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The Determination of a Matroid's Structure From Properties of Certain Large Minors.

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THE DETERMINATION OF A MATROID'S STRUCTURE
FROM PROPERTIES OF CERTAIN LARGE MINORS

A Dissertation

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This dissertation solves some problems related to the structure of matroids. In Chapter 2, we prove that if $M$ and $N$ are distinct connected matroids on a common ground set $E$, where $|E| \geq 2$, and, for every $e$ in $E$, $M\setminus e = N\setminus e$ or $M/e = N/e$, then one of $M$ and $N$ is a relaxation of the other. In addition, we determine the matroids $M$ and $N$ on a common ground set $E$ such that, for every pair of elements $\{e, f\}$ of $E$, at least two of the four corresponding minors of $M$ and $N$ obtained by eliminating $e$ and $f$ are equal. The theorems in Chapters 3 and 4 extend a result of Oxley that characterizes the non-binary matroids $M$ such that, for each element $e$, $M\setminus e$ or $M/e$ is binary. In Chapter 3, we describe the non-binary matroids $M$ such that, for every pair of elements $\{e, f\}$, at least two of the four minors of $M$ obtained by eliminating $e$ and $f$ are binary. In Chapter 4, we obtain an alternative extension of Oxley's result by changing the minor under consideration from the smallest 3-connected whirl, $U_{2,4}$, to the smallest 3-connected wheel, $M(K_4)$. In particular, we determine the binary matroids $M$ having an $M(K_4)$-minor such that, for every element $e$, $M\setminus e$ or $M/e$ has no $M(K_4)$-minor. This enables us to characterize the matroids $M$ that are not series-parallel networks, but, for every $e$, $M\setminus e$ or $M/e$ is a series-parallel network.
CHAPTER 1

INTRODUCTION

The results presented in this dissertation are matroid structure theorems. In particular, in the problems considered here, we seek to determine the structure of a matroid by analyzing its large minors. In this introduction, we describe the notation and terminology used. In addition, we establish some preliminaries that will be needed later.

The notation and terminology used here will follow Oxley [11]. In particular, the ground set, corank, and rank of a matroid $M$ will be denoted by $E(M)$, $r^*(M)$, and $r(M)$, respectively. Moreover, $\mathcal{I}(M)$, $\mathcal{B}(M)$, and $\mathcal{C}(M)$ will denote the independent sets, bases, and circuits of $M$, respectively. The fundamental circuit of the element $e$ with respect to the basis $B$ is the unique circuit contained in $B \cup e$ and is denoted by $C(e, B)$. For a subset $X$ of $E(M)$, the closure of $X$, denoted $cl(X)$, is defined to be $\{x \in E(M) : r(X \cup x) = r(X)\}$. If $cl(X) = X$, then $X$ is called a flat of $M$. A flat of $M$ of rank $r(M) - 1$ is called a hyperplane.

A subset $X$ of $E(M)$ is said to span $M$ if $cl(X) = E(M)$. If $T \subseteq E(M)$, we say that $M$ uses $T$. In addition, we denote the deletion of $T$ from $M$ by $M \setminus T$ or $M|(E(M) - T)$, while the contraction of $T$ from $M$ is denoted by $M/T$. We call an element $e$ a loop of $M$ if $\{e\}$ is a circuit of $M$. Dually, $e$ is a coloop of $M$ if $\{e\}$ is a cocircuit of $M$. If $f$ and $g$ are elements of $M$ such that $\{f, g\}$ is a circuit, then $f$ and $g$ are said to be in parallel. If $\{f, g\}$ is a cocircuit, then $f$ and $g$ are in series. A parallel class of $M$ is a maximal subset $X$ of $E(M)$ such
that any two distinct members of \( X \) are in parallel and no member of \( X \) is a loop. Series classes are defined analogously. A series or parallel class is non-trivial if it contains more than one element. The matroid \( N \) is a series extension of \( M \) if \( M = N/T \) and every element of \( T \) is in series with some element of \( M \). We call \( N \) a parallel extension of \( M \) if \( N^* \) is a series extension of \( M^* \). A matroid that is isomorphic to its dual is said to be self-dual.

Let \( X \) and \( Y \) be sets. Frequently, we will add a single element to a set \( X \) or remove a single element from \( X \). In these cases, we will often abbreviate \( X \cup \{e\} \) and \( X - \{e\} \) to \( X \cup e \) and \( X - e \), respectively. The notation \( X \cup Y \) denotes the set \( X \cup Y \) and also implies that \( X \) and \( Y \) are disjoint. The symmetric difference of the sets \( X \) and \( Y \), denoted \( X \Delta Y \), is the set \( (X - Y) \cup (Y - X) \). The following lemma (see, for example, [11, p. 304]) will be used repeatedly.

**Lemma 1.1.** A matroid \( M \) is binary if and only if the symmetric difference of distinct circuits is a disjoint union of circuits.

Let \( k \) be a positive integer. Then, for a matroid \( M \), a partition \( (X, Y) \) of \( E(M) \) is a \( k \)-separation if \( \min\{|X|, |Y|\} \geq k \) and \( r(X) + r(Y) - r(M) \leq k - 1 \). If \( M \) has a \( k \)-separation, then \( M \) is said to be \( k \)-separated. Moreover, if \( M \) is \( k \)-separated for some \( k \), then the connectivity of \( M \) is defined to be \( \min\{j: M \text{ is } j \text{-separated}\} \). Thus for a positive integer \( n \), a matroid \( M \) is \( n \)-connected if, for all positive integers \( k < n \), there is no partition \( (X, Y) \) of \( E(M) \) such that \( |X|, |Y| \geq k \) and \( r(X) + r(Y) - r(M) = k - 1 \). Our main interest is in 3-connected matroids having at least four elements. We will often use the fact that all circuits and
all cocircuits of such a matroid contain at least three elements (see, for example, [11, p. 273]).

Many of the results in this dissertation involve the following construction. A circuit-hyperplane \( X \) of a matroid \( M \) is a subset of \( E(M) \) which is both a circuit and a hyperplane of \( M \). A new matroid on \( E(M) \), denoted \( M' \), can be obtained from \( M \) by declaring the circuit-hyperplane \( X \) of \( M \) to be a basis of \( M' \), so that \( B(M') = B(M) \cup \{X\} \). We call \( M' \) a relaxation of \( M \) and say that \( M' \) has been obtained from \( M \) by relaxing the circuit-hyperplane \( X \).

![Figure 1.1](image)

**Figure 1.1.** (a) \( W_3 \); and geometric representations of (b) \( M(W_3) \) and (c) \( W^3 \).

Let the edge set of the 3-spoked wheel, denoted by \( W_3 \), be labelled as in Figure 1.1(a). The edges labelled by members of the sets \( \{1, 2, 3\} \) and \( \{4, 5, 6\} \) are referred to as the spokes and rim elements of the wheel, respectively. A geometric representation of \( M(W_3) \) is given in Figure 1.1(b). Notice that the rim is a circuit-hyperplane of \( M(W_3) \). The rank-3 whirl, denoted by \( W^3 \), is the matroid obtained by relaxing this circuit-hyperplane. It is not hard to see that \( W^3 \) is the unique relaxation of \( M(W_3) \). In general, the rank-\( r \) whirl, \( W^r \), is obtained from the cycle matroid of the \( r \)-spoked wheel, \( M(W_r) \), by relaxing its rim. Additional examples of the relaxation operation are given in Figure 1.2 where successive
relaxations of $\mathcal{W}^3$ are shown. Thus $Q_6$ is obtained from $\mathcal{W}^3$ by relaxing the circuit-hyperplane $\{1, 2, 4\}$. Notice that, if $i \in \{1, 2, 4\}$, then $\mathcal{W}^3 \setminus i = Q_6 \setminus i$. Furthermore, if $j \in \{3, 5, 6\}$, then $\mathcal{W}^3 / j = Q_6 / j$. This illustrates one of the useful properties of relaxation noted by Kahn [7]. The next lemma lists the properties of this operation that we will need.

![Diagram of successive relaxations of $\mathcal{W}^3$.](image)

**Figure 1.2.** Successive relaxations of $\mathcal{W}^3$.

**Lemma 1.2.** Let $X$ be a circuit-hyperplane of the matroid $M$ and $M'$ be the matroid obtained by relaxing $X$.

(i) If $e \in E(M) - X$, then $M/e = M'/e$ and, provided $M$ does not have $e$ as a coloop, $M'/e$ is obtained from $M\setminus e$ by relaxing the circuit-hyperplane $X$ of the latter.

(ii) If $f \in X$, then $M\setminus f = M'/f$ and, provided $M$ does not have $f$ as a loop, $M'/f$ is obtained from $M/f$ by relaxing the circuit-hyperplane $X - f$ of the latter.

(iii) $B(M') = B(M) \cup \{X\}$, and $I(M') = I(M) \cup \{X\}$.

(iv) $C(M') = (C(M) - \{X\}) \cup \{X \cup e : e \in E(M) - X\}$. 
(v) If $M$ is $n$-connected, then so is $M'$.

(vi) If $M$ is connected, then $M'$ is non-binary.

The next lemma will enable us to determine when a matroid can be obtained via relaxation. The straightforward proof is omitted.

**Lemma 1.3.** The following statements are equivalent for a rank-$r$ matroid $M$.

(i) $M$ is obtained from some matroid by relaxing the circuit-hyperplane $B$.

(ii) $M$ has a basis $B$ such that $C(e, B) = B \cup e$ for every $e$ in $E(M) - B$, and neither $B$ nor $E(M) - B$ is empty.

(iii) $M$ has a non-empty basis $B$ such that $B \neq E(M)$ and every $(r - 1)$-element subset of $B$ is a flat.

![Diagram](image)

**Figure 1.3.** (a) $F_7$. (b) $F_7^-$. (c) $F_7^{\sim}$.

If a matroid $M$ can be obtained from another matroid $N$ by relaxing two circuit-hyperplanes, then we say $M$ is a double relaxation of $N$. Consider the Fano matroid, denoted $F_7$, shown in Figure 1.3(a). The non-Fano matroid, denoted $F_7^-$, is the unique relaxation of $F_7$. A geometric representation of the non-Fano matroid is shown in Figure 1.3(b). If two circuit-hyperplanes of $F_7$ are relaxed, then, up to isomorphism, the matroid shown in Figure 1.3(c) is obtained. We will denote this
unique double relaxation of \( F \) by \( F_7^\infty \). The next lemma will enable us to identify matroids that are double relaxations.

**Lemma 1.4.** Suppose the rank-\( r \) matroid \( M \) is obtained from the matroids \( N_1 \) and \( N_2 \) by relaxing the circuit-hyperplanes \( X \) and \( Y \), respectively. Then there is a matroid \( N \) such that the relaxation of \( Y \) in \( N \) yields \( N_1 \) and the relaxation of \( X \) in \( N \) yields \( N_2 \) if and only if \(|X \cap Y| < r - 1| \).

**Proof.** First assume there is a matroid \( N \) that yields \( N_1 \) and \( N_2 \) when the circuit-hyperplanes \( Y \) and \( X \), respectively, are relaxed. As \( X \neq Y \) and \(|X| = |Y| = r \), we conclude that \(|X \cap Y| < r| \). Suppose \(|X \cap Y| = r - 1| \). Now, as \( N_2 \) is obtained from \( N \) by relaxing the circuit-hyperplane \( X \), Lemma 1.3 implies that every \((r - 1)\)-element subset of \( X \) is a flat of \( N_2 \). In particular, the set \( X \cap Y \) is a flat of \( N_2 \). Since \(|Y - X| = 1 \) and \( Y \) is a circuit of \( N_2 \), we deduce that \( Y \subseteq \text{cl}_{N_2}(X \cap Y) \). However, as \( X \cap Y \) is properly contained in \( Y \), it follows that \( X \cap Y \) is not a flat of \( N_2 \); a contradiction. We conclude that \(|X \cap Y| < r - 1| \).

Now suppose that \( M \) is obtained from the matroids \( N_1 \) and \( N_2 \) by relaxing the circuit-hyperplanes \( X \) and \( Y \), respectively. In addition, assume \(|X \cap Y| < r - 1| \). Now, Lemma 1.2(iii) implies \( B(M) = B(N_1) \cup \{X\} \) and \( B(M) = B(N_2) \cup \{Y\} \). Thus \(|X| = |Y| = r| \). Moreover, as \( X \neq Y \), we deduce that \( Y \) is a basis of \( N_1 \). Now assume \( T \) is an \((r - 1)\)-element subset of \( Y \) that is not a flat of \( N_1 \). Then there is an element \( e \) of \( E(N_1) - Y \) and a circuit \( C \) of \( N_1 \) so that \( e \in C \subseteq T \cup e \). Since \(|X \cap Y| < r - 1 \) and \(|X| = r \), it follows that \( X \) contains at least two elements of the complement of \( Y \). As \( C \) contains exactly one element, namely \( e \), of the complement.
of $Y$, we conclude that $C \neq X$. Moreover, as $C \in C(N_1) - \{X\}$, it follows from Lemma 1.2(iv), that $C$ is a circuit of $M$. Thus $T \cup e \subseteq cl_M(T)$. However, as $M$ is obtained from $N_2$ by relaxation of the circuit-hyperplane $Y$, Lemma 1.3 implies that all $(r - 1)$-element subsets of $Y$ are flats of $M$. In particular, the $(r - 1)$-element subset $T$ of $Y$ is a flat of $M$ contrary to the fact that $T \cup e \subseteq cl_M(T)$. As a result of this contradiction, we conclude that $T$ is a flat of $N_1$. Consequently, in $N$, all $(r - 1)$-element subsets of the basis $Y$ are flats. Therefore $N_1$ is obtained from some matroid $P$ via relaxation. By symmetry, $N_2$ is obtained from some matroid $Q$ via relaxation. To complete the proof of the lemma, we need only show that $P = Q$. Now $B(N_1) \cup \{X\} = B(N_2) \cup \{Y\}$. Moreover, $B(P) = B(N_1) - \{Y\} = B(N_2) - \{X\} = B(Q)$. Thus $P = Q$ and the lemma holds.

Let $M_1$ and $M_2$ be matroids having at least three elements such that $E(M_1) \cap E(M_2) = \{p\}$. Suppose neither $M_1$ nor $M_2$ has $p$ as a loop or coloop. Then the $2$-sum of $M_1$ and $M_2$, denoted $M_1 \oplus_2 M_2$, is $P(M_1, M_2) \setminus p$, where $P(M_1, M_2)$ is the parallel connection of $M_1$ and $M_2$. We call $M_1$ and $M_2$ parts of this $2$-sum. Bixby [1], Cunningham [5], and Seymour [12] independently proved the following basic link between 3-connectedness and 2-sums.

**Theorem 1.5.** A 2-connected matroid $M$ is not 3-connected if and only if $M = M_1 \oplus_2 M_2$ for some matroids $M_1$ and $M_2$, each of which is isomorphic to a proper minor of $M$.

We shall assume familiarity with other basic properties of parallel connections and 2-sums. Those needed here are contained in [11; Section 7.1]. Now, as a
consequence of Theorem 1.5, many matroid structure results can be obtained by concentrating on the structure of the 3-connected matroids in the class under consideration. Moreover, Seymour’s Splitter Theorem (see, for example [11 p. 347]) is a powerful tool for determining the structure of 3-connected matroids. In our work here, we will use the following version of the Splitter Theorem.

**Theorem 1.6.** Let $M$ and $N$ be 3-connected matroids such that $N$ is a minor of $M$. $|E(N)| \geq 4$, and if $N$ is a wheel, then $M$ has no larger wheel as a minor, while if $N$ is a whirl, then $M$ has no larger whirl as a minor. Then there is a sequence $M_0, M_1, \ldots, M_n$ of 3-connected matroids such that $M_0 \cong N$, $M_n = M$, and for all $i$ in $\{0, 1, \ldots, n-1\}$, $M_i$ is a single-element deletion or a single-element contraction of $M_{i+1}$.

In the next lemma, an extension of the Splitter Theorem due to Coullard [3: 4, Corollary 4.3], the hypothesis excluding wheels and whirls is weakened so that it applies only to the smallest 3-connected wheels and whirls.

**Theorem 1.7.** Let $N$ be a 3-connected proper minor of a 3-connected matroid $M$ such that $|E(N)| \geq 4$ and $M$ is not a wheel or a whirl. Suppose that if $N \cong W^2$, then $M$ has no $W^3$-minor, while if $N \cong M(W_3)$, then $M$ has no $M(W_4)$-minor. Then $M$ has a 3-connected minor $M_1$ and an element $e$ such that $M_1 \setminus e$ or $M_1/e$ is isomorphic to $N$.

We will often use the following result of Oxley [9, Theorem 3.1] describing non-binary 3-connected matroids. Geometric representations of the matroids appearing in the statement of the lemma are shown in Figure 1.1.
Lemma 1.8. Let $M$ be a 3-connected non-binary matroid having rank and corank at least three. Then $M$ has a minor isomorphic to one of $U_{3,6}$, $P_6$, $Q_6$, or $\mathcal{W}^3$.

The next two lemmas are structural results that relate $U_{2,4}$-minors to particular elements of a non-binary matroid. The first lemma, due to Bixby [2], provides motivation for the result of Seymour [13] which follows it.

Lemma 1.9. Let $M$ be a 2-connected non-binary matroid having a $U_{2,4}$-minor and suppose that $e \in M$. Then $M$ has a $U_{2,4}$-minor using $e$.

Lemma 1.10. Let $M$ be a 3-connected non-binary matroid having a $U_{2,4}$-minor and suppose that $e$ and $f$ are distinct elements of $M$. Then $M$ has a $U_{2,4}$-minor using $\{e,f\}$.

The smallest 3-connected whirl, $\mathcal{W}^2$, is isomorphic to the uniform matroid $U_{2,4}$. Moreover, the cycle matroid of the smallest 3-connected wheel, $M(\mathcal{W}_3)$, is clearly isomorphic to the cycle matroid of $K_4$, the complete graph on four vertices. A *series-parallel network* is a non-empty connected matroid having neither the smallest 3-connected whirl nor the cycle matroid of the smallest 3-connected wheel as a minor. In other words, a series-parallel network is a non-empty connected matroid having no $U_{2,4}$- or $M(K_4)$-minor.

The above background material on 3-connected matroids will be used extensively in Chapters 3 and 4. In Chapter 4, we determine the binary matroids $M$ having an $M(K_4)$-minor such that, for every element $e$, $M\setminus e$ or $M/e$ has no $M(K_4)$-minor. In addition, we characterize the matroids $M$ that are series-parallel networks, but, for each element $e$, $M\setminus e$ or $M/e$ is not a series-parallel
network. In Chapter 3, we describe the non-binary matroids $M$ such that, for every $\{e, f\} \subseteq E(M)$, at least two of the minors of $M$ obtained by eliminating $e$ and $f$ are binary. Each of these results is obtained by first proving it in the case where $M$ is 3-connected and then applying this result to obtain a more general one.

The theorems in Chapter 2 describe the relationship between two matroids $M$ and $N$, on a common ground set $E$, if many of the corresponding minors of $M$ and $N$ are equal. We prove that if, for every element $e$, $M\setminus e = N\setminus e$ or $M/e = N/e$, then, apart from some easily specified exceptions, the matroids $M$ and $N$ are related via relaxation. An extension of this result involving the elimination of two elements of $E$ is also proved in Chapter 2.
CHAPTER 2

RELATING TWO MATROIDS WITH MANY IDENTICAL MINORS

Recall that a circuit-hyperplane $X$ of a matroid $M$ on $E(M)$ is a subset of $E(M)$ that is both a circuit and a hyperplane of $M$. A new matroid on $E(M)$, denoted $M'$ and called a relaxation of $M$, can be obtained from $M$ by declaring the circuit-hyperplane $X$ of $M$ to be a basis of the new matroid $M'$. Moreover, Kahn [7] observed that $M \setminus e = M' \setminus e$ whenever $e \in X$, and $M/e = M'/e$ whenever $e \in E(M) - X$. Theorem 2.2, the main result of Section 2.1, is in a certain sense, a converse of Kahn's observation. It determines the relationship between two matroids on a common ground set if certain of their corresponding minors, obtained by eliminating one element, are equal. In Section 2.2, we present an extension of Theorem 2.2 that involves minors of $M$ and $N$ obtained by eliminating two elements.

2.1. The One-Element Case

In this section, we specify the relationship between two matroids $M$ and $N$ on a common ground set $E$, if, for every element $e$, we have $M \setminus e = N \setminus e$ or $M/e = N/e$. In particular, in Theorem 2.2, we show that, apart from some easily specified exceptions, any two such matroids are related via relaxation. First we note the following lemma.

Lemma 2.1. Suppose that $M$ and $N$ are matroids on $E$ and there is some element $e$ of $E$, such that $M \setminus e = N \setminus e$ and $M/e = N/e$. Then $M = N$ or $\{M, N\} = \{P \oplus U_{0,1}, P \oplus U_{1,1}\}$ where $P = M \setminus e = N \setminus e$. 

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Proof. Now, if \( e \) is a loop of neither \( M \) nor \( N \), then
\[
\mathcal{I}(M) = \mathcal{I}(M\setminus e) \cup \{I \cup e : I \in \mathcal{I}(M/e)\},
\]
\[
= \mathcal{I}(N\setminus e) \cup \{I \cup e : I \in \mathcal{I}(N/e)\},
\]
\[
= \mathcal{I}(N).
\]
Therefore \( M = N \) and the lemma holds. Moreover, if \( e \) is a loop in both \( M \) and \( N \), then \( M = (M\setminus e) \oplus U_{0,1} = (N\setminus e) \oplus U_{0,1} = N \) and the lemma clearly holds.

Thus we may assume, without loss of generality, that \( e \) is a loop of \( M \) but not a loop of \( N \). Then \( N\setminus e = M\setminus e = M/e = N/e \) which implies that \( e \) is a loop or a coloop of \( N \). We conclude that \( e \) is a coloop of \( N \) and \( M = (M\setminus e) \oplus U_{0,1} \) while \( N = (M\setminus e) \oplus U_{1,1} \).

Before stating the main result of this section, we introduce the following notation. For matroids \( P, Q, R, \) and \( S \), the set \( \{P, Q\} \) is said to be isomorphic to the set \( \{R, S\} \), denoted \( \{P, Q\} \cong \{R, S\} \), whenever \( P \neq Q \) and \( R \neq S \), but each of \( P \) and \( Q \) is isomorphic to exactly one of \( R \) and \( S \).

**Theorem 2.2.** The following two statements are equivalent for matroids \( M \) and \( N \) on an \( n \)-element set \( E \).

(i) For every element \( e \), \( M\setminus e = N\setminus e \) or \( M/e = N/e \).

(ii) (a) One of \( M \) and \( N \) is a relaxation of the other; or

(b) \( M = N \); or

(c) \( \{M, N\} \cong \{U_{0,n}, U_{1,n}\} \); or

(d) \( \{M, N\} \cong \{U_{n-1,n}, U_{n,n}\} \).

**Proof.** If (b), (c), or (d) of (ii) holds, then certainly (i) holds. Moreover, if (ii)(a) holds, then, by Lemma 1.2(i) and (ii), statement (i) holds. Now suppose that (i)
holds. Evidently (ii) holds when \( n \) equals zero or one, so we may assume \( n \geq 2 \).

Let \( C = \{ e \in E : M/e = N/e \} \) and let \( D = \{ e \in E : M\setminus e = N\setminus e \} \). Since (i) holds, it follows that every element of \( E \) is in the set \( C \cup D \).

Suppose that \( E \) contains an element \( e \) that is a loop of both \( M \) and \( N \). Then \( M\setminus e = M/e \) and \( N\setminus e = N/e \). Hence \( M = (M/e) \oplus U_{0,1} = (N/e) \oplus U_{0,1} = N \) and (ii)(b) holds.

Now suppose that there is an element \( e \) that is a loop of \( M \) but not of \( N \). Then, for all \( f \) in \( D - e \), the set \( \{ e \} \) is dependent in \( M\setminus f \). Moreover, as \( f \) is in \( D \), the matroids \( M\setminus f \) and \( N\setminus f \) are equal. Therefore \( \{ e \} \) is a circuit in \( N \) contrary to the assumption that \( e \) is not a loop of \( N \). We deduce that either \( D = \emptyset \) or \( D = \{ e \} \). In each case, \( E - e \subseteq C - e \). Therefore \( M/f = N/f \) for all \( f \) in \( E - e \).

Thus, as \( e \) is a loop in \( M/f \), we deduce that, in \( N \), every element of \( E - e \) is in parallel with \( e \). Hence \( N \cong U_{1,n} \). Now if every element of \( E - e \) is a loop of \( M \), then \( M \cong U_{0,n} \) while \( N \cong U_{1,n} \). Consequently, \( \{ M, N \} \) satisfies (ii)(c) and the theorem holds. Suppose \( f \) is an element of \( E - e \) that is independent in \( M \). Now, as \( M/g = N/g \cong U_{0,n-1} \) for all \( g \) in \( E - \{ e, f \} \), it follows that every element in \( E - \{ e, f \} \) is in parallel with \( f \) in the matroid \( M \). Thus \( M \cong U_{1,n-1} \oplus U_{0,1} \). Since \( N \cong U_{1,n} \), the matroid \( N \) is a relaxation of \( M \) and (ii)(a) holds.

We may now assume that neither \( M \) nor \( N \) has a loop. Moreover, by duality, neither \( M \) nor \( N \) has a coloop. It follows that \( r(M) = r(N) \). The rest of the proof is comprised of three cases. First note that if \( e \) is an element of a matroid \( P \) that
has no loops, then the collection of independent sets of \( P \) can be partitioned into the sets \( \mathcal{I}(P \setminus e) \) and \( \{ I \cup e : I \in \mathcal{I}(P / e) \} \). We will denote the last set by \( \mathcal{I}(P / e) \).

Case 1: Suppose \( D = E \) or \( C = E \). By duality, we may assume the former. Since neither \( M \) nor \( N \) has a coloop, \( D \) is dependent in both \( M \) and \( N \). Thus 
\[
\mathcal{I}(M) = \bigcup_{e \in D} \mathcal{I}(M \setminus e) = \bigcup_{e \in D} \mathcal{I}(N \setminus e) = \mathcal{I}(N).
\]
Hence \( M = N \) and (ii)(b) holds.

Case 2: Suppose \( e \in C \cap D \). Then \( M \setminus e = N \setminus e \) and \( M / e = N / e \). Moreover, since \( e \) is neither a loop nor a coloop of \( M \) or \( N \), Lemma 2.1 implies that \( M = N \) and (ii)(b) holds.

Case 3: Suppose \( E = C \cup D \) where neither \( C \) nor \( D \) is empty. If \( D \) is dependent in both \( M \) and \( N \), then 
\[
\mathcal{I}(M) = \bigcup_{e \in D} \mathcal{I}(M \setminus e) = \bigcup_{e \in D} \mathcal{I}(N \setminus e) = \mathcal{I}(N).
\]
Thus \( M = N \) and (ii)(b) holds. Consider the situation when \( D \) is independent in both \( M \) and \( N \). Then, for both members \( P \) of \( \{M, N\} \), the set \( \mathcal{I}(P) \) consists of all subsets of \( D \), and each independent set of \( P \) that contains some element of \( C \).

Since no element of \( C \) is a loop of \( P \), the independent sets of the latter type can be obtained by taking the union over all \( f \) in \( C \) of \( \mathcal{I}(P / f) \). Thus 
\[
\mathcal{I}(P) = \{ I : I \subseteq D \} \cup \left( \bigcup_{f \in C} \mathcal{I}(P / f) \right).
\]
Since \( M / f = N / f \) for all \( f \) in \( C \), it follows that 
\[
\mathcal{I}(M) = \mathcal{I}(N).
\]
Thus \( M = N \) and (ii)(b) holds.

Now, suppose, without loss of generality, that \( D \) is dependent in \( M \) and independent in \( N \). Then \( D \) is a circuit of \( M \). To see this, note that if \( D - e \) is dependent in \( M \) for some \( e \) in \( D \), then the set \( D - e \) is independent in \( N \setminus e \) but dependent in \( M \setminus e \). However, as \( e \) is an element of \( D \), the matroids \( M \setminus e \) and \( N \setminus e \) are equal; a contradiction. Thus \( D \) is a circuit of \( M \).
Next we show that $D$ is a basis of $N$. Certainly $D$ is independent in $N$. Suppose there is an independent set $I$ of $N$ that properly contains $D$. Then there is an element $f$ of $C$ so that $D \cup f \subseteq I$. Therefore $D$ is independent in $N/f$. Moreover, as $f$ is an element of $C$, we have that $M/f = N/f$. It follows that $D$ is independent in $M/f$; a contradiction. We conclude that $D$ is a basis of $N$. Since $I(N) = \{I : I \subseteq D\} \cup \left( \bigcup_{f \in C} \mathcal{I}(N/f) \right)$, and $I(M) = \{I : I \not\subseteq D\} \cup \left( \bigcup_{f \in C} \mathcal{I}(M/f) \right)$, we conclude that $\mathcal{B}(N) = \mathcal{B}(M) \cup \{D\}$.

To complete the proof, we need only show that $D$ is a hyperplane of $M$ and hence, $N$ is a relaxation of $M$. Since $r_M(D) = r(M) - 1 = r(N) - 1$, every basis of $M$ meets $E - D$, which equals $C$. Thus $C$ is dependent in $M^*$. Since $C$ is independent in $N^*$, we may apply the above argument using $M^*, N^*$, and $C$ in place of $M, N$, and $D$, respectively to deduce that $C$ is a circuit of $M^*$. We conclude that $D$ is a hyperplane of $M$. \qed

**Corollary 2.3.** Suppose $M$ and $N$ are distinct connected matroids on a set $E$ containing at least two elements. Then, for every element $e$, $M\setminus e = N\setminus e$ or $M/e = N/e$ if and only if one of $M$ and $N$ is a relaxation of the other.

**2.2. The Two-Element Case**

In this section, as in the preceding one, we determine the relationship between two matroids $M$ and $N$, on a common ground set $E$, if certain of their minors are equal. In particular, we present an extension of Theorem 2.2 that involves minors of $M$ and $N$ obtained by the elimination of two elements of $E$. Now, in a sense, Theorem 2.2 states that if $M$ and $N$ are matroids on $E$ so that, for every
single-element subset \{e\} of \(E\), at least half of the corresponding minors of \(M\) and \(N\) involving the elimination of \(e\) are equal, then, apart from a few easily specified exceptions, the matroids are related via relaxation. From this perspective, it is natural to attempt to generalize Theorem 2.2 by characterizing the matroids \(M\) and \(N\), on a common ground set \(E\), so that, for every \(\{e, f\} \subseteq E\), at least two of the four corresponding minors of \(M\) and \(N\) obtained by eliminating \(e\) and \(f\) are equal. In Theorem 2.5 we prove that, as before, such matroids \(M\) and \(N\) must be related via relaxation unless they belong to a small and easily described class of exceptions.

Before stating Theorem 2.5, we consider a few of the possibilities for the pair \(\{M, N\}\). Suppose that \(M\) and \(N\) are matroids on \(E\) so that, for every element \(e\) in \(E\), we have \(M\setminus e = N\setminus e\) or \(M/e = N/e\). Then, for every \(\{e, f\} \subseteq E\), we have \(M\setminus e, f = N\setminus e, f\) and \(M\setminus e/f = N\setminus e/f\), or \(M/e\setminus f = N/e\setminus f\) and \(M/e, f = N/e, f\).

In addition, for every \(\{e, f\} \subseteq E\), at least two of the four corresponding minors obtained by eliminating \(e\) and \(f\) from the pair \(\{M \oplus U_{0,1}, N \oplus U_{0,1}\}\), or the pair \(\{M \oplus U_{1,1}, N \oplus U_{1,1}\}\) will be equal. Hence, every pair of matroids listed in Theorem 2.2 naturally leads to a family of pairs of matroids listed in Theorem 2.5. In particular, if a pair of matroids \(\{M, N\}\) satisfies Theorem 2.2(i), then the pairs \(\{M, N\}, \{M \oplus U_{0,1}, N \oplus U_{0,1}\}\), and \(\{M \oplus U_{1,1}, N \oplus U_{1,1}\}\) will satisfy Theorem 2.5(i).

Now suppose that \(M\) and \(N\) are as shown in Figure 2.1. Then \(M/e_1, e_2 = N/e_1, e_2\) and \(M/e_1\setminus e_2 = N/e_1\setminus e_2\), while \(M\setminus e_1, e_3 = N/e_1, e_3\) and \(M\setminus e_1/e_3 = N\setminus e_1/e_3\). In addition \(M\setminus e_2, e_3 = N\setminus e_2, e_3\) and \(M/e_2, e_3 = N/e_2, e_3\). Therefore,
for every \(\{e, f\} \subseteq E\), at least two of the four corresponding minors of \(M\) and \(N\) involving the elimination of \(\{e, f\}\) are equal. By symmetry, the same statement holds if \(M\) and \(N\) are as depicted in Figure 2.2. Moreover, it is also true if \(\{M, N\} \cong \{U_{1,3}, U_{2,3}\}\); that is, if \(M \not\cong N\), but each of \(M\) and \(N\) is isomorphic to exactly one of \(U_{1,3}\) and \(U_{2,3}\).

Recall that \(N\) is said to be a double relaxation of \(M\) if \(N\) is obtained from \(M\) by relaxing two circuit-hyperplanes. The next lemma shows that a matroid and one of its double relaxations is another possible pair of matroids \(\{M, N\}\) satisfying Theorem 2.5(i).

**Lemma 2.4.** If \(M\) is obtained from \(N\) by relaxing two circuit-hyperplanes, then, for every \(\{e, f\} \subseteq E(M)\), at least two of \(M \setminus e, f = N \setminus e, f\); \(M \setminus e/f = N \setminus e/f\); \(M/e \setminus f = N/e \setminus f\); and \(M/e, f = N/e, f\) are true statements.
Proof. Assume \( M \) is obtained from some matroid \( P \) by relaxing the circuit-hyperplane \( C_2 \) and \( P \) is obtained from \( N \) by relaxing \( C_1 \). Now, if \( e \) or \( f \) is an element of \( C_1 \cap C_2 \), then the lemma holds. To see this, suppose that \( e \in C_1 \cap C_2 \).

Then, by Lemma 1.2(ii), \( M \setminus e = P \setminus e = N \setminus e \). Therefore \( M \setminus e, f = N \setminus e, f \) and \( M \setminus e / f = N \setminus e / f \) and the lemma holds. By symmetry, if \( e \) or \( f \) is an element of \( E(M) - (C_1 \cup C_2) \), then the lemma holds. Thus we may assume that \( \{e, f\} \subseteq C_1 \Delta C_2 \). Suppose, without loss of generality, that \( e \in C_1 - C_2 \), and \( f \in C_2 - C_1 \).

Then, by Lemma 1.2(ii), \( M \setminus e, f = P \setminus e, f = N \setminus e, f \) while \( M / e, f = P / e, f = N / e, f \). Therefore we may assume, without loss of generality, that \( e, f \in C_1 - C_2 \).

Then, as \( M \setminus e / f = P \setminus e / f = N \setminus e / f \) and \( M / e \setminus f = P / e \setminus f = N / e \setminus f \), the proof of the lemma is completed. \( \Box \)

Theorem 2.5. The following two statements are equivalent for matroids \( M \) and \( N \) on an \( n \)-element set \( E \) where \( n \geq 2 \).

(i) For every \( \{e, f\} \subseteq E \), at least two of \( M \setminus e, f = N \setminus e, f \); \( M / e, f = N / e, f \); \( M \setminus e / f = N \setminus e / f \); and \( M / e \setminus f = N / e \setminus f \) are true statements.

(ii) (a) \( |E| = 2 \); or

(b) \( M = N \); or

(c) \( \{M, N\} \cong \{U_{1,2} \oplus U_{0,1}, U_{1,2} \oplus U_{1,1}\} \) where the loop and coloop in this pair of matroids are labelled differently; or

(d) \( \{M, N\} \) is isomorphic to one of the following:

\( \{U_{1,3}, U_{2,3}\}, \{U_{0,n}, U_{1,n}\}, \{U_{n-1,n}, U_{n,n}\}, \{U_{1,n-1} \oplus U_{1,1}, U_{0,n-1} \oplus U_{1,1}\}, \)

\( \{U_{n-2,n-1} \oplus U_{0,1}, U_{n-1,n-1} \oplus U_{0,1}\}, \{U_{n-2,n-1} \oplus U_{1,1}, U_{n,n}\} \),
\{U_{1,n-1} \oplus U_{0,1}, U_{0,n}\}; 

(e) one of \(M\) and \(N\) is a relaxation of the other; or 

(f) \(M = P \oplus U_{0,1}\) and \(N = Q \oplus U_{0,1}\), where one of \(P\) and \(Q\) is a relaxation of the other; or 

(g) \(M = P \oplus U_{1,1}\) and \(N = Q \oplus U_{1,1}\), where one of \(P\) and \(Q\) is a relaxation of the other; or 

(h) one of \(M\) and \(N\) is a double relaxation of the other; or 

(i) \(M\) and \(N\) both relax to a matroid \(P\).

**Proof.** Assume that (ii) holds. If \(|E| = 2\), or \(M = N\), then clearly (i) holds. Moreover, if one of (ii)(c) through (ii)(g) hold, then, as noted in the introductory remarks at the beginning of this section, (i) holds. Moreover, Lemma 2.4 implies that if (ii)(h) holds, then (i) holds. Suppose that (ii)(i) holds; that is, the matroid \(P\) is obtained from \(M\) and \(N\) by relaxing the circuit-hyperplanes \(X_1\) and \(X_2\), respectively. Assume \({e,f}\) \(\subseteq E\). If \(e \in X_1 \cap X_2\), then, by Lemma 1.2(ii), \(M \setminus e = P \setminus e = N \setminus e\) and it follows that (i) holds. Similarly, if \(e \in E - (X_1 \cup X_2)\), then \(M / e = P / e = N / e\). and hence, (i) holds. We conclude that \({e,f}\) \(\subseteq X_1 \Delta X_2\).

Suppose \(e \in X_1 - X_2\) and \(f \in X_2 - X_1\). Therefore \(M \setminus e, f = P \setminus e, f = N \setminus e, f\) and \(M / e, f = P / e, f = N / e, f\). Thus we may assume, without loss of generality, that \({e,f}\) \(\subseteq X_1 - X_2\). Then \(M \setminus e / f = P \setminus e / f = N \setminus e / f\), while \(M / e \setminus f = P / e \setminus f = N / e \setminus f\) and (i) holds. We conclude that (ii) implies (i).

Now suppose that (i) holds. Since the theorem clearly holds if \(|E| = 2\), assume that \(|E| \geq 3\). Let \(Z = \{e \in E : M \setminus e \neq N \setminus e\} \text{ and } M / e \neq N / e\}. \) If \(Z = \emptyset\), then the
result follows by Theorem 2.2. So assume that

(2.6) \( e \) is an element of \( Z \).

The strategy of the proof is first to show that, for every element \( f \) of \( E - e \), we may assume that \((M\setminus e)\setminus f = (N\setminus e)\setminus f\) or \((M\setminus e)/f = (N\setminus e)/f\), and \((M/e)\setminus f = (N/e)\setminus f\) or \((M/e)/f = (N/e)/f\). Consequently, Theorem 2.2 implies that there are a limited number of ways that the matroids in the pairs \{\( M\setminus e, N\setminus e \)\} and \{\( M/e, N/e \)\} can be related. The last part of the proof involves analyzing these cases.

First observe the following:

(2.7) No element of \( E \) is a loop in one of \( M \) and \( N \) and a coloop in the other.

To see this, suppose that \( f \) is a loop of \( M \) and a coloop of \( N \) while \( g, h \in E - u \). Then, as \( f \) is a loop of each minor of \( M \) and a coloop of each minor of \( N \), we have \( M\setminus g, h \neq N\setminus g, h; M\setminus g/h \neq N\setminus g/h; M/g\setminus h \neq N/g\setminus h; \) and \( M/g, h \neq N/g, h; \) a contradiction.

Now, as \( e \in Z \), we have \( M\setminus e \neq N\setminus e \) and \( M/e \neq N/e \). However, if \( f \in E - e \), then at least two of \( M\setminus e, f = N\setminus e, f; M\setminus e/f = N\setminus e/f; M/e\setminus f = N/e\setminus f; \) and \( M/e, f = N/e, f \) must be true statements. In the next two lemmas we determine how \( M \) and \( N \) are related if both \( M\setminus e, f = N\setminus e, f \) and \( M\setminus e/f = N\setminus e/f \) or if both \( M/e\setminus f = N/e\setminus f \) and \( M/e, f = N/e, f \).

**Lemma 2.8.** Suppose that \( M\setminus e, f = N\setminus e, f \) and \( M\setminus e/f = N\setminus e/f \) for some \( f \) in \( E - e \). Then \( |E| = 3 \) and one of the matroids \( M \) and \( N \) is isomorphic to
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$U_{1,2} \oplus U_{1,1}$ while the other is isomorphic to $U_{1,2} \oplus U_{0,1}$ where each can be labelled as in Figure 2.1.

**Proof.** By Lemma 2.1, either $M \setminus e = N \setminus e$, or $f$ is a loop of one of $M \setminus e$ and $N \setminus e$ and a coloop of the other. The assumption that $e$ is an element of $Z$ implies that $M \setminus e \neq N \setminus e$, and hence we may assume, without loss of generality, that $f$ is a loop of $M \setminus e$ and a coloop of $N \setminus e$. Therefore $f$ is a loop of $M$, and $\{e, f\}$ contains a cocircuit of $N$ containing $f$. Suppose $f$ is a coloop of $N$. Then, as $f$ is a loop of $M$, (2.7) implies that $f$ is not a coloop of $N$. We conclude that $\{e, f\}$ is a cocircuit of $N$.

Let $g$ be an element of $E - \{e, f\}$. Then, as $f$ is a loop in both $M \setminus e, g$ and $M \setminus e / g$ while being a coloop in both $N \setminus e, g$ and $N \setminus e / g$, we have $M \setminus e, g \neq N \setminus e, g$ and $M \setminus e / g \neq N \setminus e / g$. Therefore $M / e, g = N / e, g$ and $M / e / g = N / e / g$. Furthermore, as $M / e \neq N / e$. Lemma 2.1 implies that $g$ is a loop of one of $\{M / e, N / e\}$ and a coloop of the other. Now suppose there is an element $h$ contained in $E - \{e, f, g\}$. Then, as $f$ is a loop in both $M \setminus e, h$ and $M \setminus e / h$ while being a coloop in both $N \setminus e, h$ and $N \setminus e / h$, we have $M \setminus e / h \neq N \setminus e / h$ and $M \setminus e / h \neq N \setminus e / h$. Consequently $M / e, h = N / e, h$ and $M / e / h = N / e / h$. However, $M / e / h \neq N / e / h$ since $g$ is a loop in one of the matroids and a coloop in the other. Thus $E = \{e, f, g\}$. Since $f$ is a loop in $M / e \setminus g$ which equals $N / e \setminus g$, it is evident that $\{e, f\}$ contains a circuit of $N$ containing $f$. Moreover, since $\{e, f\}$ is a cocircuit of $N$, and the intersection of a circuit and a cocircuit cannot have cardinality one, we conclude
that \( \{e, f\} \) is both a circuit and a cocircuit of \( N \). Therefore \( N \) is either \( N_1 \) or \( N_2 \) where each of these matroids is depicted in Figure 2.3.

\begin{figure}[h]
\centering
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=0.8\textwidth]{figure23a.png}
\caption{(a) \( N_1 \).}
\end{subfigure}
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=0.8\textwidth]{figure23b.png}
\caption{(b) \( N_2 \).}
\end{subfigure}
\caption{(a) \( N_1 \). (b) \( N_2 \).}
\end{figure}

Suppose \( N = N_1 \). Now by assumption, \( f \) is a loop of \( M \). Moreover, as \( M \setminus e, f = N \setminus e, f \), we deduce that \( g \) is also a loop of \( M \). It follows that \( M \) is one of the matroids \( M_1 \) and \( M_2 \) shown in Figure 2.4.

\begin{figure}[h]
\centering
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=0.8\textwidth]{figure24a.png}
\caption{(a) \( M_1 \).}
\end{subfigure}
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=0.8\textwidth]{figure24b.png}
\caption{(b) \( M_2 \).}
\end{subfigure}
\caption{(a) \( M_1 \). (b) \( M_2 \).}
\end{figure}

However, if \( N = N_1 \), and \( M \) is either of \( M_1 \) and \( M_2 \), then \( M \setminus e = N \setminus e \). Since this contradicts the assumption that \( e \in Z \), we conclude that \( N = N_2 \). Now \( g \) is independent in \( N_2 \setminus e, f \) which equals \( M \setminus e, f \). Moreover, as \( f \) is a loop of \( M \), we deduce that \( M \) is one of the matroids \( M_3, M_4, \) and \( M_5 \) shown in Figure 2.5.

Now if \( N = N_2 \), and \( M \) is either \( M_3 \) or \( M_5 \), then \( M \setminus e = N \setminus e \). Since this violates the assumption that \( e \in Z \), we conclude that \( N = N_2 \) and \( M = M_4 \).
Moreover, by relabelling $e$, $g$, and $f$ as $e_1$, $e_2$, and $e_3$, respectively, we deduce that the lemma holds.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.5.png}
\caption{(a) $M_3$ (b) $M_4$ (c) $M_5$}
\end{figure}

**Lemma 2.9.** Suppose that $M/e, f = N/e, f$ and $M/e \setminus f = N/e \setminus f$ for some element $f$ of $E - e$. Then $|E| = 3$ and one of the matroids $M$ and $N$ is isomorphic to $U_{1,2} \oplus U_{1,1}$ while the other is isomorphic to $U_{1,2} \oplus U_{0,1}$ where each can be labelled as in Figure 2.1.

**Proof.** Evidently $M^* \setminus e, f = N^* \setminus e, f$ and $M^* \setminus e / f = N^* \setminus e / f$. It follows, by Lemma 2.8. that $\{M, N\} \cong \{U_{1,2} \oplus U_{0,1}, U_{1,2} \oplus U_{1,1}\}$ where each matroid is labelled as in Figure 2.6. Then, upon interchanging $e_2$ and $e_3$, we have that one of $M$ and $N$ is isomorphic to $U_{1,2} \oplus U_{0,1}$ and the other is isomorphic to $U_{1,2} \oplus U_{1,1}$ and each is labelled as in Figure 2.1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.6.png}
\caption{The matroids $M$ and $N$.}
\end{figure}
By Lemmas 2.8 and 2.9 we may now assume that, for all \( f \) in \( E - e \),

\[(2.10)\] either \( M \setminus e, f = N \setminus e, f \) or \( M \setminus e/f = N \setminus e/f \); and

\[(2.11)\] either \( M/e \setminus f = N/e \setminus f \) or \( M/e, f = N/e, f \).

Since \( e \in Z_r \) and hence \( M \setminus e \neq N \setminus e \), Theorem 2.2 and (2.10) imply that

\[(2.12)\] (a) one of \( M \setminus e \) and \( N \setminus e \) is a relaxation of the other; or

(b) \( \{M \setminus e, N \setminus e\} \cong \{U_{n-2,n-1}, U_{n-1,n-1}\} \); or

(c) \( \{M \setminus e, N \setminus e\} \cong \{U_{1,n-1}, U_{0,n-1}\} \).

Similarly, Theorem 2.2 and (2.11) imply that

\[(2.13)\] (a) one of \( M/e \) and \( N/e \) is a relaxation of the other; or

(b) \( \{M/e, N/e\} \cong \{U_{n-2,n-1}, U_{n-1,n-1}\} \); or

(c) \( \{M/e, N/e\} \cong \{U_{1,n-1}, U_{0,n-1}\} \).

In the next two lemmas, we establish the possibilities for the pair \( \{M, N\} \) if

the matroids satisfy (2.12)(b), (2.12)(c), (2.13)(b), or (2.13)(c).

**Lemma 2.14.** If \( \{M \setminus e, N \setminus e\} \cong \{U_{1,n-1}, U_{0,n-1}\} \), then \( \{M, N\} \) is isomorphic to either \( \{U_{1,n-1} \oplus U_{1,1}, U_{0,n-1} \oplus U_{1,1}\} \), or \( \{U_{1,n-1} \oplus U_{0,1}, U_{0,n}\} \). Dually, if \( \{M/e, N/e\} \cong \{U_{n-2,n-1}, U_{n-1,n-1}\} \), then \( \{M, N\} \) is isomorphic to either \( \{U_{n-2,n-1} \oplus U_{0,1}, U_{n-1,n-1} \oplus U_{0,1}\} \), or \( \{U_{n-2,n-1} \oplus U_{1,1}, U_{n,n}\} \).

**Proof.** Suppose, without loss of generality, that \( M \setminus e \cong U_{1,n-1} \) and \( N \setminus e \cong U_{0,n-1} \).

Since \( e \) can only be adjoined to \( N \) as a loop or as a coloop, we have \( N \cong U_{0,n} \) or \( N \cong U_{0,n-1} \oplus U_{1,1} \). On the other hand, \( e \) can be adjoined to \( M \) as a loop, as a coloop, or as an element of the parallel class \( E - e \) of \( M \). Therefore \( M \) is isomorphic to one of \( U_{1,n-1} \oplus U_{0,1}; U_{1,n-1} \oplus U_{1,1} \); or \( U_{1,n} \). First suppose \( M \) is
isomorphic to $U_{1,n-1} \oplus U_{1,1}$. Now if $N$ is isomorphic to $U_{0,n}$, then $e$ is a loop in $N$ and a coloop of $M$, a contradiction to (2.7). We conclude that $N \cong U_{0,n-1} \oplus U_{1,1}$ and the lemma holds.

Now suppose that $M$ is isomorphic to $U_{1,n}$. If $N \cong U_{0,n}$, then $M/e = N/e$ which contradicts the assumption that $e \in Z$. Therefore $N \cong U_{0,n-1} \oplus U_{1,1}$. Let $\{f,g\} \subseteq E - e$. Then $e$ is a coloop in each of $N/f \setminus g; N \setminus f/g$; and $N/f, g$, while $e$ is a loop in the corresponding minors of $M$; a contradiction. We conclude that $M \not\cong U_{1,n}$.

Finally, we may assume that $M \cong U_{1,n-1} \oplus U_{0,1}$. If $N \cong U_{0,n-1} \oplus U_{1,1}$, then $e$ is a loop in $M$ and a coloop in $N$ a contradiction to (2.7). We conclude that $N \cong U_{0,n}$ and the lemma holds. □

Lemma 2.15. If $\{M/e, N/e\} \cong \{U_{1,n-1}, U_{0,n-1}\}$, then the pair $\{M,N\}$ is listed in (ii)(c), (ii)(d), or (ii)(i). Dually, if $\{M/e, N/e\} = \{U_{n-2,n-1}, U_{n-1,n-1}\}$, then the pair $\{M,N\}$ is listed in (ii)(c), (ii)(d), or (ii)(i).

Proof. Suppose that $M/e \cong U_{1,n-1}$ and $N/e \cong U_{0,n-1}$. It follows that $M \cong U_{1,n-1} \oplus U_{0,1}$, or the geometric representation of $M$ is a line that has no element in parallel with $e$. Similarly, $N$ is isomorphic to one of $U_{0,n}; U_{0,n-1} \oplus U_{1,1}; U_{1,n};$ and $U_{0,k} \oplus U_{1,j}$ where $j + k = n$, where $j$ and $k$ are positive integers that sum to $n$.

First, we assume that $N \cong U_{0,n}$. Now if $M \cong U_{1,n-1} \oplus U_{0,1}$, then $M$ and $N$ satisfy (ii)(d) and the lemma holds. If $M \cong U_{1,n-1} \oplus U_{1,1}$, then $e$ is a loop in $N$ but a coloop in $M$, which contradicts (2.7). Thus we may assume that
the geometric representation of $M$ is an $n$-element line with at least two distinct, possibly empty, parallel classes other than the point $e$. Let $f$ and $g$ be elements of $E$ that are in distinct parallel classes. Then, as $e$ is a loop in every minor of $N$, yet $e$ is independent in $M \setminus f/g; M/f \setminus g$; and $M \setminus f, g$, we arrive at a contradiction.

Now assume that $N \cong U_{0,n-1} \oplus U_{1,1}$. If $M \cong U_{1,n-1} \oplus U_{0,1}$, then $e$ is a loop in $M$ and a coloop in $N$; contrary to (2.7). Thus we assume that $M \cong U_{1,n-1} \oplus U_{1,1}$.

In this case, $M$ and $N$ satisfy (ii)(d) and the lemma holds. Now suppose that the geometric representation of $M$ is an $n$-element line with at least two distinct, but possibly empty, parallel classes other than the point $e$. In addition, suppose $f$ and $g$ are in distinct parallel classes. Now $f$ is a loop in all minors of $N$. However, of the minors of $M$ involving the elimination of $e$ and $g$, only $M/e, g$ has $f$ as a loop.

We conclude that $N \not\cong U_{0,n-1} \oplus U_{1,1}$.

Now suppose $N \cong U_{1,n}$. If $M$ is isomorphic to $U_{1,n-1} \oplus U_{0,1}$ or $U_{1,n-1} \oplus U_{1,1}$, then $M \setminus e = N \setminus e \cong U_{1,n-1}$. As this contradicts the assumption that $e \in Z$, we may suppose that the geometric representation of $M$ is an $n$-element line with at least two distinct, but possibly empty, parallel classes other than the point $e$.

Suppose that $\{e, f, g, h\} \subseteq E$. Since $\{e, h\}$ is independent in $M$ and dependent in $N$, we have that $M \setminus f, g \neq N \setminus f, g$. Moreover, $e$ is independent in both $M/f \setminus g$ and $M/f \setminus g$ but a loop in each of $N/f \setminus g$ and $N/f \setminus g$. As a result of this contradiction, we conclude that $|E| = 3$. Thus $M \cong U_{2,3}$ and $N \cong U_{1,3}$ and the lemma holds.
Finally, we assume that \( N \cong U_{0,k} \oplus U_{1,j} \) where \( j \) and \( k \) are positive and sum to \( n \). Moreover, since the case in which \( k = n - 1 \) and \( j = 1 \) has been handled previously, we may assume that \( j \geq 2 \).

Suppose that \( M \cong U_{1,n-1} \oplus U_{0,1} \). There are two cases to consider. In the first case assume that \( k \geq 2 \) and let \( f \) and \( g \) be elements of \( E \) that are loops in \( N \). However, as \( e \) is dependent in every minor of \( M \), but independent in each of \( N \setminus f, g; N \setminus f / g; N / f \setminus g; \) and \( N / f, g \), we have a contradiction. Thus we may assume that \( k = 1 \). Then, as \( M \) and \( N \) are isomorphic to \( U_{0,1} \oplus U_{1,n-1} \), both matroids relax to \( U_{1,n} \) and the lemma holds.

Now suppose that \( M \cong U_{1,n-1} \oplus U_{1,1} \). If \( k \geq 2 \), then there are elements \( f \) and \( g \) that are loops of \( N \). Now, as \( j \geq 2 \), the element \( e \) is not a coloop of \( N \setminus f, g; N \setminus f / g; N / f \setminus g; \) or \( N / f, g \). Since \( e \) is a coloop of each corresponding minor of \( M \), we have a contradiction. Thus we may assume that \( k = 1 \). Now, if \( j \geq 3 \), then there are elements \( f \) and \( g \) of \( E \) that are in parallel with \( e \) in \( N \). Then \( e \) is a loop in each of \( N \setminus f / g; N / f \setminus g; \) and \( N / f, g \). However, as \( e \) is a coloop in each minor of \( M \) it follows that \( M \setminus f, g \neq N \setminus f, g; M \setminus f / g \neq N \setminus f / g; M / f \setminus g \neq N / f \setminus g; \) and \( M / f, g \neq N / f, g; \) a contradiction. Therefore \( |E| = 3 \) and \( M \) and \( N \) satisfy (ii)(c).

Now assume that the geometric representation of \( M \) is an \( n \)-element line with at least two distinct, but possibly empty, parallel classes other than the point \( e \). Suppose that \( k \geq 1 \) and let \( f \) be an element that is a loop in \( N \). Now, there is an element \( g \), distinct from \( e \), that is not in parallel with \( f \). Therefore, although \( f \)
is a loop in every minor of $N$, the element $f$ is not a loop in $M/e\backslash g$; $M/g\backslash e$; or $M\backslash e, g$. This contradiction completes the proof of Lemma 2.9.

By (2.12) and Lemmas 2.14 and 2.15, we may now assume that $M\backslash e$ and $N\backslash e$ are related via relaxation of a circuit-hyperplane $H_d$. Analogously, by (2.13) and Lemmas 2.14 and 2.15, we may assume that $M/e$ and $N/e$ are related via relaxation of a circuit-hyperplane $H_e$. Suppose that $e$ is a loop of $M$ or $N$. Without loss of generality, we assume that $e$ is a loop of $M$. Then $\tau(N\backslash e) = \tau(M\backslash e) = \tau(M/e) = \tau(N/e)$ and hence $e$ is a loop or coloop of $N$. Moreover, as (2.7) implies that $e$ is not a coloop of $N$, we deduce that $e$ is a loop of $N$. Thus $M = (M\backslash e) \oplus U_{0,1}$ and $N = (N\backslash e) \oplus U_{0,1}$. As $M\backslash e$ and $N\backslash e$ are related via relaxation, $M = P \oplus U_{0,1}$ and $N = Q \oplus U_{0,1}$ where one of $P$ and $Q$ is a relaxation of the other. We conclude, by duality, that if $M$ or $N$ has $e$ as either a loop or a coloop, then $M$ and $N$ satisfy (ii)(f) or (ii)(g) and the theorem holds.

We may now assume that $e$ is neither a loop nor a coloop in $M$ or $N$. Since $M\backslash e$ and $N\backslash e$ are related via a relaxation of $H_d$, Lemma 1.2(iii) implies that $\mathcal{B}(M\backslash e) \Delta \mathcal{B}(N\backslash e) = \{H_d\}$. Similarly, $\mathcal{B}(M/e) \Delta \mathcal{B}(N/e) = \{H_e\}$. Moreover, as $e$ is neither a loop nor a coloop of $M$ or $N$, we have $\mathcal{B}(P) = \mathcal{B}(P\backslash e) \cup \{B \cup e : B \in \mathcal{B}(P/e)\}$ for both members $P$ of $\{M, N\}$. Thus $\mathcal{B}(M) \Delta \mathcal{B}(N) = \{H_d, H_e \cup e\}$. Without loss of generality, we may also suppose that $M\backslash e$ is obtained by relaxing the circuit-hyperplane $H_d$ of $N\backslash e$. Thus, $H_d$ is a basis of $M$ and a circuit of $N$. Also, either $H_d$ or $H_d \cup e$ is a hyperplane of $N$. First, let us assume that $H_d$ is a hyperplane of $N$. Then $H_d$ is a circuit-hyperplane of $N$ and a basis of $M$. 
Lemma 2.16. Suppose that $M\setminus e$ is obtained by relaxing the circuit-hyperplane $H_d$ of $N\setminus e$, and that $M/e$ and $N/e$ are related via relaxation of $H_c$. If $H_d$ is a circuit-hyperplane of $N$ and a basis of $M$, then both $M$ and $N$ relax to a matroid $P$, or $M$ is a double relaxation of $N$.

Proof. $H_c$ is a basis of one of $\{M/e, N/e\}$ and a circuit-hyperplane of the other.

Now we show that

(2.17) $H_c \cup e$ is a basis of one of $\{M, N\}$ and a circuit-hyperplane of the other.

To see this, assume, without loss of generality, that $H_c$ is a basis of $M/e$ and a circuit-hyperplane of $N/e$. Then $H_c \cup e$ is a basis of $M$ and a hyperplane of $N$. In addition, either $H_c$ or $H_c \cup e$ is a circuit of $N$. Suppose that $H_c$ is a circuit of $N$. Then, as $H_c \cup e$ is a hyperplane of $N$, there is a subset $X$ of $H_c$ such that $X \cup e$ is a circuit of $N$. Now $X$ is non-empty since $e$ is not a loop of $N$. Moreover, $X \neq H_c$ since we have assumed that $H_c$ is a circuit of $N$. However, if $X$ is a proper subset of $H_c$, then $H_c$ properly contains the circuit $X$ of $N/e$. Since this contradicts the assumption that $H_c$ is a circuit of $N/e$, we conclude that $H_c \cup e$ must be a circuit of $N$. By symmetry, the proof of (2.17) is complete.

Now suppose $H_c \cup e$ is a basis of $M$ and a circuit-hyperplane of $N$. Then both $H_c \cup e$ and $H_d$ are circuit-hyperplanes of $N$ and bases of $M$. Thus $B(M) = B(N) \cup \{H_c \cup e, H_d\}$ and it follows that $M$ is obtained from $N$ by relaxing the circuit-hyperplanes $H_d$ and $H_c \cup e$. Hence $M$ is a double relaxation of $N$.

We may now assume that $H_c \cup e$ is a basis of $N$ and a circuit-hyperplane of $M$. Since $H_d$ is a circuit-hyperplane of $N$ and a basis of $M$, we deduce that
\[ \mathcal{B}(N) \cup H_d = \mathcal{B}(M) \cup (H_c \cup e). \] Now \( \mathcal{B}(N) \cup H_d \) is the collection of bases of a matroid obtained from \( N \) by relaxing the circuit-hyperplane \( H_d \). Analogously, \( \mathcal{B}(M) \cup (H_c \cup e) \) is the collection of bases of a matroid obtained from \( M \) by relaxing the circuit-hyperplane \( H_c \cup e \). Thus \( M \) and \( N \) both relax to a matroid \( P \) and the lemma holds.

To complete the proof of Theorem 2.5, we now assume that \( H_d \cup e \) is a hyperplane of \( N \), while \( H_d \) is a basis of \( M/e \) and a circuit-hyperplane of \( N/e \). Then, as \( H_d \) is a circuit and \( H_d \cup e \) is a hyperplane of \( N \), we deduce that \( X \cup e \) is a circuit of \( N \) for some \( X \subseteq H_d \). Therefore \( X \) is a circuit in \( N/e \). Now, as \( N/e \) and \( M/e \) are related via a relaxation of \( H_c \), it follows, from Lemma 1.2(iii), that \( \mathcal{I}(N/e) \Delta \mathcal{I}(M/e) = \{H_c\} \). If \( X \) is independent in \( M/e \), then we deduce that \( X = H_c \), and hence, \( X \) is a circuit-hyperplane of \( N/e \) and a basis of \( M/e \). In particular, since \( X \) is a hyperplane of \( N/e \), it follows that \( X \cup e \), and hence, \( H_c \cup e \), is a hyperplane of \( N \). However, \( H_c \cup e \) is a proper subset of \( H_d \cup e \) which is a hyperplane of \( N \). Thus we conclude that \( X \) is dependent in \( M/e \). It follows, from Lemma 1.2(iv), that either \( X \) is a circuit of \( M/e \), or \( X = H_c \cup e \) where \( H_c \) is a circuit-hyperplane of \( M/e \) and a basis of \( N/e \). Suppose \( X \) is a circuit of \( M/e \). Then either \( X \) or \( X \cup e \) is a circuit of \( M \). Now, as \( X \subseteq H_d \) and \( H_d \) is a basis of \( M \), we have that \( X \) is independent in \( M \). Thus \( X \cup e \) is a circuit of \( M \). Moreover, as \( X \cup e \) and \( H_d \) are circuits of \( N \), circuit elimination implies that there is a circuit \( Y \cup e \) of \( N \) so that \( Y \cup e \subseteq (H_d - f) \cup e \). Thus \( Y \) is a circuit of \( N/e \). Suppose \( Y \) is also a circuit of \( M/e \). Then, as \( Y \) is a proper subset of the basis \( H_d \)
of $M$, we have that $Y$ is independent in $M$. Therefore $Y \cup e$ is a circuit of $M$. However, as $X \cup e$ and $Y \cup e$ are circuits of $M$ and both $X$ and $Y$ are subsets of $H_d$, circuit elimination implies that there is a circuit of $M$ contained in $H_d$. Since this contradicts the assumption that $H_d$ is a basis of $M$, we conclude that $Y$ is not a circuit of $M/e$. Moreover, if $Y$ is independent in $M/e$, then, as before, it follows from Lemma 1.2(iv) that $Y = H_c$. But then $Y \cup e$ is a hyperplane of $N$ properly contained in the hyperplane $H_d \cup e$ of $N$; a contradiction. We conclude that $Y$ is a non-minimal dependent set in $M/e$. Since $M/e$ and $N/e$ are related via relaxation, and $Y$ is a circuit of $N/e$, it follows that $Y = H_c \cup e$ where $H_c$ is a circuit-hyperplane of $M/e$ and a basis of $N/e$.

Now, as $H_c$ is a circuit-hyperplane of $M/e$, either $H_c$ or $H_c \cup e$ is a circuit of $M$. Since $H_c$ is a proper subset of the basis $H_d$ of $M$, it follows that $H_c \cup e$ is a circuit of $M$. Furthermore, $H_c \cup e$ is a hyperplane of $M$ since $H_c$ is a hyperplane of $M/e$. Thus $H_c \cup e$ is a circuit-hyperplane of $M$ and a basis of $N$. Moreover, $H_d$ is a basis of $M$ and a circuit but not a hyperplane of $N$. Therefore $\mathcal{B}(N) \cup \{H_d\} = \mathcal{B}(M) \cup \{H_c \cup e\}$. Dually, $\mathcal{B}(N^*) \cup \{E - H_d\} = \mathcal{B}(M^*) \cup \{E - (H_c \cup e)\}$. Since $H_d \cup e$ spans $M$, we have that $\{E - (H_d \cup e)\} \in \mathcal{I}(M^*)$. Moreover, as $e$ is not a coloop of $M^*$, there is a basis $B$ of $M^*$, other than $E - H_d$, that contains $\{E - (H_d \cup e)\}$. In particular, $B$ is a basis of $N^*$, and $\{E - (H_d \cup e)\} \in \mathcal{I}(N^*)$. However, as $H_d \cup e$ is a hyperplane of $N$, the set $E - (H_d \cup e)$ is a circuit of $N^*$. This contradiction completes the proof of Theorem 2.5. □
Next we list some corollaries of Theorem 2.5. Each of these follows by de-
termining which pairs of matroids in the statement of the theorem also satisfy
the more restrictive hypotheses. For instance, the Corollary 2.18 is simply a re-
statement of Theorem 2.5 with the additional assumption that both matroids are
connected.

**Corollary 2.18.** Let $M$ and $N$ be distinct connected matroids on a set $E$ contain-
ing at least two elements. For every $\{e, f\} \subseteq E$, at least two of $M\setminus e, f = N\setminus e, f$;
$M\setminus e/f = N\setminus e/f$; $M/e\setminus f = N/e\setminus f$; and $M/e, f = N/e, f$ are true statements if
and only if

(i) $\{M, N\} \cong \{U_{1,3}, U_{2,3}\}$; or

(ii) one of $M$ and $N$ is a relaxation of the other; or

(iii) one of $M$ and $N$ is a double relaxation of the other; or

(iv) $M$ and $N$ both relax to a matroid $P$.

The effort to generalize Theorem 2.2, which we have seen culminates in The-
orem 2.5, originally produced direct proofs of the next two corollaries. We omit
the straightforward proof of the first. However, as the lists of pairs of matroids
in Corollary 2.20 and Theorem 2.2 are essentially the same, we present a direct
proof of Corollary 2.20 using Theorem 2.2.

**Corollary 2.19.** Let $M$ and $N$ be matroids on an $n$-element set $E$, such that,
for every $\{e, f\} \subseteq E$, $M\setminus e, f = N\setminus e, f$; $M\setminus e/f = N\setminus e/f$; $M/e\setminus f = N/e\setminus f$; and
$M/e, f = N/e, f$. Then $|E| = 2$ or $M = N$. 
Corollary 2.20. Let $M$ and $N$ be matroids on an $n$-element set $E$ such that, for every $\{e, f\} \subseteq E$, at least three of $M \setminus e, f = N \setminus e, f$; $M \setminus e / f = N \setminus e / f$; $M / e \setminus f = N / e \setminus f$; and $M / e, f = N / e, f$ are true statements. Then

(i) $|E| = 2$; or

(ii) $M = N$; or

(iii) one of $\{M, N\}$ is a relaxation of the other; or

(iv) $\{M, N\} \cong \{U_{n-1, n}, U_{n, n}\}$; or

(v) $\{M, N\} \cong \{U_{1, n}, U_{0, n}\}$.

Proof. Since the result clearly holds if $|E| = 2$, assume that $|E| \geq 3$. Let $Z = \{e : M \setminus e \neq N \setminus e$ and $M / e \neq N / e\}$. If $Z = \emptyset$, then, by Theorem 2.2, the result holds. Thus we may suppose that $e$ is an element of $Z$, and $f$ is an element of $E - e$. Then, as at least three of the four corresponding minors of $M$ and $N$ involving the elimination of $e$ and $f$ must be equal, we have

(2.21) $(M \setminus e) \setminus f = (N \setminus e) \setminus f$ and $(M \setminus e) / f = (N \setminus e) / f$; or

(2.22) $(M / e) \setminus f = (N / e) \setminus f$ and $(M / e) / f = (N / e) / f$.

In addition, as $e \in Z$, we have that $M \setminus e$ and $N \setminus e$ are matroids on $E - e$ such that $(M \setminus e) \setminus g = (N \setminus e) \setminus g$ or $(M \setminus e) / g = (N \setminus e) / g$ for every $g$ in $E - e$. Then, by Theorem 2.2, one of $\{M \setminus e, N \setminus e\}$ is a relaxation of the other, $\{M \setminus e, N \setminus e\}$ is isomorphic to $\{U_{n-2, n-1}, U_{n-1, n-1}\}$, or $\{M \setminus e, N \setminus e\}$ is isomorphic to $\{U_{1, n-1}, U_{0, n-1}\}$. However, in each of these three cases (2.21) fails to hold. To see this, first suppose that $\{M \setminus e, N \setminus e\} \cong \{U_{1, n-1}, U_{0, n-1}\}$. Then $(M \setminus e) \setminus f \neq (N \setminus e) \setminus f$, and hence (2.14) cannot hold. Furthermore, if $\{M \setminus e, N \setminus e\}$ is isomorphic to $\{U_{n-2, n-1}, U_{n-1, n-1}\}$,
then $(M\setminus e)/f \neq (N\setminus e)/f$; a contradiction to (2.21). Now assume, without loss of
generality, that $N\setminus e$ is obtained from $M\setminus e$ by relaxing the circuit-hyperplane $C$.
If $f \in C$, then, provided $f$ is not a loop of $M\setminus e$, Lemma 1.2(ii) implies that the
matroid $N\setminus e/f$ is obtained from $M\setminus e/f$ by relaxing the circuit-hyperplane $C - f$.
In particular, $M\setminus e/f \neq N\setminus e/f$, if $f$ is independent in $M\setminus e$. If $f$ is a loop of $M\setminus e$,
then $f$ is the circuit-hyperplane $C$. Consequently $M\setminus e \cong U_{0,1} \oplus U_{1,k}$ with $f$ being
the loop, and $N\setminus e \cong U_{1,k+1}$ for some $k \geq 2$. Thus $M\setminus e/f \neq N\setminus e/f$. If, on the
other hand, $f \in E - C$, then, provided $f$ is not a coloop of $M\setminus e$, Lemma 1.2(i)
implies that $N\setminus e, f$ is obtained from $M\setminus e, f$ by relaxing the circuit-hyperplane $C$.
Thus (2.21) fails to hold, unless $f$ is a coloop in $M\setminus e$. However, if $f$ is a coloop
in $M\setminus e$, then $M\setminus e \cong U_{1,1} \oplus U_{k-1,k}$ with $f$ being the coloop, and $N\setminus e \cong U_{k,k+1}$
for some $k \geq 2$. Therefore $M\setminus e, f \neq N\setminus e, f$ and we conclude that (2.22) holds.
However, since $e \in Z$, we have that $M/e$ and $N/e$ are matroids on $E - e$ such
that $(M/e)\setminus g = (N/e)\setminus g$ or $(M/e)/g = (N/e)/g$ for every $g \in E - e$. Then, as
before, Theorem 2.2 implies that one of $\{M/e, N/e\}$ is a relaxation of the other,
$\{M/e, N/e\}$ is isomorphic to $\{U_{n-2,n-1}, U_{n-1,n-1}\}$, or $\{M/e, N/e\}$ is isomorphic
to $\{U_{1,n-1}, U_{0,n-1}\}$. By symmetry, (2.22) fails to hold in each of these cases.
However, the fact that neither (2.21) nor (2.22) holds is a contradiction, and we
conclude that $e \notin Z$. Thus $Z = \emptyset$ and Theorem 2.2 implies that the corollary
holds. \qed
CHAPTER 3
A CLASS OF NON-BINARY MATROIDS

In this chapter, we present the main result of the dissertation: a characterization of some non-binary matroids that are, in a sense, close to being binary. Tutte [15] proved that $U_{2,4}$ is the only non-binary matroid $M$ such that, for every element $e$, both $M\setminus e$ and $M/e$ are binary. Moreover, Oxley [10] characterized the non-binary matroids $M$ such that, for every element $e$ of $M$, the deletion $M\setminus e$ or the contraction $M/e$ is binary. In this chapter, we characterize those non-binary matroids $M$ such that, for all elements $e$ and $f$, at least two of $M\setminus e, f; M\setminus e/f; M/e\setminus f; \text{ and } M/e, f$ are binary.

3.1. Preliminaries

In his 1990 paper [10], Oxley proved the following generalization of Tutte's excluded minor characterization of binary matroids:

**Theorem 3.1.** The following two statements are equivalent for a matroid $M$.

(i) $M$ is non-binary and, for every element $e$, $M\setminus e$ or $M/e$ is binary.

(ii) (a) Both the rank and corank of $M$ exceed two and $M$ can be obtained from a connected binary matroid by relaxing a circuit-hyperplane; or

(b) $M$ is isomorphic to a parallel extension of $U_{2,n}$ for some $n \geq 5$; or

(c) $M$ is isomorphic to a series extension of $U_{n-2,n}$ for some $n \geq 5$; or

(d) $M$ can be obtained from $U_{2,4}$ by series extension of a subset $S$ of $E(U_{2,4})$ and parallel extension of a disjoint subset $T$ of $E(U_{2,4})$ where $S$ or $T$ may be empty.
A major step in Oxley's proof of Theorem 3.1 is the following weaker version for 3-connected matroids.

**Theorem 3.2.** The following two statements are equivalent for a matroid $M$.

(i) $M$ is non-binary, 3-connected, and, for every element $e$, $M\setminus e$ or $M/e$ is binary.

(ii) (a) $M$ is isomorphic to $U_{2,n}$ or $U_{n-2,n}$ for some $n \geq 4$; or

(b) both the rank and corank of $M$ exceed two and $M$ can be obtained from a 3-connected binary matroid by relaxing a circuit-hyperplane.

Now, considering the fact that we were able to extend Theorem 2.2, a result involving minors obtained by eliminating one element of the ground set, to Theorem 2.5, a result involving minors obtained by eliminating two elements of the ground set, it is natural to attempt to obtain an analogous extension of Theorem 3.1. Theorem 3.6 is a 3-connected version of such an extension and is proved in Section 3.2. This theorem is then used in Section 3.3 to obtain a version for matroids in general. The proof of Theorem 3.6 that we present is long. The apparent increase in the level of difficulty from obtaining an extension of Theorem 2.2 to obtaining an extension of Theorem 3.1 may be due to the fact that the former deals with equality of minors while the latter involves the presence or absence of an isomorphic copy of $U_{2,4}$ in the minors.

In the remainder of this section we list some results used in the proof of Theorem 3.6. The first lemma, due to Oxley [9], is an alternative generalization of Tutte's excluded minor characterization of binary matroids.
Lemma 3.3. Let $M$ be a non-binary matroid such that, for some element $e$, both $M \setminus e$ and $M/e$ are binary. Then $M$ is obtained from a 4-point line having ground set $\{e, e_1, e_2, e_3\}$ by a sequence of at most three 2-sums where the basepoints of these 2-sums are $e_1$, $e_2$, and $e_3$, the other part of each 2-sum is binary, and each of $e_1$, $e_2$, and $e_3$ is the basepoint of at most one of these 2-sums.

An immediate consequence of Lemma 3.3 is the following:

Corollary 3.4. If $M$ is 3-connected, non-binary and, for some element $e$, both $M \setminus e$ and $M/e$ are binary, then $M \cong U_{2,4}$.

The next lemma is used in both Sections 2 and 3. It is particularly helpful in Section 3 where it allows us to use Theorem 3.6 to obtain a stronger result.

Lemma 3.5. Let $M$ be a non-binary matroid so that, for every $\{e, f\} \subseteq E(M)$, at least two of $M \setminus e, f$; $M \setminus e/f$; $M/e \setminus f$; and $M/e, f$ are binary. If $M = M_1 \oplus M_2$ and $M_1$ is a connected binary matroid, then $M_1$ is isomorphic to $U_{1,n}$ or $U_{n-1,n}$ for some $n \geq 3$.

Proof. Since $M_1$ is connected, there is a circuit $C_1$ properly containing the basepoint $p$ of the 2-sum. Now, if $M_1$ contains no other circuits, then $E(M_1) = E(C_1)$, so $M_1$ is a circuit. Thus we may assume that $M_1$ has some circuit other than $C_1$. Choose such a circuit $C_2$ for which $|C_2 - C_1|$ is minimal. Assume $C_1 \cap C_2 = \emptyset$. Now, as $M_1$ is connected, $|C_1|$ and $|C_2|$ exceed one. Thus we may suppose that $u \in C_1 - p$, while $v, w \in C_2$. Then $M_1 \setminus (E - C_1)/(C_1 - \{p, u\})$ is a 2-element circuit consisting of $p$ and $u$. Thus $M \setminus (E - C_1)/(C_1 - \{p, u\})$ is isomorphic to the non-binary matroid $M_2$, and it follows that the matroid $M \setminus v, w$ is non-binary.
Since $C_1 \cap C_2 = \emptyset$, the set $C_2 - C_1$ equals $C_2$. Moreover, as $|C_2 - C_1|$ is minimal and $|C_2| \geq 2$, we deduce that neither $v$ nor $w$ is an element of $cl_M(C_1)$. Therefore $C_1$ is a circuit of both $M/v\backslash w$ and $M\backslash v/w$. In particular, the matroids $M/v\backslash w$ and $M\backslash v/w$ are non-binary since $M/v/(C_1 - \{p, u\}) \backslash (E - (C_1 \cup v))$ and $M\backslash (E - (C_1 \cup w))/w/(C_1 - \{p, u\})$ are isomorphic to the non-binary matroid $M_2$. Therefore three of the minors of $M$ obtained by eliminating the elements $v$ and $w$ are non-binary. As a result of this contradiction, we may assume that $C_1 \cap C_2$ is non-empty.

Let the independent set $C_2 - C_1$ be built up to a basis $B$ of $M_1|(C_1 \cup C_2)$ that avoids $p$. Now suppose $s$ and $t$ are elements of $C_1 - (B \cup p)$. Then $C_{M_1}(s, B) \supseteq C_2 - C_1$ and $C_{M_1}(t, B) \supseteq C_2 - C_1$, otherwise $|C_2 - C_1|$ is not minimal. Therefore the symmetric difference, $C_{M_1}(s, B) \Delta C_{M_1}(t, B)$, is a dependent set contained in $C_1 - p$. However, as $C_1$ is a circuit, $C_1 - p$ is independent. We conclude that $|C_1 - (B \cup p)| \leq 1$. In particular, $|C_1 - B| \leq 2$. Since $C_2 - C_1 \subseteq B$, the fact that $|C_1 - B| \leq 2$ implies that $r^*(M_1|(C_1 \cup C_2)) \leq 2$. Moreover, as $(C_1 - p) \cup (C_2 - w)$ spans $M|(C_1 \cup C_2)$ for every $w$ in $C_2 - C_1$, we have that $r^*(M_1|(C_1 \cup C_2)) = 2$.

Since $M_1$ is binary, $[M_1|(C_1 \cup C_2)]^*$ is a line consisting of at most three, possibly trivial, parallel classes. Furthermore, as $C_1$ and $C_2$ are distinct circuits with non-empty intersection, it follows that the geometric representation of $[M_1|(C_1 \cup C_2)]^*$ is a line consisting of exactly three parallel classes, $P_1$, $P_2$, and $P_3$. Let $N$ denote $[M_1|(C_1 \cup C_2)]^*$. Suppose the parallel class $P_1$ contains $p$ and is non-trivial. Then, for elements $v$ and $w$ of $P_2$ and $P_3$, respectively, each of $N\backslash v, w; N/v\backslash w;$ and $N/w\backslash v$ contains a non-trivial parallel class of $N$ containing $p$. Thus, at least
three of the minors of $M_1$ that are obtained by eliminating $v$ and $w$ have circuits properly containing $p$, and hence, are non-binary.

Now assume, without loss of generality, that the parallel class $P_2$ is non-trivial. Let $v$ and $w$ be elements of $P_2$. Then all of $N/v\backslash w$, $N/v, w$, and $N\backslash v/w$ have non-trivial parallel classes containing $p$. By duality, at least three of the minors of $M_1$ that are obtained eliminating $v$ and $w$ have circuits properly containing $p$, and hence, are non-binary. As a result of this contradiction, we conclude that $N \cong U_{2,3}$.

Now we may assume that $M_1|(C_1 \cup C_2) \cong U_{1,3}$. Since $M_1$ is connected, either $M_1 \cong U_{1,n}$, for some $n \geq 3$, or there are elements $v$ and $w$ of $E(M_1) - (C_1 \cup C_2)$ that are not in parallel with $p$. However, if the latter holds, then $p$ is in a two-element circuit of each of $M_1\backslash v,w$, $M_1\backslash v/w$, and $M_1/v\backslash w$. Consequently, each of these matroids has the non-binary matroid $M_2$ as a minor; a contradiction that completes the proof of the lemma.

\[ \Box \]

3.2. The Three-Connected Case

In this section we prove Theorem 3.6 which is the main result in the dissertation. This theorem describes the 3-connected non-binary matroids $M$ such that, for every pair of elements $\{e, f\}$ of $E(M)$, at least two of the minors of $M$ obtained by eliminating $e$ and $f$ are binary. The proof of Theorem 3.6 is long and will occupy the remainder of this section. The small matroids $P_6$, $P_7$, and $J$, shown in Figure 3.1, appear in the statement of the theorem.
Theorem 3.6. The following two statements are equivalent for a matroid $M$.

(i) $M$ is non-binary, 3-connected, and, for every $\{e, f\} \subseteq E(M)$, at least two of $M \setminus e, f; M \setminus e/f; M/e \setminus f; \text{and } M/e, f$ are binary.

(ii) (a) $M$ is isomorphic to $U_{2,n}$ or $U_{n-2,n}$ for some $n \geq 4$; or
(b) $M$ is isomorphic to one of $U_{3,6}, P_6, P_7, P_7^*$, and $J$; or
(c) both the rank and corank of $M$ exceed two and $M$ can be obtained from a 3-connected binary matroid by relaxing a circuit-hyperplane; or
(d) both the rank and corank of $M$ exceed two and $M$ can be obtained from a 3-connected binary matroid by relaxing two circuit-hyperplanes.

Proof. Assume that (ii) holds. Suppose (ii)(a) or (ii)(c) hold and $e$ is an element of $M$. By Theorem 3.2, $M$ is non-binary and 3-connected, while $M \setminus e$ or $M/e$ is binary. Thus, for every $f$ in $E(M) - e$, either $M \setminus e, f$ and $M \setminus e/f$ are binary, or $M/e \setminus f$ and $M/e, f$ are binary. Hence (i) holds.

Assume that (ii)(b) holds. In particular, suppose $M$ is isomorphic to $U_{3,6}$ or $P_6$. Then, as the deletion and contraction of every pair of elements of $M$ yield minors of corank one and rank one, respectively, we deduce that (i) holds.
Now suppose that $M$ is isomorphic to the matroid $P_7$. Since the contraction of every pair of elements of $P_7$ yields a matroid of rank one, the contraction of every pair of elements of $M$ is binary. Moreover, if $e$ denotes the element at the apex of the geometric representation of $M$ shown in Figure 3.1, then $M/e$ is binary. Therefore, for every $f$ in $E(M) - e$, both $M/e, f$ and $M/e\backslash f$ are binary. Thus we need only show that, for every $\{f, g\} \subseteq E(M) - e$, at least two minors obtained by eliminating $f$ and $g$ are binary. Notice that $M\backslash f \cong \mathcal{W}^3$ for every $f$ in $E(M) - e$. As noted in Chapter 1, the rank-3 whirl, $\mathcal{W}^3$, is the relaxation of the 3-connected binary matroid $M(K^4)$. Therefore Theorem 3.2 implies that, for every element $g$, the deletion $\mathcal{W}^3 \backslash g$ or the contraction $\mathcal{W}^3 / g$ is binary. In particular, $M\backslash f, g$ or $M\backslash f / g$ is binary for every $\{f, g\} \subseteq E(M) - e$. Since $M/f, g$ is also binary, we conclude that if $M \cong P_7$, then (i) holds. By duality, (i) also holds if $M \cong P_7^*$.

Now assume $M$ is isomorphic to the matroid $J$ and has ground set $\{1, 2, \ldots, 8\}$ as shown in Figure 3.2(a). Evidently, $M/1$ and $M\backslash 8$ are binary. Hence we need only show that, for every $\{i, j\} \subseteq E(M) - \{1, 8\}$, at least two minors obtained by eliminating $i$ and $j$ are binary. It is easy to verify that, for every element $i$ of $E(M) - \{1, 8\}$, the matroid $M/i$ is isomorphic to the matroid $N$ depicted in Figure 3.2(b). Notice that, for each element $e$ of $N$, either $N\backslash e$ or $N/e$ is binary. Thus, for every $\{i, j\} \subseteq E(M) - \{1, 8\}$, either $M/i\backslash j$ or $M/i, j$ is binary. As $J$ is self-dual, we conclude that, for every $\{i, j\} \subseteq E(M) - \{1, 8\}$, at least two minors obtained by eliminating $i$ and $j$ are binary. Hence (i) holds.
Figure 3.2. (a) M and (b) N, which is isomorphic to $M/i$, for $i \neq 1, 8$.

Now suppose that (ii)(d) holds. Then, by Lemma 1.2(v) and (vi), $M$ is 3-connected and non-binary. Moreover, by Lemma 2.4, for every $\{e, f\} \subseteq E(M)$, at least two of the minors of $M$ obtained by eliminating $e$ and $f$ are binary. We conclude that (ii) implies (i).

Now suppose that (i) holds. If $r(M) = 2$ or $r^*(M) = 2$, then, as $M$ is 3-connected, it is isomorphic to $U_{2,n}$ or $U_{n-2,n}$ for some $n \geq 4$. Thus we may assume that both the rank and corank of $M$ exceed two.

The strategy of the proof is to first show we may assume that every element $e$ in a certain subset $Z_M$ of $E(M)$ determines a partition, $\{C, D, X(e), Y(e) \cup e\}$, of the ground set. Utilizing this partition, we then show that the matroids $M\backslash e$ and $M/e$ can be obtained from binary matroids by relaxing the circuit-hyperplanes $D \cup X(e)$ and $D \cup Y(e)$, respectively. Ostensibly, the partition of $E(M)$ derived from an element $e$ could be completely unrelated to the partition of $E(M)$ obtained from another element $f$ of $Z_M$. Consequently, the circuit-hyperplanes $D \cup X(e)$ and $D \cup Y(e)$ linked to $e$ could be unrelated to the circuit-hyperplanes $D \cup X(f)$ and
DUY(f) linked to f. A significant portion of the proof is devoted to demonstrating that there is a consistent partition of the ground set. Moreover, this partition enables us to prove that M is a double relaxation of a matroid N. First, we present a lemma determining the possibilities for M if $r(M)$ or $r^*(M)$ equal three.

Recall that $F_7^\pm$ denotes the unique double-relaxation of the Fano matroid.

**Lemma 3.7.** If $r(M) = 3$ or $r^*(M) = 3$, then M is isomorphic to one of $U_{3,6}$, $P_6$, $Q_6$, $W^3$, $F_7^-$, $(F_7^-)^*$, $F_7^\pm$, and $(F_7^\pm)^*$.

**Proof.** By duality, we may assume $r(M) = 3$, otherwise replace M by $M^*$ in the argument that follows. Since M is a non-binary 3-connected matroid with rank and corank exceeding two, Lemma 1.8 implies that M has an N-minor, where N is one of $U_{3,6}$, $P_6$, $Q_6$, and $W^3$. If $M \cong N$, then the lemma clearly holds. If $M \not\cong N$, then, by Theorem 1.6, there is a sequence $M_0, M_1, \ldots, M_n$ of 3-connected matroids such that $M_0 \cong N$, $M_n = M$, and, for all $i$ in $\{0, 1, \ldots, n - 1\}$, $M_i$ is a single-element deletion or single-element contraction of $M_{i+1}$. In fact, as both M and N have rank three, it follows that, $M_i$ is a single-element deletion of $M_{i+1}$, for all $i$ in $\{0, 1, \ldots, n - 1\}$. Now, for every $\{e, f\} \subseteq E(M)$, at least two of the minors of M that are obtained by eliminating e and f are binary. Hence, for each $i$ in $\{0, 1, \ldots, n - 1\}$, and every $\{e, f\} \subseteq E(M_i)$, at least two of the minors of $M_i$ obtained by eliminating e and f are binary. Thus $M_i$ has no $U_{2,4}$-restriction.

To see this, suppose $M_i$ has a line L with at least 4 elements. Then, as $M_i$ is 3-connected and has rank three, there are elements e and f not contained in L. Therefore all of $M_i\setminus e; M_i\setminus e/f; M_i/e\setminus f$ are non-binary; a contradiction.
Now suppose $N \cong U_{3,6}$. Then $M$ has a 3-connected minor $M_1$ such that $M_1 \setminus e \cong U_{3,6}$. Suppose $M_1 \setminus e$ is labelled as in Figure 3.3(a). Then, as $(M_1 \setminus e) \setminus i$ and $(M_1 \setminus e) \setminus /i$ are non-binary, we deduce that $M_1 \setminus e \setminus i$ is binary for each $i$ in $\{1,2,\ldots,6\}$. Thus $M_1 \setminus e$ is isomorphic to the matroid shown in Figure 3.3(b) and it follows that $M_1$ has a geometric representation as shown in Figure 3.3(c).

However, if $f$ and $g$ are as labelled in the diagram, then $M_1 \setminus f,g; M_1 \setminus f/g; \text{ and } M_1 \setminus f\setminus g$ are all non-binary; a contradiction. We conclude that $N \not\cong U_{3,6}$.

![Figure 3.3. (a) $U_{3,6}$. (b) $M_1/e$. (c) $M_1$.](image)

Now assume that $N \cong P_6$. Then $M$ has a 3-connected minor $M_1$ such that $M_1 \setminus e \cong P_6$. Suppose $M_1 \setminus e$ is labelled as in Figure 3.4(a). Now, as $(M_1 \setminus e) \setminus 1/2$ and $(M_1 \setminus e) \setminus 1/2$ are non-binary, the matroid $M_1 \setminus 1,2$ must be binary. Using this and the fact that $M_1 \setminus 1,2$ contains no loops or circuits of size 2, it is not difficult to see that $M_1 \setminus 1,2$ has the geometric representation shown in Figure 3.4(b); that is, $\{x_3, e, 3\}$ is a circuit of $M_1 \setminus 1,2$, for some element $x_3$ of $\{4,5,6\}$. It follows that $\{x_3, e, 3\}$ is a circuit of $M_1$. Moreover, by symmetry, $\{x_1, e, 1\}$ and $\{x_2, e, 2\}$ are also circuits of $M_1$, for some elements $x_1$ and $x_2$ of $\{4,5,6\}$. Evidently, as $M_1$ contains no 4-point line, $x_1$, $x_2$, and $x_3$ are distinct elements of $\{4,5,6\}$. 


Suppose, without loss of generality, that \( x_1 = 6, x_2 = 5, \) and \( x_3 = 4. \) Hence the diagram in Figure 3.4(c) is a representation of \( M_1. \) However, \( M_1 \setminus 4,6; M_1 \setminus 4/6; \) and \( M_1/4 \setminus 6 \) are non-binary; a contradiction. Hence \( N \not\cong P_6. \) We conclude that \( M_1 \) is a 3-connected single-element extension of \( Q_6 \) or \( W^3, \) and has no \( U_{3,6}^- \) or \( P_6 \)-restriction.

![Figure 3.4. (a) \( P_6. \) (b) \( M_1 \setminus 1,2. \) (c) \( M_1. \)](image)

Now assume \( N \cong Q_6. \) Then \( M \) has a 3-connected minor \( M_1 \) such that \( M_1 \setminus e \cong Q_6. \) Suppose \( M_1 \setminus e \) is as labelled in Figure 3.5(a). Since \( M_1 \) is 3-connected, the addition of \( e \) to the geometric representation of \( M_1 \setminus e \) creates no loops or 2-element circuits of \( M_1. \) Moreover, we must adjoin \( e \) to the geometric representation in such a way that no \( U_{2,4}, U_{3,6}, \) or \( P_6 \)-restriction is formed. In order for \( M_1 \setminus 3 \) to avoid being isomorphic to \( U_{3,6} \) or \( P_6, \) the element \( e \) must belong to at least two 3-point lines consisting of points in the set \( \{1,2,e,4,5,6\}. \) It follows that \( e \) is contained in exactly two 3-point lines, \( L_1 \) and \( L_2, \) of the geometric representation of \( M_1 \setminus 3, \) and one element of \( \{1,2,4,5,6\} \) avoids both of these lines. First, assume that the element \( 6 \) avoids the lines \( L_1 \) and \( L_2 \) of \( M_1 \setminus e. \) We may assume, without loss of generality, that \( L_1 = \{1,e,4\} \) and \( L_2 = \{2,e,5\}. \) If \( \{3,e,6\} \) is independent,
then a geometric representation of $M_1$ is given in Figure 3.5(b). However, in this case, $M_1 \setminus e, 1; M_1 \setminus e/1; \text{ and } M_1/e \setminus 1$ are non-binary; a contradiction. Thus, \{3, e, 6\} must be a circuit of $M_1$. Then $M_1$ is a double relaxation of $F_7$, as shown in Figure 3.5(c).

![Figure 3.5](image)

**Figure 3.5.** $M_1$ may be a double relaxation of $F_7$.

Now assume $M_1 \setminus e = W^3$ where $W^3$ is labelled as in Figure 3.6. Since $M_1$ is 3-connected, the addition of $e$ to the geometric representation of $M_1 \setminus e$ creates no loops or 2-element circuits of $M_1$. Moreover, we must adjoin $e$ so as no $U_{3,6}$-or $P_6$-restriction is created. To prohibit $M_1 \setminus 1$ from being isomorphic to $P_6$, at least one of \{2, e, 6\}, \{2, e, 5\}, \{2, e, 4\}, \{6, e, 3\}, \{6, e, 4\} must be a circuit. Notice that as $M_1$ is 3-connected and contains no $U_{2,4}$-restriction, \{2, e, 5\} and \{3, e, 6\} are the only sets that can simultaneously be circuits of $M_1 \setminus 1$. Moreover, the case in which \{2, e, 6\} is a circuit of $M_1 \setminus 1$ is symmetric to the cases in which either \{2, e, 4\} or \{4, e, 6\} is a circuit.

Suppose \{2, e, 6\} is a circuit of $M_1 \setminus 1$. If \{1, e, 4\} is not a circuit of $M_1$, then it is clear that three of the minors of $M_1$ involving the elimination of the elements 1
and 2 are non-binary. Thus we may assume that \( \{1, e, 4\} \) and \( \{2, e, 6\} \) are circuits of \( M_1 \). Hence \( M_1 \) is isomorphic to the matroid \( P_7 \) shown in Figure 3.7(a).

\[ \begin{array}{ccc}
5 & 6 & 2 \\
4 & 3 & 1 \\
\end{array} \]

Figure 3.6. \( W^3 \).

Now assume \( \{2, e, 6\} \) is not a circuit of \( M_1 \setminus 1 \). By symmetry, we may assume that neither \( \{2, e, 4\} \) nor \( \{4, e, 6\} \) is a circuit of \( M_1 \setminus 1 \). Thus \( \{2, e, 5\} \) or \( \{3, e, 6\} \) is a circuit of \( M_1 \setminus 1 \). As these cases are symmetric, we may suppose, without loss of generality, that \( \{2, e, 5\} \) is a circuit of \( M_1 \setminus 1 \). Again, unless \( \{1, e, 4\} \) is a circuit of \( M_1 \), at least three of the minors of \( M_1 \) involving the elimination of 1 and 3 are non-binary. Moreover, if \( \{1, e, 4\} \) is a circuit of \( M_1 \), then \( M_1 \) is isomorphic to \( F_7^- \), the double relaxation of \( F_7 \).

Now assume that both \( \{2, e, 5\} \) and \( \{3, e, 6\} \) are circuits of \( M \). If \( \{1, e, 4\} \) is not a circuit of \( M_1 \), then \( M_1 \) is isomorphic to \( F_7^- \). On the other hand, if \( \{1, e, 4\} \) is a circuit of \( M_1 \), then \( M_1 \cong F_7^- \). We conclude that \( M_1 \) is isomorphic to \( P_7 \), \( F_7^- \), or \( F_7^\circ \). Now, if \( M \) is isomorphic to none of these matroids, then \( M \) has a 3-connected restriction, \( M_2 \), having one of these matroids as a single-element deletion. Assume \( M_2 \setminus e = N \), where \( N \) is one of the matroids shown in Figure 3.7. To prohibit \( M_2 \setminus e, 1, 7 \) from having a \( P_6 \)-restriction, one of \( \{2, e, 6\}, \{2, e, 4\}, \) and \( \{4, e, 6\} \) must be a circuit. However, it is not difficult to verify that, if any one
of these sets is a circuit in $M_2$, then there is a pair of elements such that more
than two minors obtained by eliminating the pair are non-binary; a contradiction.
Thus $M$ is isomorphic to the double relaxation of $F_7$, $P_7$, or $F_7^-$, and the lemma
holds.

(a)  
\begin{tikzpicture}
  \node (1) at (0,2) {1};
  \node (2) at (1,1) {2};
  \node (3) at (-1,1) {3};
  \node (4) at (0,0) {4};
  \node (5) at (0,-2) {5};
  \node (6) at (1,-1) {6};

  \draw (1) -- (2) -- (3) -- (1);
  \draw (1) -- (4) -- (5) -- (1);
  \draw (1) -- (6) -- (2);
\end{tikzpicture}

(b)  
\begin{tikzpicture}
  \node (1) at (0,2) {1};
  \node (2) at (1,1) {2};
  \node (3) at (-1,1) {3};
  \node (4) at (0,0) {4};
  \node (5) at (0,-2) {5};
  \node (6) at (1,-1) {6};

  \draw (1) -- (2) -- (3) -- (1);
  \draw (1) -- (4) -- (5) -- (1);
  \draw (1) -- (6) -- (2);
\end{tikzpicture}

(c)  
\begin{tikzpicture}
  \node (1) at (0,2) {1};
  \node (2) at (1,1) {2};
  \node (3) at (-1,1) {3};
  \node (4) at (0,0) {4};
  \node (5) at (0,-2) {5};
  \node (6) at (1,-1) {6};

  \draw (1) -- (2) -- (3) -- (1);
  \draw (1) -- (4) -- (5) -- (1);
  \draw (1) -- (6) -- (2);
\end{tikzpicture}

Figure 3.7. (a) $P_7$. (b) $F_7^-$. (c) $F_7^-$.

By Lemma 3.7, we may now assume that the rank and corank of $M$ ex­
ceed three. Let $C_M = \{ e \in E(M) : M/e \text{ is binary} \}$, $D_M = \{ e \in E(M) : M\setminus e \text{ is binary} \}$, and $Z_M = \{ e \in E(M) : M\setminus e \text{ and } M/e \text{ are non-binary} \}$. Since
we operate almost exclusively with the matroid $M$, we will usually omit the sub­
script denoting the matroid under consideration. The next result, due to Lemos [8],
gives valuable information regarding the cardinality of the set $Z$.

**Theorem 3.8.** If $M$ is a non-binary 3-connected matroid, then $Z$ is empty or
$|Z| \geq 3$.

Now if $Z = \emptyset$, then it follows, from Theorem 3.2, that (ii)(a) or (ii)(c) holds.
Thus we may assume that $|Z| \geq 3$. Moreover, if $e \in C \cap D$, then $M$ is non-binary,
while $M\setminus e$ and $M/e$ are binary. Thus, Corollary 3.4 implies that $M$ is isomorphic
to $U_{2,4}$. As this contradicts the assumption that $r(M) \geq 4$, we conclude that $C \cap D = \emptyset$. Therefore, the sets $C$, $D$, and $Z$ partition $E(M)$. The next lemma will allow us to refine this partition of the ground set $E(M)$.

**Lemma 3.9.** Let $e$ be an element of $Z$.

(i) If $f$ is an element of $E(M) - e$, then exactly one of the matroids $M \setminus e, f$ and $M \setminus e / f$ is binary, while exactly one of $M / e \setminus f$ and $M / e, f$ is binary.

(ii) If $f$ is an element of $Z$, then either

(a) $M \setminus e, f$ and $M / e, f$ are binary while $M \setminus e / f$ and $M / e \setminus f$ are non-binary;

or

(b) $M \setminus e / f$ and $M / e \setminus f$ are binary while $M \setminus e, f$ and $M / e, f$ are non-binary.

**Proof.** We prove statement (i) first. Suppose that $e \in Z$ and $f \in E(M) - e$. In addition, assume that $(M \setminus e) \setminus f$ and $(M \setminus e) / f$ are binary matroids. Since $e \in Z$, the matroid $M \setminus e$ is non-binary. Furthermore, for every $\{u, v\} \subseteq E(M) - e$, at least two of $(M \setminus e) \setminus u, v$; $(M \setminus e) \setminus u / v$; $(M \setminus e) / u, v$; and $(M \setminus e) / u, v$ are binary; otherwise $M$ fails to satisfy Theorem 3.6(i). On combining Lemmas 3.3 and 3.5, we deduce that $M \setminus e$ is obtained from a 4-point line having ground set $\{f, e_1, e_2, e_3\}$ by series extending a subset $S$ of $\{e_1, e_2, e_3\}$ and parallel extending a disjoint subset $T$ of $\{e_1, e_2, e_3\}$ where $S$ or $T$ may be empty. But, as $M$ is 3-connected, $M \setminus e$ cannot have a non-trivial parallel class. Thus $M \setminus e$ can be obtained from $U_{2,4}$ by series extending up to three elements. However, as $e$ is not a coloop of $M$ and $r^*(M \setminus e) = 2$, it follows that $r^*(M) = 3$. Since this contradicts the assumption that the corank of $M$ exceeds 3, we conclude that, for every element $e$ of $Z$ and
every element $f$ of $E(M) - e$, at least one of $M\setminus e/f$ and $M\setminus e, f$ is non-binary. Moreover, by duality, at least one of $M/e, f$ and $M/e\setminus f$ is non-binary. Since at least two of the four minors of $M$ involving the elimination of $e$ and $f$ must be binary, we deduce that exactly one of the matroids $M\setminus e, f$ and $M\setminus e/f$ is binary, and exactly one of $M/e\setminus f$ and $M/e, f$ is binary. Hence (i) holds. Now, if $f$ is also an element of $Z$, then, by symmetry, exactly one of the matroids $M\setminus f, e$ and $M\setminus f/e$ is binary, and exactly one of $M/f\setminus e$ and $M/f, e$ is binary. We now conclude that (ii) holds. \qed

The last lemma suggests the following notation. For each element $e$ of $Z$ define

$$X(e) = \{ x \in Z - e : M\setminus e, x \text{ and } M/e, x \text{ are binary while } M\setminus e/x \text{ and } M/e\setminus x \text{ are non-binary}\},$$

$$Y(e) = \{ y \in Z - e : M\setminus e/y \text{ and } M/e\setminus y \text{ are binary while } M\setminus e, y \text{ and } M/e, y \text{ are non-binary}\}.$$

It follows, from the definitions of $X(e)$ and $Y(e)$, that $X(e) \cap Y(e) = \emptyset$. Furthermore, Lemma 3.9 implies $X(e) \cup Y(e) = Z - e$. Thus we deduce that for every element $e$ of $Z$, the sets $C$, $D$, $X(e)$, and $Y(e) \cup e$ partition $E(M)$. Moreover, in the next lemma we show that, for each $e$ in $Z$, the matroids $M\setminus e$ and $M/e$ are obtained from binary matroids by relaxation.

**Lemma 3.10.** Suppose that $e \in Z$. Then $M\setminus e$ is obtained from a connected binary matroid by relaxing the circuit-hyperplane $X(e) \cup D$ while the matroid $M/e$
is obtained from a connected binary matroid by relaxing the circuit-hyperplane \( Y(e) \cup D \).

**Proof.** Let \( f \) be any element of \( E(M) - e \). By Lemma 3.9, we may assume that either \( (M \setminus e) \setminus f \) or \( (M \setminus e) / f \) is binary. Thus Theorem 3.1 implies that

(3.11)(a) both \( r(M \setminus e) \) and \( r^*(M \setminus e) \) exceed two and \( M \setminus e \) can be obtained from a connected binary matroid by relaxation; or

(b) \( M \setminus e \) is isomorphic to a parallel extension of \( U_{2,n} \) for some \( n \geq 5 \); or

(c) \( M \setminus e \) is isomorphic to a series extension of \( U_{n-2,n} \) for some \( n \geq 5 \); or

(d) \( M \setminus e \) can be obtained from \( U_{2,4} \) by series extension of a subset \( S \) of \( E(U_{2,4}) \) and parallel extension of a subset \( T \) of \( E(U_{2,4}) \) where \( S \) or \( T \) may be empty.

Now if (3.11)(b) holds, then \( r(M) \leq 3 \) contrary to the assumption that the rank and corank of \( M \) exceed 3. Dually, if (3.11)(c) holds, then \( r^*(M) \leq 3 \); a contradiction. Assume that (3.11)(d) holds. Now, as \( M \) is 3-connected, \( M \setminus e \) cannot have a non-trivial parallel class. Therefore, \( M \setminus e \) can be obtained from \( U_{2,4} \) by series extension of a subset \( S \) of \( E(U_{2,4}) \). Thus \( r^*(M) \leq 3 \); a contradiction.

We conclude that the matroid \( M \setminus e \) is obtained from some binary matroid \( N_d(e) \) by relaxing a circuit-hyperplane \( H_d(e) \). By duality, we may assume that \( M/e \) is obtained from some binary matroid \( N_c(e) \) by relaxing a circuit-hyperplane \( H_c(e) \).

It remains to be proved that \( H_d(e) = X(e) \cup D \) and \( H_c(e) = Y(e) \cup D \). Assume that \( x \in X(e) \cup D \). Then it follows from the definitions of \( D \) and \( X(e) \) that \( M \setminus e, x \) is binary. Suppose \( x \notin H_d(e) \). Then, by Lemma 1.2(i), \( M \setminus e / x \) equals
the binary matroid $N_d(e)/x$. Since $e$ and $x$ are elements of $Z$, this contradicts Lemma 3.9(ii), and we conclude that $x \in H_d(e)$. Thus $X(e) \cup D \subseteq H_d(e)$. Now, by Lemma 1.2(ii), for every $h$ in $H_d(e)$, the matroid $M\setminus e, h$ equals the binary matroid $N_d(e)/h$. Thus $H_d(e) \subseteq X(e) \cup D$. Consequently, $X(e) \cup D$ equals the set $H_d(e)$, which is a circuit-hyperplane of $N_d(e)$ and a basis of $M\setminus e$.

Now assume that $y \in Y(e) \cup D$. Then $M/e\setminus y$ is binary. Furthermore, if $y \notin H_c(e)$, then $M/e, y$ equals the binary matroid $N_c(e)/y$. As this contradicts Lemma 3.9(ii), we conclude that $y \in H_c(e)$. Thus $Y(e) \cup D \subseteq H_c(e)$. Conversely, $H_c(e) \subseteq Y(e) \cup D$, since, for every $h$ in $H_c(e)$, the matroid $M/e\setminus h$ equals the binary matroid $N_c(e)/h$. Therefore $Y(e) \cup D$ equals the set $H_c(e)$ which is a circuit-hyperplane of $N_c(e)$ and a basis of $M/e$. □

There are several consequences of Lemma 3.10. First, as $X(e) \cup D$ and $Y(e) \cup D$ are bases of $M\setminus e$ and $M/e$, respectively, it follows that, for every $e$ in $Z$,

(3.12) $X(e) \cup D$ and $(Y(e) \cup e) \cup D$ are bases of $M$; and

(3.13) $|X(e)| = |Y(e)| + 1$.

Since, for all $e$ in $Z$, the sets $X(e) \cup D$ and $(Y(e) \cup e) \cup D$ are bases of $M$, we deduce that, for every $\{e, f\} \subseteq Z$,

(3.14) $|X(e)| = |X(f)|$ and $|Y(e) \cup e| = |Y(f) \cup f|$.

Moreover, as $Z$ is the disjoint union of $X(e)$ and $Y(e) \cup e$, it is clear that $Z$ has even cardinality. Thus we can strengthen the assumption that $|Z| \geq 3$ to the assumption that

(3.15) $|Z| \geq 4$. while $|X(e)| \geq 2$. and $|Y(e)| \geq 1$, for every $e$ in $Z$. 
Let $e$ be an element of $Z$. Since $M\setminus e$ is obtained from a matroid by relaxing the circuit-hyperplane $Y(e) \cup D$, it follows from Lemma 1.3 that $C_{M/e}(f, Y(e) \cup D) = Y(e) \cup \{e, f\} \cup D$, for every $f$ in $E - (Y(e) \cup e \cup D)$. Thus we deduce that,

(3.16) for every $f$ in $E - (Y(e) \cup e \cup D)$, either $Y(e) \cup D \cup f$ or $Y(e) \cup D \cup \{e, f\}$ is a circuit of $M$.

Similarly, as $C_{M\setminus e}(f, X(e) \cup D) = X(e) \cup D \cup f$, for every $f$ in $E - (X(e) \cup D \cup e)$, it follows that, for each $e$ in $Z$,

(3.17) $C_M(f, X(e) \cup D) = X(e) \cup D \cup f$, for every $f$ in $E - (X(e) \cup D \cup e)$.

Indeed, Lemmas 3.21 and 3.22 will enable us to show that $C_M(e, X(e) \cup D) = X(e) \cup D \cup e$. Then, by Lemma 1.3, we will be able to conclude that the matroid $M$ is obtained from a matroid $N$ by relaxing the circuit-hyperplane $X(e) \cup D$.

The next two lemmas give valuable information concerning circuits of $M$ contained in $Z - e$. These lemmas will be used repeatedly in the remainder of the proof.

**Lemma 3.18.** Suppose that $e$ is an element of $Z$ and $C_1$ is a circuit of $M$ contained in $Z - e$.

(i) If $C_1$ contains neither $X(e)$ nor $Y(e)$, then $C_1$ contains at least two elements of each of $X(e)$ and $Y(e)$.

(ii) If $D \neq \emptyset$, then $C_1$ contains at least two elements of each of $X(e)$ and $Y(e)$.

**Proof.** Since $X(e)$ and $Y(e)$ are independent in the matroid $M$, every circuit of $M|(Z - e)$ must contain elements of both $X(e)$ and $Y(e)$. We prove statements (i) and (ii) simultaneously. First suppose $C_1 \cap Y(e) = \{y_1\}$. Then, by (3.17),
the set $X(e) \cup D \cup y_1$ is a circuit of $M$. Moreover, if $C_1$ does not contain $X(e)$, or $D$ is non-empty, then $C_1$ is properly contained in the circuit $X(e) \cup D \cup y_1$.

As a result of this contradiction, we conclude that $|C_1 \cap Y(e)| \geq 2$. Now assume $C_1 \cap X(e) = \{x_1\}$. Then, by (3.16), either $Y(e) \cup D \cup x_1$ or $Y(e) \cup D \cup \{x_1, e\}$ is a circuit of $M$. Moreover, if $C_1$ does not contain $Y(e)$, or $D$ is non-empty, then either $Y(e) \cup D \cup x_1$ or $Y(e) \cup D \cup \{x_1, e\}$ is a circuit of $M$ properly containing $C_1$; a contradiction. Thus $|C_1 \cap X(e)| \geq 2$ completing the proof of the lemma. □

**Lemma 3.19.** Let $X(e) = \{x_1, x_2, \ldots, x_n\}$ and $Y(e) = \{y_1, y_2, \ldots, y_{n-1}\}$ for some integer $n \geq 3$. Then there is at most one 2-element subset $\{x_i, x_j\}$ of $X(e)$ so that $M \setminus x_i, x_j$ is binary. Moreover, if $M \setminus x_i, x_j$ is binary, then, for every 2-element subset $\{y_k, y_l\}$ of $Y(e)$, the set $\{x_i, x_j, y_k, y_l\}$ is a circuit and a cocircuit of $M$.

![Figure 3.8.](image)

Figure 3.8. $X(e) = \{x_1, x_2, \ldots, x_n\}$ and $Y(e) = \{y_1, y_2, \ldots, y_{n-1}\}$.

**Proof.** First we show that

(3.20) if $M \setminus x_i, x_j$ is binary, then $\{x_i, x_j, y_k, y_l\}$ is a circuit of $M$ for all $\{y_k, y_l\} \subseteq \{y_1, y_2, \ldots, y_{n-1}\}$.

To see this, assume that $M \setminus x_i, x_j$ is binary. Since $x_i$ and $x_j$ are elements of $Z$, it
follows from Lemma 3.9 that the matroid $M/x_i,x_j$ is also binary. Now, by (3.17), the sets $X(e) \cup D \cup y_k$ and $X(e) \cup D \cup y_l$ are circuits of $M$ for every $\{y_k, y_l\} \subseteq Y(e)$. Hence $(X(e) - \{x_i, x_j\}) \cup D \cup y_k$ and $(X(e) - \{x_i, x_j\}) \cup D \cup y_l$ are circuits of the binary matroid $M/x_i,x_j$. By Lemma 1.1, the symmetric difference of these two circuits, $\{y_k, y_l\}$, contains a circuit of $M/x_i,x_j$. Thus $\{x_i, x_j, y_k, y_l\}$ contains a circuit of $M$. Suppose that $n = 3$. Then, by (3.12), $D \cup \{x_1, x_2, x_3\}$ is a basis of $M$. Since $r(M) \geq 4$, the set $D$ is non-empty. Thus Lemma 3.18(ii) implies that $\{x_i, x_j, y_k, y_l\}$ cannot properly contain a circuit of $M$. Moreover, if $n \geq 4$, then Lemma 3.18(i) implies that $\{x_i, x_j, y_k, y_l\}$ cannot properly contain a circuit of $M$. We conclude that $\{x_i, x_j, y_k, y_l\}$ is a circuit of $M$ completing the proof of statement (3.20). Now, as $M/x_i,x_j$ is binary, it follows that $M^* \setminus x_i,x_j$ is binary. Thus statement (3.20) implies that $\{x_i, x_j, y_k, y_l\}$ is also a cocircuit of $M$.

Now suppose that $n = 3$. Since $r(M) \geq 4$, statement (3.12) implies that $D$ is non-empty. Now assume, without loss of generality, that $M \setminus x_1, x_2$ and $M \setminus x_1, x_3$ are binary. Then, by (3.20), both $\{x_1, x_2, y_1, y_2\}$ and $\{x_1, x_3, y_1, y_2\}$ are circuits of $M$. The circuit elimination axiom implies that $\{(x_1, x_2, y_1, y_2) \cup \{x_1, x_3, y_1, y_2\}\} - y_2$ contains a circuit of $M$; that is, $\{x_1, x_2, x_3, y_1\}$ is dependent in $M$. However, $\{x_1, x_2, x_3, y_1\}$ is independent in $M$ as it is properly contained in the circuit $\{x_1, x_2, x_3, y_1\} \cup D$. We conclude that at most one of $M \setminus x_1, x_2$, $M \setminus x_1, x_3$, and $M \setminus x_2, x_3$ is binary.

Now we may assume that $n \geq 4$. Suppose without loss of generality that $M \setminus x_1, x_2$ and $M \setminus x_3, x_4$ are binary. Then, by (3.20), the sets $\{x_1, x_2, y_1, y_2\}$ and $\{x_1, x_3, y_1, y_2\}$ are circuits of $M$. By circuit elimination, the set $\{x_1, x_2, x_3, y_1\}$ is
dependent in $M$. However, \( \{x_1, x_2, x_3, y_1\} \) is independent in $M$ as it is properly contained in the circuit $X(e) \cup D \cup y_1$. We conclude that there is at most one 2-element subset $\{x_i, x_j\}$ of $X(e)$ so that $M\setminus x_i, x_j$ is binary.

The next two lemmas are the crucial steps in establishing that there is a consistent partition of the ground set of $M$. Notice that, in the last lemma, we showed there is at most one pair of elements $\{x_i, x_j\}$ of $X(e)$ for which $M\setminus x_i, x_j$ is binary. The proof of Lemma 3.21 determines the structure of the matroid $M$ if such a pair exists. Then, assuming there is no pair of elements in $X(e)$ whose deletion yields a binary matroid, we are able to show, in Lemma 3.22, that there is no such pair of elements in $Y(e)$. The fact that, for $e$ and $f$ in $Z$, the partitions $\{X(e), Y(e) \cup e\}$ and $\{X(f), Y(f) \cup f\}$ are identical will then follow easily.

**Lemma 3.21.** Let $e$ be an element of $Z$. Then either $M\setminus x_i, x_j$ is non-binary for every pair of elements $x_i, x_j$ of $X(e)$, or $M$ is isomorphic to $J$.

**Proof.** The proof will be broken into three cases. First, assume that $|X(e)| \geq 4$ and let \( \{x_1, x_2, x_3, x_4\} \subseteq X(e) \), and \( \{y_1, y_2, y_3\} \subseteq Y(e) \). Assume that $M\setminus x_1, x_2$ is binary. Then, by Lemma 3.19, the sets $\{x_1, x_2, y_1, y_2\}$ and $\{x_1, x_2, y_1, y_3\}$ are circuits of $M$. In particular, as these sets are circuits of the binary matroid $M\setminus e, x_3$, their symmetric difference, $\{y_2, y_3\}$, contains a circuit of $M\setminus e, x_3$. Thus $\{y_2, y_3\}$ is dependent in $M$. However, as (3.12) implies that $D \cup Y(e) \cup e$ is a basis of $M$, the set $\{y_2, y_3\}$ is independent in $M$. We conclude that $|X(e)| \leq 3$. Moreover, statement (3.15) implies that $|X(e)| \geq 2$. Therefore, $|X(e)| = 2$, or $|X(e)| = 3$. 

In the second case, we assume that $|X(e)| = 2$. Then, for every $e$ in $Z$, we have $|X(e)| = 2$ and $|Y(e)| = 1$. Let $X(e) = \{x_1, x_2\}$ and $Y(e) = \{y_1\}$. Now assume that $M\setminus x_1, x_2$ is binary. Since $M\setminus x_1, x_2$ and $M\setminus e, x_2$ are binary, Lemma 3.9(ii) implies that $x_1$ and $e$ are elements of $X(x_2)$. Therefore, we deduce that $X(x_2) = \{x_1, e\}$. Now consider $X(y_1)$ and $Y(y_1)$. Since $y_1 \in Y(e)$, the matroids $M\setminus e/y_1$ and $M/e\setminus y_1$ are binary. Thus $e \in Y(y_1)$, and, hence, $Y(y_1) = \{e\}$. Hence $X(y_1) = Z - (Y(y_1) \cup \{y_1\}) = \{x_1, x_2\}$. Now, as $x_2 \in X(y_1)$, the matroids $M\setminus y_1, x_2$ and $M/y_1, x_2$ are binary. This implies that $y_1 \in X(x_2)$; a contradiction. Thus we conclude that $|X(e)| > 2$.

In the third case, we assume that $|X(e)| = 3$. Let $X(e) = \{x_1, x_2, x_3\}$ and $Y(e) = \{y_1, y_2\}$. Then $E(M)$ may be partitioned as shown in Figure 3.9.

![Figure 3.9](image.png)

**Figure 3.9.** Partition of $E(M)$ derived from $e$.

Assume that $M\setminus x_1, x_2$ is binary. Now we construct a graph $G$ having vertex set $Z$ with the condition that $(f, g)$ is an edge of $G$ if and only if $M\setminus f, g$ is binary. Evidently, $X(e)$ is the set of neighbors of $e$ in the graph $G$, while $Y(e)$ consists of vertices of $G$ that are not adjacent to the vertex $e$. Thus $e$ is adjacent to each of $x_1, x_2$, and $x_3$. Furthermore, as $|X(e)| = 3$ for every $e$ in $Z$, we deduce that each
vertex of $G$ has degree 3. Since $M \setminus x_1, x_2$ is binary, $G$ contains the edge $(x_1, x_2)$. Moreover, by Lemma 3.19, $M \setminus x_1, x_3$, and $M \setminus x_2, x_3$ must be non-binary. Thus $x_1$ and $x_2$ are elements of $Y(x_3)$. Therefore $X(x_3) = \{e, y_1, y_2\}$, and $E(M)$ may be partitioned as in Figure 3.10.

![Figure 3.10](image)

**Figure 3.10.** Partition of $E(M)$ derived from $x_3$.

We conclude that $(x_3, e)$, $(x_3, y_1)$, and $(x_3, y_2)$ are edges in the graph $G$ and the graph $G'$, shown in Figure 3.11, is a subgraph of $G$.

![Figure 3.11](image)

**Figure 3.11.** The subgraph $G'$ of $G$.

Since every vertex of $G$ has degree 3 and $d(e) = d(x_3) = 3$ in $G'$, it is clear that $G'$ contains each edge of $G$ that is adjacent to either of the vertices $e$ and $x_3$. Thus the edges of $G$ not displayed in $G'$ involve the vertices $x_1$, $x_2$, $y_1$, and
Moreover, as \( d(x_1) = d(x_2) = 2 \) and \( d(y_1) = d(y_2) = 1 \) in \( G' \), we deduce that \( (y_1, y_2) \) must be an edge in \( G \). Furthermore, either \( (x_1, y_1) \) and \( (x_2, y_2) \) are edges of \( G \), or \( (x_1, y_2) \) and \( (x_2, y_1) \) are edges of \( G \). Without loss of generality, suppose that \( G \) is as depicted in Figure 3.12.

![Figure 3.12. The graph G.](image)

Now suppose \( |D| \geq 2 \) and \( \{d_1, d_2\} \subseteq D \). As \( M \) is non-binary and 3-connected, Lemma 1.10 implies that there is a \( U_{2,4} \)-minor of \( M \) using \( d_1 \) and \( d_2 \). Suppose that \( M \setminus S/T \) is such a minor. Then \( |T| = r(M) - 2 \), and, from the definition of \( C \), it is clear that \( C \cap T = \emptyset \). Since \( D \cup \{x_1, x_2, x_3\} \) is a basis of \( M \), we conclude that \( r(M) = |D| + 3 \), and hence, \( |T| = |D| + 1 \). Moreover, neither \( d_1 \) nor \( d_2 \) is an element of \( T \), since both are used in the \( U_{2,4} \)-minor. Thus \( T \) contains at least three elements of \( Z \). Assume that \( \{t_1, t_2, t_3\} \subseteq Z \cap T \). Notice that, in the graph \( G \), any collection of three vertices contains at least two vertices that are adjacent. Thus we may assume, without loss of generality, that \( (t_1, t_2) \) is an edge of \( G \). Since \( t_1 \) and \( t_2 \) are elements of \( Z \), Lemma 3.9(ii) implies that both \( M \setminus t_1, t_2 \) and \( M/t_1, t_2 \) are binary. However, as \( M \setminus S/T \cong U_{2,4} \), the matroid \( M/t_1, t_2 \) is non-binary. As a
result of this contradiction, we conclude that $|D| \leq 1$. Moreover, if $D = \emptyset$, then $r(M) = 3$; a contradiction to the assumption that the rank of $M$ exceeds 3. Thus $|D| = 1$. By symmetry, we may also assume that $|C| = 1$. Let $D = \{d\}$ and $C = \{c\}$.

We now turn our attention to compiling a list of circuits and cocircuits of $M$. This information will be displayed in Table 3.1 and will enable us to determine the structure of $M$. Now, the matroid $M$ is 3-connected, with rank and corank equal to 4, and has $\{x_1, x_2, x_3, y_1, y_2, c, d, e\}$ as its ground set. Moreover, $E(M)$ may be partitioned as shown in Figure 3.13.

Since $M\setminus x_1, x_2$ is binary, Lemma 3.19 implies that $\{x_1, x_2, y_1, y_2\}$ is a circuit and a cocircuit of $M$. Furthermore, (3.17) implies that both $\{x_1, x_2, x_3, y_1, d\}$ and $\{x_1, x_2, x_3, y_2, d\}$ are circuits of $M$. Now, from the graph $G$, we infer that $X(x_1) = \{e, x_2, y_1\}$ and $Y(x_1) = \{x_3, y_2\}$, while $X(x_2) = \{e, x_1, y_2\}$ and $Y(x_2) = \{x_3, y_1\}$. Thus $E(M)$ may be partitioned as shown in Figure 3.14. Then, as $M\setminus e, x_2$ and $M\setminus e, x_1$ are binary, Lemma 3.19 implies that $\{e, x_2, x_3, y_2\}$ and $\{e, x_1, x_3, y_1\}$
are circuits and cocircuits of $M$. Moreover, (3.17) implies that $\{x_2, x_3, y_1, d, e\}$, $\{x_2, y_1, y_2, d, e\}$, $\{x_1, x_3, y_2, d, e\}$, and $\{x_1, y_1, y_2, d, e\}$ are circuits of $M$.

Figure 3.14. Partitions of $E(M)$ determined by $x_1$ and $x_2$.

Since $\{x_2, x_3, y_2, e\}$ and $\{x_1, x_2, x_3, y_2, d\}$ are circuits of $M$, the sets $\{x_3, y_2, e\}$ and $\{x_1, x_3, y_2, d\}$ are circuits of $M/x_2 \setminus y_1$. Moreover, as $y_1 \in Y(x_2)$, the matroid $M/x_2 \setminus y_1$ is binary. By Lemma 1.1, the symmetric difference, $\{x_1, d, e\}$, is a disjoint union of circuits of $M/x_2 \setminus y_1$. It follows that $\{x_1, d, e\}$ is also a disjoint union of circuits in $M/x_2$. Now, as $M$ is 3-connected, $M/x_2$ can have no loops. Thus $\{x_1, d, e\}$ is a circuit of $M/x_2$. Therefore, either $\{x_1, d, e\}$ or $\{x_1, x_2, d, e\}$ is a circuit of $M$. Since $\{x_1, x_2, y_1, y_2\}$ is a cocircuit of $M$, we deduce that $\{x_1, d, e\}$ is not a circuit of $M$. Thus $\{x_1, x_2, d, e\}$ is a circuit of $M$.

Now, as $\{x_1, x_3, y_2, d, e\}$ and $\{x_1, x_3, y_1, e\}$ are circuits of $M$ that contain $x_3$ and avoid $x_2$, the sets $\{x_1, y_2, d, e\}$ and $\{x_1, y_1, e\}$ are circuits of $M/x_3, x_2$. Since $x_3 \in Y(x_2)$, the matroid $M/x_3 \setminus x_2$ is binary. Therefore, the symmetric difference, $\{y_1, y_2, d\}$, is a disjoint union of circuits of $M/x_3$. Moreover, as $M$ is 3-connected, $M/x_3$ has no loops. Hence $\{y_1, y_2, d\}$ is a circuit of $M/x_3$. Thus, $\{y_1, y_2, d\}$ or $\{x_3, y_1, y_2, d\}$ is a circuit of $M$. Since $\{x_2, x_3, y_2, e\}$ is a cocircuit of $M$, we conclude that $\{x_3, y_1, y_2, d\}$ is a circuit of $M$. 
### Table 3.1. A Partial List of Circuits and Cocircuits of M

<table>
<thead>
<tr>
<th>5-element circuits of M</th>
<th>(x_1 x_2 x_3 y_1 d)</th>
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<th>4-element circuits of M</th>
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<th>sets which are circuits and cocircuits of M</th>
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Now, in order to determine the matroid \(M\), we first analyze the structure of the two minors, \(M/x_1\) and \(M/x_3\), of \(M\). To analyze the structure of \(M/x_1\) we consider circuits of \(M\) that contain \(x_1\). In particular, since \(\{x_1, x_2, d, e\}\), \(\{x_1, x_2, x_3, d, y_1\}\), and \(\{x_1, x_3, y_1, e\}\) are circuits of \(M\) containing \(x_1\) and avoiding \(c\), the sets \(\{x_2, d, e\}\), \(\{x_2, x_3, d, y_1\}\) and \(\{x_3, y_1, e\}\) are circuits of \(M/x_1\setminus c\). Thus \(M/x_1\setminus y_2, c\) has the geometric representation shown in Figure 3.15(a). Now, as \(\{x_1, x_2, y_1, y_2\}\), and \(\{x_1, x_2, x_3, d, y_2\}\) are circuits of \(M\), it follows that \(\{x_2, y_1, y_2\}\) and \(\{x_2, x_3, d, y_2\}\) are circuits of \(M/x_1\setminus c\). In particular, since \(\{x_3, y_2, d\}\) is independent, we deduce that \(M/x_1\setminus c\) has a geometric representation as shown in Figure 3.15(b). Since the existence of an additional circuit would force \(M/x_1\setminus c\) to collapse to a rank-1 matroid, we conclude that \(M/x_1\setminus c\) is isomorphic to \(W^3\).
Now, as $M/x_1, c$ is binary, it is clear that $M$ is not obtained by freely adding $c$ to the geometric representation of $M/x_1\setminus c$ shown in Figure 3.15(b). Moreover, as $Y(x_1) = \{x_3, y_2\}$, the deletion of $x_3, y_2$, or $d$ from $M/x_1$ is binary. Thus $c$ cannot be added to one of the existing 3-point lines of the geometric representation of $M/x_1\setminus c$ so as to create a 4-point line. It is also evident that $c$ cannot be placed in parallel with $x_3, y_2, \text{ or } d$. Suppose $c$ is adjoined to the representation of $M/x_1\setminus c$ in parallel with $x_2$ or $e$. Then, as $M$ is 3-connected, $\{x_1, x_2, c\}$ or $\{x_1, c, e\}$ is a circuit of $M$. However, it is impossible for $\{x_1, x_2, c\}$ or $\{x_1, c, e\}$ to be a circuit since $\{x_2, x_3, y_2, e\}$ is a cocircuit of $M$. Thus we conclude that $c$ is adjoined to $M/x_1, c$ in parallel with $y_1$ or the addition of $c$ creates more 3-point lines in the geometric representation. Since $X(x_1) = \{x_2, y_1, e\}$, the contraction of $x_2, y_1, \text{ or } e$ from $M/x_1$ is binary. Consequently, $\{x_2, x_3, c\}, \{y_1, c, d\}$, and $\{y_2, c, e\}$ are circuits of $M/x_1$ and we deduce that $M/x_1$ is isomorphic to $F_7^-$. Therefore the two possible geometric representations of $M/x_1$ are as shown in Figure 3.16(a). Moreover, by an argument analogous to the one just given, we may assume that the two possible geometric representations of $M/x_3$ are as shown in Figure 3.16(b).
Figure 3.16. The two possibilities for (a) $M/x_1$ and (b) $M/x_3$.

Now assume that $M/x_1 \cong F_7^-$ and is labelled as in Figure 3.16(a). Then $\{c, y_2\}$ and $\{c, y_1\}$ are independent sets in $M/x_1, x_3$. However, if $M/x_3$ is given by either geometric representation in Figure 3.16(b), then $M/x_1, x_3$ has $\{c, y_1\}$ or $\{c, y_2\}$ as a circuit. As a result of this contradiction, we may assume that $M/x_1 \not\cong F_7^-$. Thus $\{c, y_1\}$ is a circuit of $M/x_1$ and, hence, $\{x_1, y_1, c\}$ is a circuit of $M$.

By eliminating $x_1$ from the circuits $\{x_1, y_1, c\}$ and $\{x_1, x_2, y_1, y_2\}$, we have that $\{x_2, y_1, y_2, c\}$ is dependent in $M$. Since $M$ is 3-connected, each circuit of $M$ contains at least 3 elements. Moreover, as $\{x_1, x_3, y_1, e\}$ is a cocircuit of $M$, we deduce that $\{x_2, y_2, c\}$ is a circuit of $M$.

By eliminating $x_1$ from the circuits $\{x_1, y_1, c\}$ and $\{x_1, x_3, y_1, e\}$, we have that $\{x_3, y_1, c, e\}$ is dependent in $M$. Since $\{x_1, x_2, y_1, y_2\}$ is a cocircuit of $M$, it follows that $\{x_3, c, e\}$ is a circuit of $M$. Now, as $\{x_1, y_1, c\}$, $\{x_2, y_2, c\}$, $\{x_3, c, e\}$, $\{x_1, x_2, y_1, y_2\}$, $\{x_1, x_3, y_1, e\}$, and $\{x_2, x_3, y_2, e\}$ are circuits of $M$, the diagram in Figure 3.17(a) is a geometric representation of $M \setminus d$. 
Now it is easy to see that for every subset \( \{u, v\} \) of \( \{x_1, x_2, x_3, y_1, y_2, e\} \), the set \( \{d, u, v\} \) is properly contained in one of the 5-element circuits of \( M \) listed in Table 3.1. It follows that the element \( d \) is in no 3-element circuit of \( M \). Suppose \( d \) and \( c \) are contained in a 4-element circuit of \( M \). Then \( d \) must lie on one of the three planes \( \{c, x_1, x_2, y_1, y_2\} \), \( \{c, x_1, x_3, y_1, e\} \), and \( \{c, x_2, x_3, y_2, e\} \) of \( M \setminus d \).

But the independent sets \( \{d, x_1, x_2, y_1\} \), \( \{d, x_1, x_3, y_1\} \), and \( \{d, x_2, x_3, y_2\} \) of \( M \) prevent \( d \) from lying on any of these planes. We conclude that \( d \) is not contained in a 4-element circuit of \( M \) containing \( c \). From Table 3.1 we see that \( \{x_1, x_2, d, e\} \) and \( \{x_3, y_1, y_2, d\} \) are circuits of \( M \). Moreover, it is easy to verify that, for every subset \( \{u, v, w\} \) of \( \{x_1, x_2, x_3, y_1, y_2, e\} \) other than \( \{x_1, x_2, e\} \) and \( \{x_3, y_1, y_2\} \), the set \( \{d, u, v, w\} \) is properly contained in one of the 5-element circuits of \( M \) listed in Table 3.1. Hence \( \{x_1, x_2, d, e\} \) and \( \{x_3, y_1, y_2, d\} \) are the only 4-element circuits of \( M \) containing \( d \). Therefore the geometric representation of \( M \) is as shown in Figure 3.17(b) and we conclude that \( M \cong J \).

If \( M \cong J \), then \( M \) satisfies (ii)(b) and the theorem holds. Thus, by Lemma 3.21, we may assume that \( M \setminus x_i, x_j \) is non-binary if \( x_i, x_j \in X(e) \) for some \( e \) in \( Z \).
Lemma 3.22. If $y_i, y_j \in Y(e)$ for some element $e$ of $Z$, then $M \backslash y_i, y_j$ is non-binary.

Proof. Since the lemma holds trivially if $|Y(e)| = 1$, we first assume that $|Y(e)| = 2$. Let $X(e) = \{x_1, x_2, x_3\}$ and $Y(e) = \{y_1, y_2\}$. Suppose $M \backslash y_1, y_2$ is binary. As before, we construct a graph $G$ having vertex set $\{e, x_1, x_2, x_3, y_1, y_2\}$ with the condition that $(f, g)$ is an edge of $G$ if and only if $M \backslash f, g$ is binary. Since $X(e) = \{x_1, x_2, x_3\}$, the vertex $e$ is adjacent to each of the vertices $x_1$, $x_2$, and $x_3$. Moreover, as $|X(e)| = 3$ for each element $e$ of $Z$, we deduce that each vertex of $G$ has degree exactly three. By Lemma 3.21, we may assume that $M \backslash x_1, x_2$, $M \backslash x_1, x_3$, and $M \backslash x_2, x_3$ are non-binary. Thus, for every $\{i, j\} \subseteq \{1, 2, 3\}$, the vertex $x_i$ is not adjacent to the vertex $x_j$ in the graph $G$. Consequently, each of the vertices $x_1$, $x_2$, and $x_3$ must be a neighbor of $y_1$ and a neighbor of $y_2$. Now, as $d(y_1) = d(y_2) = 3$, it is not possible for $y_1$ to be adjacent to $y_2$ in $G$. In particular, $M \backslash y_1, y_2$ is non-binary. Thus the lemma holds when $|Y(e)| = 2.$

Figure 3.18. The graph $G$. 
Now assume that $|Y(e)| \geq 3$. Let $\{x_1, x_2, x_3, x_4\} \subseteq X(e)$ and $\{y_1, y_2, y_3\} \subseteq Y(e)$. In addition, assume that $M \setminus y_1, y_2$ is binary. It follows from Lemma 3.9 that $M \setminus y_1, y_2$ is also binary. Let $x_i$ be an element of $X(e)$. Then, by (3.16), either $Y(e) \cup D \cup x_i$ or $Y(e) \cup D \cup \{e, x_i\}$ is a circuit of $M$. In particular, either $(Y(e) - \{y_1, y_2\}) \cup D \cup x_i$ or $(Y(e) - \{y_1, y_2\}) \cup D \cup \{e, x_i\}$ is a circuit of $M \setminus y_1, y_2$.

Hence, for every $\{x_i, x_j\} \subseteq X(e)$, either $\{x_i, x_j\}$ or $\{x_i, x_j, e\}$ is the symmetric difference of two circuits of the binary matroid $M \setminus y_1, y_2$. Therefore, by Lemma 1.1, either $\{x_i, x_j\}$ or $\{x_i, x_j, e\}$ is a disjoint union of circuits of $M \setminus y_1, y_2$. Moreover, (3.23) no element of $X(e) \cup e$ is a loop in $M \setminus y_1, y_2$.

To see this, suppose that $x_i$ or $e$ is a loop in $M \setminus y_1, y_2$. Then $\{x_i, y_1, y_2\}$ or $\{e, y_1, y_2\}$ contains a circuit of $M$. However, the first case violates Lemma 3.18(i), while the second contradicts the fact that $D \cup Y(e) \cup e$ is a basis. Since none of $x_i$, $x_j$, or $e$ can be a loop of $M \setminus y_1, y_2$, it follows that, for every $\{x_i, x_j\} \subseteq X(e)$, either $\{x_i, x_j\}$, or $\{x_i, x_j, e\}$, is a circuit of $M \setminus y_1, y_2$.

Now suppose $\{x_i, x_j, e\}$ is a circuit of $M \setminus y_1, y_2$ for every $\{i, j\} \subseteq \{1, 2, 3, 4\}$. Then, in particular, $\{x_1, x_2, e\}$, $\{x_1, x_3, e\}$, and $\{x_2, x_3, e\}$ are circuits of $M \setminus y_1, y_2$. It follows that the restriction of the binary matroid $M \setminus y_1, y_2$ to $\{x_1, x_2, x_3, e\}$ is isomorphic to $U_{2,4}$; a contradiction. Thus we may assume that $\{x_i, x_j\}$ is a circuit of $M \setminus y_1, y_2$ for some $\{x_i, x_j\} \subseteq X(e)$. Suppose $\{x_i, x_j\}$ and $\{x_i, x_k\}$ are circuits of $M \setminus y_1, y_2$. Then $\{x_i, x_j, y_1, y_2\}$ and $\{x_i, x_k, y_1, y_2\}$ contain circuits of $M$, and it follows, from Lemma 3.18(i), that $\{x_i, x_j, y_1, y_2\}$ and $\{x_i, x_k, y_1, y_2\}$ are circuits of $M$. Hence $\{x_i, x_j, x_k, y_1\}$ contains a circuit of $M$. Now, as $|X(e)| \geq 4$, the set
\{x_i, x_j, x_k, y_1\} is properly contained in \(D \cup X(e) \cup y_1\), which, by (3.17), is a circuit of \(M\). As a result of this contradiction, we may assume that any pair of two-element circuits of \(M/y_1,y_2\) that are contained in \(X(e)\) have empty intersection.

Now suppose that \(\{x_1, x_2\}\) and \(\{x_3, x_4\}\) are circuits of \(M/y_1,y_2\). Then, as \(\{x_1, x_2, y_1, y_2\}\) and \(\{x_3, x_4, y_1, y_2\}\) contain circuits of \(M\), Lemma 3.18(i) implies that \(\{x_1, x_2, y_1, y_2\}\) and \(\{x_3, x_4, y_1, y_2\}\) are circuits of \(M\). Thus, by circuit elimination, \(\{x_1, x_2, x_3, x_4, y_1\}\) is dependent in \(M\). Since (3.17) implies that \(D \cup X(e) \cup y_1\) is a circuit of \(M\), we conclude that, as shown in Figure 3.19, \(D = \emptyset\), while \(X(e) = \{x_1, x_2, x_3, x_4\}\) and \(Y(e) = \{y_1, y_2, y_3\}\).

![Figure 3.19. A partition of \(E(M)\).](image)

Suppose \(\{x_1, x_2, e\}\) is the symmetric difference of two circuits of the binary matroid \(M/y_1,y_2\). Then, as \(\{x_1, x_2\}\) is a circuit of \(M/y_1,y_2\), the element \(e\) is a loop of \(M/y_1,y_2\). Since this contradicts (3.23), we may assume that no two circuits of \(M/y_1,y_2\) have the set \(\{x_1, x_2, e\}\) as their symmetric difference. Since \(\{x_3, x_4\}\) is also a circuit of \(M/y_1,y_2\), it follows that no two circuits of \(M/y_1,y_2\) have the set \(\{x_3, x_4, e\}\) as their symmetric difference. Now (3.16) implies that, for every \(x_i\) in \(X(e)\), either \(\{x_i, y_3\}\) or \(\{x_i, y_3, e\}\) is a circuit of \(M/y_1,y_2\). Since
no two circuits of $M/y_1, y_2$ have symmetric difference $\{x_1, x_2, e\}$, it follows that, either $\{x_1, y_3\}$ and $\{x_2, y_3\}$ are circuits of $M/y_1, y_2$, or $\{x_1, y_3, e\}$ and $\{x_2, y_3, e\}$ are circuits of $M/y_1, y_2$. Similarly, either $\{x_3, y_3\}$ and $\{x_4, y_3\}$ are circuits of $M/y_1, y_2$, or $\{x_3, y_3, e\}$ and $\{x_4, y_3, e\}$ are circuits of $M/y_1, y_2$. Notice that if both $\{x_1, y_3\}$ and $\{x_3, y_3\}$, or both $\{x_1, y_3, e\}$ and $\{x_3, y_3, e\}$ are circuits of the binary matroid $M/y_1, y_2$, then their symmetric difference, $\{x_1, x_3\}$, is a disjoint union of circuits of $M/y_1, y_2$. Since $\{x_1, x_2\}$ is a circuit of $M/y_1, y_2$, and no pair of $2$-element circuits of $M/y_1, y_2$ intersect, we deduce that $x_1$ and $x_3$ are loops in $M/y_1, y_2$. However, since this contradicts (3.23), we may assume without loss of generality that along with $\{x_1, x_2\}$ and $\{x_3, x_4\}$, the sets $\{x_1, y_3, e\}$, $\{x_2, y_3, e\}$, $\{x_3, y_3\}$, and $\{x_4, y_3\}$ are circuits of $M/y_1, y_2$. It follows that the geometric representation of $M/y_1, y_2 \setminus C$ is a line containing the point $e$ in addition to the parallel classes $\{x_1, x_2\}$ and $\{x_3, x_4, y_3\}$. Now, as $\{x_3, x_4\}$ and $\{x_4, y_3\}$ are circuits of $M/y_1, y_2$, the sets $\{x_3, x_4, y_1, y_2\}$ and $\{x_4, y_1, y_2, y_3\}$ contain circuits of $M$. By Lemma 3.18, the set $\{x_3, x_4, y_1, y_2\}$ is a circuit of $M$. Moreover, as $D = \emptyset$ and $Y(e) = \{y_1, y_2, y_3\}$, it follows from (3.16) that $\{x_4, y_1, y_2, y_3\}$ is a circuit of $M$. Since $\{x_3, x_4, y_1, y_2\}$ and $\{x_4, y_1, y_2, y_3\}$ are circuits of the binary matroid $M\setminus e, x_1$, their symmetric difference, $\{x_3, y_3\}$, is dependent in $M\setminus e, x_1$, and hence, in $M$. However, Lemma 3.18 implies that each circuit of $M\setminus (Z - e)$ must contain at least two elements of each of $X(e)$ and $Y(e)$. As a result of this contradiction, we deduce that $M\setminus y_i, y_j$ is non-binary for every $\{y_i, y_j\} \subseteq Y(e)$. 

\[\square\]
Now recall that, for every $e$ in $Z$, the set $Z$ can be partitioned into the sets $X(e)$ and $Y(e) \cup e$. Furthermore, as a result of the last two lemmas, we are now able to show that the partitions $\{X(e), Y(e) \cup e\}$ and $\{X(f), Y(f) \cup f\}$ of $Z$ are identical for every $\{e, f\} \subseteq Z$.

**Lemma 3.24.** Let $e$ and $f$ be elements of $Z$.

(i) If $f \in X(e)$, then $X(e) = Y(f) \cup f$ and $Y(e) \cup e = X(f)$.

(ii) If $f \in Y(e)$, then $X(e) = X(f)$ and $Y(e) \cup e = Y(f) \cup f$.

**Proof.** Since $f$ is an element of $Z$ distinct from $e$, we conclude that either $f \in X(e)$ or $f \in Y(e)$. Suppose that $f \in Y(e)$. As both $\{X(e), Y(e) \cup e\}$ and $\{X(f), Y(f) \cup f\}$ are partitions of $Z$, it suffices to prove that $Y(f) \cup f = Y(e) \cup e$. Suppose $g \in Y(f) \cap X(e)$. Since $g$ is an element of $X(e)$, the matroid $M \setminus e, g$ is binary. Now, as $M \not\cong J$, Lemmas 3.21 and 3.22 imply that for every $h$ in $Z - \{e, g\}$, the set $\{e, g\}$ is contained in neither $X(h)$ nor $Y(h)$. Since $g \in Y(f)$, we deduce that $e \not\in Y(f)$. However, as $f \in Y(e)$, it follows that $e \in Y(f)$; a contradiction. We conclude that $Y(f) \cap X(e) = \emptyset$. Thus $Y(f) \cup f \subseteq Y(e) \cup e$. Now, as $|Y(f) \cup f| = |Y(e) \cup e|$, we have that $Y(f) \cup f = Y(e) \cup e$ completing the proof of (ii).

Now assume that $f \in X(e)$. If $g \in X(e) - f$, then, as $M \not\cong J$, Lemma 3.21 implies that $M \setminus f, g$ is non-binary. Therefore $g$ is in $X(f)$. We conclude that $X(e) - f \subseteq Y(f)$. Moreover, as $|X(e)| - 1 = |Y(e)| = |Y(f)|$, we deduce that $X(e) - f = Y(f)$. It follows that $X(e) = Y(e) \cup f$ and $Y(e) \cup e = X(f)$ completing the proof of (i). \qed
Now fix an element $e$ of $Z$. Lemma 3.10 implies that $M \setminus e$ is obtained from a connected binary matroid $N_d(e)$ by relaxing the circuit-hyperplane $X(e) \cup D$. It follows from Lemma 1.3 that $C_{M \setminus e}(g, X(e) \cup D) = X(e) \cup D \cup g$ for every $g$ in $E(M \setminus e) - (X(e) \cup D)$. Therefore, for every $g$ in $E(M) - (X(e) \cup D \cup e)$, we have $C_M(g, X(e) \cup D) = X(e) \cup D \cup g$. Now consider the fundamental circuit of $e$ with respect to the basis $X(e) \cup D$ of $M$. Suppose $f \in Y(e)$. Then, by Lemma 3.24(ii), $X(e) = X(f)$ and $e \in Y(f)$. Since $f \in Z$, the matroid $M \setminus f$ is obtained from some matroid by relaxing the circuit-hyperplane $X(f) \cup D$. Thus $X(f) \cup D \cup h$ is a circuit of $M \setminus f$ for every $h$ in $E(M \setminus f) - (X(f) \cup h)$. In particular, $e$ is an element of $E(M \setminus f) - (X(f) \cup D)$ since $e$ is in $Y(f)$. Therefore, the set $X(f) \cup D \cup e$ is a circuit of $M \setminus f$. Thus $X(f) \cup D \cup e$ is a circuit of $M$. Moreover, as $X(e) = X(f)$, we deduce that $C_M(e, X(e) \cup D) = X(e) \cup D \cup e$. It follows from Lemma 1.3 that $M$ is obtained from a matroid $N_1$ by relaxing the circuit-hyperplane $X(e) \cup D$.

Now, as $M/e$ is obtained from a matroid by relaxing the circuit-hyperplane $Y(e) \cup D$, the set $(Y(e) \cup e) \cup D$ is a basis of $M$. Let $h$ be an element of $E(M) - (Y(e) \cup e \cup D)$, and suppose that $g \in X(e)$. Then, by Lemma 3.24(i), $X(g) = Y(e) \cup e$. Now $M \setminus g$ is obtained from a binary matroid by relaxing the circuit-hyperplane $X(g) \cup D$. Furthermore, by the argument in the previous paragraph, we have that $X(g) \cup D \cup h$ is a circuit of $M$ for every element $h$ of $E - (X(g) \cup D)$. Since $X(g) = Y(e) \cup e$, we conclude that $(Y(e) \cup D) \cup h$ is a circuit of $M$ for every element $h$ of $E - (Y(e) \cup D \cup e)$. Therefore, by Lemma 1.3, $M$ is obtained from a matroid $N_2$ by relaxing the circuit-hyperplane $Y(e) \cup e \cup D$. 
As $M$ is obtained from the matroids $N_1$ and $N_2$ by relaxing the circuit-
hyperplanes $X(e) \cup D$ and $Y(e) \cup D \cup e$, respectively, it follows from Lemma 1.4
that there is a matroid $N$ that yields $N_1$ and $N_2$ when the circuit-hyperplanes
$Y(e) \cup D \cup e$ and $X(e) \cup D$, respectively, are relaxed. In other words, $M$ is a
double relaxation of $N$. To complete the proof of Theorem 3.6, we now show that
$N$ is 3-connected and binary.

**Lemma 3.25.** $N$ is 3-connected.

**Proof.** Suppose that $N$ is not 3-connected. Then, as $E(N) = E(M)$, there is a
partition $(S, T)$ of $E(M)$ such that $|S|, |T| \geq k$ for some $k$ in $\{1, 2\}$, and

\[(3.26) \quad r_N(S) + r_N(T) - r(N) = k - 1.\]

All subsets of $E(M)$ except $D \cup X(e)$ and $D \cup Y(e) \cup e$ have the same rank in
$N$ as they do in $M$, while $r_N(D \cup X(e)) = r_N(D \cup Y(e) \cup e) = r(N) - 1$. Since
$M$ is 3-connected, (3.26) implies that $S$ or $T$ equals $D \cup X(e)$ or $D \cup Y(e) \cup e$.

Suppose $D$ and $C$ are empty and $\{S, T\} = \{X(e), Y(e) \cup e\}$. Since $r_N(D \cup X(e)) =
\quad r_N(D \cup Y(e) \cup e) = r(N) - 1$, it follows from (3.26) that $r(N) = k + 1$, where
$k \in \{1, 2\}$. However, $r(N) = r(M) \geq 4$. As a result of this contradiction, we may
assume without loss of generality that $D \cup C$ is non-empty and $T = D \cup X(e)$. Then
$S = E - (D \cup X(e))$. Moreover, it follows from (3.26) that and $r_N(E - (D \cup X(e))) =
k$. Since $C \cup D \neq \emptyset$, the set $E - (D \cup X(e))$ has the same rank in both $M$ and $N$. In
particular, $r_M(E - (D \cup X(e))) = k$ for some $k$ in $\{1, 2\}$. Now $|E - (D \cup X(e))| \geq 4$
since $X(e) \cup D$ is a basis of $M$ and $r^*(M) \geq 4$. Thus, as $M$ is 3-connected, $k \neq 1$.

Hence $k = 2$. Therefore the set $E - (D \cup X(e))$ is a cobasis of $M$ contained in a
Lemma 3.27. \textit{\textbf{N is binary.}}

\textbf{Proof.} Suppose \(N\) is non-binary. Let \(e\) be an element of \(Z\). Then \(M\setminus e\) is obtained from the binary matroid \(N_{d}(e)\) by relaxing the circuit-hyperplane \(D \cup X(e)\). Moreover, \(M\) is also obtained from the matroid \(N_{1}\setminus e\) by relaxing the circuit-hyperplane \(D \cup X(e)\). Thus \(B(M\setminus e) = B(N_{d}(e)) \cup \{D \cup X(e)\}\) and \(B(M\setminus e) = B(N_{1}\setminus e) \cup \{D \cup X(e)\}\). Since \(B(N_{d}(e)) = B(N_{1}\setminus e)\), we have that \(N_{d}(e) = N_{1}\setminus e\).

Furthermore, Lemma 1.2(ii) implies that \(N_{1}\setminus e = N\setminus e\), and we conclude that \(N\setminus e\) is binary. Similarly, \(M\setminus e\) is obtained from both of the binary matroids \(N_{c}(e)\) and \(N_{2}/e\) by relaxing the circuit-hyperplane \(D \cup Y(e) \cup e\). Thus \(N_{c}(e) = N_{2}/e\). Moreover, Lemma 1.2(ii) implies \(N_{2}/e = N/e\), it follows that \(N/e\) is binary. Then by Corollary 3.4, \(N \cong U_{2,4}\). Since \(N\) has at least two circuit-hyperplanes and \(U_{2,4}\) has none, we have a contradiction that completes the proof of Lemma 3.27 and thereby finishes the proof of Theorem 3.6. \(\Box\)

3.3. The General Case

In this section, we determine all non-binary matroids \(M\) such that, for every pair \(\{e, f\}\) of elements of \(M\), at least two minors of \(M\) obtained by eliminating \(e\) and \(f\) are binary. Essentially, this is accomplished by combining Lemma 3.5 and Theorem 3.6.
Theorem 3.28. The following two statements are equivalent for a matroid \( M \).

(i) \( M \) is non-binary and, for every \( \{e, f\} \subseteq E(M) \), at least two of \( M \setminus e, f; M \setminus e / f; M/e \setminus f; \) and \( M/e, f \) are binary.

(ii) (a) \( M \) is isomorphic to \( U_{2,n} \) or \( U_{n-2,n} \) for some \( n \geq 4 \); or

(b) both the rank and corank of \( M \) exceed two and \( M \) can be obtained from a connected binary matroid by relaxing a circuit-hyperplane; or

(c) both the rank and corank of \( M \) exceed two and \( M \) can be obtained from a connected binary matroid by relaxing two circuit-hyperplanes; or

(d) \( M \) is isomorphic to one of \( U_{3,6}, P_6, P_7, P_7^*, \) and \( J \); or

(e) \( M \) is isomorphic to \( U_{2,4} \oplus_2 U_{2,4} \); or

(f) \( M \) is obtained from a matroid \( \tilde{M} \) described in (a) or (b) by the addition of a loop or coloop, or by series extension of a subset \( S \) of \( D_{\tilde{M}} \) or parallel extension of a subset \( T \) of \( C_{\tilde{M}} \); or

(g) \( M \) is obtained from a matroid \( \tilde{M} \) described in (a), (b), (c), or (d) by series extension of a subset \( S \) of \( D_{\tilde{M}} \) or parallel extension of a subset \( T \) of \( C_{\tilde{M}} \).

Proof. Suppose that (ii) holds. We have already seen, in the proof of Theorem 3.6, that if (ii)(a) or (ii)(d) holds, then so does (i). Furthermore, it is easy to verify that if (ii)(e) holds, then for every pair \( \{e, f\} \) of elements, \( M/e, f \) and \( M\setminus e, f \) are binary.

Suppose (ii)(b) holds and let \( e \) be an element of \( M \). By Theorem 3.1, \( M \setminus e \) or \( M/e \) is binary. Hence, for every \( f \) in \( E(M) \setminus \{e\} \), both \( M \setminus e, f \) and \( M \setminus e / f \), or both
$M/e\setminus f$ and $M/e.f$ are binary. Thus, if (ii)(b) holds, then so does (i). Moreover, if (ii)(c) holds, then it follows from Lemma 2.4 that (i) holds.

Suppose (ii)(g) holds. Let $e$ be an element of $E(M) - E(\tilde{M})$ that is in parallel with an element $f$ of $C_M$. Then $\tilde{M}/f$ and $M/f$ are binary. Moreover, as $M/e \cong M/f$, we conclude that $M/e$ is binary. Thus, for every $g$ in $E(M) - e$, the minors $M/e.g$ and $M/e\setminus g$ are binary. Dually, if $e$ is in series with an element of $D_M$, then, for every $g$ in $E(M) - e$, the matroids $M\setminus e.g$ and $M/e\setminus g$ are binary. We conclude that if (ii)(g) holds, then (i) holds.

Now assume (ii)(f) holds. By the argument in the previous paragraph, we may suppose that $M$ is obtained from $\tilde{M}$ by the addition of a loop or coloop. Let $e$ be an element of $E(M) - E(\tilde{M})$ that is a loop or coloop. Notice that, for every $f$ in $E(M) - e$, the deletion $M\setminus f$ or the contraction $M/f$ is binary. Therefore for every $f$ in $E(M) - e$, either both $M\setminus f, e$ and $M/f, e$, or both $M/f, e$ and $M/f, e$ are binary matroids. We conclude that (ii) implies (i).

Now suppose (i) holds. We argue by induction of $|E(M)|$ to show that (ii) holds. If $M$ is 3-connected, then the result follows easily from Theorem 3.6. Assume the result is true for all matroids satisfying the hypotheses and having fewer elements than $M$.

Suppose $M$ is disconnected. Then $M = M_1 \oplus M_2$ where $M_1$ or $M_2$ is non-binary. Assume, without loss of generality, that $M_2$ is non-binary. If $e, f \in E(M_1)$, then each of $M\setminus e, f; M\setminus e/f; \text{ and } M/e\setminus f$ has the non-binary matroid $M_2$ as a minor: a contradiction. We conclude that $E(M_1) = \{f\}$ for some element $f$ of $M$. 
It follows that $M$ is isomorphic to $U_{0,1} \oplus M_2$ or $U_{1,1} \oplus M_2$. Now suppose there is an $e$ in $E(M_2)$ such that $M_2 \setminus e$ and $M_2/e$ are non-binary. Then, as $f$ is a loop or coloop of $M$, the matroids $M \setminus e, f$ and $M \setminus e/f$ are isomorphic to $M_2 \setminus e$ while $M/e \setminus f$ is isomorphic to $M_2/e$. Thus at least three of the minors of $M$ involving the elimination of $e$ and $f$ are non-binary; a contradiction. Thus we may assume that for every element $g$ of $M_2$, we have $M_2 \setminus g$ or $M_2/g$ is binary. It follows from Theorem 3.1 that $M$ satisfies (ii)(f). Thus we may assume that $M$ is connected.

As $M$ is connected but not 3-connected, Theorem 1.5 implies that, for some matroids $M_1$ and $M_2$, the matroid $M = M_1 \oplus_2 M_2$ where $E(M_1) \cap E(M_2) = \{p\}$ and $|E(M_1)|, |E(M_2)| \geq 3$. Since $M$ is connected, we may assume that $M_1$ and $M_2$ are connected. Suppose $M_1$ and $M_2$ are non-binary. Then, by Lemma 1.9, both $M_1$ and $M_2$ have a $U_{2,4}$-minor using the basepoint $p$. Suppose $M_1 \not\cong U_{2,4}$. Thus there is an element $e$ of $M_1$ that is not contained in the $U_{2,4}$-minor of $M_1$ using $p$. Then, by [11; Corollary 4.3.7], $M_1$ has a connected minor $N_1$ such that $N_1 \setminus e$ or $N_1/e$ is isomorphic to $U_{2,4}$. Thus either $N_1 \cong U_{2,5}$, or $N_1$ is a parallel extension of $U_{2,4}$ by $e$. Let $f$ be an element of $E(N_1) - e$. Then, in either case, each of $N_1 \setminus e, f$; $N_1 \setminus e/f$; and $N_1/e \setminus f$ has a circuit properly containing the basepoint $p$ of the 2-sum. It follows that each of $M \setminus e, f$; $M \setminus e/f$; and $M/e \setminus f$ has the non-binary matroid $M_2$ as a minor. Since this contradicts the hypothesis that, for every $\{e, f\} \subseteq E(M)$, at least two minors of $M$ obtained by eliminating $e$ and $f$ are binary, we conclude that $M_1 \cong U_{2,4}$. By symmetry, we deduce that $M_2 \cong U_{2,4}$. Thus $M \cong U_{2,4} \oplus_2 U_{2,4}$ and the theorem holds.
Now, we may assume, without loss of generality, that $M_1$ is non-binary and $M_2$ is binary. By Lemma 3.5, $M_2$ is isomorphic to $U_{1,n}$ or $U_{n-1,n}$ for some $n \geq 3$. We assume that $M_2 \cong U_{1,n}$ for some $n \geq 3$, otherwise we replace $M$ by $M^*$ in the argument that follows. We may also suppose that $M_1$ has no elements in parallel with $p$, since any such element may be taken to be in $M_2$ rather than $M_1$. Hence $M$ is obtained from $M_1$ by replacing the basepoint $p$ by $n - 1$ elements in parallel. Moreover, $M_1/p$ is binary. To see this, suppose $M_1/p$ is non-binary and let $q$ be an element of $E(M_2) - p$. Then, as $p$ and $q$ are in parallel in $M$, $M/p \backslash q \cong M/\{p/q\} \cong M/p, q$. Moreover, $M/p, q \backslash (E(M_2) - \{p, q\})$ equals $M_1/p$. If $M_1/p$ is non-binary, then at least three of the minors of $M$ that involve the elimination of $p$ and $q$ are non-binary; a contradiction. We conclude that $M_1/p$ is binary. Thus $p \in C_{M_1}$.

By the induction assumption, one of (ii)(a)–(g) holds for $M_1$. Notice that it is impossible for $M_1$ to be isomorphic to $U_{2,4} \oplus U_{2,4}$ since $p \in C_{M_1}$, yet $U_{2,4} \oplus U_{2,4}$ has no single-element contraction that is binary. Thus one of (ii)(a)–(g), other than (ii)(c), holds for $M_1$. Now, as $M$ is obtained from $M_1$ by the parallel extension of the element $p$ of $C_{M_1}$, it is clear that (ii)(f) or (ii)(g) holds for $M$ completing the proof of the theorem. □
CHAPTER 4

MATROIDS THAT ARE ALMOST SERIES-PARALLEL NETWORKS

In matroid theory it is often important to characterize all the matroids $M$ with a certain property $P$ such that, for every element $e$ of $M$, neither $M\setminus e$ nor $M/e$ has $P$. A variant of this type of problem is to determine all such $M$ so that $M\setminus e$ or $M/e$ does not have $P$. For example, Gubser [6] characterized all graphic matroids $M$ such that, for every $e$, $M\setminus e$ or $M/e$ is the cycle matroid of a planar graph. Moreover, as noted in Chapter 3, Oxley [10] found the matroids $M$ having a $U_{2,4}$-minor such that, for every element $e$ of $M$, the deletion $M\setminus e$ or the contraction $M/e$ has no $U_{2,4}$-minor. In Sections 4.1 and 4.2, we present analogous results in which the minor under consideration has been changed from $U_{2,4}$ to $M(K_4)$; that is, from the smallest 3-connected whirl to the smallest 3-connected wheel. In Section 4.3, we characterize the matroids $M$ that are not series-parallel networks, such that, for every $e$ in $E(M)$, $M\setminus e$ or $M/e$ is a series-parallel network.

4.1. Three-Connected Binary Matroids With No $M(K_4)$-Minor

It is well known that every 3-connected matroid with at least four elements has a minor isomorphic to $U_{2,4}$ or $M(K_4)$. In particular, every binary 3-connected matroid with at least four elements has an $M(K_4)$-minor. In this section we determine the surprisingly compact list of binary 3-connected matroids $M$ with an $M(K_4)$-minor such that, for every element $e$, $M\setminus e$ or $M/e$ has no $M(K_4)$-minor. Recall that $M(\mathcal{W}_r)$ is the cycle matroid of the $r$-spoked wheel.
Theorem 4.1. The following two statements are equivalent for a binary matroid $M$.

(i) $M$ is 3-connected and has an $M(K_4)$-minor, but, for every element $e$ of $M$,

$M\setminus e$ or $M/e$ has no $M(K_4)$-minor.

(ii) (a) $M$ is isomorphic to $F_7$ or $F_7^*$; or

(b) $M$ is isomorphic to $M(W_r)$ for some $r \geq 3$.

Proof. Notice that if $M$ is a binary matroid which is isomorphic to $F_7$, $F_7^*$, or $M(W_r)$ for some $r \geq 3$, then $M$ is 3-connected and has an $M(K_4)$-minor. If $M \cong F_7$, then, for every $e$ in $E(M)$, the matroid $M/e$ has no $M(K_4)$-minor since $r(M/e) = 2$ while $r(M(K_4)) = 3$. Dually, if $M \cong F_7^*$, then, for every $e \in E(M)$, the deletion $M\setminus e$ has no $M(K_4)$-minor. Now suppose $M \cong M(W_r)$ for some $r \geq 3$. If $a$ is a rim element, then $M\setminus a$ has no $M(K_4)$-minor, while if $b$ is a spoke of the wheel, then $M/b$ has no $M(K_4)$-minor. We conclude that (ii) implies (i).

Now assume $M$ is a 3-connected binary matroid with an $M(K_4)$-minor such that, for all elements $e$ of $E(M)$, the deletion $M\setminus e$ or the contraction $M/e$ has no $M(K_4)$-minor. We will show that $M$ is isomorphic to $F_7$, $F_7^*$, or $M(W_r)$ for some $r \geq 3$.

Assume that $M$ has no $M(W_4)$-minor. Then $M \cong M(K_4)$ or, by Theorem 1.7, $M$ has a 3-connected minor $M_1$ such that $M_1\setminus e \cong M(K_4)$ or $M_1/e \cong M(K_4)$. Since $M(K_4)$ is self-dual, the statement $M_1/e \cong M(K_4)$ is equivalent to the statement $M_1^*\setminus e \cong M(K_4)$. So to determine $M_1$, we need only find the 3-connected binary single-element extensions of $M(K_4)$ and the duals of these extensions. As
$F_7$ is the only such extension of $M(K_4)$, the matroid $M_1$ is isomorphic to $F_7$ or $F_7^*$. Now if $M_1 \not\cong M$, then $M$ has a 3-connected minor $M_2$ that is a single-element extension or a single-element coextension of $F_7$ or $F_7^*$. To determine $M_2$, we need to find the 3-connected binary single-element extensions and coextensions of $F_7$. However, as $F_7$ is isomorphic to $PG(2,2)$, it has no 3-connected binary single-element extensions. Hence the 3-connected binary single-element coextensions of $F_7$ are the only possibilities for $M_2$. Seymour [14] showed that $AG(3,2)$ and $S_8$ are the only such matroids. Therefore $M_2 \cong AG(3,2)$ or $M_2 \cong S_8$. A geometric representation of each of these matroids is given in Figure 4.1.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{fig4.1.png}
\caption{(a) $AG(3,2)$. (b) $S_8$.}
\end{figure}

Notice that each matroid contains an element $e$ such that both the deletion of $e$ and the contraction of $e$ fail to destroy all the $M(K_4)$-minors of the matroid. Therefore $M$ cannot have $AG(3,2)$ or $S_8$ as a minor. Since $M_2 \not\cong AG(3,2)$ and $M_2 \not\cong S_8$, we conclude that $M \cong M_1$. Thus $M \cong F_7$ or $M \cong F_7^*$.

Now suppose $M$ has $M(W_4)$ as a minor. Assume that $M$ is not a wheel. Then, by Theorem 1.8, $M$ has a 3-connected minor $N_1$ and an element $e$ such that $N_1 \setminus e$ or $N_1/e$ is isomorphic to $M(W_4)$. Once again, as $M(W_4)$ is self-dual, we need
only consider the 3-connected single-element extensions of $M(W_4)$ to determine $N_1$. 

![Figure 4.2. The 4-wheel $W_4$.](image)

If $W_4$ is labelled as in Figure 4.2 then the following is a matrix representation for $M(W_4)$:

$$A = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1
\end{pmatrix}$$

To form a 3-connected binary extension of $M(W_4)$, the possible additions to $A$ are $(1.0.1.0)^T$, $(0.1.0.1)^T$, $(1.1.1.0)^T$, $(1.1.0.1)^T$, $(1.0.1.1)^T$, $(0.1.1.1)^T$, and $(1.1.1.1)^T$. Adding $(1.0,1.0)^T$ or $(0.1,0.1)^T$ to $A$ gives a representation of the matroid $M(K_5 - e)$, while adding $(1,1,1,1)^T$ to $A$ gives a representation of the matroid $M^*(K_{3,3})$. Upon adding any of the four remaining vectors, we obtain a matrix representation for the matroid $P_9$. So $N_1$ is isomorphic to $M(K_5 - e)$, $M^*(K_{3,3})$, or $P_9$. It is easy to verify that both $M(K_5 - e)$ and $M^*(K_{3,3})$ contain an element whose deletion and contraction fails to destroy all the $M(K_4)$-minors in the respective matroids. Moreover, if $e$ is as labelled in Figure 4.3, then both...
$P_9\setminus e$ and $P_9/e$ contain an $M(K_4)$-minor. We conclude that if $M$ has $M(W_4)$ as a minor, then $M \cong M(W_r)$ for some $r \geq 4$. Hence we have shown that, if $M$ is a binary three-connected matroid with an $M(K_4)$-minor, then $M$ is isomorphic to $F_7$, $F_7^*$, or $M(W_r)$ for some $r \geq 3$.

![Figure 4.3](image.png)

**Figure 4.3.** Both $P_9\setminus e$ and $P_9/e$ have an $M(K_4)$-minor.

### 4.2. Binary Matroids With No $M(K_4)$-Minor

In this section we utilize Theorem 4.1 to obtain an analogous result that does not require a connectivity condition. In particular, in Theorem 4.4, we characterize all binary matroids $M$ having an $M(K_4)$-minor such that, for every element $e$ of $M$, the deletion $M\setminus e$ or the contraction $M/e$ has no $M(K_4)$-minor. Moreover, as a **series-parallel network** is a non-empty connected matroid with no $U_{2,4}$- or $M(K_4)$-minor, this result describes binary matroids that are close to being series-parallel networks.

Before stating Theorem 4.4, we first note the following two lemmas. The first follows directly from the work of Seymour [12; p. 329].

**Lemma 4.2.** If $N$ is a $j$-connected minor of $M$ and $(X_1, Y_1)$ is an $m$-separation of $M$ for some $m$ with $1 \leq m < j$, then $\min\{|X_1 \cap E(N)|, |Y_1 \cap E(N)|\} \leq m - 1$. 
The proof of Theorem 4.4 will involve a matroid that is a 2-sum and contains a 3-connected minor \( N \) so that, for every element \( e \) of \( E(M) \), the deletion \( M\backslash e \) or the contraction \( M/e \) has no \( N \)-minor. Consequently, we prove the following lemma concerning the structure of such a matroid.

**Lemma 4.3.** Let \( N \) be a 3-connected matroid having at least 4 elements. Suppose \( M \) has an \( N \)-minor but, for all \( e \) in \( E(M) \), \( M\backslash e \) or \( M/e \) has no \( N \)-minor. If \( M = M_1 \oplus M_2 \), then exactly one of \( M_1 \) and \( M_2 \) has an \( N \)-minor and the other is isomorphic to \( U_{1,n} \) or \( U_{n-1,n} \) for some \( n \geq 3 \). Moreover, if \( M_i \) has an \( N \)-minor, then, for all \( e \) in \( E(M_i) \), \( M_i\backslash e \) or \( M_i/e \) has no \( N \)-minor. In particular, if \( p \) is the basepoint of the 2-sum, then \( M_i/p \) has no \( N \)-minor.

**Proof.** Suppose \( N \) is a 3-connected matroid having at least 4 elements and \( M \) has a minor that is isomorphic to \( N \). In particular, suppose \( M\backslash A/B = N_1 \) and \( N_1 \cong N \). Now, as \( M = M_1 \oplus M_2 \), we have that \( (E(M_1) - p, E(M_2) - p) \) is a 2-separation of \( M \). Moreover, Lemma 4.2 implies that

\[
\min\{|(E(M_1) - p) \cap E(N_1)|, |(E(M_2) - p) \cap E(N_1)|\} \leq 1.
\]

Therefore either \( E(M_1) \) or \( E(M_2) \) contains all but at most one element of \( N_1 \).

Without loss of generality, assume that \( |(E(M_2) - p) \cap E(N_1)| \leq 1 \). It is clear that if \( |(E(M_2) - p) \cap E(N_1)| = 0 \), then \( E(N_1) \) is contained in \( E(M_1) \) and \( M_1 \) has an \( N \)-minor. Now assume \( (E(M_2) - p) \cap E(N_1) = \{x\} \). Since \( N_1 = M\backslash A/B \) and \( M = M_1 \oplus M_2 \), we let \( A_i = A \cap (E(M_i) - p) \) and \( B_i = B \cap (E(M_i) - p) \) for \( i = 1, 2 \).

Then \( N_1 = M\backslash A/B = P(M_1\backslash A_1/B_1,M_2\backslash A_2/B_2)\backslash p \) and \( E(M_2\backslash A_2/B_2) = \{p, x\} \).

Moreover, as \( N_1 \) is 3-connected and hence contains no loops or coloops, neither \( p \)
nor \(x\) is a loop or coloop of \(M_2 \setminus A_2/B_2\). Therefore \(M_2 \setminus A_2/B_2\) is a two-element parallel class consisting of \(p\) and \(x\). Then \(N_1 = P(M_1 \setminus A_1/B_1, M_2 \setminus A_2/B_2) \setminus p = M_1 \setminus A_1/B_1\). Thus \(M_1\) has an \(N\)-minor.

Now we show that \(M_2 \cong U_{1,n}\) or \(M_2 \cong U_{n-1,n}\) for some \(n \geq 3\). The fact that \(M\) is connected means that \(M_2\) must also be connected. Now suppose \(f \in E(M_2) - p\), and let \(C_f\) and \(C_f^*\) be a maximum-sized circuit and a maximum-sized cocircuit of \(M_2\) containing \(\{p, f\}\). If both \(|C_f|\) and \(|C_f^*|\) exceed two, then each of \(M/f\) and \(M\setminus f\) has an \(M_1\)-minor. Thus both \(M\setminus f\) and \(M/f\) have an \(N\)-minor which is a contradiction. Hence, \(|C_f| = 2\) or \(|C_f^*| = 2\) for all \(f\) in \(E(M_2) - p\).

Suppose that \(f\) and \(g\) are distinct elements of \(E(M_2) - p\). If \(|C_f| = |C_g^*| = 2\), then \(C_f \cap C_g^* = \{p\}\). However, it is impossible for the cardinality of the intersection of a circuit and a cocircuit to be one. So, either \(|C_f| = 2\) for all \(f\) in \(E(M_2) - p\), or \(|C_f^*| = 2\) for all \(f\) in \(E(M_2) - p\). Therefore \(M_2 \cong U_{1,n}\) or \(M_2 \cong U_{n-1,n}\) for some \(n \geq 3\).

Next we show that for every element \(e\) of \(E(M_1)\), \(M_1 \setminus e\) or \(M_1/e\) has no \(N\)-minor and the contraction, \(M_1/p\), of the basepoint \(p\), has no \(N\)-minor. There are two cases to be considered. In the first case, we assume that \(e \neq p\). Then \(M \setminus e = P(M_1 \setminus e, M_2) \setminus p\) and \(M/e = P(M_1/e, M_2) \setminus p\). Thus \(M \setminus e\) has \(M_1 \setminus e\) as a minor and \(M/e\) has \(M_1/e\) as a minor. Since \(M \setminus e\) or \(M/e\) has no \(N\)-minor, it must be that \(M_1 \setminus e\) or \(M_1/e\) has no \(N\)-minor.

In the second case, we assume that \(e = p\). Let \(q\) be an element of \(E(M_2) - p\). Since \(M \setminus q\) has an \(N\)-minor, it must be that \(M/q\) has no such minor. But \(M/q\) is
isomorphic to \( M_1/p \oplus U_{0,n-2} \) since \( M_2 \) is isomorphic to \( U_{1,n} \). Thus \( M_1/p \) has no \( N \)-minor. \( \square \)

**Theorem 4.4.** The following three statements are equivalent for a binary matroid \( M \).

(i) \( M \) has an \( M(K_4) \)-minor, but, for every \( e \) in \( E(M) \), \( M\setminus e \) or \( M/e \) has no \( M(K_4) \)-minor.

(ii) \( M \) is not a series-parallel network, but, for every \( e \) in \( E(M) \), \( M\setminus e \) or \( M/e \) is a series-parallel network.

(iii) (a) \( M \) can be obtained from \( F_7 \) by parallel extension of a (possibly empty) subset \( S \) of \( E(F_7) \); or

(b) \( M \) can be obtained from \( F_7^* \) by series extension of a (possibly empty) subset \( T \) of \( E(F_7^*) \); or

(c) \( M \) can be obtained from \( M(K_4) \) by series extension of a subset \( S \) of \( E(M(K_4)) \) and parallel extension of a disjoint subset \( T \) of \( E(M(K_4)) \), where \( S \) or \( T \) may be empty; or

(d) \( M \) can be obtained from \( M(\mathcal{W}_r) \) for some \( r \geq 4 \) by parallel extension of a subset \( S \) of spokes or series extension of a subset \( R \) of rim elements, where \( S \) or \( T \) may be empty.

**Proof.** Statements (i) and (ii) are clearly equivalent for a binary matroid \( M \). It is not difficult to check that if any of (iii)(a), (b), (c), or (d) holds, then so does (i). Now suppose that (i) holds. We argue by induction on \( |E(M)| \) to show that (iii) holds. If \( M \) is 3-connected, then the result follows from Theorem 4.1. Assume the
result is true for all matroids satisfying the hypotheses and having fewer elements than $M$. Suppose $M$ is not connected. Then the $M(K_4)$-minor is contained in some connected component $C$ of $M$. As $M$ is disconnected, there is an element $e$ contained in $E(M) - C$. Then both $M\backslash e$ and $M/e$ have $M(K_4)$-minors which contradicts the assumption that $M\backslash e$ or $M/e$ has no $M(K_4)$-minor. Thus we may assume that $M$ is connected.

We may now assume that $M$ is connected but not 3-connected. Therefore, $M = P(M_1, M_2) \backslash p$ for some connected matroids $M_1$ and $M_2$ such that $E(M_1) \cap E(M_2) = \{p\}$, and both $|E(M_1)|$ and $|E(M_2)|$ are greater than or equal to three. Suppose $M$ has a minor $N$ that is isomorphic to $M(K_4)$. As $N$ is a 3-connected, Lemma 4.2 implies that we may assume that $M_1$ has an $M(K_4)$-minor yet for every $e \in E(M_1)$, $M_1\backslash e$ or $M_1/e$ has no $M(K_4)$-minor. Moreover, Lemma 4.2 also implies that $M_1/p$ has no $M(K_4)$-minor and that $M_2$ is isomorphic to $U_{1,n}$ or $U_{n-1,n}$ for some $N \geq 3$. Assume that $M_2 \cong U_{1,n}$ for some $n \geq 3$, otherwise substitute $M^*$ for $M$ in the argument that follows.

Now, by the induction assumption, one of (iii)(a), (b), (c), or (d) must hold for $M$. If $M_1$ satisfies (iii)(a), then clearly so does $M$. Moreover, if $p$ is in a non-trivial series class, then it follows that $M_1/p$ has an $M(K_4)$-minor; a contradiction. Thus $p$ cannot be in a non-trivial series class. Suppose $M_1$ satisfies (iii)(b); that is, $M_1$ can be obtained from $F_7^*$ by series extension. Now $p$ is an element of $F_7^*$ that is not in any non-trivial series extension used to form $M_1$. Therefore, $M_1/p$
can be obtained from $M(K_4)$ by series extension; a contradiction to the fact that $M_1/p$ has no $M(K_4)$-minor. We conclude that $M_1$ satisfies (iii)(c) or (iii)(d).

Suppose $M_1$ satisfies (iii)(c). Then $M_1$ can be obtained from $M(K_4)$ by series extension of a subset $S$ of $E(M(K_4))$ and parallel extension of a disjoint subset $T$ of $E(M(K_4))$, where $S$ or $T$ may be empty. Since $p$ cannot be in a non-trivial series class, it follows that $M$ is obtained from $M(K_4)$ by series extension of a subset $S$ of $E(M(K_4))$ and parallel extension of a disjoint subset $T \cup p$. Hence $M$ satisfies (iii)(c).

Now suppose $M_1$ satisfies (iii)(d). Then $M_1$ is obtained from $M(\mathcal{W}_r)$ for some $r \geq 4$ by parallel extension of a subset $S$ of spokes or series extension of a subset $T$ of rim elements. Now $p$ cannot be in a non-trivial series class. Moreover, if $p$ is a rim element, then $M_1/p$ has an $M(K_4)$-minor; a contradiction. Thus we may assume that $p$ is a spoke of the wheel. Therefore $M$, as well as $M_1$, satisfies (iii)(d) and the theorem holds.

4.3. Matroids That Are Almost Series-Parallel Networks

In the preceding section we presented a description of the binary matroids $M$ that are not series-parallel networks, but, for every $e$ in $E(M)$, $M\setminus e$ or $M/e$ is a series-parallel network. In this section we determine those matroids $M$, binary or not, that are not series-parallel networks, yet, for every $e$ in $E(M)$, $M\setminus e$ or $M/e$ is a series-parallel network. As before, we first prove a result relying on 3-connectivity and then utilize it to obtain a more general theorem.
Theorem 4.5. The following two statements are equivalent for a 3-connected matroid $M$.

(i) $M$ is not a series-parallel network, but, for every $e$ in $E(M)$, $M\setminus e$ or $M/e$ is a series-parallel network.

(ii) (a) $M$ is isomorphic to $U_{2,n}$ for some $n \geq 4$; or
(b) $M$ is isomorphic to $U_{n-2,n}$ for some $n \geq 4$; or
(c) $M$ is isomorphic to $F_7$ or $F_7^*$; or
(d) $M$ is isomorphic to $M(\mathcal{W}_r)$ for some $r \geq 3$; or
(e) $M$ is isomorphic to $\mathcal{W}^r$ for some $r \geq 3$.

Proof. One can check that if any of (ii)(a), (b), (c), (d), or (e) holds, then so does (i). Now suppose that (i) holds. If $M$ is binary, then as it is not a series-parallel network, $M$ must have an $M(K_4)$-minor. However, for every $e$ in $E(M)$, we have that $M\setminus e$ or $M/e$ is a series-parallel network, and hence, has no $M(K_4)$-minor. Theorem 4.1 implies that $M$ is isomorphic to one of $F_7$, $F_7^*$, or $M(\mathcal{W}_r)$ for some $r \geq 3$ and the theorem holds.

Now we may assume that $M$ is non-binary. Since, for every element $e$, we have $M\setminus e$ or $M/e$ is a series-parallel network, it must be that $M\setminus e$ or $M/e$ is binary for every element $e$ of $E(M)$. Then Theorem 3.1 implies that $M$ is isomorphic to $U_{2,n}$ or $U_{n-2,n}$ for some $n \geq 4$, or both $r(M)$ and $r^*(M)$ exceed two and $M$ can be obtained from a 3-connected binary matroid by relaxing a circuit-hyperplane. The theorem clearly holds if $M \cong U_{2,n}$ or $U_{n-2,n}$ for some $n \geq 4$. Thus we may assume that both $r(M)$ and $r^*(M)$ exceed two and $M$ can be obtained from a 3-connected
binary matroid by relaxing a circuit-hyperplane. Now, as $M$ is 3-connected and non-binary, Lemma 1.9 implies that $M$ has a minor $N$ that is isomorphic to one of $U_{3,6}$, $P_6$, $Q_6$, and $\mathcal{W}^3$. Suppose $N$ is isomorphic to $U_{3,6}$, $P_6$, or $Q_6$ and the element $e$ is as marked in Figure 4.4. Then both $N \setminus e$ and $N/e$ are non-binary, and hence, fail to be series-parallel networks. We conclude that $M$ has no minor isomorphic to $U_{3,6}$, $P_6$, or $Q_6$. Thus we may assume that $M$ has a $\mathcal{W}^3$-minor.

![Figure 4.4. (a) $U_{3,6}$. (b) $P_6$. (c) $Q_6$.](image)

Now if $M$ is not a whirl, then, by Theorem 1.8, $M$ has a 3-connected minor $M_1$ and an element $e$ such that $M_1 \setminus e$ or $M_1/e$ is isomorphic to $\mathcal{W}^3$. Since $\mathcal{W}^3$ is self-dual, the candidates for $M_1$ are the 3-connected single-element extensions of $\mathcal{W}^3$ and the duals of these extensions. Moreover, as $M_1$ is a minor of $M$, it contains no element $f$ such that both $M_1 \setminus f$ and $M_1/f$ fail to be series-parallel networks. In addition, if there were such an element in a matroid $N$ that is a possibility for $M_1$, then both $N$ and $N^*$ would be eliminated as feasible candidates for the minor $M_1$. In this way, we may focus on the single-element extensions of $\mathcal{W}^3$ to determine $M_1$. Moreover, as $M_1 \setminus e \cong \mathcal{W}^3$, the matroid $M_1/e$ must be a series-parallel network. Therefore the problem of determining the viable possibilities for $M_1$ can be viewed as a problem of characterizing the matroids $N$ with an
element \( e \) such that \( N/e \) is a series-parallel network and \( N\setminus e \cong \mathcal{W}^3 \). To determine these matroids we consider the various ways of adding the point \( e \) to the geometric representation of \( \mathcal{W}^3 \). If \( e \) is added freely, then \( N/e \) is non-binary and hence fails to be a series-parallel network. Thus \( e \) cannot be added freely to \( \mathcal{W}^3 \). Suppose \( e \) is added on one of the existing 3-point lines of the geometric representation for \( \mathcal{W}^3 \). If \( e \) is contained in only one non-trivial line, then \( N/e \) is again non-binary and fails to be a series-parallel network. Therefore \( e \) is contained in two lines and \( N \) is represented in Figure 4.5. However, if \( f \) is as marked, then both \( N\setminus f \) and \( N/f \) fail to be series-parallel networks.

\[ \text{Figure 4.5. } N\setminus f \text{ has an } M(K_4)\text{-minor and } N/f \text{ is non-binary.} \]

We conclude that \( e \) is neither added freely nor added to one of the existing non-trivial lines of the geometric representation of \( \mathcal{W}^3 \). Therefore the addition of \( e \) to the geometric representation of \( \mathcal{W}^3 \) creates one, two, or three new 3-point lines. It is easy to verify that if the addition of \( e \) creates one or two new 3-point lines, then both \( M\setminus e \) and \( M/e \) are non-binary and hence fail to be series-parallel networks. Finally, if \( e \) is added to the geometric representation of \( \mathcal{W}^3 \) so that three new 3-point lines are created, the matroid \( F_7^- \) is obtained. Moreover, if \( f \) is as marked in Figure 4.6, then both \( M\setminus f \) and \( M/f \) fail to be series-parallel networks.
Since there are no viable possibilities for $M_1$, we conclude that $M$ is a whirl and the theorem holds.

\[ \square \]

**Figure 4.6.** $F_7^−/f$ has an $M(K_4)$-minor and $F_7^−/f$ is non-binary.

**Theorem 4.6.** The following two statements are equivalent for a matroid $M$.

(i) $M$ is not a series-parallel network, but, for every $e$ in $E(M)$, $M\backslash e$ or $M/e$ is a series-parallel network.

(ii) (a) $M$ is isomorphic to a parallel extension of $U_{2,n}$ for some $n \geq 5$; or

(b) $M$ is isomorphic to a series extension of $U_{n-2,n}$ for some $n \geq 5$; or

(c) $M$ can be obtained from $U_{2,4}$ by series extension of a subset $S$ of $E(U_{2,4})$ and parallel extension of a disjoint subset $T$ of $E(U_{2,4})$ where $S$ or $T$ may be empty; or

(d) $M$ can be obtained from $F_7$ by parallel extension of a (possibly empty) subset $S$ of $E(F_7)$; or

(e) $M$ can be obtained from $F_7^*$ by series extension of a (possibly empty) subset $T$ of $E(F_7^*)$; or

(f) $M$ can be obtained from $M(K_4)$ by series extension of a subset $S$ of $E(M(K_4))$ and parallel extension of a disjoint subset $T$ of $E(M(K_4))$ where $S$ or $T$ may be empty; or

(g) $M$ can be obtained from $M(\mathcal{W}_r)$ for some $r \geq 4$ by parallel extension
of a subset $S$ of spokes or series extension of a subset $R$ of rim elements

where $S$ or $R$ may be empty; or

(h) $M$ can be obtained from $\mathcal{W}^r$ for some $r \geq 3$ by parallel extension of a

subset $S$ of spokes or series extension of a subset $R$ of rim elements where

$S$ or $R$ may be empty.

**Proof.** Clearly, if any of (ii)(a) through (ii)(h) holds then (i) holds. Now assume

that (i) holds. We argue by induction on $|E(M)|$ to show that (ii) holds. If $M$

is 3-connected, then the result follows from Theorem 4.4. Assume the result is

ture for all matroids satisfying the hypotheses and having fewer elements than $M$.

One can easily verify that $M$ must be connected. Hence, as $M$ is connected but

not 3-connected, it follows from Theorem 1.5 that $M = P(M_1, M_2) \backslash p$ for some

matroids $M_1$ and $M_2$. Moreover, $E(M_1) \cap E(M_2) = \{p\}$ and both $|E(M_1)|$ and

$|E(M_2)|$ exceed two. Since $M$ is not a series-parallel network, it must have a

minor isomorphic to $U_{2,4}$ or $M(K_4)$. As both of these matroids are 3-connected,

Lemma 4.2 implies that exactly one of $M_1$ and $M_2$ has an $N$-minor and the other

is isomorphic to $U_{1,n}$ or $U_{n-1,n}$ for some $n \geq 3$. Without loss of generality we may

assume that $M_1$ has an $N$-minor. Also suppose that $M_2 \cong U_{1,n}$; otherwise replace

$M$ by $M^*$ in the argument that follows. Lemma 4.2 also implies that $M_1/p$ has

no $N$-minor.

Now, by the induction assumption, one of (ii)(a) through (ii)(h) must hold

for $M_1$. Moreover, if $M_1$ satisfies (ii)(a) or (ii)(d), then clearly so does $M$ and the

theorem holds.
If $p$ is in a non-trivial series class, then $M_1/p$ has an $N$-minor which is a contradiction. Thus we may assume that $p$ is not in a non-trivial series class. Suppose $M_1$ satisfies (ii)(b) or (ii)(c). Then, for some $n$ greater than or equal to five, $M_1/p$ is isomorphic to a series extension of $U_{n-3,n-1}$ or a series extension of $M(K_4)$. In either case, $M_1/p$ has an $N$-minor and we conclude that $M_1$ is described by (ii)(c), (ii)(f), (ii)(g), or (ii)(h).

Now suppose that $M_1$ satisfies (ii)(c) or (ii)(f). By the choice of $M_2$, the element $p$ is not in a non-trivial parallel class of $M_1$. Thus $p$ is an element of $U_{2,4}$ or $M(K_4)$ that is not involved in any of the series or parallel extensions used to form $M_1$. Hence $M_1$ satisfies (ii)(c) or (ii)(f) and the theorem holds.

If $M_1$ satisfies (ii)(g) or (ii)(h), then, for some $r \geq 3$, $M_1$ is obtained from $M(W_r)$ or $W^n$ by parallel extension of a subset $S$ of spokes or series extension of a subset $R$ of rim elements. Now $p$ cannot be in a non-trivial series class. Moreover, if $p$ is a rim element, then $M_1/p$ has an $M(K_4)$-minor which is a contradiction. Thus we may assume that $p$ is a spoke. Then both $M_1$ and $M$ satisfy (ii)(g) or (ii)(h) and the theorem holds. $\Box$
BIBLIOGRAPHY


VITA

Allan Donald Mills, son of Stuart E. Mills and Patricia H. Mills, was born in Baton Rouge, Louisiana on October 25, 1967. In 1985, he graduated from Caddo Parish Magnet High School in Shreveport, Louisiana and enrolled in Louisiana State University. After graduating in May 1989 with a B.S., Allan remained at L.S.U. to pursue graduate studies in mathematics. He received the M.S. in May 1991 and is currently a candidate for the Doctor of Philosophy degree. Allan is married to the former Dawn Marie Ewing and they have one son, Patrick Forrest.
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