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## A Cluster Structure on the Coordinate Ring of Partial Flag Varieties

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# A CLUSTER STRUCTURE ON THE COORDINATE RING OF PARTIAL FLAG VARIETIES

A Dissertation

Submitted to the Graduate Faculty of the  
Louisiana State University and  
Agricultural and Mechanical College  
in partial fulfillment of the  
requirements for the degree of  
Doctor of Philosophy

in

The Department of Mathematics

by

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Fayadh Kadhem

This dissertation is dedicated to my father, my mother and my lovely wife, Huda Albasri.

Everything becomes possible with their love and support. I dedicate this also to my two-month-old son, Murtaja; he just came into this world and his love inspired me even before his birth!

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## Abstract

The main goal of this dissertation is to show that the (multi-homogeneous) coordinate ring of a partial flag variety  $\mathbb{C}[G/P_K^-]$  contains a cluster algebra for every semisimple complex algebraic group  $G$ . We use derivation properties and a canonical lifting map to prove that the cluster algebra structure  $\mathcal{A}$  of the coordinate ring  $\mathbb{C}[N_K]$  of a Schubert cell constructed by Goodearl and Yakimov can be lifted, in an explicit way, to a cluster structure  $\hat{\mathcal{A}}$  living in the coordinate ring of the corresponding partial flag variety. Then we use a minimality condition to prove that the cluster algebra  $\hat{\mathcal{A}}$  is equal to  $\mathbb{C}[G/P_K^-]$  after localizing some special minors, which are frozen variables.

## Chapter 1. Introduction

Cluster algebras were constructed in 2002 by Fomin and Zelevinsky [5] in a study of total positivity. They form a large family of polynomial algebras that are defined using an inductive process called “mutation”. For the last twenty years, the study of cluster algebras has been a very active area of mathematics. This is due to their connection with many other areas of mathematics including representation theory, algebraic geometry, Poisson geometry, mathematical physics, knot theory and combinatorics. On the other hand, the study of partial flag varieties plays a fundamental role in algebraic geometry, representation theory and Lie theory. The first relationship between cluster algebras and partial flag varieties appeared in Scott’s work [19] in 2006 in a project that studied the cluster algebra structures on the homogeneous coordinate rings of Grassmannians. Two years later, Geiß, Leclerc and Schröer [7] used preprojective algebras to explore the connection of cluster algebras and partial flag varieties. In that work, they conjectured that the affine coordinate ring of a Schubert cell admits a cluster-algebra structure for simply-laced semisimple complex algebraic groups. They then lift that to a cluster algebra living in the homogeneous coordinate ring of the corresponding partial flag variety. Remarkably, they proved that this lifted cluster algebra is equal to the coordinate ring of the partial flag variety in type  $A_n$  and it is equal to the coordinate ring of the partial flag variety, up to a certain localization, in type  $D_4$ . They conjectured that this lifted cluster algebra is equal to the coordinate ring of the corresponding partial flag variety in any simply-laced and non-simply-laced type, which is an important but very difficult conjecture in the area of cluster algebras. The conjecture is still wide open.

The work of Geiß, Leclerc and Schröer is divided into three main steps. In partic-



ular, they rely on the following. First, the coordinate ring of the unipotent Schubert cell admits a cluster structure  $\mathcal{A}$ . Second, the cluster algebra  $\mathcal{A}$  can be lifted to a cluster algebra  $\hat{\mathcal{A}}$  living in the corresponding homogeneous coordinate ring of the partial flag variety. Third, up to localization by the non-minuscule generalized minors, the cluster algebra  $\hat{\mathcal{A}}$  and the homogeneous coordinate ring of the partial flag variety coincide. Despite the fact that the first step was only conjectured in [7], the authors managed to prove it for all simply-laced cases in [9].

The three steps of the program in the previous paragraph can be used to generalize the work of [7] to any semisimple complex algebraic group no matter if it is simply- or non-simply-laced. The only issue is to prove each step for any semisimple complex algebraic group. The problem here is that the work of Geiß, Leclerc and Schröer was based on some categorification that is specified for the simply-laced case. But thanks to Goodearl and Yakimov, the first step can be generalized as a consequence of a long project they did in Poisson nilpotent algebras and cluster algebras [13]. This gave us the opportunity to follow the three-step path of the previous paragraph. Although the second step was generalized by Demonet in [2], we managed to prove it independently here. As a side note, it would be very interesting to discover the relationship of the generalization of  $\mathcal{A}$  by Demonet and ours. We plan to return to this in future work. We then use a minimality condition on the degree of the elements of the homogeneous coordinate ring of partial flag variety to show the third step.

Let  $I$  be the Dynkin diagram vertex set attached to a semisimple algebraic group  $G$  and  $J$  and  $K$  be nonempty sets such that  $J \sqcup K = I$ . In fact, Geiß, Leclerc and Schröer defined a homogenization map called the “tilde map” that lifts each element  $f$  in the coor-

dinate ring of a cell  $\mathbb{C}[N_K]$  uniquely to an element  $\tilde{f}$  of the corresponding coordinate ring of the partial flag variety  $\mathbb{C}[G/P_K^-]$ . They showed that, in the simply-laced case, this map can be used to lift each seed of the cluster algebra  $\mathcal{A} = \mathbb{C}[N_K]$  to a seed of a special cluster algebra  $\hat{\mathcal{A}} \subset \mathbb{C}[G/P_K^-]$ . We generalize that to any simply-laced and non-simply-laced case and prove the following.

**Theorem 1.** *Let  $\{(\mathbf{x}, B)\}$  be the collection of seeds of the cluster algebra  $\mathcal{A}$  of  $\mathbb{C}[N_K]$ .*

*Then there is a corresponding collection  $\{(\hat{\mathbf{x}}, \hat{B})\}$  of pairs, whose variables live in*

*$\mathbb{C}[G/P_K^-]$ , that forms a collection of seeds related by mutation. In other words, if*

*$(\mathbf{x}, B)$  and  $(\mathbf{x}', B')$  are two seeds of the coordinate ring of the cell  $\mathbb{C}[N_K]$  such that*

*$(\mathbf{x}', B') = \mu_k(\mathbf{x}, B)$ , then, correspondingly,  $(\hat{\mathbf{x}}', \hat{B}') = \mu_k(\hat{\mathbf{x}}, \hat{B})$ . In particular, if  $(\mathbf{x}_0, B_0)$*

*is an initial seed of  $\mathcal{A} = \mathbb{C}[N_K]$ , then  $(\hat{\mathbf{x}}_0, \hat{B}_0)$  is an initial seed of a cluster algebra*

*$\hat{\mathcal{A}} \subset \mathbb{C}[G/P_K^-]$ .*

For some of the simply-laced cases, Geiß, Leclerc and Schröer proved the equality after a localization by the minors indexed by  $J$  that are not minuscule. We omitted the second condition and proved the following for any semisimple algebraic group.

**Theorem 2.** *The localization of the homogeneous coordinate ring of the flag variety*

*$\mathbb{C}[G/P_K^-]$  by  $\Delta_{\varpi_j, \varpi_j}$ , where  $j \in J$ , equals the localization of the cluster algebra  $\hat{\mathcal{A}}$  by the same elements. In symbols,*

$$\mathbb{C}[G/P_K^-][\Delta_{\varpi_j, \varpi_j}^{-1}]_{j \in J} = \hat{\mathcal{A}}[\Delta_{\varpi_j, \varpi_j}^{-1}]_{j \in J}.$$

These two theorems together complete the second and third steps of our program and gave us the desired result.

Although the tilde map gives plays a key role in lifting the cluster algebra struc-

ture  $\mathcal{A} = \mathbb{C}[N_K]$  to a cluster algebra  $\hat{\mathcal{A}}$ , the calculations of it are not as trivial as it might seem. In fact, the lift of a generalized minor in  $\mathbb{C}[N_K]$  need not also be a generalized minor. We carry out an in-depth study of the properties of this canonical map and use them to get an explicit algorithm to calculate the image of the generalized minors. Since the initial seed of the cluster algebra  $\mathcal{A}$  consists of generalized minors only and due to the definition of  $\hat{\mathcal{A}}$ , the knowledge of the image of the generalized minors under the tilde map gives a full picture about the cluster algebra  $\hat{\mathcal{A}}$ . The second part of this thesis is dedicated to the description of this full picture of the cluster algebra  $\hat{\mathcal{A}}$ .

## Chapter 2. Lie Theory Preliminaries

### 2.1. Semisimple groups, semisimple Lie algebras

This section gives an overview about the required setup from the theory of semisimple groups and semisimple algebras and how they can be classified using Dynkin diagrams. We refer the reader to [17] and [18] for a broader background. Throughout, the ground field of all affine varieties will be  $\mathbb{C}$ .

**Definition 2.1.1.** An *algebraic group* is a group  $G$  that is also an affine variety such that the group multiplication  $m : G \times G \rightarrow G$  and the inversion  $i : G \rightarrow G$  are morphisms of varieties.

**Example 2.1.2.** The affine line  $\mathbb{C}^1$  with addition forms an additive algebraic group denoted  $\mathbb{G}_a$ . Also, the affine line without 0,  $\mathbb{C}^1 \setminus 0$  is a multiplicative algebraic group denoted  $\mathbb{G}_m$ . Moreover, the linear groups are algebraic. In particular,  $SO_n$ ,  $SL_n$  and  $GL_n$  are algebraic groups.

**Definition 2.1.3.** An (*abstract*) *torus*  $T$  is an algebraic group isomorphic to  $\mathbb{G}_m \times \dots \times \mathbb{G}_m$ . A *maximal torus* of an algebraic group  $G$  is a subgroup that is a torus and not contained in any torus.

**Definition 2.1.4.** A *unipotent subgroup*  $N$  of an algebraic group  $G$  is a group isomorphic to a closed subgroup of the group of upper triangular matrices of  $GL_n$  whose diagonal entries are all 1.

**Definition 2.1.5.** A *solvable subgroup* of an algebraic group  $G$  is a group isomorphic to a closed connected subgroup of the group of upper triangular matrices of  $GL_n$ .

*Remark 2.1.6.* The maximal connected normal solvable and unipotent subgroups exist al-

ways and are unique. The first is called the *radical* and the later is called the *unipotent radical*.

**Definition 2.1.7.** An algebraic group  $G$  is called *semisimple* if its radical is equal to the trivial group and is called *reductive* if the unipotent radical is trivial.

*Remark 2.1.8.* Obviously, each semisimple group is reductive.

**Example 2.1.9.** The groups  $SL_n$  and  $SO_n$  are semisimple, while  $GL_n$  is reductive but not semisimple.

**Definition 2.1.10.** A maximal connected solvable subgroup of an algebraic group is called a *Borel* subgroup.

**Example 2.1.11.** The subgroup of upper triangular matrices is a Borel subgroup of  $GL_n$ .

*Remark 2.1.12.* Every Borel subgroup  $B$  can be expressed as  $B = T \ltimes N$ , where  $T$  and  $N$  are maximal torus and maximal unipotent subgroup, respectively.

**Definition 2.1.13.** Let  $B$  be a Borel subgroup. A Borel subgroup  $B^-$  is called *opposite* to  $B$  if their intersection is a maximal torus.

**Example 2.1.14.** The opposite Borel subgroup of the group of upper triangular matrices of  $GL_n$  is the group of lower triangular matrices. Their intersection, the group of diagonal matrices, is a maximal torus.

**Definition 2.1.15.** A *Lie algebra* is a vector space  $\mathfrak{g}$  together with a *Lie bracket*, that is, a bilinear operation  $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  such that  $[x, x] = 0$  for all  $x \in \mathfrak{g}$  and  $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$  for all  $x, y, z \in \mathfrak{g}$ . The later axiom is called the *Jacobi identity*.

**Example 2.1.16.** Any vector space  $V$  can be endowed with a Lie algebra structure by defining the Lie bracket to be  $[x, y] = 0$  for all  $x, y \in V$ .

**Example 2.1.17.** The vector space  $M_n$  of all  $n \times n$  matrices has a Lie algebra structure

whose Lie bracket is defined by  $[x, y] = xy - yx$  for all  $x, y \in M_n$ . This Lie algebra is denoted by  $\mathfrak{gl}_n$ .

*Remark 2.1.18.* Any algebraic group  $G$  induces a Lie algebra  $\text{Lie}(G) := T_1 G$ , where  $T_x G$  is the tangent space of  $G$  at the point  $x$  whose Lie bracket is defined by  $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ . We will often denote  $\text{Lie}(G)$  by  $\mathfrak{g}$ .

**Example 2.1.19.** The Lie algebra  $\text{Lie}(GL_n)$  is isomorphic to  $\mathfrak{gl}_n$ . Also,  $\text{Lie}(SL_n) = \mathfrak{sl}_n$  is the Lie algebra of all  $n \times n$  matrices the such that sum of the diagonal of each is 0. The description of the other linear Lie algebras can be found in [17].

**Definition 2.1.20.** A *Lie subalgebra*  $\mathfrak{h}$  of a Lie algebra  $\mathfrak{g}$  is a subspace such that  $[x, y] \in \mathfrak{h}$  for all  $x, y \in \mathfrak{h}$ . The Lie subalgebra  $\mathfrak{h}$  is an *ideal* of  $\mathfrak{g}$  if  $[x, y] \in \mathfrak{h}$  for all  $x \in \mathfrak{g}$  and  $y \in \mathfrak{h}$ . A Lie algebra is called *simple* if it has no proper ideals other than 0.

**Definition 2.1.21.** A Lie algebra is called *semisimple* if it is a direct sum of simple Lie algebras.

**Example 2.1.22.** For any Lie algebra  $\mathfrak{g}$ , the trivial algebra 0 and the Lie algebra  $\mathfrak{g}$  itself are ideals of  $\mathfrak{g}$ . Also, the *derived algebra*  $[\mathfrak{g}, \mathfrak{g}] := \{[x, y] \mid x, y \in \mathfrak{g}\}$  is an ideal of  $\mathfrak{g}$ .

**Example 2.1.23.** The Lie algebra  $\mathfrak{sl}_n$  is a semisimple Lie subalgebra of  $\mathfrak{gl}_n$ .

**Definition 2.1.24.** A Lie algebra  $\mathfrak{g}$  is said to be *nilpotent* if there are ideals  $\mathfrak{a}_1, \dots, \mathfrak{a}_n$  such that

$$\mathfrak{g} = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \dots \supset \mathfrak{a}_n = 0,$$

and  $[\mathfrak{g}, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1}$ .

**Example 2.1.25.** The Lie subalgebra of strictly upper triangular matrices of  $\mathfrak{gl}_n$  is nilpotent.

## 2.2. Root system and Dynkin diagram classification

**Definition 2.2.1.** A *reflection*  $s$  of a vector space is a transformation fixing some hyperplane, that is, a subspace of codimension 1, and sending any vector orthogonal to the hyperplane to its negative. A *reflection*  $s_\alpha$  *corresponding to*  $\alpha \in V$  is a reflection such that  $s_\alpha(\alpha) = -\alpha$ .

**Definition 2.2.2.** A subset  $R$  of a  $k$ -vector space is called a *root system* if

1.  $R$  is a finite set spanning  $E$  not including 0;
2. for each  $\alpha \in R$ , there is a unique reflection  $s_\alpha$  such that  $s_\alpha(R) \subset R$ ;
3. the expression  $s_\alpha(\beta) - \beta$  is a multiple of  $\alpha$  for all  $\alpha, \beta \in R$ .

The elements of  $R$  are called *roots*. The subspace  $W$  of  $\text{GL}(V)$  generated by  $s_\alpha$  for  $\alpha \in R$  is called the *Weyl group*.

*Remark 2.2.3.* Equivalently, for an algebraic group  $G$  and a maximal torus  $T$ , the Weyl group can be defined as  $W = N(T)/T$ , where  $N(T)$  is the normalizer of  $T$ .

*Remark 2.2.4.* For  $\beta \in V$  and  $\alpha^\vee \in V^\vee$ , let  $\langle \beta, \alpha^\vee \rangle$  be the standard pair of a vector space and its dual. The root system axioms imply that

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha$$

is a unique expression of the reflection  $s_\alpha$ . Moreover, they imply that  $\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$  for all  $\alpha, \beta \in R$ , where  $\alpha^\vee$  is the element satisfying  $\langle \alpha, \alpha^\vee \rangle = 2$ . Such an is called the *coroot* of  $\alpha$ .

**Proposition 2.2.5.** *For any root system  $R$  of a vector space  $V$ , there exists an inner product  $(\cdot, \cdot)$  such that*

$$(wx, wy) = (x, y)$$

*for all  $x, y \in V$  and all  $w \in W$ .*

*Proof.* Let  $(.,.)'$  be any inner product. It is straightforward to see that the pair

$$(u, v) = \sum_{w \in W} (u, v)'$$

is an inner product satisfying the axiom of the proposition.  $\square$

**Corollary 2.2.6.** *If  $V$  is a vector space endowed with an inner product as above, then*

$$s_\alpha(\beta) = \beta - 2 \frac{(\beta, \alpha)}{(\alpha, \alpha)} \alpha,$$

*for all  $\beta \in V$ .*

**Corollary 2.2.7.** *If  $(.,.)$  is an inner product as in Proposition 2.2.5, then*

$$2 \frac{(\beta, \alpha)}{(\beta, \alpha^\vee)} = \langle \beta, \alpha^\vee \rangle,$$

*for all  $\beta \in V$  and  $\alpha \in R$ .*

*Remark 2.2.8.* If  $w = s_{i_1} \dots s_{i_n}$  is an element of the Weyl group  $W$ , where each  $s_{i_j}$  is a simple reflection, and the number of the occurrences of simple reflections cannot be reduced, then the *length*  $\ell$  of  $w$  is defined to be  $\ell(w) = n$ . Conventionally, the length of the identity element  $\ell(e) = 0$ .

*Remark 2.2.9.* The Weyl group  $W$  contains a longest element, that is, there is an element  $w_0 \in W$  such that  $\ell(w) \leq \ell(w_0)$  for any  $w \in W$ . Throughout, the longest element will always be denoted by  $w_0$ .

**Example 2.2.10.** The Weyl group  $W$  of  $G = SL_{n+1}$  is the symmetric group  $S_{n+1}$ . The longest element of  $W$  is  $w_0 = s_1 s_2 s_3 \dots s_n s_1 s_2 s_3 \dots s_{n-1} \dots s_1 s_2 s_1$ , where  $s_i$  denotes the transposition  $(i+1, i)$ .

**Definition 2.2.11.** A subset  $S$  of a root system  $R$  is called a *base* if it is a basis for  $\mathfrak{g}$  and



if every root  $\beta$  can be written as  $\beta = \sum_{\alpha \in S} n_{\alpha} \alpha$  where all  $n_{\alpha}$ 's are positive integers or negative integers.

**Theorem 2.2.12.** *Every root system  $R$  has a base  $S$ .*

**Definition 2.2.13.** Let  $R$  be a root system of a Lie algebra  $\mathfrak{g}$  whose base is  $S = \{\alpha_1, \dots, \alpha_r\}$ . The *Cartan matrix*  $A$  of the Lie algebra  $\mathfrak{g}$  is the  $r \times r$  matrix whose  $i \times j$  entry is  $\langle \alpha_i, \alpha_j^{\vee} \rangle$ .

**Definition 2.2.14.** A *Cartan subalgebra*  $\mathfrak{h}$  of a Lie algebra  $\mathfrak{g}$  is a nilpotent subalgebra with the property that if  $[x, y] \in \mathfrak{h}$  for all  $x \in \mathfrak{h}$ , then  $y \in \mathfrak{h}$ .

**Example 2.2.15.** The algebra of all diagonal matrices is a Cartan subalgebra of  $\mathfrak{gl}_n$ .

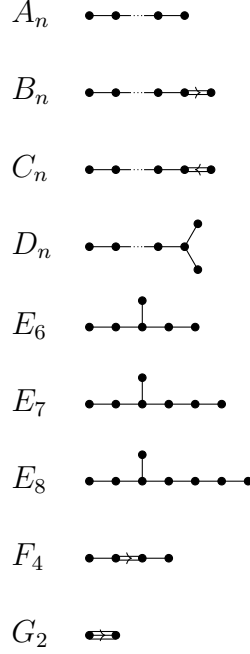
**Theorem 2.2.16.** *Every Lie algebra contains a Cartan subalgebra. Moreover, the Cartan subalgebras are all isomorphic to each other.*

**Definition 2.2.17.** The *fundamental weights* are the elements  $\varpi_i$  of  $\mathfrak{h}^{\vee}$ , where  $\mathfrak{h}$  is a fixed Cartan subalgebra, defined by the property  $\langle \varpi_i, \alpha_j^{\vee} \rangle = \delta_{ij}$ .

*Remark 2.2.18.* The information of the Cartan matrix can be encoded in a graph whose vertices are indexed by  $\alpha_1, \dots, \alpha_r$  in which the vertices  $\alpha_i, \alpha_j$  are joined by  $a_{ij}a_{ji}$  edges.

The direction of the edges goes towards  $\alpha_i$  if  $|\alpha_{ij}| > |\alpha_{ji}|$ . This graph is called the Dynkin diagram of  $\mathfrak{g}$ .

*Remark 2.2.19.* All complex semisimple Lie algebras are determined, up to isomorphism, by their root systems and Cartan matrices (and so their Dynkin diagrams). The figure below shows the list of Dynkin diagrams, where the subscript denotes the number of vertices.



**Definition 2.2.20.** An algebraic group  $G$  (and its corresponding Lie algebra) is called *simply-laced* if it is of type  $A$ ,  $D$ , or  $E$ . Otherwise it is called *non-simply-laced*.

### 2.3. Partial flag varieties

In this section, we briefly review the needed setup and results from partial flag varieties. We then review the required results of the (multi-)homogeneous coordinate rings of the partial flag varieties. The reader who is interested in a deeper overview can look to [7], [11], or [18].

*Remark 2.3.1.* From now on, the symbol  $G$  denotes a semisimple complex algebraic group whose Dynkin diagram vertex set is  $I$ .

**Definition 2.3.2.** A *parabolic* subgroup  $P$  of  $G$  is a closed subgroup containing a Borel subgroup  $B$ .

**Example 2.3.3.** Any Borel subgroup  $B$  is parabolic.

*Remark 2.3.4.* Any maximal unipotent subgroup  $N$  is generated by a distinguished collec-

tion of one parameter subgroups  $x_i(t)$ ,  $i \in I$  and  $t \in \mathbb{C}$ , called root subgroups.

**Example 2.3.5.** Let  $G = SL_{n+1}$ . Take  $N$  to be the subgroup of upper triangular matrices. The subgroups  $x_i(t)$  of the previous remark are  $x_i(t) = I + tE_{i,i+1}$ , where  $I$  is the identity matrix and  $E_{i,j}$  is the matrix whose  $i \times j$  entry is 1 and the other entries are all 0.

**Example 2.3.6.** From now on, we let  $J$  and  $K$  denote two nonempty disjoint subsets of  $I$  such that  $J \sqcup K = I$ . We fix the notation  $x_i(t)$ ,  $i \in I$  and  $t \in \mathbb{C}$  for the simple root subgroups of the unipotent radical  $N$  of  $B$  and the notation  $y_i(t)$  for the simple root subgroups of the unipotent radical  $N^-$  of  $B^-$ . The subgroup  $P_K$  generated by  $B$  and the one-parameter unipotent subgroups  $y_k(t)$ ,  $k \in K$  and  $t \in \mathbb{C}$  is a parabolic called *the standard parabolic subgroup*. Similarly, the subgroup  $P_K^-$  generated by  $B^-$  and the one-parameter unipotent subgroups  $x_k(t)$ ,  $k \in K$  and  $t \in \mathbb{C}$ , is parabolic.

**Definition 2.3.7.** A *(partial) flag variety* is a quotient  $G/P$  such that  $P$  is a parabolic subgroup of  $G$ .

*Remark 2.3.8.* Every parabolic subgroup is conjugate to a parabolic subgroup of the form  $P_K$ . In many cases, this reduces the study of partial flag varieties to the ones of the form  $G/P_K$ . For example, our focus, in this project, is on the subgroups  $P_K^-$ , but the results we get are general.

*Remark 2.3.9.* The partial flag variety  $G/P_K^-$  can be embedded naturally as a closed subset of the product of projective spaces

$$\prod_{j \in J} \mathbb{P}(L(\varpi_j)^*),$$

where  $\varpi_j$  is a fundamental weight of  $G$ , and for a dominant weight  $\lambda$ , the corresponding  $L(\lambda)$  is the finite-dimensional irreducible  $G$ -module with highest weight  $\lambda$ ; and  $L(\lambda)^*$

denotes the right  $G$ -module obtained by twisting the action of  $G$ . As a terminology, the  $L(\varpi_i)$ 's are called the *fundamental representations*.

*Remark 2.3.10.* Let  $\Pi_J \cong \mathbb{N}^J$  denote the monoid consisting of dominant integral weights  $\lambda = \sum_{j \in J} a_j \varpi_j$ , in which  $a_j$  runs in  $\mathbb{N}$ . The multi-homogeneous coordinate ring  $\mathbb{C}[G/P_K^-]$  is a  $\Pi_J$ -graded algebra defined as

$$\mathbb{C}[G/P_K^-] = \bigoplus_{\lambda \in \Pi_J} L(\lambda).$$

It is known that  $\mathbb{C}[G/P_K^-]$  can be identified with the subalgebra of  $\mathbb{C}[G/N^-]$  generated by the homogeneous elements of degree  $\varpi_j$ , where  $j \in J$ .

*Remark 2.3.11.* The set  $\Pi_J$  has a standard partial ordering given by

$$\lambda \preceq \mu \iff \mu - \lambda \text{ is an } \mathbb{N}\text{-linear combination of the fundamental weights } \varpi_j, \text{ where } j \in J.$$

*Remark 2.3.12.* Let  $\mathfrak{g}$  be the Lie algebra of  $G$ . There is a distinguished family of generators, called the *Chevalley generators* and denoted by  $e_i, f_i, h_i$ , where again  $i$  runs in  $I$ . The elements  $e_i$  generate  $\text{Lie}(N) = \mathfrak{n}$ . Consequently, it follows that  $N$  acts naturally from the left and right on  $\mathbb{C}[N]$  by the following left and right actions respectively:

$$(x \cdot g)(n) = g(nx), \quad (g \in \mathbb{C}[N] \text{ and } x, n \in N),$$

$$(g \cdot x)(n) = g(xn), \quad (g \in \mathbb{C}[N] \text{ and } x, n \in N).$$

Differentiating these two actions, we get left and right actions of  $\mathfrak{n}$  on  $\mathbb{C}[N]$ , respectively.

*Remark 2.3.13.* The right action of  $e_i$  on  $g \in \mathbb{C}[N]$  is a derivation and it is of fundamental importance in this dissertation. Throughout, it is going to be denoted by  $e_i^\dagger(g) := g \cdot e_i$ .

*Remark 2.3.14.* Let  $G$  be of type  $A$ . A *(flag) minor* is a regular irreducible function of  $\mathbb{C}[G]$  defined as follows: For each subset  $I \subset [1, n] := \{1, \dots, n\}$  and each element  $x \in G$ ,

the minor  $\Delta_I(x)$  is defined to be the determinant of the submatrix of  $x$  whose rows are indexed by  $I$  and columns are indexed by  $1, \dots, |I|$ . This notion was generalized by Fomin and Zelevinsky in [4] to the notion of (*generalized*) *minor*  $\Delta_{u\varpi_j, w(\varpi_j)}$ , where  $u, w$  belong to the Weyl group  $W$ . The notions of flag minors and generalized minors coincide in type  $A$ . However, the generalized minor notion makes sense in any type.

*Remark 2.3.15.* For each simple reflection  $s_i \in W$ , define  $\overline{s_i} := \exp(f_i) \exp(e_i) \exp(f_i)$ . The  $\overline{s_i}$ 's here are representatives in  $N(T)$ , where  $N(T)$  is as in Remark 2.2.3. If  $w = s_{i_1} \dots s_{i_r}$  with  $r$  being the length of  $w$ , then define  $\overline{w} = \overline{s_{i_1}} \dots \overline{s_{i_r}}$ . Let  $G_0 = N^- H N$  be the open set of  $G$  consisting of elements having Bruhat decomposition. Each  $x \in G_0$  is expressible uniquely as

$$x = [x]_- [x]_0 [x]_+,$$

where  $[x]_- \in N^-$ ,  $[x]_0 \in H$ ,  $[x]_+ \in N$ . Let  $V_i^+$  be the irreducible representation whose highest weight is  $\varpi_i$  and highest weight vector is  $v_i^+$ . For any  $h \in H$  one has that  $v_i^+$  is an eigenvector, that is,  $h v_i^+ = [h]^{\varpi_i} v_i^+$  and  $[h]^{\varpi_i} \in \mathbb{C} \setminus \{0\}$ . In [4], Fomin and Zelevinsky defined the following:

**Definition 2.3.16.** For  $u, v \in W$  and  $i \in I$  define the (*generalized*) *minor* to be the regular function on  $G$  given by

$$\Delta_{u\varpi_i, v\varpi_i}(x) = [\overline{u}^{-1} x \overline{v}]_0^{\varpi_i}.$$

*Remark 2.3.17.* The multi-degree of a generalized minor  $\Delta_{\varpi_j, w(\varpi_j)}$  in  $\mathbb{C}[G/P_K^-]$  is  $\varpi_j$ . It is known that the (affine) coordinate ring of the Schubert cell  $N_K$  is the homogeneous elements of degree 0 of the localization of the homogeneous coordinate ring of the corre-

sponding partial flag variety  $P_K$  by the minors  $\Delta_{\varpi_j, \varpi_j}$  where  $j \in J$ . In symbols,

$$\mathbb{C}[N_K] = \left\{ \frac{f}{\prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{a_j}} \mid f \in L\left(\sum_{j \in J} a_j \varpi_j\right) \right\}. \quad (2.1)$$

Equivalently, the generalized minors connect the coordinate ring of the cell  $N_K$  to the homogeneous coordinate ring of  $P_K^-$  by

$$\mathbb{C}[N_K] = \mathbb{C}[G/P_K^-] / (\Delta_{\varpi_j, \varpi_j} - 1)_{j \in J}. \quad (2.2)$$

*Remark 2.3.18.* By (2.2), there is a natural canonical projection map

$$\text{proj}_J : \mathbb{C}[G/P_K^-] \rightarrow \mathbb{C}[N_K]$$

$$g \mapsto g + (\Delta_{\varpi_j, \varpi_j} - 1)_{j \in J}.$$

It is known that the restriction of the projection map to each homogeneous piece  $L(\lambda)$ , for  $\lambda \in \Pi_J$ , gives an injection  $L(\lambda) \hookrightarrow \mathbb{C}[N_K]$ . Notationally, if the context is clear, we may omit the subscript  $J$  and write  $\text{proj}$  instead of  $\text{proj}_J$ .

## Chapter 3. The Homogenization Tilde Map

In this chapter, we study the definition and the properties of the homogenization map “tilde” defined by Geiß, Leclerc and Schröer [7]. We then use its properties to find an algorithm calculating it explicitly.

### 3.1. Tilde map

**Lemma 3.1.1** (GLS tilde map). *Let  $f$  be an element of the affine coordinate ring  $\mathbb{C}[N_K]$ . There is a unique homogeneous element  $\tilde{f}$  in  $\mathbb{C}[G/P_K^-]$  whose projection to  $\mathbb{C}[N_K]$  is  $f$  and whose multi-degree is minimal with respect to the partial ordering  $\preceq$  defined in Remark 2.3.11.*

*Proof (GLS).* For any  $\lambda = \sum_{i \in I} a_i \varpi_i$ , it is known that the subspace  $\text{proj}_I(L(\lambda))$  of  $\mathbb{C}[N]$  can be described as

$$\text{proj}_I(L(\lambda)) = \{f \in \mathbb{C}[N] \mid (e_i^\dagger)^{a_i+1} f = 0, \ i \in I\}.$$

This implies that  $\mathbb{C}[N_K]$  can be identified with

$$\{f \in \mathbb{C}[N] \mid e_k^\dagger f = 0, \ k \in K\} \subset \mathbb{C}[N].$$

Thus, for any  $\lambda = \sum_{j \in J} a_j \varpi_j \in \Pi_J$ , it follows that

$$\text{proj}_J(L(\lambda)) = \{f \in \mathbb{C}[N_K] \mid (e_j^\dagger)^{a_j+1} f = 0, \ j \in J\}.$$

Now, for  $f \in \mathbb{C}[N_K]$ , define

$$a_j(f) := \max \left\{ s \mid (e_j^\dagger)^s f \neq 0 \right\}, \quad (3.1)$$

and

$$\lambda(f) := \sum_{j \in J} a_j(f) \varpi_j. \quad (3.2)$$

Obviously,  $f$  is an element of  $\text{proj}_J\left(L\left(\lambda(f)\right)\right)$  where  $\lambda(f)$  is minimal with respect to  $\preceq$ , that is, if  $\lambda \in \Pi_J$  such that  $f \in \text{proj}_J(L(\lambda))$ , then  $\lambda(f) \preceq \lambda$ . On the other hand, as the restriction of  $\text{proj}_J$  to the piece  $L\left(\lambda(f)\right)$  is injective, this gives the desired uniqueness. That is, the element  $\tilde{f} \in L\left(\lambda(f)\right)$ , whose projection  $\text{proj}_J(\tilde{f}) = f$ .  $\square$

**Definition 3.1.2.** Let  $f \in \mathbb{C}[N_K]$ . The element  $\tilde{f} \in \mathbb{C}[G/P_K^-]$  will be called the *(homogeneous) lift* (or the *(homogeneous) lifting*) of  $f$  to  $\mathbb{C}[G/P_K^-]$ .

**Lemma 3.1.3.** Let  $f, g \in \mathbb{C}[N_K]$ . The lifting commutes with usual multiplication. In other words, the following diagram

$$\begin{array}{ccc} \mathbb{C}[N_K] \times \mathbb{C}[N_K] & \xrightarrow{m_{\mathbb{C}[N_K]}} & \mathbb{C}[N_K] \\ \tilde{\cdot} \times \cdot \downarrow & & \downarrow \tilde{\cdot} \\ \mathbb{C}[G/P_K^-] \times \mathbb{C}[G/P_K^-] & \xrightarrow{m_{\mathbb{C}[G/P_K^-]}} & \mathbb{C}[G/P_K^-] \end{array}$$

given by

$$\begin{array}{ccc} f \times g & \longmapsto & f \cdot g \\ \downarrow & & \downarrow \\ \tilde{f} \times \tilde{g} & \longmapsto & \tilde{f} \cdot \tilde{g} = \widetilde{f \cdot g} \end{array}$$

commutes. Moreover, if  $a_j(f + g) = \max\{a_j(f), a_j(g)\}$  for all  $j \in J$ , then

$$\widetilde{f + g} = \mu \tilde{f} + \nu \tilde{g},$$

where  $\mu$  and  $\nu$  are relatively prime monomials in the variables  $\Delta_{\varpi_j, \varpi_j}$ , ( $j \in J$ ).

*Proof (GLS).* The first property follows easily by Leibniz formula and the fact that the endomorphism  $e_j^\dagger$  is a derivation of  $\mathbb{C}[N_K]$ , for all  $j \in J$ . The additional assumption in the second statement implies the existence of relatively prime monomials  $\mu$  and  $\nu$  in  $\Delta_{\varpi_j, \varpi_j}$ 's such that the multi-degree of each of  $\mu \tilde{f}$  and  $\nu \tilde{g}$  is the same as the one of  $\widetilde{f + g}$ . This completes the proof.  $\square$



**Definition 3.1.4.** The *restricted minor*  $D_{\varpi_j, w(\varpi_j)}$  is the restriction of the generalized minor  $\Delta_{\varpi_j, w(\varpi_j)}$  to  $N$ .

*Remark 3.1.5.* The restricted minors play a fundamental role in constructing the cluster algebra structure on the coordinate ring of a Schubert cell  $\mathbb{C}[N_K]$  defined by Geiß, Leclerc and Schröer in the simply-laced case and by Goodearl and Yakimov in the general case. Their homogeneous lift is the key to constructing the cluster structure of  $\mathbb{C}[G/P_K^-]$  (see [7, 10]). As a naive guess, one may expect that the lifting of a restricted minor is a generalized minor, but this is not true in general.

In [7], Geiß, Leclerc and Schröer gave the following example of type  $A$ :

**Example 3.1.6.** Let  $G = SL_6$ , which is a group of type  $A_5$ . Let  $J = \{1, 3\}$ , and so  $K = \{2, 4, 5\}$ . The restricted minor  $D_{13,56}$  cannot be written as  $D_{12\dots m, i_1 i_2 \dots i_m}$ . Therefore,  $\widetilde{D}_{13,56}$  cannot be a flag minor. However,

$$\begin{aligned} D_{13,56} &= D_{1,2} D_{23,56} - D_{123,256} \\ &= D_{1,2} D_{123,156} - D_{123,256}. \end{aligned}$$

Hence,

$$\begin{aligned} \widetilde{D}_{13,56} &= \Delta_{1,2} \Delta_{123,156} - \Delta_{1,1} \Delta_{123,256} \\ &= \Delta_2 \Delta_{156} - \Delta_1 \Delta_{256}. \end{aligned}$$

### 3.2. Lift degree and explicit algorithm

In this section, we use the properties of the minors and the properties of the tilde map together with the relationship between  $\mathbb{C}[N_K]$  and  $\mathbb{C}[G/P_K^-]$  to find an explicit formula for lifting the restricted generalized minors by the tilde map. We start this section

by the following proposition:

**Proposition 3.2.1.** *If  $j \in J$ , then the lift of  $D_{\varpi_j, w(\varpi_j)} \in \mathbb{C}[N_K]$  is  $\widetilde{D}_{\varpi_j, w(\varpi_j)} = \Delta_{\varpi_j, w(\varpi_j)}$ .*

*Proof.* Since the tilde map gives a unique lift, we need only to show that  $\Delta_{\varpi_j, w(\varpi_j)}$  is an element of  $\mathbb{C}[G/P_K^-]$  whose projection is  $D_{\varpi_j, w(\varpi_j)}$  and whose degree is minimal. The first statement is clear. On the other hand, assume that there is an element  $g \in \mathbb{C}[G/P_K^-]$  whose projection  $\text{proj}_J(g) = D_{\varpi_j, w(\varpi_j)}$  and whose degree  $\lambda_g \preceq \varpi_j$ . The only possibility is  $\lambda_g = a_j \varpi_j$  and  $a_j \leq 1$ . This forces  $\lambda_g$  to be  $\varpi_j$ , as it is 0 otherwise. Since  $L(\lambda) \hookrightarrow \mathbb{C}[N_K]$  for each homogeneous piece  $L(\lambda)$ , it follows that  $g = \Delta_{\varpi_j, w(\varpi_j)}$ .  $\square$

**Lemma 3.2.2.** *Using equation (2.1), if*

$$\frac{f}{\prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{a_j}} = \frac{g}{\prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{b_j}}$$

*and  $a_j$  and  $b_j$  are minimal, then  $f = g$  and  $a_j = b_j$ .*

*Proof.* By (2.1), it is clear that  $\text{proj}_J(f) = \text{proj}_J(g)$ . Since  $a_j$  and  $b_j$  are minimal, we get that either  $f = g$  or  $\lambda(f)$  and  $\lambda(g)$  are incomparable. If  $\lambda(f)$  and  $\lambda(g)$  are incomparable, then the tilde map is not well-defined. Hence,  $f = g$  and consequently  $a_j = b_j$ .  $\square$

**Corollary 3.2.3.** *The equation (2.1) can be refined and expressed as:*

$$\mathbb{C}[N_K] = \left\{ \text{proj}_J(f) = \frac{f}{\prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{a_j}} \mid f \in L\left(\sum_{j \in J} a_j \varpi_j\right) \text{ and } a_j \text{ is minimal} \right\}. \quad (3.3)$$

**Corollary 3.2.4.** *If  $j \in J$ , then the restricted minor  $D_{\varpi_j, w\varpi_j}$  is given by*

$$D_{\varpi_j, w\varpi_j} = \frac{\Delta_{\varpi_j, w\varpi_j}}{\Delta_{\varpi_j, \varpi_j}}.$$

**Proposition 3.2.5.** Assume that  $\frac{f}{\prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{a_j}}$  is an element of the coordinate ring of  $N_K$  such that  $a_j$  is minimal for each  $j$ . Then the lift

$$\widetilde{\left( \frac{f}{\prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{a_j}} \right)} = f.$$

*Proof.* By (3.3) it is clear that  $f \in L\left(\sum_{j \in J} a_j \varpi_j\right)$ . Obviously,

$$\text{proj}_J(f) = \frac{f}{\prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{a_j}}.$$

The result now follows from the minimality of the  $a_j$ 's and the uniqueness of the tilde map. □

**Definition 3.2.6.** For a restricted minor  $D_{\varpi_{i_n}, w(\varpi_{i_n})}$ , define the *lift degree* to be the integer  $d_n$  in the equation:

$$s_{i_1}(s_{i_2} \dots s_{i_n})(\varpi_{i_n}) = s_{i_2} \dots s_{i_n}(\varpi_{i_n}) - d_n \alpha_{i_1}, \quad (3.4)$$

where  $w = s_{i_1} s_{i_2} \dots s_{i_n}$  and  $\alpha_{i_1}$  is the vertex of the Dynkin diagram indexed by  $i_1$ .

**Proposition 3.2.7.** Assume the setting of Definition (3.2.6). Then

1. If  $i_n = i_1$ , then  $d_n = 1$ .
2. The lift  $\widetilde{D}_{\varpi_{i_n}, w(\varpi_{i_n})}$  of the minor  $D_{\varpi_{i_n}, w(\varpi_{i_n})}$  is of degree  $d_n \varpi_{i_1}$ .

*Proof.* For (1), it follows easily from the facts that the action of a simple reflection  $s_i$  is given by  $s_i(\beta) = \beta - (\beta, \alpha_i^\vee) \alpha_i$  and  $(\varpi_j, \alpha_i^\vee) = \delta_{ij}$ . On the other hand, (2) follows from

$$\begin{aligned} a_l(D_{\varpi_{i_n}, w(\varpi_{i_n})}) &= \max \left\{ s \mid (e_l^\dagger)^s D_{\varpi_{i_n}, w(\varpi_{i_n})} \neq 0 \right\} \\ &= \begin{cases} d_n, & \text{if } l = i_1, \\ 0, & \text{otherwise;} \end{cases} \end{aligned}$$

where  $a_l$  is as introduced in Equation 3.1. In fact, the first statement is clear and the second follows from the fact that  $\bar{u}x_j(t)\bar{u}^{-1} \in N$ , when  $\ell(us_i) > \ell(u)$  and  $i \neq j$ .  $\square$

**Corollary 3.2.8.** *In the setting of Definition (3.2.6), there exists a unique  $f \in \mathbb{C}[G/P_K^-]$  of homogeneous degree  $d_n\varpi_{i_1}$  such that*

$$D_{\varpi_{i_n}, w(\varpi_{i_n})} = \frac{f}{\Delta_{\varpi_{i_1}, \varpi_{i_1}}^{d_n}}.$$

Moreover, the element  $f$  is the lift of  $D_{\varpi_{i_n}, w(\varpi_{i_n})}$  and it can be expressed explicitly by the equation

$$f = \frac{\Delta_{\varpi_{i_n}, w\varpi_{i_n}} \Delta_{\varpi_{i_1}, \varpi_{i_1}}^{d_n}}{\Delta_{\varpi_{i_n}, \varpi_{i_n}}}.$$

*Proof.* This follows immediately from Proposition 3.2.7 and Proposition 2.6 of [4].  $\square$

## Chapter 4. Cluster Algebras

### 4.1. Cluster algebra overview

In this section, we provide a background on cluster algebras and their main notions and remarks. Detailed overviews about cluster algebras and their theory can be found in [3], [5], or [11].

**Definition 4.1.1.** Throughout, the term *ambient field* will be referring to a field  $\mathcal{F}$  that is isomorphic to the field  $\mathbb{C}(x_1, \dots, x_n)(x_{n+1}, \dots, x_m)$ , where  $\{x_1, \dots, x_n, \dots, x_m\}$  is an algebraically independent generating set, that is, a set generating  $\mathcal{F}$  and does not satisfy any nontrivial polynomial equation with coefficients in  $\mathbb{C}$ .

*Remark 4.1.2.* In the ambient field  $\mathcal{F}$  of the previous definition, there is a hidden significant difference between the variables  $x_1, \dots, x_n$  and the variables  $x_{n+1}, \dots, x_m$ . This is why we write  $\mathbb{C}(x_1, \dots, x_n)(x_{n+1}, \dots, x_m)$  instead of writing  $\mathbb{C}(x_1, \dots, x_m)$ . This distinction becomes clear in the following sequence of definitions and remarks.

**Definition 4.1.3.** A *seed* is a pair  $(\tilde{\mathbf{x}}, \tilde{B})$  with the following data:  $\tilde{\mathbf{x}}$  is a tuple of algebraically independent variables  $\tilde{\mathbf{x}} = (x_1, \dots, x_n, \dots, x_m)$  generating an ambient field  $\mathcal{F}$  and  $\tilde{B}$  is an  $m \times n$  matrix whose north  $n \times n$  submatrix  $B$  is *skew-symmetrizable*, that is, can be transformed to a skew-symmetric matrix by multiplying each row  $r_i$  by some nonzero integer  $d_i$ . Terminologically, the tuple  $\tilde{\mathbf{x}}$  is called an *extended cluster*, where its first  $n$ -variables are called the *mutable* variables and the next  $(m - n)$ -variables are called the *frozen* variables. The tuple  $\mathbf{x}$  consisting of the first  $n$ -variables  $(x_1, \dots, x_n)$  is called a *cluster*. Also, the principal  $n \times n$  submatrix  $B$  of  $\tilde{B}$  is called the *exchange* matrix, while the matrix  $\tilde{B}$  is called the *extended exchange* matrix. We may use the term exchange matrix

to refer to  $\tilde{B}$  if there is no need to distinguish between  $B$  and  $\tilde{B}$ .

*Remark 4.1.4.* Sometimes it is efficient to replace the skew-symmetrizable matrix in the previous definition by a *quiver*  $Q$ , which is a directed graph with  $n$ -mutable and  $(m - n)$ -frozen vertices. This quiver has to have no loops, no 2-oriented-cycles and no edges between two frozen vertices. Indeed, each quiver gives rise to an  $m \times n$  skew-symmetrizable matrix  $\tilde{B}(Q)$  whose entries are

$$b_{ij} = \begin{cases} \#(i \rightarrow j), & \text{if } i > j, \\ 0, & \text{if } i = j, \\ -\#(i \leftarrow j), & \text{if } i < j; \end{cases}$$

where  $\#(i \rightarrow j)$  is the number of arrows from  $i$  to  $j$  and  $\#(i \leftarrow j)$  is the number of arrows from  $j$  to  $i$ .

**Definition 4.1.5.** Let  $k$  be a mutable index of a seed  $(\tilde{\mathbf{x}}, \tilde{B})$ . A *mutation*  $\mu_k$  at  $k$  is a rule exploring a new seed  $\mu_k(\tilde{\mathbf{x}}, \tilde{B}) = (\tilde{\mathbf{x}}', \tilde{B}')$  such that the entries of the matrix  $\tilde{B}'$  are given by

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}, & \text{otherwise;} \end{cases} \quad (4.1)$$

and the tuple  $\tilde{\mathbf{x}}' = (x'_1, \dots, x'_m)$ , where  $x'_i = x_i$  if  $i \neq k$  and

$$x_k x'_k = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}.$$

Two seeds are said to be *mutation equivalent* if one of them can be obtained from the other one by a sequence of mutations.

*Remark 4.1.6.* It is not hard to verify that  $\mu_k(\tilde{\mathbf{x}}, \tilde{B})$  satisfies the definition of a seed. More-

over, one can easily check that  $\mu_k$  is an *involution*, that is,

$$\mu_k(\mu_k(\tilde{\mathbf{x}}, \tilde{B})) = (\tilde{\mathbf{x}}, \tilde{B}).$$

*Remark 4.1.7.* For an ambient field  $\mathcal{F}$ , start with an *initial seed*  $(\tilde{\mathbf{x}}, \tilde{B})$ . It is known that any mutable variable can be obtained from  $(\tilde{\mathbf{x}}, \tilde{B})$  by some sequence of mutations at some mutable indices. Therefore, the initial seed gives a full picture of the list of the extended clusters.

**Definition 4.1.8.** Let  $(\tilde{\mathbf{x}}, \tilde{B})$  be a seed whose ambient field is  $\mathcal{F}$ . A *cluster algebra (of geometric type)* is the subring  $\mathcal{A}$  of  $\mathcal{F}$  generated by the frozen variables and all possible mutable variables.

*Remark 4.1.9.* Since an initial seed  $(\tilde{\mathbf{x}}, \tilde{B})$  provides full information about its corresponding cluster algebra, we shall denote the latter by  $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B})$ .

**Definition 4.1.10.** A cluster algebra  $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B})$  is said to be *of finite type* if it has a finite number of mutable variables. Otherwise it is said to be *of infinite type*.

## 4.2. A cluster structure on $\mathbb{C}[N_K]$

In [13] and [14], Goodearl and Yakimov constructed cluster algebra structures on the coordinate rings of all Schubert cells of complex semisimple algebraic groups. Thus, since the coordinate ring of any cell is the quotient of the coordinate ring of some flag variety by some generalized flag minors, it is so obvious that the result of Goodearl and Yakimov can play an important role in this paper. Their results were based on Poisson geometry and so we capture here the main elements that we need from their work. More details about the relation between Poisson geometry and cluster algebras can be found in [11] and [13].

**Definition 4.2.1.** A *Poisson bracket*  $\{-, -\}$  is a Lie bracket that is a derivation also in each variable for the associative products. A commutative algebra  $R$  together with a Poisson bracket is called *Poisson algebra*. For  $a \in R$  the *Hamiltonian associated* with  $a$  is the derivation  $\{a, -\}$ . An ideal  $I$  of  $R$  such that  $\{R, I\} \subset I$  is called *Poisson ideal*.

*Remark 4.2.2.* The Poisson bracket of a Poisson algebra  $R$  induces a Poisson bracket on any quotient of  $R$  by a Poisson ideal.

**Definition 4.2.3.** Define the *Poisson-Ore extensions* to be  $B[x; \sigma, \delta]_p$  where  $B$  is a Poisson algebra,  $B[x; \sigma, \delta]_p = B[x]$  is a polynomial ring and  $\sigma, \delta$  are suitable Poisson derivations on  $B$  such that for any  $b \in B$  we have

$$\{x, b\} = \sigma(b) + \delta(x).$$

For an iterated Poisson-Ore extension

$$R = \mathbb{K}[x_1]_p[x_2; \sigma_2, \delta_2]_p \cdots [x_m; \sigma_m, \delta_m]_p$$

and  $k \in [0, m]$ , define

$$R_k = \mathbb{K}[x_1, \dots, x_k] = \mathbb{K}[x_1]_p[x_2; \sigma_2, \delta_2]_p \cdots [x_k; \sigma_k, \delta_k]_p,$$

where  $R_0 = \mathbb{K}$ .

**Definition 4.2.4.** A *Poisson-CGL extension* is an iterated Poisson-Ore extension  $R$  as above that is endowed with a rational Poisson action of a torus  $\mathcal{H}$  in which the elements  $x_1, \dots, x_k$  are  $\mathcal{H}$ -eigenvectors, the map  $\delta_k$  is locally nilpotent on  $R_{k-1}$  for any  $k \in [2, m]$  and such that for any  $k \in [1, m]$  there is an  $h_k \in \text{Lie}\mathcal{H}$  such that  $\sigma_k = h_k|_{R_{k-1}}$  and the  $h_k$ -eigenvalue of  $x_k$  nonzero and denoted by  $\lambda_k$ .



**Definition 4.2.5.** Let  $R$  be a Noetherian Poisson domain. An element  $p \in R$  is called a *Poisson-prime element* if any of the following equivalent conditions hold: (1) the ideal  $(p)$  is a prime ideal and it is a Poisson ideal; (2) the element  $p$  is a prime element of  $R$  such that  $p \nmid \{p, -\}$ ; (3) [in the case  $\mathbb{K} = \mathbb{C}$ ]: The element  $p$  is a prime element of  $R$  and the zero locus  $V(p)$  is a union of symplectic leaves of the maximal spectrum of  $R$ .

One of the great successes is due to the work of Goodearl, Yakimov when they proved the following:

**Theorem 4.2.6.** *Every symmetric Poisson-CGL extension  $R$  such that  $\lambda_l/\lambda_j \in \mathbb{Q}_{>0}$  for all  $l, j$  has a canonical cluster algebra structure that coincides with its upper cluster algebra.*

*Remark 4.2.7.* The cluster variables in the constructions of Goodearl, Yakimov are the unique homogeneous Poisson-prime elements of Poisson-CGL (sub)extensions not belonging to smaller subextensions. The mutation matrices of their seeds can be computed using linear systems of equations that come from the Poisson structure.

A significant consequence of the work of Goodearl, Yakimov is:

**Theorem 4.2.8.** *The coordinate ring  $\mathbb{C}[N_K]$  has a canonical cluster algebra structure.*

*Proof.* Throughout this proof, the notation  $e_k$  means the  $k$ th vector of the standard basis of  $\mathbb{Z}^m$ , the notation  $a[j, k]$  is given by

$$a[j, k] := \|(w_{[j, k]} - 1)\varpi_{i_k}\|^2/4 \in \frac{1}{2}\mathbb{Z},$$

and the notation  $S(w)$  is the *support of  $w$*  and is given by

$$S(w) := \{i \in I \mid s_i \leq w\} = \{i \in I \mid i = i_k \text{ for some } k \in [1, m]\}.$$

Also, set

$$p(k) := \begin{cases} \max\{j < k \mid i_j = i_k\}, & \text{if such } j \text{ exists;} \\ -\infty, & \text{otherwise.} \end{cases}$$

$$s(k) := \begin{cases} \min\{j > k \mid i_j = i_k\}, & \text{if such } j \text{ exists;} \\ \infty, & \text{otherwise.} \end{cases}$$

From Theorem 7.3 in [14] we know that the quantum Schubert cell, denoted by  $A_q(\mathbf{n}_+(w))_{\mathcal{A}^{1/2}}$ , has the quantum cluster structure given by the equation

$$A_q(\mathbf{n}_+(w))_{\mathcal{A}^{1/2}} = \mathbf{A}(M^w, \tilde{B}^w, \emptyset)_{\mathcal{A}^{1/2}} = \mathbf{U}(M^w, \tilde{B}^w, \emptyset)_{\mathcal{A}^{1/2}},$$

where the extended cluster variables are given by

$$M^w(e_j) = q^{a[1,j]} D_{\varpi_{i_j}, w \leq j \varpi_{i_j}},$$

for all  $j \in [1, m]$ , in which

$$D_{\varpi_j, w(\varpi_j)} = \text{proj}(\Delta_{\varpi_j, w(\varpi_j)}),$$

in which the frozen variables are the ones indexed by  $j \in [1, m]$  such that  $s(j) = \infty$ . The map

$$\text{proj} : \mathbb{C}[G/P_{\bar{K}}] \rightarrow \mathbb{C}[N_K]$$

denotes the standard projection from  $\mathbb{C}[G/P_{\bar{K}}]$  to  $\mathbb{C}[N_K]$ . The exchange matrix  $\tilde{B}^w$  is of

size  $m \times (m - |S(w)|)$  and its  $j \times k$  entry is given by

$$(\tilde{B}^w)_{jk} = \begin{cases} 1, & \text{if } j = p(k), \\ -1, & \text{if } j = s(k), \\ a_{i_j i_k}, & \text{if } j < k < s(j) < s(k), \\ -a_{i_j i_k}, & \text{if } k < j < s(k) < s(j), \\ 0, & \text{otherwise;} \end{cases}$$

where the entry  $a_{i_j i_k}$  is the same  $i_j \times i_k$  entry of the Cartan matrix of the same type. Thus, by corollary 3.7 in [10], it follows that

$$\mathbb{C} \otimes A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}} \cong A(\tilde{B}^w).$$

On the other hand, by (4.7) in [21], we know that the left hand side is isomorphic to the quotient of  $A_q(\mathfrak{n}_+(w))_{\mathcal{A}^{1/2}}$  by  $(q - 1)$ . Consequently, we get the desired cluster structure in the classical case whose exchange matrix is  $\tilde{B}^w$  and cluster variables are  $D_{\varpi_{i_k}, w \leq_k \varpi_{i_k}}$ .  $\square$

## Chapter 5. Cluster Algebra Structure on $\mathbb{C}[G/P_K^-]$

### 5.1. Grassmanians and cluster algebras

Recall that a Grassmannian  $\text{Gr}_{k,n}(V)$  is the variety of  $k$ -dimensional subspaces of an  $n$ -dimensional vector space  $V$ . Throughout, we fix  $V = \mathbb{C}^n$  and we write  $\text{Gr}_{k,n}$  for  $\text{Gr}_{k,n}(\mathbb{C}^n)$ . Recall also that  $\text{Gr}_{k,n}$  is just a special case of partial flag varieties. This section describes the connection between cluster algebras and Grassmanians which was the first connection between cluster algebras and partial flag varieties in literature. Let us start with the following example.

**Example 5.1.1.** This example produces a way to equip the coordinate ring of the Grassmannian  $\text{Gr}_{2,n}$  with a cluster algebra. For a wider overview the reader is encouraged to see [3], [6] or [19]. Consider the octagon of Figure 5.1. A *triangulation* of the octagon is the maximal number of pairwise non-crossing diagonals. It is clear that any triangulation of the octagon produces exactly 5 non-crossing diagonals, one of them is the one in Figure 5.1. More generally, a *triangulation* of an  $m$ -polygon produces exactly  $m - 3$  non-crossing diagonals. Now, this octagon forms a combinatorial way of describing a seed whose frozen variables are the Plücker coordinates indexed by the sides of the octagon and whose cluster variables are the Plücker coordinates indexed by the non-crossing sides. The mutation of seeds here corresponds to diagonal flipping. For instance, flipping the diagonal  $\Delta_{58}$  to  $\Delta_{16}$  corresponds to another triangulation that is a mutation of the first one at the variable  $\Delta_{58}$ . Another example is to flip  $\Delta_{15}$  to  $\Delta_{38}$  and so on.

We now form a quiver attached to a triangulation. For a triangulation  $T$  of an  $n$ -gon, define the quiver  $Q(T)$  attached to  $T$  as follows. Let the sides of  $T$  label the frozen

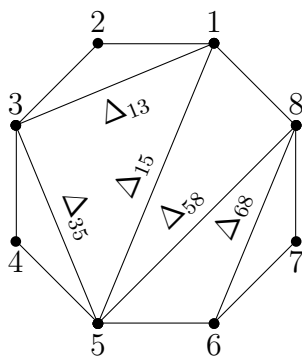


Figure 5.1. A triangulation of the octagon.

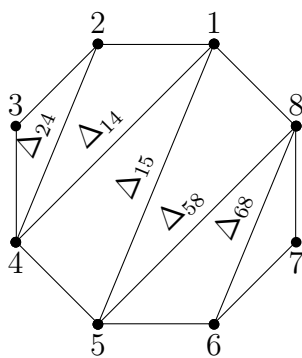


Figure 5.2. Another triangulation of the octagon.

vertices and the non-crossing diagonals label the mutable ones. Connect every two diagonals belonging to the same triangle by an edge oriented clockwise. Also, connect any diagonal and a boundary segment belonging to the same triangle by an edge oriented clockwise.

The following proposition was proved by Fomin and Zelevinsky in [6].

**Proposition 5.1.2.** *For  $n \geq 5$ , the homogeneous coordinate ring  $\mathbb{C}[\text{Gr}_{2,n}]$  is a cluster algebra whose seeds correspond to the triangulations of an  $n$ -gon. More precisely,  $\mathbb{C}[\text{Gr}_{2,n}]$  is a cluster algebra whose initial seed is given by the mutable and frozen variables and the quiver of a fixed triangulation  $T$  of the  $n$ -gon.*

In fact, this result was generalized by Scott in [19]. The construction is generalized as follows: Define the permutation  $\pi_{k,n} \in S_n$  by

$$\pi_{k,n} := \begin{pmatrix} 1 & \dots & n-k & n-k+1 & \dots & n \\ k+1 & \dots & n & 1 & \dots & k \end{pmatrix}.$$

The  $k$ -subsets that label the *boundary cells* of the  $\pi_{k,n}$ -diagram are always the intervals or boundary  $k$ -subsets

$$[1\dots k] \quad [2\dots k] \quad [3\dots k+2] \quad \dots \quad [n\dots k-1].$$

Therefore, any  $\pi_{k,n}$ -diagram contains  $n$  boundary cells exactly. Now, this setup is used to get the following results which all were proved in [19]:

**Theorem 5.1.3.** *Let  $k, n \in \mathbb{Z}_{\geq 2}$  such that  $n \geq k+2$ . There exists a  $\pi_{k,n}$ -diagram, denoted  $\mathbf{A}_{\mathbf{k},\mathbf{n}}$  whose internal even cells are quadrilateral. The family of  $k$ -subset labels that correspond to the internal cells consists of  $k$ -sets of  $[1\dots n]$  that can be expressed as a disjoint union  $I \sqcup I'$  of intervals  $I$  and  $I'$  begun with  $i$  and  $i'$  respectively, form a chord  $[ii']$  in the triangulation  $\mathbf{T}_{\mathbf{k},\mathbf{n}}$ .*

For two even regions  $I$  and  $J$  of a  $\pi_{k,n}$ -diagram  $\mathbf{A}$ , we say that  $I$  is *oriented towards*  $J$  and  $J$  is *oriented away from*  $I$  if the arrows of the region  $I$  goes towards the region  $J$ . Now, we define the matrix  $B(\mathbf{A})$  labeled by such  $I$ 's and  $J$ 's by:

$$b_{I,J} = \begin{cases} 1 & \text{if } I \text{ is oriented towards } J; \\ -1 & \text{if } I \text{ is oriented away from } J; \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 5.1.4.** *Every  $\pi_{k,n}$ -diagram  $\mathbf{A}$  gives rise to a seed of a cluster algebra  $\mathcal{A}_{k,n}$  whose mutable variables are indexed by the interior  $k$ -subsets and whose frozen variables are indexed by the  $k$ -subsets that label the boundaries and whose exchange matrix is  $B(\mathbf{A})$ .*

*Moreover, if  $\mathbf{A}'$  differs from  $\mathbf{A}$  by a single geometric exchange through a quadrilateral cell  $K$  then*

$$\mu_K(B(\mathbf{A})) = B(\mathbf{A}').$$

**Theorem 5.1.5.** *The map sending each variable of  $\mathcal{A}_{k,n}$  denoted by  $K$  to the Plücker coordinate  $\Delta_K$  of  $\mathbb{C}[\text{Gr}_{k,n}]$  is an isomorphism. Consequently,  $\mathbb{C}[\text{Gr}_{k,n}]$  is a cluster algebra.*

**Corollary 5.1.6.** *Let  $\mathbf{A}$  be a  $\pi_{k,n}$ -diagram. The multi-homogeneous coordinate ring  $\mathbb{C}[\text{Gr}_{k,n}]$  is a cluster algebra whose initial seed consists of the following data: mutable variables  $\Delta_K$  labeled by interior  $k$ -subsets, frozen variables  $\Delta_K$  labeled by boundary  $k$ -subsets and the exchange matrix  $B(\mathbf{A})$ .*

## 5.2. Simply-laced case

On a higher level of generality, the cluster algebras and partial flag varieties meet each other again in 2008 in a paper of Geiß, Leclerc and Schröer [7]. For  $G$  of type  $A$ ,  $D$  and  $E$ , they assigned a cluster algebra to the coordinate ring of the Schubert cell  $\mathbb{C}[N_K]$

and then they lift this cluster algebra to a cluster algebra living in  $\mathbb{C}[G/P_K^-]$ . The claim that  $\mathbb{C}[N_K]$  is a cluster algebra was conjectural in [7]. Later on, the same authors proved this conjecture for the simply-laced cases in [9]. As we saw in section 4.2, this conjecture has been fully proved by Goodearl and Yakimov in [13]. This section gives a summary of the results of [7].

*Remark 5.2.1.* Let  $(\tilde{\mathbf{x}}, \tilde{B})$  be a seed of the cluster algebra  $\mathcal{A} = \mathbb{C}[N_K]$ . Mutate at  $k$  to get the exchange relation

$$x_k x'_k = M_k + L_k,$$

where  $M_k, L_k$  are monomials in the variables  $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n$ . As shown in [7], this lifts to the equation

$$\widetilde{x_k x'_k} = \mu_k \widetilde{M_k} + \nu_k \widetilde{L_k},$$

where  $\mu_k$  and  $\nu_k$  are relatively prime monomials in the variables  $\Delta_{\varpi_j, \varpi_j}$ , and  $j$  runs in  $J$ .

This means that  $\mu_k$  and  $\nu_k$  can be written as

$$\mu_k = \prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{\alpha_j} \quad \text{and} \quad \nu_k = \prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{\beta_j},$$

such that  $\min\{\alpha_j, \beta_j\} = 0$  for all  $j$ .

**Theorem 5.2.2.** *Let  $G$  be simply-laced. Let  $(\mathbf{x}, B)$  be the initial seed of the cluster algebra  $\mathcal{A} = \mathbb{C}[N_K]$  defined in Theorem 4.2.8. Lift each variable of  $\mathbf{x}$  by the tilde map defined in Lemma 3.1.1 and preserve its type (mutable or frozen). Add the minors modded out in  $\mathbb{C}[N_K]$ , that is, the ones labeled by  $J$ , as frozen variables. Extend the matrix  $B$  by  $|J|$  rows labeled by  $J$  as follows:*

$$\widehat{b}_{jk} = \begin{cases} \beta_j, & \text{if } \beta_j \neq 0; \\ -\alpha_j, & \text{else,} \end{cases}$$



where  $\alpha_j$  and  $\beta_j$  are as in Remark 5.2.1. The data above defines an initial seed of a cluster algebra  $\hat{\mathcal{A}} \subset \mathbb{C}[G/P_K^-]$ .

**Conjecture 5.2.3.** *The localizations of the cluster algebra  $\hat{\mathcal{A}}$  and the coordinate ring  $\mathbb{C}[G/P_K^-]$ , by the minors indexed by  $j \in J$  such that  $\varpi_j$  is non-minuscule, are equal.*

*Remark 5.2.4.* The previous conjecture has been proved for type  $A_n$  and  $D_4$  in [7]. The other cases remained open. Indeed, since there are no non-minuscule  $\varpi_j$ 's for type  $A_n$ , actually the work of [7] showed that  $\hat{\mathcal{A}} = \mathbb{C}[G/P_K^-]$ , in type  $A$ .

### 5.3. General case

As we have seen, in the work of Geiß, Leclerc and Schröer [7], it was proved that  $\mathbb{C}[G/P_K^-]$  admits a cluster structure, up to localization, if  $G$  is simply-laced of type  $A_n$  or  $D_4$ . Their work motivates our construction here. The idea is to translate their work, which was in terms of categorification, to another language that works in the general case.

*Notation.* The cluster structure on  $\mathbb{C}[N_K]$  constructed by the work of Goodearl and Yakimov will be denoted by  $\mathcal{A}_J$ , where  $J$  and  $K$  are as defined before. We may write  $\mathcal{A}$  instead of  $\mathcal{A}_J$  if the context is clear.

**Definition 5.3.1.** A *lift* of a cluster algebra  $\mathcal{A}$  is a cluster algebra  $\tilde{\mathcal{A}}$  such that  $\mathcal{A}$  is a quotient algebra of it. Alternatively, we may say that  $\mathcal{A}$  *can be lifted* to  $\tilde{\mathcal{A}}$ .

**Definition 5.3.2.** For any seed  $(\mathbf{x}, B)$  of the cluster algebra  $\mathcal{A}_J = \mathbb{C}[N_K]$  constructed in [13], define a new pair  $(\hat{\mathbf{x}}, \hat{B})$  of  $\mathbb{C}[G/P_K^-]$  by raising each variable  $x$  of  $(\mathbf{x}, B)$  to the variable  $\tilde{x}$  (see Lemma 3.1.3) preserving the same type (mutable or frozen) and by adding the generalized minors  $\Delta_{\varpi_j, \varpi_j}$  modded out in  $\mathbb{C}[N_K]$  as frozen variables. The matrix  $\hat{B}$  of this lift is obtained as follows: Extend the matrix  $B$  of the construction of Goodearl and Yakimov

mov [13] by  $|J|$  rows labeled by the elements of  $J$  such that the entries are

$$\widehat{b}_{jk} = \begin{cases} \beta_j, & \text{if } \beta_j \neq 0; \\ -\alpha_j, & \text{else,} \end{cases}$$

where  $\alpha_j$  and  $\beta_j$  are as in Remark 5.2.1.

**Theorem 5.3.3.** *Let  $\{(\mathbf{x}, B)\}$  be the collection of seeds of the cluster algebra  $\mathcal{A}_J$  of  $\mathbb{C}[N_K]$ . The corresponding collection  $\{(\widehat{\mathbf{x}}, \widehat{B})\}$  constructed above forms a valid collection of seeds. In other words, if  $(\mathbf{x}, B)$  and  $(\mathbf{x}', B')$  are two seeds of the coordinate ring of the cell  $\mathbb{C}[N_K]$  such that  $(\mathbf{x}', B') = \mu_k(\mathbf{x}, B)$ , then correspondingly  $(\widehat{\mathbf{x}}', \widehat{B}') = \mu_k(\widehat{\mathbf{x}}, \widehat{B})$ .*

*Proof.* Let  $k$  be a mutable index in the construction of [13]. We need to show that  $\mu_k(\widehat{B}) = \widehat{B}'$ . In other words, we need to show that the matrix entries of the mutation of  $\widehat{B}$  match the ones coming from our construction (definition 5.3.2) using the mutation equations of the mutated seed  $(\mathbf{x}', B')$ . The equations are of the form

$$\widetilde{x}'_t \widetilde{x}''_t = \mu'_t \widetilde{M}'_t + \nu'_t \widetilde{L}'_t,$$

where  $x'_t$  denotes the  $t$ th variable in the mutated extended cluster in a direction  $k$  and  $x''_t$  denotes the  $t$ th variable coming from a second mutation in a direction  $t$ . Let  $\widehat{b}'_{st}$  denote the entry of position  $s \times t$  in  $\mu_k(\widehat{B})$ . Obviously, if  $s \notin J$  then  $\widehat{b}'_{st}$  equals the  $s \times t$  entry of  $\mu_k(B)$ , as the entries of  $\widehat{B}$  and  $B$  match when  $s \notin J$ . Consequently, the entries of the mutation of both of them coincide again when  $s \notin J$ . Assume now that  $s \in J$  and  $t = k$ . Then by the fact that the construction of [13] is indeed a cluster algebra, we get that  $M'_k = L_k$  and  $L'_k = M_k$ . This clearly makes  $\alpha'_j = \beta_j$  and  $\beta'_j = \alpha_j$ . Since  $\mu'_k$  and  $\nu'_t$  are relatively prime, we see easily from the construction that the entry we get is  $-\widehat{b}_{st}$  which equals  $\widehat{b}'_{st}$  by the mutation formula.

Assume now that  $t$  is a mutable index other than  $k$ . It suffices to show that in

$$\widetilde{x'_t x''_t} = \mu'_t \widetilde{M'_t} + \nu'_t \widetilde{L'_t},$$

the exponents of the minors of the monomials  $\mu'_t$  and  $\nu'_t$  match the formula of the matrix mutation. Equivalently, we may assume that  $\mu'_t$  and  $\nu'_t$  are as we want and then show that  $\mu'_t \widetilde{M'_t} + \nu'_t \widetilde{L'_t}$ , is an element whose  $\text{proj}$  is  $M'_t + L'_t$  and whose order is minimal with respect to  $\preceq$ . The first property is straightforward. Now,

$$\lambda(M'_t + L'_t) = \sum_{j \in J} a_j(M'_t + L'_t) \varpi_j = \sum_{j \in J} a_j \varpi_j \quad (5.1)$$

where

$$\begin{aligned} a_j &= a_j(M'_t + L'_t) \\ &= \max\{s \mid (e_j^\dagger)^s (M'_t + L'_t) \neq 0\} \\ &= \max \left\{ s \mid (e_j^\dagger)^s \left( \prod_{b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2} > 0} x_i'^{b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2}} + \prod_{b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2} < 0} x_i'^{-\left(b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2}\right)} \right) \neq 0 \right\} \\ &= \max \left\{ \begin{aligned} &\sum_{b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2} > 0} a_j \left( x_i'^{b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2}} \right), \\ &\sum_{b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2} < 0} a_j \left( x_i'^{-\left(b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2}\right)} \right) \end{aligned} \right\}. \end{aligned}$$

Note here that the last equality is obtained by the fact that  $a_j(fg) = a_j(f) + a_j(g)$ . Assume now, for some  $j$ , that the first sum is the maximum. Then, using the same fact once

again, we clearly get that

$$a_j = \sum_i \left( b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2} \right) a_j(x'_i).$$

Recall also that  $x'_i = x_i$  for  $i \notin \{k, t\}$ . So,

$$a_j = \sum_i \left( b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2} \right) a_j(x_i).$$

But by equation (5.1), it follows that

$$\begin{aligned} \lambda(M'_t + L'_t) &= \sum_{j \in J} a_j \varpi_j \\ &= \sum_{j \in J} \sum_i \left( \left| b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2} \right| a_j(x_i) \right) \varpi_j. \end{aligned}$$

Now, since  $\mathbb{C}[G/P_K^-]$  is graded by the set of sums of the form  $\sum_{i \in \mathbb{N}} a_i \varpi_i$ , the last equation implies that

$$\begin{aligned} L(\lambda(M'_t + L'_t)) &= L \left( \sum_{j \in J} \sum_i \left( \left| b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2} \right| a_j(x_i) \right) \varpi_j \right) \\ &\supset \prod_{j \in J} \prod_i L \left( \left| b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2} \right| a_j(x_i) \varpi_j \right) \\ &= \prod_{j \in J} \prod_i L(d_i a_j(x_i) \varpi_j), \end{aligned}$$

where

$$d_i := \left| b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2} \right|.$$

So, we get that

$$L(\lambda(M'_t + L'_t)) \supset \prod_{j \in J} \prod_i \underbrace{L(a_j(x_i) \varpi_j) \cdots L(a_j(x_i) \varpi_j)}_{d_i \text{ times}}. \quad (5.2)$$

Note here that the sum over  $i$  is the sum over the positive  $d_i$ 's only or the negative  $d_i$ 's only. A similar work with  $L\left(\lambda\left(M_k + L_k\right)\right)$  shows that

$$\begin{aligned} L\left(\lambda\left(M_k + L_k\right)\right) &= L\left(\sum_{j \in J} \sum_i \left(|b_{ik}| a_j(x_i)\right) \varpi_j\right) \\ &\supset \prod_{j \in J} \prod_i L\left(|b_{ik}| a_j(x_i) \varpi_j\right). \end{aligned}$$

This implies that

$$L\left(\lambda\left(M_k + L_k\right)\right) \supset \prod_{j \in J} \prod_i \underbrace{L(\varpi_j) \cdots L(\varpi_j)}_{|b_{ik}| a_j(x_i) \text{ times}}.$$

But since  $\Delta_{\varpi_j, \varpi_j}$  is of degree  $\varpi_j$ , it follows that the possible occurrences of the exponents of  $\Delta_{\varpi_j, \varpi_j}$  are the integers

$$0, 1, 2, \dots, \sum_i a_j(x_i) |b_{ik}|.$$

However, the minimality of

$$a_j(\lambda(M_k + L_k)) = \sum_{j \in J} \sum_i |b_i| a_j(x_i)$$

shows that the only possible solution is  $\sum_i a_j(x_i) |b_{ik}|$ , because the rest are still available in some  $L(\lambda)$ 's in which  $\lambda$  is less than  $\lambda(M_k + L_k)$ . Consequently, the only possibilities for  $\alpha_j$  and  $\beta_j$  are 0 or  $\sum_i a_j(x_i) |b_{ik}|$ . But, thank to the homogeneity of the construction of [13], there exists a unique  $j$  in which  $a_j(x_i) \neq 0$ . Hence, one of  $\alpha_{jk}$  and  $\beta_{jk}$  is  $a_j(x_i) |b_{ik}|$  and the other is 0.

Therefore, for every  $i$  there is a unique  $j$  in which one of the following must be true:

$$a_j(x_i) b_{it} = \pm \alpha_{jt}, \quad a_j(x_i) b_{ik} = \pm \alpha_{jk} \quad \text{and} \quad a_j(x_i) |b_{ik}| = \alpha_{jk},$$

or

$$a_j(x_i) b_{it} = \pm \beta_{jt}, \quad a_j(x_i) b_{ik} = \pm \beta_{jk} \quad \text{and} \quad a_j(x_i) |b_{ik}| = \beta_{jk}.$$

Now,

$$\begin{aligned} \left( \Delta_{\varpi_j, \varpi_j}^{a_j(x_i)} \widetilde{x}_i \right)^{d_i} &\in L(d_i a_j(x_i) \varpi_j) \\ \left( \Delta_{\varpi_j, \varpi_j}^{a_j(x_i)} \widetilde{x}_i \right)^{b_{it} + \frac{|b_{ik}|b_{kt} + b_{ik}|b_{kt}|}{2}} &\in L(d_i a_j(x_i) \varpi_j). \end{aligned}$$

It is not hard now to see that  $\left( \Delta_{\varpi_j, \varpi_j}^{a_j(x_i)} \widetilde{x}_i \right)^{d_i}$  forms one factor of the monomial  $\mu'_t \widetilde{M}'_t$  or the monomial  $\nu'_t \widetilde{L}'_t$ . The rest are similarly there. Since  $L(\lambda(M'_t + L'_t))$  and  $L(\lambda(M_k + L_k))$  are homogeneous ideals and since

$$\widetilde{x_k x'_k} = \widetilde{x_k} \widetilde{x'_k} = \mu_k \widetilde{M}_k + \nu_k \widetilde{L}_k \in L(\lambda(M_k + L_k)),$$

it is again not difficult to combine these information to see that

$$\widetilde{x'_t x''_t} = \widetilde{x'_t} \widetilde{x''_t} = \mu'_t \widetilde{M}'_t + \nu'_t \widetilde{L}'_t \in L(\lambda(M'_t + L'_t)).$$

This completes the proof. □

*Notation.* The cluster algebra contained in  $\mathbb{C}[G/P_K^-]$  and obtained from the preceding theorem will be denoted by  $\widehat{\mathcal{A}}_J$  or simply  $\widehat{\mathcal{A}}$  if the context is clear.

**Corollary 5.3.4.** *Let  $B$  be the matrix  $\widetilde{B}^w$  of Theorem 4.2.8. The pair*

$$\left( \left\{ \widetilde{D}_{\varpi_{i_k}, w_{\leq k} \varpi_{i_k}} \right\} \sqcup \{ \Delta_{\varpi_j, \varpi_j} \mid j \in J \}, \widehat{B} \right)$$

*is an initial seed of the cluster algebra  $\widehat{\mathcal{A}} \subset \mathbb{C}[G/P_K^-]$ . Moreover, the lift  $\widetilde{D}_{\varpi_{i_k}, w_{\leq k} \varpi_{i_k}}$  of*

*$D_{\varpi_{i_k}, w_{\leq k} \varpi_{i_k}}$  is given explicitly by the formula*

$$\widetilde{D}_{\varpi_{i_k}, w_{\leq k} \varpi_{i_k}} = \frac{\Delta_{\varpi_{i_k}, w_{\leq k} \varpi_{i_k}} \Delta_{\varpi_{i_1}, \varpi_{i_1}}^{d_k}}{\Delta_{\varpi_{i_k}, \varpi_{i_k}}}.$$

*Proof.* Apply the construction of the previous theorem to the initial seed  $(D_{\varpi_{i_k}, w_{\leq k} \varpi_{i_k}}, \tilde{B}^w)$  of  $\mathbb{C}[N_K]$  (see Theorem 4.2.8). The mutable and frozen variables are described in Definition 5.3.2. Also, the explicit formula of the lift is just a direct application of Corollary 3.2.8.  $\square$

*Remark 5.3.5.* In the simply-laced case, it is obvious that the construction of  $\widehat{\mathcal{A}}_J$  matches the one of [7].

*Remark 5.3.6.* By construction, it is clear that the extended clusters of  $\mathcal{A}$  and  $\widehat{\mathcal{A}}$  are in one-to-one correspondence. So,  $\mathcal{A}$  and  $\widehat{\mathcal{A}}$  must be of the same type (either both finite or both infinite).

**Theorem 5.3.7.** *The localization of the homogeneous coordinate ring of the flag variety  $\mathbb{C}[G/P_K^-]$  by  $\Delta_{\varpi_j, \varpi_j}$ , ( $j \in J$ ) equals the localization of the cluster algebra  $\widehat{\mathcal{A}}$  by the same elements. Namely,*

$$\mathbb{C}[G/P_K^-][\Delta_{\varpi_j, \varpi_j}^{-1}]_{j \in J} = \widehat{\mathcal{A}}[\Delta_{\varpi_j, \varpi_j}^{-1}]_{j \in J}.$$

*Proof.* Throughout the proof, for any element in  $\mathbb{C}[G/P_{I \setminus \{j\}}^-]$  the term degree will be used to refer to the homogeneous degree of it. Recall that  $\mathbb{C}[G/P_K^-] = \bigoplus_{\lambda \in \Pi_J} L(\lambda)$  is generated as a ring by the subspaces  $L(\varpi_j) \subset \mathbb{C}[G/P_{I \setminus \{j\}}^-]$ . Thus, it is generated by  $\mathbb{C}[G/P_{I \setminus \{j\}}^-]$ , where  $j \in J$ . It is proved in [7] that if  $J' \subset J$ , then  $\widehat{\mathcal{A}}_{J'}$  is a subalgebra of  $\widehat{\mathcal{A}}_J$ . Their proof is in fact true for all complex semisimple algebraic groups, not just simply-laced ones. Therefore, the result follows if localization of  $\mathbb{C}[G/P_{I \setminus \{j\}}^-]$  by  $\Delta_{\varpi_j, \varpi_j}$  is contained in the localization of  $\widehat{\mathcal{A}}_{\{j\}}$ , by the same element. We proceed by contradiction. Let  $f \in \mathbb{C}[G/P_{I \setminus \{j\}}^-]$  such that  $f \notin \widehat{\mathcal{A}}_{\{j\}}$  and its degree is minimal. Let  $g = \text{proj}(f) \in \mathbb{C}[N_{I \setminus \{j\}}]$ . Then  $\text{proj}(\tilde{g} - f) = 0$ . Thus, we have that  $\tilde{g} - f$  belongs to the principal ideal  $(\Delta_{\varpi_j, \varpi_j} - 1)$ ,

since

$$\mathbb{C}[N_{I \setminus \{j\}}] = \mathbb{C}[G/P_{I \setminus \{j\}}^-] / (\Delta_{\varpi_j, \varpi_j} - 1).$$

Consequently, there is some  $h \in \mathbb{C}[G/P_{I \setminus \{j\}}^-]$  such that

$$\tilde{g} - f = h(\Delta_{\varpi_j, \varpi_j} - 1)$$

$$\tilde{g} - f = h\Delta_{\varpi_j, \varpi_j} - h.$$

But note that the definition of  $\tilde{g}$  and the choice of  $f$  imply that the degree of the whole left-hand side is less than or equal to degree of  $f$ . On the other hand, it is obvious that the degree of the right-hand side is the degree of  $h$  plus  $\varpi_j$ . It follows that the degree of  $h$  is less than the one of  $f$ . Therefore, by minimality, we get that  $h \in \widehat{\mathcal{A}}_{\{j\}}$ . Also, since  $\Delta_{\varpi_j, \varpi_j} \in \widehat{\mathcal{A}}_{\{j\}}$ , it follows that  $h\Delta_{\varpi_j, \varpi_j} \in \widehat{\mathcal{A}}_{\{j\}}$ . Now, if the lifting  $\tilde{g} \in \widehat{\mathcal{A}}_{\{j\}}$ , we have

$$f = \underbrace{\tilde{g}}_{\in \widehat{\mathcal{A}}_{\{j\}}} - \underbrace{h\Delta_{\varpi_j, \varpi_j}}_{\in \widehat{\mathcal{A}}_{\{j\}}} + \underbrace{h}_{\in \widehat{\mathcal{A}}_{\{j\}}} \in \widehat{\mathcal{A}}_{\{j\}} \subset \widehat{\mathcal{A}}_{\{j\}}[\Delta_{\varpi_j, \varpi_j}^{-1}], \quad (5.3)$$

which is a contradiction to  $f$  being outside  $\widehat{\mathcal{A}}_{\{j\}}$ . Therefore,  $\tilde{g} \notin \widehat{\mathcal{A}}_{\{j\}}$ . Now, write  $g \in \mathbb{C}[N_{I \setminus \{j\}}]$  as  $g = \sum_{i=1}^r c_i m_i$ , where each  $m_i$  is a product of cluster variables (might not be from the same seed) and each  $c_i$  is a scalar. This forces the  $m_i$ 's to be distinct. Recall that  $\mathbb{C}[N_K]$  can be identified with

$$\mathbb{C}[N_K] = \left\{ \frac{f}{\prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{a_j}} \mid f \in L\left(\sum_{j \in J} a_j \varpi_j\right) \right\}.$$

By the uniqueness and minimality of the tilde map in Lemma 3.1.1, this can be refined to

$$\mathbb{C}[N_K] = \left\{ \frac{f}{\prod_{j \in J} \Delta_{\varpi_j, \varpi_j}^{a_j}} \mid f \in L\left(\sum_{j \in J} a_j \varpi_j\right) \text{ and the } a_j \text{'s are minimal} \right\}.$$

As  $J = \{j\}$ , we can use the second identification to write each  $m_i$  as  $\frac{f_i}{\Delta_{\varpi_j, \varpi_j}^{a_{i,j}}}$ , where  $f_i \in$



$L(a_{i,j}\varpi_j)$  and  $a_{i,j}$  is minimal with this property. Clearly, the degree  $d_i$  of  $\widetilde{m}_i$  is  $a_{i,j}\varpi_j$ . It is not hard to see that  $\widetilde{m}_i = f_i$  for all  $i = 1, \dots, r$ .

As distinct elements lift by the tilde map to distinct elements by Lemma 3.1.1, we get that the  $\widetilde{m}_i$ 's are distinct. Consequently, the  $f_i$ 's are distinct. Let  $a_j := \max \{a_{i,j} \mid i = 1, \dots, r\}$ . Now, if

$$\sum_{i=1}^r \left( c_i \Delta_{\varpi_j, \varpi_j}^{a_j - a_{i,j}} f_i \right) \quad (5.4)$$

is a lift of  $g = \sum_{i=1}^r c_i m_i$  in the minimal way, then we get that  $\tilde{g} \in \widehat{\mathcal{A}_{\{j\}}}$ , which is a contradiction. Otherwise, there is an element of lower degree in which  $\tilde{g}$  is equal to that element. Note that the multiplication of  $\tilde{g}$  by  $\Delta_{\varpi_j, \varpi_j}^{s_j}$  for some positive integer  $s_j$  gives an element whose degree is equal to the degree of the element in 5.4 and whose projection is equal to  $g$ . Since the projection of each homogeneous piece  $L(\lambda)$  to  $\mathbb{C}[N_K]$  is injective, we get that

$$\tilde{g} = \frac{\sum_{i=1}^r \left( c_i \Delta_{\varpi_j, \varpi_j}^{a_j - a_{i,j}} f_i \right)}{\Delta_{\varpi_j, \varpi_j}^{s_j}}.$$

Clearly, by (5.3) this implies again that  $f \in \widehat{\mathcal{A}_{\{j\}}}[\Delta_{\varpi_j, \varpi_j}^{-1}]$ . Consequently, the result follows, that is,

$$\mathbb{C}[G/P_K^-][\Delta_{\varpi_j, \varpi_j}^{-1}]_{j \in J} = \widehat{\mathcal{A}}[\Delta_{\varpi_j, \varpi_j}^{-1}]_{j \in J}.$$

□

**Conjecture 5.3.8.** *The homogeneous coordinate ring of the flag variety  $\mathbb{C}[G/P_K^-]$  equals the cluster algebra  $\widehat{\mathcal{A}}$ . In particular,  $\mathbb{C}[G/P_K^-]$  is a cluster algebra whose initial seed is*

$$\left( \left\{ \frac{\Delta_{\varpi_{i_k}, w \leq k \varpi_{i_k}} \Delta_{\varpi_{i_1}, \varpi_{i_1}}^{d_k}}{\Delta_{\varpi_{i_k}, \varpi_{i_k}}} \right\} \sqcup \{ \Delta_{\varpi_j, \varpi_j} \mid j \in J \}, \widehat{B} \right).$$

*Remark 5.3.9.* Using the proof of the previous theorem, the conjecture is equivalent to

proving that if  $g = \sum_{i=1}^r c_i m_i$  is written where the number of terms is minimal, then

$$\tilde{g} = \sum_{i=1}^r \left( c_i \Delta_{\varpi_j, \varpi_j}^{a_j - a_{i,j}} \widetilde{m}_i \right).$$

## Chapter 6. Theorems on the Explicit Structure of Cluster Algebras on Partial Flag Varieties

### 6.1. Explicit formulas

The results of the previous sections can be used to get these explicit examples.

**Example 6.1.1** (Type  $A$ , c.f. [7]). Let  $G$  be a type  $A_5$  semisimple algebraic group.

Namely,  $G = SL_6$ . Take  $J = \{1, 3\}$ ,  $K = \{2, 4, 5\}$  and consider the longest word

$$w_0 = s_2 s_4 s_5 s_4 s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 s_1 s_2 s_3.$$

Clearly, the subword  $w_K = s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 s_1 s_2 s_3$  generates  $N_K$ . Note that  $s(5) = s(8) = s(9) = s(10) = s(11) = \infty$ . Therefore, the list of extended cluster variables is

$$\begin{aligned} j = 1 &\implies D_{\varpi_1, s_1 \varpi_1}; & (\text{mutable}) \\ j = 2 &\implies D_{\varpi_2, s_1 s_2 \varpi_2}; & (\text{mutable}) \\ j = 3 &\implies D_{\varpi_3, s_1 s_2 s_3 \varpi_3}; & (\text{mutable}) \\ j = 4 &\implies D_{\varpi_4, s_1 s_2 s_3 s_4 \varpi_4}; & (\text{mutable}) \\ j = 5 &\implies D_{\varpi_5, s_1 s_2 s_3 s_4 s_5 \varpi_5}; & (\text{frozen}) \\ j = 6 &\implies D_{\varpi_2, s_1 s_2 s_3 s_4 s_5 s_2 \varpi_2}; & (\text{mutable}) \\ j = 7 &\implies D_{\varpi_3, s_1 s_2 s_3 s_4 s_5 s_2 s_3 \varpi_3}; & (\text{mutable}) \\ j = 8 &\implies D_{\varpi_4, s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 \varpi_4}; & (\text{frozen}) \\ j = 9 &\implies D_{\varpi_1, s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 s_1 \varpi_1}; & (\text{frozen}) \\ j = 10 &\implies D_{\varpi_2, s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 s_1 s_2 \varpi_2}; & (\text{frozen}) \\ j = 11 &\implies D_{\varpi_3, s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 s_1 s_2 s_3 \varpi_3}. & (\text{frozen}) \end{aligned}$$

The exchange matrix  $B$  is

$$B = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 6 & 7 \end{array} \\ \left( \begin{array}{cccccc} 0 & -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 1 & 1 & 0 \end{array} \right) \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 6 \\ 7 \end{array} \\ \hline \left( \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array} \right) \begin{array}{l} 5 \\ 8 \\ 9 \\ 10 \\ 11 \end{array} \end{array} .$$

Now, lifting each extended cluster variable by the tilde map, we get that the cluster

algebra  $\hat{\mathcal{A}} \subset \mathbb{C}[G/P_K^-]$  has the following extended cluster variables

$\widetilde{D}_{\varpi_1, s_1 \varpi_1};$	(mutable)
$\widetilde{D}_{\varpi_2, s_1 s_2 \varpi_2};$	(mutable)
$\widetilde{D}_{\varpi_3, s_1 s_2 s_3 \varpi_3};$	(mutable)
$\widetilde{D}_{\varpi_4, s_1 s_2 s_3 s_4 \varpi_4};$	(mutable)
$\widetilde{D}_{\varpi_5, s_1 s_2 s_3 s_4 s_5 \varpi_5};$	(frozen)
$\widetilde{D}_{\varpi_2, s_1 s_2 s_3 s_4 s_5 s_2 \varpi_2};$	(mutable)
$\widetilde{D}_{\varpi_3, s_1 s_2 s_3 s_4 s_5 s_2 s_3 \varpi_3};$	(mutable)
$\widetilde{D}_{\varpi_4, s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 \varpi_4};$	(frozen)
$\widetilde{D}_{\varpi_1, s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 s_1 \varpi_1};$	(frozen)
$\widetilde{D}_{\varpi_2, s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 s_1 s_2 \varpi_2};$	(frozen)
$\widetilde{D}_{\varpi_3, s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 s_1 s_2 s_3 \varpi_3};$	(frozen)
$\Delta_{\varpi_1, \varpi_1};$	(frozen)
$\Delta_{\varpi_3, \varpi_3}.$	(frozen)

Moreover, Corollary 3.2.4 and Corollary 3.2.8 imply that

$$\begin{aligned}
\widetilde{D}_{\varpi_1, s_1 \varpi_1} &= \Delta_{\varpi_1, s_1 \varpi_1} \\
\widetilde{D}_{\varpi_2, s_1 s_2 \varpi_2} &= f_1 \\
\widetilde{D}_{\varpi_3, s_1 s_2 s_3 \varpi_3} &= \Delta_{\varpi_3, s_1 s_2 s_3 \varpi_3} \\
\widetilde{D}_{\varpi_4, s_1 s_2 s_3 s_4 \varpi_4} &= f_2 \\
\widetilde{D}_{\varpi_5, s_1 s_2 s_3 s_4 s_5 \varpi_5} &= f_3 \\
\widetilde{D}_{\varpi_2, s_1 s_2 s_3 s_4 s_5 s_2 \varpi_2} &= f_4 \\
\widetilde{D}_{\varpi_3, s_1 s_2 s_3 s_4 s_5 s_2 s_3 \varpi_3} &= \Delta_{\varpi_3, s_1 s_2 s_3 s_4 s_5 s_2 s_3 \varpi_3} \\
\widetilde{D}_{\varpi_4, s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 \varpi_4} &= f_5 \\
\widetilde{D}_{\varpi_1, s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 s_1 \varpi_1} &= \Delta_{\varpi_1, s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 s_1 \varpi_1} \\
\widetilde{D}_{\varpi_2, s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 s_1 s_2 \varpi_2} &= f_6 \\
\widetilde{D}_{\varpi_3, s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 s_1 s_2 s_3 \varpi_3} &= \Delta_{\varpi_3, s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 s_1 s_2 s_3 \varpi_3},
\end{aligned}$$

such that  $f_i$  is the appropriate unique function of Corollary 3.2.8. To calculate the degree of the  $f_1$ , consider the equation

$$\begin{aligned}
s_1(s_2 \varpi_2) &= s_2 \varpi_2 - (s_2 \varpi_2, \alpha_1^\vee) \alpha_1 \\
&= s_2 \varpi_2 - (\varpi_2 - \alpha_2, \alpha_1^\vee) \alpha_1 \\
&= s_2 \varpi_2 - \alpha_1.
\end{aligned}$$

Thus, the left degree is 1 and by Corollary 3.2.8 the homogeneous degree of  $f_1$  is  $\varpi_1$ .

Therefore,

$$f_1 = \frac{\Delta_{\varpi_2, s_1 s_2 \varpi_2} \Delta_{\varpi_1, \varpi_1}}{\Delta_{\varpi_2, \varpi_2}}$$

Similarly, one can get that the degree of  $f_2, \dots, f_5$  is  $\varpi_1$  as well. On the other hand, a similar calculation shows that  $f_6$  is of degree 0, but this is in terms of  $\varpi_1$ . Thus,

$$\begin{aligned} f_2 &= \frac{\Delta_{\varpi_4, s_1 s_2 s_3 s_4 \varpi_4} \Delta_{\varpi_1, \varpi_1}}{\Delta_{\varpi_4, \varpi_4}}; \\ f_3 &= \frac{\Delta_{\varpi_5, s_1 s_2 s_3 s_4 s_5 \varpi_5} \Delta_{\varpi_1, \varpi_1}}{\Delta_{\varpi_5, \varpi_5}}; \\ f_4 &= \frac{\Delta_{\varpi_2, s_1 s_2 s_3 s_4 s_5 s_2 \varpi_2} \Delta_{\varpi_1, \varpi_1}}{\Delta_{\varpi_2, \varpi_2}}; \\ f_5 &= \frac{\Delta_{\varpi_4, s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 \varpi_4} \Delta_{\varpi_1, \varpi_1}}{\Delta_{\varpi_4, \varpi_4}}. \end{aligned}$$

However, it is not hard to see that

$$s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 s_1 s_2 \varpi_2 = s_3 s_4 s_5 s_2 s_3 s_4 s_1 s_2 \varpi_2.$$

Also, one can easily check that

$$s_3 s_4 s_5 s_2 s_3 s_4 s_1 s_2 \varpi_2 = s_4 s_5 s_2 s_3 s_4 s_1 s_2 \varpi_2 - \alpha_3.$$

It follows that the homogeneous degree of  $f_6$  is  $\varpi_3$ . Consequently, by Corollary 3.2.8

$$f_6 = \frac{\Delta_{\varpi_2, s_3 s_4 s_5 s_2 s_3 s_4 s_1 s_2 \varpi_2} \Delta_{\varpi_3, \varpi_3}}{\Delta_{\varpi_2, \varpi_2}}.$$

Now, the lift of the exchange relations can be calculated easily. For instance, mutating at

$k = 1$ , we get

$$\begin{aligned} D_{\varpi_1, s_1 \varpi_1} D'_1 &= D_{\varpi_2, s_1 s_2 \varpi_2} D_{\varpi_2, s_1 s_2 s_3 s_4 s_5 s_2 \varpi_2} + D_{\varpi_1, s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 s_1 \varpi_1} \\ &= \frac{f_1}{\Delta_{\varpi_1, \varpi_1}} \frac{f_4}{\Delta_{\varpi_1, \varpi_1}} + \frac{\Delta_{\varpi_1, s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 s_1 \varpi_1}}{\Delta_{\varpi_1, \varpi_1}} \\ &= \frac{f_1 f_4 + \Delta_{\varpi_1, \varpi_1} \Delta_{\varpi_1, s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 s_1 \varpi_1}}{\Delta_{\varpi_1, \varpi_1}^2}. \end{aligned}$$

Thus, by Lemma 3.1.3 and Proposition 3.2.5

$$\widetilde{D_{\varpi_1, s_1 \varpi_1} D'_1} = f_1 f_4 + \Delta_{\varpi_1, \varpi_1} \Delta_{\varpi_1, s_1 s_2 s_3 s_4 s_5 s_2 s_3 s_4 s_1 \varpi_1}.$$

The other mutation exchange relations can be lifted similarly. Finally, we get that the exchange matrix attached to the cluster algebra  $\hat{\mathcal{A}} \subset \mathbb{C}[G/P_K^-]$  is

$$\hat{B} = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 6 & 7 \end{array} \\ \left( \begin{array}{cccccc} 0 & -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 1 & 1 & 0 \end{array} \right) \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 6 \\ 7 \end{array} \\ \hline \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array} \begin{array}{l} 5 \\ 8 \\ 9 \\ 10 \\ 11 \end{array} \\ \hline \begin{array}{cccccc} -1 & 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array} \begin{array}{l} 1 \in J \\ 3 \in J \end{array} \end{array}.$$

**Example 6.1.2** (Type  $B$ , c.f. [15]). Let  $G$  be a semisimple algebraic group of type  $B_3$ , say  $G = SO_{2(3)+1} = SO_7$ ,  $J = \{3\}$  and  $K = I \setminus J = \{1, 2\}$ . Consider the longest word

$$w_0 = s_1 s_2 s_1 s_3 s_2 s_1 s_3 s_2 s_3.$$



In [15], it was shown that  $\mathbb{C}[G/P_K^-]$  has a cluster structure whose initial extended cluster is given by the variables

$$\begin{array}{ll}
\widetilde{D}_{\varpi_3, s_3 \varpi_3}; & \text{(mutable)} \\
\widetilde{D}_{\varpi_2, s_3 s_2 \varpi_2}; & \text{(mutable)} \\
\widetilde{D}_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_3}; & \text{(mutable)} \\
\widetilde{D}_{\varpi_1, s_3 s_2 s_1 \varpi_1}; & \text{(frozen)} \\
\widetilde{D}_{\varpi_2, s_3 s_2 s_1 s_3 s_2 \varpi_2}; & \text{(frozen)} \\
\widetilde{D}_{\varpi_3, s_3 s_2 s_1 s_3 s_2 s_3 \varpi_3}; & \text{(frozen)} \\
\Delta_{\varpi_3, \varpi_3}. & \text{(frozen)}
\end{array}$$

Now, using Corollary 3.2.4 and Corollary 3.2.8 it is easily seen that

$$\begin{array}{ll}
\widetilde{D}_{\varpi_3, s_3 \varpi_3} & = \Delta_{\varpi_3, s_3 \varpi_3} \\
\widetilde{D}_{\varpi_2, s_3 s_2 \varpi_2} & = f \\
\widetilde{D}_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_3} & = \Delta_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_3} \\
\widetilde{D}_{\varpi_1, s_3 s_2 s_1 \varpi_1} & = g \\
\widetilde{D}_{\varpi_2, s_3 s_2 s_1 s_3 s_2 \varpi_2} & = h \\
\widetilde{D}_{\varpi_3, s_3 s_2 s_1 s_3 s_2 s_3 \varpi_3} & = \Delta_{\varpi_3, s_3 s_2 s_1 s_3 s_2 s_3 \varpi_3},
\end{array}$$

where  $f, g, h$  are the unique suitable functions gotten by Corollary 3.2.8. Moreover, the lifting of the mutation exchange relations of the cluster algebra  $\mathcal{A}$  becomes obvious now.

In fact, exchange relation induced by mutating at 1 is

$$D_{\varpi_3, s_3 \varpi_3} D'_1 = D_{\varpi_2, s_3 s_2 \varpi_2} + D_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_3}.$$

Now, to find the lift degree of  $D_{\varpi_2, s_3 s_2 \varpi_2}$  we consider the equation

$$s_3(s_2(\varpi_2)) = s_2(\varpi_2) - 2\alpha_3.$$

This shows that the lift degree of  $D_{\varpi_2, s_3 s_2 \varpi_2}$  is 2. Thus, it follows that the lift degree of

$f \in \mathbb{C}[G/P_K^-]$  is  $2\varpi_3$ . Moreover, by Corollary 3.2.8

$$f = \frac{\Delta_{\varpi_2, s_3 s_2 \varpi_2} \Delta_{\varpi_3, \varpi_3}^2}{\Delta_{\varpi_2, \varpi_2}}$$

and

$$\begin{aligned} D_{\varpi_3, s_3 \varpi_3} D'_1 &= D_{\varpi_2, s_3 s_2 \varpi_2} + D_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_3} \\ &= \frac{f}{\Delta_{\varpi_3, \varpi_3}^2} + \frac{\Delta_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_2}}{\Delta_{\varpi_3, \varpi_3}} \\ &= \frac{f + \Delta_{\varpi_3, \varpi_3} \cdot \Delta_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_2}}{\Delta_{\varpi_3, \varpi_3}^2}. \end{aligned}$$

Hence, by Lemma 3.1.3 and Proposition 3.2.5

$$\widetilde{D_{\varpi_3, s_3 \varpi_3} D'_1} = \widetilde{D_{\varpi_3, s_3 \varpi_3}} \widetilde{D'_1} = f + \Delta_{\varpi_3, \varpi_3} \cdot \Delta_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_2}.$$

On the other hand, the exchange relation induced by mutating at 2 is

$$D_{\varpi_2, s_3 s_2 \varpi_2} D'_2 = D_{\varpi_3, s_3 s_2 s_1 s_3 \varpi_3}^2 D_{\varpi_1, s_3 s_2 s_1 \varpi_1} + D_{\varpi_3, s_3 \varpi_3}^2 D_{\varpi_2, s_3 s_2 s_1 s_2 \varpi_2}.$$

Now, it is not hard to see that

$$s_3(s_2s_1(\varpi_1)) = s_2s_1(\varpi_1) - 2\alpha_1 \quad \text{and} \quad s_3(s_2s_1s_3s_2(\varpi_2)) = s_2s_1s_3s_2(\varpi_2) - 2\alpha_3$$

Therefore, the lift degree of both  $g$  and  $h$  is  $2\varpi_3$  again. Hence, by Corollary 3.2.8, again

$$g = \frac{\Delta_{\varpi_1, s_3s_2s_1\varpi_1} \Delta_{\varpi_3, \varpi_3}^2}{\Delta_{\varpi_1, \varpi_1}}$$

$$h = \frac{\Delta_{\varpi_2, s_3s_2s_1s_3s_2\varpi_2} \Delta_{\varpi_3, \varpi_3}^2}{\Delta_{\varpi_2, \varpi_2}}$$

and

$$\begin{aligned} D_{\varpi_2, s_3s_2\varpi_2} D'_2 &= D_{\varpi_3, s_3s_2s_1s_3\varpi_3}^2 D_{\varpi_1, s_3s_2s_1\varpi_1} + D_{\varpi_3, s_3\varpi_3}^2 D_{\varpi_2, s_3s_2s_1s_2\varpi_2} \\ &= \frac{\Delta_{\varpi_3, s_3s_2s_1s_3\varpi_3}^2}{\Delta_{\varpi_3, \varpi_3}^2} \cdot \frac{g}{\Delta_{\varpi_3, \varpi_3}^2} + \frac{\Delta_{\varpi_3, s_3\varpi_3}^2}{\Delta_{\varpi_3, \varpi_3}^2} \cdot \frac{h}{\Delta_{\varpi_3, \varpi_3}^2} \\ &= \frac{\Delta_{\varpi_3, s_3s_2s_1s_3\varpi_3}^2 \cdot g + \Delta_{\varpi_3, s_3\varpi_3}^2 \cdot h}{\Delta_{\varpi_3, \varpi_3}^4} \end{aligned}$$

Hence, by Lemma 3.1.3 and Proposition 3.2.5 again

$$\widetilde{D_{\varpi_2, s_3s_2\varpi_2} D'_2} = \widetilde{D}_{\varpi_2, s_3s_2\varpi_2} \widetilde{D'_2} = \Delta_{\varpi_3, s_3s_2s_1s_3\varpi_3}^2 \cdot g + \Delta_{\varpi_3, s_3\varpi_3}^2 \cdot h.$$

The mutation relation induced by mutating at 3 lifts similarly.

The initial exchange matrix of the cluster algebra  $\mathbb{C}[N_K]$  given in Theorem 4.2.8 is

$$B = \begin{array}{ccc|c} \textcolor{blue}{1} & \textcolor{blue}{2} & \textcolor{blue}{4} & \\ \left( \begin{array}{ccc} 0 & -2 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{array} \right) & \begin{array}{c} \textcolor{blue}{1} \\ \textcolor{blue}{2} \\ \textcolor{blue}{4} \\ \textcolor{blue}{3} \\ \textcolor{blue}{5} \\ \textcolor{blue}{6} \end{array} & . \end{array}$$

Using the calculations above and Corollary 5.3.4, we get that  $\mathbb{C}[G/P_K^-]$  contains the cluster algebra  $\hat{\mathcal{A}}$  whose initial extended cluster variables are the ones in the list above and whose initial exchange matrix is

$$\hat{B} = \begin{array}{ccc|c} & \textcolor{blue}{1} & \textcolor{blue}{2} & \textcolor{blue}{4} \\ \left( \begin{array}{ccc} 0 & -2 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 0 \end{array} \right. & \textcolor{blue}{1} & & \\ & \textcolor{blue}{2} & & \\ & \textcolor{blue}{4} & & \\ \hline & \textcolor{blue}{3} & & \\ & \textcolor{blue}{5} & & \\ & \textcolor{blue}{6} & & \\ \hline & \textcolor{blue}{j \in J} & & \end{array} .$$

## 6.2. General theorems

In the previous section, we saw how one can use the results of this paper to obtain explicit examples of type  $A$  and type  $B$ . In this section, we generalize these examples to any 2-step parabolic subgroup of type  $A$  and to any maximal parabolic subgroup of type  $B$ . Other general examples can be obtained similarly to the combination of the results of this section and the previous one.

**Example 6.2.1** (2-step parabolic subgroups of type  $A$ ). Let  $G$  be of type  $A_n$ , that is,  $G = SL_{n+1}$ . Take  $J = \{j_1, j_2\}$  with  $j_1 < j_2$  and  $K = \{1, \dots, n\} \setminus \{j_1, j_2\}$ . The longest word can be determined once we know the positions of  $j_1$  and  $j_2$ . Indeed, the desired expression of the longest word starts with the longest word of  $A_{j_1-1}$  indexed by  $1, \dots, j_1 - 1$ , followed by the longest word of  $A_{j_2-j_1-1}$  indexed by  $j_1 + 1, \dots, j_2 - 1$  and then followed by the longest

word of  $A_{n-j_2}$  indexed by  $j_2 + 1, \dots, n$ , followed by the completion of that longest word by a subword generating  $N_K$ . More concretely, let

$$u_1 = s_1 s_2 \dots s_{j_1-1} s_1 s_2 \dots s_{j_1-2} \dots s_1 s_2 s_1$$

$$u_2 = s_{j_1+1} s_{j_1+2} \dots s_{j_2-1} s_{j_1+1} s_{j_1+2} \dots s_{j_2-2} \dots s_{j_1+1} s_{j_1+2} s_{j_1+1}$$

$$u_3 = s_{j_2+1} s_{j_2+2} \dots s_{n-1} s_{j_2+1} s_{j_2+2} \dots s_{n-2} \dots s_{j_2+1} s_{j_2+2} s_{j_2+1}$$

Then, there exists a subword  $u_4$  in which the longest word  $w_0$  can be expressed as  $w_0 = u_1 u_2 u_3 u_4$ . One way to construct  $u_4$  is to take  $s_1 \dots s_n$ , together with a reduced word consisting of  $j_2 - 1$  reduced subwords each of length  $n - j_2$ , followed by a reduced word consisting  $j_2 - j_1$  reduced subwords each of length  $j_1$ . In particular, we may choose  $u_4$  to be

$$u_4 = s_1 \dots s_n u_5 u_6,$$

$$u_5 = s_1 \dots s_{n-j_2} s_2 \dots s_{n+1-j_2} s_3 \dots s_{n+2-j_2} \dots s_{j_2-1} \dots s_{n-2},$$

$$u_6 = s_1 \dots s_{j_1} s_2 \dots s_{j_1+1} s_3 \dots s_{j_1+2} \dots s_{j_2-j_1} \dots s_{j_2-1}.$$

In fact, this  $u_4$  generates  $N_K$  and it is of length  $n + (n - j_2)(j_2 - 1) + j_1(j_2 - j_1)$ .

Obviously, one now can use Theorem 5.3.4 and follow the procedure of Example 6.1.1 to get the general picture of how the cluster algebra  $\hat{\mathcal{A}}$  is constructed generally from a 2-step parabolic subgroup, that is, a parabolic subgroup  $P_K^-$  with  $|J| = 2$ . Let  $M := n + (n - j_2)(j_2 - 1)$  and

$$\begin{aligned} A := & \{n - 1, n\} \\ & \cup \left\{ a \mid a \in [n + 1, M] \text{ and } i_a \geq j_2 - 1 \right\} \\ & \cup \left\{ \begin{array}{l} M + 1, \quad M + j_1 + 1, \quad M + 2j_1 + 1, \quad \dots, \quad M + (j_2 - j_1 - 1)j_1 + 1, \\ M + (j_2 - j_1 - 1)j_1 + 2, \quad M + (j_2 - j_1 - 1)j_1 + 3, \quad \dots, \quad M + (j_2 - j_1)j_1 \end{array} \right\}. \end{aligned}$$

It is not hard to see that  $s(i_k) = \infty$  if and only if  $k \in A$ . Thus, by Theorem 4.2.8, these  $k$ 's are exactly the indexes of the frozen variables of the desired initial seed of  $\mathcal{A} = \mathbb{C}[N_K]$ , while the other indexes are the ones of the mutable variables. More concretely, the frozen variables are

$$D_{\varpi_{i_k}, w \leq_k \varpi_{i_k}}, \quad (k \in A),$$

while the mutable variables are

$$D_{\varpi_{i_l}, w \leq_l \varpi_{i_l}}, \quad (l \notin A).$$

Also, by Theorem 4.2.8, the exchange matrix of this seed is

$$B_{jk} = \begin{cases} 1, & \text{if } j = p(k), \\ -1, & \text{if } j = s(k), \\ a_{i_j i_k}, & \text{if } j < k < s(j) < s(k), \\ -a_{i_j i_k}, & \text{if } k < j < s(k) < s(j), \\ 0, & \text{otherwise;} \end{cases}$$

Moreover, the frozen variables of the lift of this cluster algebra to the cluster algebra  $\hat{\mathcal{A}} \subset \mathbb{C}[G/P_K^-]$ , are  $\Delta_{\varpi_{j_1}, w(\varpi_{j_1})}$  and  $\Delta_{\varpi_{j_2}, w(\varpi_{j_2})}$  together with

$$\widetilde{D}_{\varpi_{i_k}, w \leq_k \varpi_{i_k}} = \frac{\Delta_{\varpi_{i_k}, w \leq_k \varpi_{i_k}} \Delta_{\varpi_{i_1}, \varpi_{i_1}}^{d_k}}{\Delta_{\varpi_{i_k}, \varpi_{i_k}}},$$

where  $k \in A$ . On the other hand, the mutable ones are

$$\widetilde{D}_{\varpi_{i_l}, w \leq_l \varpi_{i_l}} = \frac{\Delta_{\varpi_{i_l}, w \leq_l \varpi_{i_l}} \Delta_{\varpi_{i_1}, \varpi_{i_1}}^{d_l}}{\Delta_{\varpi_{i_l}, \varpi_{i_l}}},$$

where  $l \notin A$ . The exchange matrix attached to this is the matrix  $\hat{B}$  that extends  $B$  by the row  $\hat{b}_{j_1 k}$  for  $k \notin A$  and the row  $\hat{b}_{j_2 k}$ , as addressed in Theorem 5.3.4.

**Example 6.2.2** (Maximal parabolic subgroups of type  $B$ ). Let  $G$  be a semisimple algebraic group of type  $B_n$ . Let  $J = \{j\}$  and  $K = \{1, \dots, n\} \setminus \{j\}$ . The description of the longest word  $w_0$  here depends on the position of  $j$ . We generalize Example 6.1.2 by taking  $j = n$ . Here  $w_0$  is constructed by taking the longest word  $u$  of  $A_{n-1}$  indexed by  $1, \dots, n-1$ , completed by some subword generating  $N_k$ , for instance,

$$v = s_n s_{n-1} \dots s_1 s_n s_{n-1} \dots s_2 s_n s_{n-1} \dots s_3 s_n s_{n-1} s_n.$$

Therefore, in this case, we have  $w_0 = uv$ . Let

$$A = \left\{ n, 2n-1, 3n-3, 4n-5, \dots, \frac{n(n+1)}{2} - 1, \frac{n(n+1)}{2} \right\}.$$

Note that  $s(i_k) = \infty$  if and only if  $k \in A$ . Thus, the frozen variables are those indexed by such  $k$ 's. Same as before, the frozen variables are

$$D_{\varpi_{i_k}, w_{\leq k} \varpi_{i_k}}, \quad (k \in A),$$

while the mutable variables are

$$D_{\varpi_{i_l}, w_{\leq l} \varpi_{i_l}}, \quad (l \notin A).$$

Again, Theorem 4.2.8, gives the recipe of the exchange matrix  $B$ .

Now, the exchange matrix  $\widehat{B}$  of the lift of this seed to the cluster algebra  $\widehat{\mathcal{A}} \subset \mathbb{C}[G/P_K^-]$  is  $B$  extended by the row indexed by  $j = n$ , whose entries are obtained from Theorem 5.3.4. The frozen variables are  $\Delta_{\varpi_n, \varpi_n}$  together with

$$\widetilde{D}_{\varpi_{i_k}, w_{\leq k} \varpi_{i_k}} = \frac{\Delta_{\varpi_{i_k}, w_{\leq k} \varpi_{i_k}} \Delta_{\varpi_{i_1}, \varpi_{i_1}}^{d_k}}{\Delta_{\varpi_{i_k}, \varpi_{i_k}}},$$

where  $k \in A$ . On the other hand, the mutable ones are

$$\widetilde{D}_{\varpi_{i_l}, w_{\leq l} \varpi_{i_l}} = \frac{\Delta_{\varpi_{i_l}, w_{\leq l} \varpi_{i_l}} \Delta_{\varpi_{i_1}, \varpi_{i_1}}^{d_l}}{\Delta_{\varpi_{i_l}, \varpi_{i_l}}},$$

where  $l \notin A$ .



## Chapter 7. Future Directions

There are many of interesting questions that arise after this dissertation. We plan to return to some of them in future work. First, Demonet proved in [2] that the initial seed of the cluster algebra  $\mathbb{C}[N_K]$  given by Geiß, Leclerc and Schröer [7] can be lifted to an initial seed of  $\mathbb{C}[G/P_K^-]$ . Our work in this dissertation was independent of what Demonet did and we plan to discover the relationships between these initial seeds. Second, our work strongly relied on the work of Goodearl and Yakimov [13]. However, in this dissertation, we just dealt with classic cluster algebras, while the paper of Goodearl and Yakimov treated what is called *quantum cluster algebras*. It would be interesting to see how many of the results of this work have an analog in the quantum case. We plan to come back to this as well. Third, this project gave an explicit algorithm calculating the homogenization tilde map explicitly and showed how one could use it to describe the initial seed of the cluster algebra  $\hat{\mathcal{A}} \subset \mathbb{C}[G/P_K^-]$  explicitly. However, this special map can be so useful in the context of Lie theory, in general. We hope we can discover more about it.

Although this dissertation improved the previous results of the relationship of partial flag varieties and cluster algebras, it still shows that the cluster algebra  $\hat{\mathcal{A}}$  coincides with  $\mathbb{C}[G/P_K^-]$  after localization. It would be so interesting to prove such results with omitting this localization condition, but this does not seem trivial at all. From the first days of cluster algebras, this relationship has been growing, but it has not been fully discovered yet. We hope to see more improvements in the no-far distance.

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## **Vita**

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