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Robust Adaptive Control in H(infinity).

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Notation and Acronyms

$\mathbf{C}$:	Field of complex numbers
$\mathcal{D}$:	(open unit disk) z in $\mathbf{C}$: $\lvert z \rvert < 1$
$\mathbf{R}$:	Field of real number
$j\mathbf{R}$:	Imaginary axis
$\mathcal{H}$:	(open right half plane) s in C: $\mathrm{Re}(s) > 0$.
$\overline{\sigma}(A)$:	Largest singular value of A
$\underline{\sigma}(A)$:	Smallest singular value of A
$\rho(A)$:	Spectral radius of A
$\lambda(A)$:	Eigenvalue of A
A^*:	Complex conjugate transpose of A
A^T:	Transpose of A
A^{-1}:	Inverse of A
A^+:	Pseudo inverse of A
$G^*(s)$:	$= G^T(-s)$ (continous-time)
$G^*(z)$:	$= G^T(z^{-1})$ (discrete-time)
$\mathcal{H}_2(\mathcal{H})$:	Hilbert space of square integrable functions on $j\mathbf{R}$ which admit an analytic extension on $\mathcal{H}$
$\mathcal{H}_2(\mathcal{D})$:	Hilbert space of square integrable functions on a unit circle which admit an analytic extension to $\mathcal{D}$.

$\mathcal{H}_\infty(\mathcal{H})$: Banach space of essentially bounded functions on $j\mathbf{R}$ which admit an analytic extension to $\mathcal{H}$.

$\mathcal{H}_\infty(\mathcal{D})$: Banach space of essentially bounded functions on a unit circle which admit an analytic extension to $\mathcal{D}$.

$\|\bullet\|_\infty$: $\mathcal{L}_\infty/\mathcal{H}_\infty(\mathcal{H})$ norm

$\|\bullet\|_H$: Hankel Norm

$\mathcal{O}(n)$: Ordo(n), function tending to zero at the same rate as n.

$\mathcal{F}_L(P, K)$: Lower Linear Fractional Transformation

ARE: Algebraic Riccati Equation

DFT: Discrete-time Fourier Transform

FFT: Fast Fourier Transform

FIR: Finite Impulse Response

IIR: Infinite Impulse Response

LCF: Left Coprime Factorization

LFT: Linear Fractional Transformation

LTR: Loop Transfer Recovery

LQR: Linear Quadratic Regulator

LQG: Linear Quadratic Gaussian

LS: Least Square

MIMO: Multi-Input, Multi-Output

MRC: Model Reference Control

RCF: Right Coprime Factorization

RDE: Riccati Difference Equation

RLS: Recursive Least Square

SISO: Single-Input, Single-Output

SVD: Singular Value Decomposition

Abstract

This dissertation addresses the problem of unifying identification and control in the paradigm of $\mathcal{H}_\infty$ to achieve robust adaptive control. To achieve robust adaptive control, we employ the same approach used for identification in $\mathcal{H}_\infty$ and robust control in $\mathcal{H}_\infty$. In the modeling part, we aim not only to identify the nominal plant, but also to quantify the modeling error in $\mathcal{H}_\infty$ norm. The linear algorithm based on least-squares is used, and the upper bounds for the corresponding modeling error are derived. In the control part, we aim to achieve the performance specification in frequency domain using innovative model reference control. New algorithms are derived that minimize an $\mathcal{H}_\infty$ index function associated with the deviation between the performance of the feedback system to be designed, and that of the reference model. The results for modeling and control are then combined and applied to adaptive control. It is shown that with mild assumption on persistent excitation, the least squares algorithm in frequency domain is equivalent to the recursive least squares algorithm in time domain. Moreover, finite horizon $\mathcal{H}_\infty$ control is employed to design feedback controller recursively using the identified model that is time varying. The robust stability of the adaptive feedback system is then established.

Chapter 1

Introduction

This dissertation is a first attempt to unify identification and control in the paradigm of $\mathcal{H}_\infty$ to achieve robust adaptive control. The research is mainly motivated by the lack of robustness for adaptive control. It is shown in [61] that adaptive control systems tend to be unstable if the true plant involves unmodeled dynamics and/or its environment involves unknown disturbances. Although this is quite surprising, there are two basic reasons for the lack of stability robustness in adaptive control systems. The first lies in the identification part. In conventional adaptive modeling, physical plant is assumed to be identifiable exactly by finite dimensional models. Modeling uncertainties are largely ignored. Recall that any mathematical model is never a true representation of the physical system. The second reason lies in the control part, where the control strategy is "unsophisticated" in the sense that there is no guarantee on any measure of the stability margin. Modeling inaccuracies are ignored again. The lack of knowledge on the modeling error in the identification

part and the lack of stability margin in the control part lead inevitably to the lack of stability robustness in adaptive control systems.

This dissertation focuses on the robustness issue for adaptive control. Because robust adaptive control requires that both identification and control be capable of coping with model uncertainties, our strategy is the unification of robust identification and control in $\mathcal{H}_\infty$ for adaptive control systems. The unification is based on recent advance for identification in $\mathcal{H}_\infty$, and $\mathcal{H}_\infty$ control theory that are nonrecursive in nature. In particular, linear algorithms based on least-squares are studied for identification in $\mathcal{H}_\infty$ capable of quantifying the modeling error in frequency domain (Chapter 2). Novel algorithms for loopshaping based model reference control are investigated under the paradigm of $\mathcal{H}_\infty$ capable of tackling the modeling error (Chapter 3). The link to adaptive control is made through recursive least-squares and finite horizon $\mathcal{H}_\infty$ control (Chapter 4). Our contributions are summarized briefly as follows:

- Least-squares based algorithms for identification in $\mathcal{H}_\infty$ have been developed not only to identify the nominal plant model, but also to quantify the modeling error in frequency domain ($\mathcal{H}_\infty$-norm).

  Frequency domain error bound is crucial to ensure the stability of the feedback control system. The emphasis of the modeling error is a significant departure from the conventional identification approach where undermodeling error is often ignored. The use of least-squares algorithm in frequency domain also al-

lows us to relate it to the conventional least-squares algorithm in time-domain, and thus allows real time implementation for the purpose of adaptive control. Our results include hard bound (worst-case identification error) as well as soft bound (stochastic identification error).

- Novel algorithms for robust model reference control have been developed under the paradigm of $\mathcal{H}_\infty$ in connection with loopshaping design methodology.

  A principle reason for the rising of $\mathcal{H}_\infty$ control is due to the presence of model uncertainties. It offers worst-case stability and performance guarantees for those systems involving $\mathcal{H}_\infty$ norm bounded uncertainty. The use of $\mathcal{H}_\infty$ design methodology in model reference control allows us to tackle the model inaccuracies for adaptive control. Our results include $\mathcal{H}_\infty$ based loopshaping that is incorporated into the model reference control.

- Robust identification and control are unified under the paradigm of $\mathcal{H}_\infty$ that is applicable to adaptive control.

  Adaptive control has been an important research area for modeling and control of unknown systems because of the "self-tuning" capability. However the lack of robustness has made adaptive control less popular. Our results show that the lack of robustness can be removed with new algorithms developed for identification in $\mathcal{H}_\infty$ and robust model reference control using $\mathcal{H}_\infty$ based loopshaping. Consequently, this research has made robust adaptive control possible.

The details of this dissertation for robust adaptive control will be elaborated in the next three sections.

1.1 Overview

Theory and application of the adaptive control have progressed continuously in the last twenty years. The evolution of microcomputer technology has accelerated the availability in more realistic and wider applications since the resulting adaptive controller is implementable with microcomputer. Although adaptive control is led by microcomputer technology, it has more profound meaning for learning and adaptation in feedback control systems. Because physical systems are never known precisely, and its environment may involve unknown disturbances, of which both can be time-varying, adaptive technique has long been interested by control engineers [3] due to its "self-tuning" capability. It is not surprising to see that adaptive control is often used whenever system parameters are poorly known or system is subject to unknown disturbances. By the early 1980s, mathematical theory of adaptive control had reached its maturity, and stable adaptive control was made possible [23].

Unfortunately the success of adaptive control theory did not lead to the success in control engineering application. It was soon realized [61] that the stability of the adaptive feedback system is not ensured if undermodeling is involved and/or persistent disturbance is present that had not been taken into consideration in earlier research. The lack of robustness for adaptive control has led research for $\mathcal{H}_\infty$ robust control that aims to design a single controller to achieve both stability and

performance for a family of systems that capture the characteristic of the uncertain plant. While most researchers in the control community turned their attention to robust control in $\mathcal{H}_\infty$, research efforts in adaptive control persist. These efforts can be classified as conventional and unconventional according to their approaches. In what follows, the conventional approaches are briefly described first, and other unconventional approaches are discussed subsequently.

(1) Dead-zone method. The standard parameter update laws are modified such that the adaptation takes place only when the size of prediction error or error dynamics exceeds a certain threshold [63]. The degree of robustness for adaptive systems is limited and related to the size of dead-zone. However, one needs to know certain bound on disturbance for using this method.

(2) Adaptive law modification. This method turns off adaptation whenever the norm of the estimated controller parameters exceeds a certain value [38, 41]. Thus a backup controller is required that guarantees, in the least, the stability of the system. However, one needs to know a bound on the norm of desired controller parameter.

(3) Persistency excitation method. An external signal is introduced in control loop to produce persistency of excitation that in turn ensures exponentially stable adaptive control systems. As a consequence of exponential stability, stability is retained in the presence of bounded disturbances. However, in the presence of bounded disturbance, the amplitude and frequency richness of the external

signal should be large enough at some particular frequencies to secure persistently exciting the plant input and output to prevent the noise in the closed loop. A major shortcoming of excitation for robust design is that in many practical applications, the desired set point is not persistently exciting, and it is not usually desirable or it is impossible to inject additional probing signals into the plant [2, 56, 60].

The above methods commonly require knowledge of an upper bound on disturbance or the norm of unknown matching controller parameters that is not known in advance. Actually, a more serious problem with the conventional approach is the difficulty to tackle the problem of undermodeling that spurs recent advance on control oriented identification. Recall that the lack of robustness for adaptive control has made $\mathcal{H}_\infty$ popular where the plant is assumed to consist of a nominal model and an upper bound on the modeling error in frequency domain ($\mathcal{H}_\infty$-norm). It provides worst-case stability and performance provided such uncertain model description is available. Thus how to obtain such uncertain models from experimental data becomes the focus of the current research in the control community [66, 67]. It should be clear that this is one of the key problems for robust adaptive control. The research efforts along this line are briefly described as follows.

(1) Ellipsoid parameter bounds. These methods are based on the notions of set-membership estimation, eg., [6, 18, 64]. The model parameters are shown to lie in a set defined by a quadratic form, i.e., an ellipsoid or hyperboloid, depend-

ing on the data. A similar approach was used in [70] based on least-squares estimate. Although ellipsoid bounded plants can be incorporated with robust control, it is less consistent with the existing robust control methodology. In particular there is a considerable difficulty to relate such ellipsoid bounds to the $\mathcal{H}_\infty$ bounded uncertainty except some conservative error bounds.

(2) Worst-case identification in $\mathcal{H}_\infty$. Several researchers have considered the problem of identification using the $\mathcal{H}_\infty$ norm starting from bounded error frequency response data at a finite set of frequencies. Both linear and nonlinear algorithms have been developed and bounds on the worst-case identification error are also derived [25, 26, 31, 35, 36]. Recent development includes interpolation algorithms [8, 9, 32, 33]. This is the only identification approach that is compatible with $\mathcal{H}_\infty$ based robust control. The problem lies in the lack of time-domain algorithm. Other worst-case approaches work with bounded time domain disturbances but appear to suffer from computational difficulties.

(3) Stochastic embedding method. In contrast to the above so called " worst-case" approaches, Goodwin *et al.* adopted an "average-case" philosophy based on the stochastic embedding principal, e.g., [24]. It is assumed that both the unmodeled dynamics and noise are drawn from a probabilistic set having certain amplitude and smoothness properties. These properties are then estimated by maximum likelihood (ML) techniques. However, these bounds are "soft" in

the sense that the associated $\mathcal{H}_\infty$ control design can not guarantee the robust stability in the worst case.

Up to the present excluding, the existing adaptive control for robustness is treated to concentrate on either the identifier or the control law. This is caused by gaps between system identification and robust control. In the robust control area, the error bound associated with the identified model is assumed to be available without considering where it comes from. On the other hand, the identification area has over-emphasized the estimation of nominal model without necessary assessments for identified model quality [67]. Moreover, the interconnection between the control law selection and the identification algorithm is often ignored in research work of robust adaptive control. We need a unified approach to identification and control in the paradigm of $\mathcal{H}_\infty$. In the next section we outline our solution approach to robust adaptive control.

1.2 Solution Approach

The goal of this dissertation is to achieve robust adaptive control with greater closed loop stability margin with respect to both plant undermodeling and disturbances. This dissertation is an attempt to unify identification and control in the paradigm of $\mathcal{H}_\infty$ to achieve robust adaptive control. The analysis of the combined robust identification and robust control in $\mathcal{H}_\infty$ would possess a greater stability margin than that of the separate analysis. Although the progress is slow in this research

direction, there are some new interesting developments. For instance, Bitmead *et al.* [7] have combined a least-squares identification algorithm with receding horizon LQG control that demonstrates certain nice properties required for robust adaptive control. our solution approach is motivated by [7]. However there is a significant difference: Our approach is based on identification in $\mathcal{H}_\infty$ and robust control in $\mathcal{H}_\infty$ whereas [7] uses mean square error for both identification and control.

As discussed earlier, a serious problem with conventional adaptive control is the assumed finite dimensionality of the physical plant. With this assumption, stable adaptive control can be proven using a simple identification scheme in conjunction with an unsophisticated control method because model uncertainty does not exist. However, physical systems are often infinite-dimensional, and possibly nonlinear. This causes the instability of adaptive control in the presence of modeling uncertainties and disturbance uncertainties [7, 39, 61]. To achieve robust adaptive control, we need not only identify the nominal plant but also quantify the model uncertainty compatible with robust control in the adaptive modeling part. Furthermore, we must use both nominal plant and model uncertainty to self-tune the adaptive controller. This is our basic principle to tackle the robustness issue for adaptive control. In what follows next, we outline the solution approach used in this dissertation.

- Adaptive Modeling of Uncertain Systems.

  Our approach is to modify the algorithms from worst-case identification in $\mathcal{H}_\infty$ that produces not only nominal models, but also the modeling errors

in $\mathcal{H}_\infty$ norm. The chosen algorithm is least-squares based linear algorithms from [27, 36]. There are several reasons for choosing least-squares based linear algorithms. First, the least-squares algorithm has been used extensively in conventional identification. It is simple and effective, and can be implemented using time-domain data recursively. Second, the convergence of the least-squares algorithm from conventional parametric modeling in [23, 47] is hinged to the persistent excitation amounting to a set of non-zero spectrum lines. When the input signal is periodic with the same set of non-zero spectrum lines, the ratio of the discrete Fourier transforms between the (steady-state) input and output signals defines the frequency response data for identification in $\mathcal{H}_\infty$, and the two least-squares algorithms are then equivalent. Finally the least-squares algorithm from identification in $\mathcal{H}_\infty$ is capable of quantifying the worst-case identification error. Hence, the modeling uncertainty can be computed given certain a priori information.

- Robust Model Reference Control.

  Model reference control has been used extensively in adaptive control. However, the control algorithm did not take model uncertainty into consideration. Our approach to model reference control is the employment of loopshaping methodology. Roughly speaking, the reference model is used to represent the ideal loopshape that is in turn determined by the ideal frequency shape of sensitivity and complementary sensitivity. The control objective is to synthesize

a stabilizing feedback controller that minimizes the $\mathcal{H}_\infty$ cost associated with the reference model. The resulting design is close to the $\mathcal{H}_\infty$ loopshaping in [48]. This approach has taken the additive model uncertainty into consideration. Furthermore, the resulting feedback system also admits other robustness properties. Moreover, the resulting $\mathcal{H}_\infty$ controller has an observer-structure. The idea of incorporating ideal frequency loopshape into the reference model is novel, and has not been studied before. For robust adaptive control, finite horizon control or filtering is employed to implement robust model reference control on line. With appropriate conditions, the finite horizon $\mathcal{H}_\infty$ control converges asymptotically to $\mathcal{H}_\infty$ loopshaping based model reference control.

As it can be seen, our approach is quite different from the conventional ones. We begin with deterministic problems for robust identification and control under the paradigm of $\mathcal{H}_\infty$, and then implement the resulting algorithms in real time. This is one of the reasons why least-squares algorithm and finite horizon control algorithm are chosen. In comparison with the existing results, our approaches have three distinguishing features. The first one is the "*hard bound*" for the quantification of the modeling error. This hard bound is obtained through the use of certain *a priori* information of the infinite-dimensional system and the experimental sampled data. This is in sharp contrast with the "*soft bound*" as in [24]. It is believed that the "*hard bound* " is necessary in order to have worst case guarantees for both stability and performance of the feedback control system. The second one is in the syn

dissertation of stabilizing feedback controller through minimization of an $\mathcal{H}_\infty$ cost associated with nominal performance determined by the reference model and robust stability in presence of bounded additive uncertainty. The resulting controller has the same McMillan degree as the nominal plant model and the resulting feedback system is close to that of the reference model in terms of the frequency shape for sensitivity and complementary sensitivity if the nominal model is close to the true physical system. The last one is the consistency between identification and control for which both identification and control use $\mathcal{H}_\infty$ norm to measure the identification error and control performance cost.

1.3 Dissertation Organization

This dissertation consists of five chapters. The content of these five chapters are described briefly next.

Chapter 1 gives the introduction of the dissertation. It indicates the importance of the research work accomplished as well as the contribution of the dissertation. The existing research in the areas of system identification and adaptive control is reviewed. Different approaches to tackle robust adaptive control are described. Our solution approach adopted in this dissertation is outlined in comparison with others. Unique features of our dissertation work are also discussed.

Chapter 2 introduces the problem of identification in $\mathcal{H}_\infty$. The focus of the chapter is the least-squares based linear algorithms due to their simplicity, efficiency, and more importantly, to the applicability to on-line identification. This chapter begins

with the deterministic identification problem in frequency domain (identification in $\mathcal{H}_\infty$). The central results are the improved identification error bounds in $\mathcal{H}_\infty$ norm. It reveals some interesting features of the least-square based linear algorithms. The most important one is that the linear algorithms developed in [35, 27] for identification in $\mathcal{H}_\infty$ are exponentially convergent for nonuniformly spaced frequency response data. Moreover, it can be used in conjunction with Kung's algorithm to produce a low order state-space model that is very effective for identification of the lightly damped systems. Stochastic identification error bound is also derived for the least-square based linear algorithm in [36, 27]. Two simulation examples are used to illustrate the effectiveness of the algorithm.

Chapter 3 considers an alternative approach to model reference control (MRC). The focus is on $\mathcal{H}_\infty$ loopshaping with observer-based feedback controllers. We formulate a specific MRC problem that incorporates the reference model into the $\mathcal{H}_\infty$ optimization. The key idea is the representation of the ideal sensitivity and complementary sensitivity with a simple reference model. The $\mathcal{H}_\infty$ cost has taken additive model uncertainty into consideration. More importantly, the feedback system has performance robustness in presence of small plant perturbation. This chapter begins with analysis of observer-based feedback systems. The MRC problem is then tackled for continuous-time systems. The syn dissertation algorithm is developed that consists of design of state feedback gain, and design of output injection gain.

The same problem is tackled for discrete-time system also. State-space solutions are derived for both continuous-time and discrete-time systems.

Chapter 4 is concerned with the real time implementation for least-squares based linear algorithm for identification in $\mathcal{H}_\infty$, and robust model reference control. The frequency domain least-squares algorithm is translated into the one of time domain using Parseval's Theorem and with the assumption of persistent excitation. It is shown that the identified model converges asymptotically and exponentially to the identified model produced by least-squares based linear algorithm studied in Chapter 2. Because our adaptive control system has a feedback structure, the convergence issue for closed-loop adaptive identification is also investigated. The use of filtered input data for least-squares algorithm leads to an additive uncertainty with $\mathcal{H}_\infty$ norm error bound that is consistent with the robust model reference control. To implement robust model reference control in real time, finite horizon $\mathcal{H}_\infty$ control is introduced. The $\mathcal{H}_\infty$ performance cost associated with model reference control is converted into that of finite horizon. The feedback controller is computed iteratively based on the identified model and the real time input/output data. Finally to prevent possible instability caused by inaccurate a priori information, model validation result in [58] is employed to monitor the adaptive system.

Chapter 5 gives the concluding remarks of the dissertation. It summarizes the research achievements, and points out the unsolved research problems for robust adaptive control. Because contributions of this dissertation have been discussed in

the first chapter, this chapter focuses on the future research direction. The controller law adaptation is emphasized, and real time implementation for the finite horizon $\mathcal{H}_\infty$ control is elaborated. These problems are expected to be resolved in near future.

Finally, an appendix is used to include those background materials not available in each chapter, and those computer programs compiled for computer simulations.

Chapter 2

Least-squares Algorithms for Worst Case Identification in $\mathcal{H}_\infty$

2.1 Introduction

This chapter considers least-squares algorithms for the robust identification problem discussed in Chapter 1 that is hinged to the robustness of the adaptive feedback control system. Our objective is the development of identification algorithms that not only identifies the nominal plant, but also quantifies the identification error in the sup-norm, compatible with robust control in $\mathcal{H}_\infty$. Furthermore, this research will focus on the adaptability of the algorithm for on-line identification of the nominal plant, and on-line quantification of the model uncertainty. This leads to the least-squares based identification algorithm that is applicable to both frequency domain and time domain data.

Least-squares algorithm is well known and allegedly dates back to Gauss. It is extensively studied in the context of identification and filtering. In [47], the application of the least-squares algorithm to system identification using time domain measure-

ment data is described in great detail. Although the physical system is assumed to have an exact finite-dimensional model, the modeling error induced by measurement noise is analyzed in the stochastic framework. The modeling error caused by undermodeling is not considered in [47] until the recent work reported in [24] where the stochastic embedding is used to quantify the variance of the undermodeling error in frequency domain. A drawback with the conventional least-squares algorithm is the lack of "hard bound" on the quantification of the modeling error in frequency domain.

Our work on least-squares algorithm is motivated by robust control-oriented identification in $\mathcal{H}_\infty$ formulated by Helmicki, Jacobson and Nett [35] that aims at both identification of the nominal plant and quantification of the model uncertainty in sup-norm. Roughly speaking, the identification problem proposed in [35] can be stated as follows: given a finite number of noisy experimental frequency response data, find an algorithm which not only identifies the nominal plant model, but also quantifies the worst case identification error in $\mathcal{H}_\infty$ norm. Furthermore, the algorithm is required to have the property that the worst case identification error converges to zero as the noise level goes to zero and the number of experimental data points goes to infinity. This particular identification problem is termed as identification in $\mathcal{H}_\infty$. In the context of feedback system design, it is essential that the resulting system identification algorithm produce an identified model that converges in a topology for which feedback stability is a robust property. Such topology

is chosen as $\mathcal{H}_\infty$ that is consistent with the robust control design. The research work along this direction constitutes an important part of robust identification. In the past a few years, several effective algorithms are developed, including linear algorithms, two-stage nonlinear algorithms, and interpolation algorithms. See [1, 9, 25, 35, 36, 54, 55, 58, 71] and references cited therein.

In this chapter, we will focus on least-squares based linear algorithms for identification in $\mathcal{H}_\infty$ due to their simplicity, efficiency, and more importantly, to the applicability to on-line identification. This chapter will begin with the deterministic and stochastic identification problem in frequency domain (identification in $\mathcal{H}_\infty$), and then discuss identification of lightly damped system. Our result reveals some interesting features of the least-square based linear algorithms. The most important one is that the linear algorithms developed in [36] for identification in $\mathcal{H}_\infty$ are exponentially convergent for nonuniformly spaced frequency response data. Improved upper bounds are derived for the least-square based linear algorithm in [36]. It is also interesting to note that the linear algorithms studied in this paper, combined with the balanced model reduction, give an effective procedure for the identification of lightly damped systems. Upper bounds for modeling error are also derived for stochastic (or probablistic) case. Due to the linearity of the algorithms, such error bounds are particularly attractive. A comparison between worst-case and stochastic error bounds shows that the worst-case approach is not so pessimistic as some people think. In fact, these two error bounds are close to each other.

2.2 Problem Formulation and Preliminaries

In this section we describe the identification problem formulated by Helmicki, Jacobson, and Nett [35]. The system in consideration is assumed to be linear, discrete-time, shift-invariant, and possibly infinite-dimensional. Such system admits transfer function which corresponds to $\mathcal{Z}$-transformation:

$$G(z) := \sum_{k=0}^{\infty} g_k z^{-k}, \quad g_k \in \mathbf{R}^{p\times m} \tag{2.1}$$

where $\{g_k\}$ is the impulse response. It is further assumed that the system admits certain stability margin in the sense that

$$M = \sup_{\rho|z|>1} \overline{\sigma}(G(z)) < \infty$$

where $M > 0$, $\rho > 1$. Moreover the transfer function $G(z)$ is continuous on toroidal circle $|z| = \overline{\rho}^{-1} > \rho^{-1}$. The set of all such systems is denoted by $\mathcal{S}(\rho, M) \subset \mathcal{H}_\infty$. The value of M represents the system gain as an upper bound over all exponentially weighted sinusoidal inputs while the value of ρ represents as a lower bound on the relative stability of the system. The pair (M, ρ) characterizes the *a priori* information of the system to be identified which can be experimentally estimated. Indeed, the output response of the system at time k is given by

$$y(k) = g_k * u(k) = \sum_{i=0}^{k} g_i u(k-i)$$

by causality where $\{u(k)\}$ and $\{y(k)\}$ define the input and output sequences, and $\{g_k\}$ the impulse response. If the system input is given by

$$u(k) = \rho^{-k}\cos(k\omega) = \rho^{-k}\mathrm{Re}[e^{jk\omega}], \; k \geq 0,$$

then the output response $y(k)$ is given by

$$\begin{aligned}
y(k) &= \sum_{i=0}^{k} g_i \mathrm{Re}\left[\rho^{-(k-i)} e^{j(k-i)\omega}\right] \\
&= \mathrm{Re}\left(\left[\sum_{i=0}^{k} g_i \rho^i e^{-ji\omega}\right] \rho^{-k} e^{jk\omega}\right) \\
&= \mathrm{Re}\left(\left[\sum_{i=0}^{\infty} g_i \rho^i e^{-ji\omega}\right] \rho^{-k} e^{jk\omega} - \sum_{i=k+1}^{\infty} g_i \rho^{-k+i} e^{-j(i-k)\omega}\right) \\
&= \mathrm{Re}\left[G(\rho^{-1} e^{j\omega}) \rho^{-k} e^{jk\omega}\right] - \mathrm{Re}\left[\sum_{i=k+1}^{\infty} g_i \rho^{i-k} e^{-j(i-k)\omega}\right].
\end{aligned}$$

We thus obtain

$$\begin{aligned}
\rho^k y(k) &= \mathrm{Re}\left[G(\rho^{-1} e^{j\omega}) e^{jk\omega} - \left(\sum_{i=k+1}^{\infty} g_i \rho^i e^{-ji\omega}\right) e^{jk\omega}\right] \\
&= \mathrm{Re}\left[G(\rho^{-1} e^{j\omega}) e^{jk\omega}\right] - \mathrm{Re}\left[\left(\sum_{i=k+1}^{\infty} g_i \rho^i e^{-ji\omega}\right) e^{jk\omega}\right]
\end{aligned}$$

The hypothesis that $G(z)$ is continuous on the circle $|z| = \overline{\rho}^{-1} > \rho{-1}$ implies

$$\lim_{k\to\infty} \max_{\omega} \overline{\sigma}\left(\sum_{i=k+1}^{\infty} g_i \rho^i e^{-ji\omega}\right) = 0.$$

It follows that the quantity

$$M = \sup_{\rho|z|>1} \overline{\sigma}(G(z))$$

can be estimated from the output response $\rho^k y(k)$ asymptotically by sweeping the frequency from 0 to π. So we have the interpretation that M is the worst case gain of the system over the weighted sinusoidal input, and ρ stands for the stability margin. Because the measurement at the output can not be noise free, large ρ is prohibited. Otherwise ρ^k weights the measurement noise exponentially that will

amplify the noise considerably. Since the value of

$$R_G(k) = \overline{\sigma}\left(\sum_{i=k+1}^{\infty} g_i \rho^i e^{-ji\omega}\right)$$

depends on the time constant of the system, the appropriate value of ρ should be determined such that $\rho^k R_G(k)$ is not large.

The experimental data consists of a finite collection of frequency response samples corrupted by noise Δ. In a deterministic case, noise Δ is assumed to be available. In a stochastic case, the noise is assumed to be a random variable. Some statistical properties of the noise are needed and stochastic errors in frequency domain can be derived, but no explicit hard error bound can be established. In worst case, noise Δ is assumed to be deterministic that is bounded *a priori* by some known level δ. In the case of m (number of inputs) $\geq p$ (number of outputs), the noisy frequency response samples are given by

$$E_k = G(e^{j\omega_k}) + \Delta_k, \quad \overline{\sigma}(W(e^{j\omega_k})\Delta_k) \leq \delta, \quad k = 0, 1, ..., N-1, \tag{2.2}$$

where $W(e^{j\omega})$ is a known weighting function. Our objective is to find a causal polynomial (or FIR) model $\hat{G}(z)$ such that $\|W(G - \hat{G})\|_\infty$ is minimized. We assume temporarily that the frequency samples are uniformly spaced. For the case $m \leq p$, the measurement data is given by

$$E_k = G(e^{j\omega k}) + \Delta_k, \quad \overline{\sigma}(\Delta_k W(e^{j\omega_k})) \leq \delta, \quad k = 0, 1, ..., N-1.$$

The objective in this case is to find a causal polynomial (or FIR) model $\hat{G}(z)$ such

that $\|(G-\hat{G})W\|_\infty$ is minimized. In what follow next, we consider only the case $m \geq p$.

Since a physical system has real impulse response, its frequency response satisfies the property of conjugate symmetry. It is thus clear that we need perform frequency response experimentally for only positive frequencies. The problem of identification in $\mathcal{H}_\infty$ is to find an identification algorithm A_N which maps the experimental data (2.2), and *a priori* information (M, ρ, δ) to an approximate model $\hat{G} \in \mathcal{H}_\infty$ such that the identification error $e_n(A_N, M, \rho, \delta)$ defined by

$$e_n(A_N, M, \rho, \delta) := \sup_{G \in \mathcal{S}(\rho, M)} \left\{ \|W(G-\hat{G})\|_\infty : \ |\Delta_k| \leq \delta, 1 \leq k \leq N \right\} \qquad (2.3)$$

is suitably small. Furthermore, the modeling error is derived with an explicit worst case error bound. The algorithm A_N is said to be convergent if $e_n(A_N, M, \rho, \delta)$ approaches to zero as δ goes to zero and the number of data points N goes to infinity:

$$\lim_{\delta \to 0, N, n \to \infty} e_n(A_N, M, \rho, \delta) = 0.$$

The convergence has the same meaning as the identifiability [47] in the sense that the algorithm produces the plant model exactly if the data is complete, and noiseless.

An algorithm is said to be *tuned* if the *a priori* information of the plant model or/and noise level is used in the identification process. Since *untuned* linear algorithms are divergent in presence of the worst case noise, convergent linear algorithms are necessarily *tuned*. A class of tuned linear algorithms are those reported in [36] based on the least-squares fitting. While it remains unknown for the existence of

other types of tuned linear algorithms which are convergent, we will focus on the least-squares based linear algorithms. In what follows next, some preliminary results will be presented that are important to the development of the later sections. The following lemma clarifies the relation between frequency and time domain for the case of discrete finite data that is referred to mixed Parseval theorem.

Lemma 2.1 *Let* $P(e^{j\omega}) = p_0 + p_1 e^{-j\omega} + \ldots + p_{n-1}e^{-j(n-1)\omega}$ *where* $\{p_k\}$ *is the coefficient sequence. Denote* $W_N = e^{j2\pi/N}$. *Then, for any* $N \geq n$,

$$\begin{aligned} \sum_{i=0}^{n-1} \mathrm{tr}\,(p_i p_i^*) &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}\left(P(e^{j\omega})P^T(e^{-j\omega})\right)\, d\omega \\ &= \frac{1}{N}\sum_{k=0}^{N-1} \mathrm{tr}\left(P(W_N^k)P^T(W_N^{-j\omega})\right). \end{aligned}$$

Proof: The Fourier coefficients are given by

$$p_i = \frac{1}{2\pi}\int_0^{2\pi} P(e^{j\omega})e^{j\omega i}d\omega = \begin{cases} \frac{1}{N}\sum_{k=0}^{N-1} P(W_N^i)W_N^{ik}, & \text{for } i = 0, 1, \ldots, n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence we may write

$$P(e^{j\omega}) = \sum_{i=0}^{n-1} p_i e^{-ji\omega} = \sum_{i=-\infty}^{\infty} p_i e^{-ji\omega}.$$

It follows that

$$P(e^{j\frac{2k\pi}{N}}) = P(W_N^k) = \sum_{i=0}^{n-1} p_i W_N^{-ik}$$

We prove the lemma by direct verification using definitions:

$$\sum_{i=0}^{n-1} p_i p_i^T = \sum_{i=-\infty}^{\infty} p_i p_i^T$$

$$
\begin{aligned}
&= \sum_{i=-\infty}^{\infty}\left(\left[\frac{1}{2\pi}\int_0^{2\pi} P(e^{j\omega})e^{j\omega i}d\omega\right]\left[\frac{1}{2\pi}\int_0^{2\pi} P(e^{j\omega})e^{j\omega i}d\omega\right]^*\right) \\
&= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} P(e^{j\omega})\left[P(e^{j\tilde{\omega}})\right]^* \sum_{i=-\infty}^{\infty} e^{ji(\omega-\tilde{\omega})}\, d\omega d\tilde{\omega} \\
&= \frac{1}{2\pi}\int_0^{2\pi} P(e^{j\omega})\left(\int_0^{2\pi}\left[P(e^{j\tilde{\omega}})\right]^*\delta(\omega-\tilde{\omega})d\tilde{\omega}\right)d\omega \\
&= \frac{1}{2\pi}\int_0^{2\pi} P(e^{j\omega})\left[P(e^{j\tilde{\omega}})\right]^*\, d\omega
\end{aligned}
$$

where we have used relation

$$
\sum_{i=-\infty}^{\infty} e^{ji(\omega-\tilde{\omega})} = 2\pi\delta(\omega-\tilde{\omega}).
$$

Taking trace both sizes gives the first part of the mixed Parseval theorem. Similarly, there holds

$$
\begin{aligned}
\sum_{i=0}^{n-1}\mathrm{tr}\left[p_i p_i^*\right] &= \sum_{i=0}^{N-1}\mathrm{tr}\left(\left[\frac{1}{N}\sum_{k=0}^{N-1}P(W_N^k)W_N^{ki}\right]\left[\frac{1}{N}\sum_{l=0}^{N-1}P(W_N^l)W_N^{li}\right]^*\right) \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\mathrm{tr}\left(P(W_N^k)[P(W_N^l)]^*\frac{1}{N}\sum_{i=0}^{N-1}W_N^{i(k-l)}\right) \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\mathrm{tr}\left(P(W_N^k)[P(W_N^k)]^*\right).
\end{aligned}
$$

where we have used relation

$$
\frac{1}{N}\sum_{i=0}^{N-1}W_N^{i(k-l)} = \begin{cases} 1, & \text{if } k=l, \\ 0 & \text{otherwise.} \end{cases}
$$

Thus the second part of the mixed Parseval theorem is proven. ∎

Because least-squares algorithm minimizes mean-square error while for robust control, the modeling error has to be quantified in $\mathcal{H}_\infty$ norm, this poses a serious problem for the identification algorithm based on least-squares. The next result is useful to provide worst-case identification error bounds on the quantification of the

$\mathcal{H}_\infty$ norm in terms of the $\mathcal{H}_2$ norm. The proof is modified from [36] but has a tighter upper bound.

Lemma 2.2 *Denote $\mathcal{H}_2(\mathcal{D}_\rho)$ as the collection of functions analytic on $\mathcal{D}_\rho$, and each element absolutely square integrable on the boundary of $\mathcal{D}_\rho$. For any function $F \in \mathcal{H}_2(\mathcal{D}_\rho)$ with $\rho > 1$, define $\mathcal{H}_2$-norm by*

$$
\begin{aligned}
\|F\|_2^2 &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}\left(F(e^{j\omega})\left[F(e^{j\omega})\right]^*\right) d\omega, \\
\|F\|_{2,\rho}^2 &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}\left(F(\rho^{-1}e^{j\omega})\left[F(\rho^{-1}e^{j\omega})\right]^*\right) d\omega.
\end{aligned}
$$

Then, $F \in \mathcal{H}_\infty$, and

$$\|F\|_\infty \leq \sqrt{\frac{\rho+1}{\rho-1}}\|F\|_2^{1/2}\|F\|_{2,\rho}^{1/2}.$$

Proof: For any harmonic function $U(z)$, it can be represented by Possion's integral

$$U(re^{j\theta}) = \frac{1}{N}\int_{-\pi}^{\pi} u(t)\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}dt$$

where $u(t)$ is the boundary function. Set $F(e^{j\omega}) = U(re^{j\omega})$, $u(t) = F(\frac{1}{\sqrt{\rho}}e^t)$. Thus, Possion's integral is given by

$$F(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\sqrt{\rho}e^{jt})\left(\frac{1-\rho^{-1}}{1-2\rho^{-1/2}\cos(\omega-t)+\rho^{-1}}\right)dt.$$

Applying Schwartz inequality yields

$$\|F\|_\infty \leq \|F\|_{2,\sqrt{\rho}}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1-2\rho^{-1}}{1-2\rho^{-1/2}\cos(t)+\rho^{-1}}\right)^2 dt\right]^{1/2}.$$

We first derive an upper bound on $\|F\|_{2,\sqrt{\rho}}$. From [36], any function $G \in \mathcal{H}_2(\mathcal{D}_{r_2})$ satisfies inequality

$$\|G\|_{2,r}^{\log(r_2/r_1)} \leq \|G\|_{2,r_1}^{\log(r_2/r)}\|G\|_{2,r_2}^{\log(r/r_1)}$$

by Hadamard's theorem provided that $r_1 < r < r_2$. This is equivalent to

$$\|F\|_{2,r} \leq \|F\|_{2,r_1}^{\sigma}\|F\|_{2,r_2}^{1-\sigma},$$

where $0 < \sigma = \log(\frac{r_2}{r})/\log(\frac{r_2}{r_1}) < 1$ for $0 \leq r_1 \leq r \leq r_2$. Now taking $r_1 = 1$, $r = \sqrt{\rho}$ and $r_2 = \rho$ with G replaced by F, then $\sigma = 1/2$ and thus

$$\|F\|_{2,\sqrt{\rho}} \leq \|F\|_2^{\frac{1}{2}}\|F\|_{2,\rho}^{\frac{1}{2}}.$$

To complete the proof, we note that the Possion kernel can be written as

$$\begin{aligned}\frac{1-r^2}{1-2r\cos(\theta)+r^2} &= \sum_{n=-\infty}^{\infty} r^{|n|}e^{jn\theta} \\ &= 1+2\sum_{n=1}^{N} r^n \cos n\theta.\end{aligned}$$

It follows for $r = \rho^{-1/2}$,

$$\begin{aligned}&\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1-2\rho^{-1}}{1-2\rho^{-1/2}\cos(t)+\rho^{-1}}\right)^2 dt\right]^{1/2} \\ &= \left(\sum_{k=-\infty}^{\infty}(\sqrt{\rho})^{-2|k|}\right)^{1/2} = \left(2\sum_{k=0}^{\infty}\rho^{-|k|}-1\right)^{1/2} = \sqrt{\frac{\rho+1}{\rho-1}}.\end{aligned}$$

Therefore, we can bound the sup-norm of F using the 2-norm according to

$$\|F\|_\infty \leq \sqrt{\frac{\rho+1}{\rho-1}}\|F\|_2^{1/2}\|F\|_{2,\rho}^{1/2}.$$

The lemma is thus true. ∎

Earlier work on identification in $\mathcal{H}_\infty$ norm has used ℓ^1 norm to bound the $\mathcal{H}_\infty$ that always gives a factor $1/(\rho-1)$ in the identification error bound. As discussed earlier, the value of ρ is often close to one in order to obtain a reasonable estimate of M. This factor $1/(\rho-1)$ gives excessively large error bound. To avoid this factor, n-width approximation theory can be used. The final lemma in this section is quoted from [57]. Define $\mathcal{P}_m$ as the collection of all $(m-1)$ th order causal polynomial (or FIR) model

$$\mathcal{P}_m = \left\{P: \ P = p_0 + p_1 z^{-1} + \ldots + p_{m-1} z^{-(m-1)}\right\}. \tag{2.4}$$

The following lemma can be found in [57] (Theorem 2.1 of page 250).

Lemma 2.3 *Let $M > 0, \rho > 1$. Then for $m = 0, 1, \ldots,$*

$$\sup_{G \in \mathcal{S}(M,\rho)} \inf_{P \in \mathcal{P}_m} \|G - P\|_\infty = M\rho^{-m}.$$

Further, for any $G \in \mathcal{S}(M,\rho)$, the globally optimal approximant of G

$$P_m^o[G] := \sum_{k=0}^{m-1} \left(1 - \rho^{2(k-m)}\right) g_k z^{-k},$$

achieves the bound $M\rho^{-m}$. That is,

$$\sup_{G \in \mathcal{S}(M,\rho)} \|G - P_m^o[G]\|_\infty = M\rho^{-m}.$$

It should be mentioned that the result in [57] is stronger than what was quoted here. However, Lemma 2.3 suffices our application for identification in $\mathcal{H}_\infty$. Finally, we also note that with $P_m^o[G]$ given above, there holds

$$\|P_m^o[G]\|_{2,\rho} \leq \|G\|_{2,\rho} \leq M, \quad \rho \geq 1. \tag{2.5}$$

2.3 Modeling Uncertainty Using Least Squares

In this section, the algorithms interest to us are the *tuned* linear algorithms reported in [36]. This is called least-squares based linear algorithms. We treat the unified least-squares based linear algorithms, and derive error bounds of the modeling uncertainty applicable to the uniformly spaced frequency response data samples. The extension to nonuniformly spaced frequency response data samples will be discussed at the end of the section. For convenience, the results will be presented for scalar functions. The generalization to transfer matrix functions requires only notational change.

Let the experimental data sequence $\{E_k^N\}$ be given in (2.2) with its DFT coefficients defined by inverse DFT:

$$c_k(E^N) = \frac{1}{N}\sum_{i=0}^{N-1} E_i^N W_N^{ik}, \quad W_N = e^{j2\pi/N}, \quad k = 0, 1, ..., N-1.$$

Let the identified model be in $\mathcal{P}_m$ given by

$$\hat{G}(z) = \sum_{k=0}^{n-1} p_k z^{-k}, \quad n \leq N.$$

The objective of the linear algorithms is to determine p_k's which are linear functions of the experimental data (2.2) such that the identification error measured in $\mathcal{H}_\infty$ norm is suitably small. The class of linear algorithms in [36] are based on the solutions of certain least-squares problems. The purpose of these algorithms is to perform suitable polynomial interpolation so that a convergent algorithm is obtained in the presence of noise corruption in the data. The identified model is constrained to

satisfy only the *a priori* information in the presence of noise. Two such least-squares problems which lead to linear algorithms are constrained minimizations

$$J_1 = \min_{p_k} \left[\sum_{k=0}^{N-1} \left| p_k - c_k(E^N) \right|^2 \right]^{1/2}, \quad \text{subject to } \|\hat{G}\|_{2,\rho} \leq M$$

with the convention that $p_k = 0$ for $n \leq k \leq N-1$ if $n < N$, and

$$J_2 = \min_{\hat{G} \in \mathcal{P}_n} \left[\frac{1}{N} \sum_{k=0}^{N-1} \left| \hat{G}(W_N^k) - E_{k+1}^N \right|^2 \right]^{1/2}, \quad \text{subject to } \|\hat{G}\|_{2,\rho} \leq M.$$

These two minimization problems are treated differently in [36] and result in two different linear algorithms, as well as error bounds. It turns out that the two different linear algorithms produce the same identified model.

Proposition 2.1 *The two different constrained minimization problems J_1, and J_2 yield the same solution $\hat{G}$, and $J_1 = J_2$.*

Proof: The fact that $J_1 = J_2$ is a direct consequence of the mixed Parseval's theorem in Lemma 2.1. Since the constrained minimizations are least-squares problems, they have unique solution. The fact is thus true. ∎

The above observation is important, since one needs to consider only J_2. The next result gives an improved bound for the resulting identification error.

Theorem 2.2 *Let the noisy experimental frequency response data be given in (2.2). Let the approximated model be $\hat{G}(z) \in \mathcal{P}_n$ where the coefficients are defined by the solution of the constrained minimization problem J_2. Then, the worst-case identifi-*

cation error satisfies

$$e_N(M,\rho,\delta) \le M\rho^{-n} + 2M\sqrt{\frac{\rho+1}{\rho-1}}\left(\frac{\delta}{M}+\rho^{-n}\right)^{1/2}.$$

Proof: The constrained minimization problem given has a solution with

$$\begin{aligned}
&\left[\frac{1}{N}\sum_{k=0}^{N-1}\left|\hat{G}(W_N^k) - E_{k+1}^N\right|^2\right]^{1/2} \\
\le\ & \left(\frac{1}{N}\sum_{k=0}^{N-1}\left|\hat{G}(W_N^k) - G(W_N^k)\right|^2\right)^{1/2} + \left(\frac{1}{N}\sum_{k=0}^{N-1}|\Delta_k|^2\right)^{1/2} \\
\le\ & \delta + M\rho^{-n}
\end{aligned}$$

where the triangle inequality is used. To see this note that

$$\hat{G} = P_n^o[G] + \hat{\eta}, \quad \text{and} \quad \|\hat{\eta}\|_2 \le M\rho^{-n}$$

yields one such solution by the n-width approximation. Hence,

$$\left[\frac{1}{N}\sum_{k=0}^{N-1}\left|\hat{G}(W_N^k) - G(W_N^k)\right|^2\right]^{1/2} \le 2\delta + M\rho^{-n}$$

by the hypothesis on E^N, which in turn implies that

$$\begin{aligned}
\left\|\hat{G} - P_n^o[G]\right\|_2 &= \left[\frac{1}{N}\sum_{k=0}^{N-1}\left|\hat{G}(W_N^k) - P_n^o[G](W_N^k)\right|^2\right]^{1/2} \\
&\le 2(\delta + M\rho^{-n})
\end{aligned}$$

in light of Lemma 2.1. Note also in light of (2.5) that

$$\|P_n^o[G]\|_{2,\rho} \le \|G\|_{2,\rho} \le M, \quad \forall G \in \mathcal{S}(M,\rho).$$

Hence, $\|\hat{G} - P_n^o[G]\|_{2,\rho} \le 2M$. Using Lemma 2.2, it follows that

$$\|\hat{G} - P_n^o[G]\|_\infty \le \sqrt{\frac{\rho+1}{\rho-1}}\|\hat{G} - P_n^o[G]\|_2^{1/2}\|\hat{G} - P_n^o[G]\|_{2,\rho}^{1/2}$$

$$
\begin{aligned}
&\leq \sqrt{\frac{\rho+1}{\rho-1}}\{2(\delta+M\rho^{-n})\}^{1/2}\{2M\}^{1/2} \\
&= 2M\sqrt{\frac{\rho+1}{\rho-1}}\left(\frac{\delta}{M}+\rho^{-n}\right)^{1/2}.
\end{aligned}
$$

Now the error bound can be established by noting that

$$
\begin{aligned}
e_N(M,\rho,\delta) &\equiv \sup\left\{\|G-\hat{G}\|_\infty : G\in\mathcal{S}(M,\rho), |\Delta_k|\leq\delta, 1\leq k\leq N\right\} \\
&\leq M\rho^{-n} + \sup\left\{\|\hat{G}-P_n^o[G]\|_\infty, G\in\mathcal{S}(M,\rho), |\Delta_k|\leq\delta\right\} \\
&= M\rho^{-n} + 2M\sqrt{\frac{\rho+1}{\rho-1}}\left(\frac{\delta}{M}+\rho^{-n}\right)^{1/2}.
\end{aligned}
$$

that concludes the proof. ∎

It should be clear that the error bound in Theorem 2.2 improves the one in [36], and has a simpler form. This is due to the fact that for the case $\delta = 0$, the error bound in Theorem 2.2 decays in the order $\mathcal{O}(\rho^{-n/2})$, whereas in [36], the error decays in the order $\mathcal{O}(\rho^{-\alpha n})$ where $\alpha < 1/2$ for $\rho > 1$. Furthermore, the factor $(\rho+1)/(\rho-1)$ in [36] is replaced by $\sqrt{(\rho+1)/(\rho-1)}$ in Theorem 2.2. This is especially important because ρ is close to one in order to have good estimate of M. The constrained minimization problem means that as $N \geq n \to \infty$, the worst-case identification error decays to zero exponentially for noise free case. Because it is difficult to solve the constraint minimization problem, we will convert it into the unconstraint problem. This gives our next result.

Corollary 2.1 *Let the noisy experimental frequency response data be given by (2.2).*

Denote $\tilde{\delta} = \delta + M\rho^{-n}$. Form the approximate model

$$\hat{G}(z) = \sum_{k=0}^{n-1} \frac{c_k(E^N)}{1 + \left(\frac{\tilde{\delta}}{M}\right)^2 \rho^{2k}} z^{-k}, \quad n \leq N,$$

with $\{c_k(E^N)\}$ the DFT coefficient of $\{E_k^N\}$. Then, the worst case identification error satisfies

$$e_N(M, \rho, \delta) \leq M\rho^{-n} + (1 + \sqrt{2})M\sqrt{\frac{\rho + 1}{\rho - 1}}\left(\frac{\delta}{M} + \rho^{-n}\right)^{1/2}.$$

Proof: Consider the following unconstrained minimization problem:

$$J = \min_{\hat{G}^n \in \mathcal{P}_n} \left(\frac{1}{N}\sum_{k=0}^{N-1} \left|\hat{G}^n(W_N^k) - E_k^N\right|^2\right) + \left(\frac{\tilde{\delta}}{M}\right)^2 \left\|\hat{G}^n\right\|_{2,\rho}^2 \tag{2.6}$$

with $\hat{G}^n = p_0 + p_1 z + \ldots + p_{n-1}z^{n-1}$. It admits a solution $\hat{G}^n$ such that $J \leq 2\tilde{\delta}^2$. This can be shown by taking $\hat{G}^n = P_n^o[G]$. With the solution $\hat{G}$ for the above unconstrained minimization problem,

$$J_c = \left[\frac{1}{N}\sum_{k=0}^{N-1} \left|\hat{G}(W_N^k) - E_k^N\right|^2\right]^{1/2} \leq \sqrt{2}\tilde{\delta}.$$

It follows that

$$\begin{aligned}
\|\hat{G} - P_n^o[G]\|_2 &= \left(\frac{1}{N}\sum_{k=0}^{N-1} \left|\hat{G} - P_n^o[G] - E_k^N + E_k^N\right|^2\right)^{1/2} \\
&\leq \left[\frac{1}{N}\sum_{k=0}^{N-1} \left|\hat{G}(W_N^k) - E_k^N\right|^2\right]^{1/2} + \left(\frac{1}{N}\sum_{k=0}^{N-1} \left|P_n^o[G](W_N^k) - E_k^N\right|^2\right)^{1/2} \\
&\leq J_c + \left(\frac{1}{N}\sum_{k=0}^{N-1} \left|P_n^o[G](W_N^k) - E_k^N\right|^2\right)^{1/2} \\
&\leq (1 + \sqrt{2})(\delta + M\rho^{-n}).
\end{aligned}$$

Since $J \leq 2\tilde{\delta}^2$, there holds, by (2.6),

$$\left(\frac{\tilde{\delta}}{M}\right)^2 \left\|\hat{G}\right\|_{2,\rho}^2 \leq 2\tilde{\delta}^2 \implies \|\hat{G}\|_{2,\rho} \leq \sqrt{2}M.$$

and $\|\hat{G} - P_n^o[G]\|_{2,\rho} \le \|\hat{G} - P_n^o[G]\|_{\infty,\rho} \le M + \sqrt{2}M$. The worst case identification error bound can then be obtained following the same steps in the proof of Theorem 2.2. To obtain the explicit solution $\hat{G}$, one notices that the unconstrained minimization problem is equivalent to

$$J = \min_{p_k} \sum_{k=0}^{n-1} \left|p_k - c_k(E^N)\right|^2 + \sum_{k=n}^{N-1} \left|c_k(E^N)\right|^2 + \left(\frac{\hat{\delta}}{M}\right)^2 \left(\sum_{k=0}^{n-1} |p_k|^2 \rho^{2k}\right)$$

by mixed Parseval's theorem. The optimal solution p_k's can then be obtained by setting the partial derivatives $\partial J / \partial p_k = 0$ that gives

$$\frac{\partial J}{\partial p_k} = 2(p_k - c_k(E^N)) + 2\left(\frac{\hat{\delta}}{M}\right)^2 p_k \rho^{2k} = 0$$

for $k = 0, 1, ..., n-1$. After simplification we obtain

$$p_k \left(1 + \left(\frac{\hat{\delta}}{M}\right)^2 p_k \rho^{2k}\right) = c_k(E^N).$$

Therefore, the solution

$$p_k = \frac{c_k(E^N)}{1 + \left(\frac{\hat{\delta}}{M}\right)^2 \rho^{2k}}, \quad k = 0, 1, ..., n-1,$$

is obtained. The proof is now completed. ∎

Comparing the two upper bounds, one for constrained minimization, and the other for unconstrained minimization, we conclude that a factor of $\sqrt{2}$ is added for unconstrained one. Before concluding this section, it should be emphasized that all the linear algorithms discussed so far are derived for uniformly spaced frequency response data. Naturally, one would like to know whether or not the linear algorithms

can be adapted to the case where the frequency response data is nonuniformly spaced as studied in [1, 54]. This question will be answered for the tuned linear algorithm.

Corollary 2.2 *Let the experimental frequency response data E^N be obtained at $\{\omega_i\}_{i=0}^{N-1}$ which is not uniformly spaced. Define matrix U_l as*

$$U_l = \begin{bmatrix} 1 & e^{j\omega_0} & \dots & e^{j(l-1)\omega_0} \\ 1 & e^{j\omega_1} & \dots & e^{j(l-1)\omega_1} \\ \dots & \dots & \dots & \dots \\ 1 & e^{j\omega_{N-1}} & \dots & e^{j(l-1)\omega_{N-1}} \end{bmatrix},$$

where $0 < l \leq N$. Suppose that the identified model $\hat{G} \in \mathcal{P}_n$ is obtained from the following unconstrained minimization problem

$$J_n = \min_{\hat{G}^n \in \mathcal{P}_n} \left(\frac{1}{N} \sum_{k=0}^{N-1} \left| \hat{G}^n(e^{j\omega_k}) - E_k^N \right|^2 \right) + \left(\frac{\hat{\delta}}{M} \right)^2 \|\hat{G}^n\|_{2,\rho}^2$$

with the same $\hat{\delta}$ as in Corollary 2.1. Then, the worst case identification error satisfies

$$e_N(M, \rho, \delta) \leq M\rho^{-n} + \frac{\sqrt{N}(1+\sqrt{2})M}{\underline{\sigma}(U_N)} \sqrt{\frac{\rho+1}{\rho-1}\left(\frac{\delta}{M} + \rho^{-n}\right)}.$$

Proof: It is noted that the tuned algorithm is similar to that in Corollary 2.1 except that the frequency response data is not uniformly spaced. Same argument in Corollary 2.1 gives $J_n \leq 2\hat{\delta}^2$, and thus a solution $\hat{G} \in \mathcal{P}_n$ exists such that

$$\left[\frac{1}{N} \sum_{k=0}^{N-1} \left| \hat{G}(e^{j\omega_k}) - P_n^*(e^{j\omega_k}) \right|^2 \right]^{1/2} \leq (1+\sqrt{2})\hat{\delta}.$$

Denote $F = \hat{G} - P_n^o[G] \in \mathcal{P}_n$. Then, the above is equivalent to

$$\left[\frac{1}{N} \sum_{k=0}^{N-1} \left| F(e^{j\omega_k}) \right|^2 \right]^{1/2} = \frac{1}{\sqrt{N}} \|U_N \Gamma\|_2 \leq (1+\sqrt{2})\hat{\delta}$$

where Γ is a column vector of size N with first n elements being the n coefficients of $F(z)$, and the rest elements zero. Note that by the definition of singular values, $1 \leq \sqrt{N}\bar{\sigma}(U_N^{-1}) = \sqrt{N}/\underline{\sigma}(U_N)$. It follows that

$$\|\hat{G} - P_n^o[G]\|_2 = \|F\|_2 \leq \sqrt{N}(1+\sqrt{2})\tilde{\delta}\bar{\sigma}(U_N^{-1}) = \frac{\sqrt{N}(1+\sqrt{2})\tilde{\delta}}{\underline{\sigma}(U_N)}$$

for any $G \in \mathcal{H}(M,\rho)$. Similarly, $\|\hat{G}\|_{2,\rho} \leq \sqrt{2}M$. Thus

$$\|\hat{G} - G\|_{2,\rho} \leq (1+\sqrt{2})M \leq \frac{\sqrt{2}(1+\sqrt{2})M}{\underline{\sigma}(U_N)}.$$

Hence, the error bound can be established following the same steps in the proof of Corollary 2.1. ∎

A few comments are in order. First, denote P as a column vector of size n with coefficients of $\hat{G}$ as the elements. Then, the minimization problem in Corollary 2.2 has a matrix representation

$$\min_{P\in\mathbb{R}^n} \left\| \begin{bmatrix} \frac{1}{\sqrt{N}}U_n \\ \frac{\delta}{M}V_m\Lambda \end{bmatrix} P - \begin{bmatrix} \frac{1}{\sqrt{N}}E^N \\ 0 \end{bmatrix} \right\|_2^2 .$$

where U_n is same as in Corollary 2.2 with $l = n$, $\Lambda = \mathrm{diag}(1, \rho, ..., \rho^{(n-1)})$, and

$$V_m = \frac{1}{\sqrt{m}} \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & W_m^1 & \dots & W_m^{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & W_m^{m-1} & \dots & W_m^{(m-1)(n-1)} \end{bmatrix}, \quad E^N = \begin{bmatrix} E_0^N \\ E_1^N \\ \dots \\ E_{N-1}^N \end{bmatrix}, \quad m \geq n. \tag{2.7}$$

Hence, the solution P is easily obtained from the orthogonality condition. Second, since V_m satisfies $V_m^* V_m = I_n$ for $m \geq n$ where V_m^* denotes the conjugation transpose

of V_m,

$$P = \left(\frac{U_n^* U_n}{N} + \frac{\tilde{\delta}^2}{M^2}\Lambda^2\right)^{-1} \frac{U_n^* E^N}{N} \tag{2.8}$$

which is well defined even if $U_n^* U_n$ in singular, though in this case, the error bound is not defined. It should be clear that $U_n^* U_n$ is nonsingular if and only if all $\{\omega_i\}$'s are distinct. Finally, note that if ω_i's are uniformly spaced, then $U_n^* U_n = N I_n$ for which $U_n^* E^N / N$ is the inverse DFT of the experimental measurement data and both the solution and the error bound reduce to those of Corollary 2.1 by noting that $\overline{\sigma}(U_N) = \underline{\sigma}(U_N) = \sqrt{N}$.

2.4 Stochastic Analysis of the Least-Squares Based Identification Algorithm

The least squares method is a classical stochastic approach to determining the finite order approximant

$$\hat{G}(z) = \sum_{k=0}^{n-1} p_k \, z^{-k}, \quad p_k \in \mathbf{R}^{p \times m}. \tag{2.9}$$

The finite sequence $\{p_k\}_{k=0}^{n-1}$ represents the impulse response of $\hat{G}(z)$. The previous section has derived a worst-case identification error bound in the $\mathcal{H}_\infty$ norm. This section is devoted to the stochastic analysis of the least-squares algorithm as discussed in the previous section. As assumed before, the unknown system is a set $\mathcal{S}(\rho, M) \subset \mathcal{H}_\infty$. The noise Δ_k, $k = 1, 2, .., N$ is now assumed to consist of N independent, identically distributed, complex random variables with zero mean and

variance σ^2. The experimental data E_k^N has the same form

$$\begin{aligned} E_k^N &= G(e^{j\omega_k}) + \Delta_k & (2.10)\\ &= P_n^o[G](e^{j\omega_k}) + \left(G(e^{j\omega_k}) - P_n^o[G](e^{j\omega_k})\right) + \delta_k \\ &= \sum_{k=0}^{n-1} p_k^o\, z^{-k} + \tilde{\Delta}_k, \quad ,k = 0,1,...,N-1 & (2.11)\end{aligned}$$

where $\tilde{\Delta}$ consists of two parts: The deterministic part bounded by $M\rho^{-n}$ according to the n-width approximation theory as $P_n^o[G](z)$ is the globally optimal approximant; The stochastic part with variance σ^2. Denote

$$\tilde{V}_N = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & W_N^{-1} & \dots & W_N^{-(n-1)} \\ \dots & \dots & \dots & \dots \\ 1 & W_N^{N-1} & \dots & W_N^{-(N-1)(n-1)} \end{bmatrix}, \quad \tilde{P} = \begin{bmatrix} P_n^o[G](1) \\ P_n^o[G](W_N^{-1}) \\ .. \\ P_n^o[G](W_N^{-(n-1)}) \end{bmatrix} \qquad (2.12)$$

and use the same notation as in Corollary 2.1 and (2.7), we can write (2.10) as

$$\tilde{E} = \tilde{V}\tilde{P} + \tilde{\Delta}, \quad \tilde{\Delta}^* = \left[\begin{array}{cccc} \tilde{\Delta}_0^* & \tilde{\Delta}_1^* & \dots & \tilde{\Delta}_{N-1}^* \end{array}\right] \qquad (2.13)$$

To treat the stochastic problem, we set the cost function as (equivalent to the case of $\delta = 0$)

$$J_s := \min_{\hat{G}^n \in \mathcal{P}_n} \left(\frac{1}{N}\sum_{k=0}^{N-1}\left|\hat{G}^n(e^{j\omega_k}) - E_k^N\right|^2\right) + \rho^{-2n}\|\hat{G}^n\|_{2,\rho}^2. \qquad (2.14)$$

The above cost function is motivated by zero mean of the radom noise $\{\Delta_k\}$. The next result shows that the solution to J_s is unbiased asymptotically.

Proposition 2.3 *Let $\hat{G} \in \mathcal{P}_n$ be the solution of J_s in (2.14) where $n \le N$. Then it is an asymptotically unbiased solution in the sense that its expectation approaches*

to the true plant as the number of samples tends to infinity. There holds stochastic error bound

$$\sup_{G\in\mathcal{S}(M,\rho)} \|\mathcal{E}\left\{\hat{G}\right\} - G\|_\infty \leq M\rho^{-n} + \frac{\sqrt{N}(1+\sqrt{2})M}{\underline{\sigma}(U_N)}\sqrt{\frac{\rho+1}{\rho-1}}\rho^{-n/2}.$$

where $\mathcal{E}$ is expectation operator and U_N is the same as in Corollary 2.1 with l replaced by N. If the frequency response measurements are uniformly sampled, then there holds

$$\sup_{G\in\mathcal{S}(M,\rho)} \|\mathcal{E}\left\{\hat{G}\right\} - G\|_\infty \leq M\rho^{-n} + (1+\sqrt{2})M\rho^{-n/2}\sqrt{\frac{\rho+1}{\rho-1}}.$$

Proof: The solution to the unconstrained least-squares solution is given as in Corollary 2.1. By setting $\delta = 0$, we obtain solution

$$P = \begin{bmatrix} p_0 \\ p_1 \\ \cdot\cdot \\ p_{n-1} \end{bmatrix} = \left(\frac{U_n^* U_n}{N} + \rho^{-2n}\Lambda^2\right)^{-1}\left(\frac{U_n^* G_N}{N} + \frac{U_n^*\Delta}{N}\right)$$

where

$$G_N = \begin{bmatrix} G(1) \\ G(W_N^{-1}) \\ \cdot\cdot \\ G(W_N^{-(n-1)}) \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Delta_0 \\ \Delta_1 \\ \cdot\cdot \\ \Delta_{N-1} \end{bmatrix}.$$

Taking expectation yields

$$\mathcal{E}\left\{P\right\} = \left(\frac{U_n^* U_n}{N} + \rho^{-2n}\Lambda^2\right)^{-1}\frac{U_n^* G_N}{N}$$

This is exactly the same solution as in Corollary 2.1 with $\hat{\delta} = M\rho^{-n}$ (see the discussion after Corollary 2.1). The error bound can then be concluded using the

results in Corollary 2.1 with $\delta = 0$, or equivalently $\tilde{\delta} = M\rho^{-n}$. The error bound for uniformly spaced frequency response samples can be obtained by setting $U_n^* U_n = I_n$ and $\underline{\sigma}(U_N) = \sqrt{N}$. ∎

Although the least-square based algorithm is biased, the bias is caused by under modeling, but not the corruption noise. Because the bias approaches to zero as $N \geq n \rightarrow \infty$, it is asymptotically unbiased. In what follows next, we consider computation of variance in worst-case for scalar systems. It applies to multivariable systems with simple notational change. First we write

$$\hat{G} = \mathcal{E}\left\{\hat{G}(z)\right\} + Z(z)\left(\frac{U_n^* U_n}{N} + \rho^{-2n}\Lambda^2\right)^{-1} \frac{U_n^* \Delta}{N} = \mathcal{E}\left\{\hat{G}(z)\right\} + \hat{\Delta}(z)$$

where

$$Z(e^{j\omega}) = \begin{bmatrix} 1 & e^{-j\omega} & \dots & e^{-j(n-1)\omega} \end{bmatrix}, \quad \hat{\Delta}(z) = Z(z)\left(\frac{U_n^* U_n}{N} + \rho^{-2n}\Lambda^2\right)^{-1} \frac{U_n^* \Delta}{N}$$

Then we have the following bound for the worst-case variance in frequency domain.

Proposition 2.4 *Let $\hat{G} \in \mathcal{P}_n$ be the solution of J_s in (2.14) where $n \leq N$. Suppose that the frequency response samples are uniformly distributed, and the variance of the corruption noise is σ^2. Then the worst-case variance in frequency domain satisfies the following inequality:*

$$\begin{aligned} \sup_{G \in \mathcal{S}(M,\rho)} & \left\|\mathcal{E}\left\{\left(\hat{G} - G\right)\left(\hat{G} - G\right)^*\right\}\right\|_\infty \\ &= \sup_{G \in \mathcal{S}(M,\rho)} \left\|\mathcal{E}\left\{\hat{G}\right\} - G\right\|_\infty^2 + \left\|\mathcal{E}\left\{\hat{\Delta}\hat{\Delta}^*\right\}\right\|_\infty^2 \\ &< \left(M + (1+\sqrt{2})\sqrt{\frac{\rho+1}{\rho-1}}\right)^2 \rho^{-n} + \sigma^2. \end{aligned}$$

Proof: Because Δ_k has zero mean for each k, it follows that

$$\left\|\mathcal{E}\left\{\left(\hat{G}-G\right)\left(\hat{G}-G\right)^*\right\}\right\|_\infty = \left\|\mathcal{E}\left\{\hat{G}\right\}-G\right\|_\infty^2 + \left\|\mathcal{E}\left\{\hat{\Delta}\hat{\Delta}^*\right\}\right\|_\infty^2. \tag{2.15}$$

The first term on the right hand side satisfies

$$\left\|\mathcal{E}\left\{\hat{G}\right\}-G\right\|_\infty^2 \le \left(M+(1+\sqrt{2})\sqrt{\frac{\rho+1}{\rho-1}}\right)^2 \rho^{-n} \tag{2.16}$$

for $G \in \mathcal{S}(M,\rho)$ using the result in Proposition 2.3. To quantify the second term on the right hand side of (2.15), we use the representation

$$\hat{\Delta}(e^{j\omega}) = Z(e^{j\omega})\left(\frac{U_n^*U_n}{N}+\rho^{-2n}\Lambda^2\right)^{-1}\frac{U_n^*\Delta}{N} =$$

$$\begin{bmatrix} \frac{1}{1+\rho^{-2n}} & \frac{e^{-\omega}}{1+\rho^{-2(n-1)}} & \cdot\cdot & \frac{e^{-(n-1)\omega}}{1+\rho^{-2}} \end{bmatrix} \begin{bmatrix} 1 & e^{\omega_0} & \dots & e^{j(N-1)\omega_0} \\ 1 & e^{j\omega_1} & \dots & e^{j(N-1)\omega_1} \\ \dots & \dots & \dots & \dots \\ 1 & e^{j\omega_{n-1}} & \dots & e^{j(n-1)\omega_{N-1}} \end{bmatrix} \frac{\Delta}{N}$$

where $\omega_i = 2i\pi/N$ for uniformly distributed frequency samples. Hence, we obtain, using the fact that $U_n^*U_n = NI_n$, and $\mathcal{E}(\Delta\Delta^*) = \sigma^2$ that

$$\mathcal{E}\left\{\hat{\Delta}\hat{\Delta}^*\right\} = \frac{\sigma^2}{N}\sum_{i=0}^{n-1}\left(1+\rho^{-2(n-i)}\right)^{-2} < \sigma^2$$

Thus, substituting the above and (2.16) into the right hand side of (2.15), and taking appropriate supremum can then conclude the upper bound for the worst-case variance in frequency domain. ∎

Proposition 2.4 shows that the stochastic error (i.e., variance error) in frequency domain is comparable to the worst-case error in the previous section. The modeling

error caused by noise is not significant. A more important issue is the under modeling error that has been investigated in the previous section.

The cost of using the least squares method, or more generally any stochastic approach, is that at least some information about the noise statistics must be assumed known. Additionally, the least squares method is biased if there is a model uncertainty, although it is asymptotically unbiased. One way to improve the least squares estimate is to increase the order of the estimate $\hat{G}(z)$. As n increases, the quality of the model improves. However, the worst-case variance of the identification error caused by noise remains the same. It shows that worst-case variance is close to the worst-case identification in the sense that variance does not vanish that is the same as the worst case noise level δ.

2.5 Applications to Lightly Damped Systems

In this section, we consider identification of lightly damped systems using the least-squares based linear algorithm, in conjunction with the Kung's algorithm. This problem is significant because frequency response fitting has been the main tool for modeling of lightly damped systems such as flexible structures, while other identification algorithms from identification in $\mathcal{H}_\infty$ do not work well for such applications. For instance, the two-stage nonlinear algorithms in [35, 25] are not effective for lightly damped systems [29]. Its difficulty lies in the model reduction part of the identified model. Most model reduction algorithms such as balanced realization and

Hankel norm approximation require computations of the controllability and observability gramians. Since the identified model from the two-stage nonlinear algorithm inevitably has a high order for flexible structures and is a sum of a rational function (resulted from Nehari approximant) and a causal polynomial function, it is almost impossible to compute controllability and observability gramians, or the resulting gramians are not accurate. This problem also exists for interpolation based algorithms [8, 9, 33]. It is noted that the linear algorithms studied in this paper produce identified models having an FIR structure, and the computation for controllability and observability gramians requires only one singular value decomposition [30, 43]. Due to the reliability of singular value decomposition, the reduced order model retains dominant modes of the flexible structure. However, before we present the simulation examples, some results on Kung's algorithm will be discussed, and its relation with the algorithm developed in [30] will be investigated.

In approximation of infinite-dimensional systems, Gu, Khargonekar, and Lee have proposed an algorithm [30] that derives first an FIR approximate model, and balanced model reduction is then applied to obtain reduced model. This algorithm requires only computation of one inverse DFT, and one singular value decomposition, it is thus quite effective for obtaining low order approximate models. More importantly, upper bound on the approximation error can also be derived based on a priori information of the system, and the Hankel singular values of the FIR model. This algorithm is then adapted into the two-stage nonlinear algorithm for

identification in $\mathcal{H}_\infty$. The disadvantage of the algorithm is that full information of the FIR model is needed. We will show that this algorithm is also related to Kung's algorithm [43] that is more flexible, more suitable if the FIR model has an extremely high order. The difference lies in the fact that there is no error bound for the Kung's algorithm. Because there are some misconcept about Kung's algorithm, we will be more detail in analyzing its properties next.

2.5.1 Analysis of Kung's Algorithm

To be specific, for a given FIR model

$$\hat{G} = \sum_{k=0}^{L} \hat{g}_k \, z^{-k}, \quad \hat{g}_k \in \mathbf{R}^{p\times m},$$

a finite Hankel matrix of order q is defined by

$$H_q := [\hat{g}_{i+k-1}]_{i,k=1}^{n} = \begin{bmatrix} \hat{g}_1 & \hat{g}_2 & .. & \hat{g}_{q-1} \\ \hat{g}_2 & \hat{g}_3 & .. & \hat{g}_q \\ .. & .. & .. & .. \\ \hat{g}_{q-1} & \hat{g}_q & .. & \hat{g}_{2(q-1)} \end{bmatrix}. \tag{2.17}$$

Kung's algorithm uses $2q \leq L$ so that H_q is "full". It begins with the assumption that $\hat{g}_k = CA^{k-1}B$ for $0 \leq k \leq L-1$, and $\hat{G}$ is a truncated model of $G(z) = D + C(zI - A)^{-1}B$. In this case, the Hankel matrix in (2.17) can be written as

$$H_q = \begin{bmatrix} C \\ CA \\ .. \\ CA^{q-1} \end{bmatrix} \begin{bmatrix} B & AB & .. & A^{q-1}B \end{bmatrix}.$$

To obtain the realization of (A, B, C) based on the finite sequence $\{\hat{g}_k = g_k\}_{k=1}^{L}$, Kung suggests a singular value decomposition approach:

$$U\Sigma^{1/2} = \begin{bmatrix} C \\ CA \\ .. \\ CA^{q-1} \end{bmatrix}, \quad \Sigma^{1/2}V^T = \begin{bmatrix} B & AB & .. & A^{q-1}B \end{bmatrix}. \tag{2.18}$$

where $H_q = U\Sigma V^T$ is the singular value decomposition of H_q. If $G(z)$ has finite state-space dimension $n \le q$, then Σ has dimension $n \times n$. Kung's algorithm suggests

$$C = \begin{bmatrix} I_p & 0 \end{bmatrix} U\Sigma^{1/2}, \quad B = \Sigma^{1/2}V^T \begin{bmatrix} I_m \\ 0 \end{bmatrix}, \tag{2.19}$$

$$\begin{aligned} A &= \begin{bmatrix} C \\ CA \\ .. \\ CA^{q-2} \end{bmatrix}^{+} \begin{bmatrix} CA \\ CA^2 \\ .. \\ CA^{q-1} \end{bmatrix} = \Sigma^{-1/2}U^T \begin{bmatrix} 0 & I_{p(q-1)} \\ 0 & 0 \end{bmatrix} U\Sigma^{1/2} \qquad (2.20) \\ &= \left(\begin{bmatrix} I_{p(q-1)} & 0 \end{bmatrix} U\Sigma^{1/2} \right)^{+} \begin{bmatrix} 0 & I_{p(q-1)} \end{bmatrix} U\Sigma^{1/2}. \end{aligned}$$

A problem arises for inaccurate data $\hat{g}_k \neq g_k$. For the case $m \ge p$, we have $U, \Sigma \in \mathbf{R}^{pq \times pq}$ and $V \in \mathbf{R}^{pq \times mq}$ satisfying $U^TU = UU^T = I_{pq}$ and $V^TV = I_{mq}$. For the case that $m \le q$, $V, \Sigma \in \mathbf{R}^{mq \times mq}$ and $U \in \mathbf{R}^{pq \times mq}$ satisfying $VV^T = V^TV = I_{mq}$ and $U^TU = I_{pq}$. This is a generic case especially when the true system is infinite-dimensional of practical interest to us. Nevertheless, Kung suggests partitioning

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}, \quad V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}, \quad \Sigma = \mathrm{diag}(\Sigma_1, \Sigma_2). \tag{2.21}$$

where $U_1 \in \mathbf{R}^{p \times n}$, $V_1 \in \mathbf{R}^{m \times n}$, and $\Sigma_1 \in \mathbf{R}^{n \times n}$ with n chosen depending on the gap between Σ_1 and Σ_2.

It is noted that there is no reason to impose the assumption $2q \leq L$ if the data is inaccurate, or the true system is infinite-dimensional because then the Hankel matrix H_q may have full rank generically. We consider an extreme case when $q = L$.

Proposition 2.5 *Let H_L be the same as in (2.17) with $q = L$, and $m \geq p$. Then H_L has full rank if and only if $\hat{g}_L$ has rank p. Let $H_L = U\Sigma V^T$ be the singular value decomposition and (A, B, C) be as in (2.19). Then (A, B, C) is a balanced realization of $\hat{G}(z)$ with Σ controllability and observability gramian. Suppose U, V and Σ are partitioned as in (2.21), then direct truncation of (A, B, C) can be obtained as*

$$\hat{A} = T_r A T_r^T = \Sigma_1^{-1/2} U_1^T \begin{bmatrix} 0 & I_{p(L-1)} \\ 0 & 0 \end{bmatrix} U_1 \Sigma_1^{1/2}, \tag{2.22}$$

$$\hat{B} = T_r B = \Sigma_1^{1/2} V_1^T \begin{bmatrix} I_m \\ 0 \end{bmatrix}, \quad \hat{C} = C T_r^T = \begin{bmatrix} I_p & 0 \end{bmatrix} U_1 \Sigma_1^{1/2},$$

where $T_t = \begin{bmatrix} I_n & 0 \end{bmatrix}$ is the truncation matrix. If Σ_1 and Σ_2 have no common element, then the reduced order model is stable.

Proof: The rank of H_L is the same as the minimal dimension of state-space model of $\hat{G}$ that is determined by minimal realization of $\hat{g}_L z^{-L}$ which is pL if and only if $\hat{g}_L$ has full rank p as $p \leq m$ by assumption. To simplify the notation, we denote

$$\tilde{U}_1 = \begin{bmatrix} I_{p(L-1)} & 0 \end{bmatrix} U, \quad \tilde{U}_2 = \begin{bmatrix} 0 & I_{p(L-1)} \end{bmatrix} U, \tag{2.23}$$

$$\tilde{V}_1 = \begin{bmatrix} I_{m(L-1)} & 0 \end{bmatrix} V, \quad \tilde{V}_2 = \begin{bmatrix} 0 & I_{m(L-1)} \end{bmatrix} V. \tag{2.24}$$

Note the difference between U_i and $\tilde{U}_i$, and V_i and $\tilde{V}_i$ for $i = 1, 2$. With A, B, C as in (2.19), we have $CB = \hat{g}_1$, and

$$A^{k+1} = \Sigma^{-1/2}\tilde{U}_1^T \left(\tilde{U}_2\tilde{U}_1\right)^k \tilde{U}_2\Sigma^{1/2}.$$

From (2.23), and $UU^T = I_{pL\times pL}$, we obtain

$$\tilde{U}_2\tilde{U}_1^T = S_{p(L-1)} = \begin{bmatrix} 0 & I_{p(L-2)} \\ 0 & 0 \end{bmatrix} \in \mathbf{R}^{p(L-1)\times p(L-1)},$$

a shifting matrix. It is easy to verify that

$$S_{p(L-1)}^k = \begin{bmatrix} 0 & I_{p(L-k-1)} \\ 0 & 0 \end{bmatrix} \in \mathbf{R}^{p(L-1)\times p(q-1)}.$$

The above implies that

$$A^{k+1} = \Sigma^{-1/2}\tilde{U}_1^T \begin{bmatrix} 0 & I_{p(L-k-1)} \\ 0 & 0 \end{bmatrix} \tilde{U}_2\Sigma^{1/2}$$

Substituting C and B, we obtain (using (2.23), and $UU^T = I_{pL\times pL}$),

$$\begin{aligned} CA^{k+1}B &= \begin{bmatrix} I_p & 0 \end{bmatrix} U\tilde{U}_1^T \begin{bmatrix} 0 & I_{p(L-k-1)} \\ 0 & 0 \end{bmatrix} \tilde{U}_2\Sigma^{1/2}\tilde{U}_2\Sigma V^T \begin{bmatrix} I_m \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} I_p & 0 \end{bmatrix} \begin{bmatrix} I_{p(L-1)} \\ 0 \end{bmatrix} \begin{bmatrix} 0 & I_{p(L-k-1)} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & I_{p(L-1)} \end{bmatrix} \begin{bmatrix} \hat{g}_1 \\ \hat{g}_2 \\ \cdot\cdot \\ \hat{g}_L \end{bmatrix} \\ &= \begin{bmatrix} I_p & 0 \end{bmatrix} \begin{bmatrix} \hat{g}_{k+2} \\ \cdot\cdot \\ \hat{g}_L \end{bmatrix} = \hat{g}_{k+2}, \end{aligned} \tag{2.25}$$

if $k < L-1$. For $k \geq L-1$, we have that $CA^{k+1}B = 0$ by $S^k_{p(L-1)} = 0$ if $k \geq L-1$. It follows that

$$\hat{G}(z) = D + C(zI_L - A)^{-1}B, \quad D = \hat{g}_0.$$

Now equation (2.18) implies that

$$\Sigma = \sum_{i=0}^{q-1} (A^T)^i C^T C A^i = \sum_{i=0}^{q-1} A^i B B^T (A^T)^i$$

that verifies the claim on controllability and observability gramian. Hence $(\hat{A}, \hat{B}, \hat{C})$ is a balanced truncation of (A, B, C) that is stable if Σ_1 and Σ_2 have no common element. ∎

For lightly damped systems, the FIR model $\hat{G}(z)$ obtained from linear algorithm is likely to have high order. Thus the singular value decomposition of H_L may have difficulty. In such a situation, it is suggested to use H_q with $q < L$ so that the computation of singular value decomposition of H_q is feasible. Although in this case, the resulting (A, B, C) as in (2.19) is not a realization of $\hat{G}(z)$, and Σ is not controllability gramian, there still holds (using the same derivation earlier)

$$CB = \hat{g}_1, \quad CA^{k+1}B = \begin{cases} \hat{g}_{k+2} & \text{if } 0 \leq k < q-1, \\ 0 & \text{if } k \geq q-1. \end{cases}$$

Moreover Σ is still the observability gramian as

$$\begin{aligned} \Sigma - A^T \Sigma A &= \Sigma - \Sigma^{1/2} U^T \begin{bmatrix} 0 \\ I_{p(q-1)} \end{bmatrix} \begin{bmatrix} 0 & I_{p(q-1)} \end{bmatrix} U \Sigma^{1/2} \\ &= \Sigma^{1/2} U^T \begin{bmatrix} I_p \\ 0 \end{bmatrix} \begin{bmatrix} I_p & 0 \end{bmatrix} U \Sigma^{1/2} = C^T C. \end{aligned}$$

Hence it can be expected that if q is close to L, the reduced order model still has good approximation to $\hat{G}(z)$ in terms of sup-norm. For reduced model

$$\hat{G}_n = \hat{D} + \hat{C}(zI_n - \hat{A})^{-1}\hat{B}, \quad \hat{D} = \hat{g}_0,$$

we may also try to compute a different $\hat{B}$ and $\hat{D}$ such that it minimizes

$$\|\hat{G}_n - \hat{G}\|_2^2 = \mathrm{tr}\left(\frac{1}{2\pi}\int_0^{2\pi}\left[\hat{G}_n(e^{j\omega}) - \hat{G}(e^{j\omega})\right]\left[\hat{G}_n(e^{j\omega}) - \hat{G}(e^{j\omega})\right]^* d\omega\right).$$

We may also replace $\hat{G}$ by experimental data E and the integral with summation to simplify the computation. In this case we have a least squares solution for $\hat{B}$ and $\hat{D}$. One may also use convex programming method to minimize the sup-norm that may improve significantly the approximation error.

Remark 2.1 *The same analysis holds for Kung's algorithm if $m < p$. In this case, we can take*

$$\begin{aligned} A &= \Sigma^{1/2}V^T \begin{bmatrix} 0 & 0 \\ I_{m(q-1)} & 0 \end{bmatrix} V\Sigma^{-1/2}, \\ \hat{A} &= T_r A T_r^T = \Sigma_1^{1/2}V_1^T \begin{bmatrix} 0 & 0 \\ I_{m(q-1)} & 0 \end{bmatrix} V_1\Sigma_1^{-1/2}. \end{aligned}$$

This modification gives a similar result to Proposition 2.5. It is noted that Kung's original algorithm does not consider the difference between $m \geq p$ and $p < m$.

2.5.2 Two Illustrative Examples

The purpose of this subsection is to illustrate the effectiveness of the linear algorithm, and its application to the identification of lightly damped systems in

conjunction with the Kung's algorithm. Two simulation examples are used. The first one assumes that the true system is given by

$$G(z) = \frac{10z^2}{1 + 5z + 10z^2} \ . \tag{2.26}$$

It can be verified that $G(z) \in S(M, \rho)$ with $M = 2.1$ and $\rho = 1.5$. The experimental data is generated by uniform samples of $G(e^{j\omega})$ with corrupting noise $\Delta_k = \delta e^{j\theta}$ where θ is a uniformly distributed random variable. We have chosen $\delta = 0.2$ which is roughly one-tenth of the $\|G\|_\infty$. The simulation consists of $N = 64$ experimental data points for both uniform and nonuniform sampling cases. The identified models are obtained using least-square based linear algorithm for $n = 15$. The magnitude error responses are plotted in Figure 1 with solid line for uniformly spaced sampling case and with dashed line for nonuniformly spaced sampling case. Because nonuniform sampling takes more samples at fast variation interval and fewer samples at slow variation interval of the frequency response, it often has a better performance than that of uniform sampling although it has a larger error bound.

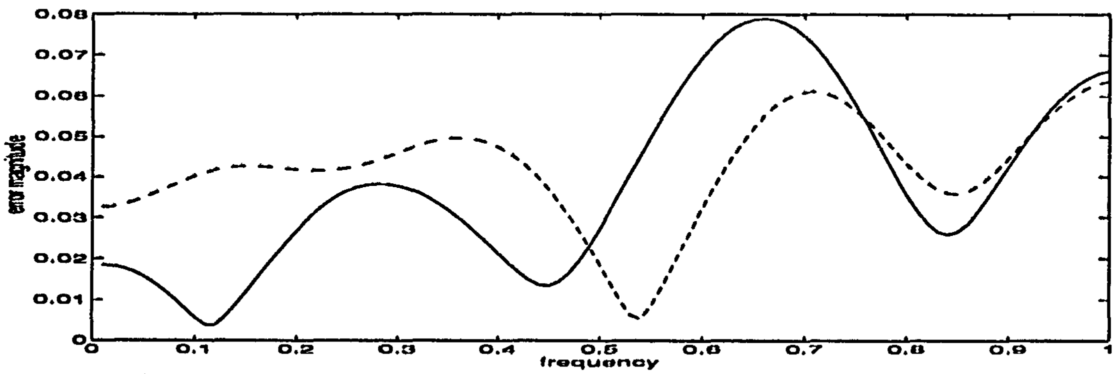


Figure 2.1: Magnitude response of the identification error for the first example

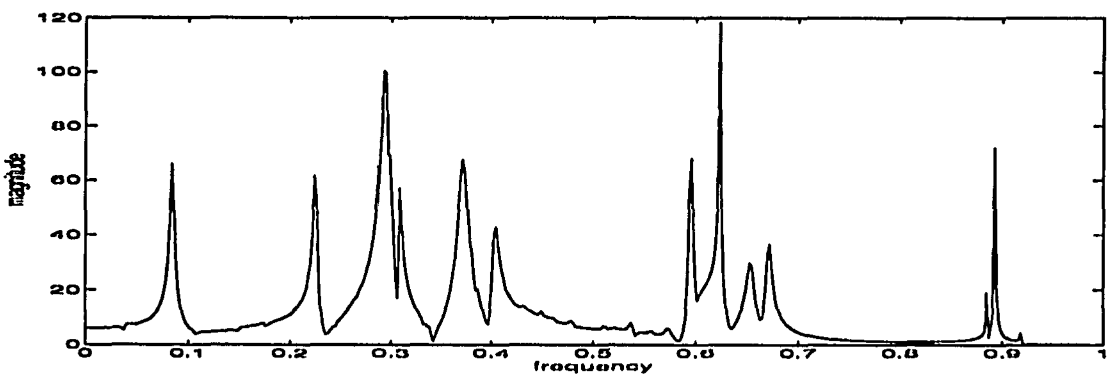


Figure 2.2: Magnitude response of the true system for the second example

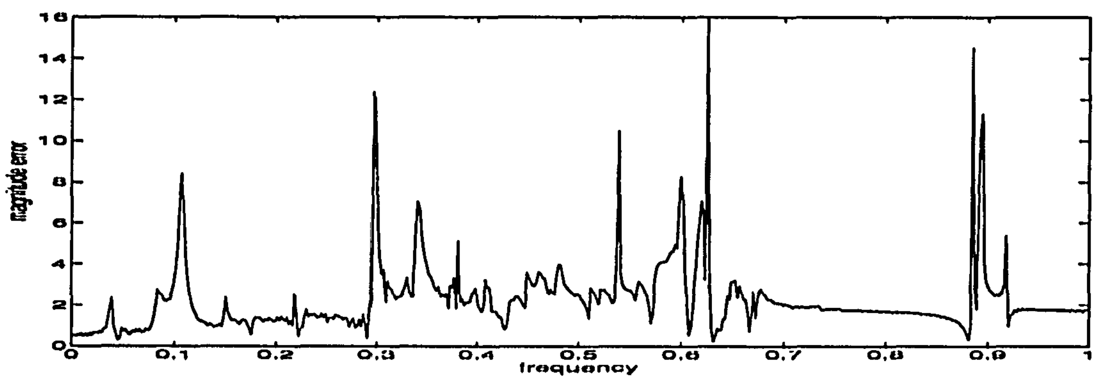


Figure 2.3: Magnitude response of the identification error for the second example

The second example is taken from [29] where the true system is a flexible structure. While the two-stage nonlinear algorithm in [35, 25] is not effective for identification of lightly damped systems [29], the linear algorithm studied in this paper, in conjunction with the Kung's algorithm, yields a very impressive result. The magnitude response of the experimental data is plotted in Figure 2 with solid line. Since no *a priori* information on M, ρ and δ are given in [29], we have used $M = 130$, $\rho = 1.01$ and $\delta = 0.5$. Corollary 2.1 is applied with $N = n = 1024$ to obtain the FIR model. Kung's algorithm is applied with $L = N - 1$, and $q = N/2$, to obtain a low order model of McMillan degree 24. Its magnitude error response is plotted in Figure 3. We would like to comment that although other algorithms can also obtain similar results with less computational effort, the linear algorithm studied in this section combined with the Kung's algorithm constitutes a convergent algorithm for identification in H_∞. Since only one singular value decomposition of a $q \times q$ matrix is involved with $q < N$, this example demonstrates that the algorithm is quite reliable. We tabulate the simulations for other values of (M, ρ, δ) and q in Table 2.1.

Table 2.1 Error Value with Different Order of Approximate Model

(M=130, ρ=1.01,N=1024,q=512)		(M=130, ρ=1.0055,N=1024,q=614)	
delta(δ)	error	delta(δ)	error
0.2	19.0157	0.2	15.4172
0.3	18.6541	0.3	15.1506
0.4	18.3343	0.4	18.1221
0.5	18.5875	0.5	19.9995
0.6	20.2805	0.6	20.4511
0.7	22.2652	0.7	20.3721
0.8	22.1595	0.8	20.3263
0.9	22.2164	0.9	20.7594
1.0	22.2864	1.0	21.1770
1.1	22.3434	1.1	21.5792

2.6 Conclusion

The least-square based linear algorithms in [36] are revisited and new error bounds are derived. It is shown that the *tuned* linear algorithms in [36] are applicable to nonuniformly spaced frequency response data which are quite different from the two-stage nonlinear algorithms as in [1]. In particular, exponential convergence for noise free case is preserved. Moreover upper bounds for stochastic error are derived that are comparable with the worst-case deterministic error bounds. This algorithm is also used for identification of lightly damped systems in conjunction

with Kung's algorithm that is very effective as shown in the example with a JPL flexible structure.

Chapter 3

Model Reference Control with $\mathcal{H}_\infty$ Loopshaping

3.1 Introduction

This chapter considers an alternative approach to model reference control (MRC). For the past two decades, MRC has been used extensively in adaptive control. However the control strategy employed in MRC is not sophisticated in the sense that the modeling error is ignored. Because MRC assumes the exact knowledge of the plant model, it is difficult to ensure the stability of the feedback control system designed with MRC in the presence of the model uncertainty. Consequently adaptive control based on MRC does not have stability robustness. As discussed in Chapter 1, robust adaptive control requires that both identification and control be capable of coping with model uncertainties. This chapter will focus on the control part. A novel approach based on $\mathcal{H}_\infty$ loopshaping will be employed to tackle MRC in the face of model uncertainties; the aim to enhance the stability robustness of the feedback control system. In this new approach, the reference model is determined by

the frequency shape of the ideal sensitivity function. Our objective is to synthesize a feedback controller that achieves the frequency shape of the ideal sensitivity specified by the reference model within a prescribed tolerance in terms of the $\mathcal{H}_\infty$ norm. This leads naturally to $\mathcal{H}_\infty$ loopshaping for MRC. Through appropriate formulation of the objective cost function, it will be shown that the resulting $\mathcal{H}_\infty$ controller has an observer structure for which only two gains, state feedback and state estimator gains, need to be synthesized. The corresponding $\mathcal{H}_\infty$ solution will also be derived and an efficient algorithm be developed for solving the $\mathcal{H}_\infty$ controller.

This chapter is organized as follows. We begin in Section 2 with problem formulation for loopshaping that is based on the frequency shape of sensitivity and complementary sensitivity functions. The loopshaping objective is to achieve the desired frequency shape of the ideal sensitivity specified by reference model. Because we are interested in observer-based controllers, some preliminary analysis is presented for observer-based feedback control systems. The $\mathcal{H}_\infty$ solution is derived in Section 3 and Section 4 for the MRC based loopshaping problem for continuous-time systems. It is shown that how the ideal sensitivity function can be synthesized based on the reference model that amounts to solving an output injection problem, and how the feedback controller can be synthesized by solving a state feedback problem. A dual case is the synthesis of the state feedback gain to achieve ideal frequency shape of the sensitivity specified by the reference model, and the synthesis of the output injection gain to recover the target loopshape. Section 5 treats

discrete-time systems. It shows that the results in Section 3 and Section 4 can be generalized to discrete-time systems, and the resulting $\mathcal{H}_\infty$ controller again has an observer form. Section 6 presents some simulation results on the $\mathcal{H}_\infty$ loopshape for MRC. In particular, a nonminimum phase system is used as an illustrative example to show the effectiveness of the proposed loopshaping algorithm.

3.2 Problem Formulation and Preliminary Analysis

The feedback system has configuration in Figure 3.1. The nominal plant has m inputs, p outputs, and admits a state-space realization

$$P(\delta) = C(\delta I_n - A)^{-1}B = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \tag{3.1}$$

where $\delta = s$ for continuous-time systems, and $\delta = z$ for discrete-time systems. If $m \leq p$ is not indicated particularly, It is implicitly assumed that $m \geq p$. The problem is to synthesize a stabilizing observer-based feedback controller

$$K(\delta) = -F(\delta I_n - A - BF - LC)^{-1}L = \left[\begin{array}{c|c} A + BF + LC & -L \\ \hline F & 0 \end{array}\right] \tag{3.2}$$

such that the closed-loop system resembles that of a given reference model in terms of the frequency shape for sensitivity and complementary sensitivity. These are equivalent to the frequency shape of the loop transfer matrix. Thus the reference model represents the ideal frequency shape of the loop transfer matrix and its state-

space realization is denoted by

$$R(\delta) = C_r(\delta I_r - A_r)^{-1}B_r = \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & 0 \end{array}\right]. \tag{3.3}$$

The ideal sensitivity and complementary sensitivity are given by

$$\begin{aligned} S_{id}(\delta) = (I_r - R(\delta))^{-1} &= I + C_r(\delta I_r - A_r - B_rC_r)^{-1}B_r \\ &= \left[\begin{array}{c|c} A_r + B_rC_r & B_r \\ \hline C_r & I_r \end{array}\right] \end{aligned} \tag{3.4}$$

$$\begin{aligned} T_{id}(\delta) = -R(\delta I_r - R(\delta))^{-1} &= -C_r(\delta I_r - A_r - B_rC_r)^{-1}B_r \\ &= \left[\begin{array}{c|c} A_r + B_rC_r & -B_r \\ \hline C_r & 0 \end{array}\right] \end{aligned} \tag{3.5}$$

respectively. We take the size of the square reference model $R(\delta)$, thus of $S_{id}(\delta)$ and $T_{id}(\delta)$, to be for $\min\{m, p\}$ for the reason to be made clear next.

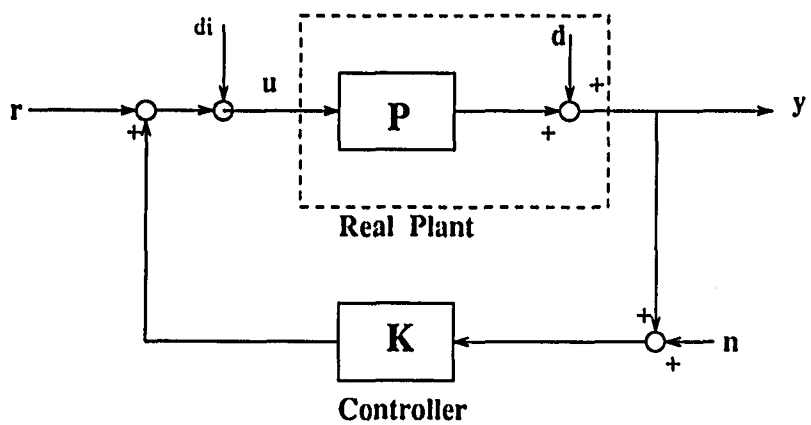


Figure 3.1: Feedback control system

Consider the feedback system in Figure 3.1. The plant P has size $p \times m$ and the controller K has size $m \times p$. The associated sensitivity functions are defined by

$$S_{in}(\delta) \;:=\; (I_m - L_{in})^{-1} = (I_m - K(\delta)P(\delta))^{-1}, \tag{3.6}$$

$$S_{out}(\delta) \;:=\; (I_p - L_{out})^{-1} = (I_p - P(\delta)K(\delta))^{-1}, \tag{3.7}$$

where $S_{in}(\delta)$ and $S_{out}(\delta)$ are the sensitivity functions at the plant input, and the plant output, respectively. The input loop gain L_{in} is obtained by breaking the loop at the input u of the plant and the output loop gain L_{out} is obtained by breaking the loop at the output y of the plant. It is well known that the frequency shape of the sensitivity functions measures the stability robustness and the performance of the feedback system [14]. For MIMO plant, $S_{in}(\delta) \neq S_{out}(\delta)$ in general, and in this case, it is difficult to synthesize a single controller $K(\delta)$ such that both sensitivity functions have desired frequency response. In fact, if $m > p$,

$$\overline{\sigma}(S_{in}(\delta)) = \frac{1}{\underline{\sigma}(I - K(\delta)P(\delta))} \geq 1$$

where the last inequality follows from the fact that $\underline{\sigma}(I - M) \leq 1 + \underline{\sigma}(M) = 1$ for the case $M = K(\delta)P(\delta)$ due to $m > p$. Hence, if the number of inputs is larger than the number of outputs, it does not make sense to minimize the sensitivity function at the plant input. Similarly, if the number of outputs is larger than the number of inputs, it does not make sense to minimize the sensitivity function at the plant output either.

The complementary sensitivity functions at the plant input and output for the feedback system in Figure 3.1 are given by

$$T_{in} = S_{in} - I_m = KP(I_m - KP)^{-1}, \;\; T_{out} = S_{out}(\delta) - I_p = PK(I_p - PK)^{-1} \tag{3.8}$$

respectively. Suppose that $m \geq p$. The performance measure for model reference

control is chosen as the relative error $\|E_{out}\|_\infty$ between the ideal sensitivity and the output sensitivity in (3.6) and (3.7) where

$$E_{out} = (S_{id} - S_{out})S_{id}^{-1} = I_p - S_{out}S_{id}^{-1} \tag{3.9}$$

Relative error has an advantage in that small relative error implies that $S_{out}(\delta)$ resembles $S_{id}(\delta)$ in frequency shape by $S_{out}(\delta) = (I_p + E_{out}(\delta))S_{id}(\delta)$. Consequently $T_{out}(\delta)$ also resembles $T_{id}(\delta)$ in frequency shape if $\|E_{out}\|_\infty$ is small due to the fact that

$$T_{out}(\delta) = S_{out} - I_p = T_{id}(\delta) + E_{out}(\delta)S_{id}(\delta)$$

More importantly, small relative error implies good loopshape properties as shown in the next result.

Proposition 3.1 *Consider the feedback system in Figure 3.1. Let $L(\delta) = P(\delta)K(\delta)$ be the loop transfer matrix of the feedback system. If $\gamma = \|E\|_\infty < 1$ where $E = I - S_{out}S_{id}^{-1}$, then there holds*

$$\bar{\sigma}(L(\delta)) \le \frac{\gamma + \bar{\sigma}(L_{id}(\delta))}{1-\gamma}, \quad \underline{\sigma}(L(\delta)) \ge \frac{\underline{\sigma}(L_{id}(\delta)) - \gamma}{1+\gamma}. \tag{3.10}$$

It follows that the frequency shape of the loop transfer function resembles that of ideal one if γ is small.

Proof: By the definition of relative error, we have

$$E = I - S_{out}S_{id}^{-1} = I - (I-L)^{-1}(I - L_{id}).$$

The argument δ is omitted for simplicity. To obtain L we rearrange left side and right side terms of the above equation as

$$I - E = (I - L)^{-1}(I - L_{id}).$$

It follows that

$$\begin{aligned} L &= I - (I - L_{id})(I - E)^{-1} \\ &= (I - E)(I - E)^{-1} - (I - L_{id})(I - E)^{-1} \\ &= (I - E - I + L_{id})(I - E)^{-1} \\ &= (L_{id} - E)(I - E)^{-1}. \end{aligned} \tag{3.11}$$

Since $\bar{\sigma}(\cdot)$ is norm, we obtain by $\gamma = \|E\|_\infty$,

$$\bar{\sigma}(L) \leq \bar{\sigma}(L_{id}(I - E)^{-1}) + \bar{\sigma}(E(I - E)^{-1}) \leq \frac{\gamma + \bar{\sigma}(L_{id})}{1 - \gamma}.$$

Furthermore, we obtain from (3.11) that

$$\underline{\sigma}(L) \geq \frac{\underline{\sigma}(L_{id} - E)}{1 + \gamma} \geq \frac{\underline{\sigma}(L_{id}(\delta)) - \gamma}{1 + \gamma}.$$

The inequalities in (3.10) are thus true. ∎

We emphasize that the use of relative error is consistent with the perturbation analysis. An important aspect for robust control is the stability and performance robustness in the face of uncertainties. Such uncertainties are often treated in forms of additive uncertainty and multiplicative uncertainty. Consider the case $m \geq p$. Let the true plant $P_t(\delta)$ be given by

$$P_t(\delta) = P(\delta) + \Delta_a(\delta)W_a(\delta), \quad \text{or} \quad P_t(\delta) = P(\delta)(I_m + \Delta_m(\delta))W_m(\delta) \tag{3.12}$$

where $P(\delta)$ is as in (3.1), Δ_a and Δ_m are unstructured uncertainty, and W_a and W_m are also weighting function or desired function in additive and multiplicative respectively as shown in Figure 3.2 and Figure 3.3. Then simple calculation yields that the true sensitivity is given by

$$S_{out} = S_t(I_p - \Delta_a W_a K S_{out}), \quad \text{or} \quad S_{out} = S_t(I_p - \Delta_m W_m P K S_{out}) \qquad (3.13)$$

for additive and multiplicative perturbed plant respectively. If the true plant $P_t(\delta)$ and the nominal plant $P(\delta)$ have the same number of unstable poles, and $\|\Delta_a\|_\infty \leq 1$, $\|\Delta_m\|_\infty \leq 1$, the robust stability conditions are given by

$$\epsilon_a = \|W_a K S_{out}\|_\infty < 1, \quad \text{and} \quad \epsilon_m = \|W_m P K S_{out}\|_\infty = \|W_m T_{out}\|_\infty < 1$$

respectively. Hence if the above conditions hold, then the relative error between the nominal and true sensitivity are given by

$$\epsilon = \|I_p - S_t^{-1} S_{out}\|_\infty = \|S_t^{-1}(S_t - S_{out})\|_\infty < 1$$

where $\epsilon = \epsilon_a = \|W_a K S_{out}\|_\infty$ for additive uncertain plant, and $\epsilon = \epsilon_m = \|W_m P K S_{out}\|_\infty$ for multiplicative uncertain plant.

For the case $m \leq p$, the performance measure is given by $\|E_{in}\|_\infty$ where

$$E_{in} = S_{id}^{-1}(S_{id} - S_{in}) = I_m - S_{id}^{-1} S_{in} \qquad (3.14)$$

and $S_{in}(\delta)$ is the sensitivity at the plant input. Similar conclusion holds for the frequency shape of the sensitivity and complementary sensitivity if $\|E_{in}\|_\infty$ is small. For perturbation analysis, the model uncertainties are assumed of the form

$$P_t(\delta) = P(\delta) + W_a(\delta)\Delta_a(\delta), \quad \text{or} \quad P_t(\delta) = (I_m + W_m(\delta)\Delta_m(\delta))P(\delta) \qquad (3.15)$$

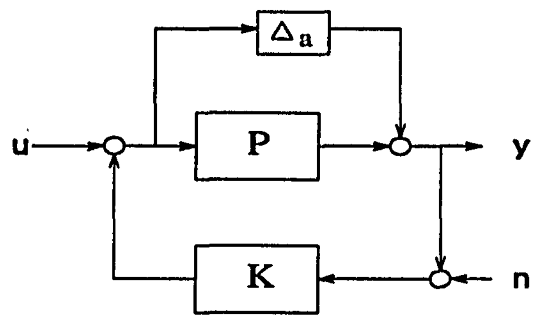


Figure 3.2: The additive uncertainty plant

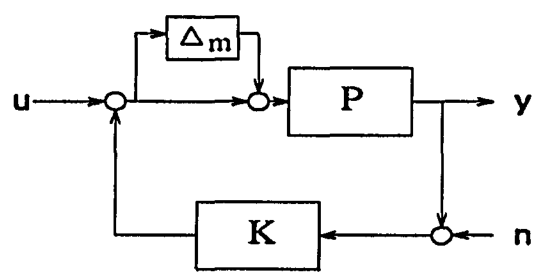


Figure 3.3: The multiplicative uncertainty plant

In this case, if the true plant $P_t(\delta)$ and the nominal plant $P(\delta)$ have the same number of unstable poles, and $\|\Delta_a\|_\infty \leq 1$, $\|\Delta_m\|_\infty \leq 1$, the robust stability conditions are given by

$$\epsilon_a = \|S_{in} K W_a\|_\infty < 1, \quad \text{and} \quad \epsilon_m = \|S_{in} K P W_m\|_\infty = \|T_{in} W_m\|_\infty < 1$$

respectively. With the above conditions, dual result

$$\epsilon = \|I_p - S_t^{-1} S_{out}\|_\infty = \|S_t^{-1}(S_t - S_{out})\|_\infty < 1$$

holds where $\epsilon = \epsilon_a = \|S_{in} K W_a\|_\infty$ for additive uncertain plant, and $\epsilon = \epsilon_m = \|S_{in} K P W_m\|_\infty$ for multiplicative uncertain plant.

Although the relative error is quite effective for loopshaping, it alone may not suffice the performance. For instance, in the case $m \geq p$, it is difficult to guarantee the frequency shape of $T_{in}(\delta)$ even if $\|E_{out}\|_\infty$ is small. This may cause stability robustness problem in the face of multiplicative uncertainty as in (3.15). A dual problem exists for the case $m \leq p$. Hence, a better performance measure as a mixed sensitivity problem is given by J_1 for $m \geq p$ and J_2 for $m \leq p$ as follows:

$$J_1 = \left\| \begin{bmatrix} (1-\lambda) E_{out} \\ \lambda T_{in} W_1 \end{bmatrix} \right\|_\infty, \quad \text{or} \quad J_2 = \left\| \begin{bmatrix} (1-\lambda) E_{in} & \lambda W_2 T_{out} \end{bmatrix} \right\|_\infty, \quad 0 < \lambda < 1. \tag{3.16}$$

The transfer matrices $W_1(\delta)$ and $W_2(\delta)$ are weighting functions that represent the inverse of the ideal $T_{in}(\delta)$ and $T_{out}(\delta)$ respectively. The parameter λ reflects the trade-off between the two cost functions of which one is relative error in sensitivity and the other is the weighted complementary sensitivity. Our problem is to synthe-

size an observer-based controller $K(\delta)$ as in (3.2) such that J_1 is minimized if $m \geq p$, or J_2 is minimized if $m \leq p$. To formalize the above discussion, we summarize the problem formulation as next.

MRC Loopshaping Problem:

Assume: the plant model has a realization in (3.1);

Given: ideal sensitivity in (3.4) specified by the reference model (3.3);

Find: an observer-based feedback controller of the form (3.2) such that

J_1 is minimized if $m \geq p$, or J_2 is minimized if $m \leq p$.

Before tackling the MRC loopshaping problem, we would like to provide some insight to the observer-based controller design, and illustrate some inherent advantages in using observers for $\mathcal{H}_\infty$ loopshaping. The following set of identities are useful. See also [52, 59, 62].

Lemma 3.1 *Let $P(\delta) = C(\delta I - A)^{-1}B$ be the plant model where (A, B) is stabilizable and (C, A) is detectable. Let $K(\delta) = -F(\delta I - A - BF - LC)^{-1}L$ be observer-based controller. Then there hold*

$$(I - P(\delta)K(\delta))^{-1}P(\delta)K(\delta) = -C(\delta I - A - BF)^{-1}BF(\delta I - A - LC)^{-1}L,$$

$$(I - P(\delta)K(\delta))^{-1}P(\delta) = C(\delta I - A - BF)^{-1}B\left(I - F(\delta I - A - LC)^{-1}B\right),$$

$$= \left(I - C(\delta I - A - BF)^{-1}L\right)C(\delta I - A - LC)^{-1}B,$$

$$(I - P(\delta)K(\delta))^{-1} = \left(I - C(\delta I - A - BF)^{-1}L\right)\left(I + C(\delta I - A - LC)^{-1}L\right),$$

$$K(\delta)(I-P(\delta)K(\delta))^{-1} = -\left(I+F(\delta I-A-BF)^{-1}B\right)F(\delta I-A-LC)^{-1}L,$$

$$= -F(\delta I-A-BF)^{-1}L\left(I+C(\delta I-A-LC)^{-1}L\right),$$

$$(I-K(\delta)P(\delta))^{-1} = \left(I+F(\delta I-A-BF)^{-1}B\right)\left(I-F(\delta I-A-LC)^{-1}B\right),$$

$$(I-K(\delta)P(\delta))^{-1}K(\delta)P(\delta) = -F(\delta I-A-BF)^{-1}LC(\delta I-A-LC)^{-1}B.$$

Proof: By stabilizability and detectability of (A,B,C), there exist left and right coprime factorizations for plant and observer-based controller

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \quad K = UV^{-1} = \tilde{V}^{-1}\tilde{U}$$

where the space realization of M and N is defined by

$$M(\delta) = \left[\begin{array}{c|c} A+BF & B \\ \hline F & I \end{array}\right], \quad N(\delta) = \left[\begin{array}{c|c} A+BF & B \\ \hline C & 0 \end{array}\right]$$

with stabilizing state feedback gain F and output injection gain L respectively. Thus there hold RCF and LCF for plant P as

$$\begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{c|c} A+BF & B \\ \hline F & I \\ C & 0 \end{array}\right], \quad \begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} = \left[\begin{array}{c|cc} A+LC & L & B \\ \hline C & I & 0 \end{array}\right].$$

Similarly, RCF and LCF of controller K are obtained by

$$\begin{bmatrix} V \\ U \end{bmatrix} = \left[\begin{array}{c|c} A+BF & -L \\ \hline C & I \\ F & 0 \end{array}\right], \quad \begin{bmatrix} \tilde{V} & \tilde{U} \end{bmatrix} = \left[\begin{array}{c|cc} A+LC & -B & -L \\ \hline F & I & 0 \end{array}\right].$$

Combining the above formulas yields

$$\begin{bmatrix} \tilde{M} & \tilde{N} \\ \tilde{U} & \tilde{V} \end{bmatrix} = \left[\begin{array}{c|cc} A+LC & L & B \\ \hline C & I & 0 \\ -F & 0 & I \end{array}\right],$$

$$\begin{bmatrix} V & -N \\ -U & M \end{bmatrix} = \left[\begin{array}{c|cc} A+BF & L & B \\ \hline -C & I & 0 \\ F & 0 & I \end{array}\right].$$

It follows that the doubly Bezout identity [50]:

$$\begin{bmatrix} \tilde{M} & \tilde{N} \\ \hat{U} & \hat{V} \end{bmatrix} \begin{bmatrix} V & -N \\ -U & M \end{bmatrix} = \begin{bmatrix} V & -N \\ -U & M \end{bmatrix} \begin{bmatrix} \tilde{M} & \tilde{N} \\ \hat{U} & \hat{V} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

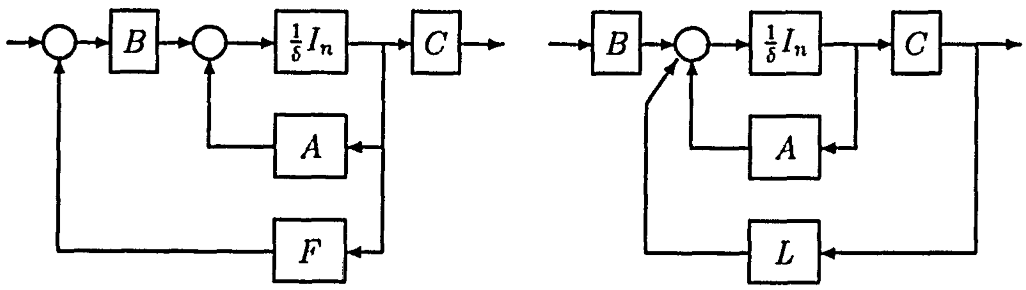
holds. Consequently,

$$\begin{aligned} (I - P(\delta)K(\delta))^{-1}P(\delta)K(\delta) &= N\hat{U} \\ (I - P(\delta)K(\delta))^{-1}P(\delta) &= N\hat{V} = V\tilde{N} \\ (I - P(\delta)K(\delta))^{-1} &= V\tilde{M} \\ K(\delta)(I - P(\delta)K(\delta))^{-1} &= M\hat{U} = U\tilde{M} \\ (I - K(\delta)P(\delta))^{-1} &= M\hat{V} \\ (I - K(\delta)P(\delta))^{-1}K(\delta)P(\delta) &= U\tilde{N} \end{aligned}$$

Substitution of the state-space realization into the above equation can then conclude the lemma. ∎

The identities in Lemma 3.1 allow us to relate the sensitivity and complementary sensitivity transfer matrices for feedback system in Figure 3.1 with those for state feedback and output injection. Consider the two feedback systems in Figure 3.4 where either state feedback, or output injection is used. The associated sensitivity functions and overall transfer function matrices are given by

$$S_{sf}(\delta) = I + F(\delta I - A - BF)^{-1}B, \quad T_{sf}(\delta) = -C(\delta I - A - BF)^{-1}B, \quad (3.17)$$



(a) (b)

Figure 3.4: State feedback and output injection

$$S_{oi}(\delta) = I + C(\delta I - A - LC)^{-1}L, \quad T_{oi}(\delta) = -C(\delta I - A - LC)^{-1}B, \quad (3.18)$$

respectively. It is interesting to note that $P = (-T_{sf})\, S_{sf}^{-1} = S_{oi}^{-1}\,(-T_{oi})$ are also left and right coprime factorizations of the plant model and thus

$$\tilde{M}(\delta) = S_{oi}(\delta), \quad \tilde{N}(\delta) = -T_{oi}(\delta), \quad M(\delta) = S_{sf}(\delta), \quad N(\delta) = -T_{sf}(\delta).$$

Furthermore by Lemma 3.1, the sensitivity and complementary sensitivity functions with observer-based output feedback controller are related to those of state feedback and output injection as follows:

$$S_{out}(\delta) = \left(I - C(\delta I - A - BF)^{-1}L\right) S_{oi}(\delta), \quad (3.19)$$

$$T_{in}(\delta) = -F(\delta I - A - BF)^{-1}LT_{oi}(\delta), \quad (3.20)$$

$$S_{in}(\delta) = S_{sf}(\delta)\left(I - F(\delta I - A - LC)^{-1}B\right), \quad (3.21)$$

$$T_{out}(\delta) = -T_{sf}(\delta)F(\delta I - A - LC)^{-1}L. \quad (3.22)$$

The above relations, though simple, are the very reason why state feedback gain

and output injection gain can be designed separately for observer-based controller. Indeed as shown in the next section, our synthesis algorithm is a two-step algorithm of which the output injection gain is synthesized in the first step for the case $m \geq p$ to achieve the ideal frequency shape for S_{oi} and T_{oi}, and then the state feedback gain is synthesized in the second step to recover the ideal frequency shape of the sensitivity and complementary sensitivity. A dual case is $m \leq p$ where the state feedback gain is synthesized in the first step to achieve the ideal frequency shape for S_{sf} and T_{sf}, and then the state feedback gain is synthesized in the second step to recover the ideal frequency shape of the sensitivity and complementary sensitivity. For this reason, we choose the weighting function $W_1(\delta)$ as

$$W_1 = T_{oi}^{+} = \begin{cases} \text{right inverse of } T_{oi} & \text{if } m \geq p, \\ \text{left inverse of } T_{sf} & \text{if } m \leq p \end{cases} \tag{3.23}$$

It follows that

$$PW_1 = S_{oi}^{-1} T_{oi} T_{oi}^{+} = S_{oi}^{-1} = S_{id}^{-1}$$

for $m \geq p$ and

$$W_1 P = T_{sf}^{+} T_{sf} S_{sf}^{-1} = S_{id}^{-1}$$

for $m \leq p$. That is, the weighting function W_1 is chosen as the generalized inverse of the ideal complementary sensitivity. In the next three sections, the state-space solutions for the $\mathcal{H}_\infty$ loopshaping problem will be derived for both continuous-time and discrete-time systems.

3.3 Observer-Based Loopshaping with $\mathcal{H}_\infty$ Criterion

Loopshaping design is one of the few direct ties between modern control theory and engineering practice. The basic idea is embedded in classical lead-lag compensator design [72]. The gain and phase margins achieved through shaping the loop transfer function are now understood in the context of robust control [14, 16] and efforts have been made to extend those classical results to multivariable systems which result in loopshaping. Thus, loopshaping can be viewed as a natural extension of the classical control system design to multivariable systems which unifies the design of SISO (single-input/single-output) and MIMO (multi-input/multi-output) feedback control systems and which enhances the robustness of the multivariable feedback systems.

There are two basic approaches to loopshaping. The first one is based on LQG control [16, 68, 62]. In this approach, a state feedback gain is synthesized first to obtain the desired loopshape, and then a state estimator gain is designed to recover the loopshape achieved with state feedback. The feedback compensator is a standard observer which has the same order as the plant model. Although LQG based loopshaping is simple and effective, it is in general difficult to obtain the desired loop shape for the state feedback case and difficult to recover the desired loop shape for nonminimum phase systems because of the use of the $\mathcal{H}_2$ norm, or mean square cost function in the LQG control. This gives rise to a second approach

which is based on $\mathcal{H}_\infty$ control [48]. In the second approach, the desired loopshape is specified by shaped plant through the use of simple weighting functions. An output feedback controller is then synthesized to minimize the $\mathcal{H}_\infty$ norm of a certain augmented feedback system to achieve the desired loopshape. It is interesting to note that the resulting feedback controller is again an observer and thus it also has an LQG interpretation. However both methods are difficult to be generalized to fit the MRC that is used extensively for adaptive control.

In this dissertation, we investigate $\mathcal{H}_\infty$ based loopshaping from a different perspective. Reference model is used to represent the frequency shape of the ideal sensitivity function. The control objective is to design a feedback controller such that it achieves the desired frequency shape specified by the reference model within a prescribed tolerance in $\mathcal{H}_\infty$ norm. This leads to the $\mathcal{H}_\infty$ loopshaping for model reference control. It is shown that the resulting $\mathcal{H}_\infty$ controller has an observer structure for which only two gains, state feedback and output injection gains, need be synthesized. We begin with our first result on the MRC loopshaping problem formulated in Section 2.

Theorem 3.2 *Let the physical plant $P(s)$ be given as in (3.1) with $m \geq p$, and J_1 be the performance index for the $\mathcal{H}_\infty$ loopshaping problem. Suppose that the ideal sensitivity and the weighting function are given by*

$$S_{id} = I + C(sI_n - A - HC)^{-1}H, \quad W_1 = P^+ S_{id}^{-1}$$

for some stabilizing H where P^+ is the right inverse of $P(s)$ that exists by $m \geq$

p. Then the resulting $\mathcal{H}_\infty$ controller has an observer form (3.2). Moreover, the stabilizing controller achieving $J_1 < \gamma$ is given by $L = H$, and $F = -\lambda^{-2}B^*X$ where X is the stabilizing solution for

$$A^*X + XA - X\left(\lambda^{-2}BB^* - \gamma^{-2}HH^*\right)X + (1-\lambda)^2C^*C = 0.$$

Proof: Since $PP^+ \equiv I_p$, there holds $T_{in}W_1 = (I_m - KP)^{-1}KS_{id}^{-1}$. From equations (3.8) and (3.9), we define transfer matrix $T_{J1}(s)$ by

$$T_{J1} := \begin{bmatrix} (\lambda-1)E_{out} \\ \lambda T_{in}W_1 \end{bmatrix} = \begin{bmatrix} (1-\lambda)I_p & 0 \\ 0 & \lambda I_m \end{bmatrix} \begin{bmatrix} (I_p - PK)^{-1}S_{id}^{-1} - I_p \\ (I_m - KP)^{-1}KS_{id}^{-1} \end{bmatrix}.$$

Then $J_1 = \|T_{J1}\|_\infty$. In order to use the state-space formulas in [15, 20], we write $T_{J1}(s) = \mathcal{F}(G, \lambda K)$ in the form of linear fraction:

$$T_{J1} = \begin{bmatrix} (1-\lambda)\left(S_{id}^{-1} - I_p\right) \\ 0 \end{bmatrix} + \begin{bmatrix} (1-\lambda)P/\lambda \\ I_m \end{bmatrix} \lambda K \left(I_p - (P/\lambda)(\lambda K)\right)^{-1} S_{id}^{-1}.$$

It follows that the generalized plant is given by

$$G = \left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right] = \left[\begin{array}{c|c} (1-\lambda)(S_{id}^{-1} - I_p) & (1-\lambda)P/\lambda \\ 0 & I_m \\ \hline S_{id}^{-1} & P/\lambda \end{array}\right].$$

A state-space realization for $G(s)$ is given by

$$\begin{aligned} G(s) &= \left[\begin{array}{c|c} 0 & 0 \\ 0 & I_m \\ \hline I_p & 0 \end{array}\right] + \left[\begin{array}{c} (1-\lambda)C \\ 0 \\ \hline C \end{array}\right] (sI - A)^{-1} \left[\begin{array}{c|c} -H & B/\lambda \end{array}\right] \\ &= \left[\begin{array}{c|c|c} A & -H & B/\lambda \\ \hline (1-\lambda)C & 0 & 0 \\ 0 & 0 & I_m \\ \hline C & I_p & 0 \end{array}\right]. \end{aligned}$$

Applying the state-space formulae in [20], we have that there exists a stabilizing controller such that $\|J_1\|_\infty < \gamma$ for some $\gamma > 0$ if and only if there exist stabilizing solutions $X \geq 0$ and $Y \geq 0$ satisfying

$$A^*X + XA - X\left(\lambda^{-2}BB^* - \gamma^{-2}HH^*\right)X + (1-\lambda)^2C^*C = 0, \quad (3.24)$$

$$(A+HC)Y + Y(A+HC)^* - \alpha^2 YC^*CY = 0, \quad (3.25)$$

where $\alpha^2 = 1 - \left(\frac{1-\lambda}{\gamma}\right)^2$ and $\rho(XY) < \gamma^2$. It is noted that $\gamma^2 > (1-\lambda)^2$ in order for the $\mathcal{H}_\infty$ problem to be solvable. As H is stabilizing, it follows that $Y = 0$ is the stabilizing solution to (3.25). In this case the condition $\rho(XY) < \gamma^2$ is automatically satisfied. Suppose that $X \geq 0$ is a stabilizing solution to (3.24), then the central controller can be easily computed according to [20]

$$K(s) = -F(sI - A - BF - LC)^{-1}L, \quad F = -\lambda^{-2}B^*X, \quad L = H$$

which is indeed an observer. ∎

We note that $W_1 = T_{oi}^+$ is the generalized inverse of the ideal complementary sensitivity in light of (3.23). Hence, with the assumption that H is stabilizing, our first result gives a simple solution to the MRC loopshaping problem. In addition Theorem 3.2 indicates that the synthesis of the $\mathcal{H}_\infty$ controller for $\mathcal{H}_\infty$ loopshaping is a two-step procedure. The first step is to synthesize a stabilizing output injection gain $L = H$ such that both $S_{id} = S_{oi}$ and T_{oi} have desired frequency shape. The second step is to synthesize the state feedback gain F such that it minimizes the performance index J_1 through solving the stabilizing solution for (3.24). This

two step procedure decouples the synthesis of the state feedback gain and output injection gain, and thus results in more efficient algorithm for $\mathcal{H}_\infty$ design. Clearly a crucial problem is the synthesis of the output injection gain $L = H$ such that it achieves the desired frequency shape for both $S_{oi}(s)$ and $T_{oi}(s)$. It is interesting to note that this can be achieved by shaping the plant model directly. Indeed suppose that the original plant $P_o(s)$ has size $p \times m$ with $p \leq m$. Suppose further that there exists $W(s)$ having size $m \times p$ such that $P(s) = P_o(s)W(s)$ has an equal number of inputs and outputs and has a desirable loopshape. Often $W(s)$ can be chosen as a PI type compensator. Our strategy is the same as in [48] by synthesizing $L = H$ such that $P(s) = S_{oi}^{-1}T_{oi}$ such that there holds power complementary condition

$$S_{oi}S_{oi}^* + T_{oi}T_{oi}^* \equiv I_p \tag{3.26}$$

Proposition 3.3 *Suppose that the shaped plant model $P(s) = P_o(s)W(s) = C(sI - A)^{-1}B$ has an equal number of inputs and outputs and has a desirable loopshape. Suppose further that $P(s)$ has a stabilizable and detectable realization (A, B, C). Then there exists a stabilizing output injection gain L such that S_{oi} and T_{oi} satisfying (3.26). Furthermore such an L achieves approximate desirable frequency shape for S_{oi} and T_{oi} and can be computed by $L = -ZC^*$ where Z is the stabilizing solution of the following algebraic Riccati equation*

$$AZ + ZA^* - ZC^*CZ + BB^* = 0. \tag{3.27}$$

Proof: For any stabilizing output injection gain L, $P(s) = \tilde{M}^{-1}\tilde{N}$ is a left

coprime factorization where

$$\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} = C(sI - A - LC)^{-1}\begin{bmatrix} L & B \end{bmatrix} + \begin{bmatrix} I_p & 0 \end{bmatrix}.$$

By [48], $(\tilde{M}, \tilde{N})$ is a pair of normalized coprime factorization satisfying (3.26) if $L = -ZC^*$ where Z is the stabilizing solution of (3.27). Since $S_{oi} = \tilde{M}$, $T_{oi} = -\tilde{N}$, and $\tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^* = I$, it follows that

$$\begin{aligned} S_{oi}^{-1}(S_{oi}^{-1})* &= \tilde{M}^{-1}(\tilde{M}^{-1})^* \\ &= \tilde{M}^{-1}(\tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^*)(\tilde{M}^{-1})^* \\ &= \tilde{M}^{-1}\tilde{M} + \tilde{M}^{-1}\tilde{N}\tilde{N}^*(\tilde{M}^{-1})^* \\ &= I + PP^* \end{aligned}$$

We conclude that S_{oi} and T_{oi} are power complementary to each other. It is easy to see that

$$\begin{aligned} \overline{\sigma}(S_{oi}(j\omega)) &= 1/\sqrt{\underline{\sigma}(I + P(j\omega)[P(j\omega)]^*)}, \\ \overline{\sigma}(T_{oi}(j\omega)) &\leq \sqrt{1 - 1/\underline{\sigma}(I + [P(j\omega)]^* P(j\omega))}. \end{aligned}$$

Hence, both S_{oi} and T_{oi} have an approximate desirable frequency shape provided that $P(s)$ has a desirable frequency loopshape. ∎

Dual results to Theorem 3.2 and Proposition 3.3 can be derived for the case $m \leq p$. Our objective is similar to the case $m \geq p$ in that a stabilizing feedback controller $K(s)$ is synthesized such that the sensitivity and the complementary sensitivity functions achieve the desired loopshape using the $\mathcal{H}_\infty$ optimization technique.

Theorem 3.4 *Let the physical plant $P(s)$ be given as in (3.1) with $m \le p$, and J_2 be the performance index for the $\mathcal{H}_\infty$ loopshaping problem. Suppose that the ideal sensitivity and the weighting function are given by*

$$S_{id} = I + H(sI_n - A - BH)^{-1}B, \quad W_2 = S_{id}^{-1}P^+$$

for some stabilizing H where P^+ is the left inverse of $P(s)$ that exists by $m \le p$. Then the resulting $\mathcal{H}_\infty$ controller has an observer form. Moreover, the stabilizing controller achieving $J_2 < \gamma$ is given by $F = H$, and $L = -\lambda^{-2}YC^*$ where Y is the stabilizing solution for

$$AY + YA^* - Y\left(\lambda^{-2}C^*C - \gamma^{-2}H^*H\right)Y + (1-\lambda)^2BB^* = 0.$$

Proof: Since $P^+P \equiv I_p$, there holds $W_2T_{out} = S_{id}^{-1}K(I_p - PK)^{-1}$. From equations (3.8) and (3.14), we define transfer matrix $T_{J2}(s)$ by

$$\begin{aligned}
T_{J2} &= \begin{bmatrix} (1-\lambda)E_{in} & \lambda W_2 T_{out} \end{bmatrix} \\
&= \begin{bmatrix} S_{id}^{-1}(I-KP)^{-1} - I_m & S_{id}^{-1}K(I_p - PK)^{-1} \end{bmatrix} \begin{bmatrix} (1-\lambda)I_m & 0 \\ 0 & \lambda I_p \end{bmatrix} \\
&= \left(\begin{bmatrix} S_{id}^{-1} - I_m & 0 \end{bmatrix} + S_{id}^{-1}K(I-PK)^{-1} \begin{bmatrix} P & I \end{bmatrix} \right) \begin{bmatrix} (1-\lambda)I_m & 0 \\ 0 & \lambda I_p \end{bmatrix}
\end{aligned}$$

Then $J_2 = \|T_{J2}\|_\infty$. In order to use the state-space formulas in [15, 20], we write $T_{J2}(s) = \mathcal{F}(G, \lambda K)$ in the form of linear fraction

$$T_{J2} = \begin{bmatrix} (S_{id}^{-1} - I_m)(1-\lambda) & 0 \end{bmatrix} + S_{id}^{-1}\lambda K(I - ((P/\lambda)(\lambda K))^{-1} \begin{bmatrix} (1-\lambda)P/\lambda & I_p \end{bmatrix}.$$

It follows that the generalized plant is given by

$$G = \left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right] = \left[\begin{array}{cc|c} (S_{id}^{-1} - I_m)(1-\lambda) & 0 & S_{id}^{-1} \\ \hline (1-\lambda)P/\lambda & I_p & P/\lambda \end{array}\right].$$

A state-space realization for $G(s)$ is given by

$$\begin{aligned} G(s) &= \left[\begin{array}{cc|c} 0 & 0 & I_m \\ \hline 0 & I_p & 0 \end{array}\right] + \left[\begin{array}{c} -H \\ \hline \lambda^{-1}C \end{array}\right] (sI - A)^{-1} \left(\left[\begin{array}{cc|c} (1-\lambda)B & 0 & B \end{array}\right]\right), \\ &= \left[\begin{array}{c|cc|c} A & (1-\lambda)B & 0 & B \\ \hline -H & 0 & 0 & I_m \\ \hline \lambda^{-1}C & 0 & I_p & 0 \end{array}\right] \end{aligned}$$

Applying the state-space formulae in [20], we have that there exists a stabilizing controller such that $\|J_2\|_\infty < \gamma$ for some $\gamma > 0$ if and only if there exist stabilizing solutions $X \geq 0$ and $Y \geq 0$ satisfying

$$(A + BH)^* X + X(A + BH) - \alpha^2 XBB^* X = 0, \tag{3.28}$$

$$YA^* + AY - Y(\lambda^{-2}C^*C - \gamma^{-2}H^*H)Y + (1-\lambda)^2 BB^* = 0, \tag{3.29}$$

respectively where $\alpha^2 = 1 - \left(\frac{1-\lambda}{\gamma}\right)^2$, and $\rho(XY) < \gamma^2$. It is noted that $\gamma^2 > (1-\lambda)^2$ in order for the $\mathcal{H}_\infty$ problem to be solvable. As F is stabilizing, it follows that $X = 0$ is the stabilizing solution to (3.28). In this case the condition $\rho(XY) < \gamma^2$ is automatically satisfied. Suppose that $Y \geq 0$ is a stabilizing solution to (3.29), then the central controller can be easily computed according to [20]

$$K(s) = -F(sI - A - BF - LC)^{-1}L, \quad L = -\lambda^{-2}YC^*, \quad F = H$$

which is indeed of observer form. ∎

We note that Theorem 3.4 is dual to Theorem 3.2 by the dual relation $L \rightarrow F^*$, $B \rightarrow C^*$, $X \rightarrow Y^*$, and $(S_{oi}, T_{oi}) \rightarrow (S_{sf}, T_{sf})$ respectively. We recognize that J_2 is the output injection problem and J_1 is the state feedback problem for standard $\mathcal{H}_\infty$ control problem. We also note that $W_2 = T_{sf}^+$ in light of (3.23). Thus the synthesis of the state feedback gain $F = H$ is crucial here. We now use the same procedure as the dual case by synthesizing H such that with $P(s) = T_{sf}^{-1} S_{sf}$ there holds power complementary condition

$$S_{sf}^* S_{sf} + T_{sf}^* T_{sf} \equiv I_m. \tag{3.30}$$

As in this case, the relation

$$\begin{aligned}
\overline{\sigma}(S_{sf}(j\omega)) &= 1/\sqrt{\underline{\sigma}\,(I + [P(j\omega)]^* P(j\omega))}, \\
\overline{\sigma}(T_{sf}(j\omega)) &\leq \sqrt{1 - 1/\underline{\sigma}\,(I + P(j\omega)[P(j\omega)]^*)}.
\end{aligned}$$

is true that implies good frequency shape for S_{sf} and T_{sf} provided that $P(s)$ can in turn be shaped directly with suitable PI compensator. The next result is dual to Proposition 3.3 and thus the proof is omitted.

Proposition 3.5 *Suppose that the shaped plant model $P(s) = Z(s)P_o(s) = C(sI - A)^{-1}B$ has an equal number of inputs and outputs and has a desirable loopshape. Suppose further that $P(s)$ has a stabilizable and detectable realization (A, B, C). Then there exists a stabilizing output injection gain F such that S_{sf} and T_{sf} satisfying (3.30). Furthermore such an F achieves approximate desirable frequency shape for S_{sf} and T_{sf} and can be computed by $F = -B^*X$ where X is the stabilizing*

solution of the following algebraic Riccati equation

$$A^*X + XA - XBB^*X + C^*C = 0. \qquad (3.31)$$

3.4 Model Reference Control with $\mathcal{H}_\infty$ Loopshaping

The purpose of this section is to study model reference control using $\mathcal{H}_\infty$ loopshaping procedure as developed in Section 3 with the objective to enhance the robust stability and performance for MRC using $\mathcal{H}_\infty$ loopshaping methodology. For this purpose, consider the plant model given by

$$P(s) = C_p(sI_n - A_p)^{-1}B_p. \qquad (3.32)$$

We consider again the case $m \geq p$ first. For the reference model in (3.3), we augment the plant according to

$$A = \begin{bmatrix} A_p & 0 \\ 0 & A_r \end{bmatrix}, \quad B = \begin{bmatrix} B_p \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_p & C_r \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ B_r \end{bmatrix}. \qquad (3.33)$$

Then there holds

$$C(sI_{n+r} - A)^{-1}H = C_r(sI_r - A_r)^{-1}B_r = R(s) \qquad (3.34)$$

and

$$C(sI_{n+r} - A)^{-1}B = C_p(sI_n - A_p)^{-1}B_p = P(s) \qquad (3.35)$$

In this case $S_{id}(s) = (I_p - R(s))^{-1} = I_p + C(sI_{n+p} - A - HC)^{-1}H$ where

$$A + HC = \begin{bmatrix} A_p & 0 \\ B_rC_p & A_r + B_rC_r \end{bmatrix}$$

It follows that the matrix $A+HC$ may not be stable if $P(s)$ is not. Thus the results in Section 3 can not be directly applied for model reference control. In fact even the existence of the $\mathcal{H}_\infty$ controller such that $J_1 < \gamma$ for some $\gamma > 0$ is questionable. The next result gives a necessary and sufficient condition for $J_1 < \gamma$ to be solvable with stabilizing controllers.

Theorem 3.6 *Let $P(s) = C(sI_n - A)^{-1}B$ with stabilizable and detectable realization. Let the weighting function in $T_{J1}(s)$ be the same as in Theorem 3.2. Suppose that $A + HC$ is unstable. Then there exists a stabilizing controller $K(s)$ such that $J_1 < \gamma$ for some $\gamma > 0$ if and only if $A + HC$ has no eigenvalues on imaginary axis. If the condition for $A+HC$ having no eigenvalues on imaginary axis holds, then (a) there exists a stabilizing solution $Y \geq 0$ for (3.25), such that the following transfer matrix*

$$V(s) = I + C(sI_{n+r} - A - L_sC)^{-1}(L_s - H), \tag{3.36}$$

is stable and all-pass where $L_s = H - \alpha^2 YC^*$; and (b) the central $\mathcal{H}_\infty$ controller achieving $J_1 < \gamma$ is given by

$$K(s) = -F\left(sI - A - \gamma^{-2}[H - Z_\infty L_s]H^*X + \lambda^{-2}BB^*X - Z_\infty L_sC\right)^{-1} Z_\infty L_s \tag{3.37}$$

where $F = -\lambda^{-2}B^*X$ with $X \geq 0$ a stabilizing solution for (3.24) and $Z_\infty = (I_n - \gamma^{-2}YX)^{-1}$.

Proof: In light of Theorem 3.2, the existence of the stabilizing controller such that $J_1 < \gamma$ is determined by the existence of the stabilizing solution X and Y

for (3.24) and (3.25) respectively, and the coupling condition $\rho(XY) < \gamma^2$. The equation (3.24) is a standard $\mathcal{H}_\infty$ algebraic Riccati equation that admits a unique solution by the given hypothesis provided that γ is sufficiently large. The coupling condition can also always be met for some $\gamma > 0$. Now the existence of the $\mathcal{H}_\infty$ controller amounts to the existence of the stabilizing solution for (3.25). Since (3.25) corresponds to standard LQR problem for dynamic system $\dot{x}(t) = A^*x(t) + C^*u(t)$ with zero state weighting, it follows from [42] that, by the hypothesis that (C, A) is detectable which is equivalent to that $(C, A + HC)$ is detectable, the existence of the stabilizing solution $Y \geq 0$ is equivalent to that the matrix $A + HC$ has no eigenvalues on imaginary axis. Indeed, by Schur decomposition, there exists a unitary matrix U such that

$$U^*(A + HC)U = \begin{bmatrix} A_u & A_{us} \\ 0 & A_s \end{bmatrix}, \quad \alpha CU = \begin{bmatrix} C_u & C_s \end{bmatrix}, \quad U^* = \begin{bmatrix} U_u^* \\ U_s^* \end{bmatrix}$$

with conformal partition, where A_s contains all the stable eigenvalues of the square matrix $A + HC$ and A_u contains all the unstable eigen values of $A + HC$. Define Z as the solution to the Lyapunov equation

$$Z(-A_u) + (-A_u^*)Z + C_u^*C_u = 0. \tag{3.38}$$

Then $Z \geq 0$ is a unique solution if and only if A_u has no eigenvalues on imaginary axis. In this case the stabilizing solution Y for (3.25) is given by $Y = U_u Z^{-1} U_u^*$ which is also unique. With the stabilizing solution $Y \geq 0$, it is noted that (3.25)

can be written as

$$Y(A+L_sC)^*+(A+L_sC)Y+\alpha^{-2}(L_s-H)(L_s-H)^*=0,\ \ (L_s-H)+\alpha^2YC^*=0.$$

By [21], the above equation implies that the square transfer matrix $V(s)$ given in (3.36) is both inner and co-inner, and thus all pass. ∎

Because the set of eigenvalues for $A+HC$ contains those for A_p as in (3.33), it is quite restrictive for A_p not to have eigenvalues on imaginary axis. In fact, for loopshaping purpose, it is often the case that integrators are employed to shape the original plant. A simple way to get rid of this problem is to decompose $A_p=$ diag (A_{p1},A_{p2}) where A_{p2} contains all the imaginary eigenvalues of A_p. The A_r matrix for the reference model is then chosen to be $A_r=$ diag (A_{r1},A_{r2}) with $A_{r1}=A_{p2}$. Moreover it is assumed that

$$P(s)=\left[\begin{array}{cc|c} A_{p1} & 0 & B_{p1} \\ 0 & A_{p2} & B_{p2} \\ \hline C_{p1} & C_{p2} & 0 \end{array}\right],\quad R(s)=\left[\begin{array}{cc|c} A_{r1} & 0 & B_{r1} \\ 0 & A_{r2} & B_{r2} \\ \hline C_{r1} & C_{r2} & 0 \end{array}\right]=\left[\begin{array}{cc|c} A_{p2} & 0 & B_{r1} \\ 0 & A_{r2} & B_{r2} \\ \hline C_{p2} & C_{r2} & 0 \end{array}\right] \tag{3.39}$$

where $C_{r1}=C_{p2}$ is satisfied. The augmentation can then be taken as

$$A=\begin{bmatrix} A_{p1} & 0 & 0 \\ 0 & A_{p2} & 0 \\ 0 & 0 & A_{r2} \end{bmatrix},\quad B=\begin{bmatrix} B_{p1} \\ B_{p2} \\ 0 \end{bmatrix},\quad C^T=\begin{bmatrix} C_{p1}^T \\ C_{p2}^T \\ C_{r2}^T \end{bmatrix},\quad H=\begin{bmatrix} 0 \\ B_{r1} \\ B_{r2} \end{bmatrix} \tag{3.40}$$

In this case, relations (3.34) and (3.35) again hold true. Furthermore with the reference model augmented in $P(s)$, we have that

$$S_{oi}(s) \;=\; S_{id}(s)=(I_p-R(s))^{-1}=I_p+C(sI_{n+r}-A-HC)^{-1}H, \tag{3.41}$$

$$T_{oi}(s) \;=\; (I_p - R(s))^{-1}P(s) = -C(sI_{n+r} - A - HC)^{-1}B. \tag{3.42}$$

Therefore, the reference model should be synthesized such that $S_{oi}(s)$ and $T_{oi}(s)$ have desired frequency shape for the ideal sensitivity and complementary sensitivity matrices so that the $\mathcal{H}_\infty$ control can achieve the desired loopshape by minimization of J_1 as in (3.16) by the fact that $W_1 = T_{oi}^{+}$.

It is noted that for unstable $A + HC$, the $\mathcal{H}_\infty$ solution for $J_1 < \gamma$ needs to solve two AREs in (3.24) and (3.25), and satisfy one coupling condition $\rho(XY) < \gamma^2$. This is numerically quite intensive if a search for the optimal γ is required. Recall that our eventual goal is to use $\mathcal{H}_\infty$ loopshaping for MRC in real time adaptive feedback control that prohibits those control algorithms require intensive computation. Moreover the resulting controller in (3.37) is not of observer form. To resolve this issue, we consider a modified performance index $\tilde{J}_1$ where

$$\tilde{J}_1 := \|\tilde{T}_{J1}\|_\infty, \;\; \tilde{T}_{J1} = \begin{bmatrix} (1-\lambda)\,(S_{out} - VS_{id}) \\ \lambda T_{in} P^{+} \end{bmatrix} (VS_{id})^{-1}, \;\; 0 < \lambda < 1 \tag{3.43}$$

and $V(s)$ is the same as in (3.36) except $\alpha = 1$ is used. Due to the all pass property for $V(s)$, VS_{id} has the same frequency shape as $S_{id}(s)$. From loopshaping point of view, there is no difference between $J_1 < \gamma$ and $\tilde{J}_1 < \gamma$. However, this modification yields a more efficient algorithm for solving the required $\mathcal{H}_\infty$ controller that is an observer.

Theorem 3.7 *Let $P(s)$ be augmented as in (3.40) where (A, B, C) is both stabilizable and detectable. Consider the performance index in (3.43). Suppose that*

$S_{id} = I_p + C(sI - A - HC)^{-1}H$ where $A + HC$ is unstable but avoids eigenvalues on imaginary axis. Then the $\mathcal{H}_\infty$ controller solves $\tilde{J}_1 < \gamma$ has an observer form (3.2) where $L_s = H - YC^*$ with Y a stabilizing solution for (3.25) in the case $\alpha = 1$. The state feedback gain is given by $F = -\lambda^{-2}B^*X_s$ where $X_s \geq 0$ is the stabilizing solution for

$$A^*X_s + X_sA - X_s\left(\lambda^{-2}BB^* - \gamma^{-2}L_sL_s^*\right)X_s + (1-\lambda)^2C^*C = 0 \tag{3.44}$$

Proof: Simple computation shows that

$$\tilde{S}_{id} = V(s)S_{id}(s) = I + C(sI - A - L_sC)^{-1}L_s$$

that is stable by $L_s = H - YC^*$ where Y is the stabilizing solution. Hence, Theorem 3.2 can be applied for solving $\tilde{J}_1 < \gamma$ with S_{id} replaced by $\tilde{S}_{id}$. The theorem is thus true. ∎

We summarize the MRC $\mathcal{H}_\infty$ loopshaping algorithm as follows for continuous-time systems in the case of $m \geq p$.

MRC Algorithm with $\mathcal{H}_\infty$ Loopshaping ($m \geq p$):

- Step 1: For the given physical plant, decompose its realization according to (3.39) where A_{p2} contains all the imaginary eigenvalues of A_p.

- Step 2: Synthesize the reference model in the form of (3.39) such that S_{oi} in (3.41) and T_{oi} in (3.42) have desired frequency shape for sensitivity and complementary sensitivity.

- Step 3: Augment the plant model according to (3.40).

- Step 4: Compute $L = L_s = H - YC^*$ where Y is the stabilizing solution for (3.25) with $\alpha = 1$. Compute $F = F_s = -\lambda^{-2}B^*X_s$ where X_s is the stabilizing solution for (3.44) for some $\gamma > 0$.

- Step 5: Set the $\mathcal{H}_\infty$ controller as the observer in (3.2) with state feedback and output injection gains given in Step 4.

In what follows next, we consider the dual case $m \leq p$. For the plant model in (3.32) and the reference model in (3.3), we augment the plant according to

$$A = \begin{bmatrix} A_p & 0 \\ 0 & A_r \end{bmatrix}, \quad B = \begin{bmatrix} B_p \\ B_r \end{bmatrix}, \quad \begin{matrix} C = \begin{bmatrix} C_p & 0 \end{bmatrix} \\ H = \begin{bmatrix} 0 & C_r \end{bmatrix} \end{matrix}. \tag{3.45}$$

Then there holds

$$H(sI_{n+r} - A)^{-1}B = C_r(sI_r - A_r)^{-1}B_r = R(s) \tag{3.46}$$

and

$$C(sI_{n+r} - A)^{-1}B = C_p(sI_n - A_p)^{-1}B_p = P(s) \tag{3.47}$$

In this case $S_{id}(s) = (I_p - R(s))^{-1} = I_p + H(sI_{n+p} - A - BH)^{-1}B$ where

$$A + BH = \begin{bmatrix} A_p & B_pC_r \\ 0 & A_r + B_pC_r \end{bmatrix}$$

It follows that the matrix $A + BH$ may not be stable if $P(s)$ is not. A parallel result to continuous-time system is given by next.

Theorem 3.8 *Let $P(s) = C(sI_n - A)^{-1}B$ with stabilizable and detectable realization. Let the weighting function in $T_{J2}(s)$ be the same as in Theorem 3.4. Suppose that $A + BH$ is unstable. Then there exists a stabilizing controller $K(s)$ such that $J_2 < \gamma$ for some $\gamma > 0$ if and only if $A + BH$ has no eigenvalues on imaginary axis. If the condition for $A + BH$ having no eigenvalues on imaginary axis holds, then (a) there exists a stabilizing solution $X \geq 0$ for (3.28), such that the following transfer matrix*

$$V(s) = I + (F_s - H)(sI_{n+r} - A - BF_s)^{-1}B, \tag{3.48}$$

is stable and all pass where $F_s = H - \alpha^2 B^* X$; and (b) the central $\mathcal{H}_\infty$ controller achieving $J_2 < \gamma$ is given by observer

$$K(s) = -F_s\left(sI - A + \alpha^2 BB^* X - BH - Z_\infty LC\right)^{-1} Z_\infty L \tag{3.49}$$

where $L = \lambda^{-2} YC^*$ with $Y \geq 0$ a stabilizing controller for (3.29) and $Z_\infty = (I_n - \gamma^{-2} YX)^{-1}$.

Proof: In light of Theorem 3.4, the existence of the stabilizing controller such that $J_2 < \gamma$ is determined by the existence of the stabilizing solution X and Y for (3.28) and (3.29) respectively, and the coupling condition $\rho(XY) < \gamma^2$. It follows from [42] that, by the hypothesis that (A, B) is stabilizable which is equivalent to that $(A + BH, B)$ is stabilizable, the existence of the stabilizing solution $X \geq 0$ is equivalent to that the matrix $A+BH$ has no eigenvalues on imaginary axis. Indeed,

by Schur decomposition, there exists a unitary matrix U such that

$$U(A+BF)U^* = \begin{bmatrix} A_u & 0 \\ A_{us} & A_s \end{bmatrix}, \quad \alpha UB = \begin{bmatrix} B_u \\ B_s \end{bmatrix}, \quad U^* = \begin{bmatrix} U_u^* & U_s^* \end{bmatrix}$$

where A_s contains all the stable eigenvalues of the square matrix $A + BH$. Define Z as the solution to the Lyapunov equation

$$Z(-A_u^*) + (-A_u)Z + B_u B_u^* = 0. \tag{3.50}$$

Then $Z \geq 0$ is a unique solution if and only if A_u has no eigenvalues on imaginary axis. In this case the stabilizing solution X for (3.28) is given by $X = U_u^* Z^{-1} U_u$ which is also unique. With stabilizing solution $X \geq 0$, it is noted that (3.28) can be written as

$$(A+BF_s)^*X + X(A+BF_s) + \alpha^{-2}(F_s - F)^*(F_s - F) = 0, \quad F_s = H - \alpha^2 B^* X = 0.$$

By [21], the above equation implies that the square transfer matrix $V(s)$ given in (3.48) is both inner and co-inner, and thus all pass. ∎

Because the set of eigenvalues for $A + BH$ contains those for A_p as in (3.33), it is quite restrictive for A_p not to have eigenvalues on imaginary axis. As discussed earlier, it is often the case that integrators are employed to shape the original plant for loopshaping purpose. A similar method is employed here. For the plant model in (3.32) and reference model in (3.3), we assume that the decomposition in (3.39) holds with $A_{p2} = A_{r1}$. Moreover it is assumed that

$$R(s) = \left[\begin{array}{cc|c} A_{r1} & 0 & B_{r1} \\ 0 & A_{r2} & B_{r2} \\ \hline C_{r1} & C_{r2} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_{p2} & 0 & B_{p2} \\ 0 & A_{r2} & B_{r2} \\ \hline C_{r1} & C_{r2} & 0 \end{array}\right] \tag{3.51}$$

where $B_{r1} = B_{p2}$ is satisfied. The augmentation is now taken as (compare with (3.40))

$$A = \begin{bmatrix} A_{p1} & 0 & 0 \\ 0 & A_{p2} & 0 \\ 0 & 0 & A_{r2} \end{bmatrix}, \quad B = \begin{bmatrix} B_{p1} \\ B_{p2} \\ B_{r2} \end{bmatrix}, \quad \begin{array}{l} C = \begin{bmatrix} C_{p1} & C_{p2} & 0 \end{bmatrix} \\ H = \begin{bmatrix} 0 & C_{r1} & C_{r2} \end{bmatrix} \end{array}. \tag{3.52}$$

It follows that relations (3.46) and (3.47) hold again. Furthermore with the reference model augmented in $P(s)$, we have that

$$S_{sf}(s) = S_{id}(s) = (I_p - R(s))^{-1} = I_p + H(sI_{n+r} - A - BH)^{-1}B, \tag{3.53}$$

$$T_{sf}(s) = (I_p - R(s))^{-1}P(s) = -C(sI_{n+r} - A - BH)^{-1}B. \tag{3.54}$$

Therefore, the reference model should be synthesized such that $S_{sf}(s)$ and $T_{sf}(s)$ have desired frequency shape for the ideal sensitivity and complementary sensitivity matrices so that the $\mathcal{H}_\infty$ control can achieve the desired loopshape by minimization of J_2 as in (3.16) by the fact that $W_2 = T_{sf}^+$.

It is also noted that for unstable $A + BH$, the $\mathcal{H}_\infty$ solution for $J_2 < \gamma$ needs to solve two AREs in (3.28) and (3.29), and satisfy one coupling condition $\rho(XY) < \gamma^2$. This is numerically quite intensive if a search for the optimal γ is required that is similar to the dual case. To resolve this issue, we consider a modified performance index $\tilde{J}_2$ where

$$\tilde{J}_2 := \|\tilde{T}_{J2}\|_\infty, \quad \tilde{T}_{J2} = VS_{id}^{-1}\begin{bmatrix} (1-\lambda)(VS_{id} - S_{in}) & \lambda P^+ T_{out} \end{bmatrix}, \quad 0 < \lambda < 1 \tag{3.55}$$

and $V(s)$ is the same as in (3.48). Due to the all pass property for $V(s)$, VS_{id} has the same frequency shape as $S_{id}(s)$. As discussed before, there is no difference

between $J_2 < \gamma$ and $\tilde{J}_2 < \gamma$ in terms of loopshaping. But this modification yields a more efficient algorithm for solving the required $\mathcal{H}_\infty$ controller that has an observer form.

Theorem 3.9 *Let $P(s)$ be augmented as in (3.52) where (A, B, C) is both stabilizable and detectable. Consider the performance index in (3.55). Suppose that $S_{id} = I_p + H(sI - A - BH)^{-1}B$ where $A + BH$ is unstable. Then the $\mathcal{H}_\infty$ controller solves $\tilde{J}_2 < \gamma$ has an observer form (3.2) where $F_s = H - B^*X$ with X a stabilizing solution for (3.28) for the case $\alpha = 1$ and $L = -\lambda^{-2}Y_sC^*$ where $Y_s \geq 0$ is the stabilizing solution for*

$$Y_sA^* + AY_s - Y_s\left(\lambda^{-2}C^*C - \gamma^{-2}F_s^*F_s\right)Y_s + (1-\lambda)^2BB^* = 0 \qquad (3.56)$$

Proof: Simple computation shows that

$$\tilde{S}_{id} = V(s)S_{id}(s) = I + F(sI - A - BF_s)^{-1}B$$

that is stable by $F_s = H - B^*X$ where X is the stabilizing solution. Hence, Theorem 3.2 can be applied for solving $\tilde{J}_2 < \gamma$ with S_{id} replaced by $\tilde{S}_{id}$. The theorem is thus true. ∎

As dual results, the following algorithm summarizes model reference control (MRC) using $\mathcal{H}_\infty$ loopshaping for the case $m \leq p$.

MRC $\mathcal{H}_\infty$ Loopshaping Algorithm ($m \leq p$):

- Step 1: For the given physical plant, decompose its realization according to (3.39) where A_{p2} contains all the imaginary eigenvalues of A_p.

- Step 2: Synthesize the reference model in the form of (3.51) such that S_{sf} in (3.53) and T_{sf} in (3.54) have desired frequency shape for sensitivity and complementary sensitivity.

- Step 3: Augment the plant model according to (3.52).

- Step 4: Compute $F = F_s = H - B^*X$ where X is the stabilizing solution for (3.28) with $\alpha = 1$. Compute $L = L_s = \lambda^{-2}Y_sC^*$ where Y_s is the stabilizing solution for (3.56) for some $\gamma > 0$.

- Step 5: Set the $\mathcal{H}_\infty$ controller as the observer in (3.2) with state feedback and output injection gains given in Step 4.

It is noted that in order for S_{sf} and T_{sf} to have desired frequency shape, the plant $P(s)$ does not require shaping but $R(s)$ does. Because the reference model $R(s)$ contains those imaginary axis poles of the plant $P(s)$, it helps the synthesis for $R(s)$ to achieve desired frequency shape for both S_{sf} and T_{sf}. Finally, if it is required to search for the optimal γ value, simple scheme such as in [65] can be used as our $\mathcal{H}_\infty$ problem does not involve the coupling condition.

3.5 MRC with $\mathcal{H}_\infty$ Loopshaping for Discrete-time Systems

This section treats $\mathcal{H}_\infty$ loopshaping problem as formulated in Section 2 for discrete-time systems. There are several reasons to consider discrete-time case. The first one is the wide use of digital computers in engineering applications. It is

a known fact that more and more digital computers are used in control systems. The second reason is that the state-space solution on $\mathcal{H}_\infty$ control for discrete-time systems is quite different from that for the continuous-time systems. Perhaps a more relevant reason for our dissertation is the adaptive control that employs $\mathcal{H}_\infty$ control algorithms. It should be clear that numerical computation of the control law has to be implemented in real time if optimal control is used for adaptive control that requires the use of digital computers. Although adaptive control has a parallel theory for continuous-time systems, it is felt that discrete-time adaptive control matches more naturally the increasing use of the digital computers in control systems. Hence there is a need for developing corresponding results on $\mathcal{H}_\infty$ loopshaping for discrete-time systems.

As in the preceding sections, there exists a parallel procedure for discrete-time loop shaping with $\mathcal{H}_\infty$ criterion. More importantly, the resulting $\mathcal{H}_\infty$ controller is again of observer form. To avoid tedious repetition, some of the background materials and proofs in this section are given in the Section 1 of the Appendix. We begin our discussion on the first result that is analog to Theorem 3.2.

Theorem 3.10 *Let the physical plant $P(z)$ be given as in (3.1) with $m \geq p$, and J_1 be the performance index for the $\mathcal{H}_\infty$ loopshaping problem. Suppose that the ideal sensitivity and the weighting function are given by*

$$S_{id} = I + C(zI_n - A - HC)^{-1}H, \quad W_1 = P^+ S_{id}^{-1}$$

for some stabilizing H where P^+ is the right inverse of $P(z)$ that exists by $m \geq p$

. *Then the resulting central $\mathcal{H}_\infty$ controller is an observer. Moreover, the stabilizing controller achieving $J_1 < \gamma$ is given by (3.2) with $L = H$, and $F = -\lambda^{-2}B^*X[I + (\lambda^{-2}BB^* - \gamma^{-2}HH^*)X]^{-1}A$ where X is the stabilizing solution for*

$$X = A^*X[I + (\lambda^{-2}BB^* - \gamma^{-2}HH^*)X]^{-1}A + (1-\lambda)^2C^*C \qquad (3.57)$$

such that $I - \gamma^{-2}H^*(I + BB^*X/\lambda^2)^{-1}H > 0$. (We shall skip the invertiability condition by including it into the condition for stabilizing solution in the future).

Proof: The proof is similar to the preceding section of continuous-time case. Because $PP^+ \equiv I_p$, we again have that $T_{in}W_1 = (I_m - KP)^{-1}KS_{id}^{-1}$. Define transfer matrix $T_{J1}(z)$ by

$$T_{J1} = \begin{bmatrix} (\lambda-1)E_{out} \\ \lambda T_{in}W_1 \end{bmatrix} = \begin{bmatrix} (1-\lambda)I_p & 0 \\ 0 & \lambda I_m \end{bmatrix} \begin{bmatrix} (I_p - PK)^{-1}S_{id}^{-1} - I_p \\ (I_m - KP)^{-1}KS_{id}^{-1} \end{bmatrix}.$$

Then $J_1 = \|T_{J1}\|_\infty$. Next we write $T_{J1}(z) = \mathcal{F}(G, \lambda K)$ and $G(z)$ in the form of linear fraction with $G(z)$ generalized plant:

$$T_{J1} = \begin{bmatrix} (1-\lambda)\left(S_{id}^{-1} - I_p\right) \\ 0 \end{bmatrix} + \begin{bmatrix} (1-\lambda)P/\lambda \\ I_m \end{bmatrix} \lambda K \left(I_p - (P/\lambda)(\lambda K)\right)^{-1} S_{id}^{-1}$$

$$G(z) = \left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right] = \left[\begin{array}{c|c} (1-\lambda)(S_{id}^{-1} - I_p) & (1-\lambda)P/\lambda \\ 0 & I_m \\ \hline S_{id}^{-1} & P/\lambda \end{array}\right]$$

A state-space realization of $G(z)$ is given by

$$G(z) = \left[\begin{array}{c|c} 0 & 0 \\ 0 & I_m \\ \hline I_p & 0 \end{array}\right] + \left[\begin{array}{c} (1-\lambda)C \\ 0 \\ \hline C \end{array}\right] (sI - A)^{-1} \left[\begin{array}{c|c} -H & B/\lambda \end{array}\right]$$

$$= \left[\begin{array}{c|cc} A & -H & B/\lambda \\ \hline (1-\lambda)C & 0 & 0 \\ 0 & 0 & I_m \\ \hline C & I_p & 0 \end{array}\right]$$

Applying the state-space formulae in [37], we have that there exists a stabilizing controller such that $\|J_1\|_\infty < \gamma$ for some $\gamma > 0$ if and only if there exist stabilizing solutions $X \geq 0$ and $Y \geq 0$ satisfying

$$X = A^* X[I + (\lambda^{-2}BB^* - \gamma^{-2}HH^*)X]^{-1}A + (1-\lambda)^2 C^* C, \tag{3.58}$$

$$Y = (A + HC)[I + \alpha^2 Y C^* C]^{-1} Y (A + HC)^*, \tag{3.59}$$

with some additional constraints where $\alpha^2 = 1 - \left(\frac{1-\lambda}{\gamma}\right)^2$ and $\rho(XY) < \gamma^2$. It is noted that $\gamma^2 > (1-\lambda)^2$ and $I + (\lambda^{-2}BB^* - \gamma^{-2}HH^*)X$ and $I + \alpha^2 C^* C Y$ are invertible in order for the H_∞ problem to be solvable. As H is stabilizing, it follows that $Y = 0$ is the stabilizing solution to (3.59). In this case the condition $\rho(XY) < \gamma^2$ is automatically satisfied. Suppose that $X \geq 0$ is a stabilizing solution to (3.58), then the central controller can be easily computed according to [37]

$$\begin{aligned} K(z) &= -F(zI - A - BF - LC)^{-1}L, \\ F &= -\lambda^{-2}B^* X[I + (\lambda^{-2}BB^* - \gamma^{-2}HH^*)X]^{-1}A, \quad L = H \end{aligned}$$

which is indeed an observer. ∎

We note that $W_1 = T_{oi}^+$ is the right inverse of the ideal complementary sensitivity by

$$PW_1 = S_{oi}^{-1} T_{oi} T_{oi}^+ = S_{oi}^{-1} = S_{id}^{-1}$$

Although Theorem 3.10 is specialized to discrete-time systems, it has the same implication as the continuous-time systems: The synthesis of the $\mathcal{H}_\infty$ controller for $\mathcal{H}_\infty$ loopshaping is a two-step procedure. The first step is to synthesize a stabilizing output injection gain $L = H$ such that both $S_{id} = S_{oi}$ and T_{oi} have desired frequency shape. The second step is to synthesize the state feedback gain F such that it minimizes the performance index J_1 through solving the stabilizing solution for (3.57). This two step procedure decouples the synthesis of the state feedback gain and output injection gain, and thus results in more efficient algorithm for $\mathcal{H}_\infty$ design. Clearly a crucial problem is the synthesis of the output injection gain $L = H$ such that it achieves the desired frequency shape for both $S_{oi}(z)$ and $T_{oi}(z)$. It is again noted that this can be achieved by shaping the plant model directly. Indeed suppose that the original plant $P_o(z)$ has size $p \times m$ with $p \leq m$. Suppose further that there exists $W(z)$ having size $m \times p$ such that $P(z) = P_o(z)W(z)$ has an equal number of inputs and outputs and has a desirable loopshape. Often $W(z)$ can be chosen as a PI type compensator. The strategy is the same as earlier by synthesizing L such that $P(z) = S_{oi}^{-1}T_{oi}$ such that there holds power complementary condition

$$S_{oi}S_{oi}^{*} + T_{oi}T_{oi}^{*} \equiv I_p \tag{3.60}$$

Proposition 3.11 *Suppose that the shaped plant model $P(z) = P_o(z)W(z) = C(zI - A)^{-1}B$ has an equal number of inputs and outputs and has a desirable loopshape. Suppose further that $P(z)$ has a stabilizable and detectable realization (A, B, C). Then there exists a stabilizing output injection gain L such that S_{oi} and T_{oi} satisfy-*

ing (3.26). Furthermore such an L achieves approximate desirable frequency shape for S_{oi} and T_{oi} and can be computed by $L = -AZC^*(I + CZC^*)^{-1}$ where Z is the stabilizing solution of the following algebraic Riccati equation

$$Z = A(I + \alpha^2 ZC^*C)^{-1} ZA^* + BB^*. \tag{3.61}$$

Proof: For any stabilizing output injection gain L, $P(z) = \tilde{M}^{-1}\tilde{N}$ is a left coprime factorization where

$$\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} = \left[\begin{array}{c|cc} A + LC & L & B \\ \hline \Omega^{-1/2}C & \Omega^{-1/2}I_p & 0 \end{array}\right], \quad \Omega = I_p + CYC^*.$$

By [48], $(\tilde{M}, \tilde{N})$ is a pair of normalized coprime factorization satisfying (3.60) if $L = -AZC^*(I + CZC^*)^{-1}$ where Z is the stabilizing solution of (3.59). Because $S_{oi} = \tilde{M}$ and $T_{oi} = -\tilde{N}$, we conclude that S_{oi} and T_{oi} are power complementary to each other. It follows that

$$\begin{aligned} \overline{\sigma}(S_{oi}(e^{j\omega})) &= 1/\sqrt{\underline{\sigma}(I + P(e^{j\omega})[P(e^{j\omega})]^*)}, \\ \overline{\sigma}(T_{oi}(e^{j\omega})) &\leq \sqrt{1 - 1/\underline{\sigma}(I + P(e^{j\omega}[P(e^{j\omega})]^*))}. \end{aligned}$$

Hence, both S_{oi} and T_{oi} have an approximate desirable frequency shape provided that $P(z)$ has a desirable frequency loopshape. ∎

Dual results to Theorem 3.10 and Proposition 3.11 can be derived for the case $m \leq p$. We are interested in synthesizing a stabilizing controller K such that sensitivity and complementary sensitivity achieve those associated with the reference model in the of $\mathcal{H}_\infty$ optimization problem.

Theorem 3.12 *Let the physical plant $P(z)$ be given as in (3.1) with $m \leq p$, and J_2 be the performance index for the $\mathcal{H}_\infty$ loopshaping problem. Suppose that the ideal sensitivity and the weighting function are given by*

$$S_{id} = I + H(zI_n - A - BH)^{-1}B, \ \ W_2 = S_{id}^{-1}P^+$$

for some stabilizing H where P^+ is the left inverse of $P(z)$ that exists by $m \leq p$. Then the resulting $\mathcal{H}_\infty$ controller is an observer. Moreover, the stabilizing controller achieving $J_2 < \gamma$ is given by (3.2) with $F = H$, and $L = -\lambda^{-2}A[I + Y(\lambda^{-2}C^*C - \gamma^{-2}H^*H)]^{-1}YC^*$ where Y is the stabilizing solution for

$$Y = A[I + Y(\lambda^{-2}C^*C - \gamma^{-2}H^*H)]^{-1}YA^* + (1-\lambda)^2BB^* \qquad (3.62)$$

such that $I - \gamma^{-2}HY(I + C^*CY/\lambda^2)^{-1}H^* > 0$. (We shall skip the invertiability condition by including it into the condition for stabilizing solution in the future).

Proof: See the Appendix. ∎

The relation $W_2 = T_{sf}^+$ holds by

$$W_2P = T_{sf}^+T_{sf}S_{sf}^{-1} = S_{sf}^{-1} = S_{id}^{-1}.$$

Hence the weighting function $W_2(z)$ as in (3.16) is the generalized inverse of the ideal complementary sensitivity. A crucial problem in the synthesis of the $\mathcal{H}_\infty$ controller is the synthesis of the state feedback gain $F = H$ such that it achieves the desired frequency shape for both $S_{sf}(z)$ and $T_{sf}(z)$. This can be achieved by shaping the plant model directly that proceeds as follows. Suppose that there exists $Z(s)$

having size $m \times p$ such that $P(z) = Z(z)P_i(z)$ has an equal number of inputs and outputs and has a desirable loopshape. Our strategy is the same as the dual case by synthesizing F such that with $P(z) = T_{sf}^{-1}S_{sf}$ there holds power complementary condition

$$S_{sf}^*S_{sf} + T_{sf}^*T_{sf} \equiv I_m \tag{3.63}$$

As in this case, there holds also

$$\begin{aligned}\overline{\sigma}(S_{sf}(e^{j\omega})) &= 1/\sqrt{\underline{\sigma}(I + [P(e^{j\omega})]^*P(e^{j\omega}))}, \\ \overline{\sigma}(T_{sf}(e^{j\omega})) &\le \sqrt{1 - 1/\underline{\sigma}(I + [P(e^{j\omega})]^*P(e^{j\omega}))}.\end{aligned}$$

Thus the sensitivity and complementary sensitivity have the desired frequency shape provided that $P(z)$ does. The following result gives the formula to synthesize such a gain $F = H$.

Proposition 3.13 *Suppose that the shaped plant model $P(z) = Z(z)P_i(z) = C(zI - A)^{-1}B$ has an equal number of inputs and outputs and has a desirable loopshape. Suppose further that $P(z)$ has a stabilizable and detectable realization (A, B, C). Then there exists a stabilizing state feedback gain F such that S_{sf} and T_{sf} satisfying (3.63). Furthermore such an F achieves approximate desirable frequency shape for S_{sf} and T_{sf} and can be computed by $F = -(I_m + B^*XB)^{-1}B^*XA$ where X is the stabilizing solution of the following algebraic Riccati equation*

$$X = A^*X(I_m + BB^*X)^{-1}A + C^*C. \tag{3.64}$$

Because our goal is to develop $\mathcal{H}_\infty$ loopshaping for MRC, in what follows next, we will derive similar results to Section 4. For this purpose, consider the plant model given by

$$P(z) = C_p(zI_n - A_p)^{-1}B_p \tag{3.65}$$

We consider again the case $m \geq p$. For the reference model in (3.3), we augment the plant according to

$$A = \begin{bmatrix} A_p & 0 \\ 0 & A_r \end{bmatrix}, \quad B = \begin{bmatrix} B_p \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_p & C_r \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ B_r \end{bmatrix}. \tag{3.66}$$

Then there hold

$$C(zI_{n+r} - A)^{-1}H = C_r(zI_r - A_r)^{-1}B_r = R(z) \tag{3.67}$$

and

$$C(zI_{n+r} - A)^{-1}B = C_p(zI_n - A_p)^{-1}B_p = P(z) \tag{3.68}$$

In this case $S_{id}(z) = (I_p - R(z))^{-1} = I_p + C(zI_{n+p} - A - HC)^{-1}H$ where

$$A + HC = \begin{bmatrix} A_p & 0 \\ B_rC_p & A_r + B_rC_r \end{bmatrix}$$

It follows that the matrix $A + HC$ may not be stable if $P(z)$ is not. The next result gives a necessary and sufficient condition for $J_1 < \gamma$ to be solvable with stabilizing controllers.

Theorem 3.14 *Let $P(z) = C(zI_n - A)^{-1}B$ with stabilizable and detectable realization. Let the weighting function in $T_{J1}(z)$ be the same as in Theorem 3.10. Suppose*

that $A + HC$ is unstable. Then there exists a stabilizing controller $K(z)$ such that $J_1 < \gamma$ for some $\gamma > 0$ if and only if $A + HC$ has no unstable eigenvalues on the unit circle. If the condition for $A + HC$ having no unstable eigenvalues on the unit circle holds, then (a) there exists a stabilizing solution $Y \geq 0$ for (3.59), such that the following transfer matrix

$$V(z) = I + C(zI_{n+r} - A - L_s C)^{-1}(L_s - H), \tag{3.69}$$

is stable and satisfies $VV^* = I_p + \alpha^2 CYC^*$ where $L_s = H - \alpha^2(A + HC)YC^*(I + \alpha^2 CYC^*)^{-1}$; and (b) the existence of stabilizing controller such that $J_1 < \gamma$ is equivalent to the existence of a stabilizing solution $X \geq 0$ for (3.57) satisfying the coupling condition $\rho(XY) < \gamma^2$ where $Y \geq 0$ is the solution to (3.59).

Proof: In light of Theorem 3.10, the existence of the stabilizing controller such that $J_1 < \gamma$ is determined by the existence of the stabilizing solution X and Y for (3.57) and (3.59) respectively, and the coupling condition $\rho(XY) < \gamma^2$. The equation (3.57) is a standard $\mathcal{H}_\infty$ algebraic Riccati equation that admits a unique solution by the given hypothesis provided that γ is sufficiently large. The coupling condition can also always be met for some $\gamma > 0$. Now the existence of the $\mathcal{H}_\infty$ controller amounts to the existence of the stabilizing solution for (3.59). Since (3.59) corresponds to standard LQR problem for dynamic system $\dot{x}(t) = A^* x(t) + C^* u(t)$ with zero state weighting, it follows from [42] that, by the hypothesis that (C, A) is detectable which is equivalent to that $(C, A + HC)$ is detectable, the existence of the stabilizing solution $Y \geq 0$ is equivalent to that the matrix $A + HC$ has no eigenvalues

on imaginary axis. Indeed, by Schur decomposition, there exists a unitary matrix U such that

$$U^*(A+HC)U = \begin{bmatrix} A_u & A_{us} \\ 0 & A_s \end{bmatrix}, \quad \alpha CU = \begin{bmatrix} C_u & C_s \end{bmatrix}, \quad U^* = \begin{bmatrix} U_u^* \\ U_s^* \end{bmatrix}$$

where A_s contains all the stable eigenvalues of the square matrix $A+HC$ and A_u contains all the unstable eigen values of $A+HC$. Define Z as the solution to the Lyapunov equation

$$Z - (A_u^{-1})^* Z A_u^{-1} = (A_u^{-1})^* C_u C_u^* A_u^{-1}. \tag{3.70}$$

Then $Z \geq 0$ is a unique solution if and only if A_u has eigenvalues within unit circle. In this case the stabilizing solution Y for (3.59) is given by $Y = U_u Z^{-1} U_u^*$ which is also unique. With stabilizing solution $Y \geq 0$, it is noted that (3.59) can be written as (see Appendix)

$$\begin{aligned} Y &= (A+L_sC)Y(A+L_sC)^* + \alpha^{-2}(L_s-H)(L_s-H)^*, \\ L_s &= H - \alpha^2(A+HC)YC^*(I+\alpha^2CYC^*)^{-1}. \end{aligned} \tag{3.71}$$

As discussed in the Appendix the above equation implies that the square transfer matrix $V(z)$ given in (3.69) satisfies $VV^* = I_p + \alpha^2 CYC^*$ that is quite similar to the all pass function. ∎

Because the set of eigenvalues for $A+HC$ contains those for A_p, it is quite restrictive for A_p not to have eigenvalues on unit circle. A simple way to get rid of this problem is to decompose $A_p = \text{diag}\,(A_{p1}, A_{p2})$ where A_2 contains all the

imaginary eigenvalues of A_p. The A_r matrix for the reference model is then chosen to be $A_r = \text{diag}\,(A_{r1}, A_{r2})$ with $A_{r1} = A_{p2}$. Moreover it is assumed that

$$P(z) = \left[\begin{array}{cc|c} A_{p1} & 0 & B_{p1} \\ 0 & A_{p2} & B_{p2} \\ \hline C_{p1} & C_{p2} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_{p1} & 0 & B_{p1} \\ 0 & A_{r1} & B_{p2} \\ \hline C_{p1} & C_{p2} & 0 \end{array}\right], \tag{3.72}$$

$$R(z) = \left[\begin{array}{cc|c} A_{r1} & 0 & B_{r1} \\ 0 & A_{r2} & B_{r2} \\ \hline C_{r1} & C_{r2} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_{p2} & 0 & B_{r1} \\ 0 & A_{r2} & B_{r2} \\ \hline C_{p2} & C_{r2} & 0 \end{array}\right],$$

where $C_{r1} = C_{p2}$ is satisfied. The augmentation can be taken as

$$A = \begin{bmatrix} A_{p1} & 0 & 0 \\ 0 & A_{p2} & 0 \\ 0 & 0 & A_{r2} \end{bmatrix}, \quad B = \begin{bmatrix} B_{p1} \\ B_{p2} \\ 0 \end{bmatrix}, \tag{3.73}$$

$$C = \begin{bmatrix} C_{p1} & C_{p2} & C_{r2} \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ B_{r1} \\ B_{r2} \end{bmatrix}.$$

In this case, relations (3.67) and (3.68) again hold true. Furthermore with the reference model augmented in $P(z)$, we have that

$$S_{oi}(z) = S_{id}(z) = (I_p - R(z))^{-1} = I_p + C(zI_{n+r} - A - HC)^{-1}H \tag{3.74}$$

$$T_{oi}(z) = (I_p - R(z))^{-1}P(z) = -C(zI_{n+r} - A - HC)^{-1}B \tag{3.75}$$

Therefore, the reference model should be synthesized such that $S_{oi}(z)$ and $T_{oi}(z)$ have desired frequency shape for the ideal sensitivity and complementary sensitivity matrices so that the $\mathcal{H}_\infty$ control can achieve the desired loopshape by minimization of J_1 as in (3.16) by the fact that $W_1 = T_{oi}^+$.

It is noted that for unstable $A + HC$, the $\mathcal{H}_\infty$ solution for $J_1 < \gamma$ needs to solve two AREs in (3.57) and (3.59), and satisfy one coupling condition $\rho(XY) < \gamma^2$. This is again numerically quite intensive if a search for the optimal γ is required. Now we employ a similar approach to the continuous-time systems by considering a modified performance index $\tilde{J}_1$ where

$$\tilde{J}_1 := \|\tilde{T}_{J1}\|_\infty, \quad \tilde{T}_{J1} = \begin{bmatrix} (1-\lambda)(S_{out} - VS_{id}) \\ \lambda T_{in} P^+ \end{bmatrix} (VS_{id})^{-1}, \quad 0 < \lambda < 1 \qquad (3.76)$$

and $V(z)$ is the same as in (3.69). Due to the almost all pass property for $V(z)$, VS_{id} has the same frequency shape as $S_{id}(z)$. It should be clear that the difference between $J_1 < \gamma$ and $\tilde{J}_1 < \gamma$ is very small. However, this modification yields a more efficient algorithm for solving the required $\mathcal{H}_\infty$ controller that is an observer.

Theorem 3.15 *Let $P(z)$ be augmented as in (3.73) where (A, B, C) is both stabilizable and detectable. Consider the performance index in (3.76). Suppose that $S_{id} = I_p + C(zI - A - HC)^{-1}H$ where $A + HC$ is unstable but avoids eigenvalues on the unit circle. Then the $\mathcal{H}_\infty$ controller solves $\tilde{J}_1 < \gamma$ has an observer form (3.2) where $L = L_s = H - (A + HC)YC^*(I + \alpha^2 CYC^*)^{-1}$ with Y a stabilizing solution for (3.59) in the case $\alpha = 1$ and $F = -\lambda^{-2}B^*X_s[I + (\lambda^{-2}BB^* - \gamma^{-2}HH^*)X_s]^{-1}A$ where $X_s \geq 0$ is the stabilizing solution for*

$$X_s = A^*X_s[I + (\lambda^{-2}BB^* - \gamma^{-2}HH^*)X_s]^{-1}A + (1-\lambda)^2C^*C. \qquad (3.77)$$

Proof: Simple computation shows that

$$\tilde{S}_{id} = V(z)S_{id}(z) = I + C(zI - A - L_sC)^{-1}L_s$$

that is stable by $L_s = H - (A + HC)YC^*(I + \alpha^2 CYC^*)^{-1}$ where Y is the stabilizing solution. Hence, Theorem 3.10 can be applied for solving $\tilde{J}_1 < \gamma$ with S_{id} replaced by $\tilde{S}_{id}$. The proof is now complete. ∎

We summarized the MRC $\mathcal{H}_\infty$ loopshaping procedure for discrete-time systems in the following algorithm.

MRC $\mathcal{H}_\infty$ Loopshaping Algorithm ($m \geq p$):

- Step 1: For the given physical plant, decompose its realization according to (3.72) where A_{p1} contains all the imaginary eigenvalues of A_p.

- Step 2: Synthesize the reference model in the form of (3.72) such that S_{oi} in (3.74) and T_{oi} in (3.75) have desired frequency shape for sensitivity and complementary sensitivity.

- Step 3: Augment the plant according to (3.73).

- Step 4: Compute $L = L_s = H - (A + HC)YC^*(I + \alpha^2 CYC^*)^{-1}$ where Y is the stabilizing solution for (3.59) with $\alpha = 1$. Compute $F = F_s = -\lambda^{-2}B^*X_s[I + (\lambda^{-2}BB^* - \gamma^{-2}HH^*)X_s]^{-1}A$ where X_s is the stabilizing solution for (3.77) for some $\gamma > 0$.

- Step 5: Set the $\mathcal{H}_\infty$ controller as the observer in (3.2) with state feedback and output injection gains given in Step 4.

In what follows next, we consider the dual case $m \leq p$. For the plant model in

(3.65) and the reference model in (3.3), we augment the plant according to

$$A = \begin{bmatrix} A_p & 0 \\ 0 & A_r \end{bmatrix}, \quad B = \begin{bmatrix} B_p \\ B_r \end{bmatrix}, \quad \begin{matrix} C = \begin{bmatrix} C_p & 0 \end{bmatrix} \\ H = \begin{bmatrix} 0 & C_r \end{bmatrix} \end{matrix}. \tag{3.78}$$

Then there holds

$$H(sI_{n+r} - A)^{-1}B = C_r(zI_r - A_r)^{-1}B_r = R(s) \tag{3.79}$$

and

$$C(zI_{n+r} - A)^{-1}B = C_p(sI_n - A_p)^{-1}B_p = P(s) \tag{3.80}$$

In this case $S_{id}(z) = (I_p - R(z))^{-1} = I_p + H(zI_{n+p} - A - BH)^{-1}B$ where

$$A + BH = \begin{bmatrix} A_p & B_pC_r \\ 0 & A_r + B_pC_r \end{bmatrix}$$

It follows that the matrix $A+BH$ may not be stable if $P(z)$ is not. The next result gives a necessary and sufficient condition for $J_2 < \gamma$ to be solvable with stabilizing controllers.

Theorem 3.16 *Let $P(z) = C(zI_n - A)^{-1}B$ with stabilizable and detectable realization. Let the weighting function in $T_{J2}(z)$ be the same as in Theorem 3.12. Suppose that $A + BH$ is unstable. Then there exists a stabilizing controller $K(z)$ such that $J_2 < \gamma$ for some $\gamma > 0$ if and only if $A + BH$ has no eigenvalues on unit circle. If the condition for $A + BH$ having no eigenvalues on unit circle holds, then (a) there exists a stabilizing solution $X \geq 0$ for*

$$X = (A + BH)^*X(I + \alpha^2BB^*)^{-1}(A + BH) \tag{3.81}$$

such that the following transfer matrix

$$V(z) = I + (F_s - H)(zI_{n+r} - A - BF_s)^{-1}B, \tag{3.82}$$

is stable and satisfies $V_* V = I_m + B^* X B$ *where* $F_s = H - B^* X(I + \alpha^2 BB^* X)^{-1}(A + BH)$*; and (b) the existence of stabilizing controller such that* $J_2 < \gamma$ *is equivalent to the existence of a stabilizing solution* $Y \geq 0$ *for (3.62) satisfying the coupling condition* $\rho(XY) < \gamma^2$ *where* $X \geq 0$ *is the solution to (3.81).*

Because the set of eigenvalues for $A + BH$ contains those for A_p, it is quite restrictive for A_p not to have eigenvalues on imaginary axis. A simple way to get rid of this problem is to decompose $A_p = \text{diag}\,(A_{p1}, A_{p2})$ where A_{p2} contains all the imaginary eigenvalues of A_p. The A_r matrix for the reference model is then chosen to be $A_r = \text{diag}\,(A_{r1}, A_{r2})$ with $A_{r1} = A_{p2}$. Moreover it is assumed that

$$\begin{aligned} P(z) &= \left[\begin{array}{cc|c} A_{p1} & 0 & B_{p1} \\ 0 & A_{p2} & B_{p2} \\ \hline C_{p1} & C_{p2} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_{p1} & 0 & B_{p1} \\ 0 & A_{r1} & B_{p2} \\ \hline C_{p1} & C_{p2} & 0 \end{array}\right], \qquad (3.83) \\ R(z) &= \left[\begin{array}{cc|c} A_{r1} & 0 & B_{r1} \\ 0 & A_{r2} & B_{r2} \\ \hline C_{r1} & C_{r2} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_{p2} & 0 & B_{p2} \\ 0 & A_{r2} & B_{r2} \\ \hline C_{r1} & C_{r2} & 0 \end{array}\right], \end{aligned}$$

where $B_{r1} = B_{p2}$ is satisfied. The augmentation can be taken as

$$\begin{aligned} A &= \begin{bmatrix} A_{p1} & 0 & 0 \\ 0 & A_{p2} & 0 \\ 0 & 0 & A_{r2} \end{bmatrix}, \; B = \begin{bmatrix} B_{p1} \\ B_{p2} \\ B_{r2} \end{bmatrix}, \qquad (3.84) \\ C &= \begin{bmatrix} C_{p1} & C_{p2} & 0 \end{bmatrix}, \; H = \begin{bmatrix} 0 & C_{r1} & C_{r2} \end{bmatrix}. \end{aligned}$$

In this case, relations (3.79) and (3.80) again hold true. Furthermore with the reference model augmented in $P(z)$, we have that

$$S_{sf}(z) = S_{id}(z) = (I_p - R(z))^{-1} = I_p + H(zI_{n+r} - A - BH)^{-1}B \quad (3.85)$$

$$T_{sf}(z) = (I_p - R(z))^{-1}P(z) = -H(zI_{n+r} - A - BH)^{-1}B \quad (3.86)$$

Therefore, the reference model should be synthesized such that $S_{sf}(z)$ and $T_{sf}(z)$ have desired frequency shape for the ideal sensitivity and complementary sensitivity matrices so that the $\mathcal{H}_\infty$ control can achieve the desired loopshape by minimization of J_2 as in (3.16) by the fact that $W_2 = T_{sf}^{+}$.

It is noted that for unstable $A + BH$, the $\mathcal{H}_\infty$ solution for $J_2 < \gamma$ needs to solve two AREs in (3.62) and (3.81), and satisfy one coupling condition $\rho(XY) < \gamma^2$. This is numerically quite intensive if a search for the optimal γ is required. Moreover the resulting controller in (3.37) is not of observer form. To resolve this issue, we consider a modified performance index $\tilde{J}_2$ where

$$\tilde{J}_2 := \|\tilde{T}_{J2}\|_\infty, \; \tilde{T}_{J2} = VS_{id}^{-1}\left[\; (1-\lambda)(VS_{id} - S_{in}) \quad \lambda P^{+}T_{out} \;\right], \; 0 < \lambda < 1 \quad (3.87)$$

and $V(z)$ is the same as in (3.82). Due to the fact that $V(z)$ is almost all pass, VS_{id} has similar frequency shape as $S_{id}(z)$. From loopshaping point of view, there is no much difference between $J_2 < \gamma$ and $\tilde{J}_2 < \gamma$. However, this modification yields a more efficient algorithm for solving the required $\mathcal{H}_\infty$ controller that is an observer. Because this is dual to Theorem 3.15, the proof is omitted.

Theorem 3.17 *Let $P(z)$ be augmented as in (3.84) where (A, B, C) is both stabilizable and detectable. Consider the performance index in (3.87). Suppose that $S_{id} = I_p + H(zI - A - BH)^{-1}B$ where $A + BH$ is unstable. Then the $\mathcal{H}_\infty$ controller solves $J_2 < \gamma$ has an observer form (3.2) where $F = F_s = H - B^*X(I + \alpha^2 BB^*X)^{-1}(A + BF)$ with X a stabilizing solution for (3.81) with $\alpha = 1$ and $L = -\lambda^{-2}A[I + Y(\lambda^{-2}C^*C - \gamma^{-2}H^*H)Y]^{-1}YC^*$ where $Y_s \geq 0$ is the stabilizing solution for*

$$Y_s = A[I + Y_s(\lambda^{-2}C^*C - \gamma^{-2}H^*H)]^{-1}Y_sA^* + (1-\lambda)^2BB^*. \qquad (3.88)$$

We summarize the results into the following algorithm for model reference control (MRC) using $\mathcal{H}_\infty$ loopshaping.

MRC $\mathcal{H}_\infty$ Loopshaping Algorithm ($m \leq p$):

- Step 1: For the given physical plant, decompose its realization according to (3.83) where A_{p1} contains all the imaginary eigenvalues of A_p.

- Step 2: Synthesize the reference model in the form of (3.83) such that S_{sf} in (3.85) and T_{sf} in (3.86) have desired frequency shape for sensitivity and complementary sensitivity.

- Step 3: Augment the plant according to (3.73).

- Step 4: Compute $F = F_s = H - B^*X(I + \alpha^2 BB^*X)^{-1}(A + BH)$ where X is the stabilizing solution for (3.81) with $\alpha = 1$. Compute $L = L_s =$

$-\lambda^{-2}A[I+Y_s(\lambda^{-2}C^*C-\gamma^{-2}F^*F)]^{-1}Y_sC^*$ where Y_s is the stabilizing solution for (3.88) for some $\gamma > 0$.

- Step 5: Set the $\mathcal{H}_\infty$ controller as the observer in (3.2) with state feedback and output injection gains given in Step 4.

It is noted that in order for S_{sf} and T_{sf} to have desired frequency shape, the plant $P(z)$ does not require shaping but $R(z)$ does. Because the reference model $R(z)$ contains those imaginary axis poles of the plant $P(z)$, it helps the synthesis for $R(z)$ to achieve desired frequency shape for both S_{sf} and T_{sf}. Finally, if it is required to search the optimal γ value, simple scheme such as in [65] can be used as our $\mathcal{H}_\infty$ problem does not involve the coupling condition.

3.6 An Illustrative Example

In this section we will illustrate our synthesis algorithm discussed in the previous sections. Consider the feedback system shown in Fig.3.1. Let the size of plant $p = m = 1$. The non-minimun phase plant $P(s)$ is given by

$$P(s) = \frac{s-9}{s(s^3+6s^2+11s+6)} = \frac{k_1}{s} + \frac{b_1s^2+b_2s+b_3}{s^3+6s^2+11s+6}$$

The realization of the plant is found as

$$\begin{aligned} Ap &= \begin{bmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_p = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \\ C_p &= \begin{bmatrix} 1.5 & 9 & 17.5 \end{bmatrix}, \end{aligned}$$

Our objective is to design a feedback controller K such that the plant achieves the reference model given by the ideal sensitivity. The performance of reference model by the step input has the requirements as follows:

- the percentage overshoot $P.O. \leq 5\%$,
- settling time $t_s \leq 3sec.$
- the steady state error $e_{ss} \leq 0.1$

We choose reference model as the prototype given by

$$R(s) = \frac{4}{s(s+2.8)}$$

that represents both the ideal time domain and frequency domain performance. Following the procedure in earlier sections, the realization of the reference model are found as

$$\begin{aligned} A_r &= \begin{bmatrix} 0 & 0 \\ 0 & -2.8 \end{bmatrix}, \quad B_r = \begin{bmatrix} 2 & -4.3 \end{bmatrix}^T, \\ C_r &= \begin{bmatrix} 2.15 & 1 \end{bmatrix}. \end{aligned}$$

It follows that

$$P(s) = C_p(sI - A_p)^{-1}B_p, \quad R(s) = C_r(sI - A_r)^{-1}B_r$$

Following the algorithm in Section 3.3, the augmented plant A, B, C and H are given by

$$A = \begin{bmatrix} -6 & -11 & -6 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.8 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & -0.0857 & -1.4 \end{bmatrix}^T$$

$$C = \begin{bmatrix} 0 & 0 & 0 & -1.5 & -1.4 \end{bmatrix}, \quad H = \begin{bmatrix} 1.5 & 9 & 17.5 & 1 & -1 \end{bmatrix}$$

Further, we choose $\lambda = 0.1$ to emphasize the performance requirement, and $\gamma = 1$ for which an $\mathcal{H}_\infty$ controller exists. Using the MATLB as a computation tool, the state feedback gain F obtained from the simulation is given by

$$F = \begin{bmatrix} 45.8 & 24.5 & 13.3 & 86.4 & -15.0 \end{bmatrix}.$$

and $L = H$. The synthesized feedback controller is obtained as

$$\begin{aligned} K(s) &= -F(sI - A - BF - LC)^{-1}L \\ &= \frac{0.1s^4 + s^3 + 3.4s^2 + 4.6s + 2.2}{s^5 + 17.7s^4 + 150s^3 + 760s^2 + 2420s + 3183} \end{aligned}$$

For comparison, the magnitude response of the sensitivity for the reference model (dashed line), and for the designed feedback system (solid line), according to the value of γ and λ, are plotted in Fig. 3.5, and the complementary sensitivity in Fig. 3.6, respectively. The step response of the the designed feedback system satisfies the design requirements specified by the the ideal model as shown in Fig. 3.7. for $\gamma = 1$ and in Fig. 3.8. for $\gamma = 2$. The smaller is λ, the better is performance.

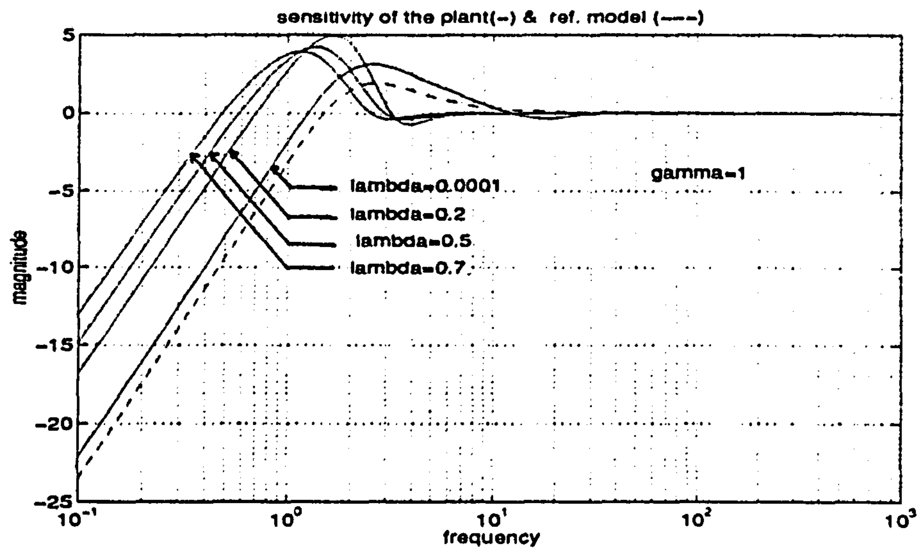


Figure 3.5: The sensitivity of the plant and the reference model.

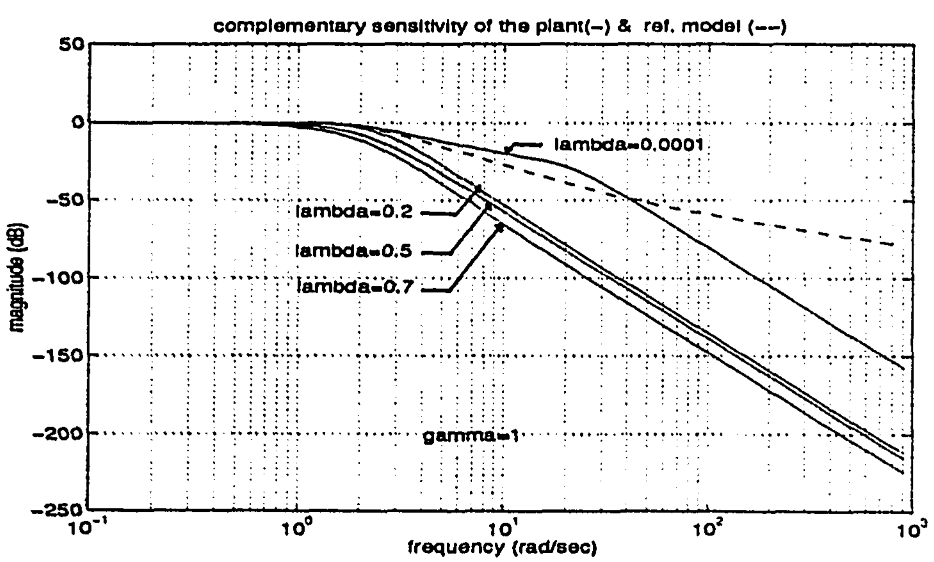


Figure 3.6: The complementary sensitivity of the plant and the reference model.

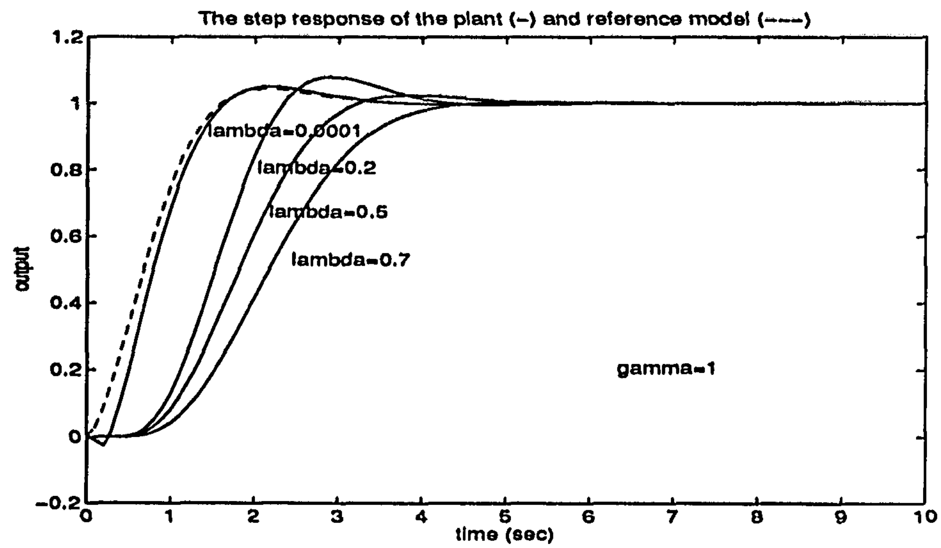


Figure 3.7: The step response of the plant and the reference model in $\gamma = 1$ case.

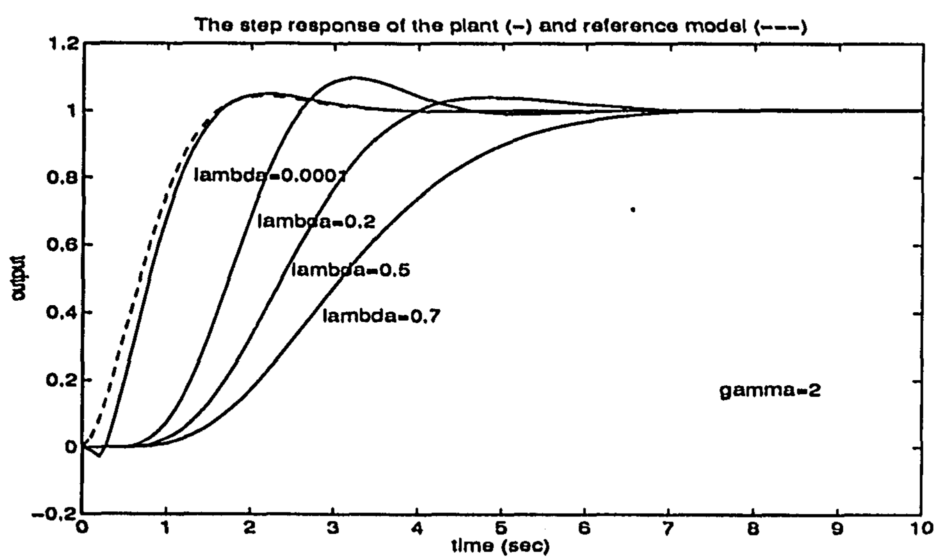


Figure 3.8: The step response of the plant and the reference model in $\gamma = 2$ case.

Chapter 4

Adaptive Identification and Control in $\mathcal{H}_\infty$

4.1 Introduction

This chapter is a unification of the previous two chapters with adaptation in both identification and control. The objective is to model and control the physical system adaptively under the paradigm of $\mathcal{H}_\infty$. As discussed earlier, to achieve robust adaptive control, we need not only identify the nominal plant, but also quantify the model uncertainty in the adaptive modeling part. Moreover, we need use both the nominal plant model and the quantification of the model uncertainty to self-tune the adaptive control law based on $\mathcal{H}_\infty$ robust control. An obvious difficulty for the unification of identification and control in $\mathcal{H}_\infty$ is the lack of recursive algorithms using real time data. This is one of the reasons why least-squares algorithm is chosen in Chapter 2 for solving the problem of identification in $\mathcal{H}_\infty$. By the Parseval Theorem, it is possible to convert the frequency domain least-squares algorithm into the time domain one. The problem is the equivalence of the two least-squares al-

gorithm, and/or under what conditions they are equivalent. The use of $\mathcal{H}_\infty$ control for the control part also gives the opportunity to implement the resulting control law adaptively because $\mathcal{H}_\infty$ norm is *induced* two norm (i.e., the size of energy). This is also manifested by the $\mathcal{H}_\infty$ performance index in time domain. The problem is clearly the computational complexity associated with $\mathcal{H}_\infty$ design that prohibits its implementation in real time. Recall that we do not know the true system except the identified model that is a function of time t. Thus the $\mathcal{H}_\infty$ controller needs to be designed for each identified plant that is simply not possible for real time implementation.

To resolve the aforementioned problems, the following strategies are used. First, a periodic signal is injected that ensures the persistent excitation at the plant input. Then it can be shown that the least-squares algorithm in frequency domain is equivalent to a specialized recursive least-squares algorithm asymptotically. Fortunately the amplitude of the periodical signal is not large that keeps the resulting performance degradation small. Second, the time domain performance index is used to convert the infinite horizon problem for $\mathcal{H}_\infty$ control into the finite horizon problem at each time instance. In this case the two algebraic Riccati equations involved in $\mathcal{H}_\infty$ control become Riccati difference equations that can be solved recursively, and thus allow the real time implementation of the robust model reference control. Under certain conditions, the finite horizon $\mathcal{H}_\infty$ control converges to infinite horizon $\mathcal{H}_\infty$ control. Hence robust adaptive control can be achieved. Because the identified

model is very inaccurate at the early stage of adaptive control, model validation is employed to monitor the closed loop system. If the system produces undesirable size of signals, the $\mathcal{H}_\infty$ controller designed for finite horizon case must be shut off. This prevents the system from suffering extremely poor performance. The block diagram of our proposed adaptive control is depicted in Figure 4.1.

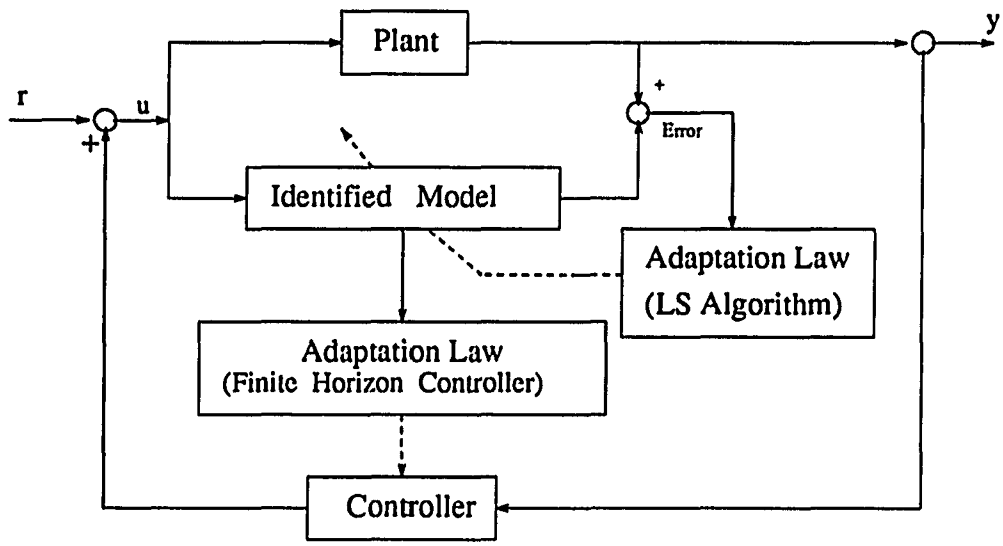


Figure 4.1: The block diagram of robust adaptive control

This chapter is organized as follows. In Section 2, the conventional recursive least-squares algorithm is reviewed first. Its frequency domain properties are discussed that lead to the equivalence between the two least-squares algorithms. In Section 3, finite horizon control for $\mathcal{H}_\infty$ case is studied for time varying systems because of the time varying model obtained from recursive least-squares algorithm. Its convergence is investigated, and the model equivalence principle is employed to achieve robust adaptive control where model validation is used as a monitor for the

closed loop system. For simplicity, the results in this chapter are derived for single-input/single-output systems. Because both recursive least-squares algorithms and finite horizon control algorithms are applicable to multivariable systems, the results in this chapter can be generalized to a multivariable case without much difficulty. There do exist some problems if the plant is unstable as in this case, and therefore, the results in Chapter 2 fail to apply. This problem will be discussed in Chapter 5.

4.2 Recursive Least-Squares Algorithm for Identification in $\mathcal{H}_\infty$

This section focuses on recursive least-squares algorithm that has been used extensively in conventional system identification [47]. Although least-squares algorithm has been investigated for undermodeling problem [70, 24], recursive least-squares algorithm remains untouched regarding to its associated identification error. This section reveals a surprising result that relates recursive least-squares algorithm to the frequency domain least-squares algorithm studied in Chapter 2. Under some mild conditions, these two algorithms are equivalent that leads to the quantification of $\mathcal{H}_\infty$ norm error for the recursive least-squares algorithm.

4.2.1 Recursive Least-Squares Algorithm

Recursive least-squares algorithm considers on-line identification of a linear discrete time system with additive disturbance modeled by the input-output relationship

$$y(t) = G(z)u(t) + v(t) \tag{4.1}$$

where $G(z)$ is the true unknown transfer function, $u(t)$ is the control input and $v(t)$ is an unmeasurable stochastic disturbance or measurement noise acting on the output $y(t)$. The conventional identification algorithm assumes

$$y(t) = G(z,\hat{\theta})u(t) + H(z,\hat{\theta})q(t) \tag{4.2}$$

for some stable $G(z)$ and $H(z)$ where $H(z)$ is a fixed transfer function that resembles the noise spectrum of $v(t)$, and thus $(H(z))^{-1}$ is also stable, and the exogenous signal $q(t)$ is white noise. In conventional system identification [47], both $G(z,\hat{\theta})$ and $H(z,\hat{\theta})$ are assumed to be FIR (or MA) model

$$G(z,\hat{\theta}) = \sum_{k=0}^{n-1} g_k z^{-k}, \quad H(z,\hat{\theta}) = \sum_{k=0}^{m-1} h_k z^{-k}.$$

The parameter $\hat{\theta}$ is defined as

$$\hat{\theta}^T = \begin{bmatrix} \hat{\theta}_1 & \hat{\theta}_2 & \dots & \hat{\theta}_{n+m} \end{bmatrix}, \tag{4.3}$$

where the following relations hold:

$$\hat{\theta}_i = \begin{cases} g_{i-1}, & \text{for } i = 1,2,\dots,n, \\ h_{i-n-1}, & \text{for } i = n+1,\dots,n+m. \end{cases}$$

For our application, the parameter vector $\hat{\theta} \in \mathcal{M}$ consisting of all possible $(n+m)$-dimensional real vectors that are bounded[1]. Recursive least-squares algorithm estimates $\hat{\theta}(t)$ at time t using past data, and past estimate. As time passes, the estimate $\hat{\theta}(t)$ is expected to be closer to the true unknown θ as more data become available.

[1]Because $|g_i| \leq M$ for each i by $G(z) \in \mathcal{S}(\rho, M)$, and $|h_i| \leq \delta$ for each i by $|\Delta_i| \leq \delta$. See Chapter 2 for details.

Because of the unknown nature of $\{q(t)\}$, the identified model at $t-1$ can only give predicted output

$$\hat{y}(t) = \sum_{k=0}^{n-1} \hat{g}_k^{t-1} u(t-k). \tag{4.4}$$

Thus the prediction error is given by

$$\begin{aligned} \epsilon(t,\hat{\theta}) &= H^{-1}(z,\hat{\theta})\left\{y(t) - \hat{y}(t,\hat{\theta})\right\} \\ &= H^{-1}(z,\hat{\theta})\left[(G(z) - \hat{G}(z,\hat{\theta}))u(t) + v(t)\right]. \end{aligned} \tag{4.5}$$

For convenience, and consistency with the least-squares algorithm in Chapter 2, $H(z) = 1$ is assumed, and thus θ is n-dimensional. In this case, the model in (4.4) has the form

$$\hat{y}(t) = \Phi^T(t-1)\hat{\theta}, \quad \Phi^T = \left[\begin{array}{cccc} u(t) & u(t-1) & \dots & u(t-n+1) \end{array}\right]. \tag{4.6}$$

The prediction error at time t is thus given by

$$\epsilon(t,\hat{\theta}) = y(t) - \hat{y}(t) = y(t) - \Phi^T(t-1)\hat{\theta}.$$

The recursive least-squares algorithm seeks $\hat{\theta}$ at $t = L$ that minimizes

$$\begin{aligned} J_L(\hat{\theta}) &= \sum_{t=1}^{L} \epsilon^2(t,\hat{\theta}) + \left(\hat{\theta} - \hat{\theta}_0\right)^T P_0^{-1}\left(\hat{\theta} - \hat{\theta}_0\right) \\ &= \sum_{t=1}^{L} (y(t) - \hat{y}(t))^T (y(t) - \hat{y}(t)) \\ &\quad + \left(\hat{\theta} - \hat{\theta}_0\right)^T P_0^{-1}\left(\hat{\theta} - \hat{\theta}_0\right) \\ &= \sum_{t=1}^{L} (y(t) - \Phi^T(t-1)\hat{\theta})^T (y(t) - \Phi^T(t-1)\hat{\theta}) \\ &\quad + \left(\hat{\theta} - \hat{\theta}_0\right)^T P_0^{-1}\left(\hat{\theta} - \hat{\theta}_0\right) \end{aligned} \tag{4.7}$$

where P_o is the covariance related to the initial estimate $\hat{\theta}_0$. A remarkable fact is that such $\hat{\theta}$ can be estimated recursively [23].

Theorem 4.1 *Suppose $\hat{\theta} = \hat{\theta}(L-1)$ minimizes $J_{L-1}(\hat{\theta})$ at $t = L-1$ using input/output data up to $t = L-1$, where $\hat{\theta}(0) = \hat{\theta}_0$. Then the minimizer $\hat{\theta} = \hat{\theta}(L)$ for $J_L(\hat{\theta})$ can be computed according to*

$$\hat{\theta}(L) = \hat{\theta}(L-1) + \frac{P(L-2)\Phi(L-1)\left(y(L) - \Phi(L-1)^T\hat{\theta}(L-1)\right)}{1 + \Phi(L-1)^T P(L-2)\Phi(L-1)}$$

where $P(-1) = P_0$ for $L = 1$, and

$$P(L-1) = P(L-2) - \frac{P(L-2)\Phi(L-1)\Phi(L-1)^T P(L-2)}{1 + \Phi(L-1)^T P(L-2)\Phi(L-1)}$$

for $L > 1$.

The modeling error for recursive least-squares algorithm has been investigated in [47] in the presence of the stochastic noise. Under mild conditions, it can be shown that the estimate $\hat{\theta}(t)$ converges to θ asymptotically that is summarized as follows.

Lemma 4.1 *Suppose $G(z) \in \mathcal{S}(\rho, M)$, and $\{y(t)\}$ and $\{u(t)\}$ are quasi-stationary for all t. Then with $J_L(\hat{\theta})$ defined in (4.7), there holds*

$$\sup_{\hat{\theta}\in\mathcal{M}} \frac{1}{L}|J_L(\hat{\theta}) - \bar{J}_L(\hat{\theta})| \rightarrow 0, \quad as \ L \rightarrow \infty$$

where $\bar{J}_L(\hat{\theta})$ is the expectation of $J_L(\hat{\theta})$ given by

$$\bar{J}_L(\theta) = \frac{1}{L}\mathcal{E}\left\{J_L\right\} = \mathcal{E}\left\{\epsilon^2(t,\theta)\right\}.$$

It is also shown that under reasonable conditions [47], the recursive least square parameter identification $\hat{\theta}^*$ converges asymptotically to a value $\hat{\theta}^*$ defined as follows:

$$\hat{\theta}(L)^* = \arg\min \ \bar{J}_L(\hat{\theta}, D^L) \tag{4.8}$$

where D^L denotes the data set as $L \to \infty$.

4.2.2 Undermodeling Error in Frequency Domain

Although recursive least-squares algorithm has nice stochastic performance, there lacks analysis for undermodeling problem. Recall that $G(z) \in \mathcal{S}(\rho, M)$ is in general infinite-dimensional. Thus it is not possible to model $G(z)$ with FIR transfer function. In this subsection, we establish the asymptotic equivalence relation, under persistent excitation condition, between recursive least-squares algorithm and the least-squares algorithm in Chapter 2, and thus establish the undermodeling error bound in frequency domain for recursive least-squares algorithm. To be consistent with stochastic analysis in [47], we define, for a signal sequence $\{s(t)\}$, correlation function by

$$R_s(t, r) = \mathcal{E}\left\{s(t)s(t-r)\right\}. \tag{4.9}$$

The signal $s(t)$ is called quasi-stationary [47] if

$$\begin{aligned} |\mathcal{E}\left\{s(t)\right\}| &\leq C, \ |R_s(t,r)| \leq C, \ \forall t, \\ R_s(\tau) &= \lim_{L\to\infty} \frac{1}{L} \sum_{t=1}^{L} R_s(t, t-\tau), \ \forall \tau \end{aligned}$$

exists where C is a finite constant. The spectrum of the signal $s(t)$ is defined by

$$\Phi_s(\omega) := \sum_{k=-\infty}^{\infty} R_s(k)e^{-j\omega k}.$$

It is not difficult to see that for deterministic $s(t)$, there holds $\Phi(\omega) = |S(\omega)|^2$ where

$$S(\omega) = \lim_{L\to\infty} \frac{1}{\sqrt{L}} \sum_{t=-L}^{L} s(t)e^{-j\omega t}.$$

By using Parseval's theorem, it is established in [47] for quasi-stationary signals $u(t)$ and $y(t)$, the least-squares criterion function $J_L(\hat{\theta})$ is given by

$$\lim_{L\to\infty} J_L(\hat{\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{\epsilon(t,\hat{\theta})}(\omega)\, d\omega + \left(\hat{\theta} - \hat{\theta}_0\right)^T P_0^{-1} \left(\hat{\theta} - \hat{\theta}_0\right). \qquad (4.10)$$

It follows from (4.8) that

$$\theta^* = \arg\ \min \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{\epsilon(t,\hat{\theta})}\, d\omega + \left(\hat{\theta} - \hat{\theta}_0\right)^T P_0^{-1} \left(\hat{\theta} - \hat{\theta}_0\right).$$

Denote $\Phi_y(\omega)$, $\Phi_u(\omega)$, and $\Phi_v(\omega)$ as spectral functions for $y(t)$,$u(t)$ and $v(t)$ respectively. Then there holds, if $u(t)$ and $v(t)$ are independent,

$$\Phi_{\epsilon(t,\hat{\theta}^*)}(\omega, \hat{\theta}^*) = |G(e^{j\omega}) - \hat{G}(e^{j\omega}, \hat{\theta}^*)|^2\Phi_u(\omega) + \Phi_v(\omega). \qquad (4.11)$$

Now suppose the input signal is persistently exciting, i.e., the input signal has the form

$$u(t) = \frac{C_0}{2} + \sum_{i=1}^{N/2} C_i \cos(\omega_i t + \alpha_i) + w(t) \qquad (4.12)$$

where $0 < \omega_i < \pi$ are all distinct for $i > 0$. In addition it is assumed that $\|W\|_\infty \leq C < \infty$ with $W(z)$ as $\mathcal{Z}$-transform of $w(t)$. It follows that [47]

$$\Phi_u(\omega) = \frac{\pi}{2}\sum_{i=0}^{N-1} |C_i|^2\delta(\omega - \omega_i)$$

where $\delta(\omega)$ satisfies

$$\text{(i)} \int_{-\infty}^{\infty} \delta(\omega)\, d\omega = 1, \quad \text{(ii)}\ \delta(\omega) = 0, \ \text{ for } \omega \neq 0.$$

It is noted that $\Phi_w(e^{j\omega}) \equiv 0$ by the definition of spectral function. The above property implies that by taking $\hat{\theta}_0 = 0$ and noting (4.11) and (4.10),

$$\begin{aligned}\lim_{L\to\infty} J_L(\hat{\theta}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{j\omega}) - \hat{G}(e^{j\omega})|^2 \Phi_u(\omega)\, d\omega + \Phi_v(\omega) + \hat{\theta}^T P_0^{-1}\hat{\theta} \\ &= \frac{1}{4}\sum_{i=0}^{N-1} |C_i|^2 |G(e^{j\omega_i}) - \hat{G}(e^{j\omega_i})|^2 + \Phi_v(\omega) + \hat{\theta}^T P_0^{-1}\hat{\theta}.\end{aligned}$$

By taking $C_i = 2/\sqrt{N}$ for all i, $\omega_i = 2i\pi/N$, and

$$P_0^{-1} = \left(\frac{\hat{\delta}}{M}\right)^2 \mathrm{diag}(1, \rho^2, \ldots, \rho^{2(n-1)}), \quad \hat{\delta} = M\rho^{-n}, \tag{4.13}$$

we arrive at

$$\begin{aligned}\lim_{L\to\infty} J_L(\hat{\theta}) &= \frac{1}{N}\sum_{i=0}^{N-1} |\hat{G}(e^{j\omega_i}) - G(e^{j\omega_i})|^2 + \Phi_v(\omega) + \hat{\theta}^T P_0^{-1}\hat{\theta} \qquad (4.14)\\ &= \left(\frac{1}{N}\sum_{k=0}^{N-1} \left|\hat{G}(W_N^k) - E_k^N\right|^2\right) + \left(\frac{\hat{\delta}}{M}\right)^2 \|\hat{G}\|_{2,\rho}^2\end{aligned}$$

that is exactly the same cost function in Chapter 2 for unconstrained least-squares algorithm for the case $\delta = 0$. Hence the results in Chapter 2 can be applied that leads to the next result.

Theorem 4.2 *Suppose the input data is persistently exciting as in (4.12). Then the estimate based on recursive least-squares converges to the solution of frequency domain least-squares solution as in Chapter 2 asymptotically. Moreover there holds error bound*

$$\sup_{G\in S(\rho,M)} \|G - \hat{G}\|_\infty \leq M\rho^{-N} + 2M\rho^{-N}\sqrt{\frac{\rho+1}{\rho-1}}$$

as $L \to \infty$.

Because the proof follows directly from the analysis in this subsection, and Corollary 2.2 of Chapter 2, it is omitted. We would like to emphasize that the assumptions used are mild. Indeed, the stochastic noise $v(t)$ does not affect the asymptotic solution $\hat{\theta}^*$ in light of (4.14). If $\|\Phi_w\|_\infty \neq 0$, then the cost function $J_L(\hat{\theta})$ converges to

$$\begin{aligned}\lim_{L\to\infty} J_L(\hat{\theta}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{G}(e^{j\omega}) - G(e^{j\omega})|^2 \Phi_w(\omega)\, d\omega \\ &+ \left(\frac{1}{N}\sum_{k=0}^{N-1}\left|\hat{G}(W_N^k) - E_k^N\right|^2\right) + \left(\frac{\tilde{\delta}}{M}\right)^2 \|\hat{G}\|_{2,\rho}^2.\end{aligned}$$

In this case, it is difficult to predict the modeling error in frequency domain with the sup-norm. But we argue that the system eventually settles at some steady-state value that can be modeled as some persistent signal whose spectrum function dominates $\Phi_w(\omega)$. One can incorporate such function in construction of the persistent signal (4.12). In that case Theorem 4.2 holds again. We would also like to mention that as $|C_i| = 2/\sqrt{N}$ for all i, the persistent signal may not affect much on the performance of the system. If does, one should scale $\{C_i\}$, and P_0^{-1} proportionally to a degree that the persistent signal is not significant.

4.2.3 Analysis for Closed-Loop Systems

The undermodeling error in frequency domain becomes more difficult to analyze for feedback systems without further assumptions on the plant model, and the persistent exciting signals. In this subsection, we consider the identification problem for closed

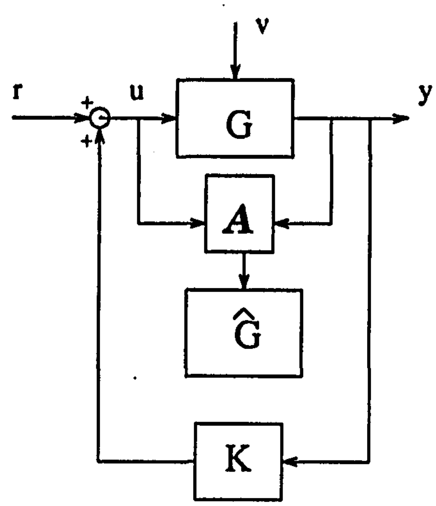


Figure 4.2: Closed-loop identification problem

loop system as in Figure 4.2, i.e., the RLS is applied to input/output data for the closed-loop system in Figure 4.2.

We assume that the system is a fixed regulator to focus the on identification part [47]. The plant input and output can be written as (see Figure 4.2)

$$u(t) = r(t) + K(z, \hat{\theta})y(t), \quad y(t) = G(z)u(t) + v(t) \tag{4.15}$$

where $r(t)$ is the reference signal, $y(t)$ is the measured plant output, $v(t)$ is the disturbance or measurement noise, and K is the transfer function of the feedback controller, parameterized by the identified parameter vector $\hat{\theta}$ of the plant model. The relation between $u(t)$ and $y(t)$ shown in (4.15) implies that

$$u(t) = \left(\frac{1}{1-KG}\right) r(t) + \left(\frac{K}{1-KG}\right) v(t). \tag{4.16}$$

Thus the prediction error as defined in (4.5) is given (by taking $H(z, \hat{\theta}) \equiv 1$) as

follows:

$$\begin{aligned}\epsilon(t,\hat{\theta}) &= \left(\frac{G-\hat{G}}{1-KG}\right)[r(t)+Kv(t)]+v(t) \\ &= \left(\frac{\Delta_G}{1-KG}\right)r(t)+\left(\frac{1-K\hat{G}}{1-KG}\right)v(t) \\ &= \left(\frac{1-K\hat{G}}{1-KG}\right)\left[\left(\frac{\Delta_G}{1-K\hat{G}}\right)r(t)-v(t)\right] \\ &= \left(\frac{1-K\hat{G}}{1-KG}\right)[\Delta_G\tilde{r}(t)+v(t)].\end{aligned} \tag{4.17}$$

where

$$\Delta_G(z)=G(z)-\hat{G}(z,\hat{\theta}),\quad \tilde{r}(t)=\left(\frac{1}{1-K\hat{G}}\right)r(t).$$

For our applications to adaptive control, both $K(z,\hat{\theta})$ and $\hat{G}(z,\hat{\theta})$ are parameterized by $\hat{\theta}$: the parameter vector that is recursively estimated using RLS algorithm as discussed earlier. It should be clear that if $\hat{G}(z,\hat{\theta})$ is close to the true unknown plant, then

$$\epsilon(t,\hat{\theta})\approx\Delta_G\tilde{r}(t)+v(t)$$

that is the same as the case for the open-loop estimation problem. The following result is obtained.

Proposition 4.3 *Suppose $\tilde{r}(t)$ is persistent exciting in the sense that*

$$\Phi_{\tilde{r}}(\omega)=\frac{2\pi}{N}\sum_{i=0}^{N-1}\delta(\omega-\omega_i),\quad \omega_i=2i\pi/N.$$

Then with $J_L(\hat{\theta})$ defined by

$$J_L(\hat{\theta})=\sum_{i=1}^{L}\epsilon^2(t,\hat{\theta})+\left(\hat{\theta}-\hat{\theta}_0\right)^T P_0^{-1}\left(\hat{\theta}-\hat{\theta}_0\right),$$

there holds asymptotically

$$\lim_{L\to\infty} J_L(\hat{\theta}) = \frac{1}{N}\sum_{i=0}^{N-1}\left|W_i\left[\hat{G}(W_N^i) - G(W_N^i)\right]\right|^2 + \left(\hat{\theta}-\hat{\theta}_0\right)^T P_0^{-1}\left(\hat{\theta}-\hat{\theta}_0\right) + \sum_{i=0}^{N-1} W_i\Phi_v(2i\pi/N)$$

where $W_i = (1-K(W_N^i)\hat{G}(W_N^i))/(1-K(W_N^i)G(W_N^i))$. *Moreover with RLS applied to closed-loop identification, the estimate* $\hat{\theta}$ *converges asymptotically to the minimization solution for* $\lim_{L\to\infty} J_L(\hat{\theta})$. *Consequently, if in addition* $W_i \equiv 1$, *there holds error bound in* $\mathcal{H}_\infty$ *norm given by*

$$\sup_{G\in S(\rho,M)} \|G-\hat{G}\|_\infty \le M\rho^{-N} + 2M\rho^{-N}\sqrt{\frac{\rho+1}{\rho-1}}$$

asymptotically.

Because the proof is similar to the derivation in Section 4.2, it is omitted. It should be clear that the above error bound may not hold as $W_i \equiv 1$ does not hold in general. However, we would like to comment that by equation (3.13) of Chapter 3, there holds

$$\frac{1-K\hat{G}}{1-KG} = \frac{1}{1-\Delta_G S_{id}^{-1}K/(1-K\hat{G})}.$$

The above equation is derived with the assumption that the control part minimizes

$$\left\|\begin{bmatrix}1-\lambda\\ \lambda K\end{bmatrix}\frac{S_{id}^{-1}}{1-K\hat{G}}\right\|_\infty =: \gamma$$

that is precisely the $\mathcal{H}_\infty$ cost for MRC based loopshaping as studied in Chapter 3. Hence there holds inequality

$$\frac{1}{1+\gamma\|\Delta_G\|_\infty/\lambda} \le \left\|\frac{1-K\hat{G}}{1-KG}\right\|_\infty \le \frac{1}{1-\gamma\|\Delta_G\|_\infty/\lambda}$$

with the assumption that $\gamma\|\Delta_G\|_\infty < \lambda$. It follows that the qualification of the identified model is also related to the control qualification for the identified model. We will assume that $W_i \approx 1$ for all i, and thus the error bound in the above proposition holds approximately. The assumption on the persistent excitation of $\tilde{r}(t)$ is mild as $K(z,\hat{\theta})$ and $\hat{G}(z,\hat{\theta})$ are known, and one can always synthesize $r(t)$ such that the spectrum of $\tilde{r}(t)$ is the same as in the above proposition.

4.3 Adaptive Control in $\mathcal{H}_\infty$ Based on Recursively Identified Model

In adaptive control, feedback controller is "tuned" according to the identified plant model that is time-varying in nature although the true unknown plant is linear and time-invariant. In traditional approach, robustness of the feedback control is not taken into account in the selection of adaptive control law. The reason lies in the assumption of exact identification for adaptive modeling, and thus model uncertainty does not exist. It is now widely known that the lack of robustness in the control part causes the adaptive control to "fail" in the presence of the modeling error. Hence, our approach is to incorporate $\mathcal{H}_\infty$ control into the adaptive control law. In particular, those results derived in Chapter 3 for model reference control will be modified, and employed in this section to achieve robust adaptive control. We begin with some known results for $\mathcal{H}_\infty$ control of time-varying systems with finite horizon performance index. These results will then be used for our adaptive

control problem. Finally, model validation results will be employed to ensure the robustness for the proposed adaptive control scheme.

4.3.1 Finite Horizon $\mathcal{H}_\infty$ Control for Time-varying Systems

Because $\mathcal{H}_\infty$ norm is an induced $\mathcal{H}_2$ norm, or ℓ^2 norm, it has a time-domain interpretation. In this section, we review some of the existing results for finite horizon $\mathcal{H}_\infty$ control with time-varying systems. Two problems will be considered: state feedback and state estimation.

In state feedback problem for finite horizon $\mathcal{H}_\infty$ control with time-varying systems, we are considering the following problem. We are given an n th order time-varying system of the form:

$$x(t+1) = A_t x(t) + B_{1t} w(t) + B_{2t} u(t), \quad \zeta(t) = C_{1t} x(t) + D_{1t} u(t), \quad x(0) = 0, \quad (4.18)$$

where $w(t) \in \mathbf{R}^{m_1}$, $u(t) \in \mathbf{R}^{m_2}$, and $\zeta(t) \in \mathbf{R}^{p_1}$. To simplify the problem, it is further assumed that $D_{1t}^T C_{1t} = 0$. For state feedback, the measurement of the state variables is available. The problem is to synthesize a state feedback controller $u(t) = \mathcal{F} x(t)$ such that the performance index

$$J_F = \sum_{t=0}^{N-1} [\zeta(t)]^T \zeta(t) - \gamma^2 \sum_{t=0}^{N-1} [w(t)]^T w(t) < 0 \quad (4.19)$$

for all possible $w(t) \in \ell^2$ and $w(t) \neq 0$ where $\gamma > 0$ is prespecified. There are two problems to be answered here. The first is the existence of such controllers, and the second is the synthesis of such controller if it exists. The following result can be found in [46].

Theorem 4.4 *Consider time varying system in (4.18), and the performance index in (4.19) with $\gamma > 0$ given. Then there exists any feedback controller $u(t) = \mathcal{F}x(t)$ such that $J_F < 0$ for all possible $w(t) \in \ell^2$, if and only if there exists a solution sequence $\{X(t)\}_{t=0}^{N-1}$ for the difference Riccati equation*

$$\begin{aligned} X(t) &= A_t^T X(t+1)(I_n + R_t X(t+1))^{-1} A_t + C_{1t}^T C_{1t}, \quad X(N) = 0, \quad (4.20) \\ R_t &= B_{2t}(D_{1t}^T D_{1t})^{-1} B_{2t}^T - B_{1t} B_{1t}^T / \gamma^2, \end{aligned}$$

such that $D_{1t}^T D_{1t} + B_{2t}^T X(t) B_{2t} > 0$, and the matrix

$$Q_t = J_t + B_t^T X(t) B_t, \quad (4.21)$$

is nonsingular, and has exactly m_1 negative eigenvalues, and m_2 positive eigenvalues where

$$J_t = \begin{bmatrix} -\gamma^2 I_{m_1} & 0 \\ 0 & D_{1t}^T D_{1t} \end{bmatrix}, \quad B_t = \begin{bmatrix} B_{1t} & B_{2t} \end{bmatrix}. \quad (4.22)$$

If the above conditions hold, then with $u(t) = F_t x(t)$ where the static state feedback gain is given by

$$F_t = -\begin{bmatrix} 0 & I_{m_2} \end{bmatrix} (J_t + B_t^T X(t) B_t)^{-1} B_t^T X(t) A_t, \quad (4.23)$$

$J_F < 0$ is satisfied for all possible $w(t) \in \ell^2$.

All controllers satisfying $J_F < 0$ for the worst case are characterized in [46], but not quoted here. For application to our particular problem, static feedback gains suffice. However we do need to know the limit of the state feedback gain as $N > t \to \infty$ under the condition that the state-space model converges to a

time-invariant system. This is directly related to the adaptive control where the identified model is convergent for the RLS algorithm as analyzed in the previous section. Moreover we will allow γ to be a function of time t because in the real time control, the exact value of γ for which the feedback controllers exist such that $\sup_{\|w\|_2=1} J_F < 0$ is not known in advance. Thus γ needs "tune" also in order that $\mathcal{H}_\infty$ control be used for adaptive control.

Corollary 4.1 *Consider time varying system in (4.18), and the performance index in (4.19) with* $\gamma_t > 0$ *given. Then there exists any feedback controller* $u(t) = \mathcal{F}x(t)$ *such that* $J_F < 0$ *for all possible* $w(t) \in \ell^2$*, if and only if there exists a solution sequence* $\{X(t)\}_{t=0}^{N-1}$ *for the difference Riccati equation (4.20) such that the matrix* Q_t *in (4.21) is nonsingular, and has* m_1 *negative eigenvalues and* m_2 *positive eigenvalues. Moreover suppose*

$$\lim_{t\to\infty} (A_t, B_{1t}, B_{2t}, C_{1t}, D_{1t}) = (A, B_1, B_2, C_1, D_{12})$$

that are constant matrices satisfying (A, B_2) *stabilizable, and*

$$\text{rank} \begin{bmatrix} A - e^{j\theta} I_n & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2 \ \forall \theta.$$

If in addition, $\lim_{t\to\infty} \gamma_t = \gamma > 0$*, and the following algebraic Riccati equation*

$$X = A^T (I_n + RX)^{-1} A + C_1^T C_1, \quad R = B_2 (D_{12}^T D_{12})^{-1} B_2^T - B_1 B_1^T / \gamma^2$$

admits a stabilizing solution $X \geq 0$ *such that* $Q = \lim_{t\to\infty} Q_t$ *is nonsingular, and has* m_1 *negative eigenvalues and* m_2 *positive eigenvalues. Then* $\lim_{N>t\to\infty} X(t) = X$.

Proof: For the first part, we simply replace B_{1t} by B_{1t}/γ_t, and apply Theorem 4.4 for the case $\gamma = 1$. For the second part, let F_t be as in (4.23). Then Theorem 4.4 implies that

$$\lim_{N>t\to\infty} \sup_{\|w\|_2\neq 0} J_F < 0$$

holds that is equivalent to $\|T_F\|_\infty < \gamma$ where

$$T_F(z) = (C_1 + D_{12}F)(zI_n - A - B_2F)^{-1}B_1$$

with $F = \lim_{N>t\to\infty} F_t$. Using the same argument as in [34], the convergence for the solution of the difference Riccati equation to the stabilizing solution of the algebraic Riccati equation can be shown but omitted. The corollary is thus true. ∎

For the numerical computation, the difference Riccati equation in (4.20) is not in a good form as it requires the computation of $(I_n + R_tX(t+1))^{-1}$ that amounts to about $\mathcal{O}(n^3)$ computational complexity. Hence, in real time computation, it is recommended to convert (4.20) into the following:

$$X(t) = A_t^TX(t+1)A_t - A_t^TX(t+1)B_t(J_t + B_t^TX(t+1)B_t)^{-1}B_t^TX(t+1)A_t + C_{1t}^TC_{1t} \tag{4.24}$$

where J_t is the same as in (4.22). In what follows next, we consider the dual problem of state estimation.

The state estimation problem considers a dual problem to state feedback. We are given an n-order time-varying system of the form:

$$x(t+1) = A_tx(t) + B_{1t}w(t), \quad x(0) = 0, \tag{4.25}$$

$$y(t) \;=\; C_{2t}x(t) + D_{2t}w(t)$$

where $w(t) \in \mathbf{R}^{m_1}$, and $y(t) \in \mathbf{R}^{p_2}$. It is assumed that $B_{1t}D_{2t}^T = 0$. The objective is to find a state estimator $\hat{\zeta}(t) = \mathcal{L}y(t)$ such that

$$J_L = \sum_{t=0}^{N-1}[\Delta_x(t)]^T\Delta_x(t) - \gamma^2\sum_{t=0}^{N-1}[w(t)]^T w(t) < 0 \tag{4.26}$$

for all possible $w(t) \in \ell^2$ where $\Delta_x(t) = \hat{\zeta}(t) - \zeta(t) \in \mathbf{R}^{p_1}$. The dual result to Theorem 4.4 is given as next.

Theorem 4.5 *Consider time varying system in (4.18), and the performance index in (4.19) with $\gamma > 0$ given. Then there exists any state estimator $\hat{\zeta}(t) = \mathcal{L}y(t)$ such that $J_L < 0$ for all possible $w(t) \in \ell^2$, if and only if there exists a solution sequence $\{Y(t)\}_{t=0}^{N-1}$ for the difference Riccati equation*

$$\begin{aligned} Y(t+1) &= A_tY(t)(I_n + \tilde{R}_tY(t))^{-1}A_t^T + B_{1t}B_{1t}^T, \quad Y(0) = 0, \\ \tilde{R}_t &= C_{2t}^T(D_{2t}D_{2t}^T)^{-1}C_{2t} - C_{1t}^TC_{1t}/\gamma^2, \end{aligned} \tag{4.27}$$

such that $D_{2t}D_{2t}^T + C_{2t}Y(t)C_{2t}^T > 0$, and the matrix

$$\hat{Q}_t = \hat{J}_t + C_tX(t)C_t^T \tag{4.28}$$

is nonsingular, and has exactly p_1 negative eigenvalues, and p_2 positive eigenvalues where

$$\hat{J}_t = \begin{bmatrix} -\gamma^2 I_{p_1} & 0 \\ 0 & D_{2t}D_{2t}^T \end{bmatrix}, \quad C_t = \begin{bmatrix} C_{1t} \\ C_{2t} \end{bmatrix}. \tag{4.29}$$

If the above conditions hold, then with $\hat{\zeta}(t) = L_t y(t)$ where the static state estimation gain is given by

$$L_t = -A_t Y(t) C_t^T (\tilde{J}_t + C_t Y(t) C_t^T)^{-1} \begin{bmatrix} 0 \\ I_{p_2} \end{bmatrix}, \tag{4.30}$$

$J_L < 0$ is satisfied for all possible $w(t) \in \ell^2$.

It is noted that the state estimation problem is slightly different from the state feedback case in that the solution of the difference Riccati equation (4.27) can be computed in real time recursively. For state feedback case, the solution of the difference Riccati equation (4.20) has to be computed backwards, and such backwards computation has to be repeated for each different N, and thus it is not suitable for real time implementation. Although there is a difference between the state feedback and estimation problems, a similar result to Corollary 4.1 holds for the case of state estimation.

Corollary 4.2 *Consider time varying system in (4.25), and the performance index in (4.26) with $\gamma_t > 0$ given. Then there exists any state estimator $\hat{\zeta}(t) = \mathcal{L}y(t)$ such that $J_L < 0$ for all possible $w(t) \in \ell^2$, if and only if there exists a solution sequence $\{Y(t)\}_{t=0}^{N-1}$ for the difference Riccati equation (4.27) such that the matrix $\tilde{Q}_t$ in (4.21) is nonsingular, and has p_1 negative eigenvalues and p_2 positive eigenvalues. Moreover suppose*

$$\lim_{t\to\infty} (A_t, B_{1t}, C_{1t}, C_{2t}, D_{2t}) = (A, B_1, C_1, C_2, D_{21})$$

that are constant matrices satisfying (C_2, A) detectable, and

$$\text{rank}\begin{bmatrix} A - e^{j\theta}I_n & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2 \ \forall\theta.$$

If in addition, $\lim_{t\to\infty}\gamma_t = \gamma > 0$, *and the following algebraic Riccati equation*

$$Y = A(I_n + \tilde{R}Y)^{-1}A^T + B_1^T B_1, \quad \tilde{R} = C_2^T(D_{21}^T D_{21})^{-1}C_2 - C_1^T C_1/\gamma^2$$

admits a stabilizing solution $Y \geq 0$ *such that* $\tilde{Q} = \lim_{t\to\infty}\hat{Q}_t$ *is nonsingular, and has* p_1 *negative eigenvalues and* p_2 *positive eigenvalues. Then* $\lim_{N>t\to\infty} Y(t) = Y$.

Because the proof is similar, it is omitted. It should be indicated that the conditions on Q and on the algebraic Riccati equation in the above lemma are both necessary and sufficient for the existence of the output injection gain L such that $\|T_L\|_\infty < \gamma$ where T_L is the transfer matrix from $w(t)$ to $\Delta_x(t) = \hat{\zeta}(t) - \zeta(t)$ given by

$$T_L(z) = C_1(zI_n - A - LC_2)^{-1}(B_1 + LD_{21})$$

with $L = \lim_{N>t\to\infty} L_t$. Again, for numerical computation, the difference Riccati equation in (4.27) is not in a good form as it requires computation of $(I_n + \tilde{R}_t Y(t))^{-1}$ that amounts to about $\mathcal{O}(n^3)$ computational complexity. Hence, in real time computation, it is recommended to convert (4.27) into the following:

$$Y(t+1) = A_t Y(t) A_t^T - A_t Y(t) C_t^T(\tilde{J}_t + C_t Y(t) C_t^T)^{-1} C_t Y(t) A_t^T + B_{1t} B_{1t}^T, \quad (4.31)$$

where $\tilde{J}_t$ is the same as in (4.29).

4.3.2 Real Time $\mathcal{H}_\infty$ Control for Finite Horizon Case

For the identified model $\hat{G}_t(z)$ at time t obtained through recursive least-squares algorithm, our control objective is to synthesize an observer-based controller $K_t(z)$ at each time t such that the closed-loop system is robustly stable, and has similar performance to that of the reference model. This problem is considered in Chapter 3 for time-invariant systems. We consider again SISO case for simplicity of the exposition although it can be easily generalized to multivariable systems. Specifically, our design problem is to synthesize $\hat{K}_t(z)$ for the feedback system as in Figure 4.3 to achieve the above mentioned stability and performance.

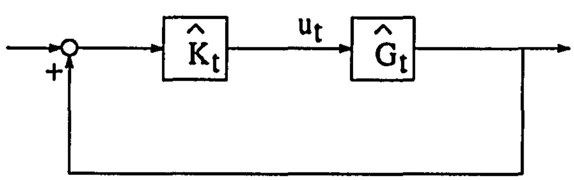


Figure 4.3: The feedback control system

This section will derive corresponding synthesis algorithm for time-varying case that is parallel to the results in Chapter 3. Since the identified model has an FIR structure of degree $n-1$, a simple state-space model can be used for $\hat{G}_t(z)$:

$$\hat{A}_t = \begin{bmatrix} 0 & 0 & .. & .. & 0 \\ 1 & 0 & 0 & .. & 0 \\ .. & .. & .. & .. & .. \\ 0 & .. & 0 & 1 & 0 \end{bmatrix}, \quad \hat{B}_t = \begin{bmatrix} 1 \\ 0 \\ .. \\ 0 \end{bmatrix}, \quad \hat{C}_t^T = \begin{bmatrix} \hat{g}_1^t \\ \hat{g}_2^t \\ .. \\ \hat{g}_{n-1}^t \end{bmatrix} \tag{4.32}$$

and $\hat{D}_t = \hat{g}_0^t$. Normally $\hat{D}_t = 0$ is assumed by strictly properness of the discretized

systems. It is noted that $\hat{A}_t$ and $\hat{B}_t$ are in fact constant matrices. For loopshaping purpose, often an integrator, or an accumulator is used to achieve both the desired loopshape, and zero steady-state error for step input. Thus, we consider the pre-compensated plant

$$\hat{P}_t(z) = \frac{\hat{G}_t(z)}{z-1} = \frac{\hat{p}_I^t}{z-1} + \sum_{i=1}^{n-1} \hat{p}_i^t z^{-i}.$$

Simple calculation yields following relation

$$\hat{p}_I^t = \sum_{k=0}^{n-1} \hat{g}_k^t, \quad \hat{p}_i^t = \sum_{k=i}^{n-1} \hat{g}_k^t, \quad i = 1, \ldots, n-1. \tag{4.33}$$

An n-order state-space realization can then be found as

$$\check{A}_t = \begin{bmatrix} \hat{A}_t & 0 \\ 0 & 1 \end{bmatrix}, \quad \check{B}_t = \begin{bmatrix} \hat{B}_t \\ \hat{p}_I^t \end{bmatrix}, \quad \check{C}_t = \begin{bmatrix} \hat{p}_1^t & \hat{p}_2^t & \ldots & \hat{p}_{n-1}^t & 1 \end{bmatrix} \tag{4.34}$$

and $\hat{P}_t = \check{C}_t(zI_n - \check{A}_t)^{-1}\check{B}_t$. The reference model is assumed of the prototype

$$\begin{aligned} R(z) &= \frac{\alpha_1}{z-1} + \frac{\alpha_2}{z-\beta} \\ &= \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & \alpha_1 \\ 0 & \beta & \alpha_2 \\ \hline 1 & 1 & 0 \end{array}\right], \quad |\beta| < 1. \end{aligned} \tag{4.35}$$

There are two reasons for choosing a prototype model. The first is its simple relation with both time domain and frequency domain performance. The second is that the augmentation as in Chapter 3 increases the order of the state-space realization by at most one. Indeed, we have the following augmentation according to Chapter 3

(using realization for $\hat{P}_t(z)$, and $\hat{G}_t(z)$):

$$A_t = \begin{bmatrix} \hat{A}_t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta \end{bmatrix}, \quad B_t = \begin{bmatrix} \hat{B}_t \\ \hat{p}_l^t \\ 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}, \quad C_t = \begin{bmatrix} \hat{C}_t & 1 & 1 \end{bmatrix}. \tag{4.36}$$

It is noted that the above realization yields

$$\hat{P}_t(z) = C_t(zI_{n+1} - A_t)^{-1}B_t, \quad R(z) = C_t(zI_{n+1} - A_t)^{-1}H$$

as required by the model reference control procedure in Chapter 3. The feedback system has a configuration in Figure 4.4 where $\hat{K}_t(z) = K_t(z)/(z-1)$.

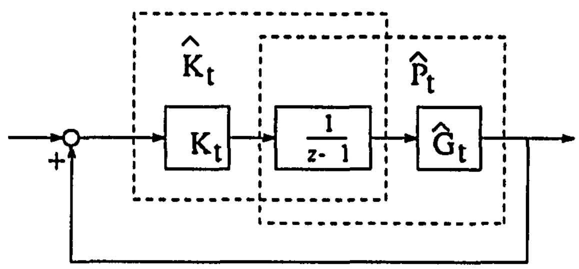


Figure 4.4: The modified feedback control system

The performance index is the induced 2-norm of the time-varying system

$$T(z) = \begin{bmatrix} (1-\lambda) \\ \lambda K_t(z) \end{bmatrix} (I - \hat{P}_t(z)K_t(z))^{-1} S_{id}^{-1} + \begin{bmatrix} (1-\lambda) \\ 0 \end{bmatrix}$$

where $S_{id}(z) = (I - R(z))^{-1} = I - C_t(zI_{n+1} - A_t + HC_t)^{-1}B_t$ is the ideal sensitivity. Strictly speaking, the above transfer matrix does not exist by time-varying nature. In that case, the above transfer matrix $T(z)$ should be interpreted as an operator that maps input $r(t)$ to the regulated output $\zeta(t)$. This problem was studied in Chapter 3 for time-invariant case. We note that with $w(t) = S_{id}r(t)$ (see Figure

4.5), there holds

$$\zeta(t) = \begin{bmatrix} (1-\lambda) \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} (1-\lambda) \\ \lambda K_t(z) \end{bmatrix} (I - \hat{P}_t(z)K_t(z))^{-1} r(t).$$

It follows that $T(z)$ maps $r(t)$ to $((1-\lambda)(w(t)+e(t)), \lambda u(t))$ by the feedback system in Figure 4.5. Because $R(z)$ can be chosen arbitrarily, S_{id} can be made stable, and has the ideal frequency shape.

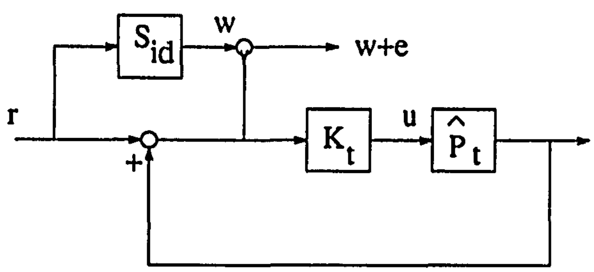


Figure 4.5: The reference feedback control system

Theorem 4.6 *Consider time varying system in (4.36), and the performance index*

$$J_\infty = \sum_{t=0}^{N-1} [\zeta(t)]^T \zeta(t) - \gamma_t^2 \sum_{t=0}^{N-1} [r(t)]^T r(t), \quad \zeta(t) = \begin{bmatrix} (1-\lambda)(w(t)+e(t)) \\ \lambda u(t) \end{bmatrix},$$

with $\gamma_t > 0$ given. Then there exists any output feedback controller $u(t) = \mathcal{K}y(t)$ such that $J_\infty < 0$ for all possible $r(t) \in \ell^2$, if and only if there exists a solution sequence $\{X(t)\}_{t=0}^{N-1}$ for the difference Riccati equation

$$\begin{aligned} X(t) &= A_t^T X(t+1)(I + R_t X(t+1))^{-1} A_t + C_t^T C_t, \quad X(N) = 0, \qquad (4.37) \\ R_t &= B_t B_t^T / \lambda^2 - H H^T / \gamma^2, \end{aligned}$$

such that the matrix

$$Q_t = J_t + \begin{bmatrix} B_t & H \end{bmatrix}^T X(t) \begin{bmatrix} B_t & H \end{bmatrix}, \quad J_t = \begin{bmatrix} -\gamma^2 I_{m_1} & 0 \\ 0 & \lambda^2 I_{m_2} \end{bmatrix},$$

is nonsingular, and has $m_1 = 1$ negative eigenvalues and $m_2 = 1$ positive eigenvalues. If the above holds, the controller achieving $\sup_{r(t)\in\ell^2} J_\infty < 0$ is given by

$$\begin{aligned} K_t(z) &= -F_t(zI_{n+1} - A_t - B_tF_t - L_tC_t)^{-1}L_t, \quad L_t = H, \qquad (4.38) \\ F_t &= -\begin{bmatrix} 0 & I_{m_2} \end{bmatrix} \left(J_t + \begin{bmatrix} B_t & H \end{bmatrix}^T X(t) \begin{bmatrix} B_t & H \end{bmatrix}\right)^{-1} \begin{bmatrix} B_t & H \end{bmatrix} X(t)A_t. \end{aligned}$$

Proof: Using the results in [46], the synthesis problem amounts to the solvability of the difference Riccati equation (4.37) with an additional constraint on Q_t, and the solvability of the difference Riccati equation

$$Y_{t+1} = (A_t + HC_t)(I_{n+1} + \alpha^2 C_t^T C Y(t))^{-1} Y(t)(A_t + HC_t)^T, \quad Y(0) = 0,$$

such that $I_{n+1} + \alpha^2 C Y(t) C_t^T > 0$ where $\alpha^2 = 1 - (1-\lambda)^2/\gamma^2 > 0$. Clearly, $Y(t) = 0$ for all t is the solution. Moreover the central controller is given as in (4.38). In this case, the output feedback reduces to the state feedback due to the identity

$$T(z) = \begin{bmatrix} (1-\lambda)C_t \\ \lambda F_t \end{bmatrix} (zI_{n+1} - A_t - B_tF_t)^{-1} H.$$

The theorem is thus true. ∎

For real time implementation, the difference Riccati equation in (4.37) can be written as

$$X(t) = A_t^T X(t+1)A_t - A_t^T X(t+1)V(t)X(t+1)A_t + C_t^T C_t$$

where

$$V(t) = \begin{bmatrix} B_t & H \end{bmatrix} \left(J_t + \begin{bmatrix} B_t & H \end{bmatrix}^T X(t+1) \begin{bmatrix} B_t & H \end{bmatrix} \right)^{-1} \begin{bmatrix} B_t & H \end{bmatrix}^T.$$

Because A_t, H are in fact time invariant, B_t has only one time varying element, and most of the entries in (A_t, B_t, H) are 1's and 0's, the iterative computation is much simpler requiring approximately $\mathcal{O}(n)$ for computational complexity.

For the dual case, we can set

$$\hat{A}_t^T = \begin{bmatrix} 0 & 0 & .. & .. & 0 \\ 1 & 0 & 0 & .. & 0 \\ .. & .. & .. & .. & .. \\ 0 & .. & 0 & 1 & 0 \end{bmatrix}, \quad \hat{B}_t = \begin{bmatrix} \hat{g}_1^t \\ \hat{g}_2^t \\ .. \\ \hat{g}_{n-1}^t \end{bmatrix}, \quad \hat{C}_t^T = \begin{bmatrix} 1 \\ 0 \\ .. \\ 0 \end{bmatrix}, \tag{4.39}$$

$$\tilde{A}_t = \begin{bmatrix} \hat{A}_t & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{C}_t^T = \begin{bmatrix} \hat{C}_t^T \\ \hat{p}_l^t \end{bmatrix}, \quad \tilde{B}_t^T = \begin{bmatrix} \hat{p}_1^t & \hat{p}_2^t & .. & \hat{p}_{n-1}^t & 1 \end{bmatrix}, \tag{4.40}$$

$$A_t = \begin{bmatrix} \hat{A}_t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta \end{bmatrix}, \quad B_t = \begin{bmatrix} \hat{B}_t \\ 1 \\ 1 \end{bmatrix}, \quad \begin{array}{l} H = \begin{bmatrix} 0 & \alpha_1 & \alpha_2 \end{bmatrix}, \\ C_t = \begin{bmatrix} \hat{C}_t & \hat{p}_l^t & 0 \end{bmatrix} \end{array} \tag{4.41}$$

where $\hat{p}_i$'s are the same as in (4.33). In this case, we have the relation

$$\hat{P}_t(z) = C_t(zI_{n+1} - A_t)^{-1}B_t, \quad R(z) = H(zI_{n+1} - A_t)^{-1}B_t.$$

Our goal is to minimize the induced 2-norm of the time-varying transfer matrix

$$\tilde{T}(z) = S_{id}^{-1}(I - K_t(z)P_t(z))^{-1} \begin{bmatrix} (1-\lambda) & \lambda K_t(z) \end{bmatrix} + \begin{bmatrix} (1-\lambda) & 0 \end{bmatrix}. \tag{4.42}$$

The transfer matrix $\tilde{T}(z)$ should be interpreted as an operator that maps the input $\tilde{r}(t)$ to the regulated output $\tilde{\zeta}(t)$. This problem was studied in Chapter 3 for time-

invariant case. The next result is parallel to the previous theorem for which the proof is omitted.

Theorem 4.7 *Consider time varying system in (4.41), and the performance index*

$$\tilde{J}_\infty = \sum_{t=0}^{N-1}[\tilde{\zeta}(t)]^T\tilde{\zeta}(t) - \gamma_t^2 \sum_{t=0}^{N-1}[\tilde{r}(t)]^T\tilde{r}(t),$$

with $\gamma_t > 0$ given. Then there exists any output feedback controller $u(t) = \mathcal{K}y(t)$ such that $\tilde{J}_\infty < 0$ for all possible $\tilde{r}(t) \in \ell^2$, if and only if there exists a solution sequence $\{X(t)\}_{t=0}^{N-1}$ for the difference Riccati equation

$$\begin{aligned} Y(t+1) &= A_tX(t)(I + \tilde{R}_tY(t))^{-1}A_t^T + B_tB_t^T, \;\; Y(0) = 0, \qquad (4.43) \\ \tilde{R}_t &= C_t^TC_t/\lambda^2 - H^TH/\gamma^2, \end{aligned}$$

such that the matrix

$$\tilde{Q}_t = \tilde{J}_t + \begin{bmatrix} C_t \\ H \end{bmatrix} Y(t) \begin{bmatrix} C_t \\ H \end{bmatrix}^T, \;\; \tilde{J}_t = \begin{bmatrix} -\gamma^2 I_{m_1} & 0 \\ 0 & \lambda^2 I_{m_2} \end{bmatrix},$$

is nonsingular, and has $m_1 = 1$ negative eigenvalues and $m_2 = 1$ positive eigenvalues. If the above holds, the controller achieving $\sup_{\tilde{r}(t)\in\ell^2} \tilde{J}_\infty < 0$ is given by

$$\begin{aligned} K_t(z) &= -F_t(zI_{n+1} - A_t - B_tF_t - L_tC_t)^{-1}L_t, \;\; F_t = H, \qquad (4.44) \\ L_t &= -A_tY(t)\begin{bmatrix} C_t \\ H \end{bmatrix}\left(\tilde{J}_t + \begin{bmatrix} C_t \\ H \end{bmatrix} Y(t) \begin{bmatrix} C_t \\ H \end{bmatrix}^T\right)^{-1} \begin{bmatrix} 0 \\ I_{p_2} \end{bmatrix}. \end{aligned}$$

For real time implementation, we convert difference Riccati equation (4.43) into the following:

$$Y(t+1) = A_tY(t)A_t - A_tY(t)\tilde{V}(t)Y(t)A_t + B_tB_t^T \qquad (4.45)$$

where

$$\tilde{V}(t) = \begin{bmatrix} C_t \\ H \end{bmatrix}^T \left(\tilde{J}_t + \begin{bmatrix} C_t \\ H \end{bmatrix} Y(t) \begin{bmatrix} C_t \\ H \end{bmatrix}^T \right)^{-1} \begin{bmatrix} C_t \\ H \end{bmatrix}.$$

For the same reason as in the dual case, the computational complexity for each iteration for solving $Y(t+1)$ requires $\mathcal{O}(n)$.

4.3.3 Robust Adaptive Control in $\mathcal{H}_\infty$

For adaptive control, the identification and control algorithms have to be combined in closed-loop and implemented in real time. A conventional approach is the model equivalence principle. However for $\mathcal{H}_\infty$ based adaptive control, there are several issues need be resolved before we can establish robustness for feedback adaptive control. The first is the quantification of the modeling error in $\mathcal{H}_\infty$ norm. Although an asymptotic error bound is derived in Section 2 for recursive least-squares algorithm, such error bound does not hold for each finite time instant, especially at the initial phase of the adaptive control. The second is the determination of the γ value for the control part. Clearly the γ value measures the performance and stability in the presence of the model uncertainty. Finally is the selection of λ value in the $\mathcal{H}_\infty$ performance index. These parameters play crucial role in the adaptive control algorithm proposed in this dissertation.

To further analyze the robustness of the feedback system, we recall the performance index in Chapter 3:

$$J_1 = \left\| \begin{bmatrix} (1-\lambda)E_{out} \\ \lambda T_{in} S_{id}^{-1} \end{bmatrix} \right\|_\infty, \text{ or } J_2 = \left\| \begin{bmatrix} (1-\lambda)E_{in} & \lambda S_{id}^{-1} T_{out} \end{bmatrix} \right\|_\infty$$

where S_{id} is the ideal sensitivity, and $T_{in}(z) = K\hat{P}(I - K\hat{P})^{-1}$ and $T_{out}(z) = \hat{P}K(I - \hat{P}K)^{-1}$. The performance index has two components: the first one involves the size of E_{out} or E_{in} that measures the performance of the feedback system in frequency domain as discussed in Chapter 3. The second component involves the size of $K(I - \hat{P}K)^{-1}$ that hinges to exactly the robust stability condition for additively perturbed plant.

Lemma 4.2 *Suppose that $J_1 < \gamma$, or $J_2 < \gamma$ for some stabilizing feedback controller where $\hat{P}(z)$ is stable except a possible pole at $z = 1$. Suppose further that $\|S_{id}\|_\infty \leq 1$ (that can always be made true because $S_{id}(z)$ can be chosen arbitrarily). Then the feedback control system in Figure 4.4 is robustly stable for all additively perturbed plant $P(z) = \hat{P}(z) + \Delta_P(z) \in \mathcal{H}_\infty$ provided that $\|\Delta_P\|_\infty \leq \lambda/\gamma$.*

Proof: For additively perturbed plant with stable uncertainty, a necessary and sufficient condition is [13]

$$\|K(I - \hat{P}K)^{-1}\|_\infty \|\Delta_P\|_\infty < 1.$$

The hypothesis $\|S_{id}\|_\infty \leq 1$, and $J_1 < \gamma$, or $J_2 < \gamma$, together with $\|\Delta_P\|_\infty \leq \lambda/\gamma$ imply that the above condition is true, and thus the closed-loop system is stable. ■

There is a gap between the robust stability condition in the above lemma and that for the feedback system in Figure 4.3 because the model uncertainty is given by

$$\Delta_G(z) = G(z) - \hat{G}(z) \in \mathcal{H}_\infty \tag{4.46}$$

whereas $\Delta_P(z) = \Delta_G(z)/(z-1)$ that may not be bounded even if $\Delta_G(z)$ is bounded. The next result is true.

Corollary 4.3 *Suppose that $J_1 < \gamma$, or $J_2 < \gamma$ for some stabilizing feedback controller where $\hat{P}(z) = \hat{G}(z)/(z-1)$ is the nominal plant with additive model uncertainty in (4.46). Suppose further that*

$$\|\hat{S}_{id}\|_\infty \leq \kappa, \quad \hat{S}_{id}(z) = S_{id}/(z-1). \tag{4.47}$$

Then the feedback control system in Figure 4.3 is robustly stable for all additively perturbed plant $G(z) = \hat{G}(z) + \Delta_G(z) \in \mathcal{H}_\infty$ provided that $\|\Delta_G\|_\infty \leq \kappa\lambda/\gamma$.

Proof: The hypothesis $J_1 < \gamma$, or $J_2 < \gamma$ implies that

$$|K(e^{j\omega})(I - \hat{P}(e^{j\omega})K(e^{j\omega}))^{-1}S_{id}^{-1}(e^{j\omega})| < \gamma/\lambda \quad \forall\omega.$$

Substituting $\hat{P}(z) = \hat{G}(z)/(z-1)$, and using the relation $\hat{K}(z) = K(z)/(z-1)$, the above is equivalent to

$$\begin{aligned}
\gamma/\lambda &> |(e^{j\omega}-1)\hat{K}(e^{j\omega})(I - \hat{G}(e^{j\omega})\hat{K}(e^{j\omega}))^{-1}S_{id}^{-1}(e^{j\omega})| \\
&= |\hat{K}(e^{j\omega})(I - \hat{G}(e^{j\omega})\hat{K}(e^{j\omega}))^{-1}\hat{S}_{id}^{-1}(e^{j\omega})| \\
&\geq |\hat{K}(e^{j\omega})(I - \hat{G}(e^{j\omega})\hat{K}(e^{j\omega}))^{-1}|/\kappa \quad \forall\omega
\end{aligned}$$

by the condition $\|\hat{S}_{id}\|_\infty \leq \kappa$. It follows form [13] that the feedback system in Figure 4.3 is stable for all $\|\Delta_G\|_\infty \leq \kappa\lambda/\gamma$. ∎

Because the reference model has a pole at $z = 1$, the value of κ in (4.47) can be made small or close to one. This condition should be taken into consideration

in the synthesis of $S_{id}(z)$ in order to obtain large stability margin. We note that the larger the value of λ, the larger the stability margin for the closed loop system as it can tolerate larger size of model uncertainty by the robust stability condition $\|\Delta_G\|_\infty \leq \kappa\lambda/\gamma$. In extreme case, the performance index with $\lambda = 1$ reduces to the robust stabilization problem without any consideration of the performance. The above analysis leads to the conclusion that the value of λ should be chosen close to one at the initial phase of the adaptive control because the plant model is poorly known at the beginning. As the model becomes more accurate, the value of λ can be decreased so that the performance of the system can be taken into account for adaptive control. The problem is clearly how to tune the value of λ, and how to determine the value of γ for which both are time varying.

It turns out that the model validation results in [58] are helpful. The following result is quoted from [58].

Theorem 4.8 *Let $\{u(t)\}_{t=0}^{L-1}$, and $\{y(t)\}_{t=0}^{L-1}$ be finite input/output response data. Then there exists a linear time varying system $P_t(z)$ capable of producing the same input/output data $\{u(t)\}_{t=0}^{L-1}$, and $\{y(t)\}_{t=0}^{L-1}$ with $\|P\|_\infty \leq \epsilon$, if and only if $\|y\|_2 \leq \epsilon\|u\|_2$.*

Because of the time varying nature for $\hat{G}_t$, the model uncertainty $\Delta_G = G - \hat{G}_t$ is in fact time varying. Thus the error signal

$$e_G(t) = y(t) - \hat{y}(t) = (G - \hat{G}_t)u(t)$$

as used in RLS ($e_G(t) = y(t) - \Phi(t-1)^T\hat{\theta}(t-1)$) is the output of a time varying system. The above theorem implies that there exists an extension of $\hat{G}_t$ such that $\|\Delta_G\|_\infty = \|\Gamma_t e_G\|_2/\|\Gamma_t u\|_2$ where Γ_t is a truncation operator that keeps only those numbers at time smaller than or equal to t. Because $e_G(t)$ is measurable, and used in RLS to estimate $\hat{G}_t$ recursively, the quantity $\hat{\gamma}_t = \|\Gamma_t e_G\|_2/\|\Gamma_t u\|_2$ can be easily computed. The effect of the noise can also be taken into account if the size of the noise is known *a priori.* For instance if ϵ_t is the truncated ℓ^2 norm of the corruption noise, then we may set

$$\hat{\delta}_t = (\epsilon_t + \|\Gamma_t e_G\|_2)/\|\Gamma_t u\|_2. \tag{4.48}$$

Although this estimate is conservative, it does not change much the value of $\hat{\gamma}_t$ if the size of noise is small. Combining the above analysis with Corollary 3.8, we conclude that the value of $\hat{\gamma}_t$ used in the control part needs to satisfy the inequality $\hat{\gamma}_t \leq \kappa\lambda/\hat{\delta}_t$.

We summarize the our proposed adaptive algorithm as follows.

$\mathcal{H}_\infty$ Based Adaptive Control Algorithm:

- Step 1: Design experiments to obtain *a priori* information (M, ρ) of the true and unknown system. Based on the *a priori* information, synthesize the reference model $R(z)$ (e.g., prototype system) to represent the desired frequency shape and time response, and to make the value of κ in (4.47) small, or close to one.

- Step 2: Using the recursive least-squares algorithm in Section 2.1 (Theorem

2.1) to identify the plant $G(z) \in \mathcal{S}(M, \rho)$ with FIR model adaptively, using $P(-1) = P_0$ as in (4.13), and $\hat{\theta}_0 = 0$.

- Step 3: At each time t, estimate the value of $\hat{\gamma}_t$, the size of the modeling error using (4.48). Choose λ close to one to start with, and set $\gamma \leq \kappa\lambda_t/\hat{\gamma}_t$.

- Step 4: For each identified FIR model $\hat{G}_t$, compute precompensated plant $\hat{P}_t$ according to (4.33), and compute realization of $\hat{P}_t$, and $F = H$ according to (4.39) – (4.41). Compute iteratively $Y(t+1)$ from the difference Riccati equation (4.45), and set the observer-based controller according to (4.44) with t replaced by $t+1$.

A few comments are in order. First at the initial stage of the adaptive control, the plant is poorly known that will result in large $\hat{\gamma}_t$, and consequently small γ_t value. Because the time interval is small at the initial stage, the difference Riccati equation (4.45) is likely to produce the required solution even though γ_t is small. As time t increases, $\hat{\gamma}_t$ tends to improve that allows large γ_t value, and thus the difference Riccati equation (4.45) is likely to continue producing the required solution even though the time interval becomes large. Second, we have used state estimation Riccati equation in order that the controller can be designed in real time. Recall that the state feedback Riccati equation has a backward structure, and is not suitable for real time implementation. Because the observer based controller has constant state feedback gain, it can be implemented as in Figure 4.6 where A, F are constant, and C has only one time varying entry. Third, the value of λ should decrease gradually

to about 0.5 as time increases in order to improve the performance of the system. The tune of λ should be based on the value of $\hat{\gamma}_t$ as discussed earlier. The adjustment of λ_t and γ_t should also take the asymptotic behavior of the identified model into consideration so that both λ_t and γ_t will converge to some fixed value.

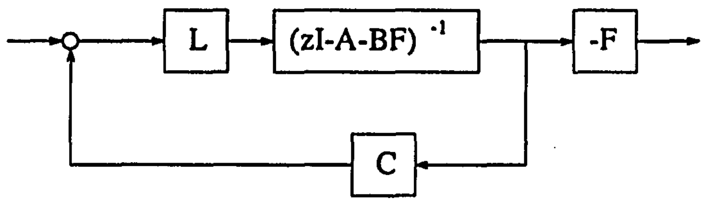


Figure 4.6: The observer-based feedback control

Theorem 4.9 *Suppose the true and unknown plant $G(z) \in S(M,\rho)$, and the reference signal is persistently exciting. Then with the $\mathcal{H}_\infty$ based Adaptive Control Algorithm, the feedback system with true plant is stable if for each time t, the difference Riccati equation (4.45) admits the required solution as in Theorem 3.6.*

Proof: The hypothesis implies that the identified model converges to $\hat{G}(z)$, an FIR model with modeling error bounded in $\mathcal{H}_\infty$ norm. Moreover the state estimator gain L_t converges to L as $t \to \infty$ that is stabilizing. It follows that the observer-based controller stabilizes the model $\hat{P}(z) = \hat{G}(z)/(z-1)$ asymptotically, and satisfies $\|\hat{K}(I-\hat{G}\hat{K})^{-1}\|_\infty < \kappa\gamma/\lambda$ where γ and λ are the limiting values of γ_t and λ_t. With $G(z) = \Delta_G(z) + \hat{G}(z)$, the feedback system in Figure 4.3 has an equivalent form (in terms of stability) in Figure 4.7. Clearly the instability of the feedback system

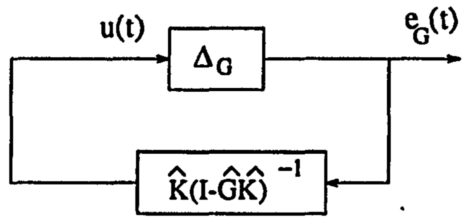


Figure 4.7: The feedback control system

implies that $\|\Delta_G \hat{K}(I - \hat{G}\hat{K})^{-1}\|_\infty \geq 1$, and the ℓ^2 norm of the signal $u(t)$ and $\|e_G\|_2$ will grow unboundedly by the persistent exciting. However this instability is clearly eliminated by the fact that

$$\|\hat{K}(I - \hat{G}\hat{K})^{-1}\|_\infty < \kappa\gamma/\lambda \leq \kappa\hat{\gamma}/\lambda \leq \kappa\|e_G\|_2/(\lambda\|u\|_2).$$

Hence the observer based controller stabilizes $G(z)/(z-1)$. ∎

A natural question is whether or not the difference Riccati equation admits the required solution for all t. Our suggest is to replace $\hat{K}_t$ with a constant gain smaller than $1/M$ if (4.45) fails to admit the required solution. Because $G(z)$ is stable, and the loop gain is smaller than one, the feedback system maintains stability. As more accurate model is obtained, the difference Riccati equation is more likely to produce the required solution. We may then turn on the controller $\hat{K}_t$.

Chapter 5

Concluding Remarks and Future Research Problems

Adaptive control has been studied for several decades because of its learning and adaptation ability. However, the conventional approach has a drawback on the assumption of a finite dimensional model for physical systems. The model uncertainty is ignored, which leads to the lack of stability robustness for adaptive systems in the presence of model uncertainty. Because physical systems can never be described exactly by finite dimensional models, adaptive control must take the model uncertainty into account in order to gain engineering applications.

This dissertation focused on robust adaptive control using an $\mathcal{H}_\infty$ approach because it offers worst-case stability and performance guarantees provided that the physical plant can be described by $\mathcal{H}_\infty$ norm bounded uncertain models. This research problem was decomposed into identification and control that was tackled separately. For the identification problem, our results showed that for a class of infinite dimensional systems, the least-squares based linear algorithm is capable of

producing the uncertain models with $\mathcal{H}_\infty$ norm bounded additive uncertainty. The upper bounds on the model uncertainty are provided for both deterministic "hard bound" and stochastic "soft bound". For the control problem, model reference control was studied using $\mathcal{H}_\infty$ based loopshaping method. Our results provide synthesis algorithms for the additive uncertain models to achieve performance close to that of the reference model. These algorithms are computationally efficient with low complexity, and are thus suitable for real time implementation. Finally the results for $\mathcal{H}_\infty$ based identification and control were employed to derive algorithms for the purpose of adaptive control. Under certain conditions, robustness for the proposed adaptive algorithm can be established.

Compared with the existing work, this dissertation provided new insight to modeling and control problems, and opened new direction for robust adaptive control. The use of $\mathcal{H}_\infty$ theory for both identification and control in adaptive systems is novel, and was not investigated by other research workers. Our results indicated that while $\mathcal{H}_\infty$ control was initially developed in frequency domain, it has applications to adaptive control using time domain data on line. Moreover our results were built on those results on identification in $\mathcal{H}_\infty$, and model validation that appeared in literature very recently. Hence robust adaptive control using $\mathcal{H}_\infty$ method is still in its infant stage. Because $\mathcal{H}_\infty$ approach is new to adaptive control, many problems have not been explored. Although our work has accomplished an important step in using $\mathcal{H}_\infty$ theory for adaptive control, we are not able to even touch some

of the problems in this research direction. We would also like to mention that the identification algorithm used in this dissertation is least-squares type. Hence it is not completely an $\mathcal{H}_\infty$ type algorithm. Thus a more interesting problem is the direct use of $\mathcal{H}_\infty$ approach to tackle the identification problem for which very little is presently known. Further research is required in this direction. We outline some of the future research problems next, and expect that these problems will be resolved in the future.

- Identification with IIR models.

  In this dissertation, we considered only those models of FIR type. In this case, least-squares algorithms are capable of producing both nominal model, and the quantification of the modeling error. However, if IIR models are used, then those results derived in Chapter 2 do not hold any more. The problem of modeling uncertainties becomes much more difficult for IIR models. Because an IIR model is more general and more effective for representation of linear systems, this problem is extremely important for system identification.

- Identification and control of unstable systems.

  For unstable plants, it is much more difficult to tackle the robustness for adaptive control systems. First, the output for an unstable system is very easy to be saturated that gives much more difficulty to obtain some useful *a priori* information. Without necessary *a priori* information, it is not possible to obtain the upper bound for the modeling error in frequency domain based on

finite measurement data. Second, as the identified model tends to be unstable also if the true plant is unstable, the control problem for such uncertain systems is more much difficult than the stable case - recall that the true plant is unknown. Moreover, the synthesis algorithm proposed in this dissertation would be more involved because in this case, both state feedback gain, and state estimation gain need to be designed.

- Finite horizon $\mathcal{H}_\infty$ control for state feedback.

For the state feedback problem, the synthesis of the state feedback gain with finite horizon $\mathcal{H}_\infty$ control can not be done on-line because of the backward iteration. As for the case that the number of inputs is larger than the number of outputs for the plant, the adaptive control algorithm developed in this dissertation has to solve state feedback gain, and thus it remains unclear how the synthesis of the state feedback gain can be implemented in real time.

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Appendix: Supplementary Data for Chapter 3

This appendix is used to include those background materials, and derivations not available in Chapter 3. This section refers to particularly those references on $\mathcal{H}_\infty$ control [14, 15, 20, 37, 48, 73].

A1. Inner and Co-inner Functions for Theorem 3.14

Corollary 5.1 *Suppose* $V(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{RH}_\infty$ *is a controller realization, then* $V(z)$ *is inner if and only if there exists a matrix* $X = X^* \geq 0$ *such that*

(a) $A^*XA - X + C^*C = 0$

(b) $D^*C + B^*XA = 0$

(c) $(D - CA^{-1}B)^*D = D^*D + B^*XB = I$

and $V(z)$ *is co-inner if and only if there exists a matrix* $Y = Y^* \geq 0$ *such that*

(d) $A^*YA - Y + BB^* = 0$

(e) $BD^* + AYC^* = 0$

(f) $D(D - CA^{-1}B)^* = DD^* + CYC^* = I$

then show that $V(z)$ *in Theorem 3.69 can be written as* $V(z)V(z)^* = I + CYC^*$.

Proof: By the Theorem 3.14, the transfer function $V(z) = I + \alpha C(zI_{n+r} - A - L_sC)^{-1}\alpha^{-1}(L_s - H)$. To further simplify, let $\bar{C} = \alpha C$, $\bar{A} = A + L_sC$ and $\bar{B} = \alpha^{-1}(L_s - H)$. Then $V(z) = I + \bar{C}(zI - \bar{A})^{-1}\bar{B}$. We need to show that $V(z)$ is co-inner. With stabilizing solution $Y \geq 0$ in Theorem 3.69, it is noted that Riccati equation (3.61) can be written as

$$\begin{aligned} Y &= (A + L_sC)Y(A + L_sC)^* + \alpha^{-2}(L_s - H)(L_s - H)^*, \\ &= \bar{A}^*Y\bar{A} + \bar{B}\bar{B}^*. \end{aligned}$$

where $L_s = H - \alpha^2(A + HC)YC^*(I + \alpha^2CYC^*)^{-1}$. To show the condition (e) above, we rearrange and multiply L_s by $(I + \alpha^2CYC^*)^{-1}$. The result is satisfied as follow.

$$\alpha^{-1}(L_s - H) + \alpha(A + L_sC)YC^* = \bar{B}D + \bar{A}Y\bar{C}^* = 0$$

and to show the condition (f), let $V_s(z) = \bar{C}(zI - \bar{A})^{-1}\bar{B}$.

$$(zI - \bar{A})Y(z^{-1}I - \bar{A}^*) = Y - \bar{A}Yz^{-1} - zY\bar{A}^* + \bar{A}Y\bar{A}^*$$

we rearrange the equation above.

$$Y = (zI - \bar{A})Y(z^{-1}I - \bar{A}^*) + \bar{A}Yz^{-1} + zY\bar{A}^* - \bar{A}Y\bar{A}^*$$

and

$$
\begin{aligned}
\bar{B}\bar{B}^* &= Y - \bar{A}Y\bar{A}^* \\
&= (zI - \bar{A})Y(z^{-1}I - \bar{A}^*) + \bar{A}Yz^{-1} + zY\bar{A}^* - \bar{A}Y\bar{A}^* \\
&= (zI - \bar{A})Y(z^{-1}I - \bar{A}^*) + \bar{A}Y(z^{-1}I - \bar{A}^*) + (zI - \bar{A})Y\bar{A}^*.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
V_sV_s^* &= \bar{C}(zI - \bar{A})^{-1}\bar{B}\bar{B}^*(z^{-1}I - \bar{A}^*)^{-1}\bar{C}^* \\
&= \bar{C}Y\bar{C}^* + \bar{C}(zI - \bar{A})^{-1}\bar{A}Y\bar{C}^* + \bar{C}Y\bar{A}^*(z^{-1}I - \bar{A}^*)^{-1}\bar{C}^*
\end{aligned}
$$

and

$$
\begin{aligned}
V(z)V(z)^* = \quad & (D + V_s)(D^* + V_s^*) \\
= \quad & DD^* + V_sD^* + DV_s^* + V_sV_s^* \\
= \quad & DD^* + \bar{C}(zI - \bar{A})^{-1}\bar{B}D^* + D\bar{B}^*(z^{-1}I - \bar{A}^*)^{-1}\bar{C}^* + \bar{C}Y\bar{C}^* \\
+ \quad & \bar{C}(zI - \bar{A})^{-1}\bar{A}Y\bar{C}^* + \bar{C}Y\bar{A}^*(z^{-1}I - \bar{A}^*)^{-1}\bar{C}^* \\
= \quad & DD^* + \bar{C}Y\bar{C}^* + \bar{C}(zI - \bar{A})^{-1}(\bar{B}D^* + \bar{A}Y\bar{C}^*) + (D\bar{B}^* \\
+ \quad & \bar{C}Y\bar{A}^*)(z^{-1}I - \bar{A}^*)^{-1}\bar{C}^* \\
= \quad & DD^* + \bar{C}Y\bar{C}^* \\
= \quad & I + \bar{C}Y\bar{C}^* \\
= \quad & I + \alpha^2 CYC^*.
\end{aligned}
$$

Therefore $V(z)V(z)^* = I + \alpha^2 CYC^*$ is proved. However, this is not exact co-inner.

∎

A2. The Derivation of Discrete-time Equation :

The state space solution in[37] yields the following equivalent conditions for $J < \gamma$:

Let A, R and $Q \in C^{n \times n}$ with Q and R Hermitian. The symplectic pairs S with $2n \times 2n$ matrices for obtaining the discrete Riccati equation

$$S = \left(\begin{bmatrix} A & 0 \\ -Q & I \end{bmatrix}, \begin{bmatrix} I & R \\ 0 & A^T \end{bmatrix} \right)$$

We now present some results on the properties of X as well as conditions under which S belongs to dom(Ric).

Definition 5.1 Discrete-time Riccati equation

Suppose that $S \in$ dom(Ric) and $X = Ric(S)$. Then

(a) $X = X^T$

(b) X satisfies $X = A^T X(I + RX)^{-1} A + Q$.

(c) The matrix $(I + RX)^{-1} A$ is stable.

Lemma 5.1 *Suppose that $R = BB^T$ and $Q = C^T C$ where (A, B) is stabilizable and (C, A) has no unobservable modes on $\{z : |z| = 1\}$. Then $S \in dom(Ric(S)) \geq 0$, and ker(X) belongs to stable unobservable subspace of (C, A).*

The discrete-time $\mathcal{H}_\infty$ control which is associated with two symplectic pairs:

$$S = \left(\begin{bmatrix} A - BR^{-1} D_{1.}^T C_1 & 0 \\ -C_1^T (I - D_{1.} R^{-1} D_{1.}^T) C_1 & I \end{bmatrix}, \begin{bmatrix} I & BR^{-1} B^T \\ 0 & (A - BR^{-1} D_{1.}^T C_1)^T \end{bmatrix} \right)$$

$$T = \left(\begin{bmatrix} (A - B_1 D_{.1}^T \tilde{R}^{-1} C)^T & 0 \\ -B_1^T (I - D_{.1}^T \tilde{R}^{-1} D_{.1}) B_1^T & I \end{bmatrix}, \begin{bmatrix} I & C^T \tilde{R}^{-1} C \\ 0 & A - B_1 D_{.1}^T \tilde{R}^{-1} C \end{bmatrix} \right)$$

Let us consider for the plant size of $m \geq p$ given by Theorem 3.10. A state-space realization of $G(z)$ is given by

$$G(z) = \left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array} \right] = \left[\begin{array}{c|c} 0 & 0 \\ 0 & I_m \\ \hline I_p & 0 \end{array} \right] + \left[\begin{array}{c} (1-\lambda)C \\ 0 \\ \hline C \end{array} \right] (zI - A)^{-1} \left[\begin{array}{c|c} -H & B/\lambda \end{array} \right]$$

$$= \left[\begin{array}{c|c|c} A & -H & B/\lambda \\ \hline (1-\lambda)C & 0 & 0 \\ 0 & 0 & I_m \\ \hline C & I_p & 0 \end{array} \right].$$

We assume that the system $G(z)$ is stable and $\bar{\sigma}(D) < 1$. First, we define some matrices:

$$R = D_{1.}^* D_{1.} - \begin{bmatrix} \gamma^2 I_m & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{1.} = [D_{11} \;\; D_{12}] = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix},$$

$$D_{1.}^* C_1 = \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D_{.1} = [D_{11}^* \;\; D_{21}^*] = \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix},$$

$$R^{-1} = \begin{bmatrix} -\gamma^{-2} I & 0 \\ 0 & I \end{bmatrix},$$

$$\tilde{R}^{-1} = \begin{bmatrix} -\gamma^{-2} I & 0 & 0 \\ 0 & -\gamma^{-2} I & \\ 0 & 0 & I \end{bmatrix}, \quad R = \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix},$$

$$B_f = [B_1 \; B_2] = \begin{bmatrix} -H & B/\lambda \end{bmatrix}, \; C_f = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \left[\begin{array}{c} (1-\lambda)C \\ 0 \\ \hline C \end{array}\right]$$

$$B_f R^{-1} D_{1.}^* C_1 = 0, \;\; D_{1.} R^{-1} D_{1.}^* = 0$$

$$I + B_f R^{-1} B_f^* X = I + (\lambda^{-2} BB^* - \gamma^{-2} HH^*)X, \;\; B_1(I - D_{.1}^* \tilde{R}^{-1} D_{.1})B_1^* = 0$$

$$A - B_1 D_{.1}^* \tilde{R}^{-1} C = A + HC, \;\; I + C_f^* \tilde{R}^{-1} C_f Y = I + [1 - (\frac{1-\lambda}{\gamma})^2] C^* C Y$$

$$\tilde{R}^{-1} C_f = \begin{bmatrix} -\frac{(1-\lambda)}{\gamma^2} C \\ 0 \\ C \end{bmatrix}$$

$$ZZ = (I + \tilde{R}^{-1} C_f Y C_f^*)^{-1} = \left[\begin{array}{cc|c} I - \alpha^2 CYC^* & 0 & -\frac{(1-\lambda)}{\gamma^2} CYC^* \\ 0 & I & 0 \\ \hline CY(1-\lambda)C^* & 0 & I + CYC^* \end{array}\right]^{-1}$$

The block element(3,3) of matrix $(I + \tilde{R}^{-1} C_f Y C_f^*)^{-1}$ can be calculated as Schur complement as follow: If A^{-1} exists, then

$$\left[\begin{array}{c|c} A & D \\ \hline C & B \end{array}\right]^{-1} = \begin{bmatrix} A^{-1} + E\Delta^{-1}F & -E\Delta^{-1} \\ -\Delta^{-1}F & \Delta^{-1} \end{bmatrix}$$

where $\Delta = B - CA^{-1}D$, $E = A^{-1}D$, and $F = CA^{-1}$.

$$\begin{aligned} ZZ_{(3,3)} &= \left(I + CYC^* + CYC^* (\frac{(1-\lambda)}{\gamma})^2 [I - (\frac{(1-\lambda)}{\gamma})^2 CYC^*]^{-1} CYC^*\right)^{-1} \\ &= \left(I + CYC^* + (-I + [I - (\frac{(1-\lambda)}{\gamma})^2 CYC^*])CYC^*\right)^{-1} \\ &= [I + (I - \beta^2 CYC^*)^{-1} CYC^*]^{-1} \\ &= I - (I - \beta^2 CYC^* + CYC^*)^{-1} CYC^* \\ &= I - (I + \alpha^2 CYC^*)^{-1} CYC^* \end{aligned}$$

If S $\in$dom(Ric) and $T \in$ dom(Ric), there exist matrix $X \geq 0$, and $Y \geq 0$. By the lemma 5.1, and lemma 5.1, there exists a matrix $X = Ric(S) \geq 0$. Then we arrange to give the Riccati equation by the lemma 5.1. The Riccati equation of the system can be represented as follows:

$$
\begin{aligned}
X = \quad & A^*X(I+RX)^{-1}A+Q \\
= \quad & (A-B_fR^{-1}D_{1.}^*C_1)^*X(I+B_fR^{-1}B_f^*X)^{-1}(A-B_fR^{-1}D_{1.}^*C_1) \\
+ \quad & C_1^*(I-D_{1.}R^{-1}D_{1.}^*)C_1 \\
= \quad & A^*X(I+[-H\ B/\lambda]\begin{bmatrix} -\gamma^{-2}I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} -H^* \\ B^*/\lambda \end{bmatrix}X)^{-1}A \\
+ \quad & \begin{bmatrix} (1-\lambda)C^* & 0 \end{bmatrix}[I-0]\begin{bmatrix} (1-\lambda)C \\ 0 \end{bmatrix} \\
= \quad & A^*X[I+(\lambda^{-2}BB^*-\gamma^{-2}HH^*)X]^{-1}A+(1-\lambda)^2C^*C.
\end{aligned}
$$

and also the state feedback matrices [37] is given by

$$
\begin{aligned}
F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad &= \quad -(R+B_f^*XB_f)^{-1}(B_f^*XA+D_{1.}^*C_1 \\
&= \quad -(R+B_f^*XB_f)^{-1}B_f^*XA \\
&= \quad -(I+R^{-1}B_f^*XB_f)^{-1}R^{-1}B_f^*XA \\
&= \quad -R^{-1}B_f^*X(I+B_fR^{-1}B^*X)^{-1}A \\
&= \quad -\begin{bmatrix} -\gamma^{-2}I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} -H^* \\ B^*/\lambda \end{bmatrix}X[I+(\lambda^{-2}BB^*-\gamma^{-2}HH^*)X]^{-1}A \\
F = F_2 \quad &= \quad -\lambda^{-2}B^*X[I+(\lambda^{-2}BB^*-\gamma^{-2}HH^*)X]^{-1}A.
\end{aligned}
$$

Similarly, there exists $Y = Ric(T)$ satisfying

$$Y = A(I + YR)^{-1}YA^{*} + Q$$

Then the stabilizing solution Riccati equation can be written as

$$\begin{aligned} Y &= A(I + YR)^{-1}YA^{*} + Q \\ &= (A - B_1 D_{.1}^{*} \tilde{R}^{-1} C_f)(I + C_f^{*} \tilde{R}^{-1} C_f Y)^{-1} Y (A - B_1 D_{.1}^{*} \tilde{R}^{-1} C_f)^{*} \\ &+ B_1 (I - D_{.1}^{*} \tilde{R}^{-1} D_{.1}) B_1^{*} \\ &= (A + HC)[I + \alpha^2 Y C^{*} C]^{-1} Y (A + HC)^{*}. \end{aligned}$$

Since $A + HC$ is stable, Y can be zero. If $Y = 0$, then $L = H$. and also the output injection matrices [37] is given by

$$\begin{aligned} L = [L_1 \ \ L_2] &= -[B_1 D_{.1}^{*} + AYC_f^{*}](\tilde{R} + C_f Y C_f^{*})^{-1} \\ &= -[B_1 D_{.1}^{*} + AYC_f^{*}](I + \tilde{R}^{-1} C_f Y C_f^{*})^{-1} \tilde{R}^{-1} \\ &= -[(\ 0 \ \ 0 \ \ -H \ ) + AYC_f^{*}](I + \tilde{R}^{-1} C_f Y C_f^{*})^{-1} \tilde{R}^{-1} \\ L = L_2 &= -([\ 0 \ \ 0 \ \ -H \ ] + AYC_f^{*})(I + \tilde{R}^{-1} C_f Y C_f^{*})^{-1} \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \end{aligned}$$

In order to get L, let us separate each step of calculation.

Step 1. consider

$$AYC_f^{*}(I + \tilde{R}^{-1} C_f Y C_f^{*})^{-1} \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} = A[I + YC_f^{*} \tilde{R}^{-1} C_f)^{-1} Y C_f \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}$$

$$= A\left[I + Y([(1-\lambda)C^* \;\; 0 \;\; C^*]\begin{bmatrix} -\gamma^{-2}I & 0 & 0 \\ 0 & -\gamma^{-2}I & \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} (1-\lambda)C \\ 0 \\ C \end{bmatrix})\right]^{-1} YC^*$$

$$= A(I + \alpha^2 YC^*C)^{-1}YC^* = AYC^*(I + \alpha^2 CYC^*)^{-1}$$

Step 2.

$$[\; 0 \;\; 0 \;\; -H \;](I + \tilde{R}^{-1}C_f Y C_f^*)^{-1}\begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} = -H[I - (I + \alpha^2 CYC^*)^{-1}CYC^*]$$

Finally, we combine two equations obtained by each steps. The output injection L is given by

$$\begin{aligned} L_2 &= -[AYC^*(I + \alpha^2 CYC^*)^{-1} - H[I - (I + \alpha^2 CYC^*)^{-1}CYC^*] \\ &= -[AYC^*(I + \alpha^2 CYC^*)^{-1} - H + HCYC^*(I + \alpha^2 CYC^*)^{-1}] \\ &= H - (A + HC)YC^*(I + \alpha^2 CYC^*)^{-1} \end{aligned}$$

Suppose that there exist two Riccati solutions, then all rational internally stabilizing controllers $K(z)$ such that $\|\mathcal{F}_L(P, K)\|_\infty$ are given by $K = \mathcal{F}_{\mathcal{L}}(K_o, \Phi)$ [37]. The central controller is given by

$$K_o(z) = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & 0 \end{array}\right]$$

where

$$\hat{A} \;=\; A + BF + \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2$$

$$
\begin{aligned}
\hat{B}_1 &= -Z_\infty^{-1}L_2 + \hat{B}_2\hat{D}_{12}^{-1}D_{11} \\
\hat{B}_2 &= Z_\infty^{-1}(B_2 + L_{12})\hat{D}_{12} \\
\hat{C}_1 &= F_2 + \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2 \\
Z_\infty &= I - \gamma^{-2}YX \\
\hat{C}_2 &= -\hat{D}_{21}(C_2 + D_{21}F_1)
\end{aligned}
$$

and we note that $Z_\infty = I$ due to $Y = 0$, and $\hat{D}_{12} = \hat{D}_{21} = I$, and $\hat{D}_{11} = 0$. The observer-based feedback controller

$$
K(z) = -F(zI_n - A - BF - LC)^{-1}L = \left[\begin{array}{c|c} A + BF + LC & -L \\ \hline F & 0 \end{array}\right]
$$

This is certainly observer form. If $A + HC$ is unstable, the Riccati solution is $Y \neq 0$. In this case, the central controller is given by

$$
K(z) = -F\left[zI - A - \gamma^{-2}(H - Z_\infty L_s)H^*X + \lambda^{-2}BB^* - Z_\infty L_s C\right]^{-1} Z_\infty L_s.
$$

where $F = -\lambda^{-2}B^*X[I + (\lambda^{-2}BB^* - \gamma^{-2}HH^*)X]^{-1}A$ and $L_s = H - \alpha^2(A + HC)YC^*(I + \alpha^2 CYC^*)^{-1}$. This is not observer form. ∎

A3. The derivation of dual case in discrete-time

By [37], let us consider for the plant size of $m \leq p$ given by Theorem 3.12. A state-space realization of $G(z)$ is given by

$$
\begin{aligned}
G(z) &= \left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right] \\
&= \left[\begin{array}{cc|c} 0 & 0 & I_m \\ \hline 0 & I_p & 0 \end{array}\right] + \left[\begin{array}{c} -H \\ \hline \lambda^{-1}C \end{array}\right](sI - A)^{-1}\left(\left[\begin{array}{cc|c} (1-\lambda)B & 0 & B \end{array}\right]\right),
\end{aligned}
$$

$$= \left[\begin{array}{c|cc|c} A & (1-\lambda)B & 0 & B \\ \hline -H & 0 & 0 & I_m \\ \hline \lambda^{-1}C & 0 & I_p & 0 \end{array}\right]$$

First, we define some matrices:

$$\begin{aligned}
R &= D_{1.}^{*}D_{1.} - \begin{bmatrix} \gamma^2 I_m & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{1.} = [D_{11}D_{12}] = \begin{bmatrix} 0 & 0 & I \end{bmatrix}, \\
D_{1.}^{*}C_1 &= \begin{bmatrix} D_{11}^{*} \\ D_{12}^{*} \end{bmatrix} C_1 = \begin{bmatrix} 0 \\ 0 \\ -H \end{bmatrix}, \quad B_f R^{-1} D_{1.}^{*} C_1 = -BH \\
I + B_f R^{-1} B_f^{*} X &= I + \alpha^2 BB^{*}X, \quad C_1^{*}(I - D_{1.}R^{-1}D_{1.}^{*})C_1 = 0 \\
\tilde{R} = \tilde{R}^{-1} &= \begin{bmatrix} -\gamma^{-2}I & 0 \\ 0 & I \end{bmatrix}, \quad B_1 D_{.1}^{*} \tilde{R}^{-1} C_f = 0 \\
I - D_{.1}^{*}\tilde{R}^{-1}D_{.1} &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1(I - D_{.1}^{*}\tilde{R}^{-1}D_{.1})B_1^{*} = (1-\lambda)^2 BB^{*} \\
D_{.1} &= [D_{11}^{*}D_{21}^{*}] = \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \quad \tilde{R}^{-1} = \begin{bmatrix} -\gamma^{-2}I & 0 \\ 0 & I \end{bmatrix}, \\
R^{-1} &= \begin{bmatrix} -\gamma^{-2}I & 0 & 0 \\ 0 & -\gamma^{-2}I & \\ 0 & 0 & I \end{bmatrix}, \quad C_f^{*}\tilde{R}^{-1}C_f = -\gamma^{-2}H^{*}H + \lambda^{-2}C^{*}C \\
I + YC_f^{*}\tilde{R}^{-1}C_f &= I + Y[-H^{*} \quad \lambda^{-1}C^{*}] \begin{bmatrix} -\gamma^{-2} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -H \\ \lambda^{-1}C \end{bmatrix} \\
&= I - \gamma^{-2}YH^{*}H + \lambda^{-2}YC^{*}C \\
B_f = [B_1 \;\; B_2] &= \begin{bmatrix} (1-\lambda)B & 0 & B \end{bmatrix}, \quad I + B_f R^{-1} B_f^{*} X = I + \alpha^2 BB^{*}X
\end{aligned}$$

The Riccati equation X and Y, and output injection gain L of the system can be represented as follows:

$$\begin{aligned}
X &= A^{*}X(I+RX)^{-1}A+Q \\
&= (A-B_fR^{-1}D_{1.}^{*}C_1)^{*}X(I+B_fR^{-1}B_f^{*}X)^{-1}(A-B_fR^{-1}D_{1.}^{*}C_1) \\
&\quad +C_1^{*}(I-D_{1.}R^{-1}D_{1.}^{*})C_1 \\
&= (A+BH)^{*}X[I+\alpha^2BB^{*})X]^{-1}(A+BH). \\
Y &= A(I+YR)^{-1}YA^{*}+Q \\
&= (A-B_1D_{.1}^{*}\tilde{R}^{-1}C_f)(I+YC^{*}\tilde{R}^{-1}C_f)^{-1}Y(A-B_1D_{.1}^{*}\tilde{R}^{-1}C_f)^{*} \\
&\quad +B_1(I-D_{.1}^{*}\tilde{R}^{-1}D_{.1})B_1^{*} \\
&= A[I+Y(\lambda^{-2}C^{*}C-\gamma^{-2}H^{*}H)]^{-1}YA^{*}+(1-\lambda)^2BB^{*}. \\
L=[L1\ \ L2] &= -(B_1D_{.1}^{*}+AYC^{*})(\tilde{R}+C_fYC_f)^{-1} \\
&= -AYC^{*}(\tilde{R}+C_fYC_f^{*})^{-1} \\
&= -AYC^{*}\tilde{R}^{-1}(I+C_fYC_f^{*}\tilde{R}^{-1})^{-1} \\
&= -A(I+YC_f^{*}\tilde{R}^{-1}YC_f)^{-1}YC_f^{*}\tilde{R}^{-1} \\
L=L2 &= -A[I+Y(\lambda^{-2}C^{*}C-\gamma^{-2}H^{*}H)]^{-1}(-\lambda^{-1}YC^{*}) \\
&= \lambda^{-1}A[I+Y(\lambda^{-2}C^{*}C-\gamma^{-2}H^{*}H)]^{-1}YC^{*}
\end{aligned}$$

In order to get F, let us separate each step of calculation. Consider that

$$\begin{aligned}
F &= \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \\
&= -(R+B_f^{*}XB_f)^{-1}(B_f^{*}XA+D_{1.}^{*}C_1)
\end{aligned}$$

$$= -R^{-1}(I + B_f^* X B_f R^{-1})^{-1} B_f^* X A + D_{1.}^* C_1$$

To simplify the calculation, let us separate the equations as follow:

$$\begin{aligned}
F &= F_2 = F_{1st} + F_{2nd} \\
F_{1st} &= R^{-1}(I + B_f^* X B_f R^{-1})^{-1} B_f^* X A \\
&= \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} (I + B_f^* X B_f R^{-1})^{-1} (B_f^* X A + D_{1.}^* C_1) \\
&= \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} (I + B_f^* X B_f R^{-1})^{-1} B_f^* X A = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} B_f^* X (I + B_f^* X B_f R^{-1})^{-1} A \\
&= B^* X (I + \alpha^2 B B^* X)^{-1} A. \\
F_{2nd} &= R^{-1}(I + B_f^* X B_f R^{-1})^{-1} D_{1.}^* C_1 \\
&= \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} (I + B_f^* X B_f R^{-1})^{-1} \begin{bmatrix} 0 \\ 0 \\ -H \end{bmatrix}
\end{aligned}$$

As the preceding proof, we apply Schur complement matrix. Then, the state feedback gain F_{2nd} is given by

$$\begin{aligned}
F_{2nd} &= \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \left[\begin{array}{cc|c} I - (1-\lambda)^2 B^* X B/\gamma^2 & 0 & (1-\lambda) B^* X B \\ 0 & I & 0 \\ \hline -B^* X (1-\lambda) B/\gamma^2 & 0 & I + B^* X B \end{array} \right]^{-1} \begin{bmatrix} 0 \\ 0 \\ -H \end{bmatrix} \\
&= \left(I + B^* X B + B^* X B \frac{(1-\lambda)^2}{\gamma^2} (I - \frac{(1-\lambda)^2}{\gamma^2} B^* X B)^{-1} B^* X B \right)^{-1} (-H) \\
&= \left(I + B^* X B + (\beta^2 B^* X B - I + I)(I - \beta^2 B^* X B)^{-1} B^* X B \right)^{-1} (-H) \\
&= \left(I + B^* X B + (-I + (I - \beta^2 B^* X B)^{-1} B^* X B \right)^{-1} (-H)
\end{aligned}$$

$$
\begin{aligned}
&= \left(I + (I - \beta^2 B^* X B)^{-1} B^* X B\right)^{-1})(-H) \\
&= [I - (I - \beta^2 B^* X B + B^* X B)^{-1} B^* X B](-H) \\
&= -H + (I + \alpha^2 B^* X B)^{-1} B^* X B H
\end{aligned}
$$

Hence, Since $F = F_{1st} + F_{2nd}$,

$$
\begin{aligned}
F = F_2 &= -\left(B^* X (I + \alpha^2 B B^* X)^{-1} A - H + (I + \alpha^2 B^* X B)^{-1} B^* X B H\right) \\
&= -\left(B^* X (I + \alpha^2 B B^* X)^{-1} A - H + B^* X (I + \alpha^2 B B^* X)^{-1} B H\right) \\
&= H - B^* X (I + \alpha^2 B B^* X)^{-1} (A + B H).
\end{aligned}
$$

∎

A4. The Supplement Proof of Theorem 3.14

Let us define L_s such that the Riccati equation Y is satisfied. The modified L_s is given by

$$L_s = H - \alpha^2 (A + HC) Y C^* (I + \alpha^2 C Y C^*)^{-1}$$

where by the matrix equivalent, $I + \alpha^2 CYC^* = (I + \alpha^2 CYC^*)^*$ and $YC^*(I + \alpha^2 CYC^*)^{-1} C = YC^*C(I + \alpha^2 YC^*C)^{-1} = Y(I + \alpha^2 CYC^*)^{-1} C^*C$, and substituting these into Y the below.

$$
\begin{aligned}
Y &= AYA^* + BB^* && (5.1) \\
&= (A + L_s C) Y (A + L_s C)^* + \alpha^{-2} (L_s - H)(L - s - H)^* \\
&= \left(A + HC - \alpha^2 (A + HC) Y C^* (I + \alpha^2 C Y C^*)^{-1} C\right) Y \\
&\quad \left(A + HC - \alpha^2 (A + HC) Y C^* (I + \alpha^2 C Y C^*)^{-1} C\right)^* nonumber && (5.2)
\end{aligned}
$$

$$+\alpha^{-2}(L_s - H)(L_s - H)^* \tag{5.3}$$

$$= (A+HC)Y(A+HC)^* - \alpha^2(A+HC)YC^*(I+\alpha^2 CYC^*)^{-1}CY(A+HC)^*$$

$$-(A+HC)Y[\alpha^2(A+HC)YC^*(I+\alpha^2 CYC^*)^{-1}C]^*$$

$$+\alpha^2(A+HC)YC^*(I+\alpha^2 CYC^*)^{-1}CY[\alpha^2(A+HC)YC^*(I+\alpha^2 CYC^*)^{-1}C]^*$$

$$+\alpha^2(A+HC)YC^*(I+\alpha^2 CYC^*)^{-2}CY(A+HC)^*$$

$$= (A+HC)Y(A+HC)^*$$

$$-(A+HC)YC^*(I+\alpha^2 C^*CY)^{-1}\alpha^2 C^*CY(A+HC)^* \tag{5.4}$$

$$-(A+HC)Y[I+\alpha^2 C^*CY]^{-1}\alpha^2 C^*CY)(A+HC)^* \tag{5.5}$$

$$+(A+HC)Y(I+\alpha^2 C^*CY)^{-1}\alpha^2 C^*CY(I+\alpha^2 C^*CY)^{-1}$$

$$\alpha^2 C^*CY(A+HC)^* \tag{5.6}$$

$$+(A+HC)Y(I+\alpha^2 CYC^*)^{-2}\alpha^2 C^*CY(A+HC)^* \tag{5.7}$$

To simplify equations, we further rearrange and combine as follows.

Step 1. Eq.(5.4) is rewritten by

$$\begin{aligned} Y_1 &= (A+HC)Y[I-(I+\alpha^2 C^*CY)^{-1}\alpha^2 C^*CY](A+HC)^* \\ &= (A+HC)Y(I+\alpha^2 C^*CY)^{-1}(A+HC)^* \end{aligned} \tag{5.8}$$

Step 2. Eq.(5.5) and Eq.(5.6) are combined by

$$Y_2 = -(A+HC)Y[I+\alpha^2 C^*CY]^{-1}\alpha^2 C^*CY)[I+\alpha^2 C^*CY]^{-1}(A+HC)^* \tag{5.9}$$

Step 3. Eq.(5.8) and Eq.(5.9) are combined by

$$Y_3 = (A+HC)Y[I+\alpha^2 C^*CY]^{-1}[I-(I+\alpha^2 C^*CY)^{-1}\alpha^2 C^*CY](A+HC)^*$$

$$= (A+HC)Y[I+\alpha^2C^*CY]^{-2}(A+HC)^* \tag{5.10}$$

Finally, Eq.(5.10) and Eq.(5.7) are combined by

$$\begin{aligned} Y_4 &= (A+HC)Y[I+\alpha^2C^*CY]^{-2}[I+\alpha^2C^*CY](A+HC)^* \\ Y = Y_4 &= (A+HC)Y[I+\alpha^2C^*CY]^{-1}(A+HC)^* \end{aligned}$$

Thus, it is verified that the modified output injection gain L_s satisfies Riccati equation Y. ∎

A5. The Supplement Proof of Lemma 2.1

Alternatively, the Lemma 2.1 can be proved by matrix factorization and equivalent fact. Also, P and K given by state space form as (3.1) and (3.2) obtain the followings equations by matrix equivalent transform:

$$\begin{aligned} (I-PK)^{-1} &= I+PK(I-PK)^{-1} = I+P(I-KP)^{-1}K \\ (I-PK)^{-1}P &= P+PK(I-PK)^{-1}P = P(I-KP) \\ K(I-PK)^{-1} &= (I-KP)^{-1}K \\ (I-PK)^{-1}PK &= PK(I-PK)^{-1} = (I-PK)^{-1}, \text{ similarly,} \\ (I-KP)^{-1} &= I+KP(I-KP)^{-1} = I+K(I-PK)^{-1}P \\ (I-KP)^{-1}P &= P+KP(I-KP)^{-1}P = P(I-PK)^{-1} \\ K(I-KP)^{-1} &= (I-PK)^{-1}K \\ (I-KP)^{-1}KP &= KP(I-KP)^{-1} = (I-KP)^{-1}-I \end{aligned}$$

Then, factorization results have

$$PK = \left[\begin{array}{cc|c} A+BF+LC & 0 & L \\ -BF & A & 0 \\ \hline 0 & C & 0 \end{array}\right]$$

$$KP = \left[\begin{array}{cc|c} A & 0 & B \\ LC & A+BF+LC & 0 \\ \hline 0 & -F & 0 \end{array}\right]$$

$$(I-KP)^{-1} = \left[\begin{array}{cc|c} A & -BF & -B \\ LC & A+BF+LC & 0 \\ \hline 0 & F & I \end{array}\right]$$

$$\begin{aligned}(I-PK)^{-1} &= \left[\begin{array}{cc|c} A+BF+LC & LC & -L \\ -BF & A & 0 \\ \hline 0 & -C & I \end{array}\right] \\ &= \left[\begin{array}{cc|c} A+LC & 0 & L \\ LC & A+BF & L \\ \hline C & -C & I \end{array}\right] \text{ by } \begin{bmatrix} 0 & I \\ I & -I \end{bmatrix} \\ &= C(\delta I-A-LC)^{-1}L(I-C(\delta I-A-BF)^{-1}L) \\ &\quad +(I-C(\delta I-A-BF)^{-1}L) \\ &= (I-C(\delta I-A-BF)^{-1}L)(I+C(\delta I-A-LC)^{-1}L), \text{ and}\end{aligned}$$

$$\begin{aligned}K(I-PK)^{-1} &= \left[\begin{array}{cc|c} A+LC & 0 & L \\ -BF & A+BF & 0 \\ \hline -F & F & 0 \end{array}\right] \\ &= -(I+F(\delta I-A-BF)^{-1}B)F(\delta I-A-LC)^{-1}L \\ &= -(F+F(\delta I-A-BF)^{-1}BF)(\delta I-A-LC)^{-1}L\end{aligned}$$

$$
\begin{aligned}
&= -F(\delta I - A - BF)^{-1}(\delta I - A - BF + BF - LC + LC) \\
&\quad (\delta I - A - LC)^{-1}L \\
&= -F(\delta I - A - BF)^{-1}L(I + C(\delta I - A - LC)^{-1}L).
\end{aligned}
$$

$$
\begin{aligned}
(I - PK)^{-1}PK &= \left[\begin{array}{cc|c} A + BF + LC & LC & -L \\ -BF & A & 0 \\ \hline 0 & -C & 0 \end{array}\right] \\
&= \left[\begin{array}{cc|c} A + LC & 0 & L \\ BF & A + BF & 0 \\ \hline C & -C & 0 \end{array}\right] \\
&= -C(\delta I - A - BF)^{-1}BF(\delta I - A - LC)^{-1}, \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
KP(I - KP)^{-1} &= \left[\begin{array}{cc|c} A & -BF & -B \\ LC & A + BF + LC & 0 \\ \hline 0 & F & 0 \end{array}\right] \\
&= \left[\begin{array}{cc|c} A + LC & 0 & B \\ LC & A + BF & 0 \\ \hline 0 & -F & 0 \end{array}\right] \\
&= -F(\delta I - A - BF)^{-1}LC(\delta I - A - LC)^{-1}B
\end{aligned}
$$

Otherwise, we omit proof. ∎

Vita

Gisoon Kim was born in Taegu, South Korea, on October 1, 1954. He received his B.E. degree in Electrical Engineering in 1977 from Kyung-Pook National University. After graduation he joined the Agency for Defense Development(ADD) as a researcher. He enrolled in the Department of Electrical Engineering at Chung-Nam National University in 1986 and received a master's degree in Electrical Engineering in 1988. The title of his thesis was " Decentralized Adaptive Control for Robot Manipulator."

He worked at ADD as a researcher from 1977 until December 1990, serving as senior researcher in 1984. His work was the development of an automatic chemical sensor and a automatic remote alarm system. As a scholarship student of ADD, he joined the Department of Electrical Engineering at LSU as a doctoral student in January 1991. Since September of 1993 he has been a teaching assistant in the Electrical Engineering Department. His current main research interests include system identification, pattern recognition, and robust adaptive control in $\mathcal{H}_\infty$. He is currently a candidate for a doctor of philosophy degree in Electrical Engineering. He is a member of the *System and Control Society* of IEEE.

DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Gisoon Kim

Major Field: Electrical Engineering

Title of Dissertation: Robust Adaptive Control in H_∞

Approved:

Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

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Date of Examination:

April 6, 1995