7-12-2022

On a Relation Between ADO and Links-Gould Invariants

Nurdin Takenov
Louisiana State University and Agricultural and Mechanical College

ON A RELATION BETWEEN ADO AND LINKS-GOULD INVARIANTS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Nurdin Takenov
B.S., Moscow Institute of Physics and Technology, 2008
M.S., University of Southern California, 2014
August 2022

This dissertation is dedicated to my mother.

Below, the whole Universe twinkled at Rincewind. There was Great A'Tuin, huge and ponderous and pocked with craters. There was the little Disc moon. There was a distant gleam that could only be the Potent Voyager. And there were all the stars, looking remarkably like powdered diamonds spilled on black velvet, the stars that lured and ultimately called the boldest towards them...

—Terry Pratchett
The Colour of Magic

Acknowledgments

I would like to thank my advisor, Professor Oliver Dasbach, for guiding my research and patience.

I would also like to thank Professor Vela-Vick, for supporting my research and providing a lot of motivation.

I would also like to thank Ignat Soroko for very helpful discussions about algebraic questions related to the research.

Finally I would like to thank my classmates, especially Tamanna and Shashika, without their help and company, life would be much harder.

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Abstract

In this thesis we consider two knot invariants: Akutsu-Deguchi-Ohtsuki(ADO) invariant and Links-Gould invariant. They both are based on Reshetikhin-Turaev construction and as such share a lot of similarities. Moreover, they are both related to the Alexander polynomial and may be considered generalizations of it. By experimentation we found that for many knots, the third order ADO invariant is a specialization of the Links-Gould invariant. The main result of the thesis is a proof of this relation for a large class of knots, specifically closures of braids with five strands.

Chapter 1. Introduction

Here we will introduce the main concepts.

1.1. Knots and links

Definition 1.1.1. *A **knot** is a piecewise-linear embedding of S^1 into $\mathbb{R}^3 \subset S^3$. A **link** with n components is a piecewise-linear embedding of n disjoint copies of S^1 into $\mathbb{R}^3 \subset S^3$.*

Note, that knots are special case of links, links with one component. We will consider oriented knots (respectively links), that is two knots with different orientations would be considered as different knots.

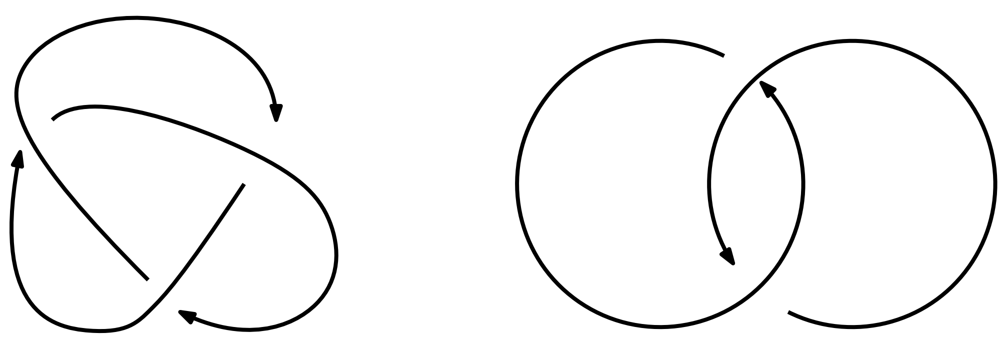

Figure 1.1. Examples of a knot and a link. On the left is trefoil - an example of a knot. On the right - a Hopf link, with two components.

Definition 1.1.2. *Two knots(links) L_1 and L_2 are **equivalent** if there is a piecewise linear orientation-preserving homeomorphism $h : S^3 \to S^3$, such that $h(L_1) = L_2$.*

Definition 1.1.3. *Let $\mathbb{L}$ be the set of links in S^3. We say that the function $f : \mathbb{L} \to P$ is a **link invariant** if $f(L_1) = f(L_2)$ for any pair of equivalent links L_1 and L_2.*

From now on, we will consider equivalent knots to be equal.

Definition 1.1.4. *A knot (link) projection is called **regular** if no three points are projected to the same point and no endpoint of a linear segment is projected to the same point as any other point on the knot(link). A knot(link) projection with specified under-strands*

*and over-strands is a **knot(link) diagram**.*

We can see link diagrams on a picture above, for a trefoil and a Hopf link. Of course, sometimes different knot(link) diagrams represent the same knot(link). We can consider the following **Reidemeister** moves, which preserve equivalence class of a knot(link).

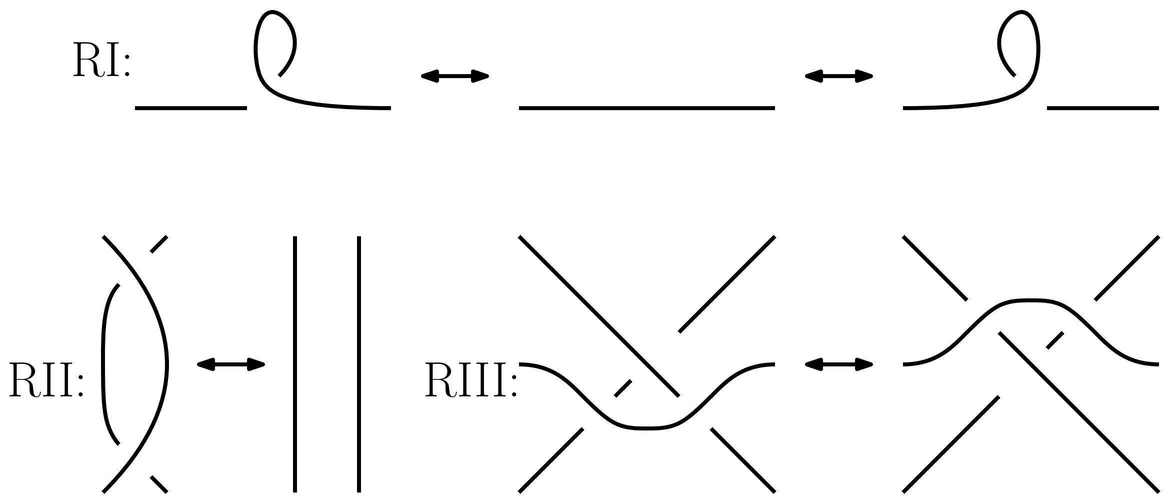

Figure 1.2. Reidemeister moves RI, RII and RIII.

Theorem 1.1.5. ***Reidemeister theorem****(e.g. [13]). Two link diagrams correspond to equivalent links if and only if one can be obtained from the other by a finite sequence of Reidemeister moves and plane isotopies.*

Now we need to introduce closely related concept of a braid.

Definition 1.1.6. *A **braid** on n strands is a disjoint union of n strands embedded in* $\mathbb{R}^2 \times [0,1]$ *such that*

1) Each strand is ascending monotonically,

2) The endpoints of strands correspond to the points $(i,0,0)$ and $(i,0,1)$.

Two braids are equivalent if they are related by an isotopy of $\mathbb{R}^2 \times [0,1]$ *that preserves the vertical coordinate and its boundary. We consider braids up to equivalence.*

The set of equivalence classes of braids in n strands is denoted as B_n. It can be given a group structure, the product of two braids β and γ being the braid γ put on top of braid β. Let us consider elementary braids σ_i, $1 \leq i \leq n-1$.

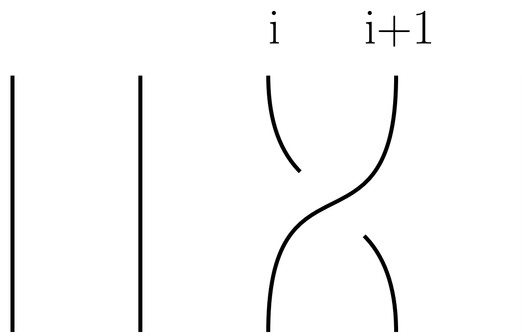

Figure 1.3. Elementary braid σ_i.

Theorem 1.1.7. ***Artin's theorem*** *(e.g. [13]). The braid group B_n is generated by the elementary braids σ_1, ..., σ_{n-1} that satisfy the relations:*

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \ \text{if } |i-j| \geq 2$$

The braids are important for us because they allow us to represent links. We will define the **closure** of a braid: take representation of a braid and connect upper and lower endpoints of strands as shown in the figure 1.4.

Theorem 1.1.8. ***Alexander's braiding theorem*** *(e.g. [13]). Any link(in particular any knot) is the closure of some braid.*

Any link can be represented as the closure of infinitely different braids. The next theorem tells us when two braids correspond to the same link:

Theorem 1.1.9. ***Markov's theorem*** *(e.g. [13]). The closures of two braids are equivalent if and only if one braid can be taken to another by a finite sequence of Markov moves:*

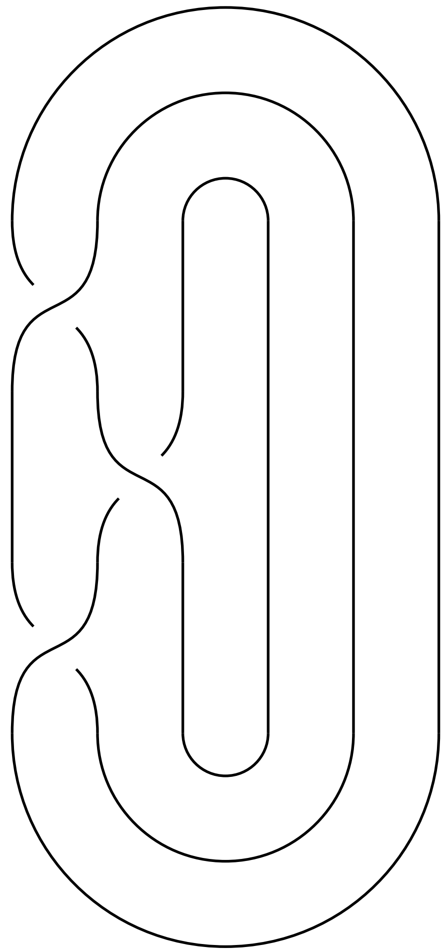

Figure 1.4. Braid closure of a braid $\sigma_1\sigma_2^{-1}\sigma_1$.

1) $\beta\gamma \leftrightarrow \gamma\beta$, *where* $\beta, \gamma \in B_n$

2) $\beta \leftrightarrow \beta\sigma_n^{\pm 1}$, *where* $\beta \in B_n$ *and* $\beta\sigma_n^{\pm 1} \in B_{n+1}$.

1.2. Reshetikhin-Turaev construction

The Reshetikhin-Turaev construction is a powerful tool for constructing link invariants. All link invariants in this work are variations of this construction.

Let us pick a vector space V and its dual V^*, with a ground field k. We define elementary tangles and associated linear maps as shown in Figure 1.5.

Then, for a given link, we consider a **sliced diagram** - i.e. a diagram of a knot, with a set of horizontal lines that slice it into a finite number of elementary tangles. Any

link has a sliced diagram. Example of a sliced diagram is given in Figure 1.6. To each sliced diagram we associate a sequence of linear maps, generated by elementary tangles.

As we can see, eventually we have a linear map, $f : k \to k$, which has to be a linear map, $f(x) = cx$. With a proper choice of linear maps for elementary tangles, the value of this scalar c would be constant under Reidemeister moves and thus be a link invariant. For example, in order to satisfy Reidemeister move RII, the maps shown in Figure 1.7 should be inverses of each other.

A full set of conditions can be found in [12], Chapters 3 and 4. Eventually, it turns out that to have a link invariant we need to specify a map $R : V \otimes V \to V \otimes V$ and the map $h : V \to V$, where R is the map on the left in Figure 1.7, and h specifies the other map in Figure 1.5. The maps, R and h should satisfy several conditions, the most important of them is the Yang-Baxter equation:

$$(\mathrm{Id}_V \otimes R)(R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R) = (R \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes R)(R \otimes \mathrm{Id}_V)$$

which stems from the invariance of a link invariant under the Reidemeister move RIII. Invariants obtained throught Reshetikhin-Turaev construction are called operator invariants.

1.3. Alexander polynomial

The Alexander polynomial is a classical knot invariant, with many equivalent definitions, using covering spaces, skein relations, etc. Here we will however define the Alexander polynomial through the Reshetikhin-Turaev construction. We need to slightly modify the definition. Instead of regular knots, we will be using long knots, that is knots with one strand going up to infinity and another strand going down to infinity. An example of a long knot is shown in Figure 1.8.

In this modification, the overall linear map would be $f : V \to V$, but given a proper choice of maps it would be a scalar map $f(v) = cv$ and the scalar c would be a link invariant. For the Alexander invariant the proper choice of V, R and h is the following. The vector space V is a two-dimensional vector space with basis $\{e_0, e_1\}$. The maps R and h are given by the matrices

$$R = \begin{pmatrix} \frac{1}{\sqrt{t}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{\sqrt{t}} - \sqrt{t} & 0 \\ 0 & 0 & 0 & -\sqrt{t} \end{pmatrix}, h = \begin{pmatrix} \sqrt{t} & 0 \\ 0 & -\sqrt{t} \end{pmatrix}$$

in the bases $\{e_0 \otimes e_0, e_0 \otimes e_1, e_1 \otimes e_0, e_1 \otimes e_1\}$ and $\{e_0, e_1\}$ respectively. We will denote the Alexander polynomial for a link L as $\Delta_L(t)$. For example, for a trefoil we have $\Delta_K(t) = t - 1 + t^{-1}$.

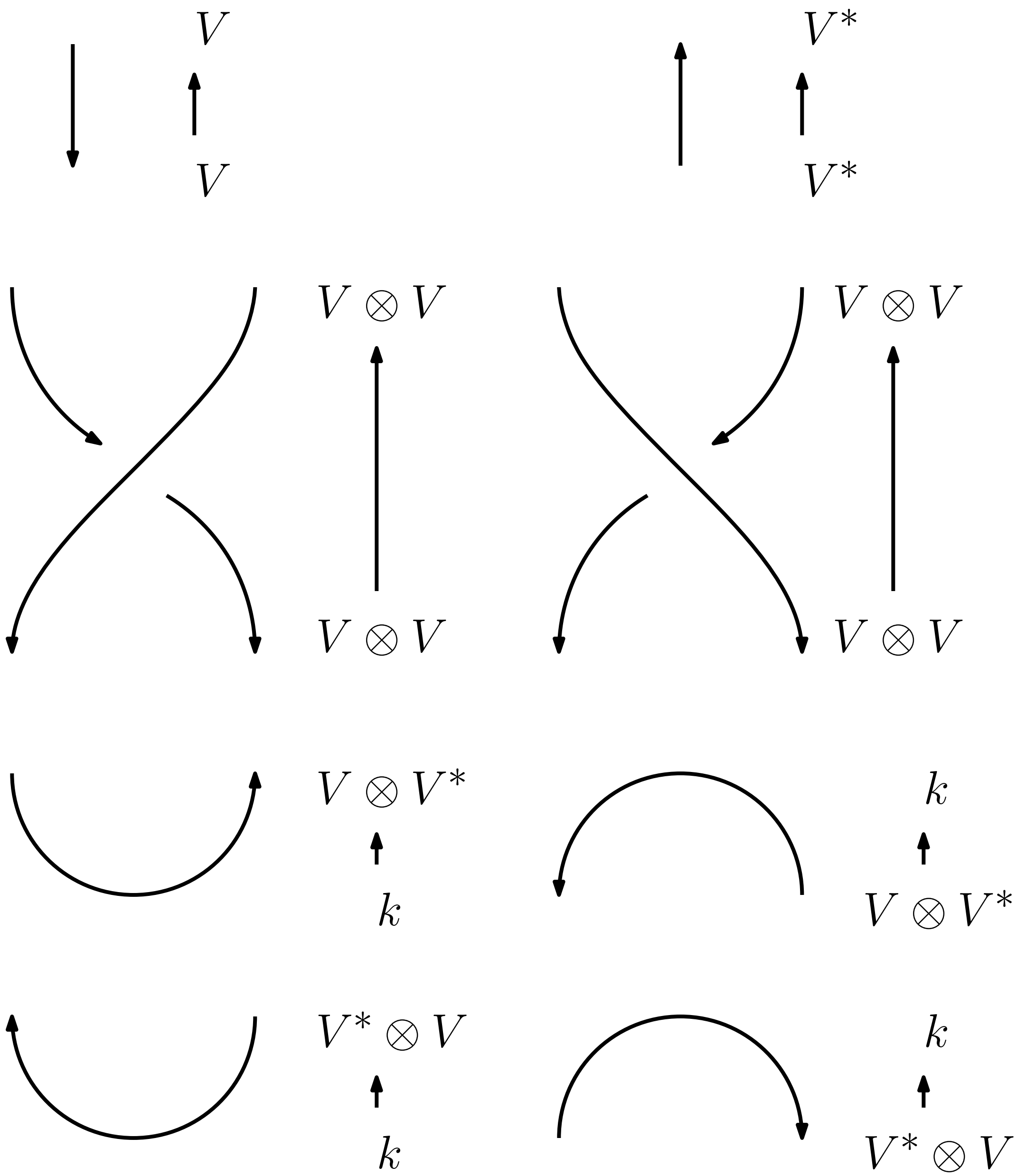

Figure 1.5. Elementary tangles and associated maps.

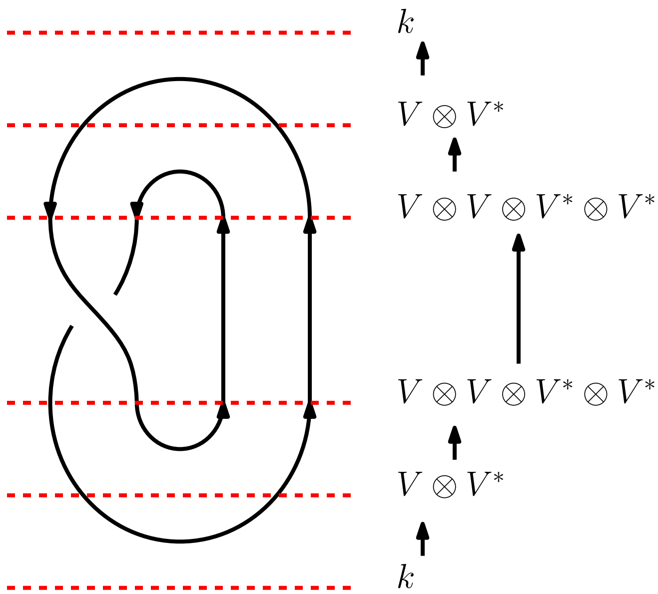

Figure 1.6. Example of a sliced diagram.

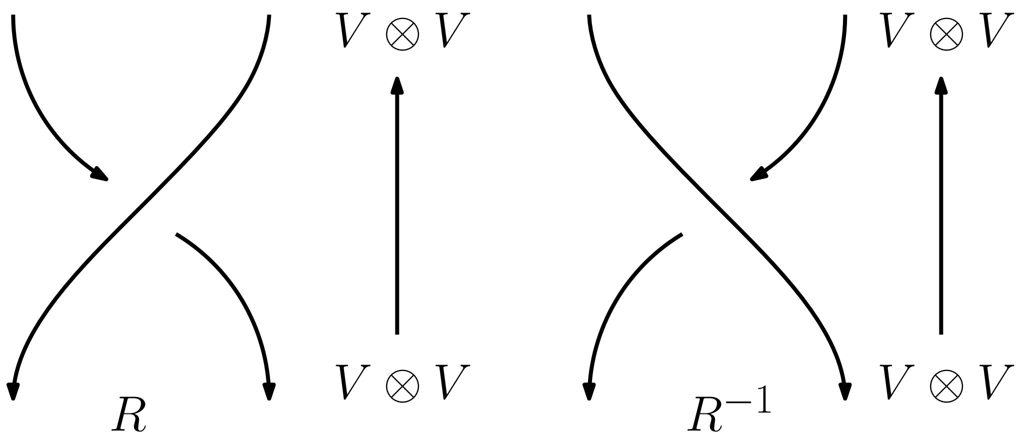

Figure 1.7. R and inverse map R^{-1}.

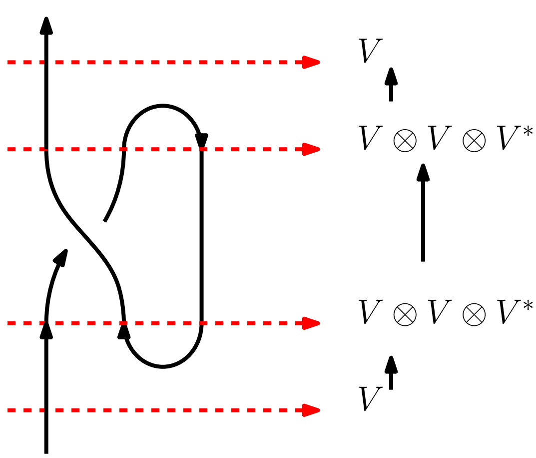

Figure 1.8. Long knot and associated linear maps.

Chapter 2. ADO and Links-Gould Invariants

Both ADO and Links-Gould invariants are closely related to the Alexander invariant and share many properties with it. Here we will describe them and their properties.

2.1. ADO invariant

ADO invariant, also known as Colored Alexander polynomial, was defined by Akustu, Deguchi and Ohtsuki in 1992. We will follow the description given by Jun Murakami in 2008 [11](with some modifications). The ADO invariants form a sequence, $ADO_2, ADO_3, ADO_4, \ldots$. Here, N is a dimension of the vector space in Reshetikhin-Turaev construction. When $N = 2$, we obtain the regular Alexander polynomial (up to some change of variable).

To get the R-matrix for the ADO invariants we need to use the universal R-matrix. Consider the quantum algebra

$$U_q(sl(2)) = \Big\langle K, K^{-1}, E, F \Big| KK^{-1} = K^{-1}K = 1, KEK^{-1} = q^2E, \\ KFK^{-1} = q^{-2}F, [E,F] = \frac{K - K^{-1}}{q - q^{-1}} \Big\rangle$$

Pick a finite-dimensional representation of $U_q(sl(2))$. Then the R-matrix is given by the formula (here $\{a\} = q^a - q^{-a}$):

$$R(x \otimes y) = q^{\frac{1}{2}H \otimes H} \sum_{n=0}^{\infty} \frac{\{1\}^{2n}}{\{n;n\}} q^{n(n-1)/2} (E^n \otimes F^n)(y \otimes x)$$

To obtain the ADO-invariants we need to fix $q = \exp(\pi i/N)$ and consider the fol-

lowing N-dimensional representation of $U_q(sl(2))$:

$$Ev_i = \frac{\{\lambda - i + 1\}}{\{1\}} v_{i-1}, Ev_0 = 0;$$
$$Fv_i = \frac{\{i+1\}}{\{1\}} v_{i+1}, Fv_{N-1} = 0;$$
$$Kv_i = q^{\lambda - 2i} v_i$$

Thus we can obtain explicit formulas for the R-matrix for the ADO_N invariant:

$$R(v_i \otimes v_j) = \tag{2.1}$$
$$\sum_n q^{\frac{1}{2}(\lambda-2i-2n)(\lambda-2j+2n)+n(n-1)/2} \frac{\{i+n;n\}\{\lambda-j+n;n\}}{\{n;n\}} v_{j-n} \otimes v_{i+n}$$

where

$$\{x;n\} = \begin{cases} \prod_{i=0}^{n-1} \{x-i\}, n > 0 \\ 1, n = 0. \end{cases}$$

The map h is then given by $h(v_i) = q^{(N-1)\lambda+2i} v_i$. The construction described above gives a framed knot invariant. For a convenience, we will use the modified R-matrix, which doesn't fundamentally change the invariant, but helps to avoid problems with framing corrections:

$$R(v_i \otimes v_j) = \tag{2.2}$$
$$q^{(N-1)\lambda-(i+j)\lambda} \sum_n q^{2(i+n)(j-n)+n(n-1)/2} \frac{\{i+n;n\}\{\lambda-j+n;n\}}{\{n;n\}} v_{j-n} \otimes v_{i+n}.$$

Then if a link L is a closure of a $(1,1)$ tangle T, and $O_T^N(\lambda)$ is the corresponding operator, the ADO-invariant is defined by

$$ADO_N(L;\lambda)\mathrm{Id}_V = O_T^N(\lambda).$$

This is the version that we will be using from now on. $ADO_N(L;\lambda)$ is a Laurent polynomial in q and q^λ, so we will work instead with the variable $t = q^\lambda$. We can note that $ADO_N(L;\lambda)$ takes values in $\mathbb{Z}[q, t^{\pm 1}]$. Here are examples of ADO-invariants:

$$
\begin{aligned}
ADO_2(3_1, t) &= t^2 + \frac{1}{t^2} - 1 \\
ADO_3(3_1, t) &= \frac{1}{2} i \left(\sqrt{3} + i\right) t^4 + \frac{1}{t^4} + t^2 + \frac{i\left(\sqrt{3}+i\right)}{2t^2} - i\sqrt{3} \\
ADO_4(3_1, t) &= -t^6 + \frac{1}{t^6} + it^4 + \frac{i}{t^4} + (1+i)t^2 - \frac{1+i}{t^2} + (1-2i) \\
ADO_2(4_1, t) &= -t^2 - \frac{1}{t^2} + 3
\end{aligned}
$$

2.2. Links-Gould invariant

First Links-Gould invariant $LG(L; t_0, t_1)$ was introduced in 1992 by Links and Gould [7] and more thoroughly studied by De Wit, Kauffman and Links in 1999 [3]. It is the first in a series of Links-Gould invariants $LG^{n,m}$ and denoted as $LG^{2,1}$ to distinguish it from others. For example $LG^{1,1}$ is Alexander polynomial [14]. We will however only work with the first Links-Gould invariant, so we will refer to it just as LG. We can also note that Links-Gould polynomial takes values in $\mathbb{Z}[t_0^{\pm 1}, t_1^{\pm 1}]$ (Theorem 1, [5]).

In case of a Lanks-Gould polynomial, the vector space W is 4-dimensional, with basis $\{e_0, e_1, e_2, e_3\}$. The map h is given by the matrix

$$
\begin{pmatrix}
\frac{1}{t_0} & 0 & 0 & 0 \\
0 & -t_1 & 0 & 0 \\
0 & 0 & -\frac{1}{t_0} & 0 \\
0 & 0 & 0 & t_1
\end{pmatrix},
$$

To write the R-matrix for Links-Gould polynomial we denote $Y = \sqrt{(t_0 - 1)(1 - t_1)}$. Then the R-matrix for Links-Gould polynomial is

$$
\begin{pmatrix}
t_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{t_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{t_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & \sqrt{t_0} & 0 & 0 & t_0-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & t_0t_1-1 & 0 & 0 & -\sqrt{t_0t_1} & 0 & 0 & -\sqrt{t_0t_1}Y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{t_1} & 0 & 0 \\
0 & 0 & \sqrt{t_0} & 0 & 0 & 0 & 0 & 0 & t_0-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{t_0t_1} & 0 & 0 & 0 & 0 & 0 & Y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{t_1} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -\sqrt{t_0t_1}Y & 0 & 0 & Y & 0 & 0 & Y^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{t_1} & 0 & 0 & 0 & 0 & 0 & t_1-1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{t_1} & 0 & 0 & t_1-1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1
\end{pmatrix}
$$

Then the Links-Gould polynomial is calculated according to general description given above for operator invariants.

The Links-Gould invariant has close relation to the Alexander invariant [6]:

$$LG(L;t,-1) = \left(\Delta_L(t^2)\right)^2.$$

2.3. Common properties of ADO and Links-Gould invariants

In this section we will list common properties of ADO and Links-Gould invariants, some of which we will use to prove the main result.

2.3.1. Cubic skein relation

By direct computation, it can be checked that R_{ADO-3}, R-matrix for ADO polynomial when $N = 3$ satisfied the cubic relation

$$R^3 = -\omega^2 \mathrm{Id}_9 + R^2\left(\frac{\omega^2}{t^2} - 1 + t^2\right) + R\left(\frac{\omega^2}{t^2} - \omega^2 + t^2\right).$$

where

$$\omega = e^{\pi i/3} = \frac{1 + i\sqrt{3}}{2}.$$

In fact this relation follows from a minimal polynomial for this particular matrix. This relation for R-matrices gives us cubic skein relation:

$$\begin{aligned} ADO_3(L_3;t) = -\omega^2 ADO_3(L_0;t) + \left(\frac{\omega^2}{t^2} - \omega^2 + t^2\right) ADO_3(L_1;t) + \\ + \left(\frac{\omega^2}{t^2} - 1 + t^2\right) ADO_3(L_2;t) \end{aligned} \tag{2.3}$$

where L_k are links that look identically except for some ball, inside which two strands make k twists. Pictorially it is shown in Figure 2.1.

Links-Gould polynomial famously satisfies skein relation

$$\begin{aligned} LG(L_3) + (1 - t_0 - t_1)LG(L_2) + \\ + (t_0 t_1 - t_0 - t_1)LG(L_1) + t_0 t_1 LG(L_0) = 0 \end{aligned} \tag{2.4}$$

This relation is given in [3], p.170, but we use different set of variables, the one is used in [4]. The crucial fact is that skein relations 2.3 and 2.4 become the same if we specify variables in Links-Gould variable to $t_0 = t^2$, $t_1 = \omega^2 t^{-2}$. Therefore ADO polynomial

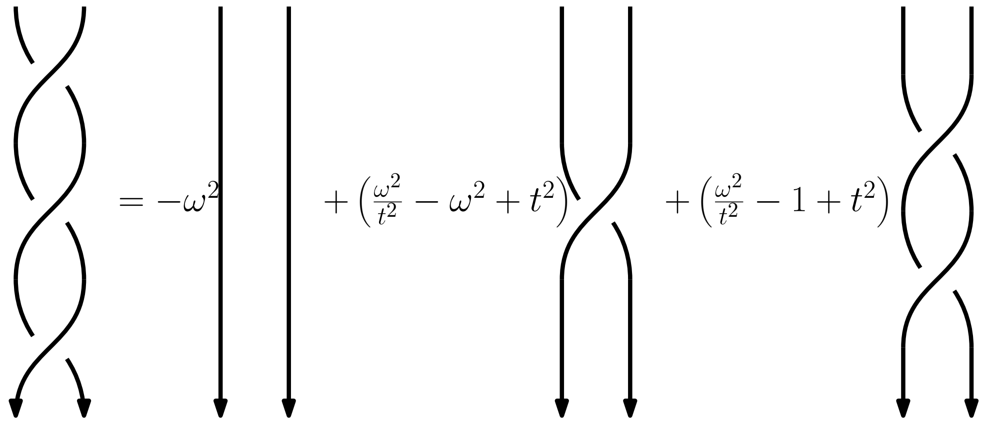

Figure 2.1. Cubic skein relation for ADO-3 invariant.

$ADO_3(L;t)$ and the specialized Links-Gould polynomial $LG(L;t^2,\omega^2t^{-2})$ satisfy the same cubic skein relation, the one shown on the picture above.

2.3.2. Three strand relation

Ishii in [4], Proposition 1, described several three-strand relations for Links-Gould polynomial. Let's consider one of them, namely relation (3.4) in Proposition 1. Ishii introduced the new operators Q_i given by

$$Q_0 = \frac{t_0}{t_0 - t_1}R + \frac{t_0(1-t_1)}{t_0 - t_1}\mathrm{id}_{V\otimes V} - \frac{t_0t_1}{t_0 - t_1}R^{-1},$$
$$Q_1 = \frac{t_1}{t_1 - t_0}R + \frac{t_1(1-t_0)}{t_1 - t_0}\mathrm{id}_{V\otimes V} - \frac{t_0t_1}{t_1 - t_0}R^{-1}.$$

Then the following relation holds:

$$(Q_0 \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes Q_1)(Q_1 \otimes \mathrm{id}_V) = \frac{t_1(t_0-1)}{t_0\,(1-t_1)}(Q_0 \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes Q_0)(Q_1 \otimes \mathrm{id}_V)$$

By direct calculation, it can be shown that if we specialize this relation, namely put $t_0 = t^2$, $t_1 = \omega^2t^{-2}$, the same relation holds for the ADO polynomial for $N = 3$. The same is true for other relations given by Ishii in Proposition 1.

2.3.3. Invariants vanish on split links

Another common property of the ADO and Links-Gould invariants is that they both vanish on split links. This follows from the fact that the operator invariant for $(0,0)$ tangle is zero for both ADO and Links-Gould construction (the result for vanishing invariant of $(0,0)$-tangle can be found in [1], Proposition 4.3 and [2], section 3.5). The exact argument for the LG invariant can be found in [2], section 3.5, but the same argumentation works for the ADO-invariant.

Chapter 3. The Main Result

The common properties for the ADO and Links-Gould invariants imply that there should be a relation between them. After some experimentation, a conjectured relation was found:

Conjecture 3.0.1. *For any link L we have*

$$ADO_3(L;t) = LG(L;t^2,\omega^2 t^{-2}), \tag{3.1}$$

$$ADO_3(L;1) = ADO_3(L;\omega) = \begin{cases} 1, & \textit{if } L \textit{ is a knot,} \\ 0, & \textit{otherwise.} \end{cases} \tag{3.2}$$

that is, ADO_3 invariant is a special case of Links-Gould invariant. Here is an example for a figure eight knot:

$$ADO_3(K,t) = \frac{-1-i\sqrt{3}}{2}t^4 + \frac{-1+i\sqrt{3}}{2t^4} + \frac{-3+3i\sqrt{3}}{2}t^2 + \frac{-3-3i\sqrt{3}}{2t^2} + 5$$
$$LG(K;t_0,t_1) = 2t_1t_0 + \frac{2}{t_0t_1} + \frac{t_0}{t_1} + \frac{t_1}{t_0} - 3t_0 - 3t_1 - \frac{3}{t_1} - \frac{3}{t_0} + 7$$
$$ADO_3(K;t) = LG(K;t^2,\omega^2 t^{-2})$$

We managed to prove this result for a large class of links:

Theorem 3.0.2. *If a link L is a closure of a 5-braid, then*

$$ADO_3(L;t) = LG(L;t^2,\omega^2 t^{-2}), \tag{3.3}$$

$$ADO_3(L;1) = ADO_3(L;\omega) = \begin{cases} 1, & \textit{if } L \textit{ is a knot,} \\ 0, & \textit{otherwise.} \end{cases} \tag{3.4}$$

In the next sections we present the proof.

3.0.1. Quotient of braid group algebra RB_5 by cubic relation

We are considering the knots that are closures of 5-braids. Since both ADO and specialized Links-Gould polynomials satisfy the same cubic skein relation, we only need to consider the quotient of group of 5-braids by the cubic relation. This quotient is finite [8], so we can check the values of the ADO and LG invariant on all of them. This part is done by computer calculations.

This section closely follows Marin's paper [8] and Marin and Wagner's paper [9]. Let us consider braid group algebra RB_n, where $R = \mathbb{Z}[\omega^2, \omega^2/t^2 + t^2] = \mathbb{Z}[\omega, \omega^2/t^2 + t^2]$. Then we can consider A_n, quotient of RB_n by cubic relation

$$s_i^3 - \left(\frac{\omega^2}{t^2} - 1 + t^2\right) s_i^2 - \left(\frac{\omega^2}{t^2} - \omega^2 + t^2\right) s_i + \omega^2 = 0, \tag{3.5}$$

where s_i are Artin generators of a braid group. We want to use results from [8], namely Theorems 1.2 and 4.1. The problem is that in our case, the coefficients a, b, c (in Marin's notation) of a cubic relation are not independent, for example there are relations $a - b + c + 1 = 0$, $c^3 = 1$. Still, most of the arguments can be adapted to our case, the only major difference would be that A_n, instead of a being free R-module, will be quotient of a free R-module generated by elements described in theorems 1.2 and 4.1 of [8]. If we define this generating set as S_n, we have:

$$S_2 = \{1, s_1, s_1^{-1}\},$$

$$S_3 = S_2 \sqcup S_2 s_2^{\pm 1} S_2 \sqcup S_2 s_2^{-1} s_1 s_2^{-1}$$

Using proposition 4.8 of [8], we also get the description of S_4:

$$S_4 = US_3, \text{ where}$$

$$U = \{1, s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}, s_3s_2^{-1}s_1s_2^{-1}s_3, s_3^{\pm 1}, s_3^{\pm 1}s_2^{\pm 1}, s_3^{\pm 1}s_2^{\pm 1}s_1^{\pm 1},$$

$$s_3^{\pm 1}s_2^{-1}s_1s_2^{-1}, s_3s_2^{-1}s_3, s_3s_2^{-1}s_3s_1^{\pm 1}, s_3s_2^{-1}s_3s_1s_2^{-1}s_1, s_3s_2^{-1}s_3s_1^{\pm 1}s_2^{\pm 1}\}$$

3.0.2. Extending knot invariants to RB_n and A_n

Any knot invariant can be defined as a function on B_n by assigning to an element of the braid group the value of a knot invariant of a braid closure. Then, it can be linearly extended to RB_n. Since both $ADO_3(L;t)$ and $LG(L;t^2,w^2t^{-2})$ invariants satisfy the cubic Relation 3.5, it means that they are well-defined on A_n. Abusing the notation, we will use the same notation for extended functions on A_n. We want to show that these link invariants coincide on B_5. In light of the previous discussion, equivalently we can try to prove that these link invariants coincide on A_5. But, since A_5 is finitely generated module over R, we only need to show that the $ADO_3(L;t)$ and $LG(L;t^2,w^2t^{-2})$ invariants coincide on generating set for A_5. We denote this generating set as S_5. We will proceed slightly differently: we will prove the statement for S_4(and consequently for A_4 and B_4) and then we will use this to prove the result for A_5.

3.0.3. Checking the statement of Main Theorem for A_5

Now we need to check the statement of Main Theorem for A_5. We will try to simplify calculations, for that we need first to take a look at S_4. Using the description of S_4 given above, we can generate all elements of S_4 and check the statement of conjecture for them. The corresponding calculations are performed on a computer, the details are given in an Appendix.

A description of A_5 is given in Theorem 6.21 from [8]. We have:

$$A_5 = A_4 + A_4 s_4 A_4 + A_4 s_4^{-1} A_4 + A_4 s_4 s_3^{-1} s_4 A_4 + A_4 s_4^{-1} s_3 s_2^{-1} s_3 s_4^{-1} A_4 +$$
$$+ A_4 s_4 s_3^{-1} s_2 s_3^{-1} s_4 A_4 + A_4 s_4^{-1} w^+ s_4^{-1} A_4 + A_4 s_4 w^- s_4 A_4 + A_4 s_4^{-1} w^- s_4^{-1} A_4 +$$
$$+ A_4 s_4 w^+ s_4 A_4 + s_4 w^- s_4 w^- s_4 A_4 + s_4 w^+ s_4^{-1} w^+ s_4 A_4 + s_4^{-1} w^- s_4 w^- s_4^{-1} A_4,$$

where $w^+ = s_3 s_2^{-1} s_1 s_2^{-1} s_3$, $w^- = s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1}$. If one can prove the main conjecture for separate subsets(like A_4, $A_4 s_4 A_4$, etc) then this proves the main conjecture for the whole of A_5. Let us consider the subsets separately.

For $A_4 \subset A_5$: we consider here the inclusion of A_4 into A_5, which basically means adding one additional strand, unconnected to others. Then, each element of A_4 is a linear combination of 5-braids with one strand not interacting with others. If we take the closure of such braid, we would get a split link, for which both link invariants are zero. Therefore for all elements of $A_4 \subset A_5$ our invariants coincide.

For elements of the type $A_4 s_4^{\pm 1} A_4$ we can use the second Markov move to reduce them to elements of A_4.

Namely, every element of the type $A_4 s_4^{\pm 1} A_4$ can be written as a linear combination of elements of the form $b_1 s_4^{\pm 1} b_2$, where b_1 and b_2 are some 4-braids. When taking the closure, using the second Markov move we can remove the middle term $s_4^{\pm 1}$, and we are left with a closure of a 4-braid $b_1 b_2$. But we already proved the statement of the theorem for closures of 4-braids, so it proves the statement for elements of the type $A_4 s_4^{\pm 1} A_4$.

Now let us consider the elements of the type $A_4 w A_4$, where w is a 5-braid. Using the first Markov move we can show that they can be reduced to elements of the type $A_4 w$. Namely, every element of the type $A_4 w A_4$ can be written as linear combination of the el-

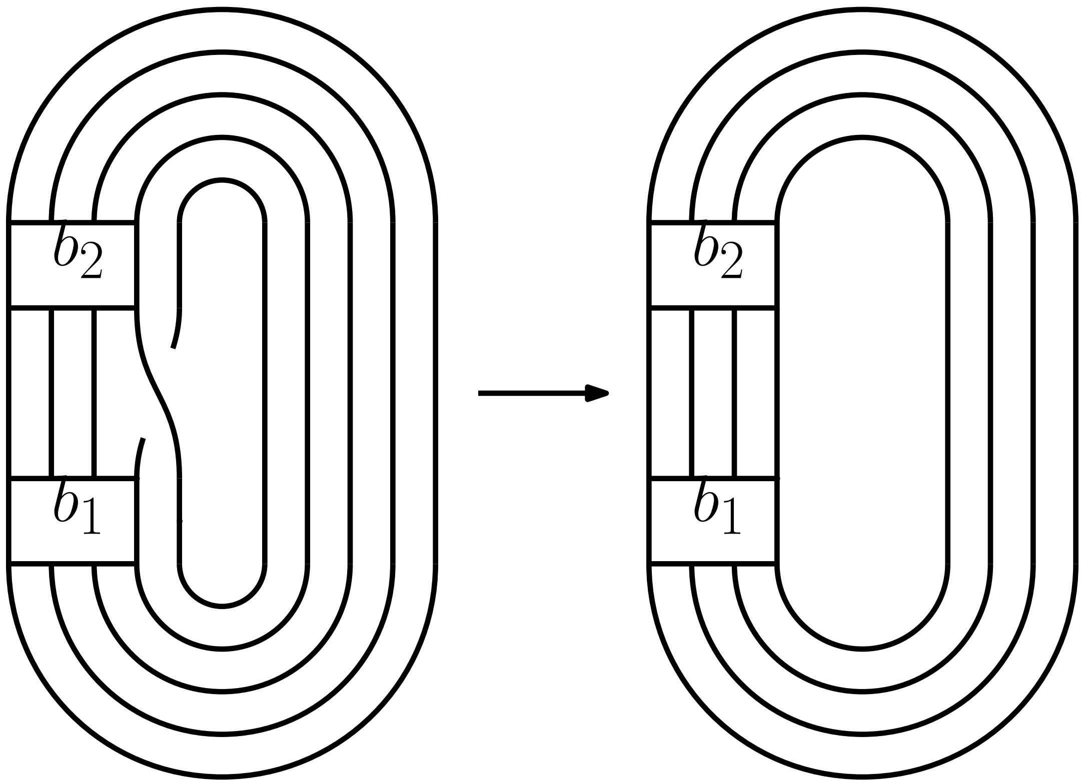

Figure 3.1. Second Markov move.

emenst of the form b_1wb_2, where b_1, b_2 - some 4-braids. We need to show that ADO_3 and the specialized Links-Gould polynomial coincide for the closure of a braid b_1wb_2. Applying the first Markov move, we note that the closure of a braid b_1wb_2 is the same as a closure of a braid b_2b_1w. But $b_2b_1w \in A_4w$. So it is enough to show that ADO_3 and specialized Links-Gould polynomials coincide on A_4w.

Since A_4w as an R-module is generated by S_4w, it is enough to check the ADO_3 and the specialized Links-Gould polynomials for for the set S_4w. Given the description of A_5 above, it means that in order to prove the statement of the theorem for A_5 it is enough

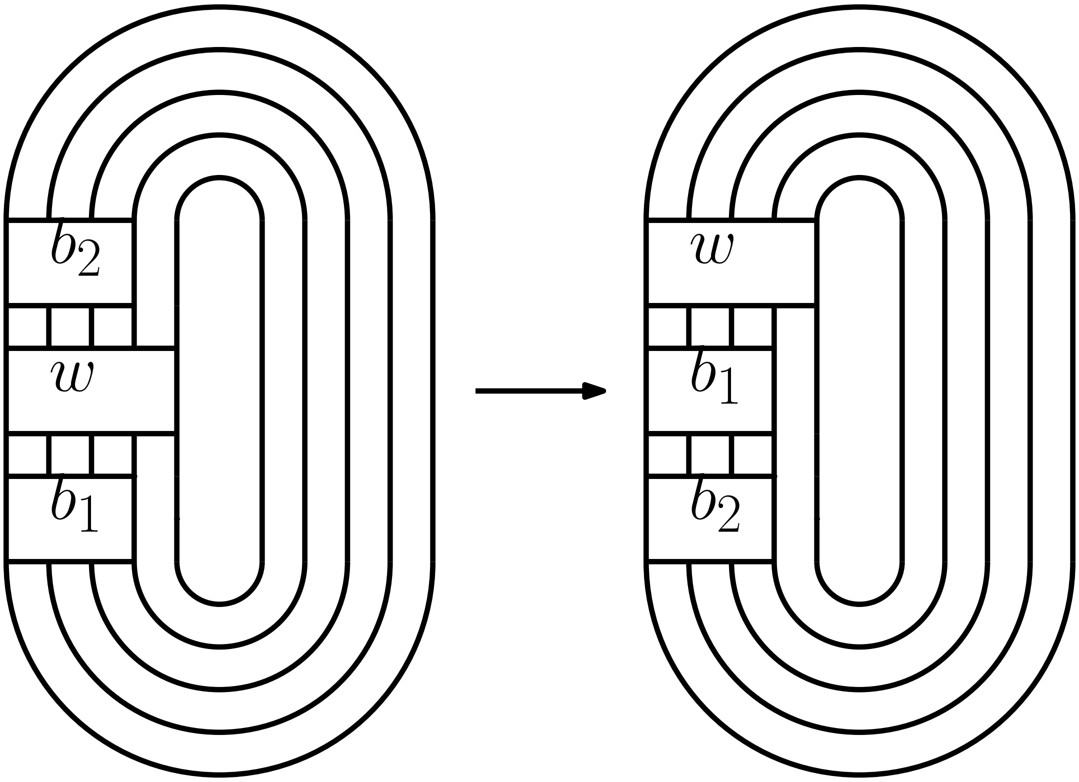

Figure 3.2. First Markov move.

to prove it for the sets:

$$S_4s_4s_3^{-1}s_4, S_4s_4^{-1}s_3s_2^{-1}s_3s_4^{-1}, S_4s_4s_3^{-1}s_2s_3^{-1}s_4, S_4s_4^{-1}w^+s_4^{-1}, S_4s_4w^-s_4,$$

$$S_4s_4^{-1}w^-s_4^{-1}, S_4s_4w^+s_4, s_4w^-s_4w^-s_4S_4, s_4w^+s_4^{-1}w^+s_4S_4, s_4^{-1}w^-s_4w^-s_4^{-1}S_4,$$

where $w^+ = s_3s_2^{-1}s_1s_2^{-1}s_3$, $w^- = s_3^{-1}s_2s_1^{-1}s_2s_3^{-1}$.

The calculations are done with the help of a computer. The details are given in the Appendix.

3.1. Proof of the corollary

The corollary is a direct consequence of properties of a Links-Gould polynomial. According to Theorem 7 from [5], which states that for any link L:

$$LG(L; t_0, 1) = LG(L; 1, t_1) = \begin{cases} 1, & \text{if } L \text{ is a knot,} \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, we have:

$$ADO_3\,(L; 1) = LG\left(L; 1, \omega^2\right) = \begin{cases} 1, & \text{if } L \text{ is a knot,} \\ 0, & \text{otherwise,} \end{cases}$$

$$ADO_3\,(L; \omega) = LG\left(L; \omega^2, 1\right) = \begin{cases} 1, & \text{if } L \text{ is a knot,} \\ 0, & \text{otherwise.} \end{cases}$$

3.2. Concluding remarks

Extending the methods described here to six or more strands is difficult, since A_n does not have finite basis for $n > 5$. Another way to prove conjecture in full extent would be to properly study the R-matrices for the Links-Gould and ADO polynomials, similar to [6].

One of the most promising ways is to use the main result of [9]. In this paper Marin and Wagner showed that Links-Gould polynomial is completely defined by 3 skein-like relations, r_1, r_2 and r_3. The first relation, r_1 is a cubic Relation 2.4. As we have seen, the ADO_3 polynomial satisfies a specialization of this cubic Relation, 2.3.

The second relation, r_2 is one of the relations (3.4)-(3.6) from [4]. As we mentioned in a subsection about three strand relations, by direct computation in Mathematica we can show that ADO_3 satisfied the same relation with the same specialization.

If we could show that ADO_3 satisfies the specialized relation r_3 this would prove the conjecture in full extent. Unfortunately, the explicit form of relation r_3 is still not known.

Another fact that implies validity of the main conjecture is the fact that the ADO invariant is almost-symmetric:

$$ADO_3(L;t) = ADO_3\left(L;\omega t^{-1}\right).$$

It was proved recently by Martel and Willetts [10] in more general form, but it also can be shown to be consequence of the main conjecture and properties of a Links-Gould invariant:

$$\begin{aligned}
ADO_3\left(L;\omega t^{-1}\right) &= LG\left(L;\left(\omega t^{-1}\right)^2,\omega^2\left(\omega t^{-1}\right)^{-2}\right) = \\
&= LG\left(L;\omega^2 t^{-2},t^2\right) = LG\left(L;t^2,\omega^2 t^{-2}\right) = \\
&= ADO_3\left(L;t\right)
\end{aligned}$$

Appendix. Details of Computer Calculations

All the code is available upon request

Files with ".nb" extension are executed in Mathematica. Files with ".py" extension are executed in Python.

All knots are represented as closures of braids. Braids represented in the following format:

`{number of strands, {twist 1, twist 2, ...}}`

Generator s_k is represented as k, s_k^{-1} is represented as $-k$, the braid is read from bottom up.

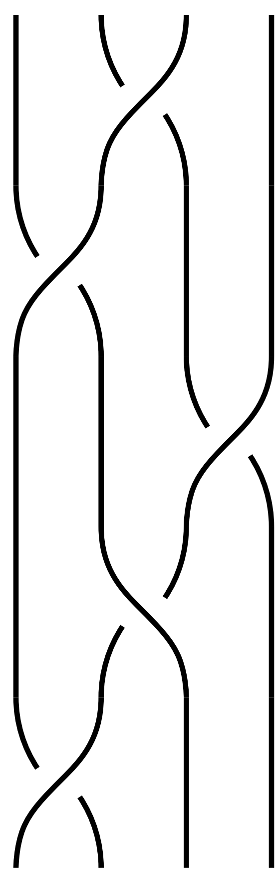

Figure A.3. The braid {4,{1,-2,3,1,2}}.

Here is the list of the files and their description:

`checking_identities.nb` - this file checks cubic and Ishii relations for ADO_3 invariant R-matrix, Section 2.3.

`s4_elements_generator.py` - this Python code generates elements of S_4, Subsection 3.0.1. The results are listed in the file `s4elements.txt`.

`checking_s4.nb` - this file checks the difference between ADO_3 and specialized Links-Gould invariant for elements of S_4, Subsection 3.0.1.

Regarding S_5: we need to check 6480 elements, which are broken into 10 types, Subsection 3.0.3:

Type 1: $S_4 s_4 s_3^{-1} s_4$.

Type 2: $S_4 s_4^{-1} s_3 s_2^{-1} s_3 s_4^{-1}$.

Type 3: $S_4 s_4 s_3^{-1} s_2 s_3^{-1} s_4$.

Type 4: $S_4 s_4^{-1} w^+ s_4^{-1}$.

Type 5: $S_4 s_4 w^- s_4$.

Type 6: $S_4 s_4^{-1} w^- s_4^{-1}$.

Type 7: $S_4 s_4 w^+ s_4$.

Type 8: $s_4 w^- s_4 w^- s_4 S_4$.

Type 9: $s_4 w^+ s_4^{-1} w^+ s_4 S_4$.

Type 10: $s_4^{-1} w^- s_4 w^- s_4^{-1} S_4$.

Here $w^+ = s_3 s_2^{-1} s_1 s_2^{-1} s_3$, $w^- = s_3^{-1} s_2 s_1^{-1} s_2 s_3^{-1}$.

In order to perform calculations faster, the separate code was written for each type of a braid. For each type `k` we have three files:

`calculate_beginning_for_ADO_S5_type_k.nb`

`calculate_beginning_for_LGS_S5_Type_k.nb`

`checking_S5_type_k.nb`

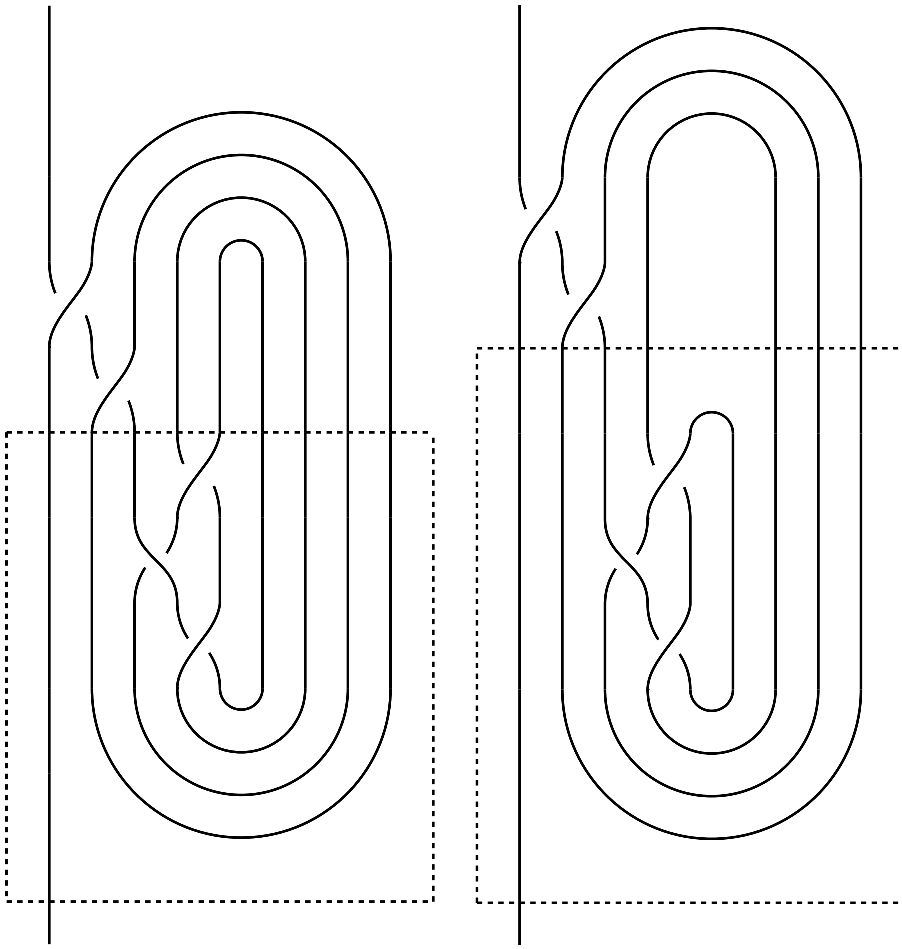

Figure A.4. The long knot corresponding to the braid {5,{4,-3,4,2,1}}.

Here is the purpose of these files: `checking_S5_type_k.nb` is the main file, that checks equality of specialized Links-Gould and ADO_3 invariants for the braids of type `k`. However, it needs two other files. They calculate the operator corresponding to the beginning of the braid.

Let's consider an example. Suppose we want to check braids of the type 1, $S_4 s_4 s_3^{-1} s_4$ and in particular {5,{2,1,4,-3,4}}. Using first Markov move we can change it into braids of the type $s_4 s_3^{-1} s_4 S_4$, our particular braid will become {5,{4,-3,4,2,1}}. Then we need to consider a long knot corresponding to this braid. It is shown on the pic-

ture on the left. We can isotope the knot and get the picture on the right. The auxiliary files `calculate_beginning_for_ADO_S5_type_k.nb` and `calculate_beginning_for_LGS_S5_Type_k.nb` calculate operators corresponding to the tangle encircled by dashed lines. It is the same for the knots of the specific type. This information the is used in the file `checking_S5_type_k.nb` to speed up calculations. Also, since the beginning for knots of the specific type is the same, the common part is not listed.

For example to check whether specialized Links-Gould and ADO_3 invariants coincide for the knot corresponding to the braid `{5,{4,-3,4,2,1}}`, we run the command `LGSMADO[{4, {2, 1}}]` in the file `checking_S5_type_k.nb`. We are skipping the part `{4,-3,4}` because it's the common part for the braids of the type 1. The blank file without pre-calculated beginnings is `checking_blank_s5.nb`.

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Vita

Nurdin Takenov was born in Bishkek, Kyrgyzstan. He finished his undergraduate studies at Moscow Institute of Physics and Technology in 2010. He earned a Master of Science degree in Mathematics in 2014 at University of Southern California. In August 2016, he came to Louisiana State University to pursue a doctorate degree in mathematics. He is currently a candidate for the degree of Doctor of Philosophy in mathematics.