Mathematical Problems in Elasticity and Quantum Mechanics.

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MATHEMATICAL PROBLEMS IN ELASTICITY AND QUANTUM MECHANICS

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

The Department of Mathematics

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ABSTRACT

This thesis is concerned with new mathematical contributions to this area of mathematical physics. It consists of two parts. The first part discusses Pauli uniqueness problem in the content of locally compact abelian groups. We get improved results for $\mathbb{R}^n$, new results for $T^m$ and $\mathbb{Z}_n$. In the second part which is the main part of the thesis, we develop a new approach to the scattering for Nth order factored equations based on an abstract version of d'Alembert's formula. We show that the asymptotic equivalence (or scattering theory) of the pair of higher order factored equations reduces to the asymptotic equivalence of the first order equations. Our approach is clean, direct and easy to use. It also enable us to recover and unify old results on acoustic waves as well as to obtain new results on elastic waves. Moreover, it follows from our approach that elastic wave theory is a corollary of the acoustic wave theory.
INTRODUCTION

Despite their obvious differences, classical physics and quantum physics have much in common mathematically. The framework for each is a suitable Schrödinger equation on an underlying Hilbert space of state vectors. The unitarity of the governing dynamical group reflects a fundamental conservation principle. Thus with classical wave propagation, energy is concerned while in quantum mechanics, probability is concerned. Frequently the Hilbert space is a function space involving Euclidean spaces, and the Fourier transform plays a role in the interpretation of various quantities.

This thesis is concerned with new mathematical contributions to this area of mathematical physics.

The first chapter deals with Fourier transforms and wave function interpretation. If $f$ is a vector in $L^2(\mathbb{R}^n)$ and $\hat{f}$ is its Fourier transform, then $|f(x)|^2$ (resp. $|\hat{f}(k)|^2$) is the position (resp. momentum) probability density of the quantum mechanical state (or wave function) $f$. To what extent do $|f|^2$ and $|\hat{f}|^2$ determine the joint position-momentum probability distribution of $f$? This is an improperly posed question, and the joint distribution does not exist. This reflects one of the special features of quantum mechanics. The probabilistic interpretation is only a partial one. All information about the state is contained in the complex valued wave function $f$. In any event, one can raise the question of Pauli uniqueness: if $|f|^2 = |g|^2$ and $|\hat{f}|^2 = |\hat{g}|^2$, does $f = cg$ for some unimodular constant $c$? The answer is no in general.
and yes under some additional hypotheses. The same question (with a different scattering theoretic interpretation) arises when the underlying group \( \mathbb{R}^n \) is replaced by the circle group \( T = \{ z \in \mathbb{C} : |z| = 1 \} \). In Chapter 1 we replace \( \mathbb{R}^n \) by a general locally compact abelian group \( G \) and consider the Pauli uniqueness question in that context. We get improved results for \( G = \mathbb{R}^n \), and new results for \( G = T^n \) and \( G = \mathbb{Z}_n \), the (finite) cyclic group of order \( n \).

The main part of the thesis deals with factored equations of the form

\[
\prod_{j=1}^{N} \left( \frac{d}{dt} + iA_j \right) u(t) = 0
\]

with \( A_1, \ldots, A_N \) commuting self-adjoint operators in a Hilbert space \( \mathcal{H} \). Examples of such \( N \)th order equations are the acoustic wave equations (\( N = 2 \)) and the elastic wave equations (\( N = 4 \)). We let \( N \) be an arbitrary positive integer.

Many papers contribute to the scattering theory for the above factored higher order equations. One of the recent works is Kondoyannidis' Ph.D dissertation. In his work, Kondoyannidis treats factored equation problems with commuting normal operators in a general context, using multiparameter spectral theory and distribution theory for these higher order equations. Our approach is quite direct. Using an abstract version of d'Alembert's formula, we show that the asymptotic equivalence (or scattering theory) of the pair of equations

\[
\prod_{j=1}^{N} \left( \frac{d}{dt} + iA_j^{(\ell)} \right) u(t) = 0 \quad \ell = 0, 1
\]
reduces to the asymptotic equivalence of the first order equations

\[ \left( \frac{d}{dt} + iA_j^{(k)} \right) u(t) = 0 \]

for \( k = 0, 1 \) and \( j = 1, \cdots, N \). Chapter 2 presents the theory. Chapter 3 gives a variety of interpretations.
CHAPTER 1

FOURIER ANALYSIS
AND ITS INTERPRETATION IN SCATTERING

§1.1 Introduction

Suppose $\mathbb{R}^n$ is the n-dimensional Euclidean space, and $f \in L^2(\mathbb{R}^n)$ a vector with Fourier transform

$$\hat{f}(k) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ikx} f(x) dx, \quad k \in \mathbb{R}^n.$$  

(1)

One may ask whether $f$ can be determined up to a constant multiple of modulus one by $|f|$ and $|\hat{f}|$. This problem, referred to the problem of Pauli uniqueness by [5], was first raised by Pauli in the context of quantum mechanics: suppose $f(x)$ is the wave function for a single quantum mechanical particle in one dimension, can the state $f$ be determined by the position probability density $|f|^2$ and the momentum probability density $|\hat{f}|^2$? This is a significant problem because it is interesting to classify all functions which are Pauli unique from either a mathematical or a physical point of view. Several authors made contributions to it (see [3], [5], [9], [10], etc.). Fried- man [9] showed that if either $f$ or $\hat{f}$ has a non-negative representative, then $f$ is Pauli unique. Ashbaugh [3] gave a non-trivial example about the case of non-uniqueness. Goldstein [10] raised this problem again for functions on the circle group $T$ in order to interpret the measurements of scattering experiments. A generalized version of the question follows:
Let \( G \) be a locally compact abelian group and define \( C \) by
\[
C = \{ c | c : G \to \{ z : z \text{ is a complex number with } |z| = 1 \} \}. \tag{2}
\]

Given \( \beta \in L^2(G) \), for which \( c \) in \( C \) is it true that \( |\hat{\beta}| = |c\beta| \) on \( \hat{G} \)?

However this problem is not so simple as one might think at the first glance, and a complete solution is still not available.

In next section we will discuss some properties of this problem in the context of locally compact abelian groups.

§1.2 Pauli Uniqueness

Let \( G \) be a locally compact abelian group. Then \( G \) has a Haar measure \( dx \). \( f \in L^2(G) \) means that \( f \) is a complex measurable function on \( G \) such that
\[
\int_G |f|^2 dx < \infty. \tag{3}
\]

A complex measurable function \( \gamma \) on \( G \) is called a character of \( G \) if \( |\gamma(x)| = 1 \) for all \( x \in G \) and \( \gamma(x + y) = \gamma(x)\gamma(y) \) for all \( x, y \in G \). Clearly \( \gamma(-x) = \overline{\gamma(x)} \) for any character \( \gamma \). The set of all continuous characters of \( G \) forms a group \( \hat{G} \), the dual group of \( G \), if the operation, call it addition, is defined by \( (\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x) \). The Fourier transform of \( f \) is defined by
\[
\hat{f}(\gamma) = \int_G \gamma(-x)f(x)dx, \quad \text{for } \gamma \in \hat{G}. \tag{4}
\]

By the Plancherel Theorem we know the Fourier transform is unitary from \( L^2(G) \) to \( L^2(\hat{G}) \). Here we assume \( dx \) and \( d\gamma \) are suitably normalized. For
any pair of Borel functions \( f \) and \( g \) on \( G \) we define their convolution \( f * g \) by

\[
(f * g)(x) = \int_G f(x - y)g(y)dy
\]

for \( x \in G \), provided that \( \int_G |f(x - y)g(y)|dy < \infty \). It is easy to check \( \hat{f * g} = \hat{f} \hat{g} \) for any \( f, g \in L^2(G) \).

Suppose \( f \in L^2(G) \), then \( \{ |f|, |\hat{f}| \} \) is called the Pauli data of \( f \). \( f \) is Pauli unique if it can be uniquely determined up to modulus one by its Pauli data, i.e., for any \( c \in \mathbb{C} \), that \( |\hat{f}| = |c| |f| \) on \( \hat{G} \) a.e. (with respect to Haar measure on \( \hat{G} \)) implies that \( c \) is a constant function on the support of \( f \).

\( f \) is a representative of an integrable function \( g \in L^2(G) \) if \( f = cg \) almost everywhere for some constant \( c \) of modulus one.

**Proposition 1.2.1.** Suppose \( G \) is a locally compact abelian group and \( f \in G \). For any \( c \in \mathbb{C} \),

\[
|\hat{f}| = |c| |f| \iff < f(\cdot + y), f > = < c(\cdot + y)f(\cdot + y), cf >
\]

for a.e. \( y \in G \).

**Proof.** Let \( g(x) = \overline{f(-x)} \). Then, omitting the argument \( \gamma \) in \( \hat{f}(\gamma) \),

\[
|\hat{f}|^2 = \hat{f}\hat{f} = \hat{f}g = f * g = \int_G \gamma(-x)f * gdx = \int_G \gamma(-x) \int_G f(x - y)g(y)dydx.
\]
Hence \(|f| = |c| f|\) if \(f(x - y)f(-y)dy = f(x - y)f(x - y)c(-y)f(-y)dy\)
which is equivalent to \(<f(x + y), f > = <c(x + y)f(x + y), cf>\) for a.e. \(y\).
This completes the proof.

**Theorem 1.2.2.** Suppose \(f\) or \(\widehat{f}\) has a non-negative representative. Then \(f\) is Pauli unique.

**Proof.** It is similar to Theorem 1 in [9]. But for completeness, we give a short argument as follows: without loss of generality, we assume that \(f\) is non-negative. Let \(c \in \mathbb{C}\) and \(|c| f| = |f|\). Then by (6) we have
\[
\int_G (1 - c(x + y)c(x))f(x + y)\overline{f(x)}dx = 0
\]
\[
\int_G (1 - c(x + y)c(x))f(x + y)\overline{f(x)}dx = 0
\]
for a.e. \(y \in G\) which implies that \(c\) is constant on support of \(f\) since \(f\) is non-negative. Hence \(f\) is Pauli unique.

Let \(\alpha = (\alpha_1, \ldots, \alpha_n)\) be a multi-index. Define \(|\alpha| = \sum_{j=1}^{n} \alpha_j\), \(x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}\), and \(D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}\), where \(D_j = -i \frac{\partial}{\partial x_j}\). If \(P\) is the polynomial defined by \(P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha\), then define \(P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha\).
One may consult [11] for the details of these notations.

**Theorem 1.2.3.** Suppose \(f \in L^2(T^n)\) has a non-negative representative, then \(f\) is Pauli unique, where \(T^n\) is the \(n\)-torus.

**Proof.** It follows from Theorem 1.2.2 immediately.

**Theorem 1.2.4.**

(i) Suppose \(f \in L^2(\mathbb{R}^n)\) is a non-negative function, then \(f\) is Pauli unique.
(ii) Let \( P(\xi) = \sum_{\alpha \neq 0} a_\alpha \xi^\alpha + a_0(x) \) be a nonzero polynomial with \( a_\alpha \) a constant function for all \( \alpha \neq 0 \). Suppose \( f, g \in L^2(\mathbb{R}^n) \) are such that 
\[
(\sum_{\alpha \neq 0} a_\alpha \xi^\alpha) \hat{f}(\xi) \in L^2(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} a_0(x)|f|^2 \text{ exists, and } c \in \mathbb{C}.
\]
Then 
\[
< P(D)f, f > = < P(D)(cf), cf >.
\]
In particular, if both \( f, cf \) are eigenfunctions of \( P(D) \), then they are in the same eigenspace.

(iii) Suppose \( f, cf \) are bound states of a time-independent Schrödinger operator \( H = -\Delta + V(x) \) on \( L^2(\mathbb{R}^n) \), where \( V(x) \) is a suitable real potential. If \( c \in \mathbb{C} \), then \( f, cf \) are in the same eigenspace of \( H \).

(iv) Suppose \( f \in L^2(\mathbb{R}^n) \cap C^2(\mathbb{R}^n) \) is a real function. If \( D^\alpha f \in L^2(\mathbb{R}^n) \) for 
\[
|\alpha| \leq 2,
\]
then for any \( c \in \mathbb{C} \cap C^2(\mathbb{R}^n) \), 
\[
|\hat{f}| = |\hat{cf}|
\]
implies that \( c \) must be a constant on each connected component of the support of \( f \).

Proof. (i) This follows from Theorem 1.2.2 where \( G = \mathbb{R}^n \).

(ii) Let \( g = cf \). Then
\[
< P(D)f, f > = < (\sum_{\alpha \neq 0} a_\alpha D^\alpha)f, f > + < a_0(x)f, f >
\]
\[
= < (\sum_{\alpha \neq 0} a_\alpha \xi^\alpha)\hat{f}, \hat{f} > + < a_0(x)f, f >
\]
\[
= \int_{\mathbb{R}^n} P_0(\xi)|\hat{f}|^2 + \int_{\mathbb{R}^n} a_0(x)|f|^2
\]
\[
= \int_{\mathbb{R}^n} P_0(\xi)|\hat{g}|^2 + \int_{\mathbb{R}^n} a_0(x)|g|^2
\]
\[
= < P(D)g, g >,
\]
where \( P_0(\xi) = P(\xi) - a_0(\xi) \). Now if \( P(D)f = \lambda_1 f \) and \( P(D)g = \lambda_2 g \), then
\[
\lambda_1 \|f\|^2 = < \lambda_1 f, f > = < P(D)f, f > = < P(D)g, g > = \lambda_2 \|g\|^2.
\]
Hence \( \lambda_1 = \lambda_2 \) since \( \|f\|^2 = \|g\|^2 > 0 \).
(iii) This is a special case of (ii).

(iv) Let \( c(x) = e^{i\theta(x)} \) for some \( \theta(x) \in C^2(R^n) \) and \( g(x) = c(x)f(x) \).

Then
\[
\frac{\partial g(x)}{\partial x_j} = \frac{\partial f(x)}{\partial x_j} e^{i\theta(x)} + if(x) e^{i\theta(x)} \frac{\partial \theta(x)}{\partial x_j}
\]
\[
\frac{\partial^2 g(x)}{\partial x_j^2} = \frac{\partial^2 f(x)}{\partial x_j^2} e^{i\theta(x)} + 2i \frac{\partial f(x)}{\partial x_j} e^{i\theta(x)} \frac{\partial \theta(x)}{\partial x_j}
\]
\[
- f(x) e^{i\theta(x)} \left( \frac{\partial \theta(x)}{\partial x_j} \right)^2 + i f(x) e^{i\theta(x)} \frac{\partial^2 \theta(x)}{\partial x_j^2}.
\]

By (i) By (ii), \( \int_{R^n} \frac{\partial^2 f(x)}{\partial x_j^2} f(x) = \int_{R^n} \frac{\partial^2 g(x)}{\partial x_j^2} g(x) \). So we can conclude
\[
\int_{R^n} |f|^2 \left( \frac{\partial \theta(x)}{\partial x_j} \right)^2 = 0
\]
for \( j = 1, \ldots, n \).

Now let \( E = \{ x : f(x) \neq 0 \} \). Then \( E \) is dense in \( \text{Supp}(f) \), the support of \( f \). By (iv) \( \frac{\partial \theta(x)}{\partial x_j} = 0 \) for all \( x \in E \) since \( f \) is continuous. Because \( \theta \in C^1(R^n) \) we conclude \( \frac{\partial \theta(x)}{\partial x_j} = 0 \) on \( \text{Supp}(f) \) for all \( j = 1, \ldots, n \). Hence \( \theta(x) \) is constant on each connected component of \( \text{Supp}(f) \). It is clear that \( g(x) = e^{i\theta_0} f(x) \) holds for some \( \theta_0 \in R \) and all \( x \in R^n \), and \( c \) is constant on \( \text{Supp}(f) \) provided \( \text{Supp}(f) \) is connected.

Let \( Z_n = \{0, 1, \ldots, n-1\} \), under + (mod n) be the cyclic group of order \( n \). Then \( Z_n \) can be viewed as a compact abelian group. It is of interest to characterize Pauli uniqueness of \( Z_n \) as indicated in [10].

**Theorem 1.2.5.**

(i) Suppose \( f \in L^2(Z_n) \) has a non-negative representative, then \( f \) is Pauli unique.
(ii) \( f \in L^2(Z_2) \) is Pauli unique if and only if \( f \) has a real representative.

(iii) Suppose \( f \in L^2(Z_3) \). If \( f(k) = 0 \) for some \( k \in Z_3 \), then \( f \) is Pauli unique.

**Proof.** (i) This follows from Corollary 1.2.2 where \( G = Z_n \).

(ii) Suppose \( f \) is Pauli unique. Let \( g(j) = \overline{f(j)} \). Then it is elementary to verify that \( |f| = |g| \) and \( |\overline{f}| = |g| \). Hence \( g(j) = cf(j) \) for some constant \( c \) with \( |c| = 1 \) by Pauli uniqueness of \( f \), \( j = 0, 1 \). It follows that

\[
\overline{f(0)} = cf(0) \text{ and } \overline{f(1)} = cf(1).
\]

Let \( f(j) = |f(j)|e^{i\theta(j)} \). If \( f(0)f(1) = 0 \), then clearly \( f \) has a real representative. Now suppose \( f(0)f(1) \neq 0 \), then

\[
e^{-2i\theta(0)} = \frac{\overline{f(0)}}{f(0)} = c = \frac{\overline{f(1)}}{f(1)} = e^{-2i\theta(1)}.
\]

Therefore \( e^{2i\theta(0)} = e^{2i\theta(1)} \). Hence either \( e^{i\theta(0)} = e^{i\theta(1)} \) or \( e^{i\theta(0)} = -e^{i\theta(1)} \), each of which implies that \( f \) has a real representative.

Conversely, suppose \( f \) has a real representative. Without the loss of generality, we assume that \( f \) is real. Let \( c \in C \) in (2) where \( G = Z_2 \). By Proposition 1.2.1, \( |\overline{f}| = |c\overline{f}| \) if and only if

\[
f(1)\overline{f(0)}(1 - c(1)c(0)) = 0.
\]

If either \( f(0) = 0 \) or \( f(1) = 0 \), then clearly \( f \) is Pauli unique. So we may assume \( f(0)f(1) \neq 0 \). Then \( 1 - c(0)c(1) = 0 \), and it follows that \( c(0) = c(1) \).

Hence \( c \) is constant on the support of \( f \), and \( f \) is Pauli unique.
(iii) The argument is very similar to (ii). Let $c \in C$ in (2) where $G = \mathbb{Z}_3$.

By Proposition 1.2.1, $|f| = |c|f|$ if and only if

$$f(1)f(0)(1 - c(1)c(0)) + f(2)f(1)(1 - c(2)c(1))$$

$$+ f(0)f(2)(1 - c(0)c(2)) = 0.$$ 

Without loss of generality, we may assume $f(1) = 0$, then the above reduces to

$$f(0)f(2)(1 - c(0)c(2)) = 0.$$ 

If $f(0) = 0$ or $f(2) = 0$, then clearly $f$ is Pauli unique. So we assume $f(0)f(2) \neq 0$. Then $1 - c(0)c(2) = 0$, and it follows that $c(0) = c(2)$. Hence $c$ is constant on the support of $f$, and $f$ is Pauli unique.

§1.3 Examples of Pauli Non-uniqueness

A variety of examples of Pauli non-uniqueness can be obtained from the following proposition.

Proposition 1.3.1. Suppose $G$ is a locally compact abelian group, $f \in L^2(G)$, and $|f|$ is symmetric about $x = b$, i.e., $|f(-x + b)| = |f(x)|$. Let $g(x) = f(-x + b)$. Then $|f| = |g|$, and $|f| = |\hat{g}|$.

Proof. Suppose $\gamma$ is an arbitrary character in $L^2(\hat{G})$. Then

$$\hat{g}(\gamma) = \int_G \gamma(-x)g(x)dx$$

$$= \int_G \gamma(-x)f(-x + b)dx$$

$$= \int_G \gamma(x - b)f(\overline{x})dx$$

$$= \gamma(b)\hat{f}(\gamma).$$

The conclusion follows from the above immediately.
Corollary 1.3.2 [9]. Suppose $f$ and $g$ are nonzero real valued square integrable functions on $\mathbb{R}$ with disjoint support and such that $f$ is even and $g$ is odd. Then $f + g$ is not Pauli unique.

**Proof.** Let $h(x) = f + g$ and $k(x) = h(-x)$. Then it is straightforward to verify that $k$ is a representative of $h$, but there is no constant $c$ with modulus one such that $k = ch$.

**Example 1.** Suppose $T$ is the circle group. Here we view $T$ as the interval $[0, 1)$ under addition modulo 1. Let $f(x) = e^{-ix^2} \sin \pi x$ for any $x \in T$. Then $f$ is not Pauli unique. As a matter of fact,

$$|f(-x + 0)| = |f(1 - x)|$$

$$= |e^{-i(1-x)^2} \sin \pi (1 - x)|$$

$$= |f(x)|$$

for any $x \in T$. Let $g(x) = \overline{f(-x)}$. Then by Proposition 1.3.1, $f$, $g$ have the same Pauli data, but obviously $g$ is not a representative of $f$.

We can also modify the above example to easily obtain a counterexample in $L^2(\mathbb{R})$ as follows.

**Example 2.** Let $f(x) = \frac{1}{x^2 + 1} e^{-ix^2} \sin \pi x$ for any $x \in \mathbb{R}$. It is clear that $f \in L^2(\mathbb{R})$ and

$$|f(-x + 0)| = |f(x)|.$$

By Proposition 1.3.1, $f$ and $g(x) = \overline{f(-x + 0)}$ have the same Pauli data, but clearly $f \neq cg$ for any constant $c$. 
Note that from the above two examples we can quickly obtain a class of counterexamples in $L^2(\mathbb{R}^n)$ and $L^2(T^n)$. The following is an example of Pauli non-uniqueness in $L^2(\mathbb{Z}_2)$.

Example 3. Let $f(0) = 1$ and $f(1) = i$. Then $f \in L^2(\mathbb{Z}_2)$. By Theorem 1.2.5 (ii) $f$ is not Pauli unique.

For other Pauli non-unique examples one may consult Ashbaugh [3] or Vogt [34].
CHAPTER 2
SCATTERING THEORY
FOR NTH ORDER EQUATIONS

Lax and Phillips made the first study of obstacle scattering for the (Dirichlet) acoustical wave equations (with the obstacle $O$ star-shaped and the dimension $n$ odd). This was sketched in [21]. Wilcox [37] studied obstacle scattering for acoustic waves with the generalized Neumann boundary condition. This work spurred much additional research, a sample of which is covered by Weder [35]. There is also a number of papers published on scattering by elastic waves, see for instance Dassions et al [7] and [8]. Many papers also contribute to the scattering theory for higher order equations or differential operators. For example, Pickett [31] discussed the scattering for $2^N$th order equations. One of the recent works is Kondoyannidis’ Ph.D dissertation [20]. In his work, Kondoyannidis treats factored equation problem with commuting normal operators in a general context, using multiparameter spectral theory and distribution theory for these higher order equations. In this chapter, we will focus on the scattering for Nth order equations. Based on the abstract d’Alembert formula in §2.3, we define vector wave operators which are similar to the usual wave operators, and then we discuss the existence, completeness of these operators. Our approach here is simpler compared with Kondoyannidis’ method, and it is clean, direct and easy to use. It also enable us to recover and unify old results on acoustic waves as well as to obtain new results on elastic waves. Moreover, it follows from
our approach that the elastic wave theory is a corollary of the acoustic wave theory. Sections 2.1 and 2.2 review some basic facts about scattering theory. Section 2.3 provides another proof of the d’Alembert’s formula. Section 2.4 introduces the vector wave operators, and discusses the existence and completeness of these operators.

§2.1 Absolutely Continuous Subspaces

We first recall some basic concepts regarding absolutely continuous subspaces which are necessary for developing scattering theory.

A Borel measure $M_1$ on real line $R$ is said to be absolutely continuous relative to another Borel measure $M_2$ if $M_1(X) = 0$ whenever $M_2(X) = 0$ for any Borel set $X \subseteq R$. Measures $M_1$ and $M_2$ are said to be mutually singular if there are two Borel sets $X_1$ and $X_2$ such that $M_1(R \setminus X_1) = M_2(R \setminus X_2) = 0$ and $X_1 \cap X_2 = 0$.

Clearly, a nonzero measure $M_1$ cannot simultaneously be absolutely continuous with respect to $M_2$ and also singular with respect to it.

Suppose $H$ is a selfadjoint operator with domain $D(H)$ in a separable Hilbert space $H$. We denote by $E_H(X)$, briefly $E(X)$ if no confusion arises, the spectral measure of the operator $H$, where $X$ varies over Borel sets of $R$. We use the notation $E(\lambda) := E((-\infty, \lambda))$, so that $E(\lambda)$ is the resolution of the identity corresponding to the spectral measure $E(X)$.

An element $f \in H$ is called absolutely continuous (singular) relative to $H$ if the measure $(E(\cdot)f, f)$ is absolutely continuous (singular) with respect
to Lebesgue measure on $\mathbb{R}$. We denote the set of absolutely continuous (singular) elements by $\mathcal{H}_{ac}(H)$ ($\mathcal{H}_s(H)$). It is known that the sets $\mathcal{H}_{ac}(H)$ and $\mathcal{H}_s(H)$ form closed subspaces. These subspaces are orthogonal complements, i.e. $\mathcal{H} = \mathcal{H}_{ac}(H) \oplus \mathcal{H}_s(H)$. (See [11], [17], [38].)

§2.2 Wave Operators

For $j = 0, 1$, let $H_j$ be a selfadjoint operator on a separable Hilbert space $\mathcal{H}_j$. The abstract Schrödinger equation $i \frac{d}{dt} u = H_j u$ is governed by the $(C_0)$ unitary group $\{e^{-itH_j} : t \in \mathbb{R}\}$; the unique solution $u$ of the Schrödinger equation with initial data $u(0) = f$ is given by $u(t) = e^{-itH_j} f$. This solution is classical and in $C^1(\mathbb{R}, \mathcal{H}_j)$ if $f \in \mathcal{D}(H_j)$. Otherwise $u$ is called a mild solution.

One is to think of the subscript 1 (resp. 0) as describing "perturbed" (resp. "free") motion, and in some sense, the two groups are expected to be equivalent at $t = \pm \infty$. For definitness, suppose $\mathcal{H}_0 = \mathcal{H}_1 = \mathcal{H}$. We suppose that the perturbed solution $e^{-itH_1} f$ looks like a free solution $e^{-itH_0} f_\pm$ as $t \to \pm \infty$ in the sense that $e^{-itH_1} f - e^{-itH_0} f_\pm \to 0$ as $t \to \pm \infty$. Then the operators defined by

$$\mathcal{W}_\pm g = \lim_{t \to \pm \infty} e^{itH_1} e^{-itH_0} g$$

exist (for $g = f_\pm$). These operators are called the wave operators. Here all limits are in the norm topology of $\mathcal{H}$.

Let $P_j$ be the orthogonal projection onto the absolutely continuous subspace $\mathcal{H}_{ac}(H_j)$ for $H_j$, $j = 0, 1$. Suppose the wave operators $\mathcal{W}_\pm$ exist on
all of \( P_0 \mathcal{H} = \mathcal{H}_{ac}(H_0) \). Then their ranges satisfy \( \text{Ran}(\mathcal{W}_\pm) \subset \mathcal{H}_{ac}(H_1) \). \( \mathcal{W}_\pm \) are called **complete** if

\[
\text{Ran}(\mathcal{W}_\pm) = \mathcal{H}_{ac}(H_1).
\]  

(2)

The *scattering operator* \( S = \mathcal{W}_+^* \mathcal{W}_- \) is then a unitary operator from \( \mathcal{H}_{ac}(H_0) \) to \( \mathcal{H}_{ac}(H_1) \), sending \( f_- \) to \( f_+ \); \( f_- \) describes how the perturbed motion \( e^{-itH_1}f \) looks like a free motion near \( t = -\infty \), thus \( f_- \) is the incoming data. Similarly, \( f_+ \) is the outgoing data; it describes how the perturbed motion looks free near \( t = +\infty \). The scattering operator \( S \) maps \( f_- \) to \( f_+ \); this describes the result of a scattering experiment which "sees" the incoming and outgoing solutions.

When \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \) are different spaces, one needs an identification operator \( J_0 : \mathcal{H}_0 \to \mathcal{H}_1 \). \( J_0 \) is to be a bounded linear operator. Then \( e^{-itH_1}f \) should approximately equal \( J_0 e^{-itH_0}f_\pm \) as \( t \to \pm \infty \). The *wave operators* are

\[
\mathcal{W}_\pm g = \mathcal{W}_\pm(H_1, H_0; J_0)g
\]

\[
= \lim_{t \to \pm \infty} e^{itH_1} J_0 e^{-itH_0} g
\]

(for \( g = f_\pm \)). Suppose \( \mathcal{W}_\pm \) exist on \( \mathcal{H}_{ac}(H_0) \). Call \( \mathcal{W}_\pm \) **complete** if

\[
\text{Ran}(\mathcal{W}_\pm) = \mathcal{H}_{ac}(H_1)
\]

(4)

and \( \mathcal{W}_\pm \) is injective on \( \mathcal{H}_{ac}(H_0) \). Then, as before, the *scattering operator* 

\( S = \mathcal{W}_+^* \mathcal{W}_- \) is unitary from \( \mathcal{H}_{ac}(H_0) \) to \( \mathcal{H}_{ac}(H_1) \), provided \( \mathcal{W}_\pm \) are isometric on \( \mathcal{H}_{ac}(H_0) \), where \( \mathcal{W}_\pm^* \) are the dual operators of \( \mathcal{W}_\pm \).

As an example consider the wave equation with a potential,

\[
v_{tt} = \Delta v - V(x)v,
\]
where \( x \in \mathbb{R}^n \) and \( 0 \leq V \in L^\infty(\mathbb{R}^n) \). (The hypotheses on \( V \) can be greatly relaxed.) For the free equation, rewrite \( v_{tt} = \Delta v \) as

\[
\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = -iH_0u.
\]

\( H_0 \) is selfadjoint on

\[
\mathcal{H}_0 = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : \|(-\Delta)^{1/2} f_1\|_2^2 + \|f_2\|_2^2 < \infty \right\},
\]

which is a Hilbert space in the obvious way; here \( \| \cdot \|_2 \) refers to the \( L^2(\mathbb{R}^n) \) norm. Similarly, \( H_1 = \begin{pmatrix} 0 & I \\ \Delta - V & 0 \end{pmatrix} \) is selfadjoint on \( \mathcal{H}_1 \), which is normed by

\[
\| \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \| = \left\{ \|(-\Delta + V)^{1/2} f_1\|_2^2 + \|f_2\|_2^2 \right\}^{1/2}.
\]

\( \mathcal{H}_0 \) and \( \mathcal{H}_1 \) are equal as sets but they have different (but equivalent) norms. Let \( J: \mathcal{H}_0 \to \mathcal{H}_1 \) be the identity operator from \( \mathcal{H}_0 \) to \( \mathcal{H}_1 \). \( J \) is bounded but not unitary. This \( J \) is appropriate for scattering theory in this context.

For obstacle scattering by sound waves, let \( \mathcal{O} \) be, say, a smooth star shaped bounded region in \( \mathbb{R}^n \). The free group is as above. The perturbed group governs

\[
v_{tt} = \Delta v \text{ for } x \in \mathbb{R}^n \setminus \mathcal{O}.
\]

Here associate either Dirichlet or Neumann conditions with \( \Delta \), acting on \( L^2(\mathbb{R}^n \setminus \mathcal{O}) \). Then

\[
\| \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \|_{\mathcal{H}_1}^2 = \| \begin{pmatrix} (-\Delta)^{1/2} f_1 \\ f_2 \end{pmatrix} \|_{\mathcal{H}_1}^2
\]
where $\| \cdot \|_2$ refers to the $L^2(R^n \setminus \mathcal{O})$ norm. Here $J \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_1 |_{\mathcal{O}} \\ f_2 |_{\mathcal{O}} \end{pmatrix}$ has norm one. But $J$ is not injective. Nonetheless, $e^{-itH_0}f$ is transporting the wave (described by $f$) out to infinity since

$$
\int_{|x|<R} |(e^{itH_0}f)(x)|^2 dx \to 0
$$

as $t \to \pm \infty$ for each $R > 0$ (local energy decay.) Thus $J$ "acts like the identity" on $e^{-itH_0}f$ for $|t|$ large. In this sense, $J$ is "morally" injective.

Note that $J$ can be replaced by any bounded linear operator $K : H_0 \to H_1$ satisfying

$$
\| (J - K)e^{-itH_0}f_0 \| \to 0
$$

as $t \to \pm \infty$ for any $f_0 \in H_0$. In this sense, $J$ is not uniquely determined.

The following are some facts about the wave operators $\mathcal{W}_\pm(H_1, H_0; J_0)$, where the $H_1$, $H_0$, and $J_0$ are defined as before. For more details about the scattering theory, see the books and papers listed in the Reference section.

Proposition 2.2.1. $\overline{\text{Ran}(\mathcal{W}_\pm)} \subset \mathcal{H}_{ac}(H_1)$, where the $\overline{\text{Ran}(\mathcal{W}_\pm)}$ are the closures of $\text{Ran}(\mathcal{W}_\pm)$ in $H_1$.

Proposition 2.2.2 (Chain Rule). Suppose $J_1$ is another bounded operator from $H_1$ to a Hilbert space $H_2$.

(i) If the wave operators $\mathcal{W}_\pm(H_1, H_0; J_0)$ and $\mathcal{W}_\pm(H_2, H_1; J_1)$ exist, then for $J = J_1 J_0$ the wave operators $\mathcal{W}_\pm(H_2, H_0; J)$ also exist, and

$$
\mathcal{W}_\pm(H_2, H_0; J) = \mathcal{W}_\pm(H_2, H_1; J_1) \mathcal{W}_\pm(H_1, H_0; J_0).
$$

(5)
(ii) If the wave operators $\mathcal{W}_\pm(H_1, H_0; J_0)$ and $\mathcal{W}_\pm(H_2, H_1; J_1)$ are complete, then for $J = J_1J_0$ the wave operators $\mathcal{W}_\pm(H_2, H_0; J)$ are also complete.

**Proposition 2.2.3.** Let the wave operators $\mathcal{W}_\pm(H_1, H_0; J_0)$ exist and for some $J_1: \mathcal{H}_1 \rightarrow \mathcal{H}_0$, $s-\lim_{t\to\pm\infty}(J_1J_0 - I)U_0(t)P_0 = 0$. Then the completeness of $\mathcal{W}_\pm(H_1, H_0; J_0)$ is equivalent to the existence of $\mathcal{W}_\pm(H_0, H_1; J_1)$ and the condition

$$s-\lim_{t\to\pm\infty}(J_1J_0 - I)U_0(t)P_1 = 0. \quad (6)$$

In this case the wave operators $\mathcal{W}_\pm(H_0, H_1; J_1)$ are also complete, and the wave operators $\mathcal{W}_\pm(H_1, H_0; J_0)$ and $\mathcal{W}_\pm(H_0, H_1; J_1)$ are mutually invertible mappings between $\mathcal{H}_{ac}(H_0)$ and $\mathcal{H}_{ac}(H_1)$, i.e.

$$\mathcal{W}_\pm(H_0, H_1; J_1)\mathcal{W}_\pm(H_1, H_0; J_0) = P_0,$$

$$\mathcal{W}_\pm(H_1, H_0; J_0)\mathcal{W}_\pm(H_0, H_1; J_1) = P_1. \quad (7)$$

**Proposition 2.2.4 (Cook-Kuroda Criterion).** Suppose $J_0$ is a bounded operator taking the domain $\mathcal{D}(H_0)$ of the operator $H_0$ into $\mathcal{D}(H_1)$. Suppose that for some set $D_0 \subset \mathcal{D}(H_0) \cap \mathcal{H}_{ac}(H_0)$ dense in $\mathcal{H}_{ac}(H_0)$, for any $f \in D_0, \pm \int_0^{\pm\infty} ||(H_1J_0 - J_0H_0)U_0(t)f||dt < \infty. \quad (8)$

Then the wave operators $\mathcal{W}_\pm(H_1, H_0; J_0)$ exist.

**Proposition 2.2.5 (Birman-Kato Invariance Principle).** Assume that $H_1$, $H_0$ and $J_0$ satisfy

$$J_0\mathcal{D}(H_0) \subset \mathcal{D}(H_1), \text{ and } J_0^*\mathcal{D}(H_1) \subset \mathcal{D}(H_0).$$
Moreover, let \( \{I_m\} \) be a family of disjoint bounded open intervals such that
\[
\bigcup_{m=1}^{\infty} I_m = \mathbb{R} - Z,
\]
where \( Z \) is a Lebesgue null set, and assume that for \( m = 1, 2, 3, \ldots \),
\[
(H_1 J_0 - J_0 H_0) E_0(I_m) \text{ is in trace-class,}
\]
\[
(J_0^* J_0 - I) E_0(I_m) \text{ is a bounded operator,}
\]
\[
(J_0 J_0^* - I) E_1(I_m) \text{ is a bounded operator.}
\]
Then \( \mathcal{W}_\pm = \mathcal{W}_\pm(H_1, H_0; J_0) \) exist. Moreover, \( \mathcal{W}_\pm : \mathcal{H}_0 \to \mathcal{H}_1 \) is partially isometric with initial set \( \mathcal{H}_{ac}(H_0) \) and final set \( \mathcal{H}_{ac}(H_1) \), and the invariance principle holds:
\[
\mathcal{W}_\pm = \mathcal{W}_\pm(\phi(H_1), \phi(H_0); J_0)
\]
for all continuous monotone increasing functions \( \phi(\lambda) \).

**Remark.** The idea behind the invariance principle is simple. Let \( \phi : \mathbb{R} \to \mathbb{R} \) be continuous and piecewise linear with positive slope away from the corners. Then
\[
e^{it\phi(H_1)} J_0 e^{-it\phi(H_0)} = e^{i\tau H_1} J_0 e^{-i\tau H_0}
\]
for a suitable \( \tau = \tau(t) \), where \( \tau \to \pm \infty \) as \( t \to \pm \infty \). So the invariance principle holds for such \( \phi \). The general \( \psi \) can be approximated by these \( \phi' \)'s; thus the invariance principle is clearly plausible (even if the rigorous proof is nontrivial). For proof of these propositions one may consult [33] and [38].
§2.3 The Abstract D’Alembert’s Formula

The classical d’Alembert’s formula says the solution of the equation $u_{tt} = u_{xx}$ for $x, t \in \mathbb{R}$ is of the form $u = F(x + t) + G(x - t)$. This result was nicely generalized to Nth order factored equations by Goldstein and Sandefur [13], and a lot of applications of this generalized formula have already been found (see [13], [14]). In this section, we provide another proof of d’Alembert’s formula in the context of Hilbert space with less assumptions. On the other hand, the range assumptions made in [13] in the Banach space case are not required here.

**Theorem 2.3.1.** Let $A_j = A_j^*$ be selfadjoint on $\mathcal{H}$ for $1 \leq j \leq N$ and suppose that $A_1, \ldots, A_N$ commute in the sense that $[e^{itA_j}, e^{isA_k}] = 0$ for all $s, t, j, k$. If also $A_j - A_k$ is injective for $j \neq k$, then every mild solution $u$ of

$$\prod_{j=1}^{N} \left( \frac{d}{dt} + iA_j \right) u(t) = 0 \quad (10)$$

is of the form

$$u(t) = \sum_{j=1}^{N} e^{-itA_j} f_j \quad (11)$$

(where $f_j \in \mathcal{H}$).

**Proof.** By the spectral theorem for commuting selfadjoint operators, there is a unitary operator $U$ from $\mathcal{H}$ to some $L^2(\Omega, \Sigma, \mu)$ and real $\Sigma$-measurable

---

* $e^{-itA} f$ is a strong solution of $u' + iAu = 0$ if $f \in \mathcal{D}(A)$. It is a mild solution if $f \in \overline{\mathcal{D}(A)} = \mathcal{H}$. Similarly for higher order equations.
functions $a_j$ on $\Omega$ such that $U A_j U^{-1}$ is multiplication by $a_j$ (with maximal domain) on $L^2(\Omega, \Sigma, \mu)$ for $j = 1, \ldots, N$. Moreover, by the injectivity hypothesis, $N_{jk} = \{ \omega \in \Omega; a_j(\omega) \neq a_k(\omega) \}$ is a $\mu$-null set whenever $j \neq k$.

In the representation in $L^2(\Omega, \Sigma, \mu)$, (10) becomes

$$\prod_{j=1}^{N} \left( \frac{d}{dt} + i a_j(\omega) \right) v(t, \omega) = 0, \quad \omega \in \Omega.$$ 

The general solution is given by

$$v(t, \omega) = \sum_{j=1}^{N} e^{-it a_j(\omega)} g_j(\omega)$$

for all $t \in \mathbb{R}$ and $\omega \in \Omega \setminus N_0$, $N_0 = \bigcup_{j,k=1}^{N} N_{jk}$. Here $g_j \in L^2(\Omega, \Sigma, \mu)$.

This is because for $\omega \in \Omega \setminus N_0$, this $N$th order constant coefficient ODE has distinct roots. Translating back to $u(t)$ in $\mathcal{H}$ yields (11) (where $f_j = U^{-1} g_j$). This ends the proof.

**Remark.** Note that the injectivity of $A_j - A_k$ is a necessary condition, since if $(A_j - A_k)f = 0$ (for $j \neq k$ and $f \neq 0$), then $u(t) = te^{-itA_j}f$ is a solution of (10) which is not of the form (11).

§2.4 Scattering Theory Based on D'Alembert's Formula

Suppose there are two systems, an unperturbed one and a perturbed one, governed by the following factored equations of $N$th order in time:

$$\prod_{j=1}^{N} \left( \frac{d}{dt} + i A_j^{(k)} \right) u(t) = 0 \quad (12)$$

for $k = 0, 1$ respectively, where $A_1^{(k)}, \ldots, A_N^{(k)}$ are selfadjoint operators on $\mathcal{H}_k$ such that they all commute, i.e. $[\exp(it A_j^{(k)}), \exp(is A_i^{(k)})] = 0$ for all
One concern about these two systems is to understand the perturbed system through the unperturbed one because the latter is much simpler. In particular, one may be interested in the following:

*Given a state $f$ of the perturbed system, is there any state $g$ of the unperturbed system such that these two states are very similar in the far past or the far future?*

In this section, we will answer this question by constructing vector wave operators which are similar to the usual wave operators, and we also discuss the existence and completeness of these operators.

Since in a lot of cases, the above two systems are in two different Hilbert spaces, we need identification operators. Let $J_j \in B(\mathcal{H}_0, \mathcal{H}_1)$, the space of bounded linear operators from $\mathcal{H}_0$ to $\mathcal{H}_1$, $1 \leq j \leq N$; and let $P_{kj}$ be the orthogonal projection onto $\mathcal{H}_{kj} := \mathcal{H}_{ac}(A_j^{(k)})$ for all $j$, $k$, and $P_k = (P_{k1}, P_{k2}, \ldots, P_{kN})$. For any $f = (f_1, f_2, \ldots, f_N) \in \mathcal{H}_1^N$ and $g = (g_1, g_2, \ldots, g_N) \in \mathcal{H}_0^N$, define

\[
A(t)f := \sum_{j=1}^{N} \exp(-itA_j^{(1)})P_{1j}f_j,
\]

\[
B(t)g := \sum_{j=1}^{N} J_j \exp(-itA_j^{(0)})P_{0j}g_j.
\]

**Proposition 2.4.1.** Suppose $A(t)$ and $B(t)$ are defined as above, then:

(i) The operators $A(t) : \mathcal{H}_1^N \to \mathcal{H}_1$ and $B(t) : \mathcal{H}_0^N \to \mathcal{H}_1$ are bounded
operators, uniformly in $t \in R$.

(ii) $\lim_{t \to \pm \infty} A(t)P_1 = 0$, i.e. for any $f \in \mathcal{H}_1^N$ and $h \in \mathcal{H}_1$ we have

$$< A(t)P_1 f, h > \to 0 \text{ as } t \to \pm \infty.$$ 

(iii) If $J_i$ is a compact operator for $i = 1, 2, \ldots, N$, then

$$s - \lim_{t \to \pm \infty} B(t)P_0 = 0,$$ 

i.e. $\|B(t)P_0 f\| \to 0$ as $t \to \pm \infty$ for any $f \in \mathcal{H}_1^N$.

\textbf{Proof.} (i)

\begin{align*}
\|A(t)f\|^2 &= \| \sum_{j=1}^{N} \exp(-itA_j^{(1)})P_{ij}f_j\|^2 \\
&\leq N \sum_{j=1}^{N} \| \exp(-itA_j^{(1)})P_{ij}f_j\|^2 \\
&= N \sum_{i=1}^{N} \|f_j\|^2 = N\|f\|^2.
\end{align*}

Hence $\{A(t) : t \in R\}$ is a family of uniformly bounded operators. Similarly $\{B(t) : t \in R\}$ is also a family of uniformly bounded operators. We emphasize that $N$ is fixed here.

(ii) For $f \in \mathcal{H}_1^N$ and $h \in \mathcal{H}_1$, we have

\begin{align*}
\lim_{t \to \pm \infty} < A(t)P_1 f, h > &= \lim_{t \to \pm \infty} \sum_{j}^{N} \exp(-it\lambda_j)P_{1j}f_j, h > \\
&= \sum_{j}^{N} \lim_{t \to \pm \infty} \int_{-\infty}^{+\infty} \exp(-it\lambda_j) d < E_j(\lambda_j)P_{1j}f_j, h > \\
&= \sum_{j}^{N} \lim_{t \to \pm \infty} \int_{-\infty}^{+\infty} \exp(-it\lambda_j) \frac{d}{d\lambda_j} E_j(\lambda_j)P_{1j}f_j, h > d\lambda_j \\
&= 0.
\end{align*}
The third equality holds because $P_{ij}f_j$ is absolutely continuous; and the last equality is a corollary of the Riemann-Lebesgue Lemma.

(iii) From the proof of (ii), $w - \lim_{t \to \pm \infty} \exp(-itA_j^{(0)})P_{0j} = 0$. The conclusion then follows from the fact that compact operators map weakly convergent sequences to strongly convergent sequences.

Remark. Proposition 2.4.1 holds in the more general context in which $\mathcal{H}_j^N$ is replaced by $\mathcal{H}_j \oplus \cdots \oplus \mathcal{H}_j^N$.

Now for the convenience of discussion we let

$$A_\pm = \{ f \in \mathcal{K}_1 : j=1 \bigoplus \mathcal{H}_{1j} : \lim_{t \to \pm \infty} \| A(t)f \| = 0 \},$$

$$B_\pm = \{ g \in \mathcal{K}_0 : j=1 \bigoplus \mathcal{H}_{0j} : \lim_{t \to \pm \infty} \| B(t)g \| = 0 \}. \quad (14)$$

Note that these expressions define $\mathcal{K}_0$ and $\mathcal{K}_1$.

**Proposition 2.4.2.** $A_\pm, B_\pm$ are closed subspaces of $\mathcal{K}_1$ and $\mathcal{K}_0$ respectively.

**Proof.** It is clear that $A_+$ is a subspace of $\mathcal{K}_1$. Now suppose $\{ f_n : n = 1, 2, \ldots \} \subset \mathcal{K}_1$ and $s - \lim_{n \to \infty} f_n = f$. Then

$$\lim_{t \to \infty} \| A(t)f \| = \lim_{t \to \infty} \lim_{n \to \infty} \| A(t)f_n \| = \lim_{n \to \infty} \lim_{t \to \infty} \| A(t)f_n \| = 0;$$

the second equality holds because, by Proposition 2.4.1, $\{ A(t) : t \in \mathbb{R} \}$ is uniformly bounded. Hence $f \in A_+$, and $A_+$ is closed. Similarly, we can prove that the subspaces $A_-, B_\pm$ are also closed.
We say that the system \( S = (\overrightarrow{A}, \overrightarrow{A}^{(0)}; J) \)
\[= \left( A_1^{(1)}, \ldots, A_N^{(1)}, A_1^{(0)}, \ldots, A_N^{(0)}; J_1, \ldots, J_N \right) \] (15)
has the wave operator existence property [WOEP] if and only if for all \( g \in \mathcal{K}_0^* \) there are \( f_\pm \in \mathcal{K}_1^* \) such that
\[
\| A(t) f_\pm - B(t) g \| \to 0 \quad \text{as} \quad t \to \pm \infty. \tag{16}
\]
as \( t \to \pm \infty \). The wave operator existence and uniqueness property [WOEUP] for \( S \) means that, in addition, \( f_\pm \) are unique. Furthermore, \( S \) has the wave operator semi-completeness property [WOSCP] means \( S \) has WOEP and for all \( f \in \mathcal{K}_1 \), there exist \( g_\pm \in \mathcal{K}_0 \) such that \( \| A(t) f - B(t) g_\pm \| \to 0 \) as \( t \to \pm \infty \).

The wave operator completeness property [WOCP] means that, in addition to all of the above, \( g_\pm \) are unique.

**Proposition 2.4.3.**

(i) \( \text{WOEUP} \iff \text{WOEP} \) and \( A_\pm = \{0\} \);

(ii) \( \text{WOCP} \iff \text{WOSCP} \) and \( B_\pm = \{0\} \).

**Proof.** (i) Suppose the system \( S \) has WOEUP and \( f_0 \in A_+ \). Also suppose \( f \in \mathcal{K}_1 \) and \( g \in \mathcal{K}_0 \) are such that
\[
\| A(t) f - B(t) g \| \to 0 \quad \text{as} \quad t \to \infty.
\]
Let \( f' = f_0 + f \). Then \( f', g \) also satisfy
\[
\| A(t) f' - B(t) g \| \to 0 \quad \text{as} \quad t \to \infty.
\]

* \( \mathcal{K}_0, \mathcal{K}_1 \) are defined in (14).
By the uniqueness of WOEUP, we must have $f' = f$. Hence $f_0 = 0$ and $A_+ = \{0\}$.

Conversely, suppose $A_+ = \{0\}$ and let $S$ have the WOEP. Then if $f, f' \in \mathcal{K}_1, g \in \mathcal{K}_0$ are such that

$$
\|A(t)f - B(t)g\| \to 0 \quad \text{as} \quad t \to \infty,
$$

$$
\|A(t)f' - B(t)g\| \to 0 \quad \text{as} \quad t \to \infty.
$$

Then it is elementary to check that $\|A(t)(f-f')\| \to 0$ as $t \to \infty$. Hence $f-f' \in A_+$ and $f = f'$. We can similarly prove the case for $t \to -\infty$.

(ii) Similar to (i).

After the above preparation, we can finally choose our wave operators as follows:

**Definition 2.4.4.** The linear operators $\hat{W}_\pm : \mathcal{K}_0 \to \mathcal{K}_1$ are called **vector wave operators** of the system $S$ if they satisfy

$$
\|A(t)\hat{W}_\pm g - B(t)g\| \to 0
$$

as $t \to \pm\infty$ for $g \in \mathcal{K}_0$.

Note that when $N$ is 1, the above definition reduces to that of wave operators. Hence we believe that the vector wave operators are good candidates for studying the asymptotic equivalence of the systems involving two Nth order factored equations.

**Proposition 2.4.5.** Suppose $S$ is defined as in (15), then:

(i) $S$ has WOEP if and only if $S$ has vector wave operators;
(ii) $S$ has WOEUP if and only if it has vector wave operators $\hat{W}_\pm$, and $A_\pm = \{0\}$, i.e. if and only if $\hat{W}_\pm$ exist and are unique.

(iii) $S$ has WOCP if and only if the vector wave operators $\hat{W}_\pm$ exist and have inverses.

**Proof.** (i) For any $g \in \mathcal{K}_0$, by the WOEP, there exists $f \in \mathcal{K}_1$ such that $\|A(t)f - B(t)g\| \to 0$ as $t \to \infty$. Since $A_+$ is a closed subspace in $\mathcal{K}_1$, we can decompose $f$ into $f = f_1 \oplus f_2$ where $f_1 \in \mathcal{K}_1 \odot A_+$ and $f_2 \in A_+$. Define

$$\hat{W}_+: \mathcal{K}_0 \to \mathcal{K}_1$$

$$\hat{W}_+(g) = f_1.$$

Then $\hat{W}_+$ is well-defined. In fact, suppose $f' = f'_1 \oplus f'_2 \in (\mathcal{K}_1 \odot A_+) \odot A_+ = \mathcal{K}_1$ is another element such that $\|A(t)f - B(t)g\| \to 0$ as $t \to \infty$; then $\|A(t)(f - f')\| \to 0$ as $t \to \infty$. Hence $\|A(t)(f_1 - f'_1)\| \to 0$ as $t \to \infty$, which implies that $f_1 - f'_1 \in A_+$. But by the construction, we know that $f_1 - f'_1 \in \mathcal{K}_1 \odot A_+$. Hence $f_1 - f'_1 = 0$ and $f_1 = f'_1$. Similarly, we can show that $\hat{W}_+$ is linear. Hence $\hat{W}_+$ is indeed a vector wave operator of the system $S$. The same argument works for $\hat{W}_-$.

(ii) From (i) and Proposition 2.4.4(i).

(iii) From (i) and the definition of WOCP.

Next we state the main theorem for the vector wave operators. This result is not difficult to prove, but it has many applications (see Chapter
3). Before that, we define the Riemann-Lebesgue class. Let \( A = A^* \) on \( \mathcal{H} \).

Then \( A \) is in the Riemann-Lebesgue class iff \( e^{itA} \to 0 \) as \( t \to \pm \infty \) in the weak operator topology (see [12]). Writing \( A \) as \( A = \int_{-\infty}^{\infty} \lambda dE(\lambda) \), then \( A \) is in the Riemann-Lebesgue class iff for all \( f, g \in \mathcal{H} \),

\[
\int_{-\infty}^{\infty} e^{it\lambda} d\lambda \langle E(\lambda)f, g \rangle \to 0
\]

as \( t \to \pm \infty \). By the Riemann-Lebesgue lemma, this holds provided \( \mathcal{H}_{ac}(A) = \mathcal{H} \). (But the converse is not true.)

**Theorem 2.4.6.** For \( k = 0, 1 \), let \( A_1^{(k)}, \ldots, A_N^{(k)} \) be commuting selfadjoint operators on \( \mathcal{H}_k \) such that \( A_j^{(k)} - A_l^{(k)} \) is injective for \( j \neq l \). Let \( J_j \in B(\mathcal{H}_0, \mathcal{H}_1) \) and suppose the wave operators \( \mathcal{W}_{\pm,j} = \mathcal{W}_{\pm}(A_j^{(1)}, A_j^{(0)}; J_j) \) exist for \( j = 1, \ldots, N \). Then

(i) \( S = (A^{(1)}, A^{(0)}; J) \) has bounded vector wave operators \( \vec{W}_{\pm} \) which are given by \( \vec{W}_{\pm} g = \text{Diag}(\mathcal{W}_{\pm,j}g_j) \) for all \( g = (g_1, \ldots, g_N) \in \mathcal{K}_0 \), where \( \text{Diag}(\mathcal{W}_{\pm,j}) \) is the matrix \( (a_{ij}) \) with \( a_{ij} = \delta_{ij}\mathcal{W}_{\pm,j} \);

(ii) If for any \( j \neq l \), \( A_j^{(1)} - A_l^{(1)} \) is in the Riemann-Lebesgue class, then the vector wave operators are unique.

(iii) \( \mathcal{W}_{\pm,j} \) is complete for each \( j \) implies that \( S \) has the WOSCP; Furthermore, if \( \mathcal{A}_\pm = \{0\} \), then \( S \) has the WOCP;

(iv) If \( \mathcal{A}_\pm = \{0\} \) and each \( \mathcal{W}_{\pm,j} \) is partially isometric and complete, then \( \vec{W}_{\pm} : \mathcal{K}_0 \to \mathcal{K}_1 \) are unitary.

**Proof.** (i) Suppose \( g = (g_1, \ldots, g_N) \in \mathcal{K}_0 \). Let

\[
f_{\pm} = (\mathcal{W}_{\pm 1}(g_1), \ldots, \mathcal{W}_{\pm N}(g_N)).
\]
Then \( f_\pm \in \mathcal{K}_1 \) because \( \mathcal{W}_{\pm j} g_j \in \mathcal{H}_{ac}(A_j) \) and

\[
\|A(t)f_\pm - B(t)g\| \leq \sum_{1}^{N} \|\exp(-itA_j^{(1)})W_\pm, j(g_j) - J_j\exp(-itA_j^{(0)})g_j\|
\]

\[
= \sum_{1}^{N} \|W_\pm, j(g_j) - \exp(itA_j^{(1)})J_j\exp(-itA_j^{(0)})g_j\|
\]

\[
\rightarrow 0 \tag{20}
\]

as \( t \rightarrow \pm \infty \) by the definition of \( \mathcal{W}_{\pm}, j(A_j^{(1)}, A_j^{(0)}; J_j) \). Hence by Proposition 2.4.5(i), \( S \) has vector wave operators. By (19) and (20) we see that \( \hat{\mathcal{W}}_{\pm} g = \text{Diag}(\mathcal{W}_{\pm,j} g_j) \) exist for all \( g \in \mathcal{K}_0 \). Clearly

\[
\|\hat{\mathcal{W}}_{\pm} g\| = \|\text{Diag}(\mathcal{W}_{\pm,j})\| \leq N\max\{\|\mathcal{W}_{\pm,j}\| : j = 1, \ldots, N\}.
\]

Hence \( \hat{\mathcal{W}}_{\pm} \) are bounded.

(ii) Suppose \( h \in \mathcal{K}_1 \), then

\[
\|A(t)h\|^2 = \|\sum_{j=1}^{N} \exp(-itA_j^{(1)})h_j\|^2
\]

\[
= \sum_{j=1}^{N} \|h_j\|^2 + 2\text{Re} \sum_{j < t} \langle \exp(-itA_j^{(1)})h_j, \exp(-itA_t^{(1)})h_t \rangle
\]

\[
= \sum_{j=1}^{N} \|h_j\|^2 + 2\text{Re} \sum_{j < t} \langle \exp(-it(A_j^{(1)} - A_t^{(1)}))h_j, h_t \rangle \rightarrow \|h\|^2
\]

as \( t \rightarrow \pm \infty \) by the Riemann-Lebesgue property. This implies that \( A_{\pm} = \{0\} \). Hence by (i) and Proposition 2.4.5(ii) \( S \) has unique vector wave operators \( \hat{\mathcal{W}}_{\pm} \).

(iii) and (iv) These follow from the proofs of (i) and (ii) and Proposition 2.4.5.
Basically the above theorem reduces scattering theory for (12) to scattering theory for the pair \((A_j^{(1)}, A_j^{(0)})\) for \(1 \leq j \leq N\). This is the main feature of our approach which can reduce the scattering for higher order factored equations to the scattering for the corresponding first order equations.
CHAPTER 3
APPLICATIONS
OBSTACLE SCATTERING FOR ELASTIC WAVES

This chapter focuses on the applications of our approach to scattering theory through three typical examples. Through the explanation one may see that the approach developed in the last chapter can be applied to a variety of cases.

Example 1 (Acoustical scattering in $C^1$ exterior domains). Let $\mathcal{O} \subset \mathbb{R}^n$ be a $C^1$ exterior domain, that is, $\mathbb{R}^n \setminus \mathcal{O}$ is compact and $\partial \mathcal{O}$ is $C^1$. Let

$$H^{2,m}(\mathcal{O}) = L^2(\mathcal{O}) \cap \{ u : D^\alpha u \in L^2(\mathcal{O}), \text{ for } |\alpha| \leq m \}$$

$$H^2(\Delta, \mathcal{O}) = L^2(\mathcal{O}) \cap \{ u : \Delta u \in L^2(\mathcal{O}) \}$$

$$H^{2,1}(\Delta, \mathcal{O}) = H^2(\Delta, \mathcal{O}) \cap H^{2,1}(\mathcal{O}).$$

Then these spaces are Hilbert spaces with respect to the inner products

$$\langle u, v \rangle_m = \sum_{|\alpha| \leq m} \langle D^\alpha u, D^\alpha v \rangle$$

$$\langle u, v \rangle_\Delta = \langle u, v \rangle + \langle \Delta u, \Delta v \rangle$$

$$\langle u, v \rangle_{1,\Delta} = \langle u, v \rangle + \langle \Delta u, \Delta v \rangle,$$

where $\langle u, v \rangle = \int_\mathcal{O} u(x)v(x)dx$.

A function $u \in H^{2,1}(\Delta, \mathcal{O})$ is said to satisfy the Neumann boundary condition if for all $v \in H^{2,1}(\mathcal{O})$,

$$\int_{\mathcal{O}} (\Delta u)v + \int_{\mathcal{O}} \nabla u \cdot \nabla v = 0. \quad (1)$$

Note that (1) is equivalent to $\frac{\partial u}{\partial n} = 0$ on $\partial \mathcal{O}$ when $u$ and the boundary of $\mathcal{O}$ are smooth enough, but it is a more general condition when $\partial \mathcal{O}$ is not so
smooth. See [37] for an example. Denote

\[ H^2_N(\Delta, \mathcal{O}) = H^2_1(\Delta, \mathcal{O}) \cap \{ u : u \text{ satisfies (1)} \}. \]

Suppose we have a perturbed wave equation

\[ \frac{d^2 u(t)}{dt^2} - c^2 \Delta u(t) = 0, \quad x \in \mathcal{O}, \quad (2) \]

where \( c > 0 \). Here the term "perturbed" refers to the assumption that \( \mathcal{O} \neq \mathbb{R}^n \). Consider the linear operator \( B_1 : L^2(\mathcal{O}) \to L^2(\mathcal{O}) \) defined by

\[ D(B_1) = H^2_N(\Delta, \mathcal{O}) \]

\[ B_1 u = -\Delta u \quad \text{for all } u \in D(B_1). \]

Then one can show that \( B_1 \) is a non-negative selfadjoint operator on the Hilbert space \( L^2(\mathcal{O}) \), and its absolutely continuous subspace \( \mathcal{H}_{ac}(B_1) \) exactly equals \( L^2(\mathcal{O}) \) (see [37]).

Similarly, for the unperturbed wave equation

\[ \frac{d^2 u(t)}{dt^2} - c^2 \Delta u(t) = 0, \quad x \in \mathbb{R}^n \quad (3) \]

one can define the linear operator \( B_0 \) by

\[ D(B_0) = H^2(\Delta, \mathbb{R}^n) = H^2_1(\mathbb{R}^n) \]

\[ \text{and } B_0 = -\Delta u \text{ for all } u \in D(B_0). \]

Then \( B_0 \) is also non-negative selfadjoint operator and its absolutely continuous subspace \( \mathcal{H}_{ac}(B_0) \) is \( L^2(\mathbb{R}^n) \).

Now we write equations (2) and (3) in the factored forms:

\[ \left( \frac{d}{dt} + i A_1^{(k)} \right) \left( \frac{d}{dt} + i A_2^{(k)} \right) u(t) = 0, \quad k = 0, 1, \]
where $A_1^{(k)} = cB_k^{(1)} = -A_2^{(k)}$. By the abstract d'Alembert's formula in §2.3, their mild solutions are given by

$$u^{(k)}(t) = e^{-itA_1^{(k)}} f_1^{(k)} + e^{-itA_2^{(k)}} f_2^{(k)}$$

for some $f_1^{(k)}, f_2^{(k)} \in \mathcal{H}_{ac}(A_k), k = 0, 1$. From [37], we know the wave operators $\mathcal{W}_\pm(A_1^{(1)}, A_1^{(0)}; J_\mathcal{O})$ and $\mathcal{W}_\pm(A_2^{(1)}, A_2^{(0)}; J_\mathcal{O})$ exist and are unitary, where $J_\mathcal{O} : L^2(R^n) \to L^2(\mathcal{O})$ is defined by $J_\mathcal{O} f = f|_\mathcal{O}$. Furthermore, the difference $A_2^{(k)} - A_1^{(k)}$ is injective, selfadjoint, and absolutely continuous, hence in the Riemann-Lebesgue class. By Theorem 2.4.6 we can conclude that the vector wave operators $\mathcal{W}_\pm$ exist, and are unitary.

**Example 2 (Elastic scattering).** Let $x \in R^n$; usually $n = 3$, but we allow $n$ to be arbitrary. Let $u = (u_1, \ldots, u_n)$ represent the displacement vector for an elastic wave in $R^n$; here $u = u(t, x)$ with $t \in R$ and $x \in R^n$. Let $\lambda, \mu$ be the Lamé parameters and $\rho$ the density of the medium; these we take to be three positive constants. Then for $i = 1, \ldots, n$, $u_i$ satisfies

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \mu \Delta u_i + (\lambda + \mu) \frac{\partial}{\partial x_i} (\text{div}(u)).$$

This is a coupled system of second order (in $t$) equations. Each component $v = u_i$ satisfies

$$\prod_{j=1}^2 \left( \frac{d^2}{dt^2} - \mu_j \Delta \right) v = 0$$

where $0 < \mu_1 = \frac{\mu}{\rho} < \mu_2 = (\lambda + 2\mu)/\rho$ (see [16]). Thus, we take as our basic equation

$$\prod_{j=1}^4 \left( \frac{d}{dt} + ic_j A^{(k)} \right) u(t) = 0$$

(6)
where
\[ c_1 = -c_2 = \sqrt{\mu_1}, \quad c_3 = -c_4 = \sqrt{\mu_2}, \]
\[ A^{(0)} = (-\Delta)^{\frac{1}{2}} \quad \text{on } L^2(\mathbb{R}^n), \]
\[ A^{(1)} = (-\Delta)^{\frac{1}{2}} \quad \text{on } L^2(\mathcal{O}). \]

The superscript 0 (resp. 1) refers to the free [resp. perturbed] elastic wave equation. The obstacle \( \mathcal{O} \) is an exterior domain and assumed to be smooth (as in Example 1). Thus we view (5) (with superscripts (0),(1)) as our given pair of equations which we replace by (6). As in Example 1, suitable boundary conditions (e.g. Dirichlet or Neumann) must be assigned to \( \Delta \) on \( \mathcal{O} \), i.e. to \(-A^{(1)}\). As was shown in Example 1, the system
\[ \mathcal{S} = (ic_3A^{(1)}, ic_4A^{(1)}, ic_1A^{(0)}, ic_2A^{(0)}; J_O, J_O) \]
has vector wave operators \( \tilde{W}_\pm \) which are also unitary, where \( J_Of = f|\mathcal{O} \).

Example 3 (Inverse Problems). Using the notation in Example 1, let
\[ v(t) = \begin{pmatrix} iA_1^{(k)}u(t) \\ u'(t) \end{pmatrix}. \]
Then the acoustic wave equation \( u_{tt} + c^2B_ku = 0 \) is equivalent to
\[ \frac{d}{dt}v(t) = i \begin{pmatrix} 0 & A_1^{(k)} \\ A_1^{(k)} & 0 \end{pmatrix}v(t) \quad (7) \]
on
\[ \mathcal{H}_k = \begin{cases} L^2(\mathbb{R}^n)^2 & \text{if } k = 0, \\ L^2(\mathbb{R}^n \setminus \mathcal{O})^2 & \text{if } k = 1. \end{cases} \]
Define a unitary operator \( Q_1 \) on \( \mathcal{H}_1 \) by \( Q = \sqrt{2} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \). Then
\[ Q_1^* \begin{pmatrix} A_1^{(k)} & 0 \\ 0 & -A_1^{(k)} \end{pmatrix} Q_1 = \begin{pmatrix} 0 & A_1^{(k)} \\ A_1^{(k)} & 0 \end{pmatrix} \]
and for $Jf = f\chi_{\mathbb{R}^n\setminus\mathcal{O}}$ and $\tilde{J} = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$, using $Q_i \tilde{J} Q_i^* = \tilde{J}$, $A_j = A_j^{(j)}$, we have

$$
\exp \left[ it \begin{pmatrix} 0 & A_1 \\ A_1 & 0 \end{pmatrix} \right] \tilde{J} \exp \left[ -it \begin{pmatrix} 0 & A_0 \\ A_0 & 0 \end{pmatrix} \right] = Q_i^* \exp \left[ it \begin{pmatrix} A_1 & 0 \\ 0 & -A_1 \end{pmatrix} \right] Q_i \tilde{J} Q_i^* \exp \left[ -it \begin{pmatrix} A_0 & 0 \\ 0 & -A_0 \end{pmatrix} \right] Q_i
$$

$$
= Q_i^* \left( e^{itA_1} \begin{pmatrix} 0 & 0 \\ 0 & e^{-itA_1} \end{pmatrix} \right) \left( \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \right) \left( e^{-itA_0} \begin{pmatrix} 0 & 0 \\ 0 & e^{itA_0} \end{pmatrix} \right) Q_i
$$

$$
= Q_i^* \left( e^{itA_1 J e^{-itA_0}} \begin{pmatrix} 0 & 0 \\ 0 & e^{-itA_1 J e^{itA_0}} \end{pmatrix} \right) Q_i
$$

$\rightarrow Q_i^* \Omega_{1\pm} Q_i$ strongly as $t \rightarrow \pm \infty$, where

$$
\Omega_{1\pm} = \begin{pmatrix} W_{\pm}(A_1, A_0; J) & 0 \\ 0 & W_{\mp}(A_1, A_0; J) \end{pmatrix}
$$

Let $W_\pm = Q_i^* \Omega_{1\pm} Q_i$. Then the corresponding (acoustic) scattering operator satisfies

$$
S_{1a} = W_+^* W_- = Q_i^* \Omega_{1+}^* Q_i Q_i^* \Omega_{1-} Q_i
$$

$$
= Q_i^* (\Omega_{1+}^* \Omega_{1-}) Q_i
$$

$$
= Q_i^* S_1 Q_i
$$

where $S_1 = \Omega_{1+}^* \Omega_{1-}$. Since $Q_i$ is known explicitly, $S_{1a}$ and $S_1$ determine one another; similarly for $\Omega_\pm$ and $W_\pm$. Thus with $J$ fixed, the inverse problem is solvable for the pair $A_1, A_0$ iff it is solvable for the pair

$$
\begin{pmatrix} 0 & A_1 \\ A_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & A_0 \\ A_0 & 0 \end{pmatrix}
$$

The latter pair corresponds to the equations

$$
u'' + A_2^2 u = \prod_{j=1}^2 \left( \frac{d}{dt} + i(-1)^j A_k \right) u = 0, \ k = 0, 1.$$
Similarly the fourth order equation

$$\prod_{j=1}^{2} \left( \frac{d^2}{dt^2} + c_j^2 B_k \right) u(t) = 0$$

(8)

is equivalent to $\frac{d}{dt} v(t) = iH_k v(t)$ where

$$H_k = \frac{1}{2} \left( \begin{array}{cccc}
0 & (c_1 + c_2)B_k^\frac{1}{2} & 0 & (c_1 - c_2)B_k^\frac{1}{2} \\
(c_1 + c_2)B_k^\frac{1}{2} & 0 & (c_1 - c_2)B_k^\frac{1}{2} & 0 \\
0 & (c_1 - c_2)B_k^\frac{1}{2} & 0 & (c_1 + c_2)B_k^\frac{1}{2} \\
(c_1 - c_2)B_k^\frac{1}{2} & 0 & (c_1 + c_2)B_k^\frac{1}{2} & 0 \\
\end{array} \right),$$

$k = 0, 1$. (Cf. [14].)

Let $\vec{J} = \text{Diag}_{4 \times 4}(J)$ and $Q_2 = \frac{1}{2} \left( \begin{array}{cc}
Q_1 & Q_1 \\
Q_1 & -Q_1 \\
\end{array} \right).$ Then $W_{\pm}$

$= W_{\pm}(H_1, H_0; J)$ is the limit of

$$\exp(itH_1) \vec{J} \exp(-itH_0)$$

$$= Q_2^* \left( \begin{array}{ccc}
W_{1\pm} & W_{1\mp} & 0 \\
0 & W_{2\mp} & W_{2\mp} \\
\end{array} \right) Q_2$$

$$= \left( \begin{array}{cc}
Q_1^* \Omega_{1\pm} Q_1 & 0 \\
0 & Q_1^* \Omega_{2\pm} Q_1 \\
\end{array} \right)$$

where $W_{j\pm} = W_{\pm}(c_k B_1^\frac{1}{2}, c_k B_0^\frac{1}{2}; J), j = 1, 2, \Omega_{2\pm} = \left( \begin{array}{cc}
W_{2\pm} & 0 \\
0 & W_{2\mp} \\
\end{array} \right)$, and

$Q_1, \Omega_{1\pm}$ are defined as before.

Thus the elastic scattering operator $S_\varepsilon$ of the pair of the fourth order equation is given by

$$S_\varepsilon = W_{+}^* W_- = \left( \begin{array}{cc}
Q_1^* S_{1\alpha} Q_1 & 0 \\
0 & Q_1^* S_{2\alpha} Q_1 \\
\end{array} \right).$$

Hence the scattering operator for the pair (8) can be expressed in terms of the scattering operator for the pair (7). (For details see [14], [31].)
Thus if the map $O \to S_a$ from the obstacle (modulo congruence) to the scattering operator for the acoustic wave equation is injective, then so is the map $O \to S_e$, where $S_e$ is the scattering operator for the elastic wave equation (with the same boundary condition).

For more general domains the inverse problem for obstacle scattering by sound waves was solved by the efforts of many. Principal contributors were A. Majda, R. Melrose and M. Taylor.
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