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A New Perspective on a Polynomial Time Knot Polynomial

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A NEW PERSPECTIVE ON A POLYNOMIAL TIME KNOT POLYNOMIAL

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A great discovery solves a great problem but there is a grain of discovery in the solution of any problem.

—George Pólya
How to solve it

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Abstract

In this work we consider the $Z_1(K)$ polynomial time knot polynomial defined and described by Dror Bar-Natan and Roland van der Veen in their 2018 paper "A polynomial time knot polynomial". We first look at some of the basic properties of $Z_1(K)$, and develop an invariant of diagrams $\Psi_m(D)$ related to this polynomial. We use this invariant as a model to prove how $Z_1(K)$ acts under the connected sum operation. We then discuss the effect of mirroring the knot on $Z_1(K)$, and described a geometric interpretation of some of the building blocks of the invariant. We then use these to develop state sum interpretation of $Z_1(K)$. We describe a base set of knots which can be used to build the $Z_1(K)$, or rather its normalization $\rho_1(K)$, showcasing some of its symmetry properties. Finally, we use this idea to give an explicit expansion of $\rho_1(K)$ for the family of $T(2, 2p + 1)$ torus knots in terms of this base set of knot invariants.

Chapter 1. Introduction

A *knot* can be defined as a copy of S^1 smoothly embedded in $\mathbb{R}^3$ or S^3 (the compactification of $\mathbb{R}^3$). To imagine this one can think of a string wrapped around itself and tangled up, but with the ends of the string either stuck together to form a closed loop. A *link* is a collection of one or more disjoint knots. When we draw the projection of such an object in a 2-dimensional plane we often replace the intersection points that appear with pictures that indicate which strand of the diagram is on top of the other - we call these *crossings*. We also require that there are only a finite number such intersection points (that is to say, we will have the arcs in the diagram intersect transversely)

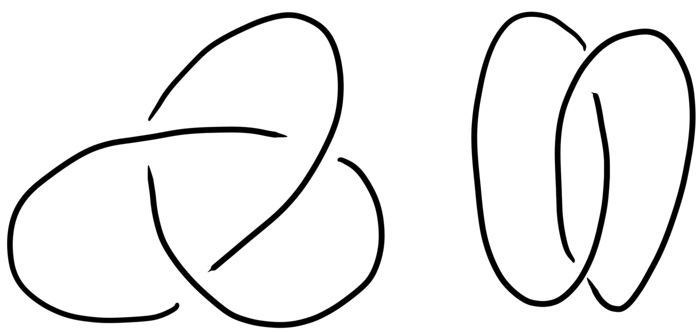

Figure 1.1. The Trefoil and the Hopf link

One of the most basic questions one can ask when handed two different diagrams of knots is: "Do these represent the same knot or do they represent different knots?" It turns out that this simple question can actually be a really difficult one to answer! It is, however, an interesting question and was a driving force in the creation of modern knot theory.[1]

If one considers what can be done to the aforementioned knotted string, one can

[1]Peter Guthrie Tait began classifying knot diagrams in response to the conjecture by William Thompson that stated atoms were composed of knotted vortex cubes. [1]

see that moving the string around, stretching it (for it is a very stretchy string), juggling it, and twisting it up will not change the underlying knot - only what it looks like. This notion captures the idea of planar isotopies and something called the *Reidemeister moves* (see Figure 1.2).

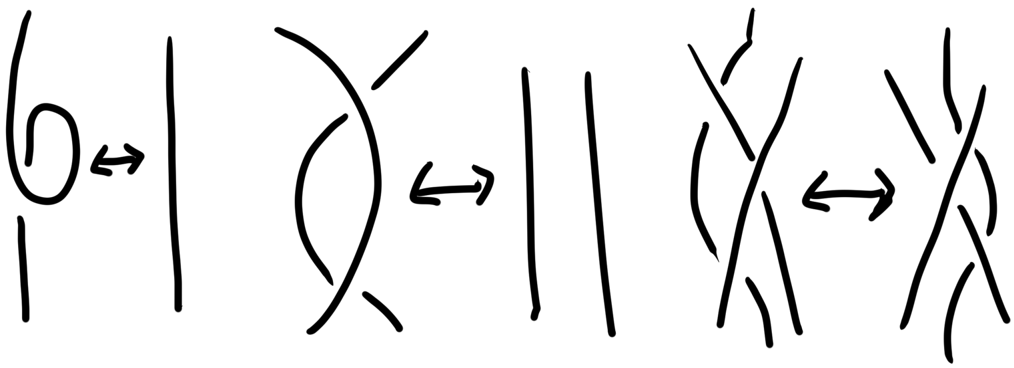

Figure 1.2. The R1, R2, and R3 Reidemeister moves

These operations do not change the underlying knot so we quickly see that there are an infinite number of diagrams that represent a given knot, that look wildly different from each other. What one can do however, is consider knot diagrams up to their *equivalence classes* - that is, if two diagrams represent the same knot, we put them in the same equivalence class and just pick a representative diagram to work with.

We can add more information to knot diagrams by specifying a direction on the diagram (resulting in an *oriented* knot diagram). One can think of this as a way of specifying what direction one would travel along the knot. The crossings in such a diagram now give more information; they are either “positive” or “negative” depending on the directions of the arrows.

One particularly interesting family of knots (or rather, of links) are torus links.

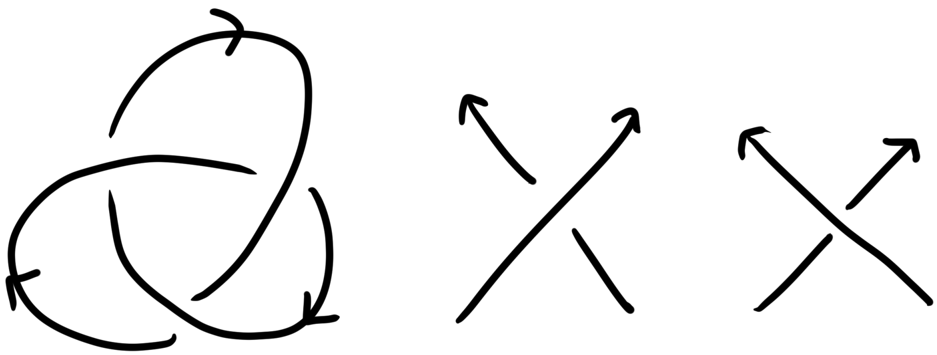

Figure 1.3. An oriented trefoil, a positive crossing, and a negative crossing

These are defined as links that lie on the surface of an unknotted torus in $\mathbb{R}^3$. Typically, we write these as $T(p, q)$ where p is the number of times the link wraps around the outside of the torus, and q is the number of times the link goes through the center hole.

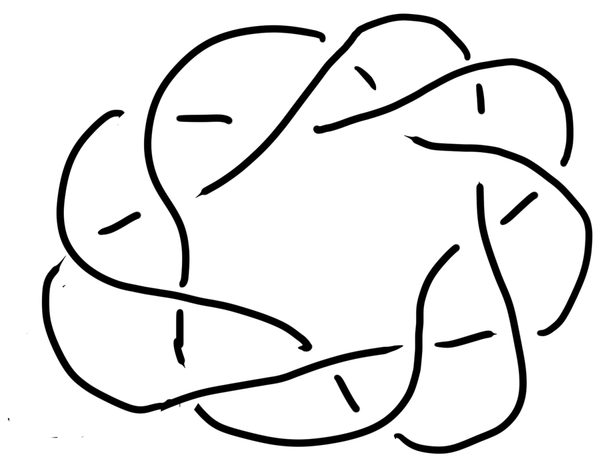

Figure 1.4. The T(3,-8) torus link

This family of links, while interesting, does not contain every possible link. For example, the figure-eight knot (see Figure 1.5) cannot be embedded in the surface of a torus, thus it is not a torus knot.

There are many different way to represent knots and links, but one of particular

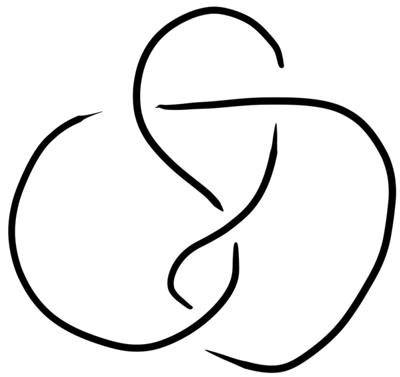

Figure 1.5. The Figure-8 knot

importance to us is known as the *braid representation.* For this representation we consider a cylinder with n dots identified at the top and bottom of the cylinder. We will take n strings, each anchored to a different dot on the bottom, and connect each of them to a different dot on the top while allowing them to weave back, fourth, and around (but always travelling upwards). By labelling each crossing involving strand i (on the left hand side) with a σ_i for positive crossings or a σ_{-i} for negative crossings, we can assign to a braid a *braid word*.

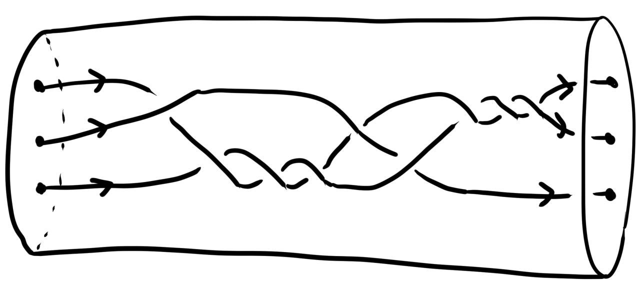

Figure 1.6. A braid

If we take a braid and connect each of the points on the top to the corresponding point directly below it (outside the cylinder and without letting them cross each other) we

can convert a braid into a knot (or link). We call this representation a *closed braid* representation. If we take one of the outside arcs (say, the first one) and instead send each end out towards infinity we have something called a *long knot*. An interesting result of Alexander is that every link can be represented as a closed braid [2].

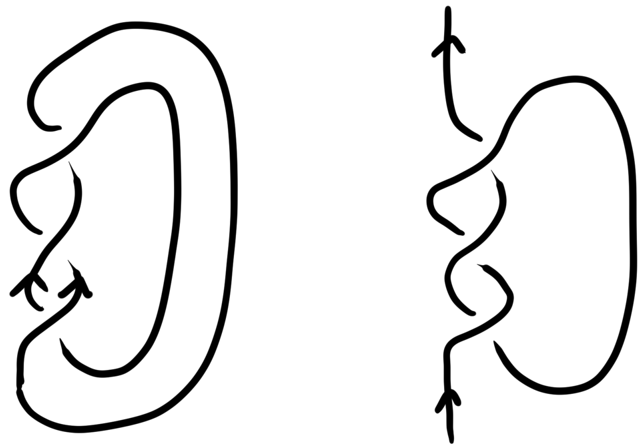

Figure 1.7. The closure of a braid and a long knot

Braids are particularly nice to work with as they carry a group structure. This *braid group* $B(n)$ is generated by σ_i for $i \in \{1, ..., n-1\}$ subject to the following relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \qquad |i-j| \geq 2$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \qquad 1 \leq i \leq n-2$$

It is worth noting that the second of these relations corresponds to the third Reidemeister move on knot diagrams. This leads to the natural question: are there analogues to the other Reidemeister moves for braids?

In 1935, Markov showed [3] that if two closed braids represent the same link one can transform one of braids into the other using a series of elementary moves called Markov moves. The first of these moves is called *conjugation*, which takes a braid

$\beta \in B(n)$ and sends it to $\sigma_i \beta \sigma_i^{-1}$. This move corresponds to the second Reidemeister move on knot diagrams. The second Markov move is called *stabilization* and the inverse of this move is called *destabilization*. The stabilization move takes a braid $\beta \in B(n)$ and sends it to $\beta\sigma_n^{\pm 1} \in B(n+1)$, and, when used in conjunction with the first Markov move, corresponds to the first of the Reidemeister moves.

Another useful construction is called the *connected sum* of two knots which is, in some sense, a way to add two knots together in a well defined manner. Intuitively, one can think of this as using the string to tie one knot, then the other on the same piece of string. Diagrammatically, this corresponds to removing a small arc from two oriented knots, then connecting the endpoints between the knot in an orientation preserving way. For long knots, this corresponds to stacking the diagrams on top each other. For a much more extensive discussion of all these ideas, one can refer to the excellent book *An Introduction to Knot Theory* by Lickorish [4].

In Chapter 2 of this work we will introduce the notion of polynomial invariants of knots and describe the two main invariants that we will be working with. Chapter 3 dives into the basic properties of a particular polynomial invariant and we explore how it acts under basic operations. In Chapter 4 we interpret the polynomial geometrically, and build a state sum model for the invariant. Finally, in Chapter 5, we will see how he invariant we are considering relates to other polynomial invariants.

Chapter 2. Background

A useful tool to help classify knots and links (and to do other things as well) is that of the *knot polynomial*. Generally, these are invariants of knots and links which assign a particular polynomial to a knot or link, a polynomial whose coefficients and degree encode useful information about the knot or link. These polynomials are independent of the chosen diagram, making them a useful tool for distinguishing knots (in addition to the other information they may encode). In this paper we consider two such polynomials in particular, diving deeply into their structures, and find that they are strongly related.

2.1. The Alexander Polynomial

A well known and understood invariant of knots is the Alexander Polynomial $\Delta_K(t)$, which Alexander computes as the determinate of a matrix associated with an oriented link diagram [5]. He shows that the polynomial is invariant up to sign and multiplication by a power of the variable by showing how it acts under the Reidemeister moves.

In brief, place two dots just to the left of the undercrossing arc in each crossing, on either side of the overcrossing arc, viewing the undercrossing arc as a vertical arc. Starting with the top dot, label each of the four regions A, B, C, and D (see Firgue 2.1). To each of these crossings, Alexander associates the equation

$$xA - xB + C - D = 0$$

while simplifying that equality if two regions are the same. Alexander then gives a matrix M_k whose rows correspond to the crossings and the coefficients of the equations they give, and whose columns correspond to the regions in the diagram. Let R and R' be two

adjacent regions of the diagram, and let $M_k[R, R']$ be the matrix with the corresponding columns removed. Then the Alexander polynomial is given by $\Delta_K(x) = \det(M_K[R, R'])$.

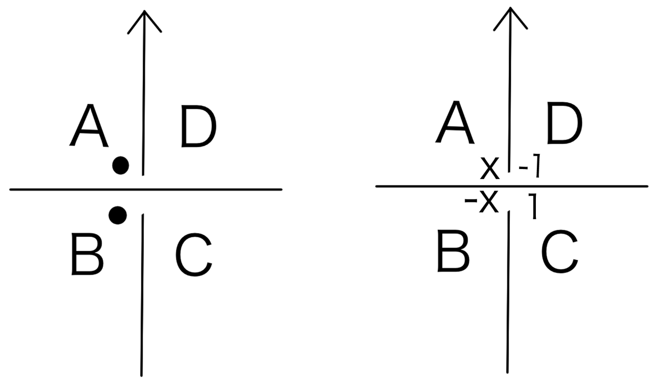

Figure 2.1. Alexander labeling

In [6], Kauffman reformulates the Alexander polynomial as a state summation. He gives a formula for the Alexander polynomial which is a sum of evaluations of combinatorial configurations (or states) related to the link diagram. The state sum actually computes the Conway normalized version of the Alexander polynomial, which removes the question of sign an multiplication by a power of the variable. Not only this, but it satisfies a *skein relation*:

$$\nabla_{L_+}(z) - \nabla_{L_-}(z) = z\nabla_{L_0}(z)$$

which describes a relation between diagrams for the knot K where the diagram is the same everywhere except for at a chosen crossing L, with L_+, L_-, and L_0 corresponding to a positive crossing, a negative crossing, and the remove of a crossing as shown in Figure 2.2.

This polynomial, $\nabla_K(z)$, is known as the Conway-normalized Alexander polynomial and is related the the Alexander polynomial $\Delta_K(t)$ via the relation

$$\Delta_K(t^2) = \nabla_K(t - t^{-1}) = \nabla_K(z)$$

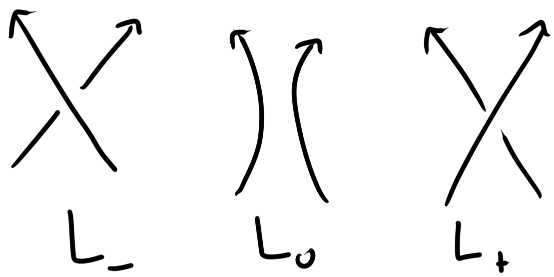

Figure 2.2. The Skein objects L_-, L_0, L_+

where $\Delta_K(t)$ is normalized by sign and multiplication by t to satisfy the skein relation

$$\Delta_{L_+}(t) + \Delta_{L_-}(t) = (t^{1/2} - t^{-1/2})\Delta_{L_0}(t)$$

The Alexander polynomial has several interesting properties including that $\Delta_K(t) \doteq \Delta_K(t^{-1})$ where $\doteq$ means equal up to multiplication by a unit in $Z[t, t^{-1}]$. It also has the property that $\Delta_K(1) = \pm 1$. Furthermore, every integral Laurent polynomial with these two properties is the Alexander polynomial of some knot [7]. If one looks at the Conway normalized version of the Alexander polynomial, the symmetry properties becomes even nicer: the polynomial is symmetric for all knots K; that is, $\Delta_K(t) = \Delta_K(t^{-1})$. If we have the connected sum of two knots, $K_1 \# K_2$, then we have

$$\Delta_{K_1 \# K_2} = \Delta_{K_1}\Delta_{K_2}.[4]$$

This polynomial also exhibits some important geometric properties. If the knot K bounds a smoothly embedded disk in the 4-ball D^4, we call such knots *slice*, then Fox and Milnor showed in [8]:

$$\Delta_K(t) \doteq f(t)f(t^{-1})$$

for some $f(t) \in Z[t, t^{-1}]$. Moreover, if a knot K has genus g then twice the knot genus is bounded below by the degree of the Alexander polynomial [4]. That is, $2g \geq$ breadth

$\Delta_K(t)$.

2.2. A Polynomial Time Knot Polynomial

In order to build the second polynomial invariant of interest $Z_1(K)$ we will follow the process described by Bar-Natan and van der Veen in [9], so we first consider the notion of a *snarl diagram* which can be used to represent long knots.

DEFINITION 2.2.1. *A snarl diagram is a finite set L together with a finite oriented graph $G = (V, E)$ and functions $\sigma : V \to \{\pm 1\}$ and $\rho : E \to \mathbb{Z}$. The edges E are assumed to be a disjoint union of oriented paths and each path is labelled by an element of L. Furthermore, the edges around any vertex are ordered cyclically such that two adjacent edges enter and two exit each vertex that is not an endpoint of a path.[9]*

Bar-Natan and van der Veen explain that one can think of the weighted vertices as crossings and endpoints of pieces of the knot, the paths L as connected components, and the map ρ as keeping track of rotation numbers of tangent vectors on the edges (see figure 2.3). In order to build knots from disjoin unions of these pieces, one needs the notion of "stitching".

DEFINITION 2.2.2. *For $i \neq j \in L$ and $k \notin L - \{i, j\}$ define the snarl diagram $m_k^{ij}(G)$ to be the graph obtained from G by connecting the endpoint of component i to the start of component j, erasing the vertex in the middle. For the newly created edge e we define $\rho(e)$ to be the sum of the values of ρ on the edges that disappear. The newly created component is labeled k so the label set is $L - \{i, j\} \cup \{k\}$. [9]*

Bar-Natan and van der Veen showed that every long knot can be represented by such a snarl diagram, that two snarl diagrams representing isotopic knots are equivalent,

and that snarl diagrams can be made into algebras. Recall the definition of a *snarl-algebra* also given in [9]:

DEFINITION 2.2.3. *A snarl-algebra is an algebra $\mathcal{A}$ together with invertible elements $X \in \mathcal{A}^{\otimes\{1,2\}}$ $\alpha \in \mathcal{A}$ such that the equations below are satisfied. For any diagram D with label set L denoted by $Z(D) \in \mathcal{A}^{\otimes L}$ the unique element characterized by $Z(X^{\pm}_{ij}) = X^{\pm}_{ij}$ and $Z(\alpha^r_i) = \alpha^r_i$ and $Z(D \bigsqcup D') = Z(D) \otimes Z(D')$ and $Z(m^{ij}_k D) = m^{ij}_k Z(D)$. $Z = Z_{\mathcal{A},X,\alpha}$ is called the snarl invariant corresponding to $\mathcal{A}$.*

$$Z(X^{\pm}_{13}\alpha^{\mp 1}//m^{(123)}) = Z(\alpha^0_1) = Z(X^{\pm}_{31}\alpha^{\pm 1}//m^{(123)})$$

$$Z(X^{-}_{12}X^{+}_{34}\alpha_5\alpha_6^{-1}//m^{(13)(4526)}) = Z(\alpha^0_1\alpha^0_2) = Z(X^{+}_{12}X^{-}_{34}\alpha_5\alpha_6^{-1}//m^{(5163)(42)})$$

$$Z(X^{\pm}_{12}X^{\pm}_{34}X^{\pm}_{56}//m^{(13)(25)(46)}) = Z(X^{\pm}_{12}X^{\pm}_{34}X^{\pm}_{56}//m^{(35)(16)(24)})$$

$$Z(X^{\pm}_{12}) = Z(\alpha_1^{\pm 1}\alpha_2^{\pm 1}X^{\pm}_{34}\alpha_5^{\mp 1}\alpha_6^{\mp 1}//m^{(135)(246)})$$

Of note they prove that if one takes the Weyl algebra, $\mathcal{W}$, to be the snarl algebra in question, then the corresponding invariant for a knot K is $Z_{\mathcal{W}}(K) = \Delta^{-1}(K)$, where $\Delta(K)$ is the Alexander polynomial of the knot. If one then generalizes this to the $q-$Weyl algebra, restricting to $q = 1 + \epsilon$ and $\epsilon^2 = 0$, then one obtains a knot invariant $Z_{\mathcal{W}q}(K)$. The coefficient of ϵ in the constant part of $Z_{\mathcal{W}q}(K)$ is equal to $Z_1(K)/\Delta^2_K$, which is what we will concern ourselves with. Thankfully, this can be computed using a polynomial time algorithm.

Take a diagram for a long knot K and split it into a collection of crossings $\chi^{\sigma}_{i,j}$, where σ is $\pm$ for a positive or negative crossing and i,j are the labels of the strands, and cuaps u^{σ}_i, a strand on the diagram with label i where the orientation is pointed in the positive x direction and σ is given by the direction of its rotation in the plane (see figure 2.3).

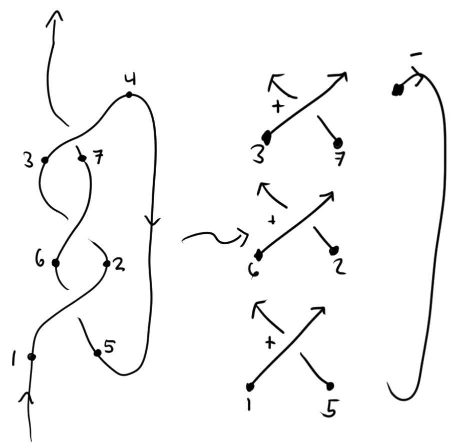

Figure 2.3. The trefoil and its basic pieces: $K = \chi^+_{1,5}\chi^+_{6,2}\chi^+_{3,7}u^-_4$

One should see that this is exactly the construction we would use to turn the diagram of a knot into a snarl diagram, so long as we insure that the tangent vectors at each crossing are pointed upwards. To remember how all these pieces fit together we record the stitching information $m^{(i_1,\ldots,i_n)}$, which tells us to connect strand i_1 to strand i_2 and so on. For our purposes, we will label the diagram following its orientation so that the stitching is $m^{(1,2,\ldots,n)}$, what we will call the "normal stitching". Thus our knot can be represented as a collection of crossings, cuaps, and stitchings:

$$K = \chi^{\sigma_1}_{i_1,j_1}\ldots\chi^{\sigma_n}_{i_n,j_n}u^{\sigma_{n+1}}_{i_{n+1}}\ldots u^{\sigma_{n+s}}_{i_{n+s}}//m^{(1,\ldots,n,n+1,\ldots n+s)}$$

To arrive at Z_1 Bar-Natan and van der Veen define some basic matrices, which correspond to simplified versions of the objects used to compute $Z_{\mathcal{W}q}$.

$$Q = \sum_{\chi^\sigma_{i,j}} \sigma t^{\frac{\sigma}{2}}(E^j_j - E^i_j) \qquad W = \sum_{i<j} E^i_j \qquad c = \prod_{\chi^\sigma_{i,j}, u^\sigma_i} t^{-\sigma}$$

Where E^i_j is the elementary $n \times n$ matrix with a 1 at position (i, j).

They then define

$$B = I - (t^{1/2} - t^{-1/2})WQ \qquad G = Q\text{adj}(B) \qquad H = \text{adj}(B)W$$

and the polynomials

$$Z_G = (t - t^{-1}) \sum_{j=2}^{n} \sum_{a,b<j} G_a^j \left(\frac{1}{2} G_a^j + \sum_{g>j} G_b^g \right)$$

$$Z_H = \sum_{\chi_{i,j}^\sigma} \frac{\sigma}{2} \left((1 - t^\sigma) H_i^j \right)^2 - \frac{\sigma}{2} \left((1 + t^\sigma) H_j^j \right)^2 + \sigma t^\sigma \left(H_i^j H_j^i + H_i^i H_j^j \right)$$
$$+ t^\sigma (1 - t) H_i^j \left((1 + \sigma) H_j^j + (1 - \sigma) H_i^i \right) + \det(B) \sum_{u_i^\sigma} \sigma H_i^i$$

Finally, they compute Z_1 as

$$Z_1 = c(Z_G + Z_H)$$

As an example, consider the trefoil $K = \chi_{1,5}^+ \chi_{6,2}^+ \chi_{3,7}^+ u_4^-$ together with the normal stitching $m^{(1,2,...,n)}$.

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 1-t & 0 & 0 \\ 0 & t & 0 & 0 & 1-t & 0 & 0 \\ 0 & t-1 & 1 & 0 & 1-t & 0 & 0 \\ 0 & t-1 & 0 & 1 & 1-t & 0 & 1-t \\ 0 & t-1 & 0 & 0 & 1 & 0 & 1-t \\ 0 & 0 & 0 & 0 & 0 & 1 & 1-t \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad c^{1/2} \det(B) = t^{-1} - 1 + t = \Delta_{3_1}(t)$$

$$ZH = -\frac{1}{2} - t + \frac{7}{2} t^2 - 3t^3 + t^4, \quad ZG = \frac{1}{2} - \frac{3}{2} t^2 + t^4$$

$$Z_1(3_1) = c(ZH + ZG) = 2 - \frac{1}{t} - 3t + 2t^2$$

We can also see that $Z_1(3_1) = \Psi_0(3_1) + (\Delta_{3_1}(t))^2 - 1$, where $\Psi_0(K) = t \frac{\mathrm{d}}{\mathrm{dt}} (\Delta_K(t)) \Delta_K(t)$. We will explore this property more in Chapter 5, but first we must examine $\Psi_0(K)$.

Chapter 3. Basic Properties

3.1. Defining and working with Ψ_0

When one begins exploring the basic properties of the Z_1 polynomial, one can observe it is possible to construct a function on diagrams which acts *almost* invariant; the variance depends on how many turn around points there are in the chosen diagram. Let the knot polynomial $Z_1(K)$, the matrix B containing crossing information, and correction term $c = t^{2m}$ for the chosen diagram D of the knot K be defined as in [9].

Let

$$\Psi_m(D) = t\Delta_K^2 \mathrm{Tr}(B_D^{-1} \frac{d}{dt}[B_D])$$

where $\Delta_K = c^{\frac{1}{2}} \det(B_D)$ is the (normalized) Alexander polynomial of K. Choosing a few different diagrams for the trefoil, we can see that the value of $\Psi_m(D)$ changes based on the diagram, but in a predictable way (see figure 3.1).

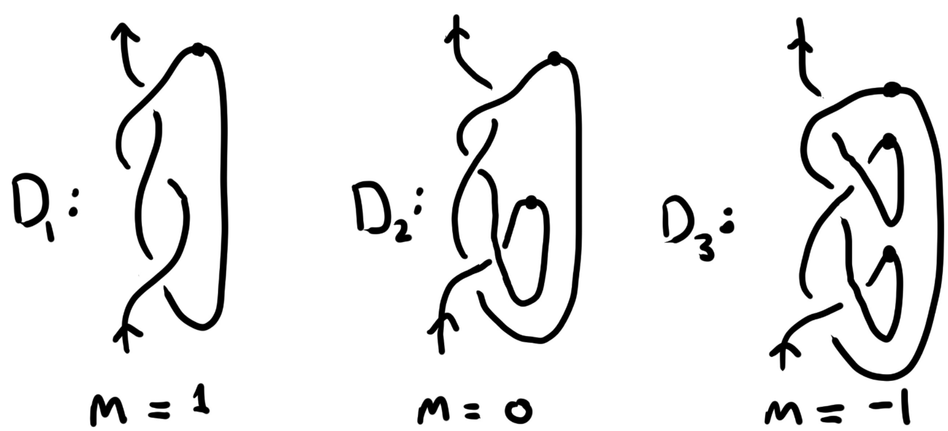

Figure 3.1. The several diagrams for the right handed trefoil.

Computing $\Psi_m(D_i)$ we find

$$\Psi_1(D_1) = -t^{-1}+3-3t+2t^2, \quad \Psi_0(D_2) = -t^{-2}+t^{-1}-t+t^2, \quad \Psi_{-1}(D_3) = -2t^{-2}+3t^{-1}-3+t.$$

That is, choosing diagrams that adjust the value of m by ± 1 shift the values of $\Psi_m(D)$ by $\pm(\Delta_K)^2$. Notice that in general we have

$$
\begin{aligned}
\Psi_m(D) + m(\Delta_K)^2 &= t(\Delta_K)^2 Tr(B_D^{-1}\frac{\mathrm{d}}{\mathrm{dt}}[B_D]) + m(c\det(B_D)^2) \\
&= tc^{\frac{1}{2}}(\Delta_K)\det(B)Tr(B_D^{-1}\frac{\mathrm{d}}{\mathrm{dt}}[B_D]) + mt^{2m}\det(B_D)^2 \\
&= tc^{\frac{1}{2}}(\Delta_K)\frac{\mathrm{d}}{\mathrm{dt}}(\det(B_D)) + tt^m\det(B_D)^2\frac{\mathrm{d}}{\mathrm{dt}}(t^m) \\
&= tc\det(B_D)\frac{\mathrm{d}}{\mathrm{dt}}(\det(B_D)) + tc^{\frac{1}{2}}\det(B_D)^2\frac{\mathrm{d}}{\mathrm{dt}}(c^{\frac{1}{2}}) \\
&= tc^{\frac{1}{2}}\det(B_D)\frac{\mathrm{d}}{\mathrm{dt}}(c^{\frac{1}{2}}\det(B_D)) \\
&= t\Delta_K\frac{\mathrm{d}}{\mathrm{dt}}\Delta_K
\end{aligned}
$$

That is to say, in some sense, we can consider diagrams with $m = 0$ to be a good prospect for a normalized diagram, because in this case $\Psi_m(D) = t\Delta_K\frac{\mathrm{d}}{\mathrm{dt}}\Delta_K = \frac{t}{2}\frac{\mathrm{d}}{\mathrm{dt}}(\Delta_K^2)$. Let this be called $\Psi_0(K)$. So we can actually calculate

$$\Psi_m(D) = \Psi_0(K) - m(\Delta_K)^2$$

for a given diagram D and the sum

$$\Psi_0(K) = \Psi_m(D) + m(\Delta_K)^2$$

which is an invariant of the knot itself.

We get some interesting results from this. For a diagram D with $c \neq 1$ we have:

$$\frac{\Psi_m(D) - \Psi_{-m}(-D)}{2m_D} = \Delta_K^2$$

and

$$\frac{\Psi_m(D) + \Psi_{-m}(-D)}{t} = \frac{\mathrm{d}}{\mathrm{dt}}(\Delta_K^2).$$

Looking at how this quantity works under the connected sum operation, we discover:

THEOREM 3.1.1. *Suppose that we have a knot $K_1\#K_2$. Then*

$$\Psi_0(K_1\#K_2) = \Delta^2_{K_2}\Psi_0(K_1) + \Delta^2_{K_1}\Psi_0(K_2)$$

Proof. Notice that

$$\begin{aligned}
\Psi_0(K_1\#K_2) &= \frac{t}{2}\frac{\mathrm{d}}{\mathrm{dt}}(\Delta^2_{K_1\#K_2}) \\
&= t\Delta_{K_1}\Delta_{K_2}(\Delta_{K_2}\frac{\mathrm{d}}{\mathrm{dt}}\Delta_{K_1} + \Delta_{K_1}\frac{\mathrm{d}}{\mathrm{dt}}\Delta_{K_2}) \\
&= \Delta^2_{K_2}(t\Delta_{K_1}\frac{\mathrm{d}}{\mathrm{dt}}\Delta_{K_1}) + \Delta^2_{K_1}(t\Delta_{K_2}\frac{\mathrm{d}}{\mathrm{dt}}\Delta_{K_2})) \\
&= \Delta^2_{K_2}\Psi_0(K_1) + \Delta^2_{K_1}\Psi_0(K_2).
\end{aligned}$$

3.2. The connected sum $Z_1(K_1\#K_2)$

To see how the Z_1 polynomial operates under connected sums, one first computes some examples. Using the simple example of a connected sum between two copies of the same knot, we quickly observe that $Z_1(K\#K) = 2\Delta^2_K Z_1(K)$. For example:

$$\begin{aligned}
Z_1(3_1\#3_1) &= -2t^{-3} + 8t^{-2} - 20t^{-1} + 32 - 36t + 28t^2 - 14t^3 + 4t^4 \\
&= 2\Delta^2_{3_1}Z_1(3_1) \\
Z_1(4_1\#4_1) &= -2t^{-4} + 18t^{-3} - 58t^{-2} + 72t^{-1} - 72t + 58t^2 - 18t^3 + 2t^4 \\
&= 2\Delta^2_{4_1}Z_1(4_1) \\
Z_1(5_2\#5_2) &= -72t^{-4} + 376t^{-3} - 930t^{-2} + 1392t^{-1} - 1364 + 888t - 370t^2 + 88t^3 - 8t^4 \\
&= 2\Delta^2_{5_2}Z_1(5_2)
\end{aligned}$$

However, if we try to compute $Z_1(3_1\#4_1)$ we see

$$Z_1(3_1\#4_1) = -t^{-4} + 4t^{-3} - t^{-2} - 18t^{-1} + 48 - 66t + 51t^2 - 20t^3 + 3t^4$$

has odd coefficients so there must be something else going on. Motivated by Theorem 3.1.1, one might wonder if the polynomial Z_1 acts in a similar manner, since $\Psi_m(D)$ and $Z_1(K)$ seem to share many of the same properties. Treating $Z_1(3_1\#4_1)$ the same way reveals this may in fact be the case.

$$\begin{aligned} Z_1(3_1\#4_1) &= -t^{-4} + 4t^{-3} - t^{-2} - 18t^{-1} + 48 - 66t + 51t^2 - 20t^3 + 3t^4 \\ &= \Delta_{4_1}^2 Z_1(3_1) + \Delta_{3_1}^2 Z_1(4_1) \end{aligned}$$

With some careful work, one can expand these examples to a general theorem which tells us exactly how $Z_1(K)$ operates under the connected sum operation:

THEOREM 3.2.1. *For a knot $K_1\#K_2$, we have*

$$Z_1(K_1\#K_2) = \Delta_{K_2}^2 Z_1(K_1) + \Delta_{K_1}^2 Z_1(K_2).$$

Proof. Notice that by construction, $Q_{K_1\#K_2} = Q_{K_1} \bigoplus Q_{K_2}$. Since $B = I - (t^{1/2} - t^{-1/2})WQ$ and W is just ones above the diagonal, we can see that $B_{K_1\#K_2} = B_{K_1} \bigoplus B_{K_2}$. Considering the classical adjoint of this matrix we find that

$$\operatorname{adj}(B_{K_1\#K_2}) = \det(B_{K_1}) \det(B_{K_2})(B_{K_1}^{-1} \bigoplus B_{K_2}^{-1}).$$

Recall that $H = \text{adj}(B)W$ so we have

$$H(K_1\#K_2) = \begin{pmatrix} \det(B_{K_2})\text{adj}(B_{K_1}) & 0 \\ 0 & \det(B_{K_1})\text{adj}(B_{K_2}) \end{pmatrix} \begin{pmatrix} 0 & 1 & \dots & 1 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

$$= \left(\begin{array}{c|c} \det(B_{K_2})H_{K_1} & * \\ \hline 0 & \det(B_{K_1})H_{K_2} \end{array}\right)$$

The computation for ZH is a sum over the crossings so we can do the calculation separately on each block, since for a crossing $X^{\sigma}_{i,j}$ the sum only depends on entries of H indexed by i and j. We see that in each piece, we have two entries in H multiplied together so for $H(K_1\#K_2)$ we will have each crossing from K_α contributing a $(\det(B_{K_\beta}))^2$ where $\beta \neq \alpha$.For the last piece of the sum, we sum over the cuaps and we have

$$\begin{aligned} \det(B_{K_1\#K_2}) \sum \sigma H^i_i &= \det(B_{K_1}) \det(B_{K_2}) \sum \sigma * \det(B_{K_\beta}) H(K_\alpha)^i_i \\ &= (\det(B_{K_\beta}))^2 \det(B_{K_\alpha}) \sum \sigma H(B_{K_\alpha})^i_i \end{aligned}$$

So in the end we have

$$ZH_{K_1\#K_2} = (\det(B_{K_2}))^2 ZH_{K_1} + (\det(B_{K_1}))^2 ZH_{K_2}.$$

Now let us consider $G = Q\text{adj}(B)$.

We can see that

$$
\begin{aligned}
G(K_1\#K_2) &= Q_{K_1\#K_2}\operatorname{adj}(B_{K_1\#K_2}) \\
&= \begin{pmatrix} Q_{K_1} & 0 \\ 0 & Q_{K_2} \end{pmatrix} \begin{pmatrix} \det(B_{K_2})\operatorname{adj}(B_{K_1}) & 0 \\ 0 & \det(B_{K_1})\operatorname{adj}(B_{K_2}) \end{pmatrix} \\
&= \begin{pmatrix} \det(B_{K_2})G_{K_1} & 0 \\ 0 & \det(B_{K_1})G_{K_2} \end{pmatrix}
\end{aligned}
$$

The polynomial ZG sums columns under the diagonal and multiplies by entries under the diagonal, so once again one can do the calculation block-wise and see that for K_α each contributes a $(\det(B_{K_\beta}))^2$. Thus we have

$$ZG_{K_1\#K_2} = (\det(B_{K_2}))^2 ZG_{K_1} + (\det(B_{K_1}))^2 ZG_{K_2}.$$

Hence, we arrive at the equality we desire:

$$
\begin{aligned}
Z_1(K_1\#K_2) &= c_1c_2(ZH_{K_1\#K_2} + ZG_{K_1\#K_2}) \\
&= c_1c_2(\det(B_{K_2})^2 ZH_{K_1} + \det(B_{K_1})^2 ZH_{K_2} \\
&\qquad + \det(B_{K_2})^2 ZG_{K_1} + \det(B_{K_1})^2 ZG_{K_2}) \\
&= c_1c_2(\det(B_{K_2})^2(ZH_{K_1} + ZG_{K_1}) + \det(B_{K_1})^2(ZH_{K_2} + ZG_{K_2})) \\
&= c_2\det(B_{K_2})^2 c_1(ZH_{K_1} + ZG_{K_1}) + c_1\det(B_{K_1})^2 c_2(ZH_{K_2} + ZG_{K_2}) \\
&= \Delta_{K_2}^2 Z_1(K_1) + \Delta_{K_1}^2 Z_1(K_2) \qquad \square
\end{aligned}
$$

In [9], Bar Natan and van der Veen defined the following normalization of $Z_1(K)$:

$$\rho_1(K) = \frac{-t}{(1-t^2)}\left(Z_1(K) - t\Delta_K(t)\frac{d}{dt}\Delta_K(t)\right)$$

Now that we know what the connected sum operation does to $Z_1(K)$ we can consider what happens to its normalization $\rho_1(K)$.

THEOREM 3.2.2. *For a knot $K_1\#K_2$, we have*

$$\rho_1(K_1\#K_2) = \Delta_{K_2}^2\rho_1(K_1) + \Delta_{K_1}^2\rho_1(K_2).$$

Proof. Consider $K_1\#K_2$.

$$\begin{aligned}
\rho_1(K_1\#K_2) &= \frac{-t}{(1-t^2)}\left(Z_1(K_1\#K_2) - t\Delta_{K_1\#K_2}(t)\frac{d}{dt}\Delta_{K_1\#K_2}(t)\right)\\
&= \frac{-t}{(1-t^2)}\left(\Delta_{K_2}^2 Z_1(K_1) + \Delta_{K_1}^2 Z_1(K_2) - t\Delta_{K_1}(t)\Delta_{K_2}(t)\frac{d}{dt}\left(\Delta_{K_1}(t)\Delta_{K_2}(t)\right)\right)\\
&= \frac{-t}{(1-t^2)}\left(\Delta_{K_2}^2 Z_1(K_1) - t\Delta_{K_1}(t)\Delta_{K_2}^2(t)\frac{d}{dt}\Delta_{K_1}(t)\right)\\
&+ \frac{-t}{(1-t^2)}\left(\Delta_{K_1}^2 Z_1(K_2) - t\Delta_{K_1}^2(t)\Delta_{K_2}(t)\frac{d}{dt}\Delta_{K_2}(t)\right)\\
&= \frac{-t}{(1-t^2)}\Delta_{K_2}^2\left(Z_1(K_1) - t\Delta_{K_1}(t)\frac{d}{dt}\Delta_{K_1}(t)\right)\\
&+ \frac{-t}{(1-t^2)}\Delta_{K_1}^2\left(Z_1(K_2) - t\Delta_{K_2}(t)\frac{d}{dt}\Delta_{K_2}(t)\right)\\
&= \Delta_{K_2}^2\rho_1(K_1) + \Delta_{K_1}^2\rho_1(K_2) \qquad \square
\end{aligned}$$

3.3. $Z_1(K)$ and the effect of mirroring

In this section we will review a conjecture regarding the effects of mirroring on the polynomial, and obtain some progress towards a proof.

CONJECTURE 3.3.1. *Let K be a knot and $m(K)$ its mirror. Define $Z_1(K;t)$ as in [9]. Then*

$$Z_1(m(K);t) = -Z_1(K;t^{-1}).$$

A direct proof of this conjecture has been difficult to obtain, but we have been able to show the following:

THEOREM 3.3.2. *Let K be a knot and $m(K)$ its mirror. Then*

$$Z_1(m(K);t) = -Z_1(K;t^{-1})$$

if and only if

$$\sum_{\chi} \dot{H}_i^j (t^{-\sigma} - 1)(\dot{H}_i^i - t^{-\sigma} \dot{H}_j^j) = 0$$

where $\dot{H}$ is the matrix obtained by computing H as in [9] and sending $t \to t^{-1}$.

Proof. In the knot diagrams, the mirroring operation sends $\sigma_i \to -\sigma_i$ for each crossing χ and cuap u, that is it switches the sign of each crossing and cuap. We define $\bar{\mathbb{S}}_t$ to be some symbol $\mathbb{S}$ where σ has been negated and which uses variable t.

Now we have

$$\bar{Q}_t = \sum_{\chi^{\sigma}_{i,j}} -\sigma t^{\frac{-\sigma}{2}} E^j_j - E^i_j = -Q_{1/t}$$

$$\bar{W}_t = \sum_{i<j} E^i_j = W_{1/t} = W_t$$

$$\bar{c}_t = \prod_{\chi^{\sigma}_{i,j}, u^{\sigma}_i} t^{\sigma} = t^{2(-m)} = c_{1/t}$$

$$\bar{B}_t = I - (t^{1/2} - t^{-1/2})\bar{W}_t\bar{Q}_t = I - (t^{-1/2} - t^{1/2})W_{1/t}Q_{1/t} = B_{1/t}$$

$$\bar{G}_t = \bar{Q}_t \mathrm{adj}(\bar{B}_t) = -Q_{1/t}\mathrm{adj}(B_{1/t}) = -G_{1/t}$$

$$\bar{H}_t = \mathrm{adj}(\bar{B}_t)\bar{W}_t = \mathrm{adj}(B_{1/t})W_{1/t} = H_{1/t}$$

For ease of use let $\dot{G} = G_{1/t}$. For the polynomials we have

$$\begin{aligned} Z_{\bar{G}_t} &= (t - t^{-1}) \sum_{j=2}^{n} \sum_{a,b<j} \bar{G}^j_a \left(\frac{1}{2}\bar{G}^j_a + \sum_{g>j} \bar{G}^g_b \right) \\ &= (t - t^{-1}) \sum_{j=2}^{n} \sum_{a,b<j} -\dot{G}^j_a \left(\frac{1}{2}(-\dot{G}^j_a) + \sum_{g>j} -\dot{G}^g_b \right) \\ &= -(t^{-1} - t) \sum_{j=2}^{n} \sum_{a,b<j} \dot{G}^j_a \left(\frac{1}{2}(\dot{G}^j_a) + \sum_{g>j} \dot{G}^g_b \right) \\ &= -Z_{\dot{G}} \end{aligned}$$

We now consider Z_H. Again, for ease of use let $\dot{H} = H_{1/t}$. Given a particular $X^{\sigma}_{i,j}$ we have, in $Z_{\bar{H}}$:

$$\frac{-\sigma}{2}\left((1-t^{-\sigma})\bar{H}_i^j\right)^2 = -\frac{\sigma}{2}\left((1-t^{-\sigma})\dot{H}_i^j\right)^2$$
$$-\frac{-\sigma}{2}\left((1+t^{-\sigma})\bar{H}_j^j\right)^2 = \frac{\sigma}{2}\left((1+t^{-\sigma})\dot{H}_j^j\right)^2$$
$$-\sigma t^{-\sigma}(\bar{H}_i^j\bar{H}_j^i + \bar{H}_i^i\bar{H}_j^j) = -\left(\sigma t^{-\sigma}(\dot{H}_i^j\dot{H}_j^i + \dot{H}_i^i\dot{H}_j^j)\right)$$
$$\det(\bar{B})\sum_{u_i^\sigma} -\sigma\bar{H}_i^i = -\det(\dot{B})\sum_{u_i^\sigma}\sigma\dot{H}_i^i$$

So far we have shown that the polynomial Z_G acts as expected, and that 4 of the 5 pieces of Z_H act as expected. One would hope that we could make the same argument for that last term in the sum $t^\sigma(1-t)H_i^j((1+\sigma)H_j^j + (1-\sigma)H_i^i)$ but this is not the case. This last piece relies on summing over all of the crossings. Applying the same method to the last piece of Z_H we see that we would need

$$\sum_{\chi_{i,j}^\sigma} t^{-\sigma}(1-t)\bar{H}_i^j\left((1-\sigma)\bar{H}_j^j + (1+\sigma)\bar{H}_i^i\right) = -\sum_{\chi_{i,j}^\sigma} t^{-\sigma}(1-t^{-1})\dot{H}_i^j\left((1+\sigma)\dot{H}_j^j + (1-\sigma)\dot{H}_i^i\right)$$

Or, taken by crossings, (recall that $\dot{H}_j^i = \bar{H}_j^i$)

$$\sum_{\sigma^+}\dot{H}_i^j\left((t^{-1}-1)\dot{H}_i^i + (t^{-1}-t^{-2})\dot{H}_j^j\right) + \sum_{\sigma^-}\dot{H}_i^j\left((t-t^2)\dot{H}_j^j + (t-1)\dot{H}_i^i\right) = 0.$$

We can rewrite this as,

$$\sum_{\chi}\delta_{\sigma,1}\dot{H}_i^j\left((t^{-\sigma}-1)\dot{H}_i^i + (t^{-\sigma}-t^{-2\sigma})\dot{H}_j^j\right) + \delta_{\sigma,-1}\dot{H}_i^j\left((t^{-\sigma}-t^{-2\sigma})\dot{H}_j^j + (t^{-\sigma}-1)\dot{H}_i^i\right) = 0$$

or rather,

$$\sum_{\chi}\dot{H}_i^j\left((t^{-\sigma}-1)\dot{H}_i^i + (t^{-\sigma}-t^{-2\sigma})\dot{H}_j^j\right) = \sum_{\chi}\dot{H}_i^j(t^{-\sigma}-1)(\dot{H}_i^i - t^{-\sigma}\dot{H}_j^j) = 0$$

So, if the last term of Z_H has that property, then we get the equality we desire.

That is,

$$Z_1(m(K);t) = -Z_1(K;t^{-1})$$

if and only if

$$\sum_{\chi} \dot{H}_i^j (t^{-\sigma} - 1)(\dot{H}_i^i - t^{-\sigma} \dot{H}_j^j) = 0$$

as desired. $\square$

Chapter 4. A State Sum Model

We will now turn our attention to something more abstract; what we wish to do is develop a state sum model for this polynomial. To do this we will look at a modified Gauss diagram and a geometric interpretation of one of the main pieces of the polynomial.

4.1. A quick geometric interpretation of B

Following the notation of [9] a knot K given by a collection of crossings $\chi^{\sigma}_{i,j}$ and cuaps u^{σ}_{i} with a trivial stitching $m^{(12\ldots n)}$ gives rise to the following matrices:

$$Q = \sum_{\chi^{\sigma}_{i,j}} \sigma t^{\frac{\sigma}{2}} (E^{j}_{j} - E^{i}_{j}) \qquad W = \sum_{i<j} E^{i}_{j} \qquad B = I - (t^{1/2} - t^{-1/2}) W Q$$

We understand the matrix Q to contain the crossing information and the matrix W to contain the stitching information. The matrix B can be understood in the following way: each row of B tells how far along the knot one has progressed in an oriented manner, and which crossings have been started but not completed. The diagonal elements record the progression along the knot, according to the labelling of the diagram presented, and the $\pm(1 - t^{\sigma})$ entries (in the columns corresponding to the under crossings) record which crossings have been partially traversed up to that point (see figures 2 and 3).

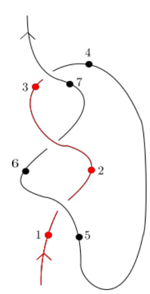

Figure 4.1. The trefoil traversed up to the third point.

$$\begin{pmatrix} t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ t^{-1}-1 & 1 & 0 & 0 & 0 & 1-t^{-1} & 0 \\ t^{-1}-1 & 0 & t^{-1} & 0 & 0 & 1-t^{-1} & 0 \\ t^{-1}-1 & 0 & t^{-1}-1 & 1 & 0 & 1-t^{-1} & 0 \\ 0 & 0 & t^{-1}-1 & 0 & 1 & 1-t^{-1} & 0 \\ 0 & 0 & t^{-1}-1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure 4.2. The matrix B, with the third row highlighted. We see that we can read off which crossings have been partially traversed.

4.2. The state sum setup

A knot K given by a collection of crossings $\chi^{\sigma}_{i,j}$ and cuaps u^{σ}_{i} with a trivial stitching $m^{(12\ldots n)}$ gives rise to a weighted graph D with the following properties:

- There are vertices v_i labeled from 1 to n connected in increasing order
- For each crossing $\chi^{\sigma}_{i,j}$ there is a directed edge (with color σ) from i to j with weight $w(v_i) = 1$ and $w(v_j) = t^{\sigma}$ if $i < j$ (or $w(v_i) = t^{\sigma}$ and $w(v_j) = 1$ if $j < i$)
- For each cuap u^{σ}_{i}, the vertex i has color σ

For example, the graph associated to the (left-handed) trefoil given by

$$\chi^{-1}_{5,1}\chi^{-1}_{2,6}\chi^{-1}_{7,3}u^{-1}_{4}//m^{(1234567)}$$

is:

$D :=$

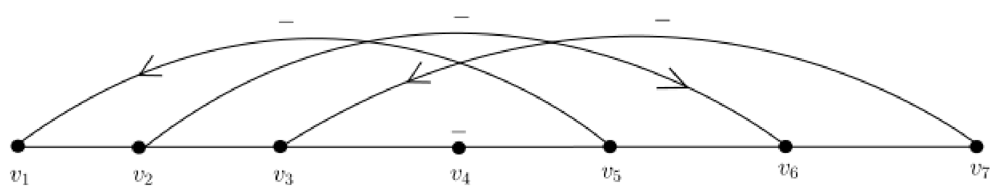

Figure 4.3. One can think of the vertex weights for a particular edge as 1 if the vertex is the first one (ordered from left to right) and t^{σ} for the second vertex in the edge.

We define ϕ^i to be a "toggle" function that toggles directed edges in the graph D on and off up to vertex v_i. $\phi^i(D)$ returns a graph where each directed edge $e_{\alpha,\beta}$ from v_α to v_β in D is removed unless either $\alpha \leq i$ or $\beta \leq i$ but not both. We consider these toggled diagrams to be states.

We define $\tilde{\phi}_j(S)$ to be an evaluation of a state S at vertex v_j given by $\tilde{\phi}_j(S) = w(i) - w(j)$ if v_j is at the end of a directed edge $e_{i,j}$ and zero otherwise.

For convenience we also define $\phi_j(S)$, given a state $S = \phi^i(D)$, to be $\tilde{\phi}_j(\phi^i(D))$ if $i \neq j$ and $\tilde{\phi}_j(\phi^i(D)) + 1$ if $i = j$.

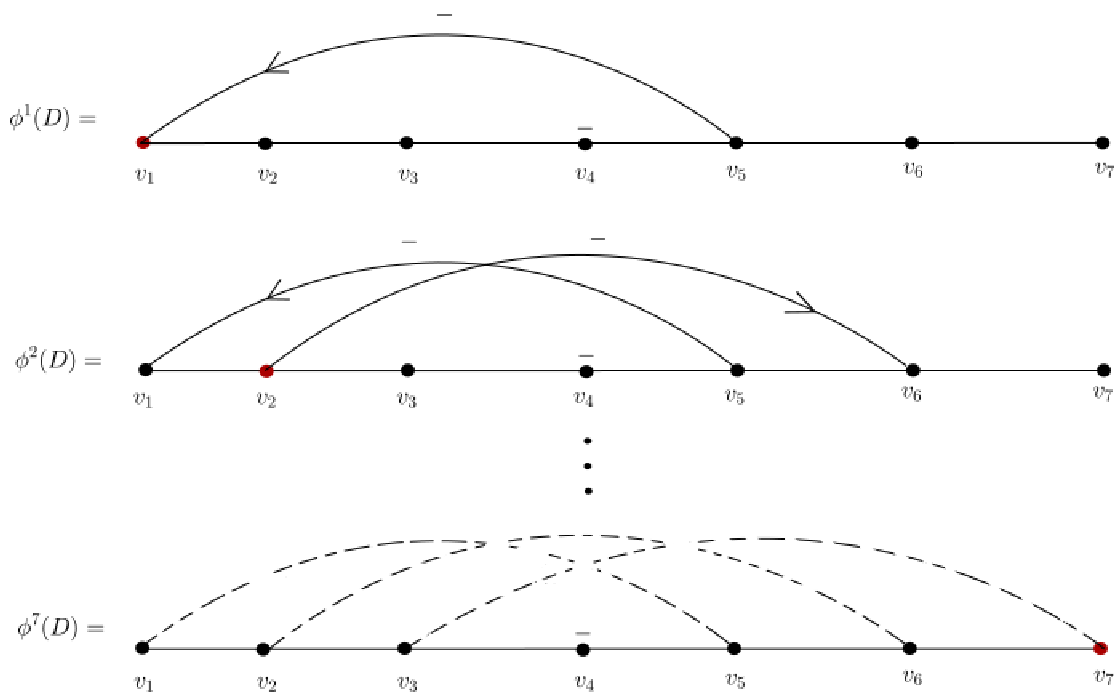

Figure 4.4. As the designated vertex moves from left to right, we toggle arcs on and off.

With the above definitions we can see that, given the geometric interpretation of the matrix B, by construction we have

$$B = \sum_{i,j}^{n} \phi_j \phi^i(D) E_j^i$$

where E_j^i is the elementary matrix with non-zero entry 1 in the (i,j) entry.

4.3. The state sum

We can interpret the derivative of B, which from [9] we see is the Alexander Polynomial (up to multiplication by t), as

$$\det(B) = \sum_{p \in (1,2,\ldots,n)} (-1)^{\epsilon_p} \prod_{k=1}^{n} \phi_{p(k)} \phi^k(D) \doteq \Delta_K(t).$$

Recall that $B = Id - (t^{1/2} - t^{-1/2})WQ$ and consider $A = (t^{1/2} - t^{-1/2})WQ = -(B - Id)$. We are interested in $\text{adj}(B) = \det(B)B^{-1}$ which we can think of as its expansion $\text{adj}(B) = \det(B)(Id + A + A^2 + \ldots)$. Let $\mathcal{O}_i$ be the set of vertices in $S = \phi^i(D)$ with

non-zero evaluations $\phi_j(S)$. Picking any particular entry (ω_0, j) of A^m (denoted $(A^m)^{\omega_0}_j$) we find

$$(A^m)^{\omega_0}_j = (A)^{\omega_0}(A^{m-1})_j = (-(B-Id))^{\omega_0}(A^{m-1})_j = -\sum_{\omega_{1,\alpha}\in\mathcal{O}_{\omega_0}} (\widetilde{\phi}_{\omega_{1,\alpha}}\phi^{\omega_0}(D))(A^{m-1})^{\omega_{1,\alpha}}_j$$

since the multiplication of a row in $(B-Id)$ by a column in A^{m-1} corresponds to summing over the non-zero evaluations of a particular state, each multiplied by the corresponding entry in the column of A^{m-1}.

This tells us that a particular entry $(A^m)^{\omega_0}_j$ is:

$$(A^m)^{\omega_0}_j = (-1)^m(\prod_{i=0}^{m-1}\sum_{\omega_{i+1,\alpha}\in\mathcal{O}_{\omega_i}}\widetilde{\phi}_{\omega_{i+1,\alpha}}\phi^{\omega_i}(D))\delta^{\omega_m}_j$$

and that

$$\begin{aligned}
\mathrm{adj}(B) &= \det(B)B^{-1}\\
&= \det(B)(Id-A)^{-1}\\
&= \det(B)(\sum_{s=0}^{\infty}A^s)\\
&= \det(B)(\sum_{s=0}^{\infty}\sum_{i,j}^{n}(A^s)^i_j * E^i_j)\\
&= \det(B)\sum_{s=0}^{\infty}(\sum_{i,j}^{n}(-1)^s(\prod_{\tau=0,\omega_0=i}^{s-1}\sum_{\omega_{\tau+1,\alpha}\in\mathcal{O}_{\omega_\tau}}\widetilde{\phi}_{\omega_{\tau+1,\alpha}}\phi^{\omega_\tau}(D))\delta^{\omega_s}_j)E^i_j
\end{aligned}$$

where

$$\det(B) = \sum_{p\in(1,2,\ldots,n)}(-1)^{\epsilon_p}\prod_{k=1}^{n}\phi_{p(k)}\phi^k(D).$$

Thus the two variable function on the graph D

$$F(i,j) = (\sum_{p\in(1,2,\ldots,n)}(-1)^{\epsilon_p}\prod_{k=1}^{n}\phi_{p(k)}\phi^k(D))(\sum_{s=0}^{\infty}((-1)^s(\prod_{\tau=0,\omega_0=i}^{s-1}\sum_{\omega_{\tau+1,\alpha}\in\mathcal{O}_{\omega_\tau}}\widetilde{\phi}_{\omega_{\tau+1,\alpha}}\phi^{\omega_\tau}(D))\delta^{\omega_s}_j))$$

is a state sum formula for the (i, j) entry in $\mathrm{adj}(B)$, which lets us understand the matrices H and G as state sums which, in turn, lets us understand $Z = c(Z_H + Z_G)$ in terms of state sums.

4.3.1. The Polynomial as a State Sum

Recall that $H = \mathrm{adj}(B)W$. Multiplication by W makes each entry of H a sum over the entries in a row of $\mathrm{adj}(B)$ up to a certain entry. That is,

$$H_j^i = \sum_{k=1}^{j-1} F(i,k).$$

Then

$$\begin{aligned}
Z_H &= \sum_{\chi_{i,j}^\sigma} \frac{\sigma}{2}((1-t^\sigma)H_i^j)^2 - \frac{\sigma}{2}((1+t^\sigma)H_j^j)^2 + \sigma t^\sigma (H_i^j H_j^i + H_i^i H_j^j)) \\
&+ t^\sigma(1-t)H_i^j((1+\sigma)H_j^j + (1-\sigma)H_i^i) + \det(B)\sum_{u_i^\sigma} \sigma H_i^i \\
&= \sum_{\chi_{i,j}^\sigma} \frac{\sigma}{2}((1-t^\sigma)\sum_{k=1}^{i-1} F(j,k))^2 - \frac{\sigma}{2}((1+t^\sigma)\sum_{k=1}^{j-1} F(j,k))^2 \\
&+ \sigma t^\sigma((\sum_{k=1}^{i-1} F(j,k))(\sum_{k=1}^{j-1} F(i,k)) + (\sum_{k=1}^{i-1} F(i,k))(\sum_{k=1}^{j-1} F(j,k)))) \\
&+ t^\sigma(1-t)\sum_{k=1}^{i-1} F(j,k)((1+\sigma)\sum_{k=1}^{j-1} F(j,k) + (1-\sigma)\sum_{k=1}^{i-1} F(i,k)) \\
&+ (\sum_{p \in (1,2,\ldots,n)} (-1)^{\epsilon_p} \prod_{k=1}^{n} \phi_{p(k)}\phi^k(D)) \sum_{u_i^\sigma} \sigma \sum_{k=1}^{i-1} F(i,k)
\end{aligned}$$

Recall that $G = Q\mathrm{adj}(B)$. We can re-write G as a sum over crossings as

$$G = \sum_{\chi_{o,u}^\sigma} \sum_{\alpha \in \{o,u\}} \sum_{k=1}^{n} (\delta_o^\alpha - \delta_u^\alpha)\sigma t^{\sigma/2} F(o,k) * E_k^\alpha$$

This gives us a formula for each entry:

$$G^i_j = \sum_{\chi^{\sigma}_{o,u}} (\delta^i_o - \delta^i_u)\sigma t^{\sigma/2} F(o,j)$$

Note that

$$(G^i_j)^2 = \sum_{\chi^{\sigma}_{o,u}} (\delta^i_o + \delta^i_u) t^{\sigma} (F(o,j))^2$$

and that

$$G^j_a G^g_b = \sum_{\chi^{\sigma_\alpha}_{o_\alpha,u_\alpha}} \sum_{\chi^{\sigma_\beta}_{o_\beta,u_\beta}} (\delta^j_{o_\alpha} - \delta^j_{u_\alpha})(\delta^g_{o_\beta} - \delta^g_{u_\beta})\sigma_\alpha\sigma_\beta t^{\frac{\sigma_\alpha+\sigma_\beta}{2}} F(o_\alpha,a)F(o_\beta,b)$$

Then

$$\begin{aligned}
Z_G &= (t-t^{-1})\sum_{j=2}^{n}\sum_{a,b<j} G^j_a(\frac{1}{2}G^j_a + \sum_{g>j} G^g_b) \\
&= (t-t^{-1})\sum_{j=2}^{n}\sum_{a,b<j} (\frac{1}{2}(G^j_a)^2 + G^j_a \sum_{g>j} G^g_b) \\
&= (t-t^{-1})\sum_{j=2}^{n}\sum_{a,b<j} (\frac{1}{2}(\sum_{\chi^{\sigma}_{o_\alpha,u_\alpha}} (\delta^j_{o_\alpha} + \delta^j_{u_\alpha}) t^{\sigma} (F(o_\alpha,a))^2) \\
&+ \sum_{g>j}(\sum_{\chi^{\sigma_\alpha}_{o_\alpha,u_\alpha}} \sum_{\chi^{\sigma_\beta}_{o_\beta,u_\beta}} (\delta^j_{o_\alpha} - \delta^j_{u_\alpha})(\delta^g_{o_\beta} - \delta^g_{u_\beta})\sigma_\alpha\sigma_\beta t^{\frac{\sigma_\alpha+\sigma_\beta}{2}} F(o_\alpha,a)F(o_\beta,b)) \\
&= (t-t^{-1})\sum_{j=2}^{n}\sum_{a,b<j\leq g} (\sum_{\chi^{\sigma_\alpha}_{o_\alpha,u_\alpha}} \sum_{\chi^{\sigma_\beta}_{o_\beta,u_\beta}} (\delta^j_{o_\alpha} - \delta^j_{u_\alpha})(\delta^g_{o_\beta} - \delta^g_{u_\beta})\sigma_\alpha\sigma_\beta t^{\frac{\sigma_\alpha+\sigma_\beta}{2}} F(o_\alpha,a)((1-\delta^g_j)F(o_\beta,b) \\
&+ \frac{1}{2}\delta^g_j F(o_\beta,a)))
\end{aligned}$$

Chapter 5. A Base Set of Knots

For a knot K with Conway polynomial $\nabla_K = 1 + a_1 z^2 + a_2 z^4 + ... + a_n z^{2n}$ we want to investigate the properties of the following normalization: $\frac{1}{z^2}(\nabla_K^2 - 1)$.

First, we have:

$$\begin{aligned}(\nabla_K)^2 &= (1 + a_1 z^2 + a_2 z^4 + a_3 z^6 ... + a_{n-1} z^{2n-2} + a_n z^{2n})^2 \\ &= 1 + (2a_1)z^2 + (a_1^2 + 2a_2)z^4 + (2a_3 + 2a_1 a_2)z^6 + ... + (2a_n a_{n-1})z^{4n-2} + (a_n)^2 z^{4n}\end{aligned}$$

In particular we have:

$$\frac{1}{z^2}(\nabla_K^2 - 1) = 2a_1 + (a_1^2 + 2a_2)z^2 + (2a_3 + 2a_1 a_2)z^4 + ... + (2a_n a_{n-1})z^{4n-4} + (a_n)^2 z^{4n-2}$$

Notice that for the z^{2i} terms, where i is even, the coefficient is also even.

What we wish to do now is consider the coefficients of this normalization as a vector, and find a collection of knots for which the coefficients of their (normalized) Conway polynomials form a basis. In particular, we want a collection $\mathcal{K}$ of knots such that the coefficients of the (normalized) Conway polynomial of these form a basis for the vector space of coefficients given by all knots. For added challenge, given a knot K, we want to use basis elements consisting of knots with no more crossings than K itself.

With that in mind, let us consider the following collection $\mathcal{K}$ of knots:

- The torus knots given by: $T(2, 2n + 1)$
- The Figure-8 knot: 4_1
- The connected sums given by: $T(2, 3)\#T(2, 2n + 1)$

From the axiomatic definition of the Conway polynomial, we can quickly see that for a knot with $2n+1$ crossings, the maximum degree of the polynomials must be less than or equal to $2n$ (recall that we gain a z each time we smooth out a crossing, and can only do so a maximum of $2n$ times before we must reduce a diagram via Reidemeister moves).

Moreover since Conway polynomials have even degree, the maximum degree of the polynomials must be less than or equal to $2n$ for knots with $2n + 2$ crossings as well.

The observant reader might think to ask at this point: "How do we know T(2,2n+1) is only knot with $2n + 1$ crossings with Conway polynomial of degree $2n$?" Recall that based on our definition, to achieve the maximum degree poissible, we must switch between knots and links as we smooth out the crossings (without losing any crossings to Reidemeister moves along he way). Thus the second to last step will yield a Hopf link with a coefficient of z^{2n}. going backwards to see what diagrams we could have come from, the only possibility is to add a crossing in the same direction as the others on the same strands (any other choice either would have been removed via a Reidemeister move or requires a Reidemeister move to set up, which would have lowered our overall degree.) Following this process backwards yields the knot $T(2, 2n + 1)$, because of the restriction on crossing number.

In [10] Kauffman showed that the Conway polynomials for the $T(2, q)$ torus links were given by the Fibonacci polynomials

$$F_q[z] = \sum_{j=0}^{\lfloor \frac{q-1}{2} \rfloor} C(q - j - 1, j) z^{q-2j-1}$$

where C(x,y) is the binomial coefficient. We can see that when $q = 2n + 1$, the polynomial is given by $z^{2n} + (2n - 1)z^{2n-2} + \mathcal{O}(z^{2n-4})$. In our normalization, we have:

$$\frac{1}{z^2}(\nabla^2_{(T(2,2n+1)} - 1) = z^{4n-2} + (4n - 2)z^{4n-4} + \mathcal{O}(z^{4n-6})$$

Moreover, if we consider the connected sum $T(2, 3)\#T(2, 2n - 1)$ we compute

$$\frac{1}{z^2}(\nabla^2_{T(2,3)\#T(2,2n-1)} - 1) = z^{4n-2} + (4n - 4)z^{4n-4} + \mathcal{O}(z^{4n-6})$$

So, given a knot K and its normalization $\frac{1}{z^2}(\nabla_K^2 - 1)$, we can take care of the top degree z^{4n-2} term with the appropriate choice of monic polynomial $\frac{1}{z^2}(\nabla_{T(2,2n+1)}^2 - 1)$. Since the z^{2i} (with i even) term has even coefficient, we can remove the z^{4n-4} with the appropriate linear combination of $\frac{1}{z^2}(\nabla_{T(2,2n+1)}^2 - 1)$ and $\frac{1}{z^2}(\nabla_{T(2,3)\#T(2,2n-1)}^2 - 1)$, since the coefficients of that term differ by 2 in these and they are monic themselves.

Given this, we can reduce any given $\frac{1}{z^2}(\nabla_K^2 - 1)$ to something that looks like

$$\frac{1}{z^2}(\nabla_K^2 - 1) = \frac{1}{z^2}\sum C_\alpha(\nabla_{K_\alpha}{}^2 - 1) + az^2 + 2b, \qquad a, b, C_\alpha \in \mathbb{Z}, \quad K_\alpha \in \mathcal{K}$$

Now, consider the trefoil and figure-8 knots. They have Conway polynomials $1 + z^2$ and $1 - z^2$ respectively. This means that, under our normalization, we have

$$\frac{1}{z^2}(\nabla_{K_{3_1}}{}^2 - 1) = 2 + z^2$$

and

$$\frac{1}{z^2}(\nabla_{K_{4_1}}{}^2 - 1) = -2 + z^2.$$

The constant term differs by 4, not 2, so we may not *immediately* see that we will be able to use these to finish off the remaining terms. Thankfully we see that, given a knot K, our normalization gives

$$\frac{1}{z^2}(\nabla_K^2 - 1) = 2a_1 + (a_1^2 + 2a_2)z^2 + \mathcal{O}(z^4)$$

and, after reducing by a linear combination of polynomials from $\mathcal{K}$ to get rid of the $\mathcal{O}(z^4)$ part, we have

$$2a_1 + \sum C_\alpha(2b_{\alpha,1}) + (a_1^2 + 2a_2 + \sum C_\alpha(b_{\alpha,1}^2 + 2b_{\alpha,2}))z^2$$

and a quick computation shows that this is equal to

$$\left(\frac{a_1^2+2a_2+a_1+\sum C_\alpha(b_{\alpha,1}^2+2b_{\alpha,2}+b_{\alpha,1})}{2}\right)(2+z^2)$$
$$+\left(\frac{a_1^2+2a_2-a_1+\sum C_\alpha(b_{\alpha,1}^2+2b_{\alpha,2}-b_{\alpha,1})}{2}\right)(-2+z^2).$$

Note that these coefficients are integers.

Thus, for any K we can express our normalization as a linear combination of normalizations of knots in $\mathcal{K}$ (using knots with no more crossings than K itself):

$$\frac{1}{z^2}(\nabla_K^2-1)=\frac{1}{z^2}\sum C_\alpha(\nabla_{K_\alpha}^{\ 2}-1), \quad C_\alpha\in\mathbb{Z}, \quad K_\alpha\in\mathcal{K}$$

Remark 5.0.1. *Our choice of knots in $\mathcal{K}$ allows us to index them by crossing number, and we will do so from here on out. Moreover we also could have chosen the $T(2,-(2n+1)$ knots and would gotten the same result due to squaring.*

5.1. Expanding $\rho_1(K)$

Once again, using the Alexander polynomial as a model, we can ask if the $Z_1(K)$ polynomial, or rather it's normalization $\rho_1(K)$, works in a similar way. That is, can we write it as a sum of polynomials from the base set of knots $\mathcal{K}$?

CONJECTURE 5.1.1. *For an n crossing knot K with symmetric polynomial $\rho_1(K;t)$ as defined in [9] and Conway polynomial $\nabla(K;z)$ we can write:*

$$\rho_1(K;z=t^{1/2}-t^{-1/2})=\frac{1}{z^2}\sum_{\alpha=3}^{n}C_\alpha(\nabla^2(K_\alpha;z)-1).$$

where $C_\alpha\in\mathbb{Z}$ and each K_α is the α crossing knot in $\mathcal{K}$.

This conjecture is made for $\rho_1(K;t)$ that are symmetric with respect to $t\to t^{-1}$ (so that we can use that normalization). It is conjectured that all $\rho_1(K;t)$ have this property [9]. One could also write the preceding conjecture as:

CONJECTURE 5.1.2. *For an n crossing knot K with polynomial $\rho_1(K;t)$ as defined in [9] and Alexander polynomial $\Delta(K;t)$ we can write:*

$$\rho_1(K;t) = \frac{1}{(t^{1/2} - t^{-1/2})^2} \sum_{\alpha=3}^{n} C_\alpha(\Delta^2(K_\alpha;t) - 1)$$

where $C_\alpha \in \mathbb{Z}$ and each K_α is the α crossing knot in $\mathcal{K}$.

Equivalently, one could formulate this for the original knot invariant $Z_1(K;t)$ as:

CONJECTURE 5.1.3. *For an n crossing knot K with polynomial $Z_1(K;t)$ as defined in [9] and (normalized) Alexander polynomial $\Delta(K;t)$ we can write:*

$$Z_1(K;t) = t\Delta(K;t)(\frac{d}{dt}\Delta(K;t)) + \sum_{\alpha=3}^{n} C_\alpha(\Delta^2(K_\alpha;t) - 1)$$

where $C_\alpha \in \mathbb{Z}$ and each K_α is the α crossing knot in $\mathcal{K}$.

At the end of this paper we list all of the knots through nine crossings, and their expansions in the manner described above.

Speaking to the greater context around this problem for a moment, while these conjectures would not prove Conjecture 3 from [9], if that conjecture were true it would imply that the $P^{(1)}$ term in Rosansky's expansion of the colored Jones polynomial defined in [11], otherwise known as the 2-loop invariant, is equal to a linear combination of the sums of squares of Alexander polynomials, given our conjecture.

We may also make a remark regarding the coefficients given in our expansion, in the form of a conjecture.

CONJECTURE 5.1.4. *For a knot K with coefficients $\{C_\alpha\}$ as defined above, the mirror of K has coefficients $\{-C_\alpha\}$.*

Since $t\Delta(K)(\frac{d}{dt}\Delta(K))$ is anti-symmetric with respect to $t \rightarrow t^{-1}$ and does not change under mirroring, this is true if and only if $Z_1(m(K);t) = -Z_1(K;t^{-1})$.

As an extended example, let us consider the formula given in [9] for the family of alternating torus knots $T(2, 2p+1)$.

THEOREM 5.1.5. *Conjecture 5.1.1 holds for the family of alternating torus knots* $T(2, 2p+1)$ *with formula*

$$\rho_1\left(T(2, \pm(2p+1)); t\right) = \mp \sum_{k=0}^{p-1} \frac{1}{2}(p-k)(p+k+1)(t^{2k+1} + t^{-2k-1})$$

Proof. We immediately see that this is a symmetric polynomial, and that every other coefficient in the polynomial is even. In fact, these coefficients are equal to zero which means that the coefficient of the second highest degree term after the substitution $z = t^{1/2} - t^{-1/2}$ is completely determined by the coefficient of the highest degree term and the degree (this will be useful later). This insures the structure of $\rho_1(T(2, \pm(2p+1)); t)$ meets the basic requirements for our conjecture.

We notice that the highest degree term (in variable t) will be $2p-1$ so in variable z the highest degree term will be $4p-2$, which is the same degree that $\frac{1}{z^2}(\nabla^2(T(2, 2p+1); z) - 1)$ has. Thus we know that we will be able to use this to remove the highest degree term in the expansion, and all that is left to show is that it will also remove the second highest degree term (this will ensure that no knots with more that $2p+1$ crossings are required).

So, we know that

$$\frac{1}{z^2}(\nabla^2(T(2, 2p+1); z) - 1) = z^{4p-2} + (4p-2)z^{4p-4} + \mathcal{O}(z^{4p-6})$$

and that

$$\rho_1(T(2, \pm(2p+1)); t) = \mp p t^{2p-1} + \mathcal{O}(t^{2p-3})$$

Since we are considering the substitution $z = t^{1/2} - t^{-1/2}$, or rather $z^2 = t - 2 + t$, we see that for $\rho_1(T(2, \pm(2p+1)); t)$ we will need to use the term

$$p(t - 2 + t)^{2p-1} = pt^{2p-1} + p(-2(2p-1))t^{2p-2} + \mathcal{O}(z^{2p-3}).$$

Since the coefficient in front of the t^{2p-2} term in $\rho_1(T(2, \pm(2p+1)); t)$ is zero, a variable substitution for the $\mp pt^{2p-1}$ term would add a $\mp p(-4p+2))t^{2p-2}$ term which we must cancel out with a $\mp p(4p-2))t^{2p-2}$ term.

Thus, $\mp p$ copies of $\frac{1}{z^2}(\nabla^2(T(2, 2p+1); z) - 1)$ is exactly enough to take care of the two highest degree terms, insuring that we will use only knots with at most $2p+1$ crossings in our expansion, as required by the conjecture. □

Now that we know this family of knots has the right properties, we would like to know exactly how to expand them. Thankfully, we may do so.

THEOREM 5.1.6. *For the family of alternating torus knots $T(2, 2p+1)$ we have:*

$$\rho_1(T(2, \pm(2p+1)); t) = \frac{\mp 1}{(t-2+t)}\left(p\left(\Delta^2(T(2,(2p+1)); t) - 1\right) - \sum_{k=0}^{p-1}\left(\Delta^2(T(2,(2k+1)); t)\right) - 1\right)$$

Proof. Let us first examine $\rho_1(T(2, \pm(2p+1)); t)$. We notice that

$$
\begin{aligned}
\rho_1(T(2,\pm(2p+1));t) &= \mp\sum_{k=0}^{p-1}\frac{1}{2}(p-k)(p+k+1)(t^{2k+1}+t^{-2k-1}) \\
&= \mp\sum_{k=0}^{p-1}\frac{1}{2}(p^2+p-k^2+k)(t^{2k+1}+t^{-2k-1}) \\
&= \mp\sum_{k=0}^{p-1}\frac{1}{2}(p(p+1)-k(k+1))(t^{2k+1}+t^{-2k-1}) \\
&= \mp(p\sum_{k=0}^{p-1}(t^{2k+1}+t^{-2k-1})+(p-1)\sum_{k=0}^{p-2}(t^{2k+1}+t^{-2k-1})+...+(t+t^{-1})) \\
&= \mp(p\sum_{k=0}^{p-1}(t^{2k+1}+t^{-2k-1})+\rho_1(T(2,\pm(2p-1));t))
\end{aligned}
$$

Now we consider:

$$
\begin{aligned}
&(t-2+t^{-1})\rho_1(T(2,\pm(2p+1));t) \\
=&\mp(t(p\sum_{k=0}^{p-1}(t^{2k+1}+t^{-2k-1})+(p-1)\sum_{k=0}^{p-2}(t^{2k+1}+t^{-2k-1})+...) \\
&-2\mp(p\sum_{k=0}^{p-1}(t^{2k+1}+t^{-2k-1})+(p-1)\sum_{k=0}^{p-2}(t^{2k+1}+t^{-2k-1})+...) \\
&+t^{-1}\mp(p\sum_{k=0}^{p-1}(t^{2k+1}+t^{-2k-1})+(p-1)\sum_{k=0}^{p-2}(t^{2k+1}+t^{-2k-1})+...)) \\
=&\mp(p\sum_{k=0}^{p-1}(t^{2k+2}-2t^{2k+1}+t^{2k}+t^{-2k}-2t^{-2k-1}+t^{-2k-2}) \\
&+(p-1)\sum_{k=0}^{p-2}(t^{2k+2}-2t^{2k+1}+t^{2k}+t^{-2k}-2t^{-2k-1}+t^{-2k-2})+...)
\end{aligned}
$$

Now we collect terms, and examine coefficients:

$$\begin{aligned}
= \quad & \mp (p(t^{2p} + t^{-2p}) + (-2p)(t^{2p-1} + t^{-2p+1}) \\
& + (p + p + (p-1))(t^{2p-2} + t^{-2p+2}) + (-2p - 2(p-1))(t^{2p-3} + t^{-2p+3}) \\
& + (p + p + (p-1) + (p-1) + (p-2))(t^{2p-4} + t^{-2p+4}) \\
& + (-2p - 2(p-1) - 2(p-2))(t^{2p-5} + t^{-2p+5}) + ... + p(p+1))
\end{aligned}$$

$$= \mp (\sum_{k=0}^{p}((2k+1)p - k^2)(t^{2p-2k} + t^{-2p+2k}) + \sum_{k=0}^{p}(-(2k+2)p + k(k+1))(t^{2p-2k-1} + t^{-2p+2k+1})$$

Now we must consider the other side of things. Let us examine

$$p\Delta^2(T(2,(2p+1));t) - \sum_{k=0}^{p-1}(\Delta^2(T(2,(2k+1));t)).$$

First, we recall that for torus knots [4]:

$$\Delta(T(p,q)) = \frac{(t^{qp}-1)(t-1)}{(t^p-1)(t^q-1).}$$

For our purposes, we may simplify to

$$\Delta(T(2,2p+1)) = \frac{t^{2p+1}+1}{t+1}.$$

Expanding this as a power series gives us

$$\frac{t^{2p+1}+1}{t+1} = 1 - t + t^2 - t^3 + ... + t^{2p}.$$

However, we want to work with the symmetric version so we scale by $\frac{1}{t^p}$:

$$\Delta(T(2,2p+1)) = t^{-p} - t^{-p+1} + ... - t^{p-1} + t^p$$

Next, we want to examine

$$\Delta^2(T(2,2p+1))-1 = t^{-2p}-2t^{-2p+1}+3t^{-2p+2}...-2pt^{-1}+2p-2pt+...+3t^{2p-2}-2t^{2p-1}+t^{2p}$$

Finally, we consider:

$$\begin{aligned}
& \mp (p(\Delta^2(T(2,2p+1))-1)-(\Delta^2(T(2,2p-1))-1)-(\Delta^2(T(2,2p-3))-1)-...) \\
= & \mp (p(t^{2p}+t^{-2p})+(-2p)(t^{2p-1}+t^{-2p+1}) \\
& +(3p-1)(t^{2p-2}+t^{-2p+2})+(-4p+2)(t^{2p-3}+t^{-2p+3}) \\
& +(5p-3-1)(t^{2p-4}+t^{-2p+4}) \\
& +(-6p+4+2)(t^{2p-5}+t^{-2p+5})+... \\
& +2p^2-(2p-2)-(2p-4)-...-2)
\end{aligned}$$

$$=\mp (\sum_{k=0}^{p}((2k+1)p-k^2)(t^{2p-2k}+t^{-2p+2k})+\sum_{k=0}^{p}(-(2k+2)p+k(k+1))(t^{2p-2k-1}+t^{-2p+2k+1})$$

Thus, we have shown

$$\rho_1(T(2,\pm(2p+1));t) = \frac{\mp 1}{(t-2+t)}(p(\Delta^2(T(2,(2p+1));t)-1)-\sum_{k=0}^{p-1}(\Delta^2(T(2,(2k+1));t))-1)$$

as required. $\square$

Now that we have some proof that $\mathcal{K}$ might be a good choice for what we want to do, we can turn to programming to compute whichever examples we desire. For my purposes I used Mathematica, starting with the code for computing $\rho_1(k)$ provided by Bar-Natan and van der Veen at http://www.rolandvdv.nl/MLA/. In brief, we first record our base set $\mathcal{K}$ as a matrix of coefficients of the normalization we are using:

```
BMat = {
  {2, 1, 0, 0, 0, 0, 0, 0},
  {-2, 1, 0, 0, 0, 0, 0, 0},
  {6, 11, 6, 1, 0, 0, 0, 0},
  {4, 6, 4, 1, 0, 0, 0, 0},
  {12, 46, 62, 37, 10, 1, 0, 0},
  {8, 24, 34, 24, 8, 1, 0, 0},
  {20, 130, 314, 367, 230, 79, 14, 1},
  {14, 71, 166, 207, 146, 58, 12, 1}
  };
```

Then, we write a function that takes a knot K and the base set of knot coefficients, and solves a system of equations to see what coefficients are needed to expand $\rho_1(K)$ in terms of the base set of knots. The ZNorm[p(t), z^2] function computes the variable substitution $t^{1/2} - t^{-1/2} \to z$, and the \[Rho][K] function computes $\rho_1(K)$.

```
ConwayExpandSolve[K_] := Module[{Sp, V},
 Sp = BMat; V = Total[
   Flatten[{CoefficientList[ZNorm[\[Rho][K]], z^2], {0, 0, 0, 0, 0, 0, 0, 0}},
     {{2}, {1}}], {2}];
 LinearSolve[Transpose[Sp], V]
 ]
```

This will compute the expansion of a given knot up through 10 crossings. If one

would want higher crossing knots, one would need to add the proper coefficients from $\mathcal{K}$ to the base set and expand the vectors in the functions to the new expected dimension. If we want to compute more than one at a time, we can run through a list of knots such as the one from the KnotTheory package:

```
Ans = {#, CES = ConwayExpandSolve[#]; CES[[1]], CES[[2]], CES[[3]], CES[[4]],
    CES[[5]], CES[[6]], CES[[7]], CES[[8]]} & /@AllKnots[{3, 6}];
Ans = Prepend[
   Ans, {"", "3_1", "4_1", "5_1", "3#3", "7_1", "3#5", "9_1", "3#7"}];
Ans // MatrixForm
```

The above code, with the omitted functions, would produce the following table (Figure 5.1) that tells us how $\rho_1(K)$ can be expanded in terms of the base set of knots $\mathcal{K}$. The numbers listed are the coefficients C_α from our conjectures. A more comprehensive list is given in Appendix A.

	3_1	4_1	5_1	3♯3	7_1	3♯5	9_1	3♯7
Knot[3, 1]	1	0	0	0	0	0	0	0
Knot[4, 1]	0	0	0	0	0	0	0	0
Knot[5, 1]	-1	0	2	0	0	0	0	0
Knot[5, 2]	4	1	0	0	0	0	0	0
Knot[6, 1]	0	1	0	0	0	0	0	0
Knot[6, 2]	-3	-1	-1	2	0	0	0	0
Knot[6, 3]	0	0	0	0	0	0	0	0

Figure 5.1. Example Output

Chapter 6. Closing Remarks

In the previous chapter we saw that using the coefficients of a normalized version of the square of the Alexander polynomials of the base set of knots $\mathcal{K}$ let us give an explicit expansion of $\rho_1(K)$ for a large family of examples, the $T(2, 2n + 1)$ torus knots. We hope that this method can be used to find expansions for other families of knots, and to find a general formula for expanding $\rho_1(K)$ in terms of these knots. It may be possible to define $\rho_1(K)$ in terms of a recursive relation, such as

$$\rho_1(K; t) = \rho_1(K'; t) + \mu(k; t)$$

where K' is K with a full twist removed and $\mu(K; t)$ is a particular evaluation on the crossings in K (this relation pops up when working with the $T(2, 2n + 1)$ torus knots for example). It may also be of use to consider knots which have a particular type of expansion, such as $3_1, 5_2, 6_1, 7_2, 7_4$, and 7_7 which expand as

$$\rho_1(K; z) = C_3(\nabla^2_{3_1}(z) - 1) + C_4(\nabla^2_{4_1}(z) - 1)$$

for some $C_3, C_4 \in Z$ for each knot K in that list.

Appendix. Polynomial Expansion Coefficients

Knot	*Polynomial*	3_1	4_1	5_1	$3_1\#3_1$	7_1	$3_1\#5_1$	9_1
3_1	$\rho(K;z)$	1						
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	1						
4_1	$\rho(K;z)$	0	0					
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	0	1					
5_1	$\rho(K;z)$	-1	0	2				
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	0	0	1				
5_2	$\rho(K;z)$	4	1	0				
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	3	1	0				
$3_1\#3_1$	$\rho(K;z)$	-2	0	0	2			
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	0	0	0	1			
6_1	$\rho(K;z)$	0	1	0	0			
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	1	3	0	0			
6_2	$\rho(K;z)$	-3	-1	-1	2			
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-2	0	-1	2			
6_3	$\rho(K;z)$	0	0	0	0			
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	1	1	-1	2			
$3_1\#4_1$	$\rho(K;z)$	1	0	-2	3	0		
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	1	1	-2	3	0		

Knot	*Polynomial*	3_1	4_1	5_1	$3_1\#3_1$	7_1	$3_1\#5_1$	9_1
7_1	$\rho(K;z)$	-1	0	-1	0	3		
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	0	0	0	0	1		
7_2	$\rho(K;z)$	10	4	0	0	0		
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	6	3	0	0	0		
7_3	$\rho(K;z)$	15	3	-9	8	0		
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-5	0	2	2	0		
7_4	$\rho(K;z)$	-16	-8	0	0	0		
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	10	6	0	0	0		
7_5	$\rho(K;z)$	-16	-5	9	-16	0		
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-4	-0	0	4	0		
7_6	$\rho(K;z)$	2	-1	1	-8	0		
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	5	3	-3	4	0		
7_7	$\rho(K;z)$	-1	-2	0	0	0		
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	6	6	-3	4	0		
$3_1\#5_1$	$\rho(K;z)$	-1	0	-1	-1	0	3	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	0	0	0	0	0	1	
$3_1\#5_2$	$\rho(K;z)$	-9	1	-4	13	0	0	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-2	1	-2	6	0	0	
$4_1\#4_1$	$\rho(K;z)$	0	0	0	0	0	0	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	10	10	-4	5	0	0	

Knot	*Polynomial*	3_1	4_1	5_1	$3_1\#3_1$	7_1	$3_1\#5_1$	9_1
8_1	$\rho(K;z)$	1	4	0	0	0	0	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	3	6	0	0	0	0	
8_2	$\rho(K;z)$	8	1	0	-6	-2	4	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	3	1	0	-2	-1	2	
8_3	$\rho(K;z)$	0	0	0	0	0	0	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	6	10	0	0	0	0	
8_4	$\rho(K;z)$	-9	-3	1	4	0	0	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-9	0	-2	6	0	0	
8_5	$\rho(K;z)$	0	0	1	2	2	-4	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-2	0	-1	1	-1	2	
8_6	$\rho(K;z)$	-10	-2	-5	10	0	0	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-5	1	-4	8	0	0	
8_7	$\rho(K;z)$	7	1	-2	0	3	-2	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-4	0	0	2	-1	2	
8_8	$\rho(K;z)$	-2	-2	1	-2	0	0	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	1	3	-4	8	0	0	
8_9	$\rho(K;z)$	0	0	0	0	0	0	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-6	0	-2	4	-1	2	
8_{10}	$\rho(K;z)$	7	1	-1	-2	1	-2	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-6	0	-1	5	-1	2	

$Knot$	$Polynomial$	3_1	4_1	5_1	$3_1\#3_1$	7_1	$3_1\#5_1$	9_1
8_{11}	$\rho(K;z)$	-6	-1	-7	12	0	0	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	0	3	-6	10	0	0	
8_{12}	$\rho(K;z)$	0	0	0	0	0	0	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	15	15	-5	6	0	0	
8_{13}	$\rho(K;z)$	-3	-3	1	2	0	0	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	5	6	-6	10	0	0	
8_{14}	$\rho(K;z)$	2	3	-9	14	0	0	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	6	6	-8	12	0	0	
8_{15}	$\rho(K;z)$	-21	3	-11	32	0	0	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-5	3	-6	15	0	0	
8_{16}	$\rho(K;z)$	-14	-2	1	5	-2	3	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-13	0	-2	10	-2	3	
8_{17}	$\rho(K;z)$	0	0	0	0	0	0	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-12	1	-4	12	-2	3	
8_{18}	$\rho(K;z)$	0	0	0	0	0	0	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-19	1	-5	19	-3	4	
8_{19}	$\rho(K;z)$	-10	-2	4	2	-3	0	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	5	1	-1	-1	1	0	
8_{20}	$\rho(K;z)$	3	1	0	0	0	0	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	0	0	0	1	0	0	

$Knot$	$Polynomial$	3_1	4_1	5_1	$3_1\#3_1$	7_1	$3_1\#5_1$	9_1
8_{21}	$\rho(K;z)$	1	1	-3	4	0	0	
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	1	1	-2	3	0	0	
9_1	$\rho(K;z)$	-1	0	-1	0	-1	0	4
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	0	0	0	0	0	0	1
9_2	$\rho(K;z)$	20	10	0	0	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	10	6	0	0	0	0	0
9_3	$\rho(K;z)$	-11	2	11	8	-7	-6	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	3	1	-3	-2	2	2	0
9_4	$\rho(K;z)$	-39	-3	9	14	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-14	0	3	6	0	0	0
9_5	$\rho(K;z)$	-40	-25	0	0	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	21	25	0	0	0	0	0
9_6	$\rho(K;z)$	7	2	-7	-10	1	12	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	2	1	-2	-2	0	4	0
9_7	$\rho(K;z)$	-30	-1	-5	28	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-9	1	-3	12	0	0	0
9_8	$\rho(K;z)$	0	-1	-5	8	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	6	6	-8	12	0	0	0
9_9	$\rho(K;z)$	-11	0	-7	0	1	12	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-6	0	-2	2	0	4	0

Knot	*Polynomial*	3_1	4_1	5_1	$3_1\#3_1$	7_1	$3_1\#5_1$	9_1
9_{10}	$\rho(K;z)$	68	6	-4	-36	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-24	0	0	16	0	0	0
9_{11}	$\rho(K;z)$	26	2	-11	-2	6	-8	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-13	1	4	4	-3	4	0
9_{12}	$\rho(K;z)$	13	7	-13	18	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	13	10	-10	14	0	0	0
9_{13}	$\rho(K;z)$	60	4	6	-46	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-20	1	-4	20	0	0	0
9_{14}	$\rho(K;z)$	-10	-11	3	-4	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	16	15	-10	14	0	0	0
9_{15}	$\rho(K;z)$	-24	-14	15	-20	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	21	15	-12	16	0	0	0
9_{16}	$\rho(K;z)$	22	0	6	-6	5	-18	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-10	0	-2	5	-2	6	0
9_{17}	$\rho(K;z)$	-27	-6	5	4	-3	4	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-24	0	0	12	-3	4	0
9_{18}	$\rho(K;z)$	-50	1	-14	54	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-15	3	-8	24	0	0	0
9_{19}	$\rho(K;z)$	4	2	-3	4	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	23	21	-12	16	0	0	0

$Knot$	$Polynomial$	3_1	4_1	5_1	$3_1\#3_1$	7_1	$3_1\#5_1$	9_1
9_{20}	$\rho(K;z)$	-34	-3	1	18	-6	8	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-18	0	-2	14	-3	4	0
9_{21}	$\rho(K;z)$	-32	-19	17	-22	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	30	21	-14	18	0	0	0
9_{22}	$\rho(K;z)$	26	6	-3	-7	3	-4	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-24	0	-3	17	-3	4	0
9_{23}	$\rho(K;z)$	-37	8	-24	64	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-9	6	-12	28	0	0	0
9_{24}	$\rho(K;z)$	7	3	-2	2	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-19	1	-5	19	-3	4	0
9_{25}	$\rho(K;z)$	12	12	-23	35	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	15	15	-18	27	0	0	0
9_{26}	$\rho(K;z)$	35	5	3	-20	3	-4	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-23	1	-6	22	-3	4	0
9_{27}	$\rho(K;z)$	6	4	-3	4	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-19	3	-8	24	-3	4	0
9_{28}	$\rho(K;z)$	-24	0	-6	22	-3	4	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-21	3	-9	27	-3	4	0
9_{29}	$\rho(K;z)$	-26	-6	0	12	-3	4	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-21	3	-9	27	-3	4	0

$Knot$	$Polynomial$	3_1	4_1	5_1	$3_1\#3_1$	7_1	$3_1\#5_1$	9_1
9_{30}	$\rho(K;z)$	3	3	-3	5	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-18	6	-11	29	-3	4	0
9_{31}	$\rho(K;z)$	-23	2	-9	28	-3	4	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-18	6	-12	32	-3	4	0
9_{32}	$\rho(K;z)$	46	5	7	-32	4	-5	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-34	3	-12	38	-4	5	0
9_{33}	$\rho(K;z)$	8	5	-4	5	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-27	6	-14	40	-4	5	0
9_{34}	$\rho(K;z)$	5	5	-6	9	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-20	15	-22	52	-4	5	0
9_{35}	$\rho(K;z)$	54	36	0	0	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	28	21	0	0	0	0	0
9_{36}	$\rho(K;z)$	31	3	-7	-9	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-16	0	1	9	-3	4	0
9_{37}	$\rho(K;z)$	5	3	-3	4	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	31	28	-14	18	0	0	0
9_{38}	$\rho(K;z)$	-59	11	-40	102	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-14	10	-20	45	0	0	0
9_{39}	$\rho(K;z)$	-35	-25	30	-42	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	36	28	-24	33	0	0	0

Knot	*Polynomial*	3_1	4_1	5_1	$3_1\#3_1$	7_1	$3_1\#5_1$	9_1
9_{40}	$\rho(K;z)$	-52	0	-17	54	-5	6	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-40	10	-25	65	-5	6	0
9_{41}	$\rho(K;z)$	17	17	-7	10	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	21	12	-18	27	0	0	0
9_{42}	$\rho(K;z)$	4	2	0	-1	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-4	0	0	1	0	0	0
9_{43}	$\rho(K;z)$	-18	-4	-2	11	2	-4	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	9	3	1	-5	-1	2	0
9_{44}	$\rho(K;z)$	4	2	-1	1	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	3	3	-2	3	0	0	0
9_{45}	$\rho(K;z)$	16	8	-6	7	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	10	6	-4	5	0	0	0
9_{46}	$\rho(K;z)$	0	3	0	0	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	1	3	0	0	0	0	0
9_{47}	$\rho(K;z)$	17	2	-5	1	2	-3	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-10	1	2	2	-2	3	0
9_{48}	$\rho(K;z)$	-15	-7	5	-6	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	16	10	-5	6	0	0	0
9_{49}	$\rho(K;z)$	34	4	-2	-19	0	0	0
	$\frac{1}{z^2}(\nabla^2(K;z)-1)$	-12	0	0	9	0	0	0

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Vita

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