamma(X)$, and, since $M_2 \in \mathcal{M}$, that the other meets $\gamma(X)$ in at most one element. If $cl_{M_2}(Y_2) \supseteq \gamma(X)$, then (3.2.22) implies that $M$ has a simple decomposition with component $N_1 \cong F_7$. Therefore, we may assume that $cl_{M_2}(Y_1) \supseteq \gamma(X)$. Hence $\gamma(X) \subseteq N_1$. Since each triangle of $F_7$ meets every other triangle of $F_7$, we have that $|\gamma(X) \cap \gamma(Y)| = 1$. By (3.2.12), $Z = \{Z_1, Z_2\}$ is a split of $M$, where $Z_2 = Y_2 \cap X_2$ and $Z_1 = (Y_1 \cap X_2) \cup X_1$. By (3.2.13), $\gamma(Z) = \gamma(Y)$.

Let $\{H_1, H_2\}$ be the decomposition of $M$ generated by $Z$. By (3.2.22), $H_1$ has a simple decomposition $\{M_1, N_1\}$ generated by a split $W$. By (4.5.4), $W$ is a good split. Since $\gamma(W) = \gamma(X)$ we have that $|\gamma(W) \cap \gamma(Z)| = 1$. This contradicts (4.4.6).

Hence $cl_{M_2}(Y_1) \not\supseteq \gamma(X)$. Hence, the result follows if $N_1 \cong F_7$.

Suppose that $M_2$ has a split $Y = \{Y_1, Y_2\}$ which generates the decomposition $\{N_1, N_2\}$, where $N_1 \cong M(K_4)$. Observe that $\gamma(Y) \subseteq N_1$. We may assume that the closure in $M_2$ of either $Y_1$ or $Y_2$ contains $\gamma(X)$, and, since $M_2 \in \mathcal{M}$, that the closure of the other meets $\gamma(X)$ in at most one element. If $cl_{M_2}(Y_2) \supseteq \gamma(X)$, then (3.2.22) implies that $M$ has a simple decomposition with component $N_1 \cong M(K_4)$.

Therefore, we may assume that $cl_{M_2}(Y_1) \supseteq \gamma(X)$. Hence $\gamma(X) \subseteq N_1$. Since every
two distinct triangles in $M(K_4)$ have exactly one common element and there are six elements in $M(K_4)$, we have that $E(N_1) = \gamma(Y) \cup \gamma(X) \cup \{y\}$ for some element $y$ of $Y_1$. Let $\{x\} = \gamma(Y) \cap \gamma(X)$. By (3.2.12), $Z = \{Z_1, Z_2\}$ is a split of $M$ where $Z_1 = (Y_1 \cap X_2) \cup X_1$ and $Z_2 = Y_2 \cap X_2$. By (3.2.13), $\gamma(Z) = \gamma(Y)$. Let $\{H_1, H_2\}$ be the decomposition of $M$ generated by $Z$. By (3.2.22), $H_1$ has a simple decomposition $\{M_1, N_1\}$ generated by a split $W = \{W_1, W_2\}$. By (4.5.4), either $W$ is a good split, or $|\delta(W)| \leq 1$. Moreover, by (3.2.13), $\gamma(W) = \gamma(X)$. Therefore, $|\gamma(W) \cap \gamma(Z)| = 1$. Hence, (4.4.6) implies that $W$ is not a good split. Therefore, $|\delta(W)| \leq 1$. Since $\gamma(Z) \subseteq H_1$, we have that $\delta(W) = \{x\}$. By (4.5.5), $X_1 = S \cup T$, where $r(S) = 3$, $r(T) = 2$, and, for every subset $R$ of $M$, $3 \leq r(S \cup R) \leq r(R) + 2$; and $r(T \cup R) = r(R) + 2$. Therefore, $r(S \cup Y_2) \leq r(Y_2) + 2$, and $r(T \cup \{y\}) = 3$. Observe that

$$E(M) = X_1 \cup X_2$$

$$= X_1 \cup (Y_1 \cup Y_2)$$

$$\subseteq X_1 \cup [\{y\} \cup \delta(X) \cup \delta(Y)) \cup Y_2)].$$

But, $\delta(X) \subseteq X_1$ and $\delta(Y) \subseteq Y_2$. Hence,

$$E(M) \subseteq X_1 \cup Y_2 \cup \{y\}$$

$$= (X \cup Y_2) \cup T \cup \{y\}.$$ 

We have that

$$r(S \cup Y_2) + r(T \cup \{y\}) = (r(Y_2) + 2) + 3$$

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\[ r(Y_2) + 5 = r(M) + 2. \]

Hence, \( U = \{ cl_M(S \cup Y_2), cl_M(T \cup \{y\})\} \) is a split of \( M \). Let \( \{G_1, G_2\} \) be the decomposition generated by \( U \). Suppose that \( e \in cl_M(T \cup \{y\}) - (T \cup \{y\}) \). Then there is an element \( f \) in \( T \) such that \( \{f, e, y\} \) is a triangle of \( M \). Therefore, \( \{f, e, y\} \) is contained in either \( X_1 \) or \( X_2 \). Since \( T \cap \gamma(X) \) is empty, \( \{f, e, y\} \) is contained in \( X_1 \). But \( y \not\in \gamma(X) \), hence \( \{f, e, y\} \not\subset X_1 \); a contradiction. Therefore \( cl_M(T \cup \{y\}) = T \cup \{y\} \). Since \( r(T \cup \{y\}) = 3 \), \( G_2 \) is isomorphic to either \( M(K_4) \) or \( F_7 \). In either case, \( \gamma(U) \) must meet \( T \). Hence \( |E(G_2)| \leq 6 \). Thus \( G_2 \cong M(K_4) \). □

We are now prepared to prove the main results of this chapter. For convenience, we restate each of the theorems.

**Theorem 4.1.1.** Suppose that \( M \in \mathcal{M} \) and \( M \) has no decomposition with a component isomorphic to either \( M(K_4) \) or \( F_7 \). Then \( M \cong M^*(K_{3,n}) \), for some \( n \geq 4 \).

**Proof.** Let \( M \) be a counterexample where \( |E(M)| \) is minimal. By (3.2.12), we may assume that \( M \) has a decomposition \( \{M_1, M_2\} \) such that \( M_1 \) has no splits. Let \( X = \{X_1, X_2\} \) be the split of \( M \) which generates \( \{M_1, M_2\} \). By (4.4.2) and (4.4.7), we have that \( \delta(X) = \emptyset \). By (4.4.11), we have that \( M_1 \cong M^*(K_{3,3}) \). If \( M_2 \not\in \mathcal{M} \), then (4.4.13) implies that either \( M_2 \) has no split, or \( M_1 \) has a triad. Since \( M^*(K_{3,3}) \) has no triad, we have that \( M_2 \) has no split. By (4.4.11), \( M_2 \cong M^*(K_{3,3}) \). Hence, \( M \cong M^*(K_{3,4}) \). Therefore, we may assume that \( M_2 \in \mathcal{M} \). By (4.5.6), \( M_2 \) does not have a component isomorphic to either \( M(K_4) \) or \( F_7 \). By the minimality of \( M \),
we have that $M_2 \cong M^*(K_{3,k})$ for some $k \geq 4$. Hence, $M \cong M^*(K_{3,(k+1)})$. This completes the proof. □

**Theorem 4.1.2.** Suppose that $M \in \mathcal{M}$ and that $X = \{X_1, X_2\}$ is a split of $M$ with decomposition $\{M_1, M_2\}$ such that $M_1$ is a maximal wheel of $M$. Then $|\delta(X)| \leq 2$ and

\begin{enumerate}[(i)]
  \item if $|\delta(X)| = 2$, then $M_2 \in \mathcal{M}$; and
  \item if $|\delta(X)| \leq 1$, then $M_2 \in \mathcal{M} \cup \mathcal{N} \cup \{F_7, M(K_4)\}$.
\end{enumerate}

**Proof.** By Proposition 4.3.8, $|\delta(X)| \leq 2$. Suppose that $|\delta(X)| = 2$. If $M_2 \notin \mathcal{M}$, then (4.4.5) implies that $M_2 \cong M(K_4)$. Since $M \in \mathcal{M}$, (4.2.3) implies that $M$ is a wheel, contradicting the assumption that $M_1$ is a maximal wheel of $M$. Hence $M_2 \in \mathcal{M}$. Suppose that $|\delta(X)| \leq 1$. Then (4.4.10) and (4.4.14) imply that the result holds. □

**Theorem 4.1.3.** Suppose that $M \in \mathcal{M}$ and that $X = \{X_1, X_2\}$ is a split of $M$ with decomposition $\{M_1, M_2\}$ such that $M_1$ is a maximal spike of $M$. Then $|\delta(X)| \leq 1$ and

\begin{enumerate}[(i)]
  \item if $|\delta(X)| = 1$, then $M_2 \in \mathcal{M} \cup \{M(K_4)\}$; and
  \item if $\delta(X) = \emptyset$, then $M_2 \in \mathcal{M} \cup \{M(K_4), F_7\}$.
\end{enumerate}

Moreover, if $M_2 \cong F_7$, then $M$ is a tipless spike.

**Proof.** By (4.5.1) (i), $|\delta(X)| \leq 1$. Suppose that $\delta(X) = \{x\}$. By (4.5.1) (ii), $x$ is a tip of $M_1$. If $M_2 \cong F_7$, then (4.2.5) says that $x$ is a tip of the spike $M_2$. By (4.2.6),
$M$ is a spike, contradicting the assumption that $M_1$ is a maximal spike of $M$. Hence $M_2 \not\subseteq F_7$. Suppose that $M_2 \not\subseteq M$ and $M_2 \not\subseteq M(K_4)$. Then (4.4.10) implies that $M_1$ has a triad. But $F_7$ has no triad. Therefore, assertion (i) holds.

Suppose that $\delta(X) = \emptyset$. Then (4.4.14) implies that $M_2 \cong M^*(K_{3,3})$, or $M_1$ has a triad, or (ii) holds. Since $F_7$ has no triad, we may assume that $M_2 \cong M^*(K_{3,3})$. But this contradicts (4.5.1) (iii). Hence, assertion (ii) holds.

Now suppose that $M_2 \cong F_7$. We have shown that $\delta(X) = \emptyset$ in this situation. By (4.2.5), there is an element $x \in \delta(X)$ which is a tip for $M_1$. Let $M' = P|(E(M) \cup \{x\})$. By (3.3.5), $X' = \{X_1 \cup \{x\}, X_2 \cup \{x\}\}$ is a split of $M'$. This split generates the same decomposition $\{M_1, M_2\}$. By (4.2.5), $M'$ is a spike with the unique tip $x$. But $M = M'\setminus e$, so $M$ is a tipless spike. □
CHAPTER 5

CONTRACTION-MINIMALLY 3-CONNECTED BINARY MATROIDS

5.1 Introduction

In this chapter we show that if $M$ is a contraction-minimally 3-connected binary matroid and $N$ is a component in the minimal decomposition of $M$, then $N$ is a contraction-minimally 3-connected binary matroid. The main result of this chapter is the following theorem, describing in more detail, the components of such a decomposition. As in the last two chapters, we shall assume throughout this chapter that $M$ is a 3-connected binary matroid.

(5.1.1) Theorem. Suppose that $M$ is contraction-minimally 3-connected and that $N$ is a component in the minimal decomposition of $M$. Then

(i) $N$ is a vertically 4-connected matroid that is a union of triangles;

(ii) $N \in \{\mathcal{W}_r : r \geq 4\}$;

(iii) $N \in \{\mathcal{S}_r : r \geq 4\}$; or

(iv) $N \in \{M^*(K_{3,n}) : n \geq 4\}$.

The above result generalizes to binary matroids the following graph result of Coullard, Gardner, and Wagner (1993, Theorem 1.1).

(5.1.2) Theorem. Every minimally 3-connected graph has a unique minimal decomposition with the property that every member is either cyclically 4-connected, a twirl, or a wheel.
As was shown in Section 3.3, if the 3-connected graph $G$ has the minimal decomposition $\{G_1, G_2, \ldots, G_n\}$ using the theory of Coullard, Gardner, and Wagner, then $M^*(G)$ will have the minimal decomposition $\{M^*(G_1), M^*(G_2), \ldots, M^*(G_n)\}$ using our more general decomposition for binary matroids. Therefore, in terms of the present decomposition, Theorem 5.1.2 translates as follows.

**(5.1.3) Theorem.** Suppose $M$ is a contraction-minimally 3-connected cographic matroid, and that $N$ is a component in the minimal decomposition of $M$. Then

(i) $N$ is a vertically 4-connected matroid which is a union of triangles;

(ii) $N \in \{W_r : r \geq 4\}$; or

(iii) $N \in \{M^*(K_{3,n}) : n \geq 4\}$.

In light of Theorem 5.1.3, our main result Theorem 5.1.1 is equivalent to the statement that a contraction-minimally 3-connected binary matroid that has no good splits is either cographic or a spike. Recall that if $M$ is a 3-connected matroid, then $M$ is minimally 3-connected if for all $e \in E(M)$, the matroid $M\setminus e$ is not 3-connected. The matroid $M$ is contraction-minimally 3-connected if for all $e \in E(M)$, the matroid $M/e$ is not 3-connected.

### 5.2 The Components

In this section we prove the main result of the chapter, Theorem 5.1.1. We begin by showing that the components of a binary contraction-minimally 3-connected matroid are themselves contraction-minimally 3-connected. The next result is the key to this.
(5.2.1) Suppose that $M$ is contraction-minimally 3-connected and $X = \{X_1, X_2\}$ is a good split of $M$ generating the simple decomposition $\{M_1, M_2\}$. Then $M_1$ and $M_2$ are contraction-minimally 3-connected.

**Proof.** Suppose there is an element $e \in E(M_1)$ such that $M_1/e$ is 3-connected. Since $\gamma(X)$ is a triangle of $M_1$, the element $e$ belongs to $E(M)$. By (4.2.2), $e$ is not in a triangle of $M$. Since $M/e$ is not 3-connected, there is a 2-separation $\{Z'_1, Z'_2\}$ of $M/e$. Therefore, $r_{M/e}(Z'_1) + r_{M/e}(Z'_2) = r(M/e) + 1$ and $|Z'_i| \geq 2$ for each $i \in \{1, 2\}$.

Suppose that $r_{M/e}(Z'_2) = r_M(Z'_2)$. Since $r_M(Z'_i \cup \{e\}) = r_{M/e}(Z'_i) + 1$ for each $i \in \{1, 2\}$, we have that

$$r_M(Z'_i \cup \{e\}) + r_M(Z'_2) = r_{M/e}(Z'_1) + r_{M/e}(Z'_2) + 1$$

$$= r(M/e) + 2$$

$$= r(M) + 1.$$ 

This contradicts the 3-connectivity of $M$. Therefore, we may assume that $r_{M/e}(Z'_i) = r_M(Z'_i) - 1$ for each $i \in \{1, 2\}$. This implies that $e \in cl_M(Z'_i)$. Moreover,

$$r_M(Z'_i \cup \{e\}) + r_M(Z'_2) = r_M(Z'_1) + r_M(Z'_2 \cup \{e\})$$

$$= r_{M/e}(Z'_1) + r_{M/e}(Z'_2) + 2$$

$$= r(M/e) + 3$$

$$= r(M) + 2.$$ 

If $|Z_i| = 2$, then $e$ is in the triangle $Z_i \cup \{e\}$. Hence, $Z = \{cl_M(Z'_1 \cup \{e\}), cl_M(Z'_2 \cup \{e\})\}$ is a split of $M$. Let $Z_i = cl_M(Z'_i \cup \{e\})$ for each $i \in \{1, 2\}$. Observe that

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$e \in \gamma(Z)$. Since $X$ is a good split of $M$, the splits $X$ and $Z$ are compatible. Moreover, since $e \in X_1 - \gamma(X)$, we may assume that $Z_1 \subseteq X_1$. By (3.2.11), there is a split $Y' = \{Y'_1, Y'_2\}$ of $M_1$, where $Y'_1 = c_{M_1}(Z_1)$ and $Y'_2 = (Z_2 \cap X_1) \cup \gamma(X)$. Since $e \in \gamma(Z)$, it also belongs to $\gamma(Y')$. Therefore, $M_1/e$ is not 3-connected; a contradiction. The result follows by symmetry. □

The next result is an immediate corollary of (5.2.1).

(5.2.2) Suppose $N$ is a component in the minimal decomposition of a contraction-minimally 3-connected matroid $M$. Then $N$ is contraction-minimally 3-connected.

The following is a result of Oxley (1981c, Theorem 2.9).

(5.2.3) Let $M$ be a contraction-minimally 3-connected matroid having at least four elements. Then, for all elements $e$ such that $e$ is not in a triangle, $M\sslash e$ is contraction-minimally 3-connected.

What follows is a sequence of technical results that will be used in the proof of Theorem 5.1.1.

(5.2.4) Suppose that $X = \{X_1, X_2\}$ is a split of $M$ generating the decomposition $\{M_1, M_2\}$ where $M_1$ is a wheel. If $|\delta(X)| \leq 1$, then $X_1 - \delta(X)$ contains an element which is not in a triangle of $M$.

Proof. Suppose that an element $e \in X_1 - \delta(X)$ is not in a triangle of $M\sslash X_1$. If $e$ is in a triangle of $M$, then at least one of the other elements of this triangle must be in $X_1 - \delta(X)$. Therefore, this triangle is in $X_1$; a contradiction. Therefore, it is enough to show that there is an element in $X_1 - \delta(X)$ which is not in a triangle of $M\sslash X_1$. 

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Since $|\delta(X)| \leq 1$, the matroid $M|X_1$ is a wheel with either two or all three of the elements deleted from one of its triangles. If $M_1 \cong M(K_4)$, then $M|X_1$ has either three or four elements and has rank three. In both cases, there is clearly an element not contained in a triangle of $M|X_1$. Suppose that $M_1 \cong \mathcal{W}_r$ where $r \geq 4$. Then $E(M_1)$ is the disjoint union of two sets $Y = \{y_1, y_2, \ldots, y_n\}$ and $Z = \{z_1, z_2, \ldots, z_n\}$, such that each triangle of $M_1$ is of the form $\{z_i, y_i, z_{i+1}\}$ where the subscripts are taken modulo $n$. By symmetry, we may assume that $\gamma(X) = \{z_2, y_2, z_3\}$ and that $X_1 = E(M_1) - \{y_2, z_3\}$, or $X_1 = E(M_1) - \{z_2, z_3\}$, or $X_1 = E(M_1) - \{z_2, y_2, z_3\}$. In every case, $y_3$ is not in a triangle of $M|X_1$. This completes the proof. □

(5.2.5) Suppose that $M$ is contraction-minimally 3-connected, that $X = \{X_1, X_2\}$ is a split of $M$ generating the decomposition $\{M_1, M_2\}$, and that $M \in \mathcal{M}$. Further, suppose that $M_1$ is a maximal wheel. Then $|\delta(X)| = 2$.

Proof. By Theorem 4.1.2, $|\delta(X)| \leq 2$. Suppose that $|\delta(X)| \leq 1$. By (5.2.4), there is an element $e$ of $X_1 - \delta(X)$ such that $e$ is not in a triangle of $M$. By (5.2.3), $M\backslash e$ is contraction-minimally 3-connected. By (3.3.6), $\{X_1 - \{e\}, X_2\}$ is a split of $M\backslash e$ with connection $\gamma(X)$. Hence, $M_1\backslash e$ is a component of the decomposition generated by $\{X_1 - \{e\}, X_2\}$. By (3.2.7), $M_1\backslash e$ is 3-connected; a contradiction. Hence $|\delta(X)| = 2$. □

(5.2.6) Suppose that $M$ is contraction-minimally 3-connected and $M\backslash e$ is a wheel. Then $e$ is in a triangle of $M$.

Proof. Since $M/e$ is not 3-connected, there is a 2-separation $\{Z'_1, Z'_2\}$ of $M/e$ such that $e \in cl_M(Z'_i)$ for each $i \in \{1, 2\}$, and $r_M(Z'_1) + r_M(Z'_2) = r(M) + 2.$

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Hence, either $Z_i' \cup \{e\}$ is a triangle of $M$ for some $i \in \{1, 2\}$, or there is a split $Z = \{cl_M(Z_1'), cl_M(Z_2')\}$ of $M$. Therefore, we may assume that there is a split $Z = \{Z_1, Z_2\}$ generating a decomposition $\{M_1, M_2\}$ such that $e \in \gamma(Z)$. By (3.3.6), $W = \{Z_1 - \{e\}, Z_2 - \{e\}\}$ is a split of $M \setminus e$ generating the same decomposition, where $\gamma(W) = \gamma(Z)$. Since $M \setminus e$ is a wheel, $|\delta(W)| = 2$. Since $\delta(Z) = \delta(W) \cup \{e\}$, we have that $\delta(Z)$ is a triangle of $M$. This completes the proof. □

(5.2.7) Suppose that $X = \{X_1, X_2\}$ and $Y = \{Y_1, Y_2\}$ are distinct splits of $M$, where $X$ generates the decomposition $\{M_1, M_2\}$. Further, suppose that $M_1$ is a wheel and $Y$ is a good split of $M$. Then, either $Y_1 \supset X_1$ or $Y_2 \supset X_1$.

Proof. Since $Y$ is a good split of $M$, we may assume that either $Y_1 \supset X_1$ or $Y_2 \supset X_2$. The latter case implies that $Y_2 \subset X_1$. If $Y_2 \subset X_1$, then (3.2.16) implies that $M_1$ has a good split, contradicting (4.2.1). Hence the statement is proved. □

(5.2.8) Suppose that $X = \{X_1, X_2\}$ and $Y = \{Y_1, Y_2\}$ are distinct splits of $M$, where $X$ generates the decomposition $\{M_1, M_2\}$. Further, suppose that $M_1$ is a spike and $Y$ is a good split of $M$. Then, either $Y_1 \supset X_1$ or $Y_2 \supset X_1$.

Proof. Since $Y$ is a good split of $M$, we may assume that either $Y_1 \supset X_1$ or $Y_2 \supset X_2$. The latter case implies that $Y_2 \subset X_1$. If $Y_2 \subset X_1$, then (3.2.16) implies that $M_1$ has a good split, contradicting (4.2.5). Hence the statement is proved. □

The next two results are direct consequences of the definition and characterization of spikes given by Definition 4.2.4 and (4.2.5).
(5.2.9) Suppose that \( X = \{X_1, X_2\} \) is a split of \( M \) generating the decomposition \( \{M_1, M_2\} \) where \( M_1 \) is a spike. If \( \delta(X) = 0 \), then \( X_1 \) contains elements \( e \) and \( f \) where neither \( e \) nor \( f \) belong to a triangle of \( M \), and \( e \) and \( f \) are in a single triangle of \( M_1 \).

(5.2.10) Suppose that \( M \) is a spike and \( e \) and \( f \) are elements of \( M \) belonging to the same triangle of \( M \), neither of which is the tip. Then \( M \setminus \{e, f\} \) is not 3-connected.

(5.2.11) Suppose that \( M \) is contraction-minimally 3-connected, that \( X = \{X_1, X_2\} \) is a split of \( M \) generating the decomposition \( \{M_1, M_2\} \), and that \( M \in M \). Further, suppose that \( M_1 \) is a maximal spike. Then \( |\delta(X)| = 1 \).

Proof. By Theorem 4.1.3, \( |\delta(X)| \leq 1 \). Suppose that \( |\delta(X)| = 0 \). By (5.2.9), there are elements \( e \) and \( f \) of \( X_1 - \delta(X) \) such that \( e \) and \( f \) belongs to the same triangle of \( M_1 \), but neither of them belong to a triangle of \( M \). By applying (5.2.3) twice, we get that \( M \setminus \{e, f\} \) is contraction-minimally 3-connected. By applying (3.3.6) twice, we get that \( \{X_1 - \{e, f\}, X_2\} \) is a split of \( M \setminus \{e, f\} \) with connection \( \gamma(X) \). Hence, \( M_1 \setminus \{e, f\} \) is a component of the decomposition generated by \( \{X_1 - \{e, f\}, X_2\} \). By (3.2.7), \( M_1 \setminus \{e, f\} \) is 3-connected, contradicting (5.2.10). Hence \( |\delta(X)| = 1 \). \( \square \)

(5.2.12) Suppose that \( M \) is contraction-minimally 3-connected and \( M \setminus e \) is a spike. Then \( M \) has a good split.

Proof. Observe that the rank of \( M \) is at least four. Let \( E(M \setminus e) = \bigcup_{i=1}^{n} L_i \), where \( L_1, L_2, \ldots, L_n \) are the three-point lines of the spike \( S_n \) with tip \( p \). Then
\[
n = r(M). \quad \text{Since } M/e \text{ is not 3-connected, there is a 2-separation } \{Z_1', Z_2'\} \text{ of } M/e \text{ such that } e \in cl_M(Z_i') \text{ for each } i \in \{1, 2\}, \text{ and } r_M(Z_1') + r_M(Z_2') = r(M) + 2.
\]

Hence, either \(Z_i' \cup \{e\}\) is a triangle of \(M\) for some \(i \in \{1, 2\}\), or there is a split \(Z = \{cl_M(Z_1'), cl_M(Z_2')\}\) of \(M\). Suppose that \(e\) is in a triangle \(T\) of \(M\). Then, we may assume that \(T = \{e, a_1, a_2\}\) where \(a_i \in L_i - \{p\}\) for \(i \in \{1, 2\}\). Since \(r(L_1 \cup L_2 \cup \{e\}) + r(L_3 \cup L_4 \cup \cdots \cup L_n) = 3 + (n - 1) = r(M) + 2\), either \(X = \{L_1 \cup L_2 \cup \{e\}, L_3 \cup L_4 \cup \cdots \cup L_n\}\) or \(\{L_1 \cup L_2 \cup \{e\}, L_3 \cup L_4 \cup \cdots \cup L_n \cup \{e\}\}\) is a split of \(M\). The former cannot be a split because if it were, then \(S(X) = \{p\}\) and
\[
|cl_P(L_1 \cup L_2 \cup \{e\})| \geq |L_1 \cup L_2 \cup \{e\}| + 2 = 8,
\]
a contradiction. Hence, \(X = \{L_1 \cup L_2 \cup \{e\}, L_3 \cup L_4 \cup \cdots \cup L_n \cup \{e\}\}\) is a split of \(M\), where \(S(X) = \{p, e\}\). By (4.5.1), \(X\) is a good split of \(M\).

Suppose that \(Z = \{cl_M(Z_1'), cl_M(Z_2')\}\) is a split of \(M\) generating the decomposition \(\{M_1, M_2\}\). We have that \(e \in \gamma(Z)\). By (3.3.6), \(W = \{cl_M(Z_1') - \{e\}, cl_M(Z_2') - \{e\}\}\) is a split of \(M\setminus e\) generating the same decomposition, where \(\gamma(Z) = \gamma(W)\). Since \(M\setminus e\) is a spike, \(|\delta(W)| = 1\). Since \(\delta(Z) = \delta(W) \cup \{e\}\), we have that \(|\delta(Z)| = 2\). By (4.5.1), \(Z\) is a good split of \(M\). This completes the proof. \(\Box\)

Finally, we prove our main result. For convenience, we also restate it.

**Theorem 5.1.1.** Suppose that \(M\) is contraction-minimally 3-connected and that \(N\) is a component in the minimal decomposition of \(M\). Then

(i) \(N\) is a vertically 4-connected matroid which is a union of triangles;

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(ii) \( N \in \{ W_r : r \geq 4 \}; \)

(iii) \( N \in \{ S_r : r \geq 4 \}; \) or

(iv) \( N \in \{ M^*(K_{3,n}) : n \geq 4 \}. \)

**Proof.** By (5.2.2), \( N \) is contraction-minimally 3-connected. Hence, if \( N \) is vertically 4-connected, then \( N \) satisfies (i). Suppose that \( N \) is not vertically 4-connected; that is, suppose that \( N \notin M \). By Theorem 4.1.1, we may assume that \( N \) has a component isomorphic to either \( M(K_4) \) or \( F_7 \). Suppose that \( N \) is the smallest matroid in \( M \) which is contraction-minimally 3-connected but is not a wheel, not a spike, and not isomorphic to any \( M^*(K_{3,n}) \) with \( n \geq 4 \).

Suppose that \( N \) has a component isomorphic to \( M(K_4) \). Let \( X = \{X_1, X_2\} \) be a split of \( N \) generating the decomposition \( \{M_1, M_2\} \) where \( M_1 \) is a maximal wheel of \( N \). By (5.2.5), \( |\delta(X)| = 2 \). By Theorem 4.1.2, \( M_2 \in M \). If \( M_2 \) is not contraction-minimally 3-connected, then there is an element \( e \in E(M_2) \) such that \( M_2/e \) is 3-connected. This implies that \( e \) is not in a triangle of \( M_2 \). By (4.2.2), \( e \in X_2 - \delta(X) \) and \( e \) is not in a triangle of \( N \). By (5.2.3), \( N \setminus e \) is contraction-minimally 3-connected. By (4.1.5), \( Y = \{Y_1, Y_2\} \) is a split of \( N \setminus e \) where \( Y_1 = X_1 \) and \( Y_2 = X_2 - \{e\} \). Moreover, this split generates the decomposition \( \{M_1, M_2 \setminus e\} \). If \( N \setminus e \in M \), then the minimality of \( N \) implies that \( N \setminus e \) is a wheel, a spike, or the dual of a twirl. By (4.2.6) and (4.2.9), \( N \setminus e \) must be a wheel. Therefore, (4.2.3) implies that \( M_2 \setminus e \) is a wheel. But this contradicts (5.2.6). Therefore, we may assume that \( N \setminus e \notin M \). Since \( N \setminus e \) has a split, it must have a good split. Let \( Z = \{Z_1, Z_2\} \) be a
good split of $N \setminus e$ generating the decomposition $\{H_1, H_2\}$. By (5.2.7), for any such split, we may assume that $Z_1 \supset Y_1$. Let $Z$ be such that $|Z_1|$ is minimal. Therefore, $H_1$ has no good splits. Hence $H_1 \in \mathcal{M}$. By (5.2.1), $H_1$ is contraction-minimally 3-connected. By the minimality of $N$, the matroid $H_1$ is a wheel, a spike, or the dual of a twirl. By (4.2.6) and (4.2.9), $H_1$ is a wheel, contradicting the maximality of $M_1$. Therefore, we may assume that $N$ has no component isomorphic to $M(K_4)$.

Suppose that $N$ has a component isomorphic to $F_7$ but does not have a component isomorphic to $M(K_4)$. Let $X = \{X_1, X_2\}$ be a split of $N$ generating the decomposition $\{M_1, M_2\}$ where $M_1$ is a maximal spike of $N$. By (5.2.11), $|\delta(X)| = 1$. By Theorem 4.1.3, $M_2 \in \mathcal{M}$. If $M_2$ is not contraction-minimally 3-connected, then there is an element $e \in E(M_2)$ such that $M_2/e$ is 3-connected. This implies that $e$ is not in a triangle of $M_2$. By (4.2.2), $e \in X_2 - \delta(X)$ and $e$ is not in a triangle of $N$. By (5.2.3), $N \setminus e$ is contraction-minimally 3-connected. By (4.1.5), $Y = \{Y_1, Y_2\}$ is a split of $N \setminus e$ where $Y_1 = X_1$ and $Y_2 = X_2 - \{e\}$. Moreover, this split generates the decomposition $\{M_1, M_2 \setminus e\}$. If $N \setminus e \in \mathcal{M}$, then the minimality of $N$ implies that $N \setminus e$ is a wheel, a spike, or the dual of a twirl. By (4.2.3) and (4.2.9), $N \setminus e$ must be a spike. Therefore, (4.2.6) implies that $M_2 \setminus e$ is a spike. But this contradicts (5.2.12). Therefore, we may assume that $N \setminus e \not\in \mathcal{M}$. Since $N \setminus e$ has a split, it must have a good split. Let $Z = \{Z_1, Z_2\}$ be a good split of $N \setminus e$ generating the decomposition $\{H_1, H_2\}$. By (5.2.8), for any such split, we may assume that $Z_1 \supset Y_1$. Let $Z$ be chosen such that $|Z_1|$ is minimal. Therefore, $H_1$ has no good splits. Hence $H_1 \in \mathcal{M}$. By (5.2.1), $H_1$ is contraction-minimally 3-connected. By the minimality of $N$, the
matroid $H_1$ is a wheel, a spike, or the dual of a twirl. By (4.2.3) and (4.2.9), $H_1$ is a spike, contradicting the maximality of $M_1$. This completes the proof. \qed
REFERENCES


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John William Leo, son of Gerald and Dorothy Leo, was born in Seattle, Washington, on January 14, 1962. In 1980, he graduated from Shoreline High School in Seattle. In 1984, he received his bachelor of arts degree in mathematics from Whitman College in Walla Walla, Washington. After serving as an officer in the United States Navy, he enrolled at Louisiana State University in August, 1990. He received his master of science degree from Louisiana State University in May, 1992, and will receive his doctor of philosophy degree in August, 1996. John is married to the former Kimberly Anne Holbrook. They have a son, Magnus Anthony.
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