Commutative C*-Algebras Generated by Toeplitz Operators on the Fock Space

Vishwa Nirmika Dewage
Louisiana State University and Agricultural and Mechanical College

Follow this and additional works at: https://digitalcommons.lsu.edu/gradschool_dissertations

Part of the Analysis Commons, and the Harmonic Analysis and Representation Commons

Recommended Citation
https://digitalcommons.lsu.edu/gradschool_dissertations/5930
COMMUTATIVE $C^*$-ALGEBRAS GENERATED BY TOEPLITZ OPERATORS ON THE FOCK SPACE

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

Vishwa Nirmika Dewage
B.S., University of Colombo, 2016
August 2022
To my loving parents, Anula and Aruna.
Acknowledgments

I would like to thank my adviser, Professor Gestur Ólafsson who made this work possible. I am thankful for the guidance, expertise and encouragement I received from him throughout my journey as a graduate student. I thank professors, Hwang Lee, Phuc Cong Nguyen and Stephen Shipman for taking the time to be on my committee. I would also like to thank the people that make the department of mathematics, LSU, for making my experience a better one.

I thank Matthew Dawson for math discussions and insights. I thank professors, Wolfram Bauer, Egor Maximenko, Raul Quiroga-Barranco and Nikolai Vasilevski for taking an interest in my research, providing me with feedback and for pointing to possible consequences.

I thank my friend, Nisal Kevin Kotinkaduwa for being an awesome mentor. I thank my teachers and professors, Chandani Bandara, Dayal Dharmsena, Romaine Jayawardene, W. Ramasinghe and Manoj Sonlangaarchchi.

I thank my parents, Anula and Aruna, for the unwavering support and for many sacrifices they made. I thank my father for inspiring me creatively. I thank my husband, Dulith, for being my companion through this journey.
**Table of Contents**

Acknowledgments ........................................ iv

Abstract .................................................. vii

Chapter 1. Introduction .................................. 1
  1.1. Toeplitz matrices .................................. 1
  1.2. Toeplitz operators on spaces of analytic functions .... 3
  1.3. Contributions discussed in the dissertation ........... 5

Chapter 2. Preliminaries ................................. 7
  2.1. Fock space ......................................... 7
  2.2. C*-algebras ........................................ 10
  2.3. Representation theory ............................... 12
  2.4. Special functions ................................... 13

Chapter 3. Toeplitz Operators with Quasi-Radial Symbols .... 14
  3.1. Toeplitz operators with bounded $k$-quasi-radial symbols .... 14
  3.2. A classification of the class of symbols $L^\infty(\mathbb{C}^n)^G$ .... 15
  3.3. Diagonalizing $T_\phi$ by Schur’s lemma ................ 16
  3.4. Computing eigenvalue functions ........................ 21

Chapter 4. The Space $C_{b,u}(\mathbb{N}_0^k, \rho_k)$ .............. 26
  4.1. Examples ........................................... 28
  4.2. Invariance of $C_{b,u}(\mathbb{N}_0^k, \rho_k)$ under shifts ....... 29
  4.3. A comparison of $C_{b,u}(\mathbb{N}_0^k, \rho_k)$ with $C_{b,u}(\mathbb{N}_0, \rho_1) \otimes \cdots \otimes C_{b,u}(\mathbb{N}_0, \rho_1)$ .... 31
  4.4. Extending functions in $C_{b,u}(\mathbb{N}_0^k, \rho_k)$ to $\mathbb{R}_+^k$ ....... 34

Chapter 5. The C*-Algebra Generated by Toeplitz Operators with $k$-Quasi-Radial Symbols .......................... 43
  5.1. Uniform continuity of eigenvalue functions with respect to the square-root metric .......................... 43
  5.2. Density of $\mathcal{G}_1$ in $C_{b,u}(\mathbb{N}_0^k, \rho_k)$ .................. 47
  5.3. The C*-algebra generated by $\mathcal{G}_n$ .................. 57
  5.4. The C*-algebra $C_{b,u}(\mathbb{N}_0, \rho_1) \otimes \cdots \otimes C_{b,u}(\mathbb{N}_0, \rho_1)$ .......................... 58

Chapter 6. Future Research Directions .................... 62
  6.1. Separately-radial Toeplitz operators on the Fock space ....... 62
  6.2. Toeplitz operators with dilation-invariant symbols .......... 62

Appendix. Copyright Information .......................... 64
Abstract

The Fock space $\mathcal{F}(\mathbb{C}^n)$ is the space of holomorphic functions on $\mathbb{C}^n$ that are square-integrable with respect to the Gaussian measure on $\mathbb{C}^n$. This space plays an essential role in several subfields of analysis and representation theory. In particular, it has for a long time been a model to study Toeplitz operators. Grudsky and Vasilevski showed in 2002 that radial Toeplitz operators on $\mathcal{F}(\mathbb{C})$ generate a commutative $C^*$-algebra $\mathcal{T}^G$, while Esmeral and Maximenko showed that $C^*$-algebra $\mathcal{T}^G$ is isometrically isomorphic to the $C^*$-algebra $C_{b,u}(\mathbb{N}_0, \rho_1)$. In this thesis, we extend the result to $k$-quasi-radial symbols acting on the Fock space $\mathcal{F}(\mathbb{C}^n)$. We calculate the spectra of the said Toeplitz operators and show that the set of all eigenvalue functions is dense in the $C^*$-algebra $C_{b,u}(\mathbb{N}_0^k, \rho_k)$ of bounded functions on $\mathbb{N}_0^k$ which are uniformly continuous with respect to the square-root metric. In fact, the $C^*$-algebra generated by Toeplitz operators with quasi-radial symbols is $C_{b,u}(\mathbb{N}_0^k, \rho_k)$. 
Chapter 1. Introduction

The theory of Toeplitz operators originated from a collection of ideas developed in the latter half of the 20th century by mathematicians such as Hermann Hankel, Eberhard Hopf, Gábor Szegő, Otto Toeplitz, and Norbert Wiener, among many [N20]. Toeplitz operators have been studied extensively over the last few decades and have far reaching applications in modern mathematics, having applications in singular integral equations, prediction theory, moment problems, quantization, and so on (see, [BC86, C94, EU11, GQV06]). However, the story of Toeplitz operators started with Toeplitz matrices and a intriguing change of perspective.

1.1. Toeplitz matrices

An infinite Toeplitz matrix, named after Otto Toeplitz who contributed to the mathematical ideas surrounding the early stages of the theory, is an infinite matrix that has constant diagonals. Hence it has the form:

$$
T = \begin{pmatrix}
    a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\
    a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\
    a_2 & a_1 & a_0 & a_{-1} & \cdots \\
    a_3 & a_2 & a_1 & a_0 & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

On the other hand, the counterparts of the Toeplitz matrices, the Hankel matrices, are infinite matrices with constant off-diagonals. Toeplitz and Hankel matrices were a main topic of discussion in the 20th century mathematics and among others, mathematicians
such as Bernhard Riemann, David Hilbert, Norbert Wiener, George Birkhoff, and Otto Toeplitz have contributed to the theory [N20].

We can consider $T$ as an operator acting on $\ell^2$. Then arises the question: For which sequences $\{a_k\}_{k \in \mathbb{Z}}$, the Toeplitz matrix given above defines a bounded operator? It was discovered in the mid 20th century that the above matrix defines a bounded operator if and only if there exists a function $\varphi \in L^\infty(\mathbb{T})$ such that $a_k = \hat{\varphi}(k)$ where $\hat{\varphi}(k)$ are the Fourrier coefficients of $\varphi$. The sufficiency of the statement can be seen without much difficulty by adapting the language of Toeplitz operators.

Let $L^2(\mathbb{T}, d\sigma)$ be the space of square integrable functions on $\mathbb{T}$ with respect to the unique rotational invariant probability measure $\sigma$ on $\mathbb{T}$. Then the Hardy space $H^2(\mathbb{T})$ over the unit circle, can be understood as the closure of the linear span of $\{e^{in\theta}| n \in \mathbb{N}_0\}$ in $L^2(\mathbb{T}, d\sigma)$. $H^2$ is a closed subspace of $L^2(\mathbb{T}, d\sigma)$ and moreover, it is a reproducing kernel Hilbert space. For $\varphi \in L^\infty(\mathbb{T})$, we define the Toeplitz operator $T_\varphi$ by

$$T_\varphi f = P(\varphi f), \quad f \in H^2$$

where $P : L^2(\mathbb{T}, d\sigma) \to H^2$ denotes the projection operator. Then $T_\varphi$ is a bounded operator and the matrix representation of $T_\varphi$ with respect to the monomial basis

$$\{z^n| n \in \mathbb{N}_0\}$$

is given by the Toeplitz matrix $T$ with $a_k = \hat{\varphi}(k)$.

In modern analysis, we use the same set-up to define Toeplitz operators on a variety of spaces.
1.2. Toeplitz operators on spaces of analytic functions

$L^2$ spaces of analytic functions such as Bergman spaces, Hardy spaces and Fock spaces are well researched objects and has a wide variety of applications in many sub-fields of analysis, representation theory and in mathematical physics. These reproducing kernel Hilbert spaces also serve as a model on which to define Toeplitz operators. Let $\mathcal{D}$ denote a complex domain endowed with a probability measure $d\lambda$. Denote by $\mathcal{A}^2(\mathcal{D})$ the $L^2$ space of analytic functions on $\mathcal{D}$ and let

$$P : L^2(\mathcal{D}) \to \mathcal{A}^2(\mathcal{D})$$

denote the Bergman projection which is a norm decreasing bounded operator. The Toeplitz operator $T_\varphi$ with the symbol $\varphi \in L^\infty(\mathcal{D})$, acting on $\mathcal{A}^2(\mathcal{D})$, is defined by

$$T_\varphi f = P(\varphi f), \quad f \in \mathcal{A}^2(\mathcal{D}).$$

The Toeplitz operator $T_\varphi$ is a bounded operator with $\|T_\varphi\| \leq \|\varphi\|_\infty$.

In the scope of this thesis we focus on Toeplitz operators on the Fock space.

1.2.1. Fock spaces

The Fock space $\mathcal{F}(\mathbb{C}^n)$, also known as the Segal-Bargmann space, is the space of all analytic functions on $\mathbb{C}^n$ that are square-integrable with respect to the Gaussian measure $d\lambda$ given by

$$d\lambda_n(z) = \frac{1}{\pi^n} e^{-|z|^2} dz$$

where $dz$ denotes the Lebesgue measure on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. The Fock space stands out among other similar spaces of analytic functions due to the Euclidean metric on the domain.
Also, when the Toeplitz operators on Fock spaces are taken into account, constants are the only essentially bounded analytic symbols. In [V00], Vasilevski introduced poly-Fock spaces and proved that the $L^2$ space over $\mathbb{C}^n$ with respect to the Gaussian measure can be written as a direct sum of true-poly-Fock spaces.

1.2.2. Commutative operator algebras generated by Toeplitz operators

The commutativity of operator algebras, $C^*$ or otherwise, generated by families of Toeplitz operators is one of the main problems in the theory of Toeplitz operators. Typically, commuting families of Toeplitz operators are found by considering essentially bounded symbols that are invariant under the action of a group. If the group $G$ acts on the domain $\mathcal{D}$, then, the group acts on the space of symbols by

$$g \cdot \varphi(z) = \varphi(g^{-1}z), \quad z \in \mathcal{D}, \; \varphi \in L^\infty(\mathcal{D}).$$

The symbol $\varphi$ is $G$-invariant if $g \cdot \varphi = \varphi$ for all $g \in G$. Several examples of families of commuting Toeplitz operators can be found in [DÓQ18, DQ18, GKV03, V08, V10a, V10b]. If the $C^*$-algebra $\mathcal{T}^G$ generated by the Toeplitz operators under consideration is commutative, then, the next question is to determine the spectra of the Toeplitz operators and describe the $C^*$-algebra $\mathcal{T}^G$. Spectra of the Toeplitz operators are usually computed by constructing an analog of the Bargmann transform that maps the Toeplitz operators under consideration to multiplier operators.

For the Bergman space over the unit ball $\mathbb{B}^n$, it was discovered that the $C^*$-algebras generated by Toeplitz operators with essentially bounded symbols invariant under maximal abelian subgroups of the group of biholomorphisms, are commutative [QV07].
with the use of representation theory, a more general theory of commutative $C^*$-algebras generated by Toeplitz operators was discussed for Toeplitz operators on Bergman spaces over bounded symmetric domains, with symbols invariant under suitable subgroups (including maximal compact subgroups) of the group of biholomorphisms, in [DÓQ15]. Moreover, the commutative $C^*$-algebras generated by Toeplitz operators with radial symbols acting on the Bergman space over the unit ball $\mathbb{B}^n$ was shown to be isomorphic to the slowly oscillating sequences on $\mathbb{N}_0$ in [GMV13].

Further, the Toeplitz operators on the Fock space $\mathcal{F}^2(\mathbb{C})$ with horizontal symbols also generate a commutative $C^*$-algebra [EV16]. The $C^*$-algebras generated by Toeplitz operators on the Fock space $\mathcal{F}^2(\mathbb{C})$ with radial symbols generate a commutative $C^*$-algebra which is isomorphic to the square-root-slowly oscillating sequences on $\mathbb{N}_0$ [EM16, GV02]. Several families of commuting Toeplitz operators on Fock spaces and poly-Fock spaces were discussed in [BI12, BL11, EV16, SGLA18].

1.3. Contributions discussed in the dissertation

1.3.1. Spectra of quasi-radial Toeplitz operators and the commutativity of the Toeplitz algebra

Grudsky and Vasilevski [GV02] showed that the eigenvalue sequences of radial Toeplitz operators on the Fock space $\mathcal{F}(\mathbb{C})$ over $\mathbb{C}$ are of the form:

$$\gamma_{1,a}(m) = \frac{1}{m!} \int_0^\infty a(\sqrt{r}) r^m e^{-r} dr$$

and that the $C^*$-algebra generated by Radial Toeplitz operators is commutative. By means of representation theory we prove that for $\mathbf{n} \in \mathbb{N}^k$, the Toeplitz operators with $k$-quasi-
radial symbols diagonalize with eigenvalue functions given by
\[ \gamma_{n,a}(m) = \frac{1}{(m+n-1)!} \int_{\mathbb{R}^k_+} a(\sqrt{r_1}, \ldots, \sqrt{r_k}) r_1^{m+n-1} e^{-(r_1 + \cdots + r_k)} dr \]
for any \( m \in \mathbb{N}_0^k \) and that the \( C^* \)-algebra generated by Toeplitz operators with \( k \)-quasi-radial symbols is a commutative \( C^* \)-algebra.

As a corollary, the eigenvalue sequences of radial Toeplitz operators on \( \mathcal{F}(\mathbb{C}^n) \) are of the form
\[ \gamma_{n,a}(m) = \frac{1}{(m+n-1)!} \int_{0}^{\infty} a(\sqrt{r}) r^{m+n-1} e^{-r} dr \]
which are the \((n-1)^{th}\) left shift of \( \gamma_{1,a} \).

1.3.2. \( C^* \)-algebra generated by eigenvalue functions

Motivated by Gelfand-Naimark theory, we characterize the \( C^* \)-algebra \( \mathcal{T}^G \) generated by \( k \)-quasi radial Toeplitz operators. We show that the set of all eigenvalue functions form a is a dense subset of \( C_{b,u}(\mathbb{N}_0^k, \rho_k) \) the \( C^* \)-algebra of bounded functions on \( \mathbb{N}_0^k \) that are uniformly continuous with respect to the square-root metric \( \rho_k \). As a consequence \( \mathcal{T}^G \) is isometrically isomorphic to \( C_{b,u}(\mathbb{N}_0^k, \rho_k) \).

This is consistent with the results discussed in [EM16] for radial Toeplitz operators on the Fock space \( \mathcal{F}(\mathbb{C}) \) over \( \mathbb{C} \). The techniques used in the proof are, to some extent, a generalization of ideas in [EM16].

Moreover, to debunk the misleading idea that \( \mathcal{T}^G \) could be isometrically isomorphic to \( C_{b,u}(\mathbb{N}_0, \rho_1) \otimes \cdots \otimes C_{b,u}(\mathbb{N}_0, \rho_1) \), we show that the inclusion
\[ C_{b,u}(\mathbb{N}_0, \rho_1) \otimes C_{b,u}(\mathbb{N}_0, \rho_1) \hookrightarrow C_{b,u}(\mathbb{N}_0^2, \rho_2). \]
is proper by means of a counter example. This work was first published as [DO22].
Chapter 2. Preliminaries

2.1. Fock space

We now introduce some notations and recall the following well known facts about the Fock space. Denote by $\mathcal{F}(\mathbb{C}^n)$, the Fock space of all holomorphic functions on $\mathbb{C}^n$ that are square integrable with respect to the Gaussian measure $d\lambda_n$. The Fock space, $\mathcal{F}(\mathbb{C}^n)$ is a closed subspace of $L^2(\mathbb{C}^n, \lambda_n)$. Moreover, the point-evalutation maps $f \mapsto f(z)$ are continuous and hence $\mathcal{F}(\mathbb{C}^n)$ is a reproducing kernel Hilbert space with the reproducing kernel $K$ given by

$$K(z, w) = e^{w\bar{z}}, \quad \text{for } (z, w) \in \mathbb{C}^{2n} \text{ (here } \bar{z}w \text{ denotes the scalar product } w_1\bar{z}_1 + \cdots + w_k\bar{z}_k).$$

Thus for $z \in \mathbb{C}^n$ and $f \in \mathcal{F}(\mathbb{C}^n)$ we have

$$f(z) = \langle f, K_z \rangle$$

with the inner product of $L^2(\mathbb{C}^n, \lambda_n)$.

The Bergman projection $P : L^2(\mathbb{C}^n) \rightarrow \mathcal{F}(\mathbb{C}^n)$ is given by

$$Pf(z) = \langle f, K_z \rangle.$$ 

$P$ is a bounded linear operator with $\|P\| = 1$.

**Proposition 2.1.1.** Let $n \in \mathbb{N}$. For $\alpha \in \mathbb{N}^n$ define the monomial $p_\alpha$ by

$$p_\alpha(z) = z^{\alpha} := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$$

$z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Then

$$\|p_\alpha\|_{\mathcal{F}}^2 = \alpha! = \alpha_1! \cdots \alpha_n!$$

and the collection of normalized monomials given by

$$\left\{ q_\alpha = \frac{1}{\sqrt{\alpha!}}p_\alpha \right\}_{\alpha \in \mathbb{N}^n}$$
forms an orthonormal basis for $\mathcal{F}(\mathbb{C}^n)$.

Define the Bargmann transform $\mathcal{B} : L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{C}^n, \lambda_n)$ by

$$\mathcal{B} f(z) = \pi^{-n/4} \int_{\mathbb{R}^n} f(x) e^{\sqrt{2}x \cdot z - \frac{x^2 + z^2}{2}} dx, \quad z \in \mathbb{C}^n. \quad (2.1.1)$$

**Theorem 2.1.2.** The Bargmann transform is an isometric isomorphism from $L^2(\mathbb{R}^n, dx)$ to $\mathcal{F}(\mathbb{C}^n)$. Then inverse of the Bargmann transform, $\mathcal{B}^{-1} : \mathcal{F}(\mathbb{C}^n) \to L^2(\mathbb{R}^n, dx)$ is given by

$$\mathcal{B}^{-1} f(x) = \pi^{-n/4} \int_{\mathbb{C}^n} f(w) e^{\sqrt{2}x \cdot \bar{w} - \frac{x^2 + |w|^2}{2}} d\lambda_n(w), \quad x \in \mathbb{R}^n \quad (2.1.2)$$

It is interesting to note that the Hermite function basis of $L^2(\mathbb{R}^n, dx)$ gets mapped to the monomial basis $\mathcal{F}(\mathbb{C}^n)$, under the Bargmann transform.

We point to [Z12] for more details.

**2.1.1. Toeplitz operators on the Fock space**

Given $\varphi \in L^\infty(\mathbb{C}^n)$, we define the Toeplitz operator $T_\varphi : \mathcal{F}(\mathbb{C}^n) \to \mathcal{F}(\mathbb{C}^n)$ by

$$T_\varphi f(z) = P(\varphi f)(z) = \int_{\mathbb{C}^n} \varphi(w) f(w) \overline{K_z(w)} d\lambda_n(w).$$

As the multiplier operator $M_\varphi : L^2(\mathbb{C}^n, \lambda) \to L^2(\mathbb{C}^n, \lambda)$ is bounded of norm $\|\varphi\|_\infty$ and $P$ is bounded, it follows that $T_\varphi$ is bounded and $\|T_\varphi\| \leq \|\varphi\|_\infty$. The function $\varphi$ is called the symbol of the Toeplitz operator $T_\varphi$.

**Lemma 2.1.3.** The $C^*$-algebra generated by the set of all Toeplitz operators with essentially bounded symbols, is not commutative.

**Proof.** We construct two noncommuting Toeplitz operators as counter examples. Define $\varphi_1, \varphi_2 \in L^\infty(\mathbb{C}^n)$ by

$$\varphi_1(w) = \chi_{B_n}(w), \quad \varphi_2(w) = w \chi_{B_n}(w); \quad w \in \mathbb{C}^n.$$
Let $f$ be the constant function $1$ on $\mathbb{C}^n$. Since monomials form a basis for $\mathcal{F}(\mathbb{C}^n)$, and by polar coordinates formula (see Lemma 3.4.1 for details), there exists a constant $C_n > 0$ s.t.

$$T_{\varphi_1} f(z) = \int_{B_n} e^{z\bar{w}} d\lambda_n(w)$$

$$= \int_{B_n} 1 \, d\lambda_n(w)$$

$$= C_n \int_0^1 r^{2n-1} e^{-r^2} dr \quad \text{(polar coordinates formula)}$$

$$= C_n \int_0^1 r^{n-1} e^{-r} dr \quad \text{(by a change of variable)}$$

and similarly,

$$T_{\varphi_2} f(z) = \int_{B_n} we^{z\bar{w}} d\lambda_n(w)$$

$$= z \int_{B_n} |w|^2 \, d\lambda_n(w)$$

$$= zC_n \int_0^1 r^n e^{-r} dr$$

for $z \in \mathbb{C}^n$. Then,

$$T_{\varphi_2} T_{\varphi_1} f(z) = \left( C_n^2 \int_0^1 r^{n-1} e^{-r} dr \int_0^1 r^n e^{-r} dr \right) z$$

and

$$T_{\varphi_1} T_{\varphi_2} f(z) = C_n \int_0^1 r^n e^{-r} dr \int_{B_n} we^{z\bar{w}} d\lambda_n(w)$$

$$= \left( C_n \int_0^1 r^n e^{-r} dr \right)^2 z$$
for $z \in \mathbb{C}^n$. Note that $\int_0^1 r^n e^{-r}dr \neq \int_0^1 r^{n-1} e^{-r}dr$ (by Lemma 2.4.1) and hence $T_{\varphi_1}$ and $T_{\varphi_2}$ do not commute.

\[ \square \]

## 2.2. C*-algebras

Here we collect a few basic facts about Banach algebras and $C^*$-algebras. We point to [BR79, D72, DE09, R91] for a more extensive theory.

Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}$ is unital, the spectrum $\sigma_{\mathcal{A}}(x)$ of an element $x \in \mathcal{A}$ is defined to be the set

$$\sigma_{\mathcal{A}}(x) := \{ \lambda \mid x - \lambda \text{id is not invertible} \}.$$  

If $\mathcal{A}$ does not contain a unity, we construct the Banach algebra $\mathcal{A}^{id}$ by adjoining a unity. Then $\mathcal{A} \subset \mathcal{A}^{id}$ and we define the spectrum $\sigma_{\mathcal{A}}(x)$ by setting

$$\sigma_{\mathcal{A}}(x) = \sigma_{\mathcal{A}^{id}}(x) \text{ for } x \in \mathcal{A}.$$  

### 2.2.1. Gelfand-Naimark theory

Let $\Delta_\mathcal{A}$ be the set of all nonzero continuous algebra homomorphisms (multiplicative linear functionals) on $\mathcal{A}$, and let $\mathcal{A}'$ be the dual space of $\mathcal{A}$. The set $\Delta_\mathcal{A}$, endowed with the topology induced by the weak-* topology on the dual $\mathcal{A}'$, is called the maximal ideal space. This is due to the one-to-one correspondence between $\Delta_\mathcal{A}$ and the set of all maximal ideals of $\mathcal{A}$.

For each $x \in \mathcal{A}$, define the evaluation map $\hat{x}$ on $\Delta_\mathcal{A}$ by

$$\hat{x}(\varphi) = \varphi(x), \quad \varphi \in \Delta_\mathcal{A}.$$  

10
Define the set \( \hat{\mathcal{A}} \) by
\[
\hat{\mathcal{A}} := \{ \hat{x} \mid x \in \mathcal{A} \}.
\]

The map \( \Phi : \mathcal{A} \to \hat{\mathcal{A}} \), given by
\[
\Phi(x) = \hat{x}, \ x \in \mathcal{A},
\]
is called the \textit{Gelfand transform}.

Recall that given a locally compact Hausdorff space \( Y \), we say a function \( g \) on \( Y \) vanishes at infinity if for each \( \epsilon > 0 \), there exists a compact set \( K \) such that \( |f(y)| < \epsilon \) for all \( y \in Y/K \). Denote by \( C_0(Y) \) the space of continuous functions on \( Y \) that vanish at infinity. The space \( C_0(Y) \) is a \( C^* \)-algebra with pointwise multiplication and the sup norm.

**Proposition 2.2.1.** Let \( \mathcal{A} \) be a commutative Banach algebra. Then

(i) For all \( x \in \mathcal{A} \),
\[
\sigma_{\mathcal{A}}(x) = \hat{x}(\Delta_{\mathcal{A}}).
\]

(ii) \( \Delta_{\mathcal{A}} \) is a locally compact Hausdorff space.

(iii) \( \Phi \) is a continuous algebra homomorphism into \( C_0(\Delta_{\mathcal{A}}) \) s.t.
\[
\|\Phi(x)\| \leq \|x\|.
\]

(iv) \( \Phi \) is an isometry if and only if \( \|x\|^2 = \|x^2\| \) for all \( x \in \mathcal{A} \).

Given a locally compact Hausdorff space \( X \), denote by \( C_0(X) \) the space of all continuous functions that vanish at infinity. It is a \( C^* \)-algebra with the uniform norm.

**Theorem 2.2.2** (Gelfand-Naimark). Let \( \mathcal{A} \) be a \( C^* \)-algebra. Then the Gelfand transform is an isometric isomorphism onto the \( C^* \)-algebra \( C_0(\Delta_{\mathcal{A}}) \) of continuous functions on \( \Delta_{\mathcal{A}} \) that vanish at infinity. Moreover, \( \Delta_{\mathcal{A}} \) is compact if and only if \( \mathcal{A} \) is unital.
2.2.2. Tensor products with bounded continuous functions

**Definition 2.2.3.** If $X$ is a locally compact Hausdorff space and if $A$ is a $C^*$-algebra equipped with $\| \cdot \|_A$, let $C^b(X, A)$ be the set of all continuous bounded functions $f : X \to A$ equipped with the norm $\| \cdot \|_\infty$ given by

$$\| f \|_\infty = \sup_{x \in X} \| f(x) \|_A.$$  

**Lemma 2.2.4.** The space $C^b(X, A)$ is $C^*$-algebra.

We consider algebraic tensor product $C^b(X) \odot A$ as a subset of $C^b(X, A)$. Then the restriction of $\| \cdot \|_\infty$ to $C^b(X) \odot A$ is a $C^*$-norm.

**Definition 2.2.5.** The tensor product $C^b(X) \otimes A$ is the completion of $C^b(X) \odot A$ with respect to the $\| \cdot \|_\infty$.

When $A = C^b(Y)$ where $Y$ is a locally compact Hausdorff space, $C^b(X) \otimes C^b(Y)$ can be defined as the closure of the algebraic tensor product $C^b(X) \odot C^b(Y)$ in $C^b(X \times Y)$. This definition is consistent with Definition 2.2.5.

2.3. Representation theory

Let $\pi$ and $\pi'$ be unitary representations of a topological group $H$. A continuous linear map $T : V_\pi \to V_\pi'$ is an *intertwining operator* if for all $A \in H$ commutating relation $T \pi(A) = \pi'(A)T$ holds. The algebra of all operators that intertwine with $\pi$ and $\pi'$ is denoted by $\text{Hom}(\pi, \pi')$. The representations $\pi$ and $\pi'$ are equivalent if there exists an unitary isomorphism that intertwines $\pi$ and $\pi'$. The Schur’s lemma says that

$$\text{Hom}(\pi, \pi') \simeq \begin{cases} 
\mathbb{C}, & \text{if } \pi \simeq \pi' \\
\{0\}, & \text{if } \pi \not\simeq \pi'.
\end{cases}$$
2.4. Special functions

2.4.1. Gamma function

The gamma function is defined by

\[ \Gamma(z) = \int_0^\infty r^{z-1}e^{-r}dr \]  

(2.4.1)

for \( z \in \mathbb{C} \) with \( \text{Re}(z) > 0 \). We also have

\[ \Gamma(n) = (n-1)!, \quad n \in \mathbb{N}. \]

Lemma 2.4.1. Let \( b > 0 \). Then for any \( m \in \mathbb{N}_0 \),

\[ \int_0^b \frac{r^m e^{-r}}{m!}dr = 1 - \sum_{i=0}^m \frac{b^i e^{-b}}{i!} \]

Proof. The proof follows from induction on \( m \) as a consequence of the integration by parts equation: for any \( m \in \mathbb{N} \)

\[ \int \frac{r^{m+1}e^{-r}}{(m+1)!}dr = -\frac{b^{m+1}e^{-b}}{(m+1)!} + \int \frac{r^m e^{-r}}{m!}dr. \]

\[ \square \]

2.4.2. Stirling’s approximation

We recall the well-known Stirling’s approximation:

\[ n! \sim \sqrt{2\pi nn^ne^{-n}} \]

We also have the following upper and lower bounds that holds for \( n \in \mathbb{N} \):

\[ n^n \sqrt{2\pi ne^{-n}} \leq n! \leq n^n \sqrt{2\pi ne^{-n} + \frac{1}{12n}} \]  

(2.4.2)
Chapter 3. Toeplitz Operators with Quasi-Radial Symbols

3.1. Toeplitz operators with bounded $k$-quasi-radial symbols

In this section, we introduce quasi-radial symbols and diagonalize the Toeplitz operators with quasi-radial symbols.

We begin with some notations. Let $k \in \mathbb{N}$ and let $n = n(k) = (n_1, \ldots, n_k) \in \mathbb{N}^k$ and let $n = n_1 + \cdots + n_k$. Let $\mathbb{C}^n = \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_k}$. We identify $z \in \mathbb{C}^n$ by $z = (z_1, \cdots, z_k) \in \mathbb{C}^n$ where $z_j \in \mathbb{C}^{n_j}$. Consider the group $G = U_{n_1} \times \cdots \times U_{n_k}$ where $U_{n_i}$ are $n_i \times n_i$ unitary matrices. They act on $f$ defined on $\mathbb{C}^n$ by

$$(A_1, \ldots, A_k)f(z) = (A_1, \ldots, A_k)f(z_1, \cdots, z_k) := f(A_1^{-1}z_1, \ldots, A_k^{-1}z_k)$$

which is the usual action of matrices we get by considering $G$ as a subgroup of $U_n$. The action leaves the Gaussian measure $d\lambda_n(z) = d\lambda_{n_1}(z_1) \cdots d\lambda_{n_k}(z_k)$ invariant: for $E \subset \mathbb{C}^n$ and $A = (A_1, \ldots, A_k) \in G$,

$$\int_{\mathbb{C}^n} \chi_E(A^{-1}w)d\lambda_n(w) = \int_{\mathbb{C}^n} \chi_E(A^{-1}w) \frac{e^{-|w|^2}}{\pi^{n}} dw$$

$$= \int_{\mathbb{C}^{n_1}} \cdots \int_{\mathbb{C}^{n_k}} \chi_E(A_1^{-1}w_1, \ldots, A_k^{-1}w_k) \frac{e^{-|w(1)|^2}}{\pi^{n_1}} dw(1) \cdots \frac{e^{-|w(k)|^2}}{\pi^{n_k}} dw(k)$$

$$= \int_{\mathbb{C}^{n_1}} \cdots \int_{\mathbb{C}^{n_k}} \chi_E(w_1, \ldots, w_k) \frac{e^{-|w(1)|^2}}{\pi^{n_1}} dw(1) \cdots \frac{e^{-|w(k)|^2}}{\pi^{n_k}} dw(k)$$

(by a change of variable)

$$= \frac{1}{\pi^n} \int_{\mathbb{C}^n} \chi_E(w) e^{-|w|^2} dw$$

$$= \int_{\mathbb{C}^n} \chi_E(w) d\lambda_n(w).$$

Definition 3.1.1. Let $\varphi : \mathbb{C}^n \to \mathbb{C}$ be measurable. The function $\varphi$ is said to be $G$-invariant if for all $(A_1, \ldots, A_k) \in G$ and $z \in \mathbb{C}^n$ we have $(A_1, \ldots, A_k)\varphi(z) = \varphi(A_1^{-1}z(1), \ldots, A_k^{-1}z(k)) = \varphi(z)$.

Define the set of all $k$-quasi-radial symbols, denoted by $L_\infty^G(\mathbb{C}^n)$, to be the set of all essentially bounded functions on $\mathbb{C}^n$ that are $G$-invariant. The case $k = 1$ corresponds to radial symbols and the case $n = k$ corresponds to separately radial symbols. Now we take into consideration the $C^*$-algebra $T_G$ generated by Toeplitz operators with symbols in $L_\infty^G(\mathbb{C}^n)$. The main objective of this section is to diagonalize the Toeplitz operators with symbols in $L_\infty^G(\mathbb{C}^n)$.

3.2. A classification of the class of symbols $L_\infty^G(\mathbb{C}^n)$

Definition 3.2.1. Let $\varphi : \mathbb{C}^q \to \mathbb{C}$ be measurable. The function $\varphi$ is said to be,

1. radial if there exists a measurable function $a_\varphi : \mathbb{R}_+ \to \mathbb{C}$ such that $\varphi(z) = a_\varphi(|z|)$ for $z \neq 0$.

2. $U_q$-invariant if for all $A \in U_q$ and $z \in \mathbb{C}^q$ we have $\varphi(Az) = \varphi(z)$.

Following lemma is a well-known fact.

Lemma 3.2.2. The measurable function $\varphi : \mathbb{C}^q \to \mathbb{C}$ is $U_q$-invariant if and only if there exists $a : \mathbb{R}_+ \to \mathbb{C}$ such that $\varphi(z) = a(|z|)$.

Proof. As $|Az| = |z|$ for $A \in U_q$, it follows that any function of the form $\varphi(z) = a(|z|)$ is $U_q$-invariant. As $z \mapsto |z|$ is continuous it follows that $\varphi$ is measurable if and only if $a$ is measurable.

Assume that $\varphi$ is $U_q$-invariant. Define $a_\varphi(r) = \varphi(re_1)$. Then $a_\varphi$ is measurable. Let $z \in \mathbb{C}^q$. Then, as $U_q$ acts transitively on the sphere $S^{2q-1}$, there exist $A \in U_q$ such that
Thus \( \varphi(z) = \varphi(Az) = \varphi(|ze_1|) = a_\varphi(|z|). \)

Thus \( \varphi(z) = a_\varphi(|z|). \)

The following lemma is a consequence of Lemma 3.2.2.

\textbf{Lemma 3.2.3.} The measurable function \( \varphi : \mathbb{C}^n \to \mathbb{C} \) is \( G \)-invariant if and only if there exists \( a : \mathbb{R}^k_+ \to \mathbb{C} \) such that \( \varphi(z) = a(|z_{(1)}|, \ldots, |z_{(k)}|) \).

3.3. Diagonalizing \( T_\varphi \) by Schur’s lemma

In this section diagonalize the Toeplitz operators with \( k \)-quasi-radial symbols and show that the Toeplitz algebra is commutative. Since the action of \( G \) is defined by the action of \( U_{n_i} \) on \( F^2(\mathbb{C}^{n_i}) \), \( i = 1, \ldots, k \), first we consider the action of \( U_q \) on \( F^2(\mathbb{C}^q) \), with \( q \in \mathbb{N} \) fixed.

Denote by \( P^m[\mathbb{C}^q] \) the space of homogeneous holomorphic polynomials on \( \mathbb{C}^q \) of degree \( m \). Note that the space of holomorphic polynomials is dense in \( F(\mathbb{C}^q) \) and any holomorphic polynomial can be written in a unique way as a direct sum of homogeneous polynomials. In fact recall that by Proposition 2.1.1,

\[
\left\{ q_\alpha(z) = \frac{z^\alpha}{\sqrt{\alpha!}} \right\}_{\alpha \in \mathbb{N}^q}
\]

forms an orthonormal basis for \( F(\mathbb{C}^q) \).

Define the representation \( \pi^q \) of \( U_q \) acting on \( L^2(\mathbb{C}^n, \lambda_n) \) by

\[
\pi^q(A)f(z) = f(A^{-1}z); \quad A \in U_q.
\]

Above representation is well-known and \( F(\mathbb{C}^q) \) is an invariant subspace. Hereafter we will
define the subrepresentation that act on $\mathcal{F}(\mathbb{C}^q)$ by $\pi^q$. Indeed for $f \in \mathcal{F}(\mathbb{C}^q)$ and $A \in U_q$, we have $\pi^q(A)f \in \mathcal{F}(\mathbb{C}^q)$ as

$$P(\pi^q(A)f)(z) = \langle \pi^q(A)f, K_z \rangle$$

$$= \frac{1}{\pi q} \int_{\mathbb{C}^q} f(A^{-1}w)e^{z\bar{w}}e^{-|w|^2} dw$$

$$= \frac{1}{\pi q} \int_{\mathbb{C}^q} f(A^{-1}w)e^{A^{-1}zA^{-1}w}e^{-|w|^2} dw$$

$$= \frac{1}{\pi q} \int_{\mathbb{C}^q} f(w)e^{z\bar{w}}e^{-|w|^2} dw \quad \text{(by a change of variable)}$$

$$= P(\pi^q(A)f)(z).$$

We present the proof of the following lemma for the sake of completeness.

**Lemma 3.3.1.** $\pi^q$ defines a unitary representation of $U_q$.

**Proof.** Since the Gaussian measure $d\lambda_q$ is invariant under the action of $U_q$ defined by $\pi^q$ it is easy to check that $\pi^q$ defines a group homomorphism into the group of unitary operators on $L^2(\mathbb{C}^q, \lambda_q)$. We show that $A \mapsto \pi^q(A)f$ is continuous for all $f \in L^2(\mathbb{C}^q, \lambda_q)$. Let $\epsilon > 0$ and let $f \in L^2(\mathbb{C}^q, \lambda_q)$. Since $\pi^q$ is a group homomorphism, it is sufficient to show that there exists a neighborhood $V$ of id s. t.

$$\|\pi^q(A)f - f\|_2 < \epsilon$$

for all $A \in V$. Since the space of continuous functions with compact support on $\mathbb{C}^q$, denoted $C_c(\mathbb{C}^q)$, is dense in $L^2(\mathbb{C}^q, \lambda_q)$, there exists $g \in C_c(\mathbb{C}^q)$, s. t.

$$\|f - g\|_2 < \frac{\epsilon}{4}.$$
Let $V_0$ be a fixed compact neighborhood of id. Then $V_0^{-1} := \{ A^{-1} \mid A \in V_0 \}$ is a compact neighborhood of id and

$$V_0^{-1} \cdot \text{supp}(g) := \{ A^{-1}z \mid z \in \text{supp}(g) \}$$

is of finite measure. Assume $g \neq 0$. Then $|V_0^{-1} \cdot \text{supp}(g)| \neq 0$ as $\text{supp}(g) \subset V_0^{-1} \cdot \text{supp}(g)$. Moreover, $g$ is uniformly continuous as a consequence of Heine’s theorem. Then there exists $\delta > 0$ s.t. whenever $|z - w| < \delta$,

$$|g(z) - g(w)| < \frac{\epsilon}{2|V_0^{-1} \cdot \text{supp}(g)|^2}.$$  

Choose a neighborhood $V_1 \subset V_0^{-1}$ of id s.t.

$$|A^{-1}z - z| < \delta$$

for all $A^{-1} \in V_1$ and $z \in V_0^{-1} \cdot \text{supp}(g)$. Let $V := V_1^{-1} = \{ A \mid A^{-1} \in V_1 \}$. Then $V$ is a neighborhood of id and for $A \in V$, $z \in V_0^{-1} \cdot \text{supp}(g)$,

$$|g(A^{-1}z) - g(z)| < \frac{\epsilon}{2|V_0^{-1} \cdot \text{supp}(g)|^2}.$$

Hence for $A \in V$,

$$\|\pi^q(A)g - g\|_2^2 = \int_{V_0^{-1} \cdot \text{supp}(g)} |g(A^{-1}w) - g(w)|^2 d\lambda_n(w) \leq \frac{\epsilon^2}{4}. $$

Note that above inequality still holds as if $g = 0$, $\|\pi^q(A)g - g\|_2 = 0$. 

18
Thus for \( A \in V \),

\[
\| \pi^q(A)f - f \|_2 \leq \| \pi^q(A)f - \pi^q(A)g \|_2 + \| f - g \|_2 + \| \pi^q(A)g - g \|_2 \\
\leq 2\| f - g \|_2 + \| \pi^q(A)g - g \|_2 \\
< \epsilon.
\]

The representation \( \pi^q \) is reducible as the spaces of homogeneous polynomials \( P^m[\mathbb{C}^q] \) are invariant under the action of matrices. The corresponding subrepresentations, denoted by \( \pi^q_m \), are irreducible (see section 11.4, Example 2 in [F08]). Hence the decomposition of \( \mathcal{F}(\mathbb{C}^q) \) in irreducible \( U_q \) representations is given by

\[
\mathcal{F}(\mathbb{C}^q) = \bigoplus_{m=0}^{\infty} P^m[\mathbb{C}^q].
\]

Moreover, the representations \( \pi^q_m \) and \( \pi^q_k \) are inequivalent if \( m \neq k \). This can be seen by the fact \( d_m = \text{dim}_\mathbb{C} P^m[\mathbb{C}^q] \neq d_k \) if \( m \neq k \) for \( q > 1 \). For \( q = 1 \), \( d_m = 1 \) and \( \mathbb{I} \) acts on \( P^m[\mathbb{C}] \) by the character \( z \mapsto z^{-m}\text{id} \).

**Definition 3.3.2.** Let \( \pi^{n_i} \) be the unitary representations of \( U_{n_i} \), acting on \( \mathcal{F}(\mathbb{C}^{n_i}) \), \( i = 1, \ldots, k \), as in the above discussion. The outer tensor product of the representations \( \pi_{n_i} \), denoted \( \pi^{n_1} \otimes \cdots \otimes \pi^{n_k} \) is defined by

\[
\pi^{n_1} \otimes \cdots \otimes \pi^{n_k}(A_1, \ldots, A_k) = \pi^{n_1}(A_1) \otimes \cdots \otimes \pi^{n_k}(A_k)
\]

for all \( (A_1, \ldots, A_k) \in U_{n_1} \times \cdots \times U_{n_k} = G \). Here \( \pi^{n_1}(A_1) \otimes \cdots \otimes \pi^{n_k}(A_k) \) is the tensor product of the operators \( \pi^{n_i}(A_i) \) acting on the Hilbert spaces \( \mathcal{F}(\mathbb{C}^{n_i}) \).

The outer tensor product \( \pi^{n_1} \otimes \cdots \otimes \pi^{n_k} \) is a unitary representation of \( G \) acting on \( \mathcal{F}(\mathbb{C}^{n_1}) \otimes \cdots \otimes \mathcal{F}(\mathbb{C}^{n_k}) = \mathcal{F}(\mathbb{C}^n) \). For more details see [FÓ14]. To understand the equality
\( \mathcal{F}(\mathbb{C}^n) \otimes \cdots \otimes \mathcal{F}(\mathbb{C}^n) = \mathcal{F}(\mathbb{C}^n) \), the reader may consider the monomial bases for \( \mathcal{F}(\mathbb{C}^n) \)'s and \( \mathcal{F}(\mathbb{C}^n) \).

**Lemma 3.3.3.** Let \( \varphi \in L^\infty(\mathbb{C}^n)^G \). Then \( T_\varphi \) intertwines with \( \pi^{n_1} \otimes \cdots \otimes \pi^{n_k} \).

**Proof.** Let \( A_i \in U_{n_i} \) and \( f_i \in \mathcal{F}(\mathbb{C}^{n_i}) \), \( i = 1, \ldots, m \). Notice that

\[
T_\varphi(\pi^{n_1} \otimes \cdots \otimes \pi^{n_k})(A_1, \ldots, A_k) f_1 \otimes \cdots \otimes f_k(z) = T_\varphi f(A_1^{-1} z_1, \ldots, A_k^{-1} z_k)
\]

\[
= \int_{\mathbb{C}^n} \varphi(w) f(A_1^{-1} z_1) \cdots f(A_k^{-1} z_k) k_z(w) d\lambda_n(w)
\]

\[
= \int_{\mathbb{C}^n} \varphi(A_1 w_1, \ldots, A_k w_k) f_1 \otimes \cdots \otimes f_k(w) k_z(A_1 w_1, \ldots, A_k w_k) d\lambda_n(w)
\]

(by the change of variable \( (w_1, \ldots, w_k) \rightarrow (A_1 w_1, \ldots, A_k w_k) \)).

Since

\[
k_z(A_1 w_1, \ldots, A_k w_k) = e^{A_1 w_1 z_1 + \cdots + A_k w_k z_k}
\]

\[
= e^{w_1 (A_1^{-1} z_1) + \cdots + w_k (A_k^{-1} z_k)}
\]

\[
= e^{w_1 (A_1^{-1} z_1) + \cdots + w_k (A_k^{-1} z_k)}
\]

(as \( A_i \) are unitary)

\[
= k_z(A_1^{-1} z_1, \ldots, A_k^{-1} z_k)(w)
\]

and \( \varphi \) is \( G \)-invariant,

\[
T_\varphi(\pi^{n_1} \times \cdots \times \pi^{n_k})(A_1, \ldots, A_k) f_1 \otimes \cdots \otimes f_k(z)
\]

\[
= \int_{\mathbb{C}^n} \varphi(w) f_1 \otimes \cdots \otimes f_k(w) k_z(A_1^{-1} z_1, \ldots, A_k^{-1} z_k)(w) d\lambda_n(w)
\]

\[
= (\pi^{n_1} \otimes \cdots \otimes \pi^{n_k})(A_1, \ldots, A_k) T_\varphi f(z).
\]

Since the tensor product of irreducible representations is irreducible (see Proposition 6.75 in [FÓ14] for a proof), \( \mathcal{F}(\mathbb{C}^n) \) can be decomposed by irreducible sub-
representations of $\pi^n \otimes \cdots \otimes \pi^k$ as
\[
F(C^n) = \bigoplus_{m_1, \ldots, m_k = 0}^{\infty} P^{m_1}[C^{n_1}] \otimes \cdots \otimes P^{m_k}[C^{n_k}].
\]

Moreover, this decomposition is multiplicity free. This can be seen by noting that
the center of $G, T^k$ acts on $P^{m_1}[C^{n_1}] \otimes \cdots \otimes P^{m_k}[C^{n_k}]$ by $z^{-m}\text{id}$, $z \in T^k$. And hence
any two irreducible subrepresentations of $\pi^n \otimes \cdots \otimes \pi^k$ given above, corresponding to $m, m' \in \mathbb{N}^k_0$ are inequivalent whenever $m \neq m'$. Therefore we have the following lemma
as a consequence of Schur’s lemma.

**Lemma 3.3.4.** If an operator $T$ intertwines with the representation $\pi^n \otimes \cdots \otimes \pi^k$, then
\[
T|_{P^{m_1}[C^{n_1}] \otimes \cdots \otimes P^{m_k}[C^{n_k}]} = \lambda \text{id}
\]
for some $\lambda \in \mathbb{C}$. Moreover, the algebra of all operators that intertwine with $\pi^n \otimes \cdots \otimes \pi^k$, denoted $\text{Hom}(\pi^n \otimes \cdots \otimes \pi^k)$, is commutative.

Hence if $\varphi \in L^\infty(C^n)^G$, $T_\varphi|_{P^{m_1}[C^{n_1}] \otimes \cdots \otimes P^{m_k}[C^{n_k}]} = \gamma_{n,\varphi}(m)\text{id}$ for some $\gamma_{n,\varphi}(m) \in \mathbb{C}$
for all $m = (m_1, \ldots, m_k) \in \mathbb{N}^k_0$ by Lemma 3.3.3. As a consequence of Lemma 3.2.3, we identify the class of symbols $L^\infty(C^n)^G$ with essentially bounded functions $a : \mathbb{R}_+^k \rightarrow \mathbb{C}$ and
denote the corresponding Toeplitz operator and its spectrum by $T_{n,a}$ and $\gamma_{n,a}$. We also
have the following corollary by Lemma 3.3.3.

**Corollary 3.3.5.** The $C^*$-algebra generated by Toeplitz operators with $G$-invariant symbols
is commutative.

### 3.4. Computing eigenvalue functions

We denote by $S^q = \{x \in \mathbb{R}^{q+1} \mid |x| = 1\}$ the $q$-dimensional sphere in $\mathbb{R}^{q+1}$. We
denote by $\sigma$ the unique $U_{q+1}$-invariant measure on $S^q$ that satisfy the polar coordinates
formula given by
\[
\int_{\mathbb{R}^{q+1}} f(x)dx = \int_0^\infty \int_{S^q} f(r\omega)d\sigma(\omega)r^qdr
\]
for any \( f \in L^1(\mathbb{R}^{q+1}, dx) \) where \( dx \) is the Lebesgue measure on \( \mathbb{R}^{q+1} \). See section 2.7 in [F99] for more details.

**Lemma 3.4.1.** Let \( f \in L^1(\mathbb{C}^q, dz) \). Then
\[
\int_{\mathbb{C}^q} f(z)dz = \int_0^\infty \int_{S^{2q-1}} f(r\omega)d\sigma(\omega)r^{2q-1}dr.
\]
In particular if \( f(z) = a(r) \) is \( U(q) \)-invariant then
\[
\int_{\mathbb{C}^q} f(z)dz = \sigma(S^{2q-1}) \int_0^\infty a(r)r^{2q-1}dr.
\]

We recall the following well known fact:

**Lemma 3.4.2.** Let \( f,g \in F^2(\mathbb{C}^q) \) and \( \varphi \in L^\infty(\mathbb{C}^q) \). Then
\[
\langle T_\varphi f, g \rangle = \langle \varphi f, g \rangle = \frac{1}{\pi^q} \int_{\mathbb{C}^q} \varphi(z)f(z)\overline{g(z)}e^{-|z|^2}dz.
\]
In particular for \( f = g = p_\alpha \) we get
\[
\langle T_\varphi p_\alpha, p_\alpha \rangle = \frac{1}{\pi^q} \int_{\mathbb{C}^q} \varphi(z) \prod_{j=1}^n |z_j|^{2\alpha_j}e^{-|z_j|^2}dz_1 \cdots dz_n.
\]

**Proof.** Let \( P : L^2(\mathbb{C}^q) \rightarrow F^2(\mathbb{C}^q) \) be the orthogonal projection. Then, as \( g \in F^2(\mathbb{C}^q) \), we get
\[
\langle T_\varphi f, g \rangle = \langle Pf, g \rangle = \langle \varphi f, g \rangle.
\]

The following is well known and can be found for more general situations in [FK94] and other places. We also point to [F01] for a general discussion on how to integrate polynomials over spheres:
Lemma 3.4.3. Let $p_{\alpha} \in P^m(\mathbb{C}^q)$. Then $\|p_{\alpha}|_{S^{2q-1}}\|^2 = \frac{2\pi^q}{\Gamma(q + m)} \|p_{\alpha}\|^2$. In particular

$$\|q_{\alpha}|_{S^{2q-1}}\|^2 = \frac{2\pi^q}{\Gamma(q + m)}.$$  

Proof. We have $p_{\alpha}(r\omega) = r^m p(\omega)$, $r > 0$. Hence:

$$\|p_{\alpha}\|^2 = \frac{1}{\pi^q} \int_{S^{2q-1}} |p_{\alpha}(r\omega)|^2 d\sigma(\omega) e^{-r^2} r^{2q-1} dr$$

$$= \frac{1}{2\pi^q} \left( \int_{S^{2q-1}} |p_{\alpha}(\omega)|^2 d\sigma(\omega) \right) \left( \int_0^{\infty} r^{2m+2q-2} e^{-r^2} (2rdr) \right)$$

$$= \frac{1}{2\pi^q} \|p_{\alpha}|_{S^{2q-1}}\|^2 \int_0^{\infty} u^{m+q-1} e^{-u} du$$

$$= \frac{1}{2\pi^q} \|p_{\alpha}|_{S^{2q-1}}\|^2 \Gamma(m + q). \quad \square$$

We recall that the $C^*$-algebra $T_G$ generated by $G$-invariant bounded symbols is commutative and acts by scalars on each of the spaces $P^{m_1}[\mathbb{C}^{n_1}] \otimes \cdots \otimes P^{m_k}[\mathbb{C}^{n_k}]$.

Theorem 3.4.4. Assume $\varphi \in L^\infty(\mathbb{C}^n)^G$ let $m = (m_1, \ldots, m_k) \in \mathbb{N}_0^k$. Then

$$T_\varphi|_{P^{m_1}[\mathbb{C}^{n_1}] \otimes \cdots \otimes P^{m_k}[\mathbb{C}^{n_k}]} = \gamma_{n,\varphi}(m) \text{id} = \gamma_{n,a}(m) \text{id}$$

where

$$\gamma_{n,a}(m) = \frac{1}{(m + n - 1)!} \int \cdots \int \sqrt{r}^m a_{\varphi}(\sqrt{r}) r^{m+n-1} e^{-(r_1 + \cdots + r_k)} dr_1 \cdots dr_k,$$

where we use the notations $\sqrt{r} = (\sqrt{r_1}, \ldots, \sqrt{r_k})$ and $1 = (1, \ldots, 1) \in \mathbb{N}^k$.

Proof. Assume that $\varphi(z) = a(|z_1|, \ldots, |z_k|) = a(r_1, \ldots, r_k)$. Let $\alpha_j \in \mathbb{N}_0^{n_j}$ such that $|\alpha_j| = m_j$ and let $q_{\alpha_j} \in P^{m_j}[\mathbb{C}^{n_j}]$ be the normalized monomials defined in Lemma 3.4.3. Recall from Lemma 3.4.3 that

$$\|q_{\alpha_j}|_{S^{2n_j-1}}\|^2 = \frac{2\pi^{n_j}}{\Gamma(n_j + m_j)}$$
and hence
\[ \prod_{j=1}^{k} \|q_{\alpha_j}|_{S^{2n_j-1}}\|^2 = \frac{2^k\pi^n}{(m+n-1)!}. \]

Let \( \alpha = (\alpha_1, \ldots, \alpha_k) \) and define the normalized monomial \( q_\alpha \) by
\[ q_\alpha(z) = \prod_{j=1}^{k} q_{\alpha_j}(z(j)). \]

Then \( q_\alpha \in P^{m_1}[C^{n_1}] \otimes \cdots \otimes P^{m_k}[C^{n_k}] \) and hence
\[ \gamma_{n,\varphi}(m) = \langle T_\varphi q_\alpha, q_\alpha \rangle. \]

As \( \varphi \) is \( G \)-invariant and bounded we can use polar-coordinates on \( C^{n_j} \), Fubini’s theorem, Lemma 3.2.3, Lemma 3.4.2 and the fact that \( q_{\alpha_j} \) are homogeneous of order \( m_j \), to write
\[ \langle T_\varphi q_\alpha, q_\alpha \rangle = \frac{1}{\pi^n} \int_{C^n} \varphi(z)|q_\alpha(z)|^2 e^{-|z|^2} dz \]
\[ = \frac{1}{\pi^n} \int_{C^{n_1}} \cdots \int_{C^{n_k}} a(|z(1)|, \ldots, |z(k)|) \prod_{j=1}^{k} |q_{\alpha_j}(z(j))|^2 e^{-|z(j)|^2} dz(1) \cdots dz(k) \]
\[ = \frac{1}{\pi^n} \int_0^\infty \cdots \int_0^\infty a(r) \prod_{j=1}^{k} r_j^{2m_j+2n_j-1} e^{-r_j^2} dr_1 \cdots dr_k \prod_{j=1}^{k} \|q_{\alpha_j}|_{S^{2n_j-1}}\|^2 \]
\[ = \frac{2^k}{(m+n-1)!} \int_0^\infty \cdots \int_0^\infty a(r) \prod_{j=1}^{k} r_j^{2m_j+2n_j-1} e^{-r_j^2} dr_1 \cdots dr_k \]
\[ = \frac{1}{(m+n-1)!} \int_0^\infty \cdots \int_0^\infty a(\sqrt{r}) \prod_{j=1}^{k} r_j^{m_j+n_j-1} e^{-r_j^2} dr_1 \cdots dr_k, \]

where we in the last line used the substitutions \( u = r_j^2 \) and hence \( du = 2r_jdr_j \).

\[ \square \]

**Corollary 3.4.5.** If \( n = (n_1, \ldots, n_k) \) and \( m = (m_1, \ldots, m_k) \), then
\[ \gamma_{n,\alpha}(m) = \gamma_{1,\alpha}(m_1 + n_1 - 1, \ldots, m_k + n_k - 1) = \gamma_{1,\alpha}(m + n - 1). \]
Note that eigenvalue functions $\gamma_{1,a}$ given above corresponds to Toeplitz operators with separately radial symbols. In particular we obtain the following special cases:

**Corollary 3.4.6** (Grudsky and Vasilevski [GV02]). Assume that $n = 1$

$$\gamma_{1,a}(m) = \frac{1}{m!} \int_0^\infty a(\sqrt{r}) r^m e^{-r} dr.$$

If $k = 1$, then $n = (n)$ and we have the following corollary.

**Corollary 3.4.7.** If $k = 1$ (radial symbols) then

$$\gamma_{n,a}(m) = \frac{1}{(m+n-1)!} \int_0^\infty a(\sqrt{r}) r^{m+n-1} e^{-r} dr = \gamma_{1,a}(m+n-1).$$

If $n = k$, then $n = (1, \ldots, 1) = 1$ and we have the following corollary.

**Corollary 3.4.8.** If $n = k$ (separately radial symbols) then

$$\gamma_{1,a}(m) = \frac{1}{m!} \int_{\mathbb{R}_+^k} a(\sqrt{r}) r^m e^{-(r_1+\cdots+r_k)} dr_1 \cdots dr_k.$$

**Corollary 3.4.9.** If $a(r_1, \ldots, r_k) = \prod_{j=1}^k a_j(r_j)$ then

$$\gamma_{n,a}(m_1, \ldots, m_k) = \prod_{j=1}^k \gamma_{1,a_j}(m_j + n_j - 1).$$
Chapter 4. The Space $C_{b,u}(\mathbb{N}_0^k, \rho_k)$

In this chapter we introduce the C*-algebra $C_{b,u}(\mathbb{N}_0^k, \rho_k)$, the set of all bounded functions on $\mathbb{N}_0^k \times \mathbb{N}_0^k$ that are uniformly continuous with respect to the square-root metric and collect some of its properties. These are sometimes called Square root-slowly oscillating functions.

For $k \in \mathbb{N}$, let $\rho_k : \mathbb{N}_0^k \times \mathbb{N}_0^k \to [0, \infty)$ be given by

$$
\rho(m, m') = |\sqrt{m_1} - \sqrt{m'_1}| + \cdots + |\sqrt{m_k} - \sqrt{m'_k}|
$$

for all $m = (m_1, \ldots, m_k), m' = (m'_1, \ldots, m'_k) \in \mathbb{N}_0^k$. Then $\rho_k$ is a metric on $\mathbb{N}_0^k$. Modulus of continuity with respect to the metric $\rho_k$ of a function $\sigma \in l^\infty(\mathbb{N}_0^k)$ is the function $\omega_{\rho_k, \sigma} : [0, \infty) \to [0, \infty)$ given by

$$
\omega_{\rho_k, \sigma}(\delta) = \sup\{|\sigma(m) - \sigma(m')| : \rho(m, m') \leq \delta\}.
$$

$C_{b,u}(\mathbb{N}_0^k, \rho_k)$ is the set of all bounded functions on $\mathbb{N}_0^k$ that are uniformly continuous with respect to the square-root metric $\rho_k$:

$$
\{ \sigma \in l^\infty(\mathbb{N}_0^k) : \lim_{\delta \to 0} \omega_{\rho_k, \sigma}(\delta) = 0 \}.
$$

Lemma 4.0.1. $C_{b,u}(\mathbb{N}_0^k, \rho_k)$ is a C*-subalgebra of $l^\infty(\mathbb{N}_0^k)$.

Proof. It is enough to show that $C_{b,u}(\mathbb{N}_0^k, \rho_k)$ is a closed *-subalgebra of $l^\infty(\mathbb{N}_0^k)$. The fact that $C_{b,u}(\mathbb{N}_0^k, \rho_k)$ is a subalgebra follows from the inequalities: for $f, g \in C_{b,u}(\mathbb{N}_0^k, \rho_k)$ and

This chapter previously appeared in the following journal article: V. Dewage, G. Ólafsson, Toeplitz operators on the Fock space with quasi-radial symbols. First published in Complex Anal. Oper. Theory 16 (2022), no. 4, Paper No. 61, 32 pp. 47B35 (32A37 47B32), by Springer Nature.
\( \lambda \in \mathbb{C}, \)

\[
|(f + \lambda g)(n) - (f + \lambda g)(m)| \leq |f(n) - f(m)| + |\lambda||g(n) - g(m)|
\]

and

\[
|(fg)(n) - f g(m)| \leq \|f\|_\infty|g(n) - g(m)| + \|g\|_\infty|f(n) - f(m)|.
\]

Also \( C_{b,u}(\mathbb{N}_0^k, \rho_k) \) is selfadjoint as \( |\bar{f}| = |f| \). To show that \( C_{b,u}(\mathbb{N}_0^k, \rho_k) \) is closed, let \( f \in C_{b,u}(\mathbb{N}_0^k, \rho_k) \) s.t.

\[
\lim_{n \to \infty} f_n = f
\]

uniformly for some \( f \in l^\infty(\mathbb{N}_0^k) \). To show that \( f \in C_{b,u}(\mathbb{N}_0^k, \rho_k) \), let \( \epsilon > 0 \). Then there exists \( N \in \mathbb{N} \) s.t. for all \( n \geq N \),

\[
\|f - f_n\|_\infty < \frac{\epsilon}{4}.
\]

Since \( f_N \) is uniformly continuous there exists \( \delta > 0 \) s.t. whenever \( \rho_k(n, m) < \delta \)

\[
|f_N(n) - f_N(m)| < \frac{\epsilon}{2}.
\]

Then for \( n, m \in \mathbb{N}_0^k \) s.t. \( \rho_k(n, m) < \delta \)

\[
|f(n) - f(m)| < |f(n) - f_N(n)| + |f_N(n) - f_N(m)| + |f(m) - f_N(m)|
\]

\[
\leq 2\|f - f_N\|_\infty + |f_N(n) - f_N(m)|
\]

\[
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Therefore \( C_{b,u}(\mathbb{N}_0^k, \rho_k) \) is closed. \( \Box \)
4.1. Examples

Here we present some members of $C_{b,u}(\mathbb{N}_0^k, \rho_k)$. The proofs of following two lemmas are trivial.

**Lemma 4.1.1.** Let $f$ be a constant function. Then $f \in C_{b,u}(\mathbb{N}_0^k, \rho_k)$.

**Lemma 4.1.2.** Let $f : \mathbb{R}_+ \to \mathbb{C}$ be a bounded uniformly continuous function. Define $g : \mathbb{N}_0^k \to \mathbb{C}$ by

$$g(n_1, \ldots, n_k) = f(\sqrt{n_1} + \cdots + \sqrt{n_k}), \quad (n_1, \ldots, n_k) \in \mathbb{N}_0^k.$$ 

**Example 4.1.3.** Define $f : \mathbb{N}_0^k \to \mathbb{C}$ by

$$f(n_1, \ldots, n_k) = \sin(\sqrt{n_1} + \cdots + \sqrt{n_k}).$$

Then $f \in C_{b,u}(\mathbb{N}_0^k, \rho_k)$ by lemma 4.1.2.

**Lemma 4.1.4.** Suppose $\lim_{|n| \to \infty} f(n)$ exists. Then $f \in C_{b,u}(\mathbb{N}_0^k, \rho_k)$. Here we use the notation

$$|n| := n_1 + \cdots + n_k.$$ 

**Proof.** By lemma 4.1.1 it is sufficient to assume $f$ vanish at $\infty$. Let $\epsilon > 0$. Then there exist $N \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}_0^k$ with $n_i > N$ for some $i$, 

$$|f(n)| < \frac{\epsilon}{2}.$$ 

Let $\delta = \sqrt{N+1} - \sqrt{N}$. Suppose $\rho_k(n, m) < \delta$. Then since square-root function is increasing, we have that $n_i > N$ or $m_i > N$ for all $i = 1, \ldots, k$. We claim that there exists $i,j$ s.t.
\(n_i, m_j > N\). On the contrary, suppose that \(m_i \leq N\) for all \(i\). Then \(n_i \geq N\) for all \(i\) and
\[
\rho_k(n, m) \geq \rho_k((N + 1, \ldots, N + 1), (N, \ldots, N))
\]
\[
= k(\sqrt{N + 1} - \sqrt{N})
\]
which is a contradiction as \(\rho_k(n, m) < \delta\) and \(k \geq 1\). Hence there exists \(i, j\) s.t. \(n_i, m_j > N\) and hence
\[
|f(n) - f(m)| \leq |f(n)| + |f(m)|
\]
\[
< \epsilon.
\]
\(\square\)

### 4.2. Invariance of \(C_{b,u}(\mathbb{N}_0^k, \rho_k)\) under shifts

**Definition 4.2.1.** Let \(\sigma \in C_{b,u}(\mathbb{N}_0^k, \rho_k)\) and let \(s = (s_1, \ldots, s_k) \in \mathbb{N}_0^k\). We define the left and right shift operators on \(C_{b,u}(\mathbb{N}_0^k, \rho_k)\) with respect to \(s\), denoted \(\tau^s_L\) and \(\tau^s_R\) respectively, by
\[
\tau^s_L \sigma(m) = \sigma(m + s) \quad \text{and} \quad \tau^s_R \sigma(m) = \begin{cases} 
\sigma(m - s) & ; m_i \geq s_i \ \forall \ i = 1, \ldots, k \\
0 & ; \text{otherwise}
\end{cases}
\]
for all \(m = (m_1, \ldots, m_k) \in \mathbb{N}_0^k\).

Notice that to make the \(\rho(m, m')\) arbitrarily small, \(m_k, m'_k\) has to be necessarily large. Hence the uniform continuity of a function on \(\mathbb{N}_0^k\) is invariant under shifts as stated in the following two lemmas.

**Lemma 4.2.2.** Let \(\sigma \in C_{b,u}(\mathbb{N}_0^k, \rho_k)\) and \(s = (s_1, \ldots, s_k) \in \mathbb{N}_0^k\). Then \(\tau^s_L \sigma \in C_{b,u}(\mathbb{N}_0^k, \rho_k)\).
Proof. Boundedness of $\tau_R^s \sigma$ follows easily from the boundedness of $\sigma$. Let

$$ m = (m_1, \ldots, m_k), \quad m' = (m'_1, \ldots, m'_k) \in \mathbb{N}_0^k. $$

By observing the square-root function, we have,

$$ |\sqrt{m_i + s_i} - \sqrt{m'_i + s_i}| \leq |\sqrt{m_i} - \sqrt{m'_i}|, \quad \text{for } i = 1, \ldots, k. $$

Hence $\omega_{\rho_k, \tau_R^s \sigma}(\delta) \leq \omega_{\rho_k, \sigma}(\delta)$. \hfill \Box

**Lemma 4.2.3.** Let $\sigma \in C_{b,u}(\mathbb{N}_0^k, \rho_k)$ and $s = (s_1, \ldots, s_k) \in \mathbb{N}_0^k$. Then $\tau_R^s \sigma \in C_{b,u}(\mathbb{N}_0^k, \rho_k)$.

Proof. Again boundedness of $\tau_R^s \sigma$ follows from the boundedness of $\sigma$. Let

$$ s = \max\{s_1, \ldots, s_k\}. $$

Assume $\delta < \sqrt{2s} - \sqrt{2s - 1}$ and let $m = (m_1, \ldots, m_k), m' = (m'_1, \ldots, m'_k) \in \mathbb{N}_0^k$ such that $\rho_k(m, m') < \delta$. Then $|\sqrt{m_i} - \sqrt{m'_i}| < \sqrt{2s} - \sqrt{2s - 1}$ and since

$$ \min_{j,l \in \mathbb{N}_0, i,j \leq 2s} |\sqrt{j} - \sqrt{l}| = \sqrt{2s} - \sqrt{2s - 1}, $$

we have $m_i, m'_i > 2s \geq 2s_i$ for all $i = 1, \ldots, k$. Notice that

$$ m_i, m'_i > 2s_i = \frac{s_i}{1 - \frac{1}{2}} \implies \frac{1}{2} m_i < m_i - s_i, \quad \frac{1}{2} m'_i < m'_i - s_i $$

$$ \implies \frac{1}{\sqrt{2}} \sqrt{m_i} < \sqrt{m_i - s_i}, \quad \frac{1}{\sqrt{2}} \sqrt{m'_i} < \sqrt{m'_i - s_i} $$

$$ \implies \frac{1}{\sqrt{2}} (\sqrt{m_i} + \sqrt{m'_i}) < \sqrt{m_i - s_i} + \sqrt{m'_i - s_i} $$

$$ \implies \frac{|m_i - m'_i|}{\sqrt{m_i - s_i} + \sqrt{m'_i - s_i}} < \frac{1}{\sqrt{2}} \frac{|m_i - m'_i|}{\sqrt{m_i} + \sqrt{m'_i}} $$

$$ \implies |\sqrt{m_i} - s_i - \sqrt{m'_i} - s_i| < \frac{1}{\sqrt{2}} |\sqrt{m_i} - \sqrt{m'_i}|. $$

Hence $\omega_{\rho_k, \tau_R^s \sigma}(\delta) \leq \omega_{\rho_k, \sigma}(\sqrt{2}\delta)$ for all $\delta < \sqrt{2s} - \sqrt{2s - 1}$. \hfill \Box
4.3. A comparison of $C_{b,u}(\mathbb{N}_0^k, \rho_k)$ with $C_{b,u}(\mathbb{N}_0, \rho_1) \otimes \cdots \otimes C_{b,u}(\mathbb{N}_0, \rho_1)$

Denote by $C_{b,u}(\mathbb{N}_0, \rho_1) \otimes \cdots \otimes C_{b,u}(\mathbb{N}_0, \rho_1)$ the closure of the algebraic tensor product $C_{b,u}(\mathbb{N}_0, \rho_1) \otimes \cdots \otimes C_{b,u}(\mathbb{N}_0, \rho_1)$ in $C^b(\mathbb{N}_0^k)$. In this section we compare the $C^*$-algebra $C_{b,u}(\mathbb{N}_0^k, \rho_k)$ with the $C^*$-tensor product $C_{b,u}(\mathbb{N}_0, \rho_1) \otimes \cdots \otimes C_{b,u}(\mathbb{N}_0, \rho_1)$.

We have the inclusions

$$C_{b,u}(\mathbb{N}_0^{k-1}, \rho_{k-1}) \otimes C_{b,u}(\mathbb{N}_0, \rho_1) \hookrightarrow C_{b,u}(\mathbb{N}_0^k, \rho_k).$$

In fact

$$C_{b,u}(\mathbb{N}_0, \rho_1) \otimes \cdots \otimes C_{b,u}(\mathbb{N}_0, \rho_1) \hookrightarrow C_{b,u}(\mathbb{N}_0^k, \rho_k).$$

However, the above inclusions are not necessarily isomorphisms. We present a counter example for $k = 2$.

4.3.1. A counter example

Here we construct a counter example to show that $C_{b,u}(\mathbb{N}_0^2, \rho_2)$ is strictly larger than $C_{b,u}(\mathbb{N}_0, \rho_1) \otimes C_{b,u}(\mathbb{N}_0, \rho_1)$. Let

$$I_i : [\sqrt{i}, \sqrt{i} + \pi) ; i \in \mathbb{N}_0.$$ Define $g : \mathbb{N}_0^2 \to \mathbb{C}$ by

$$g(i, j) = \sin(\sqrt{j} - \sqrt{i}) \chi_{I_i}(\sqrt{j}) ; (i, j) \in \mathbb{N}_0^2.$$ 

**Lemma 4.3.1.** Let $g$ be the function defined above. Then $g \in C_{b,u}(\mathbb{N}_0^2, \rho_2)$.

**Proof.** Clearly $g$ is bounded. First we show that if $|\sqrt{j} - \sqrt{j'}| < \pi$,

$$|g(i, j) - g(i, j')| < |\sqrt{j} - \sqrt{j'}|.$$
If \( \sqrt{j}, \sqrt{j'} \notin I_i \),

\[
|g(i, j) - g(i, j')| = 0 < |\sqrt{j} - \sqrt{j'}|.
\]

W.l.o.g. assume \( \sqrt{j} \in I_i \). Then \( \sqrt{j'} \in (\sqrt{i} - \pi, \sqrt{i} + 2\pi) \) as \( |\sqrt{j} - \sqrt{j'}| < \pi \). Note that if \( \sqrt{j'} \in I_i \), \( \sin(\sqrt{j'} - \sqrt{i}) = g(j', i) \) and if \( \sqrt{j'} \in (\sqrt{i} - \pi, \sqrt{i}) \cup (\sqrt{i} + \pi, \sqrt{i} + 2\pi) \), \( \sin(\sqrt{j'} - \sqrt{i}) < 0 \). Hence

\[
|g(i, j) - g(i, j')| \leq |\sin(\sqrt{j} - \sqrt{i}) - \sin(\sqrt{j'} - \sqrt{i})|
\]

\[
\leq |\sqrt{j} - \sqrt{j'}|
\]

as required. Next we prove that if \( |\sqrt{i} - \sqrt{i'}| < \pi \),

\[
|g(i, j) - g(i', j)| < |\sqrt{i} - \sqrt{i'}|.
\]

If \( \sqrt{j} \notin I_i \cup I_{i'} \), \( |g(i, j) - g(i', j)| = 0 < |\sqrt{i} - \sqrt{i'}| \). W.l.o.g., assume that \( \sqrt{j} \in I_i \). Then \( \sqrt{j} \in (\sqrt{i} - \pi, \sqrt{i} + 2\pi) \) as \( |\sqrt{i} - \sqrt{i'}| < \pi \). Therefore

\[
|g(i, j) - g(i', j)| \leq |\sin(\sqrt{j} - \sqrt{i}) - \sin(\sqrt{j} - \sqrt{i'})|
\]

\[
\leq |\sqrt{i} - \sqrt{i'}|,
\]

proving the inequality.

Now notice that for all \((i, j), (i', j') \in \mathbb{N}_0^2\) such that \( \rho_2((i, j), (i', j')) < \pi \),

\[
|g(i, j) - g(i', j')| \leq |g(i, j) - g(i', j)| + |g(i', j) - g(i', j')|
\]

\[
\leq |\sqrt{i} - \sqrt{i'}| + |\sqrt{j} - \sqrt{j'}|
\]

\[
< \rho_2((i, j), (i', j')),
\]

proving that \( g \in C_{b,u}(\mathbb{N}_0^2, \rho_2) \). \( \square \)
Recall that a subset of a topological space is said to be precompact if its closure is compact. The following theorem from Williams [W03] describes a criterion to check whether an element in the $C^*$-algebra $C^b(X, A)$ belongs to the possibly smaller $C^*$-algebra $C^b(X) \otimes A$. We will use the Theorem 4.3.2 to show that $g \notin C_{b,u}(\mathbb{N}_0, \rho_1) \otimes C_{b,u}(\mathbb{N}_0, \rho_1)$.

**Theorem 4.3.2** (Williams [W03]). *If $X$ is a locally compact Hausdorff space and if $A$ is a $C^*$-algebra, then $f \in C^b(X, A)$ is in $C^b(X) \otimes A$ if and only if the range of $f$, $R(f) := \{f(x) : x \in X\}$, is precompact.*

In order to use Theorem 4.3.2 we present several lemmas about $g$. The proof of Lemma 4.3.3 is trivial as $\mathbb{N}_0$ has discrete topology and $g$ is bounded.

**Lemma 4.3.3.** Let $g$ be the function defined above. Define $\tilde{g} : \mathbb{N}_0 \to C_{b,u}(\mathbb{N}_0, \rho_1)$ by

$$\tilde{g}(i) = g(i, \cdot).$$

Then $\tilde{g} \in C^b(\mathbb{N}_0, C_{b,u}(\mathbb{N}_0, \rho_1))$.

**Lemma 4.3.4.** Let $\tilde{g} : \mathbb{N}_0 \to C_{b,u}(\mathbb{N}_0, \rho_1)$ be defined as in Lemma 4.3.3. Then $\|\tilde{g}(i)\|_{\infty} \geq \frac{1}{2}$ for all $i \in \mathbb{N}_0$.

**Proof.** Notice that for all $i \in \mathbb{N}_0$ and for all $p \in \mathbb{N}$,

$$\sqrt{i + p} - \sqrt{i + p - 1} \leq 1 < \frac{2\pi}{3}$$

because $\sqrt{i + p} - \sqrt{i + p - 1}$ attains its maximum when $i + p = 1$. Fix $i \in \mathbb{N}_0$. Then the sequence $\{\sqrt{i + p}\}_{p=1}^{\infty}$ contains a point in any interval of length $\frac{2\pi}{3}$ and, in particular, it contains a point in $(\sqrt{i + \frac{5\pi}{6}}, \sqrt{i + \frac{5\pi}{6}})$. Denote that point by $\sqrt{i + p_0}$. Then $\sqrt{i + p_0} - \sqrt{i} \in$

33
\((\frac{\pi}{6}, \frac{5\pi}{6})\)

\[\|\tilde{g}(i)\|_\infty \geq |(\tilde{g}(i))(i + p_0)|\]

\[= \sin(\sqrt{i + p_0 - \sqrt{i}})\]

\[\geq \frac{1}{2}.\]

\[\square\]

**Lemma 4.3.5.** Let \(\tilde{g} : \mathbb{N}_0 \to C_{b,u}(\mathbb{N}_0, \rho_1)\) be defined as in Lemma 4.3.3. Then the range of \(\tilde{g}\) is not precompact.

**Proof.** Whenever \(|\sqrt{i_1} - \sqrt{i_2}| > \pi, I_{i_1} \cap I_{i_2} = \emptyset\) and hence

\[\|\tilde{g}(i_1) - \tilde{g}(i_2)\|_\infty = \max\{\|\tilde{g}(i_1)\|_\infty, \|\tilde{g}(i_2)\|_\infty\} \geq \frac{1}{2}.\]

Thus there exits a sequence \(\{\tilde{g}(i_s)\}_{s=1}^\infty\) s.t. \(\|\tilde{g}(i_{s_1}) - \tilde{g}(i_{s_2})\|_\infty \geq \frac{1}{2}\) whenever \(s_1 \neq s_2\). It follows that the range of \(\tilde{g}\) is not totally bounded and hence it is not precompact. \(\square\)

By Lemma 4.3.3, Lemma 4.3.5, and Theorem 4.3.2, we have that \(g \notin C_{b,u}(\mathbb{N}_0, \rho_1) \otimes C_{b,u}(\mathbb{N}_0, \rho_1)\). Also by Lemma 4.3.1, \(g \in C_{b,u}(\mathbb{N}_0^2, \rho_2)\). Hence we have the following proposition.

**Proposition 4.3.6.** \(C_{b,u}(\mathbb{N}_0^2, \rho_2)\) is strictly larger than \(C_{b,u}(\mathbb{N}_0, \rho_1) \otimes C_{b,u}(\mathbb{N}_0, \rho_1)\).

### 4.4. Extending functions in \(C_{b,u}(\mathbb{N}_0^k, \rho_k)\) to \(\mathbb{R}^k_+\)

The metric \(\rho_k\) can be extended to \(\mathbb{R}^k_+\) and we denote the slowly oscillating functions on \(\mathbb{R}^k_+\) by \(C_{b,u}(\mathbb{R}^k_+, \rho_k)\). Now we prove that any \(\sigma \in C_{b,u}(\mathbb{N}_0^k, \rho_k)\) can be extended to some \(f \in C_{b,u}(\mathbb{R}^k_+, \rho_k)\).
Lemma 4.4.1. Let $\sigma \in C_{b,u}(\mathbb{N}_0^k, \rho_k)$. Define $f$ on $\mathbb{R}_+$ by

$$f(x) = f_m(x) := \sigma(m) + \sum_{l=1}^k \sum_{i_1,\ldots,i_l \in \{1,\ldots,k\}} a_{i_1,\ldots,i_l}(m) \prod_{q=1}^l \frac{\sqrt{x_{i_q}} - \sqrt{m_{i_q}}}{\sqrt{m_{i_q} + 1} - \sqrt{m_{i_q}}}$$

where $m_i = \lfloor x_i \rfloor$, $i = 1,\ldots,k$ and the coefficients $a_{i_1,\ldots,i_l}(m)$ are given by

$$a_{i_1,\ldots,i_l}(m) = \sum_{q=0}^l (-1)^{l-q} \sum_{j_1,\ldots,j_q \in \{i_1,\ldots,i_l\}} \sigma(m + e_{j_1} + \cdots + e_{j_q})$$

where $e_i$ denotes the standard basis of $\mathbb{N}_0^k$.

Then $f \in C_{b,u}(\mathbb{R}_+^k, \rho_k)$ and $f|_{\mathbb{N}_0^k} = \sigma$. Moreover $\|f\|_\infty = \|\sigma\|_\infty$.

The function $f$ defined above is motivated by the interpolation formula for the case $k = 1$, given by

$$f(x) = \sigma(m) + (\sigma(m + 1) - \sigma(m)) \frac{\sqrt{x} - \sqrt{m}}{\sqrt{m + 1} - \sqrt{m}}$$

for $\sigma \in C_{b,u}(\mathbb{N}_0, \rho_1)$, where $m = \lfloor x \rfloor$ and $x \in \mathbb{R}_+$. The proof of Lemma 4.4.1 is a generalization of the proof of Lemma 3.3 in [EM16]. Rest of this section is committed for the proof of Lemma 4.4.1.

It is easy to see that $f|_{\mathbb{N}_0^k} = \sigma$, from the definition of $f$. Hence $\|f\|_\infty \geq \|\sigma\|_\infty$.

4.4.1. The uniform norm of $f$

To show that $\|f\|_\infty = \|\sigma\|_\infty$, we will use the following lemma.

Lemma 4.4.2. Let $s \in \mathbb{N}$ and let $a_i \in (0,1)$ for all $i = 1,\ldots,s$. Then

$$1 + \sum_{l=1}^s (-1)^l \sum_{i_1,\ldots,i_l \in \{1,\ldots,s\}} \prod_{q=1}^l a_{i_q} \geq 0$$
Proof. Notice that the statement is true for $s = 1$. Assume the result is true for $s$. Then

$$1 + \sum_{l=1}^{s+1} (-1)^l \sum_{i_1, \ldots, i_l \in \{1, \ldots, s+1\}} \prod_{q=1}^{l} a_{i_q} = 1 + \sum_{l=1}^{s} (-1)^l \sum_{i_1, \ldots, i_l \in \{1, \ldots, s\}} \prod_{q=1}^{l} a_{i_q}$$

$$- a_{s+1} \left( 1 + \sum_{l=1}^{s} (-1)^l \sum_{i_1, \ldots, i_l \in \{1, \ldots, s\}} \prod_{q=1}^{l} a_{i_q} \right)$$

$$= (1 - a_{s+1}) \left( 1 + \sum_{l=1}^{s} (-1)^l \sum_{i_1, \ldots, i_l \in \{1, \ldots, s\}} \prod_{q=1}^{l} a_{i_q} \right)$$

$$\geq 0.$$  

Hence the statement is true by induction on $s$.

Note that $f_m$ can also be written as

$$f_m(x) = \sigma(m) B_0(x, m) + \sum_{\substack{j_1, \ldots, j_s \in \{1, \ldots, k\} \atop j_1 < \cdots < j_s}} \sigma(m + e_{j_1} + \cdots + e_{j_s}) B_{j_1, \ldots, j_s}(x, m)$$

where

$$B_{j_1, \ldots, j_s}(x, m) = \prod_{p=1}^{s} \frac{\sqrt{x_{j_p}} - \sqrt{m_{j_p}}}{\sqrt{m_{j_p} + 1} - \sqrt{m_{j_p}}} \left( 1 + \sum_{l=1}^{k-s} (-1)^l \sum_{i_1, \ldots, i_l \in \{1, \ldots, k\}\backslash\{j_1, \ldots, j_s\}} \prod_{q=1}^{l} \frac{\sqrt{x_{i_q}} - \sqrt{m_{i_q}}}{\sqrt{m_{i_q} + 1} - \sqrt{m_{i_q}}} \right)$$

and

$$B_0(x, m) = 1 + \sum_{l=1}^{k} (-1)^l \sum_{i_1, \ldots, i_l \in \{1, \ldots, k\}} \prod_{q=1}^{l} \frac{\sqrt{x_{i_q}} - \sqrt{m_{i_q}}}{\sqrt{m_{i_q} + 1} - \sqrt{m_{i_q}}}.$$  

The coefficients $B_{j_1, \ldots, j_s}(x, m)$ are computed by summing the products

$$(-1)^{l-s} \prod_{q=1}^{l} \frac{\sqrt{x_{j_q}} - \sqrt{m_{j_q}}}{\sqrt{m_{j_q} + 1} - \sqrt{m_{j_q}}}.$$
over \( \{i_1, \ldots, i_l\} \) such that \( i_1 < \cdots < i_l \) and \( \{j_1, \ldots, j_s\} \subset \{i_1, \ldots, i_l\} \).

Consider the sum

\[
B_0(\mathbf{x}, \mathbf{m}) + \sum_{j_1, \ldots, j_s \in \{1, \ldots, k\}} \sum_{j_1 < \cdots < j_s} B_{j_1, \ldots, j_s}(\mathbf{x}, \mathbf{m}).
\]

Let \( l \in \mathbb{N} \) and let \( i_1, \ldots, i_l \in \{1, \ldots, k\} \) such that \( i_1 < \cdots < i_l \). Note that

\[
(-1)^{l-s} \prod_{q=1}^{l} \frac{\sqrt{x_{i_q}} - \sqrt{m_{i_q}}}{\sqrt{m_{i_q} + 1 - \sqrt{m_{i_q}}}}
\]

is a term in \( B_{j_1, \ldots, j_s}(\mathbf{x}, \mathbf{m}) \) whenever \( \{j_1, \ldots, j_s\} \subset \{i_1, \ldots, i_l\} \). Therefore in the above sum, the coefficient of \( \prod_{q=1}^{l} \frac{\sqrt{x_{i_q}} - \sqrt{m_{i_q}}}{\sqrt{m_{i_q} + 1 - \sqrt{m_{i_q}}}} \) is given by

\[
\sum_{s=0}^{l} (-1)^{l-s} \binom{l}{s} = (1 - 1)^l = 0.
\]

Hence

\[
B_0(\mathbf{x}, \mathbf{m}) + \sum_{j_1, \ldots, j_s \in \{1, \ldots, k\}} \sum_{j_1 < \cdots < j_s} B_{j_1, \ldots, j_s}(\mathbf{x}, \mathbf{m}) = 1.
\]

Also, if \( \mathbf{x} \in \prod_{i=1}^{k} [m_i, m_i + 1) \), \( B_{j_1, \ldots, j_s}(\mathbf{x}, \mathbf{m}) \geq 0 \) by Lemma 4.4.2 and hence

\[
|f_{m}(\mathbf{x})| \leq \|\sigma\|_{\infty} \left( B_0(\mathbf{x}, \mathbf{m}) + \sum_{j_1, \ldots, j_s \in \{1, \ldots, k\}} \sum_{j_1 < \cdots < j_s} B_{j_1, \ldots, j_s}(\mathbf{x}, \mathbf{m}) \right)
\]

\[
= \|\sigma\|_{\infty}.
\]

Therefore \( \|f\|_{\infty} = \|\sigma\|_{\infty} \) as \( f|_{\mathbb{N}_0^k} = \sigma \).

4.4.2. Some useful lemmas

The following lemma is quite useful in the proofs that follow.
Lemma 4.4.3. Let $l, s \in \{1, \ldots, k\}$. Suppose $i_1, \ldots, i_l \in \{1, \ldots, k\}$ such that $i_1 < \cdots < i_l$.

Then

$$a_{i_1, \ldots, i_l+1}^{l+1} (m) = a_{i_1, \ldots, i_l}^l (m + e_s) - a_{i_1, \ldots, i_l}^l (m)$$

Proof. To keep the notations simple, we will assume $s > i_l$ and we label $s$ by $i_{l+1}$.

$$a_{i_1, \ldots, i_{l+1}}^{l+1} (m) = \sum_{q=0}^{l+1} (-1)^{l+1-q} \sum_{j_1, \ldots, j_q \in \{i_1, \ldots, i_{l+1}\}} \sum_{j_{l+1} < \cdots < j_q} \sigma (m + e_{j_1} + \cdots + e_{j_q})$$

$$= (-1)^{l+1} \sigma (m) + \sum_{q=1}^{l} (-1)^{l+1-q} \left( \sum_{j_1, \ldots, j_q \in \{i_1, \ldots, i_l\}} \sum_{j_{l+1} < \cdots < j_q} \sigma (m + e_{j_1} + \cdots + e_{j_q}) + \sigma (m + e_{i_1} + \cdots + e_{i_{q+1}}) \right) + \sigma (m + e_{i_1} + \cdots + e_{i_{l+1}})$$

$$= - \sum_{q=0}^{l} (-1)^{l-q} \sum_{j_1, \ldots, j_q \in \{i_1, \ldots, i_l\}} \sum_{j_{l+1} < \cdots < j_q} \sigma (m + e_{j_1} + \cdots + e_{j_q})$$

$$+ \sum_{q=1}^{l+1} (-1)^{l+1-q} \sum_{j_1, \ldots, j_{q-1} \in \{i_1, \ldots, i_l\}} \sum_{j_{l+1} < \cdots < j_q} \sigma (m + e_{i_{q+1}} + e_{j_1} + \cdots + e_{j_q})$$

$$= a_{i_1, \ldots, i_l}^l (m) + \sum_{q=0}^{l} (-1)^{l-q} \sum_{j_1, \ldots, j_q \in \{i_1, \ldots, i_l\}} \sum_{j_{l+1} < \cdots < j_q} \sigma (m + e_{i_{q+1}} + e_{j_1} + \cdots + e_{j_q})$$

$$= a_{i_1, \ldots, i_l}^l (m + e_{l+1}) - a_{i_1, \ldots, i_l}^l (m).$$

Notice that $f$ is defined on $\prod_{i=1}^k [m_i, m_i + 1]$ by $f_m$. But the following lemma allows $f$ to be defined on $\prod_{i=1}^k [m_i, m_i + 1]$ by $f_m$.

Lemma 4.4.4. Let $x \in \mathbb{R}^k_+$, $m \in \mathbb{N}_0^k$ and let $s \in \{1, \ldots, k\}$. Suppose $[x_i] = m_i$ for all $i$. 38
Then

\[ f(x + (1 + m_s - x_s)e_s) = f_m(x + (1 + m_s - x_s)e_s). \]

**Proof.** Note that the \( s \)th coordinate of \( x + (1 + m_s - x_s)e_s \) is \( m_s + 1 \). Therefore

\[
\begin{align*}
f_m(x + (1 + m_s - x_s)e_s) &= \sigma(m) + \sum_{l=1}^{k-1} \sum_{i_1, \ldots, i_l \in \{1, \ldots, k\} \setminus \{s\} \atop i_1 < \cdots < i_l} a^l_{i_1, \ldots, i_l}(m) \prod_{q=1}^{l} \frac{\sqrt{m_{i_q}^2 - m_{i_q}}}{\sqrt{m_{i_q} + 1} - \sqrt{m_{i_q}}} \\
&\quad + a^1_s(m) + \sum_{l=2}^{k} \sum_{i_1, \ldots, i_l \in \{1, \ldots, k\} \setminus \{s\} \atop i_1 < \cdots < i_l} a^l_{i_1, \ldots, i_l}(m) \prod_{q=1}^{l} \frac{\sqrt{m_{i_q}^2 - m_{i_q}}}{\sqrt{m_{i_q} + 1} - \sqrt{m_{i_q}}}
\end{align*}
\]

\[ = \sigma(m + e_s) + \left( \sum_{l=1}^{k-1} \sum_{i_1, \ldots, i_l \in \{1, \ldots, k\} \setminus \{s\} \atop i_1 < \cdots < i_l} (a^l_{i_1, \ldots, i_l}(m) + a^{l+1}_{i_1, \ldots, i_l}(m)) \right)
\]

\[ \times \prod_{q=1}^{l} \frac{\sqrt{m_{i_q}^2 - m_{i_q}}}{\sqrt{m_{i_q} + 1} - \sqrt{m_{i_q}}}, \]

by reindexing the last summand and because \( \sigma(m + a^1_s(m)) = \sigma(m + e_s) \). Also,

\[ a^l_{i_1, \ldots, i_l}(m) + a^{l+1}_{i_1, \ldots, i_l}(m) = a^l_{i_1, \ldots, i_l}(m + e_s) \]

by Lemma 4.4.3. Then

\[
\begin{align*}
f_m(x + (1 + m_s - x_s)e_s) &= \sigma(m + e_s) \\
&\quad + \sum_{l=1}^{k-1} \sum_{i_1, \ldots, i_l \in \{1, \ldots, k\} \setminus \{s\} \atop i_1 < \cdots < i_l} a^l_{i_1, \ldots, i_l}(m + e_s) \prod_{q=1}^{l} \frac{\sqrt{m_{i_q}^2 - m_{i_q}}}{\sqrt{m_{i_q} + 1} - \sqrt{m_{i_q}}} \\
&\quad = f_{m+e_s}(x + (1 - x_s)e_s) \\
&\quad = f(x + (1 + m_s - x_s)e_s).
\end{align*}
\]

The second equality in the above computation holds as the \( s \)th coordinate of \( x + (1 + m_s - x_s)e_s \) is \( m_s + 1 \) and hence any term in \( f_{m+e_s}(x + (1 + m_s - x_s)e_s) \) indexed by \( \{i_1, \ldots, i_l\} \) would vanish if \( s \in \{i_1, \ldots, i_l\} \).
As a consequence of above lemma, we have the following corollary.

**Corollary 4.4.5.** Let \( m \in \mathbb{N}_0^k \) and let \( x \in \prod_{i=1}^{k} [m_i, m_i + 1] \). Then

\[
f(x) = f_m(x).
\]

**Lemma 4.4.6.** Let \( l \in \mathbb{N} \) and let \( a_i, b_i \in \mathbb{C} \) such that \( |a_i|, |b_i| \leq 1 \) for all \( i = 1, \ldots, l \). Then

\[
\left| \prod_{i=1}^{l} a_i - \prod_{i=1}^{l} b_i \right| \leq \sum_{i=1}^{l} |a_i - b_i|.
\]

**Proof.** Clearly, the result holds for \( l = 1 \). Assume that the result is true for \( l \). Then by triangle inequality,

\[
\left| \prod_{i=1}^{l+1} a_i - \prod_{i=1}^{l+1} b_i \right|
\leq |a_{l+1} - b_{l+1}| \left| \prod_{i=1}^{l} a_i \right| + |b_{l+1}| \left| \prod_{i=1}^{l} a_i - \prod_{i=1}^{l} b_i \right|
\leq |a_{l+1} - b_{l+1}| + \left| \prod_{i=1}^{l} a_i - \prod_{i=1}^{l} b_i \right|
\leq \sum_{i=1}^{l+1} |a_i - b_i|.
\]

Hence, the result is true by induction on \( l \).

**4.4.3. The uniform continuity of \( f \) with respect to the square-root metric**

Let \( x, x' \in \mathbb{R}_+^k \) s.t. \( \rho_k(x, x') < \delta \). Let \( m = (m_1, \ldots, m_k) \) and \( m' = (m'_1, \ldots, m'_k) \) where \( m_i = \lfloor x_i \rfloor \) and \( m'_i = \lfloor x'_i \rfloor \) for \( i = 1, \ldots, k \).

Case I. Assume \( x'_i \in [m_i, m_i + 1] \) for all \( i \). Then by Corollary 4.4.5
\[ |f(x) - f(x')| \leq \sum_{l=1}^{k} \sum_{i_1, \ldots, i_l \in \{1, \ldots, k\}\atop i_1 < \cdots < i_l} |a^l_{i_1, \ldots, i_l}(m)| \times \left| \prod_{q=1}^{l} \frac{\sqrt{x'_{i_q}} - \sqrt{m_{i_q}}}{\sqrt{m_{i_q} + 1} - \sqrt{m_{i_q}}} - \prod_{q=1}^{l} \frac{\sqrt{x_{i_q}} - \sqrt{m_{i_q}}}{\sqrt{m_{i_q} + 1} - \sqrt{m_{i_q}}} \right| \]

\[ \leq \sum_{l=1}^{k} \sum_{i_1, \ldots, i_l \in \{1, \ldots, k\}\atop i_1 < \cdots < i_l} |a^l_{i_1, \ldots, i_l}(m)| \sum_{q=1}^{l} \left| \frac{\sqrt{x'_{i_q}} - \sqrt{x_{i_q}}}{\sqrt{m_{i_q} + 1} - \sqrt{m_{i_q}}} \right| \]

by Lemma 4.4.6.

Fix \( l \in \{1, \ldots, k\} \) and \( i_1, \ldots, i_l \in \{1, \ldots, k\} \) s.t. \( i_1 < \cdots < i_l \). If \( \sqrt{m_{i_q} + 1} - \sqrt{m_{i_q}} \geq \sqrt{\delta} \) for all \( q = 1, \ldots, l \),

\[ |a^l_{i_1, \ldots, i_l}(m)| \sum_{q=1}^{l} \left| \frac{\sqrt{x'_{i_q}} - \sqrt{x_{i_q}}}{\sqrt{m_{i_q} + 1} - \sqrt{m_{i_q}}} \right| \leq |a^l_{i_1, \ldots, i_l}(m)| \frac{l\delta}{\sqrt{\delta}} \]

\[ \leq 2^l \|\sigma\|_\infty \sqrt{\delta} \]

as \( |a^l_{i_1, \ldots, i_l}(m)| \leq 2^l \|\sigma\|_\infty \).

Assume \( \sqrt{m_{i_q} + 1} - \sqrt{m_{i_q}} < \sqrt{\delta} \) for some \( q \in \{1, \ldots, l\} \). W.l.o.g. assume \( \sqrt{m_{i_q} + 1} - \sqrt{m_{i_q}} < \sqrt{\delta} \). If \( l \geq 2 \),

\[ |a^l_{i_1, \ldots, i_l}(m)| \]

\[ = |a^{l-1}_{i_1, \ldots, i_{l-1}}(m + e_i) - a^{l-1}_{i_1, \ldots, i_{l-1}}(m)| \]

\[ \leq \sum_{j_1, \ldots, j_q \in \{i_1, \ldots, i_{l-1}\}\atop j_1 < \cdots < j_q} |\sigma(m + e_{j_1} + \cdots + e_{j_q} + e_i) - \sigma(m + e_{j_1} + \cdots + e_{j_q})| \]

\[ \leq 2^{l-1} \omega_{\rho, \sigma}(\sqrt{\delta}). \]
Hence

\[
|f(x) - f(x')| \leq \sum_{l=1}^{k} \sum_{i_1, \ldots, i_l \in \{1, \ldots, k\} \atop i_1 < \cdots < i_l} 2^{l-1} l \max\{2\|\sigma\|_{\infty} \sqrt{\delta}, \omega_{\rho_k, \sigma}(\sqrt{\delta})\}
\]

\[
= A_k \max\{2\|\sigma\|_{\infty} \sqrt{\delta}, \omega_{\rho_k, \sigma}(\sqrt{\delta})\}
\]

where \( A_k = k! \sum_{l=1}^{k} \frac{2^{l-1}}{(l-1)! (k-l)!} \).

Case II. Suppose \( x'_i \notin [m_i, m_i + 1] \) for some \( i \). Define \( p = (p_1, \ldots, p_k), p' = (p'_1, \ldots, p'_k) \in \mathbb{N}_0^k \) by

\[
p_i = m_i + 1, \quad p'_i = m'_1 \quad \text{if } x_i < x'_i
\]

\[
p_i = m_i, \quad p'_i = m'_1 + 1 \quad \text{if } x_i > x'_i
\]

\[
p_i = m_i = m'_1 = p'_i \quad \text{if } x_i = x'_i
\]

for \( i = 1, \ldots, k \). Then \( p_i \in [m_i, m_i + 1] \) and \( p'_i \in [m'_i, m'_1 + 1] \) for all \( i \). Hence by case I and because \( \rho_k(p, p') \leq \rho_k(x, x') < \delta \),

\[
|f(x) - f(x')| \leq |f(x) - f(p)| + |\sigma(p) - \sigma(p')| + |f(p') - f(x')|
\]

\[
\leq 2A_k \max\{2\|\sigma\|_{\infty} \sqrt{\delta}, \omega_{\rho_k, \sigma}(\sqrt{\delta})\} + \omega_{\rho_k, \sigma}(\delta).
\]

In both cases,

\[
|f(x) - f(x')| \leq 2A_k \max\{2\|\sigma\|_{\infty} \sqrt{\delta}, \omega_{\rho_k, \sigma}(\sqrt{\delta})\} + \omega_{\rho_k, \sigma}(\delta).
\]

Therefore \( f \in C_{b,u}(\mathbb{R}_+^k, \rho_k) \). This completes the proof of Lemma 4.4.1.
Chapter 5. The $C^*$-Algebra Generated by Toeplitz Operators with $k$-Quasi-Radial Symbols

In this chapter, we show that the set of all eigenvalue functions is a dense subset of $C_{b,u}(\mathbb{N}_0^k, \rho_k)$. It follows that the $C^*$-algebra generated by Toeplitz operators with $k$-quasi-radial symbols is isometrically isomorphic to $C_{b,u}(\mathbb{N}_0^k, \rho_k)$.

5.1. Uniform continuity of eigenvalue functions with respect to the square-root metric

In this section, we show that the eigenvalue functions $\gamma_{n,a}$ belong to $C_{b,u}(\mathbb{N}_0^k, \rho_k)$.

**Definition 5.1.1.** Let $\mathcal{G}_n$ be the set of all eigenvalue functions:

$$\mathcal{G}_n := \{\gamma_{n,a} \mid \varphi \in L^\infty(\mathbb{C}^n)^G\}.$$

**Proposition 5.1.2.** The eigenvalue functions $\gamma_{n,a}$ are bounded functions that are uniformly continuous with respect to the square-root metric $\rho_k$, i.e.,

$$\mathcal{G}_n \subseteq C_{b,u}(\mathbb{N}_0^k, \rho_k).$$

The boundedness of $\gamma_{n,a}$ follows from the boundedness of the symbol. We proceed to prove the uniform continuity of $\gamma_{n,a}$ with respect to the metric $\rho_k$. The Lemma 5.1.3 is Lemma 4.3 in [EM16] which is used by Esmeral and Maximenko to prove the above theorem for $n = 1$. However, we present the proof for the sake of completeness. Then we prove Lemma 5.1.4 and Lemma 5.1.5 which are used to prove Proposition 5.1.6. Theorem 5.1.2 follows as a corollary of Proposition 5.1.6.


43
Lemma 5.1.3. For any \( m \in \mathbb{N} \),

\[
\int_0^\infty \left| \frac{r^m}{m!} - \frac{r^{m-1}}{(m-1)!} \right| e^{-r} \, dr \leq \sqrt{\frac{2}{\pi m}}.
\]

Proof. Notice that for \( r \in [0, \infty) \), \( \frac{r^m}{m!} - \frac{r^{m-1}}{(m-1)!} > 0 \) if and only if \( r > m \). Hence by Lemma 2.4.1,

\[
\int_0^\infty \left| \frac{r^m}{m!} - \frac{r^{m-1}}{(m-1)!} \right| e^{-r} \, dr = \int_0^\infty \left( \frac{r^m}{m!} - \frac{r^{m-1}}{(m-1)!} \right) e^{-r} \, dr + 2 \int_0^m \left( \frac{r^{m-1}}{(m-1)!} - \frac{r^m}{m!} \right) e^{-r} \, dr
\]

\[
= 1 - 1 + 2e^{-m} \left( \sum_{i=0}^m \frac{m^i}{i!} - \sum_{i=0}^{m-1} \frac{m^i}{i!} \right)
\]

\[
= \frac{2m^m e^{-m}}{m!}
\]

\[
\leq \sqrt{\frac{2}{\pi m}}
\]

where the last inequality follows from inequality 2.4.2:

\[
\sqrt{2\pi mm^m e^{-m}} \leq m!.
\]

As a consequence of the above lemma we prove the following:

**Lemma 5.1.4.** For any \( m, m' \in \mathbb{N}_0 \),

\[
\int_0^\infty \left| \frac{r^m}{m!} - \frac{r^{m'}}{m'!} \right| e^{-r} \, dr \leq 2 \sqrt{\frac{2}{\pi}} |\sqrt{m} - \sqrt{m'}|.
\]
Proof. Let \( \underline{m} = \min\{m, m'\} \) and \( \overline{m} = \max\{m, m'\} \). Notice that
\[
\int_0^\infty \left| \frac{r^m}{m!} - \frac{r^{m'}}{m'!} \right| e^{-r} dr \leq \sum_{i=m+1}^{\overline{m}} \int_0^\infty \left| \frac{r^i}{i!} - \frac{r^{i-1}}{(i-1)!} \right| e^{-r} dr
\leq \sum_{i=m+1}^{\overline{m}} \sqrt{\frac{2}{\pi i}} \quad \text{(by Lemma 5.1.3)}
= \sqrt{\frac{2}{\pi}} \sum_{i=m+1}^{\overline{m}} \frac{2\sqrt{i}(\sqrt{i} - \sqrt{i-1})}{\sqrt{i}} = 2\sqrt{\frac{2}{\pi}} |\sqrt{m} - \sqrt{m'}|.
\]

Lemma 5.1.5. Suppose \( f_i, g_i : \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{C} \) satisfy \( \int_{\mathbb{R}_+} |f_i(m, r)| dr, \int_{\mathbb{R}_+} |g_i(m, r)| dr \leq 1 \) for \( i = 1, \ldots, k \). Then for all \( m_i, m'_i \in \mathbb{N}_0 \),
\[
\int_{\mathbb{R}_+^k} \left| \prod_{i=1}^k f_i(m_i, r_i) - \prod_{i=1}^k g_i(m'_i, r_i) \right| dr_1 \cdots dr_k \leq \sum_{i=1}^k \int_{\mathbb{R}_+} |f_i(m_i, r_i) - g_i(m'_i, r_i)| dr_i.
\]

Proof. Notice that the statement is true for \( k = 1 \). Assume the statement is true for \( k \).

Then
\[
\int_{\mathbb{R}_+^{k+1}} \left| \prod_{i=1}^{k+1} f_i(m_i, r_i) - \prod_{i=1}^{k+1} g_i(m'_i, r_i) \right| dr_1 \cdots dr_{k+1}
\leq \int_{\mathbb{R}_+} |f_{k+1}(m_{k+1}, r_{k+1}) - g_{k+1}(m_{k+1}, r_{k+1})| dr_{k+1} \left| \int_{\mathbb{R}_+^k} \prod_{i=1}^k f_i(m_i, r_i) dr_1 \cdots dr_k \right|
+ \int_{\mathbb{R}_+} |g_{k+1}(m_{k+1}, r_{k+1})| dr_{k+1} \int_{\mathbb{R}_+^k} \left| \prod_{i=1}^k f_i(m_i, r_i) - \prod_{i=1}^k g_i(m'_i, r_i) \right| dr_1 \cdots dr_k
\leq \sum_{i=1}^{k+1} \int_{\mathbb{R}_+} |f_i(m_i, r_i) - g_i(m'_i, r_i)| dr_i
\]
by the assumption and because
\[
\left| \int_{\mathbb{R}_+^k} \prod_{i=1}^k f_i(m_i, r_i) dr_1 \cdots dr_k \right| \leq \prod_{i=1}^k \int_{\mathbb{R}_+} |f_i(m_i, r_i)| dr_i
\leq 1.
\]

45
Hence the result holds by induction on $k$. 

**Proposition 5.1.6.** The eigenvalue functions $\gamma_{n,a}$ are Lipchitz with respect to the square-root metric $\rho_k$: There exists $A_\varphi > 0$ such that

$$|\gamma_{n,a}(m) - \gamma_{n,a}(m')| \leq A_\varphi \rho_k(m, m')$$

for all $m = (m_1, \ldots, m_k), m' = (m'_1, \ldots, m'_k) \in \mathbb{N}_0^k$.

**Proof.** Define $K : \mathbb{N}_0 \times \mathbb{R}_+ \to [0, \infty)$ by $K(m, r) = \frac{m}{m!} e^{-r}$. Then $\int_0^\infty K(m, r)dr = 1$ and

$$\gamma_{1,a}(m) = \int_{\mathbb{R}_+^k} a_\varphi(\sqrt{r_1}, \ldots, \sqrt{r_k}) \prod_{i=1}^k K(m_i, r_i)dr_1 \cdots dr_k$$

where $1 = (1, \ldots, 1) \in \mathbb{N}_0^k$. Notice that

$$|\gamma_{1,a}(m) - \gamma_{1,a}(m')|$$

$$\leq \|a_\varphi\|_\infty \int_{\mathbb{R}_+^k} \left| \prod_{i=1}^k K(m_i, r_i) - \prod_{i=1}^k K(m'_i, r_i) \right| dr_1 \cdots dr_k$$

$$\leq \|a_\varphi\|_\infty \sum_{i=1}^k \int_0^\infty |K(m_i, r_i) - K(m'_i, r_i)|dr_i \quad \text{(by Lemma 5.1.5)}$$

$$\leq 2 \sqrt{\frac{2}{\pi}} \|a_\varphi\|_\infty \rho_k(m, m') \quad \text{(by Lemma 5.1.4)}.$$ 

Hence by Corollary 3.4.5,

$$|\gamma_{n,a}(m) - \gamma_{n,a}(m')| = |\gamma_{1,a}(m + n - 1) - \gamma_{1,a}(m' + n - 1)|$$

$$\leq 2 \sqrt{\frac{2}{\pi}} \|a_\varphi\|_\infty \rho_k(m + n - 1, m' + n - 1)$$

$$\leq 2 \sqrt{\frac{2}{\pi}} \|a_\varphi\|_\infty \rho_k(m, m').$$
Therefore the function $\gamma_{n,a}$ are uniformly continuous with respect to the metric $\rho_k$, proving Theorem 5.1.2.

5.2. Density of $\mathfrak{G}_1$ in $C_{b,u}(\mathbb{N}_0^k, \rho_k)$

In the previous section we showed that $\mathfrak{G}_n \subseteq C_{b,u}(\mathbb{N}_0^k, \rho_k)$. To show that the $C^*$-algebra generated by $\mathfrak{G}_n$ is $C_{b,u}(\mathbb{N}_0^k, \rho_k)$, it is enough to show that $\mathfrak{G}_n$ is dense in $C_{b,u}(\mathbb{N}_0^k, \rho_k)$ which will be proved in all generality in section 5.3. In this section we focus on the case $k = n$, i.e., $n = 1 = (1, \ldots, 1) \in \mathbb{N}^n$. In other words we will show that $\mathfrak{G}_1$, the set of eigenvalue functions we get for the case $n = 1$, is dense in $C_{b,u}(\mathbb{N}_0^k, \rho_k)$. Note that in the above case, the Toeplitz operators under consideration are the Toeplitz operators on $\mathcal{F}(\mathbb{C}^n)$ with separately radial symbols.

This discussion is a generalization of the proof of density for the case $n = (1)$, presented in [EM16].

5.2.1. Approximation by convolutions

Main goal of this subsection is to prove Proposition 5.2.6, where we approximate spectral functions by convolutions.

By a change of variable $(r_1, \ldots, r_k) \rightarrow (r_1^2, \ldots, r_k^2)$, we have

$$\gamma_{1,a}(m) = \int_{\mathbb{R}^k_+} a(r_1, \ldots, r_k) \prod_{i=1}^{k} g(m_i, r_i) dr_1 \cdots dr_k$$

where $g(m, r) = \frac{2^{m+1} e^{-r^2}}{m!}$. Let $h(x) = \sqrt{\frac{2}{\pi}} e^{-2x^2}$.

We have the following lemma from [EM16] (Lemma 6.5). However, we present a slightly different proof, through a series of lemmas: 5.2.2,5.2.3 and 5.2.4.
Lemma 5.2.1. Let the functions \( g \) and \( h \) be as above. Then

\[
\lim_{m \to \infty} \int_0^\infty |g(m, r) - h(\sqrt{m} - r)|dr = 0.
\]

In order to prove Lemma 5.2.1, we require several estimates. We start by defining,

\[
\tilde{g}(m, r) := \sqrt{\frac{2}{\pi}} \frac{r^{2m+1}e^{m-r^2}}{m^{m+\frac{1}{2}}}; m \in \mathbb{N}.
\]

Lemma 5.2.2. We have

\[
\lim_{m \to \infty} \int_0^\infty |g(m, r) - \tilde{g}(m, r)|dr = 0.
\]

Proof. Notice that for \( m \in \mathbb{N} \),

\[
0 \leq \tilde{g}(m, r) - g(m, r) \leq \sqrt{\frac{2}{\pi}} \frac{r^{2m+1}e^{m-r^2}}{m^{m+\frac{1}{2}}} (1 - e^{-\frac{1}{12m}})
\]

by the inequality, 2.4.2. Hence for \( m \in \mathbb{N} \),

\[
0 \leq \tilde{g}(m, r) - g(m, r) \leq \frac{2r^{2m+1}e^{-r^2}}{m!} (1 - e^{-\frac{1}{12m}})
\]

and

\[
\int_0^\infty |g(m, r) - \tilde{g}(m, r)|dr \leq (e^{\frac{1}{12m}} - 1) \int_0^\infty \frac{2r^{2m+1}e^{-r^2}}{m!}dr
\]

\[
= (e^{\frac{1}{12m}} - 1).
\]

This completes the proof as

\[
\lim_{m \to \infty} e^{\frac{1}{12m}} - 1 = 0.
\]
Due to Lemma 5.2.2, it is enough to show that
\[ \lim_{m \to \infty} \int_{0}^{\infty} \left| \tilde{g}(m, r) - h(\sqrt{m} - r) \right| dr = 0, \]
to prove Lemma 5.2.1. We will achieve this by Lebesgue dominated convergence theorem.

First notice that for \( m \in \mathbb{N} \),
\[ \int_{0}^{\infty} \left| \tilde{g}(m, r) - h(\sqrt{m} - r) \right| dr = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \left| \frac{r^{2m+1} e^{r^2} - e^{-2(r - \sqrt{m})^2}}{m^{m+\frac{1}{2}}} \right| dr 
= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{(2m+1)(\ln r - \ln \sqrt{m})+m-r^2} - e^{-2(r - \sqrt{m})^2} \right| dr 
= \sqrt{\frac{2}{\pi}} \int_{-\sqrt{m}}^{\infty} \left| e^{(2m+1)\ln(\frac{u}{\sqrt{m}})+1-u^2-2u\sqrt{m}} - e^{-2u^2} \right| dr \quad (5.2.1) \]
by the change of variable \( u = r - \sqrt{m} \).

**Lemma 5.2.3.** For all \( u \in [-\sqrt{m}, \infty) \)
\[ \lim_{m \to \infty} \left| e^{(2m+1)\ln(\frac{u}{\sqrt{m}})+1-u^2-2u\sqrt{m}} - e^{-2u^2} \right| = 0. \]

**Proof.** Note that it is enough to show that
\[ \lim_{m \to \infty} (2m + 1) \ln(\frac{u}{\sqrt{m}} + 1) - 2u\sqrt{m} = -u^2 \]
for all \( u \in [-\sqrt{m}, \infty) \). By L’Hospital’s rule,
\[ \lim_{m \to \infty} (2m + 1) \ln(\frac{u}{\sqrt{m}} + 1) - 2u\sqrt{m} = \lim_{m \to \infty} \frac{\ln(\frac{u}{\sqrt{m}} + 1) - 2u\sqrt{m}(2m + 1)^{-1}}{(2m + 1)^{-1}} \]
\[ = \lim_{m \to \infty} \frac{\frac{-u}{2m(u+\sqrt{m})} - um^{-\frac{1}{2}}(2m+1)^{-1} + 4u\sqrt{m}(2m+1)^{-2}}{-2(2m+1)^{-2}} \]
\[ = \lim_{m \to \infty} \frac{u(2m+1)^2}{4m(u + \sqrt{m})} + \frac{um^{-\frac{1}{2}}(2m+1)}{2} - 2u\sqrt{m} \]
\[ = \lim_{m \to \infty} \frac{u(4m + 1)}{4m(u + \sqrt{m})} + \frac{u}{2\sqrt{m}} - \frac{u^2\sqrt{m}}{u + \sqrt{m}} \]
\[ = -u^2. \]
Lemma 5.2.4. For all \( m \in \mathbb{N} \),

\[
e^{(2m+1)\ln(\frac{u}{\sqrt{m}}+1)-u^2-2u\sqrt{m}} \leq e^{u-u^2}
\]

Proof. Since \( \ln(x+1) \leq x \) for all \( x \in \mathbb{R} \), we have that

\[
(2m+1)\ln(\frac{u}{\sqrt{m}}+1) - u^2 - 2u\sqrt{m} \leq (2m+1)\frac{u}{\sqrt{m}} - u^2 - 2u\sqrt{m}
\]

\[
+ \frac{u}{\sqrt{m}} - u^2
\]

\[
\leq u - u^2.
\]

Since \( e^{u-u^2} \) is integrable on \([-\sqrt{m}, \infty)\), by lemmas, 5.2.3,5.2.4, by equation 5.2.1, and by Lebesgue dominated convergence theorem, we have

\[
\int_0^\infty \left| g(m, r) - h(\sqrt{m} - r) \right| dr = 0.
\]

Hence Lemma 5.2.1 is proved as a consequence of Lemma 5.2.2.

Let \( H(x_1, \ldots, x_k) = \prod_{i=1}^k h(x_i) = \left(\frac{2}{\pi}\right)^\frac{k}{2} e^{-2(x_1^2 + \cdots + x_k^2)} \).

Lemma 5.2.5. Let \( \epsilon > 0 \). Then there exists \( N \) such that for all \( m_i > N, i = 1, \ldots, k \)

\[
\int_{\mathbb{R}_+^k} \left| \prod_{i=1}^k g(m_i, r_i) - H(\sqrt{m_1} - r_1, \ldots, \sqrt{m_k} - r_k) \right| dr_1 \ldots dr_k < \epsilon.
\]

Proof. By Lemma 5.2.1, there exists \( N \) such that for all \( m > N, \int_0^\infty |g(m, r) - h(\sqrt{m} - r)|dr < \frac{\epsilon}{k} \). Also notice that \( \int_0^\infty g(m, r)dr = 1 \) and \( \int_0^\infty h(\sqrt{m} - r)dr \leq \int_\mathbb{R} h(\sqrt{m} - r)dr = \int_\mathbb{R} h(r)dr = 1 \) for any \( m \in \mathbb{N}_0 \). Then if \( m_i > N, i = 1, \ldots, k \),
\[
\int_{\mathbb{R}^k_+} \left| \prod_{i=1}^k g(m_i, r_i) - H(\sqrt{m_1} - r_1, \ldots, \sqrt{m_k} - r_k) \right| dr_1 \ldots dr_k \\
= \int_{\mathbb{R}^k_+} \left| \prod_{i=1}^k g(m_i, r_i) - \prod_{i=1}^k h(\sqrt{m_i} - r_i) \right| dr_1 \ldots dr_k \\
\leq \sum_{i=1}^k \int_0^\infty |g(m_i, r_i) - h(\sqrt{m_i} - r_i)| dr_i \quad \text{(by Lemma 5.1.5)} \\
< k \frac{\epsilon}{k} = \epsilon. \]

Let \( f \in L^1(\mathbb{R}^k, dx) \) and \( g \in L^\infty(\mathbb{R}^k, dx) \) where \( dx \) denotes the Lebesgue measure on \( \mathbb{R}^k \). Recall that the convolution of \( f \) and \( g \), denoted \( f * g \), is given by

\[
f * g(x) = g * f(x) = \int_{\mathbb{R}^k} f(x-y)g(y)dy, \quad x \in \mathbb{R}^k.
\]

The following proposition presents an approximation of the eigenvalue functions by convolutions.

**Proposition 5.2.6.** Let \( a \in L^\infty(\mathbb{R}^k_+) \) and let \( \epsilon > 0 \). Define \( H * a \), by considering \( a \) as a function on \( \mathbb{R}^k \) whose support is \( \mathbb{R}^k_+ \). Then there exists \( N \) such that for all \( m_i > N, i = 1, \ldots, k, \)

\[
|\gamma_{1,a}(m_1, \ldots, m_k) - (H * a)(\sqrt{m_1}, \ldots, \sqrt{m_k})| < \epsilon.
\]

**Proof.** Since support of \( a \) is \( \mathbb{R}^k_+ \),

\[
|\gamma_{1,a}(m_1, \ldots, m_k) - (H * a)(\sqrt{m_1}, \ldots, \sqrt{m_k})| \\
= \left| \int_{\mathbb{R}^k_+} \left( \prod_{i=1}^k g(m_i, r_i) - H(\sqrt{m_1} - r_1, \ldots, \sqrt{m_k} - r_k) \right) a(r)dr_1 \ldots dr_k \right| \\
\leq \|a\|_\infty \int_{\mathbb{R}^k_+} \left| \prod_{i=1}^k g(m_i, r_i) - H(\sqrt{m_1} - r_1, \ldots, \sqrt{m_k} - r_k) \right| dr_1 \ldots dr_k.
\]
Hence the result follows from Lemma 5.2.5

**Lemma 5.2.7.** Let \( b \in L^\infty(\mathbb{R}^k) \) and \( a = \chi_{\mathbb{R}^k_+} b \). Let \( \epsilon > 0 \). Then there exists \( M \) such that for all \( x_i > M, i = 1, \ldots, k \),

\[
|H * a(x_1, \ldots, x_k) - H * b(x_1, \ldots, x_k)| < \epsilon.
\]

**Proof.** Notice that for \( x = (x_1, \ldots, x_k) \in \mathbb{R}^k_+ \),

\[
|H * a(x) - H * b(x)| \leq \int_{\mathbb{R}^k} |a(y) - b(y)|H(x - y)dy
\]

\[
\leq \|b\|_\infty \int_{\mathbb{R}^k_+} H(x - y)dy
\]

\[
\leq \|b\|_\infty \sum_{j=1}^k \int_{\mathbb{R}_-} h(x_j - y_j)dy_j \prod_{i=1 \atop i \neq j}^k \int_{\mathbb{R}} h(x_i - y_i)dy_i
\]

\[
= \|b\|_\infty \sum_{j=1}^k \int_{\mathbb{R}_-} h(x_j - y_j)dy_j
\]

\[
= \|b\|_\infty \sum_{j=1}^k \int_{x_j}^{\infty} h(y_j)dy_j \quad \text{(by a change of variable)}.
\]

The lemma holds as \( \int_{x_j}^{\infty} h(y_j)dy_j \) approaches zero as \( x_j \) goes to \( \infty \).

**5.2.2. Approximating uniformly continuous bounded functions by convolutions**

The main goal of this subsection is to prove Proposition 5.2.9. For this we present a series of propositions and lemmas.

**Proposition 5.2.8.** Let \( g \in C_{b,u}(\mathbb{R}^k) \) and let \( \epsilon > 0 \). Then there is \( b \in L^\infty(\mathbb{R}^k) \) such that

\[
\|H * b - g\|_\infty < \epsilon.
\]

**Proof.** Let \( \tilde{h} \) denote the bump function (any compactly supported smooth function would
\[
\tilde{h}(\zeta) = e^{-\frac{1}{1-|\zeta|^2}} X_B(\zeta), \quad \zeta \in \mathbb{R}^k
\]

where \( B \) is the open unit ball in \( \mathbb{R}^k \). Let \( h \) denote the normalized Fourier inverse of \( \tilde{h} \), i.e.,

\[
h = \frac{\mathcal{F}^{-1}(\tilde{h})}{C}
\]

where \( C = \| \mathcal{F}^{-1}(\tilde{h}) \|_1 \). Define the approximate identity \( h_t \) on \( \mathbb{R}^k \) by

\[
h_t(x) = \frac{h(x/t)}{t}, \quad x \in \mathbb{R}^k.
\]

Notice that \( \lim_{t \to 0} \| h_t * g - g \|_\infty = 0 \). Choose \( h_{t_0} \) s.t. \( \| h_{t_0} * g - g \|_\infty < \epsilon \). Let \( \hat{l} := \frac{\mathcal{F}(h_{t_0})}{\mathcal{F}(H)} \).

Then

\[
\hat{l}(\zeta) = \frac{\tilde{h}(t_0 \zeta) e^{\zeta^2/2}}{C}, \quad \zeta \in \mathbb{R}^k.
\]

Since \( \hat{l} \) is a Schwartz function and \( \mathcal{F}(h_{t_0}) = \mathcal{F}(H) \hat{l} \), \( h_{t_0} = H * l \) where \( l = \mathcal{F}^{-1}(\hat{l}) \). Let \( b = l * g \). Then \( b \in L^\infty(\mathbb{R}^k) \) and \( \| H * b - g \|_\infty = \| h_{t_0} * g - g \|_\infty < \epsilon \).

Proposition 5.2.8 can also be proved using Wiener’s division lemma, as in the proof of Proposition 5.4 in [EM16].

5.2.3. A partial approximation by spectral functions

The main goal of this subsection is to prove Proposition 5.2.9, that partly fulfills our intention to approximate functions in \( C_{b,a}(\mathbb{N}_0^k, \rho_k) \) by spectral functions.

Proposition 5.2.9. Let \( \sigma \in C_{b,a}(\mathbb{N}_0^k, \rho_k) \) and let \( \epsilon > 0 \). Then there exists \( a \in L^\infty(\mathbb{R}_+^k) \) and \( N \) such that for all \( m = (m_1, \ldots, m_k) \in \mathbb{N}_0^k \) with \( m_i > N \),

\[
|\sigma(m) - \gamma_{1,a}(m)| < \epsilon.
\]
Proof. This proof is similar to the proof of Proposition 6.8 in [EM16].

By Lemma 4.4.1, there exists \( g \in C_{b,u}(R^k) \) such that \( g|_{\mathbb{N}_0^k} = \sigma \) and \( \|g\|_\infty = \|\sigma\|_\infty \).

Define \( \tilde{g} \) on \( R^k \) by \( \tilde{g}(x_1, \ldots, x_k) = g(x_2^2, \ldots, x_k^2) \). Then \( \tilde{g} \in C_{b,u}(R^k) \). Then by Proposition 5.2.8 there exists \( b \in L^\infty(R^k) \) such that

\[
\|H \ast b - \tilde{g}\|_\infty < \frac{\epsilon}{3}.
\]

Define \( a \) on \( R^k_+ \) by \( a = b|_{R^k_+} \). Then \( a \in L^\infty(R^k_+) \). Then by Proposition 5.2.6 and Lemma 5.2.7, there exists \( N_1, N_2 \) such that for all \( m_i > N_1 \), \( i = 1, \ldots, k \)

\[
|\gamma_{1,a}(m_1, \ldots, m_k) - (H \ast a)(\sqrt{m_1}, \ldots, \sqrt{m_k})| < \frac{\epsilon}{3}
\]

and for all \( m_i > N_2 \),

\[
|H \ast a(\sqrt{m_1}, \ldots, \sqrt{m_k}) - H \ast b(\sqrt{m_1}, \ldots, \sqrt{m_k})| < \frac{\epsilon}{3}.
\]

Then for all \( m = (m_1, \ldots, m_k) \) with \( m_i > N = max\{N_1, N_2\} \)

\[
|\gamma_{1,a}(m) - \sigma(m)| \leq |\gamma_{1,a}(m) - (H \ast a)(\sqrt{m_1}, \ldots, \sqrt{m_k})| \\
+ |H \ast a(\sqrt{m_1}, \ldots, \sqrt{m_k}) - H \ast b(\sqrt{m_1}, \ldots, \sqrt{m_k})| \\
+ \|H \ast b - \tilde{g}\|_\infty \\
< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]

5.2.4. The density of \( \mathfrak{G}_1 \) in \( C_{b,u}(\mathbb{N}_0^k, \rho_k) \)

The proof of density (Theorem 5.2.12) requires induction on \( k \). Hence to indicate dependency on \( k \), we identify \( \mathfrak{G}_1 \) by \( \mathfrak{G}_1^k \) and \( \gamma_{1,a} \) by \( \gamma_{1,a}^k \) as needed in this subsection.
Lemma 5.2.10. Let $k > 1$ and assume $\mathfrak{S}_1^{k-1}$ is dense in $C_{b,u}(\mathbb{N}_0^{k-1}, \rho_{k-1})$. For $i = 1, \ldots, k$ and $m_o \in \mathbb{N}_0$, define the set $K_i(m_o) \subset \mathbb{N}_0^k$

$$K_i(m_o) = \{ m = (m_1, \ldots, m_k) \in \mathbb{N}_0^k \mid m_i = m_o \}.$$ 

Let $\sigma \in C_{b,u}(\mathbb{N}_0^k, \rho_k)$ and let $\epsilon > 0$. Then there exists $a \in L^\infty(\mathbb{R}_+^k)$ s.t.

$$\|\sigma \chi_{K_i(m_o)} - \gamma_{1,a}^k\|_\infty < \epsilon.$$ 

Proof. Identify $K_i(m_o)$ with $\mathbb{N}_0^{k-1}$ by the map

$$p_{i,m_o} : (m_1, \ldots, m_{i-1}, m_o, m_{i+1}, \ldots, m_k) \mapsto (m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_k).$$

Define $\sigma_{m_o} : \mathbb{N}_0^{k-1} \to \mathbb{C}$ by

$$\sigma_{m_o}(m_1, \ldots, m_{k-1}) = \sigma \chi_{K_i(m_o)}(p_{i,m_o}^{-1}(m_1, \ldots, m_{k-1})).$$

Then $\sigma_{m_o} \in C_{b,u}(\mathbb{N}_0^{k-1}, \rho_{k-1})$. By the assumption there is $b \in L^\infty(\mathbb{R}_+^{k-1})$ s.t.

$$\|\sigma_{m_o} - \gamma_{1,b}^{k-1}\| < \frac{\epsilon}{2}.$$ 

Since $\chi_{\{m_o\}} \in C_{b,u}(\mathbb{N}_0, \rho_1)$, there is $c \in L^\infty(\mathbb{R}_+)$ s.t.

$$\|\chi_{\{m_o\}} - \gamma_{1,c}\| < \|b\|_\infty \frac{\epsilon}{2}.$$ 

Now define $a \in L^\infty(\mathbb{R}_+^k)$ by

$$a(x) = b(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)c(x_i) \quad \forall \ x \in \mathbb{R}_+^k.$$
Then $\gamma_{1,a}^{k}(m) = \gamma_{1,b}^{k-1}(p_i(m))\gamma_{1,c}(m_i)$ for all $m \in \mathbb{N}_0^k$. Let $\nu \in C_{b,u}(\mathbb{N}_0^k, \rho_k)$ be defined by $\nu(m) = \gamma_{1,b}^{k-1}(p_i(m))\gamma_{1,c}(m_i)$ for all $m \in \mathbb{N}_0^k$. Then

$$
\|\sigma \chi_{K_i(m_0)} - \gamma_{1,a}^{k} \|_{\infty} \leq \|\sigma \chi_{K_i(m_0)} - \nu\|_{\infty} + \|\nu - \gamma_{1,a}^{k}\|_{\infty} 
$$

$$
\leq \|\sigma_{m_0} - \gamma_{1,b}^{k-1}\|_{\infty} + \|\gamma_{1,b}^{k-1}\|_{\infty} \|\gamma_{1,c} - \chi_{\{m_0\}}\|_{\infty} 
$$

$$
< \frac{\epsilon}{2} + \frac{\epsilon}{2\|b\|_{\infty}} = \epsilon.
$$

\[\square\]

**Lemma 5.2.11.** Let $k > 1$ and assume $\mathbf{G}_{i}^{k-1}$ is dense in $C_{b,u}(\mathbb{N}_0^{k-1}, \rho_{k-1})$. Let $K_N \subset \mathbb{N}_0^k$ be defined by

$$
K_N := \{m = (m_1, \ldots, m_k) \in \mathbb{N}_0^k | m_i \leq N \text{ for some } i\}
$$

Let $\sigma \in C_{b,u}(\mathbb{N}_0^k, \rho_k)$ and let $\epsilon > 0$. Then there exists $a \in L^\infty(\mathbb{R}_+^k)$ s.t.

$$
\|\sigma \chi_{K_N} - \gamma_{1,a} \|_{\infty} < \epsilon.
$$

**Proof.** For $i = 1, \ldots, k$ and $j = 0, 1, \ldots, N$, let the sets $K_i(j)$ be as defined in Lemma 5.2.10. Then there exists $\sigma_{i,j} \in C_{b,u}(\mathbb{N}_0^k, \rho_k)$ s.t. $\sigma_{i,j} \chi_{K_i(j)} = \sigma_{i,j}$ and

$$
\sigma \chi_{K_N} = \sum_{i=1}^{k} \sum_{j=1}^{N} \sigma_{i,j}.
$$

The functions $\sigma_{i,j}$ can be constructed by considering $\sigma \chi_{K_i(j)}$ and removing up to finitely many points from its support. By Lemma 5.2.10, there exists $a_{i,j} \in L^\infty(\mathbb{R}_+^k)$ s.t.

$$
\|\sigma_{i,j} - \gamma_{1,a_{i,j}}\|_{\infty} < \frac{\epsilon}{kN}.
$$

Define $a \in L^\infty(\mathbb{R}_+^k)$ by $a = \sum_{i=1}^{k} \sum_{j=1}^{N} a_{i,j}$. Then $\gamma_{1,a} = \sum_{i=1}^{k} \sum_{j=1}^{N} \gamma_{1,a_{i,j}}$ and

$$
\|\sigma \chi_{K_N} - \gamma_{1,a}\|_{\infty} \leq \sum_{i=1}^{k} \sum_{j=1}^{N} \|\sigma_{i,j} - \gamma_{1,a_{i,j}}\|_{\infty} \leq \frac{\epsilon}{kN} \leq \epsilon.
$$

\[\square\]
Theorem 5.2.12. $\mathcal{G}_1$ is dense in $C_{b,u}(\mathbb{N}_0^k, \rho_k)$. 

Proof. The proof is by induction on $k$. The result for $k = 1$ was proved in [EM16]. Assume the result is true for $k - 1$. Let $\sigma \in C_{b,u}(\mathbb{N}_0^k, \rho_k)$ and let $\epsilon > 0$. By Proposition 5.2.9 there is $N$ and $b \in L^\infty(\mathbb{R}_+^k)$ s.t. for all $m \in \mathbb{N}_0^k$ with $m_i > N$,

$$|\sigma(m) - \gamma_{1,b}(m)| < \frac{\epsilon}{2}.$$ 

By the induction hypothesis and Lemma 5.2.11, there exists $c \in L^\infty(\mathbb{R}_+^k)$ s.t.

$$\|\sigma - \gamma_{1,b}\chi_{K_N} - \gamma_{1,c}\| < \frac{\epsilon}{2}.$$ 

Let $a = b + c$. Then $\gamma_{1,a} = \gamma_{1,b} + \gamma_{1,c}$ and

$$\|\sigma - \gamma_{1,a}\|_\infty = \|(\sigma - \gamma_{1,b})\chi_{K_N} + (\sigma - \gamma_{1,b})\chi_{K_N} - \gamma_{1,c}\|_\infty \leq \|(\sigma - \gamma_{1,b})\chi_{K_N}\|_\infty + \|(\sigma - \gamma_{1,b})\chi_{K_N} - \gamma_{1,c}\|_\infty < \sup_{m \in \mathbb{N}_0^k, m_i > N} |\sigma(m) - \gamma_{1,b}(m)| + \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \square$$

5.3. The $C^*$-algebra generated by $\mathcal{G}_n$

In this section we prove the density theorem in all generality: the $C^*$-algebra generated by $\mathcal{G}_n$, where $n = (n_1, \ldots, n_k)$, is the $C^*$-algebra $C_{b,u}(\mathbb{N}_0^k, \rho_k)$. We already noticed that $\gamma_{n,a} \in C_{b,u}(\mathbb{N}_0^k, \rho_k)$ in Proposition 5.1.2. Recall the shift operators in definition 4.2.1. Following lemma is a restatement of Corollary 3.4.5

Lemma 5.3.1. The eigenvalue function $\gamma_{n,a}$ is the $(n - 1)^{th}$ left shift of $\gamma_{1,a}$, i.e.,

$$\gamma_{n,a} = \tau^{n-1}_L \gamma_{1,a}.$$
Proposition 5.3.2. The set of eigenvalue functions \( \mathfrak{F}_n \) is dense in the \( C^* \)-algebra 
\( C_{b,u}(N_0^k, \rho_k) \) for any \( n = (n_1, \ldots, n_k) \in N_0^k \).

Proof. Let \( \sigma \in C_{b,u}(N_0^k, \rho_k) \) and let \( \epsilon > 0 \). By Lemma 4.2.3, \( \sigma' := \tau^{n-1}_R \sigma \in C_{b,u}(N_0^k, \rho_k) \).

Also \( \mathfrak{F}_1 \) is dense in \( C_{b,u}(N_0^k, \rho_k) \) by Theorem 5.2.12. Hence there exists \( a \in L^\infty(\mathbb{R}_+^k) \) such that \( \|\sigma' - \gamma_{1,a}\|_\infty < \epsilon \). Then

\[
\|\sigma - \gamma_{n,a}\|_\infty = \|\tau^{n-1}_L \sigma' - \tau^{n-1}_L \gamma_{1,a}\|_\infty \\
= \|\tau^{n-1}_L\| \|\sigma' - \gamma_{1,a}\|_\infty < \epsilon.
\]

as \( \|\tau^{n-1}_L\| < 1 \). □

It follows that the \( C^* \)-algebra generated by \( \mathfrak{F}_n \) is the \( C^* \)-algebra \( C_{b,u}(N_0^k, \rho_k) \).

5.4. The \( C^* \)-algebra \( C_{b,u}(N_0^1, \rho_1) \otimes \cdots \otimes C_{b,u}(N_0^1, \rho_1) \)

Denote by \( L^\infty(\mathbb{R}_+^k) \) the algebraic tensor product of \( k \)-many spaces of \( L^\infty(\mathbb{R}_+) \), which is contained in \( L^\infty(\mathbb{R}_+^k) \). We denote the closure of \( L^\infty(\mathbb{R}_+^1) \otimes \cdots \otimes L^\infty(\mathbb{R}_+^1) \) in \( L^\infty(\mathbb{R}_+^k) \) by \( L^\infty(\mathbb{R}_+^1) \otimes \cdots \otimes L^\infty(\mathbb{R}_+^1) \). The following proposition is a consequence of Corollary 3.4.9 and the density of \( \mathfrak{F}_1 \) in \( C_{b,u}(N_0^1, \rho_1) \). However, we present the details.

Proposition 5.4.1. Let

\[
\tilde{\mathfrak{F}}_n = \{\gamma_{n,a}| a \in L^\infty(\mathbb{R}_+^1) \otimes \cdots \otimes L^\infty(\mathbb{R}_+)\}.
\]

Then \( \tilde{\mathfrak{F}}_n \) is dense in \( C_{b,u}(N_0^1, \rho_1) \otimes \cdots \otimes C_{b,u}(N_0^1, \rho_1) \).

The proof of the above proposition is presented in the form of several lemmas.

Lemma 5.4.2. The set \( \tilde{\mathfrak{F}}_n \) is contained in \( C_{b,u}(N_0^1, \rho_1) \otimes \cdots \otimes C_{b,u}(N_0^1, \rho_1) \).
Proof. Let $a \in L^\infty(\mathbb{R}_+) \otimes \cdots \otimes L^\infty(\mathbb{R}_+)$ and let $\epsilon > 0$. Then there exists $b \in L^\infty(\mathbb{R}_+) \otimes \cdots \otimes L^\infty(\mathbb{R}_+)$ s.t.

$$\|a - b\|_\infty < \epsilon.$$  

Then for any $m \in \mathbb{N}_0^k$,

$$|\gamma_{n,a}(m) - \gamma_{n,b}(m)| \leq \frac{1}{(m + n - 1)!} \int_{\mathbb{R}_+^k} |a(\sqrt{r}) - b(\sqrt{r})| r^{m+n-1} e^{-(r_1+\cdots+r_k)} dr_1 \cdots dr_k$$

$$\leq \frac{1}{(m + n - 1)!} \|a - b\|_\infty \int_{\mathbb{R}_+^k} r^{m+n-1} e^{-(r_1+\cdots+r_k)} dr_1 \cdots dr_k$$

$$= \|a - b\|_\infty$$

and hence

$$\|\gamma_{n,a} - \gamma_{n,b}\|_\infty < \epsilon.$$  

Also, by Corollary 3.4.9, $\gamma_{n,b} \in C_{b,u}(\mathbb{N}_0) \otimes \cdots \otimes C_{b,u}(\mathbb{N}_0)$. Hence $\gamma_{n,a}$ is in $C_{b,u}(\mathbb{N}_0, \rho_1) \otimes \cdots \otimes C_{b,u}(\mathbb{N}_0, \rho_1)$. \hfill \Box

In order to show density, we start with the following lemma.

**Lemma 5.4.3.** Let $a_j, b_j \in \mathbb{C}$, $j = 1, \ldots, k$. Then for $m \in \mathbb{N}_0^k$

$$\left| \prod_{j=1}^{k} a_j - \prod_{j=1}^{k} b_j \right| \leq \sum_{j=1}^{k} |a_j - b_j| \prod_{s=1}^{j-1} |a_s| \prod_{s=j+1}^{k} |b_s|$$

**Proof.** To use induction on $k$, assume the inequality is true for $k$. Then by triangle inequality and the assumption,
\[ \left| \prod_{j=1}^{k+1} a_j - \prod_{j=1}^{k+1} b_j \right| \leq |a_{k+1} - b_{k+1}| \prod_{j=1}^{k} |a_j| + |b_{k+1}| \prod_{j=1}^{k} |a_j| - \prod_{j=1}^{k} b_j \]

\[ \leq |a_{k+1} - b_{k+1}| \prod_{j=1}^{k} |a_j| + \sum_{j=1}^{k} |a_j - b_j| \prod_{s=1}^{j-1} |a_s| \prod_{s=j+1}^{k+1} |b_s| \]

\[ = \sum_{j=1}^{k+1} |a_j - b_j| \prod_{s=1}^{j-1} |a_s| \prod_{s=j+1}^{k+1} |b_s| \]

as needed. \( \square \)

**Lemma 5.4.4.** Let \( \sigma \in C_{b,u}(\mathbb{N}_0, \rho_1) \otimes \cdots \otimes C_{b,u}(\mathbb{N}_0, \rho_1) \) and let \( \epsilon > 0 \). Then there exists \( a \in L^\infty(\mathbb{R}_+^k) \) s.t.

\[ \| \sigma - \gamma_n,a \|_\infty < \epsilon. \]

**Proof.** Then there exists \( \tilde{\sigma} \in C_{b,u}(\mathbb{N}_0) \otimes \cdots \otimes C_{b,u}(\mathbb{N}_0) \) s.t.

\[ \| \tilde{\sigma} - \sigma \|_\infty < \epsilon. \]

Then there exists \( \tilde{\sigma} \) can be written as a finite sum of simple tensors:

\[ \tilde{\sigma} = \sum_{i=1}^{N} g_{i,1} \otimes \cdots \otimes g_{i,k} \]

where \( g_{i,j} \in C_{b,u}(\mathbb{N}_0) \) and \( N \in \mathbb{N} \). Recall that the simple tensors \( g_{i,1} \otimes \cdots \otimes g_{i,k} \) are given by

\[ g_{i,1} \otimes \cdots \otimes g_{i,k}(m) = g_{i,1}(m_1) \cdots g_{i,k}(m_k), \quad m \in \mathbb{N}_0^k. \]

By the density of \( \mathfrak{S}(n) \) in \( C_{b,u}(\mathbb{N}_0, \rho_1) \), choose \( a_{i,j} \in L^\infty(\mathbb{R}_+^k) \) through induction on \( j \) s.t.

\[ \| g_{i,j} - \gamma_{n,a_{i,j}} \|_\infty < \frac{\epsilon}{2kN \prod_{s=1}^{j-1} \| a_{i,s} \|_\infty \prod_{s=j+1}^{k} \| g_s \|_\infty} \]

60
and hence by Lemma 5.4.3

\[ |g_{i,1} \otimes \cdots \otimes g_{i,k}(m) - \gamma_{n,a_{i,1}} \otimes \cdots \otimes \gamma_{n,a_{i,k}}(m)| \]

\[ \leq \sum_{j=1}^{k} |g_{i,j}(m_j) - \gamma_{n,a_{i,j}}(m_j)| \prod_{s=1}^{j-1} \|\gamma_{n,a_{i,s}}\|_\infty \prod_{s=j+1}^{k} \|g_s\|_\infty \]

\[ \leq \frac{\varepsilon}{2N}. \]

Define the symbol \( a \in L^\infty(\mathbb{R}_k^+) \) by

\[ a = \sum_{i=1}^{N} a_{i,1} \otimes \cdots \otimes a_{i,k}. \]

Then by Corollary 3.4.9 and the linearity of \( a \mapsto \gamma_{n,a} \)

\[ \gamma_{n,a} = \sum_{i=1}^{N} \gamma_{n,a_{i,1}} \otimes \cdots \otimes \gamma_{n,a_{i,k}} \]

and hence by the triangle inequality

\[ \|\tilde{\sigma} - \gamma_{n,a}\| < \frac{\varepsilon}{2}. \]

Therefore

\[ \|\sigma - \gamma_{n,a}\|_\infty < \|\sigma - \tilde{\sigma}\|_\infty + \|\tilde{\sigma} - \gamma_{n,a}\|_\infty \]

\[ < \varepsilon. \]

The proof of Proposition 5.4.1 is complete by lemmas, 5.4.2 and 5.4.4. \( \square \)

Above proposition warrants a comparison of \( C_{b,u}(\mathbb{N}_0, \rho_1) \otimes \cdots \otimes C_{b,u}(\mathbb{N}_0, \rho_1) \) with \( C_{b,u}(\mathbb{N}_0^k, \rho_k) \). And from the discussion in Section 4.3, we have that the inclusion

\[ C_{b,u}(\mathbb{N}_0, \rho_1) \otimes \cdots \otimes C_{b,u}(\mathbb{N}_0, \rho_1) \hookrightarrow C_{b,u}(\mathbb{N}_0^k, \rho_k) \]

could be proper.
Chapter 6. Future Research Directions

6.1. Separately-radial Toeplitz operators on the Fock space

The Bergman space over unit ball, denote by $A^2(\mathbb{B}^n)$ is the space analytic functions on $\mathbb{B}^n$ that are square-integrable with respect to the normalized Lebesgue measure on $\mathbb{B}^n$. The spectra of Toeplitz operators on the Bergman space with separately radial symbols were computed in [Q16]:

$$\gamma_a(m) = \frac{(2 + |m|)!}{m!} \int_\Delta a(r) r^m dr, \quad m \in \mathbb{N}_0^n$$

where $\Delta := \{ r \in \mathbb{R}_+^n \mid r_1 + \cdots + r_m < 1 \}$ and $a \in L^\infty(\Delta)$.

It is known that the $C^*$-algebra generated by radial Toeplitz operators is isometrically isomorphic to the space of slowly-oscillating sequences (bounded sequences that are uniformly continuous with respect to the logarithmic metric). However it is still an open problem to understand the $C^*$-algebra generated by Toeplitz operators with separately-radial symbols as a space of uniformly continuous functions.

6.2. Toeplitz operators with dilation-invariant symbols

The underlying groups for radial and quasi-radial symbols, are unitary groups and products of unitary groups which are subgroups of the metaplectic group. And the Fock model of the metaplectic representation acts on the Fock space $\mathcal{F}(\mathbb{C}^n)$ (see [F89]). For unitary groups, the action of the metaplectic representation on $\mathcal{F}(\mathbb{C}^n)$ is simplified to

$$\pi(A)f(z) = f(A^{-1}z).$$
We consider another subgroup of the metaplectic group, $\mathbb{R}^*$. Then $\mathbb{R}^*$ acts on $\mathcal{F}(\mathbb{C}^n)$ via the metaplectic representation $\nu$ given by

$$\nu(t)f(z) = \int_{\mathbb{C}^n} f(w) K_t(z, \bar{w}) d\lambda(w), \quad z \in \mathbb{C}^n, \quad t \in \mathbb{R}^*$$

$$K_t(z, \bar{w}) = A_t e^{B_t z^2 + C_t \bar{z} \bar{w} - B_t \bar{w}^2}$$

$$A_t = \left( \frac{2t}{t^2 + 1} \right)^{\frac{n}{2}}, \quad B_t = \frac{t^2 - 1}{2(t^2 + 1)}, \quad C_t = \frac{2t}{t^2 + 1}.$$

We say a symbol $\varphi$ is $\mathbb{R}^*$-invariant if the corresponding Toeplitz operator intertwines with the representation $\nu$. Note that in this case, the $\mathbb{R}^*$-invariance of symbols is not defined through an action of the group on the domain $\mathbb{C}^n$, but rather by forcing the Toeplitz operators to intertwine with the representation $\nu$. Moreover, the representation $\nu$ cannot be written as

$$\nu(t)f(z) = j(t^{-1}, z)f(t^{-1}z)$$

where $j$ is a cocycle. Therefore this example does not fit into the framework discussed in [DÓQ15]; this makes the problem far more interesting. Also, understanding the class of symbols is complicated problem.
Appendix. Copyright Information

Copyright information for Vishwa Dewage and Gestur Ólafsson (for chapters 3-5)

<table>
<thead>
<tr>
<th>Licence to Publish</th>
<th>Springer Nature</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Licensee:</strong></td>
<td>Springer Nature Switzerland AG</td>
</tr>
<tr>
<td><strong>Journal Name:</strong></td>
<td>Complex Analysis and Operator Theory</td>
</tr>
<tr>
<td><strong>Manuscript Number:</strong></td>
<td>CAOT-D-21-00239RL</td>
</tr>
<tr>
<td><strong>Proposed Title of Article:</strong></td>
<td>Toeplitz Operators on the Fock Space with Quasi-Radial Symbols</td>
</tr>
<tr>
<td><strong>Authors:</strong></td>
<td>Vishwa Dewage, Gestur Ólafsson</td>
</tr>
<tr>
<td><strong>Corresponding Author Name:</strong></td>
<td>Vishwa Dewage</td>
</tr>
</tbody>
</table>

1 Grant of Rights

   a) For good and valuable consideration, the Author hereby grants to the Licensee the perpetual, exclusive, world-wide, assignable, sublicensable and unlimited right to: publish, reproduce, copy, distribute, communicate, display publicly, sell, rent and/or otherwise make available the article identified above, including any supplementary information and graphic elements therein (e.g. illustrations, charts, moving images) (the “Article”) in any language, in any versions or editions in any and all forms and/or media of expression (including without limitation in connection with any and all end-user devices), whether now known or developed in the future. Without limitation, the above grant includes: (i) the right to edit, alter, adapt, adjust and prepare derivative works; (ii) all advertising and marketing rights including without limitation in relation to social media; (iii) rights for any training, educational and/or instructional purposes; (iv) the right to add and/or remove links or combinations with other media/works; and (v) the right to create, use and/or license and/or sublicense content data or metadata of any kind in relation to the Article (including abstracts and summaries) without restriction. The above rights are granted in relation to the Article as a whole or any part and with or in relation to any other works.

   b) Without limiting the rights granted above, Licensee is granted the rights to use the Article for the purposes of analysis, testing, and development of publishing and research-related workflows, systems, products, projects, and services; to confidentially share the Article with select third parties to do the same; and to retain and store the Article and any associated correspondence/file forms to maintain the historical record, and to facilitate research integrity investigations. The grant of rights set forth in this clause (b) is irrevocable.

   c) The Licensee will have the right, but not the obligation, to exercise any or all of the rights granted herein. If the Licensee elects not to publish the Article for any reason, all publishing rights under this Agreement as set forth in clause 1.a) above will revert to the Author.

2 Copyright

Ownership of copyright in the Article will be vested in the name of the Author. When reproducing the Article or extracts from it, the Author will acknowledge and reference first publication in the journal.

3 Use of Article Versions

   a) For purposes of this Agreement: (i) references to the “Article” include all versions of the Article; (ii) “Submitted Manuscript” means the version of the Article as first submitted by the Author; (iii) “Accepted Manuscript” means the version of the Article accepted for publication, but prior to copy-editing and typesetting; and (iv) “Version of Record” means the version of the Article published by the Licensee, after copy-editing and typesetting. Rights to all versions of the Manuscript are granted on an exclusive basis, except for the Submitted Manuscript, to which rights are granted on a non-exclusive basis.

   b) The Author may make the Submitted Manuscript available at any time and under any terms (including, but not limited to, under a CC BY licence), at the Author’s discretion. Once the Article has been published, the Author will include an acknowledgement and provide a link to the Version of Record on the publisher’s website: “This preprint has not undergone peer review (when applicable) or any post-submission improvements or corrections. The Version of Record of this article is published in [insert journal title], and is available online at [insert DOI]”.

   c) The Licensee grants to the Author (i) the right to make the Accepted Manuscript available on their personal, self-maintained website immediately on acceptance. (ii) the right to make the Accepted Manuscript available for public release on any of the following twelve (12) months after first publication (the “Embargo Period”); their employer’s internal website; their institutional and/or funder repositories. Accepted Manuscripts may be deposited in such repositories immediately upon acceptance, provided they are not made publicly available until after the Embargo Period.
The rights granted to the Author with respect to the Accepted Manuscript are subject to the conditions that (i) the Accepted Manuscript is not enhanced or substantially reformatted by the Author or any third party, and (ii) the Author includes on the Accepted Manuscript an acknowledgement in the following form, together with a link to the published version on the publisher's website: "This version of the article has been accepted for publication, after peer review (when applicable) but is not the Version of Record and does not reflect post-acceptance improvements, or any corrections. The Version of Record is available online at: http://dx.doi.org/[insert DOI]. Use of this Accepted Version is subject to the publisher's Accepted Manuscript terms of use https://www.springernature.com/gp/open-research/policies/accepted-manuscript-terms". Under no circumstances may an Accepted Manuscript be shared or distributed under a Creative Commons or other form of open access licence.

d) The Licensee grants to the Author the following non-exclusive rights to the Version of Record, provided that, when reproducing the Version of Record or extracts from it, the Author acknowledges and references first publication in the Journal according to current citation standards. As a minimum, the acknowledgement must state: “First published in [journal name, volume, page number, year] by Springer Nature”.

i. to reuse graphic elements created by the Author and contained in the Article, in presentations and other works created by them;

ii. the Author and any academic institution where they work at the time may reproduce the Article for the purpose of course teaching (but not for inclusion in course pack material for onward sale by libraries and institutions);

iii. to reuse the Version of Record or any part in a thesis written by the same Author, and to make a copy of that thesis available in a repository of the Author(s)’ awarding academic institution, or other repository required by the awarding academic institution. An acknowledgement should be included in the citation: “Reproduced with permission from Springer Nature”;

iv. to reproduce, or to allow a third party to reproduce the Article, in whole or in part, in any other type of work (other than thesis) written by the Author for distribution by a publisher after an embargo period of 12 months.

4 Warranties & Representations

Author warrants and represents that:

a) i. the Author is the sole copyright owner or has been authorised by any additional copyright owner(s) to grant the rights defined in clause 1,

ii. the Article does not infringe any intellectual property rights (including without limitation copyright, database rights or trade mark rights) or other third party rights and no licence from or payments to a third party are required to publish the Article,

iii. the Article has not been previously published or licensed, nor has the Author committed to licensing any version of the Article under a licence inconsistent with the terms of this Agreement,

iv. if the Article contains materials from other sources (e.g. illustrations, tables, text quotations), Author has obtained written permissions to the extent necessary from the copyright holder(s), to license to the Licensee the same rights as set out in clause 1 but on a non-exclusive basis and without the right to use any graphic elements on a stand-alone basis and has cited any such materials correctly;

b) all of the facts contained in the Article are according to the current body of research true and accurate;

c) nothing in the Article is obscene, defamatory, violates any right of privacy or publicity, infringes any other human, personal or other rights of any person or entity or is otherwise unlawful and that informed consent to publish has been obtained for any research participants;

d) nothing in the Article infringes any duty of confidentiality owed to any third party or violates any contract, express or implied, of the Author;

e) all institutional, governmental, and/or other approvals which may be required in connection with the research reflected in the Article have been obtained and continue in effect;

f) all statements and declarations made by the Author in connection with the Article are true and correct; and
g) the signatory who has signed this agreement has full right, power and authority to enter into this agreement on behalf of all of the Authors.

5 Cooperation

a) The Author will cooperate fully with the Licensee in relation to any legal action that might arise from the publication of the Article, and the Author will give the Licensee access at reasonable times to any relevant accounts, documents and records within the power or control of the Author. The Author agrees that any Licensee affiliate through which the Licensee exercises any rights or performs any obligations under this Agreement is intended to have the benefit of and will have the right to enforce the terms of this Agreement.

b) Author authorises the Licensee to take such steps as it considers necessary at its own expense in the Author’s name(s) and on their behalf if the Licensee believes that a third party is infringing or is likely to infringe copyright in the Article including but not limited to initiating legal proceedings.

6 Author List

Changes of authorship, including, but not limited to, changes in the corresponding author or the sequence of authors, are not permitted after acceptance of a manuscript.

7 Post Publication Actions

The Author agrees that the Licensee may remove or retract the Article or publish a correction or other notice in relation to the Article if the Licensee determines that such actions are appropriate from an editorial, research integrity, or legal perspective.

8 Controlling Terms

The terms of this Agreement will supersede any other terms that the Author or any third party may assert apply to any version of the Article.

9 Governing Law

This Agreement will be governed by, and construed in accordance with, the laws of Switzerland. The courts of Zug, Switzerland will have exclusive jurisdiction.

Signed for and on behalf of the Author(s)
Corresponding Author: Vishwa Dewage
Email: vdwag1@isu.edu
IP Address: 
Time Stamp: 2022-04-24 05:21:34
Bibliography


Vita

Vishwa Dewage was born in a coastal city in southern Sri Lanka and grew up in Colombo. Her interest in mathematics grew as she started thinking of mathematics as art as much as it is science. She completed her bachelor’s degree in mathematics at University of Colombo in 2016. Upon her graduation, she was an assistant lecturer at University of Colombo, briefly, before coming to Louisiana State University to pursue a doctorate in mathematics. She plans to graduate in August, 2022 and join Clemson University as a postdoctoral researcher.