4-7-2022

Characterizations of Certain Classes of Graphs and Matroids

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CHARACTERIZATIONS OF CERTAIN CLASSES OF GRAPHS AND MATROIDS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
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B.S. - M.S., Indian Institute of Science Education and Research, Mohali, 2016
M.S., Louisiana State University, 2018
May 2022

This work is dedicated to my parents.

Acknowledgments

This is the part of my thesis where words might not suffice to convey what I intend them to. I feel extremely grateful that I had the chance to learn from my advisor, James Oxley. James has been a friend and a mentor to me who not only taught me to do research, be a better teacher, and write mathematics effectively, but also offered his help and advice whenever I needed it. He has been so kind, patient, and generous to me through out my Ph.D. that the Australian cricket team is my second favourite cricket team out of my loyalty for James. Judith has been so warm and welcoming as well. Both James and Judith have made me feel at home in Baton Rouge, and I am thankful to have them as my academic parents.

I feel fortunate to have the constant support of the Mathematics Department at LSU. In particular, Guoli Ding has always encouraged my questions and proposed the project on 2-cographs that forms a chapter of this dissertation. I am glad that I had the chance to work with Charles Semple who has been very welcoming. My Computer Science project advisor, Nash Mahmoud, has been really supportive and helped me finish my computer science master's degree in time. I am thankful to have Tara Fife and Zach Gershkoff as my academic siblings. My friends Tara, Kaju, and Irfan Alam were there to help when I needed.

My family has always been my support. I am indebted to my parents for the sacrifices they have made to see me get here and to my sister who has always encouraged her younger brother. I would also like to acknowledge my cousin Pavinder Singh who has been an inspiration to me since my childhood. I am grateful to all my friends who have helped me in my path to this degree. My friends from the MS11 batch at IISER Mohali who al-

ways kept in touch. In particular, my friends Partha Sankar and Deepanshu Sisodia, who were always willing to listen to me and offer help and advice. I also wish to acknowledge my school friends Atul Attri, Rahil Singh, and Muvesh Kumar who have been my trusted confidants since my school days. My friends in Baton Rouge made living so far away from home in a different country much easier. In particular, Divya offered me delicious home-cooked food to make me feel at home in Baton Rouge, and Amrat Gandhi, who was always able to lift my spirits.

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Abstract

"If a theorem about graphs can be expressed in terms of edges and cycles only, it probably exemplifies a more general theorem about matroids." Most of my work draws inspiration from this assertion, made by Tutte in 1979.

In 2004, Ehrenfeucht, Harju and Rozenberg proved that all graphs can be constructed from complete graphs via a sequence of the operations of complementation, switching edges and non-edges at a vertex, and local complementation. In Chapter 2, we consider the binary matroid analogue of each of these graph operations. We prove that the analogue of the result of Ehrenfeucht et. al. does not hold for binary matroids. However, we introduce a fourth operation that does enable the construction of all binary matroids from projective geometries.

A graph in which every connected induced subgraph has a disconnected complement is called a cograph. Such graphs are precisely the graphs that do not have the 4-vertex path as an induced subgraph. In Chapter 3, we define a 2-cograph to be a graph in which the complement of every 2-connected induced subgraph is not 2-connected. The class of 2-cographs is closed under induced minors. We characterize the class of non-2-cographs for which every proper induced minor is a 2-cograph. We further find the finitely many members of this class whose complements are also induced-minor-minimal non-2-cographs. Chapter 4 introduces binary comatroids, a matroid analogue of cographs. We identify all binary non-comatroids for which every proper flat is a binary comatroid. In addition, we extend our results to ternary matroids.

Chapter 1. Introduction

In this dissertation, the terminology will follow Oxley [18] except where indicated otherwise. Terminology for graph theory will follow Diestel [7]. The reader is assumed to have basic familiarity with graph theory and matroid theory. This introductory chapter reviews basic definitions of graph theory and matroid theory that will be used throughout the rest of the dissertation. The treatment here follows [7] and [18].

1.1. Basic definitions

A *graph* G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of *vertices* together with a multiset $E(G)$ of *edges*, each of which consists of an unordered pair of vertices. If $e \in E(G)$ and $e = \{u, v\}$, where u and v are in $V(G)$, then we say that u and v are *neighbours* or are *adjacent*, and that e is *incident* with u and v. We call $V(G)$ and $E(G)$ the *vertex set* and *edge set*, respectively, of the graph G. An edge that joins a vertex to itself is called a *loop*, and edges that join the same pair of distinct vertices are called *parallel edges*. A graph is *simple* if it has no loops and no parallel edges. In this dissertation, we mainly consider simple graphs. A *complete graph* is a graph in which every pair of distinct vertices is joined by a single edge. Two graphs G_1 and G_2 are *isomorphic* if there are bijections $\psi : V(G_1) \to V(G_2)$ and $\theta : E(G_1) \to E(G_2)$ such that a vertex $v \in V(G_1)$ is incident to an edge $e \in E(G_1)$ if and only if $\psi(v)$ is incident to $\theta(e)$ in G_2. In this case, we write $G_1 \cong G_2$.

A graph H is a *subgraph* of a graph G if $V(H)$ and $E(H)$ are subsets of $V(G)$ and $E(G)$, respectively. If V' is a subset of $V(G)$, then $G[V']$ denotes the subgraph of G whose vertex set is V' and whose edge set consists of those edges of G that have both endpoints

in V'. We say that $G[V']$ is the subgraph of G *induced* by V'.

A *walk* in a graph is a sequence $v_0e_1v_1e_2\dots v_{k-1}e_kv_k$ such that $v_0, v_1, \dots, v_k$ are vertices, $e_1, e_2, \dots, e_k$ are edges, and each vertex or edge in the sequence, except v_k, is incident with its successor in the sequence. Now suppose that the vertices $v_0, v_1, \dots, v_k$ are distinct. Then $e_1, e_2, \dots, e_k$ are also distinct and the walk is a *path*. The *end-vertices* or *ends* of this path are v_0 and v_k, and the path is said to be a (v_0, v_k)*-path* or to *join* v_0 and v_k. If P is a (u, v)-path in a graph G and e is an edge of G that joins u to v but is not in P, then the subgraph of G whose vertex set is $V(P)$ and whose edge set is $E(P) \cup \{e\}$ is called a *cycle*. The *length* of a cycle is the number of edges it contains.

In 1935, Whitney and Nakasawa independently introduced matroids to capture abstractly the fundamental properties of dependence that are common to graphs and matrices. A *matroid* is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E, the *ground set*, and a collection $\mathcal{I}$ of subsets of E, called *independent sets*, having the following three properties:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(I3) If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

If $e \in E$, then we say that e is an *element* of M and we write $e \in E(M)$. By (I3), all maximal independent sets have the same size. Such sets are the *bases* of M and the cardinality of a basis is the *rank* of M. A set is called *dependent* if it is not independent, and a *circuit* of M is a minimal dependent set. If $\{e\}$ is a circuit, then e is called a *loop*, and if $\{e\}$ is not contained in any circuits, then e is a *coloop*. If $\{e, f\}$ is a circuit, then we say that e and f are *in parallel*. If every circuit containing e also contains f, and e is not a

coloop, then e and f are *in series*. We call a matroid *simple* if every set with at most two elements is in $\mathcal{I}$. In this dissertation, all matroids are simple unless specified otherwise. A subset X of $E(M)$ is called a *flat* or a *closed set* of M if there is no circuit C of M such that $|C - X| \neq 1$. A *hyperplane* of M is a flat of rank $r(M) - 1$. The smallest closed set cl(X) of M containing the subset X of $E(M)$ is called the *closure* of X in M.

Given a matroid M having ground set E and having $\mathcal{I}$ as its collection of independent sets, there is another matroid, M^*, having the same ground set whose collection, $\mathcal{I}^*$, of independent sets consists of all subsets I^* of $E(M)$ such that $E(M) - I^*$ contains a basis of M. We call M^* the *dual* of M. The circuits of M^* are the *cocircuits* of M.

Let G be a graph with edge set E. Let $\mathcal{I}$ be the collection of subsets X of E such that no cycle of G has its edge set contained in X. Then $(E, \mathcal{I})$ is a matroid, the *cycle matroid*, $M(G)$, of G. A matroid is *graphic* if it is isomorphic to the cycle matroid of some graph. Let A be a matrix over a field $\mathbb{F}$, and let E be the set of column labels of A. Then $(E, \mathcal{I})$, where $\mathcal{I}$ is the collection of subsets I of E such that the columns of I are linearly independent, is a matroid. Such a matroid is called an $\mathbb{F}$-*representable matroid*. In particular, the matroids derived from matrices over the fields $GF(2)$ and $GF(3)$ are called *binary* and *ternary matroids*, respectively. Two matroids M_1 and M_2 are *isomorphic* if there is a bijection $\psi : E(M_1) \to E(M_2)$ such that, for all $X \in E(M_1)$, the set $\psi(X)$ is independent in M_2 if and only if X is independent in M_1. In this case, we write $M_1 \cong M_2$.

1.2. Minors

Recall that a graph H is a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(H)$. We define two particular subgraphs. For $v \in V(G)$, the *deletion* of v from G, de-

noted by $G - v$, is the graph $(V - v, E')$, where E' is the set of edges of G not incident to v. For $e \in E(G)$, the subgraph $(V, E - e)$ is the *deletion* of e from G and is denoted by $G \backslash e$. The *contraction* of e, denoted by G/e, is the graph obtained by identifying the vertices incident to e and removing e from the edge set of G. In particular, if e is a loop, then G/e is equal to $G \backslash e$. The simple graph si(G) associated with a given graph G is obtained from G by deleting all the loops and, for each parallel class of edges, identifying all the edges of the class as a single edge. We call this operation *simplification* of G. It is well known that, for any $e, f \in E$, we have $G/e \backslash f = G \backslash f/e$ and $G \backslash e \backslash f = G \backslash f \backslash e$, as well as $G/e/f = G/f/e$. So, for disjoint sets $X \subseteq E$ and $Y \subseteq E$, we can denote by $G/X \backslash Y$ the graph that is derived from G by, in any order, contracting all the edges of X and deleting all the edges of Y. A vertex is *isolated* if it is not incident to any edges. If H is obtained from $G/X \backslash Y$, for some X and Y, by deleting any number of isolated vertices, then H is a *minor* of G. A minor H of G is a *proper minor* if $H \neq G$. A simple graph H is an *induced minor* of a simple graph G if H can be obtained from G by a sequence of operations each of which consists of deletion of a vertex, or contraction of an edge and then simplifying.

As in graphs, we can remove elements of a matroid M either by deletion or by contraction. For the matroid $M = (E, \mathcal{I})$, the *deletion* $M \backslash e$ of e from M is the matroid $(E - e, \mathcal{I} - e)$ where $\mathcal{I} - e$ is the set of independent sets of M avoiding e. Suppose e is not a loop. Then the *contraction* M/e of e from M is the matroid $(E - e, \mathcal{I}')$ where $\mathcal{I}'$ is the set $\{I \subseteq E - e : I \cup e \in \mathcal{I}\}$. If e is a loop, we define $M/e = M \backslash e$. For a given matroid M, if we delete all the loops from M and then, for each parallel class X, distinguish one element and delete all the other elements of X, the matroid we obtain is uniquely determined up to a renaming of the distinguished elements. We denote this ma-

troid by si(M) and call it the *simplification* of M. As for graphs, for disjoint subsets X and Y of E, the matroid $M/X\backslash Y$ is well-defined and is obtained by contracting the elements of X and deleting the elements of Y from M, in any order. The matroid $M/X\backslash Y$ is a *minor* of M; a minor N of M is a *proper minor* if $N \neq M$. For $X \subseteq E$, the *restriction* of M to X is $M|X = M\backslash(E - X)$ and the *rank* $r_M(X)$ of X is the rank of $M|X$. If there is no ambiguity, we will often write $r(X)$ instead of $r_M(X)$. Deletion and contraction in graphs correspond to deletion and contraction in matroids in that $M(G)\backslash e = M(G\backslash e)$ and $M(G)/e = M(G/e)$. A simple matroid N is an *induced minor* of a simple matroid M if N can be obtained from M by a sequence of operations each of which consists of restricting to a flat, or contracting an element and then simplifying.

1.3. Graph and matroid connectivity

A graph is *connected* if each pair of distinct vertices is joined by a path. A graph that is not connected is *disconnected*. In any graph G, the maximal connected subgraphs are called *(connected) components*. A graph having no cycles is a *forest*, while a connected forest is a *tree*. A *spanning tree* of a connected graph G is a subgraph T of G such that T is a tree and $V(T) = V(G)$.

A subset X of the vertex set if a graph G is called a *vertex cut* if $G - X$ has more connected components than G, where $G - X$ is obtained from G by deleting the vertices in X and all incident edges. If a vertex-cut X contains a single vertex v, then v is called a *cut-vertex* of G. For a connected graph G that has at least one pair of distinct non-adjacent vertices, the *(vertex) connectivity* $\kappa(G)$ of G is the smallest integer j for which G has a j-element vertex cut. When G is connected, but has no pair of distinct non-adjacent

vertices, we take $\kappa(G)$ to be $|V(G)|-1$. Finally, if G is disconnected, we let $\kappa(G)=0$. For a positive integer k, a graph G is said to be k-connected if $\kappa(G) \geq k$.

A matroid is *connected* if, for every distinct pair of distinct elements e and f, there is a circuit containing $\{e, f\}$. For an integer n exceeding one, the notion of n-connectivity is defined in terms of the matroid connectivity function λ. Let M be a matroid with ground set E. If $X \subseteq E$, then

$$\lambda_M(X) = r(X) + r(E-X) - r(M).$$

A *k-separation* is a pair $(X, E-X)$ for which $\lambda_M(X) < k$ and $\min\{|X|, |E-X|\} \geq k$. A matroid is n-connected if it has no k-separations for all positive integers $k < n$. Note that a matroid is 2-connected if and only if it is connected in the sense defined above.

1.4. Projective geometries

Let V be a vector space over a field $\mathbb{F}$. The *projective geometry* $PG(V)$ associated with V is constructed from V by first deleting the zero vector and then, for each 1-dimensional subspace, deleting all but one of the remaining elements. If $V = V(n+1, \mathbb{F})$, then $PG(V)$ has *dimension* n and we denote this projective geometry by $PG(n, \mathbb{F})$. When $\mathbb{F}$ is the finite field $GF(q)$, we usually write $PG(n, q)$ for $PG(n, \mathbb{F})$. In view of the following result [18, Theorem 6.1.3], we see that a study of $\mathbb{F}$-representable simple matroids is a study of the restrictions of projective geometries over $\mathbb{F}$. For every finite subset S of $PG(m-1, \mathbb{F})$, there is a matroid induced on S by linear independence over $\mathbb{F}$. We shall denote this matroid by $PG(m-1, \mathbb{F})|S$.

Theorem 1.4.1. *Let M be a simple rank-r matroid and $\mathbb{F}$ be a field. The following state-*

ments are equivalent:

(i) M *is* $\mathbb{F}$*-representable.*

(ii) $PG(r-1,\mathbb{F})$ *has a finite subset* T *such that* $M \cong PG(r-1,\mathbb{F})|T$.

(iii) *For some* $m \geq r$*, there is a finite subset* S *of* $PG(m-1,\mathbb{F})$ *such that* $M \cong PG(m-1,\mathbb{F})|S$.

Just as every n-vertex simple graph can be obtained from K_n by deleting edges, so too every rank-r simple $\mathbb{F}$-representable matroid can be obtained from the projective geometry $PG(r-1,\mathbb{F})$ by deleting elements.

Chapter 2. Complementation, Local Complementation and Switching in Binary Matroids

2.1. Introduction

The results in this chapter are based on joint work with James Oxley [22]. In this chapter, the only graphs we consider are simple. Ehrenfeucht et al. consider three natural operations on a graph $G = (V, E)$ in [9]. Let K_V be the complete graph on the vertex set V. The *complement* $\omega(G)$ of G is $(V, E(K_V) - E)$. Let x be a vertex of G and $E_x(G)$ be the set of edges of G meeting x. The graph $\sigma_x(G)$, the *switching* of G at x, is $(V, E \triangle E_x(K_V))$. Thus, at x, we interchange edges and non-edges. Let $N_G(x)$ be the set of neighbours of x. The *local complementation* $\lambda_x(G)$ of G at x is $(V, E \triangle E(K_{N_G(x)}))$, that is, in the neighbourhood of x, we interchange edges and non-edges.

Ehrenfeucht et al. [9] showed that complementation can be obtained by a sequence of the operations of switching and local complementation. Their main result is the following.

Theorem 2.1.1. *Every graph on the vertex set V can be obtained from K_V via a sequence of switchings and local complementations.*

In this chapter, we try to generalize this theorem to binary matroids. Just as every n-vertex simple graph is a subgraph of K_n, every simple binary matroid of rank at most r is a restriction of $PG(r-1, 2)$. Moreover, the three graph operations defined above have natural analogues for binary matroids.

Throughout this chapter, we denote the rank-r binary projective geometry $PG(r-1, 2)$ by P_r. We call cocircuits and hyperplanes of P_r *projective cocircuits* and

This chapter was previously published as J. Oxley and J. Singh, “Complementation, local complementation and switching in binary matroids”, *Adv. in Appl. Math.* **126** (2021), 102041, 13pp.

projective hyperplanes, respectively. For a given binary matroid M, we fix a binary projective geometry P_r of which M is a restriction. The *complement* $\omega(M)$ of M in P_r is the matroid on the ground set $E(M) \triangle E(P_r)$, that is, we change the elements of P_r present in the ground set of M to non-elements and vice versa. Observe that the switching of the graph G with respect to a vertex x is obtained by complementing inside the vertex bond of K_V at x. Since bonds in graphs correspond to cocircuits in matroids, a natural binary-matroid generalization of the switching operation in graphs is to complement inside a cocircuit of P_r. The *switching* $\sigma_{C^*}(M)$ of M in P_r with respect to a cocircuit C^* of P_r is the matroid on the ground set $E(M) \triangle C^*$, that is, we change the elements of C^* present in the ground set of M to non-elements and vice versa. The *local complementation* $\lambda_{C^*}(M)$ of M in P_r with respect to a projective cocircuit C^* is the matroid on the ground set $E(M) \triangle (\mathrm{cl}_{P_r}(C^* \cap E(M)) - C^*)$, that is, we complement inside $\mathrm{cl}_{P_r}(C^* \cap E(M)) - C^*$, where cl_{P_r} denotes projective closure, that is, closure in P_r. For example, $\lambda_{C^*}(P_r)$ is the rank-r binary affine geometry, $AG(r-1, 2)$, which we will write as A_r here.

In Section 3, we shall observe that the operations of complementation and switching commute with each other and that a composition of switchings is a switching. We use these observations to characterize the matroids in the same orbit as P_r under the action of the operations of switching and complementation. In Section 4, we show that the switchings with respect to row-cocircuits of some fixed standard representation of P_r generate all switchings.

In Section 5, we show that complementation can be written in terms of switchings and local complementations and that not all binary matroids of rank at most r can be obtained from P_r via a sequence of the operations of complementation, switching, and local

complementation.

In Section 6, we introduce the pointed-swap operation and show that this new operation along with the operations of switching and complementation are enough to transform P_r into any binary matroid of rank at most r. Moreover, we show that any non-empty binary matroid other than $U_{1,1}$ can be obtained from P_r via the operations of local complementation and pointed swaps.

2.2. Preliminaries

Let $\mathcal{M}_r$ denote the set of all binary matroids that are restrictions of a fixed copy of P_r and let $\mathrm{Sym}(\mathcal{M}_r)$ denote the symmetric group on $\mathcal{M}_r$. For any subset A of $E(P_r)$ and a matroid $M \in \mathcal{M}_r$, let $M \triangle A$ denote the matroid on the ground set $E(M) \triangle A$, the symmetric difference of $E(M)$ and A. With the above notation, our three operations with respect to P_r can be written in the following way:

1. **Complementation:** $\omega(M) = M \triangle E(P_r)$.
2. **Switching:** $\sigma_{C^*}(M) = M \triangle C^*$.
3. **Local Complementation:** $\lambda_{C^*}(M) = M \triangle [\mathrm{cl}_{P_r}(E(M) \cap C^*) - C^*]$.

From the above notation, it is clear that $\omega^2, \sigma^2_{C^*}$, and $\lambda^2_{C^*}$ are all identity operations, that is, $\omega^2(M) = \sigma^2_{C^*}(M) = \lambda^2_{C^*}(M) = M$. Note that the operations of switching, complementation, and local complementation generate a subgroup of $\mathrm{Sym}(\mathcal{M}_r)$. The following lemma is an immediate consequence of the fact that the operation of symmetric difference is commutative. It implies that the operations of complementation and switching commute.

Lemma 2.2.1. *For a binary matroid M of rank at most r and $X, Y \subseteq E(P_r)$,*

$$M \triangle (X \triangle Y) = (M \triangle Y) \triangle X.$$

The following is a straightforward consequence of the last lemma.

Lemma 2.2.2. *Let M be a restriction of P_r and C^* be a projective cocircuit. Then $\omega\sigma_{C^*}(M) = M \triangle (E(P_r) - C^*)$.*

We shall refer to the operation $M \triangle (E(P_r) - C^*)$ as *complementation inside the projective hyperplane* $E(P_r) - C^*$. It may appear that the matroid generalization of the switching operation is stronger than the switching operation for graphs since, in graphs, we do complementation inside a vertex bond of the complete graph while, in matroids we do complementation inside any projective cocircuit. The following well-known result shows that this is not the case.

Lemma 2.2.3. *A set B of edges of a complete graph K_V on V is a bond if and only if it can be written as the symmetric difference of some set of vertex bonds of K_V.*

This implies that we can do complementation inside any bond of the complete graph via a sequence of switchings.

2.3. Switching and complementation

In this section, we characterize all the matroids obtainable from P_r using just the operations of complementation and switching. The corresponding result for graphs is already known [8, 10, 27].

Theorem 2.3.1. *The graphs obtainable from a complete graph on n vertices using the operations of complementation and switching consist of all complete bipartite graphs on n vertices together with their complements.*

For binary matroids, we start with the following elementary result.

Lemma 2.3.2. *Let C_1^* and C_2^* be two distinct cocircuits of P_r. Then $C_1^* \triangle C_2^*$ is also a cocircuit of P_r.*

Proof. Note that $C_1^* \triangle C_2^*$ is a disjoint union of cocircuits of P_r. Since $|C_1^* \triangle C_2^*| = 2^{r-1}$ and the number of elements in any projective cocircuit is 2^{r-1}, we deduce that $C_1^* \triangle C_2^*$ is a cocircuit of P_r. □

Recall that we are denoting the operation of complementation by ω. We use ι to denote the identity operator.

Lemma 2.3.3. *Every product of a sequence of the operations of complementation and switching is equal to ι, to ω, to σ_{C^*} for some cocircuit C^* of P_r, or to $\omega\sigma_{C^*}$.*

Proof. Observe that $\sigma_{C_1^*}\sigma_{C_2^*}\ldots\sigma_{C_k^*} = \sigma_{C_1^* \triangle C_2^* \triangle \cdots \triangle C_k^*}$. Since the operations of complementation and switching commute, and both have order two, the result follows. □

This lemma immediately implies the following result.

Proposition 2.3.4. *The binary matroids obtainable from P_r using the operations of complementation and switching are all of the matroids that are isomorphic to one of P_r, $U_{0,0}$, P_{r-1}, and A_r.*

By combining this result with Lemma 2.2.2, we obtain the following.

Corollary 2.3.5. *Two distinct binary matroids M_1 and M_2 are in the same orbit under the action of the group generated by the operations of switching and complementation on $\mathcal{M}_r$ if and only if one of the following holds:*

(i) M_1 can be obtained from M_2 via complementation inside a cocircuit of P_r;

(ii) M_1 can be obtained from M_2 via complementation inside a hyperplane of P_r; or

(iii) M_1 can be obtained from M_2 via complementation inside P_r.

2.4. Local complementation and switching with respect to row cocircuits

In this section, let M denote a restriction of P_r. For local complementation in M, we do the following in order:

(i) Fix a cocircuit C^* of P_r.

(ii) Find the intersection D of C^* with $E(M)$.

(iii) Find the projective closure $\mathrm{cl}_{P_r}(D)$ of D.

(iv) Complement inside $\mathrm{cl}_{P_r}(D) - C^*$.

Note that $\mathrm{cl}_{P_r}(D) - C^*$ is a flat of P_r and is therefore a smaller projective geometry. Thus local complementation in a matroid takes the complement inside a smaller projective geometry just as, in a graph, local complementation takes the complement inside a smaller complete graph.

Next, we show that, in performing a local complementation in M, we can focus initially on any disjoint union of cocircuits of M.

Lemma 2.4.1. *Let D be a non-empty disjoint union of cocircuits of M. Then there is a projective cocircuit C^* such that $C^* \cap E(M) = D$.*

Proof. Observe that $M = P_r \backslash T$ for some $T \subseteq E(P_r)$. Let D be the disjoint union of cocircuits $D_1^*, D_2^*, \dots, D_k^*$ of M. Note that, for all $1 \leq i \leq k$, there is a projective cocircuit C_i^* of P_r such that $D_i^* = C_i^* - T$. Then $D = D_1^* \triangle D_2^* \triangle \cdots \triangle D_k^* = (C_1^* - T) \triangle (C_2^* - T) \triangle \cdots \triangle (C_k^* - T) = (C_1^* \triangle C_2^* \triangle \cdots \triangle C_k^*) - T$. The result follows. □

In view of this lemma, every local complementation in M can be achieved as follows:

(i) Find a non-empty disjoint union D of cocircuits in M.

(ii) Find $\mathrm{cl}_{P_r}(D)$.

(iii) Find a hyperplane H of $\mathrm{cl}_{P_r}(D)$ avoiding D.

(iv) Complement inside H.

Switchings in the graph case are only done with respect to the vertex bonds. However, by Lemma 2.2.3, via those switchings, we can complement inside any arbitrary bond of the complete graph. In the matroid case, we have been allowing switchings relative to any projective cocircuit. Next, we note that, just as for graphs, we can restrict ourselves to doing switchings with respect to a small number of cocircuits.

Fix a standard representation $[I_r|D]$ of P_r with respect to the basis B. Row i of this matrix is the incidence vector of a cocircuit of P_r, a fundamental cocircuit with respect to the cobasis $E(P_r) - B$. We call such a cocircuit a *row-cocircuit.* The following lemma is well-known (see, for example, [18] Proposition 9.2.2).

Lemma 2.4.2. *Every cocircuit C of P_r can be written as a symmetric difference of row-cocircuits.*

This lemma has the following immediate consequence.

Proposition 2.4.3. *Switching with respect to an arbitrary cocircuit C^* of P_r can be expressed in terms of switchings with respect to row-cocircuits of P_r.*

2.5. Not all binary matroids are obtainable

In this section, we show that, unlike for graphs, we cannot obtain all binary matroids of rank at most r from P_r using the three generalized operations.

The following result shows that the effect on a matroid M of a local complementa-

tion with respect to a projective cocircuit C^* is the same as performing complementation in the complementary projective hyperplane $E(P_r) - C^*$ if $E(M) \cap C^*$ has rank r.

Lemma 2.5.1. *Let C^* be a cocircuit of P_r and M be a restriction of P_r such that $E(M) \cap C^*$ has rank r. Then*

$$\lambda_{C^*}(M) = \omega\sigma_{C^*}(M).$$

Proof. As $r(E(M) \cap C^*) = r$, we see that $\mathrm{cl}_{P_r}(E(M) \cap C^*) = E(P_r)$. Thus $\lambda_{C^*}(M) = M\triangle(E(P_r) - C^*)$. By Lemma 2.2.2, the last matroid is $\omega\sigma_{C^*}(M)$. □

The next result shows that the operation of complementation is redundant and can be expressed in terms of the operations of switching and local complementation. Recall that complementation and switching commute.

Proposition 2.5.2. *Let N be a matroid that can be obtained from P_r using the operations of complementation, switching, and local complementation. Then the operations of switching and local complementation are enough to obtain N from P_r.*

Proof. It is enough to show that complementation of an arbitrary restriction M of P_r can be written in terms of switchings and local complementation. Suppose $r(M) < r$. Then there is a projective cocircuit C^* of P_r such that $C^* \cap E(M) = \emptyset$. Hence $C^* \cap E(\sigma_{C^*}(M)) = C^*$. As every cocircuit of P_r is spanning, by Lemma 2.5.1, we have $\lambda_{C^*}(\sigma_{C^*}(M)) = \omega\sigma_{C^*}(\sigma_{C^*}(M))$. Thus $\lambda_{C^*}(\sigma_{C^*}(M)) = \omega(M)$. We may now assume that $r(M) = r$. Then M contains a basis B of P_r. Since every projective cocircuit contains a basis of P_r, by the symmetry of P_r, the basis B is contained in a projective cocircuit, say C^*. Then, using Lemma 2.5.1 again, we get $\omega(M) = \sigma_{C^*}\lambda_{C^*}(M)$. □

The following is the main result of this section.

Theorem 2.5.3. *For fixed r exceeding seven, not all binary matroids of rank at most r can be obtained from P_r using the operations of complementation, local complementation, and switching.*

Proof. Consider a restriction M of P_r. We say that M has *Property 1* if, for every two distinct cocircuits C^* and D^* of P_r, both $(C^* - D^*) \cap E(M)$ and $(C^* - D^*) - E(M)$ have rank $r - 1$. Instead, if, for every cocircuit C^* of P_r, both $C^* \cap E(M)$ and $C^* - E(M)$ have rank r, then we say that M has *Property 2*.

2.5.3.1. *If M has Property 1, then it has Property 2.*

Let C^* be an arbitrary cocircuit of P_r. Let $e \in C^* \cap E(M)$. Observe that there is a different projective cocircuit D^* such that $e \in C^* \cap D^*$. Since M has Property 1, $r((C^* - D^*) \cap E(M)) = r - 1$. As e is not in the projective closure of $C^* - D^*$, we deduce that $r((C^* - D^*) \cap E(M)) < r(((C^* - D^*) \cap E(M)) \cup e)$, so $r(C^* \cap E(M)) = r$. Similarly, $r(C^* - E(M)) = r$ unless $E(M) \supseteq C^* \cap D^*$. In the exceptional case, since, by Lemma 2.3.2, $C^* \triangle D^*$ is a cocircuit of P_r, and $C^* \cap D^* = C^* - (C^* \triangle D^*)$, we get a violation of Property 1. Hence 2.5.3.1 holds.

We will use probabilistic methods for the rest of the proof. Independently colour each element of P_r green or red with equal probability. Let G and R denote the sets of green and red elements, respectively. This gives $2^{2^r - 1}$ members of a sample space, $\mathbb{S}$, of all possible 2-colourings of P_r.

2.5.3.2. *There is a 2-colouring, say X, of P_r such that, for all distinct cocircuits C^* and D^* of P_r, both $(C^* - D^*) \cap G$ and $(C^* - D^*) \cap R$ have rank $r - 1$.*

Call a 2-colouring of P_r bad if, for some pair of distinct cocircuits, C^* and D^*, of P_r, either $(C^* - D^*) \cap G$ or $(C^* - D^*) \cap R$ has rank less than $r-1$. Note that $P_r|(C^* - D^*)$ is isomorphic to A_{r-1}. Moreover, $(C^* - D^*) \cap G$ or $(C^* - D^*) \cap R$ has rank less than $r-1$ if and only if it is contained in an A_{r-2}. Thus a 2-colouring is bad if and only if there is a monochromatic A_{r-2}. Therefore, the number of bad colourings is at most twice the product of the number of copies of A_{r-2} in P_r and the number of subsets of $E(P_r) - E(A_{r-2})$. The number of copies of A_{r-2} in P_r is the product of the number of copies of P_{r-2} in P_r and the number of hyperplanes of P_{r-2}. Thus the number of bad colourings is at most

$$\frac{2(2^r-1)(2^{r-1}-1)(2^{r-2}-1)(2^{2^r-1-2^{r-3}})}{3}.$$

Thus the probability that P_r has a bad colouring is at most

$$\frac{2(2^r-1)(2^{r-1}-1)(2^{r-2}-1)}{3(2^{2^{r-3}})}.$$

For $r > 7$, this probability is less than 1, so 2.5.3.2 holds.

Now, for the 2-colouring X, let $P_r|G = M$. Then, by 2.5.3.1 and 2.5.3.2, M satisfies Properties 1 and 2. Note that $\omega(M)$ satisfies Properties 1 and 2; and $\sigma_{C^*}(M)$ satisfies Property 2 for all projective cocircuits C^*. Moreover, since M satisfies Property 2, by Lemma 2.5.1, $\lambda_{C^*}(M) = \omega\sigma_{C^*}(M)$ and so $\lambda_{C^*}(M)$ also satisfies Property 2. Now suppose that α is a sequence of n operations, each a complementation, a switching, or a local complementation. We show next that

2.5.3.3. *$\alpha(M)$ is M, $\omega(M)$, $\sigma_{C^*}(M)$, or $\sigma_{C^*}\omega(M)$, for some projective cocircuit C^*, and so $\alpha(M)$ satisfies Property 2.*

We obtain this from Lemma 2.3.3 by showing, by induction on n, that α can be written as a sequence of switchings and complementations. This was noted above for $n = 1$. Assume it holds for $n < k$ and let $n = k \geq 2$. Then $\alpha = \gamma\beta$ where β is the product of the first $k-1$ operations and γ is the k^{th} operation. By the induction assumption, $\beta(M)$ has the specified form and satisfies Property 2. Thus, so does $\alpha(M)$ unless $\gamma = \lambda_{D^*}$. But, in the exceptional case, by Lemma 2.5.1, $\alpha(M) = \lambda_{D^*}\beta(M) = \omega\sigma_{D^*}\beta(M)$ and the assertion again follows by Lemma 2.3.3. We conclude that 2.5.3.3 holds.

It follows that any matroid that can be obtained from M by a sequence of the operations of complementation, switching, and local complementation satisfies Property 2. Since each of these operations has order two and P_r cannot be obtained from M, we see that M cannot be obtained from P_r. $\square$

It is not difficult to check that, for r in $\{1, 2, 3\}$, all binary matroids of rank at most r can be obtained from P_r using the given operations. The next proposition establishes that this is also true when r is 4. By Theorem 3.1.3, the corresponding result fails when r is at least 8. We do not know what happens when $r \in \{5, 6, 7\}$.

Proposition 2.5.4. *All binary matroids of rank at most four can be obtained from P_4 using the operations of complementation, switching, and local complementation.*

Proof. Because we can use complementation inside of projective hyperplanes, to see that every matroid of rank at most three can be obtained, it suffices to show that a member of each of the following pairs can be obtained from P_4: $\{P_3, U_{0,0}\}$, $\{M(K_4), U_{1,1}\}$, $\{P(U_{2,3}, U_{2,3}), U_{2,2}\}$ $\{U_{3,4}, U_{2,3}\}$, $\{U_{2,3} \oplus U_{1,1}, U_{3,3}\}$. Now $\sigma_{C^*}(P_4) = P_3$. A local complementation in P_3 gives $U_{3,4}$. Doing a local complementation using a 2-cocircuit of $U_{3,4}$ gives

$M(K_4\backslash e)$. In this matroid, there is 3-cocircuit and a 4-element set that is a disjoint union of two 2-cocircuits. Local complementation with respect to the first of these sets gives $U_{3,3}$; local complementation with respect to the second gives $M(K_4)$. We conclude that every matroid of rank at most three is obtainable from P_4.

Assume the result fails and let M be a minimum-sized matroid that cannot be obtained from P_4. Observe that $r(M) = 4$, $|E(M)| \leq 7$, and that every hyperplane of M has exactly three elements. First, we show that M is not isomorphic to $U_{4,4}$. Observe that there is a projective cocircuit C^* such that $|C^* \cap E(U_{4,4})| = 3$. Performing a local complementation with respect to C^* transforms $U_{4,4}$ to $U_{1,1} \oplus M(K_4)$. Now, complementation in the hyperplane containing the ground set of $M(K_4)$ gives us $U_{2,2}$. Since $U_{2,2}$ can be obtained from P_4, so too can $U_{4,4}$.

Observe that if M has a coloop, then $M \cong U_{4,4}$. Assume all cocircuits of M have at least three elements. Then M^* is simple of corank four and rank at most three. Thus $M \cong F_7^*$. Note that $\omega(F_7^*) = F_7 \oplus U_{1,1}$. Now, complementation in the hyperplane F_7 gives us $U_{1,1}$ and therefore, F_7^* can be obtained from P_4 using the given operations.

We may now assume that M has a 2-element cocircuit. Since M has no coloops, $M \cong U_{4,5}$. Using local complementation with respect to a 2-cocircuit, we can transform $U_{4,5}$ into $P(U_{3,4}, U_{2,3})$. Now, doing a complementation in the hyperplane containing the ground set of $U_{3,4}$ gives us $U_{2,3} \oplus U_{2,2}$. Finally, performing a local complementation using a 2-cocircuit of $U_{2,3} \oplus U_{2,2}$, we obtain $U_{4,4}$ implying that $U_{4,5}$ is obtainable via the given operations, a contradiction. $\square$

2.6. More operations

In this section, we introduce a new operation. Let M be a restriction of P_r. Colour an element e of P_r green if $e \in E(M)$ and red otherwise. Denote the sets of green and red elements of P_r by G and R, respectively. Note that $E(M) = G$. Let f be an element of P_r. For every line $L = \{f, f', f''\}$ passing through f, we consider $\{f', f''\}$, and swap their colours if these colours are different. We call this operation a *pointed swap with respect to* f. It is an *on-element swap* if $f \in G$ and an *off-element swap* otherwise. We use $\psi_f^+(M)$ and $\psi_f^-(M)$ to denote the matroids obtained from M by doing on-element and off-element swaps with respect to f.

This new operation can also be viewed in terms of matrices. Let A be an $r \times t$ matrix over $GF(2)$ representing M such that the columns of A correspond to the green elements of P_r. Let v be a binary vector of length r corresponding to an element f of P_r. By $A \hat{+} v$, we denote the matrix obtained by adding v to every column of A that is distinct from v. Observe that $M[A \hat{+} v]$ is the matroid obtained from $M[A]$ by a pointed swap with respect to f. This operation is an on-element swap if $f \in G$ and an off-element swap otherwise.

Lemma 2.6.1. *Let M be a t-element matroid that is a restriction of P_r. Then every t-element restriction of P_r can be obtained from M using pointed swaps.*

Proof. Let $A = [v_1, v_2, \dots, v_t]$ be a matrix representing M, so the columns of A correspond to the green elements of P_r. Note that it is enough to show that, for any vector w corresponding to a red element e of P_r and an arbitrary green element f of P_r corresponding to the column v_k, we can use pointed swaps to obtain a matroid M' from M such that

$M' = P_r|(E(M) \triangle \{e, f\})$. Doing an off-element swap on M with respect to w, we get

$$[v_1, \ldots, v_k, \ldots, v_r] \xrightarrow{\psi_w^-} [v_1 + w, \ldots, v_k + w, \ldots, v_r + w].$$

Now, doing an on-element swap with respect to $v_k + w$, we get

$$[v_1 + w, \ldots, v_r + w] \xrightarrow{\psi_{v_k+w}^+} [v_1 + v_k, \ldots, v_k + w, \ldots, v_r + v_k].$$

Finally, doing an off-element swap with respect to v_k, we get

$$[v_1 + v_k, \ldots, v_k + w, \ldots, v_r + v_k] \xrightarrow{\psi_{v_k}^-} [v_1, \ldots, w, \ldots, v_r].$$

□

Next we show that the operation of complementation can be obtained by three complementations inside of projective hyperplanes.

Lemma 2.6.2. *For $r > 1$, let H_1, H_2, H_3 be three projective hyperplanes that contain a fixed rank-$(r-2)$ flat of P_r. Then $\omega(M) = M \triangle H_1 \triangle H_2 \triangle H_3$.*

Proof. By Lemma 2.2.1, $M \triangle H_1 \triangle H_2 \triangle H_3 = M \triangle (H_1 \triangle H_2 \triangle H_3)$. But $H_1 \triangle H_2 \triangle H_3 = E(P_r)$, so $M \triangle H_1 \triangle H_2 \triangle H_3 = \omega(M)$. □

Theorem 2.6.3. *For $r > 1$, all binary matroids of rank at most r can be obtained from P_r via a sequence of the operations of pointed swaps and complementation inside projective hyperplanes.*

Proof. Assume the theorem fails and let M be a matroid with minimum-sized ground set that cannot be obtained from P_r via the specified operations. First, we show that $U_{1,1}$ can be obtained from P_r, and so M has at least two elements. Doing a complementation inside a hyperplane of P_r, we obtain a matroid N that is isomorphic to A_r and thus has 2^{r-1} elements. As $P_{r-1} \oplus U_{1,1}$ also has 2^{r-1} elements, Lemma 2.6.1 implies we can get a matroid isomorphic to $P_{r-1} \oplus U_{1,1}$ from N via pointed swaps. Now, complementation inside the hyperplane P_{r-1} gives us $U_{1,1}$.

Let x, y be two distinct elements in the ground set of M and let C^* be a projective cocircuit containing x and y. Let $H = E(P_r) - C^*$. Recall that x and y are coloured green in P_r. We may assume that H has at least one red element, say z, otherwise, by complementation inside of H, we reduce the number of green elements. Note that the colours of any pair of differently coloured elements of P_r can be interchanged by a single pointed swap. Thus we can swap the colours of x and z to get a matroid M' such that $E(M') = E(M) \triangle \{x, y\}$. Lemma 2.6.2 allows us to perform complementation on M' followed by complementation inside the hyperplane H. Observe that, after this operation, y becomes a red element and x becomes a green element. Again, using pointed swaps, we swap the colours of y and z. Finally, doing switching with respect to C^* gives us a matroid M'' such that $E(M'') = E(M) - \{x, y\}$. Since $|E(M'')| < |E(M)|$, M'' is obtainable from P_r using the allowed operations and so, M is obtainable as well, a contradiction. □

We now show that the operations of on-element and off-element swaps are linked via complementation.

Lemma 2.6.4. *Let M be a restriction of P_r and u be an element of P_r. Then*

$$\psi_u^+(M) = \omega\psi_u^-\omega(M) \quad \textit{when } u \in E(M),$$

and

$$\psi_u^-(M) = \omega\psi_u^+\omega(M) \quad \textit{when } u \notin E(M).$$

Proof. To show the first part, it suffices to show that $\omega\psi_u^+(M) = \psi_u^-\omega(M)$. In considering $\psi_u^+(M)$, we note that u must be green. The following statements are equivalent for an element e of P_r:

(i) e is green in $\omega\psi_u^+(M)$;

(ii) e is red in $\psi_u^+(M)$;

(iii) $e+u$ is red in M;

(iv) $e+u$ is green in $\omega(M)$;

(v) e is green in $\psi_u^-\omega(M)$.

Thus the first part holds.

For the second part, let u be red in P_r. Then u is green in $\omega(M)$. Thus, by the first part, $\psi_u^+(\omega(M)) = \omega\psi_u^-\omega(\omega(M))$, so $\psi_u^+\omega(M) = \omega\psi_u^-(M)$. Hence $\omega\psi_u^+\omega(M) = \psi_u^-(M)$ and the second part holds. □

By combining the last three results we obtain the following.

Theorem 2.6.5. *Let P_r be a binary projective geometry of rank r greater than one. Then all binary matroids of rank at most r can be obtained from P_r via a sequence of the operations of pointed swaps and complementation in hyperplanes, where either all such swaps are on-element swaps, or they are all off-element swaps.*

Proof. By the last lemma, when one has the operation of complementation, one needs only

on-element or off-element swaps but not both. By Lemma 2.6.2, complementation can be achieved via a sequence of complementations inside of projective hyperplanes. The theorem follows from Theorem 2.6.3. □

Theorem 2.6.6. *Let P_r be a binary projective geometry of rank r and M be a binary matroid of rank at most r such that M is not isomorphic to $U_{0,0}$ or $U_{1,1}$. Then M can be obtained from P_r via a sequence of the operations of local complementation and pointed swaps.*

Our proof of this theorem will use the following two lemmas.

Lemma 2.6.7. *Let M be a k-element restriction of P_r such that e and f are coloops of M. Then there is a $(k+1)$-element matroid M' that can be obtained from M via local complementation.*

Proof. Let $L = \{e, f, g\}$ be the line containing e and f in P_r. By Lemma 2.4.1, there is a projective cocircuit C^* such that $C^* \cap E(M) = \{e, f\}$. Since $|C^* \cap L|$ cannot be odd, $g \notin C^*$. Note that $g \notin E(M)$ and $E(\lambda_{C^*}(M)) = E(M) \cup g$. Therefore, the result holds. □

Lemma 2.6.8. *Let $r \geq 3$. For an integer k in $[2, 2^{r-2}+1]$, there is a restriction M of P_r such that $|E(M)| = k$ and M has two coloops.*

Proof. Let F be a flat of P_r of rank $r-2$ and M be a restriction of P_r such that $|E(M)| = |E(M|F)| + 2$ and $r(M) = r(M|F) + 2$. Note that M has two coloops. The maximum number of elements M can have is $|F| + 2$, that is, $2^{r-2} + 1$. Therefore, our result follows.

□

Proof of Theorem 2.6.6. We can check that the result is true for $r = 1, 2,$ and 3. There-

fore, we may assume that $r \geq 4$. We say $M_1 \sim M_2$ if M_1 is obtainable from M_2 via the given operations. Since both the operations have order two, the relation $\sim$ is symmetric and so is an equivalence relation.

First, we show that all binary matroids with at least two and at most $2^{r-2}+2$ elements can be obtained from $U_{2,2}$ via the operations above. Given a matroid M with at least two coloops and at most $2^{r-2}+1$ elements altogether, Lemma 2.6.7 implies that we can obtain a matroid with $|E(M)|+1$ elements. Thus, for all integers k in $[2, 2^{r-2}+2]$, we can construct a k-element matroid starting with $U_{2,2}$. Hence, by Lemma 2.6.1, using pointed swaps, we can construct every k-element matroid from $U_{2,2}$.

Let M be a matroid with $2^{r-2}+2$ elements. Via pointed swaps, we can obtain a rank-r matroid M' from M such that $E(M') \subseteq C^*$ for a projective cocircuit C^*. Note that $\lambda_{C^*}(M')$ has size $2^{r-2}+2^{r-1}+1$ and is obtainable from $U_{2,2}$.

We now show the following.

2.6.6.1. *For $0 \leq k \leq 2^{r-1}-r$, every binary matroid with $(2^r-1)-2k$ elements can be obtained from P_r using the given operations.*

Recall that an element e of P_r is coloured green if e is in the ground set of the present matroid and red otherwise. We start with P_r and do a local complementation with respect to a projective cocircuit C^* to obtain a matroid N whose ground set is C^*. Let B be a basis of P_r contained in C^*. Pick k-element subsets of each of C^*-B and $E(P_r)-C^*$ and, by Lemma 2.6.1, pointed swaps enable us to interchange the colours on these $2k$ elements to get a matroid N'. Now, we do a local complementation in N' with respect to C^* to obtain a matroid N'' with $(2^r-1)-2k$ elements. Thus 2.6.6.1 holds.

The above implies that all matroids of odd size between $2^r - 1$ and $2r - 1$ can be obtained from P_r using the allowed operations. Therefore, the $(2^{r-2} + 2^{r-1} + 1)$-element matroid $\lambda_{C^*}(M')$, which was constructed above from $U_{2,2}$, can also be constructed from P_r. Since $\sim$ is an equivalence relation, from P_r, we can obtain all matroids with t elements for all t in $[2, 2^{r-2} + 2]$. This implies that all matroids other than $U_{1,1}$ that have an odd number of elements are obtainable from P_r.

Finally, we show that all matroids with a non-zero even number of elements are obtainable from P_r. Since $|C^*| - 1$ is odd for a projective cocircuit C^*, we can obtain a matroid N from P_r such that $E(N) = C^* - \{e\}$. Let k be an integer in $[0, 2^{r-1} - r - 1]$. Let B be a basis of P_r contained in $C^* - \{e\}$. Pick k-element subsets of each of $C^* - (B \cup e)$ and $E(P_r) - C^*$ and, using pointed swaps, interchange the colours on these $2k$ elements to get a matroid N'. Now, we do a local complementation in N' with respect to C^* to obtain a matroid N'' of size $(2^r - 2) - 2k$. Observe that, when $k = 2^{r-1} - r - 1$, the matroid N'' has $2r$ elements. Since $r \geq 4$ and all matroids having at most $2^{r-2} + 2$ elements are obtainable from P_r, the matroids having even size less than $2r$ are also obtainable from P_r. This completes the proof. $\square$

Chapter 3. 2-Cographs

3.1. Introduction

The results in this chapter are based on joint work with James Oxley [23]. All of the graphs considered in this chapter are simple. A *cograph* is a graph in which every connected induced subgraph has a disconnected complement. By convention, the graph K_1 is taken to be a cograph. Replacing connectedness by 2-connectedness, we define a graph G to be a 2-*cograph* if G has no induced subgraph H such that both H and its complement, $\overline{H}$, are 2-connected. Note that K_1 is a 2-cograph. Cographs have been extensively studied over the last fifty years (see, for example, [12, 28, 6]). They are also called P_4-free graphs due to following characterization [5].

Theorem 3.1.1. *A graph G is a cograph if and only if G does not contain the path P_4 on four vertices as an induced subgraph.*

A consequence of the fact that we consider only simple graphs here is that, when we write G/e for an edge e of a graph G, we mean the graph that we get from the graph obtained by contracting the edge e by deleting all but one edge from each class of parallel edges. In Section 2, we show that 2-cographs can be recursively defined, that every induced minor of a 2-cograph is also a 2-cograph, and that the complement of every 2-cograph is also a 2-cograph. In addition, we correct a result of Akiyama and Harary [1] that had claimed to characterize when the complement of a 2-connected graph is 2-connected.

Because the class of 2-cographs is closed under induced minors, our initial goal was to find all non-2-cographs with the property that every proper induced minor is a 2-

cograph. But, as we show in Section 3, in contrast to Theorem 3.1.1, there are infinitely many such non-2-cographs. However, we were able to determine all infinite families of such graphs. For all $k \geq 1$, let M_k and N_k be the graphs shown in Figures 3.3 and 3.4, respectively. Let M_k' and N_k' be obtained from M_k and N_k by adding the edge st. Further, let N_k'' be the graph obtained from N_k' by adding the edge uz; let L_k be the graph shown in Figure 3.2; and, for all $j \geq 0$, let F_j be the graph shown in Figure 3.1. The next two theorems are the main results of the chapter.

Theorem 3.1.2. *Let G be a graph that is not a 2-cograph such that every proper induced minor of G is a 2-cograph. Then*

(i) $|V(G)| \leq 16$*; or*

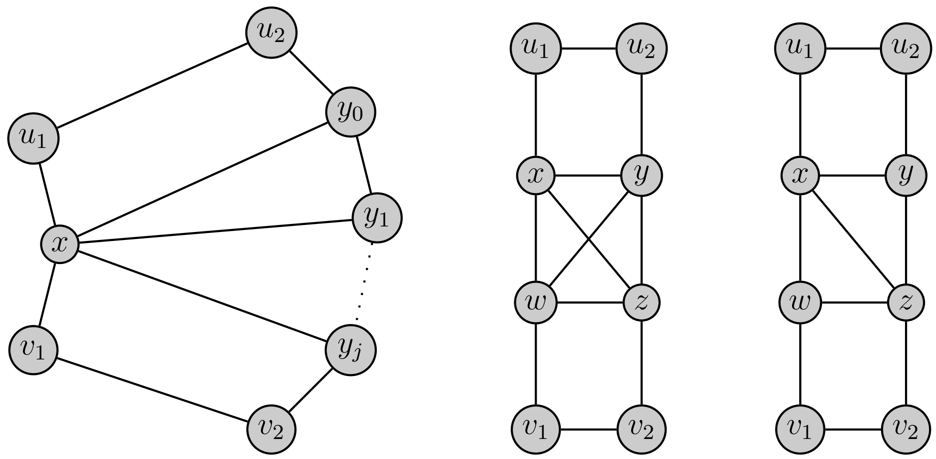
(ii) G is the complement of a cycle of length at least five; or

(iii) for some positive integer k, the complement of G is isomorphic to F_{k-1}, L_k, M_k, M_k', $N_k, N_k', or N_k''$.

As we were unable to improve this bound of 16 vertices and the task of finding induced-minor-minimal non-2-cographs with at most 16 vertices seemed computationally infeasible, we were prompted to try to determine those graphs G for which both G and $\overline{G}$ are induced-minor-minimal non-2-cographs. The following theorem proves that, up to isomorphism, there are finitely many such graphs G. Its proof occupies most of Section 4.

Theorem 3.1.3. *Let G be a graph. Suppose that G is not a 2-cograph but that every proper induced minor of each of G and $\overline{G}$ is a 2-cograph. Then $5 \leq |V(G)| \leq 10$.*

The unique 5-vertex graph satisfying the hypotheses of the last theorem is C_5, the 5-vertex cycle. In the final section, we list all of the other graphs that satisfy these hypotheses.



F_j ($j \geq 0$) H_1 H_2

Figure 3.1. The complements of the induced-minor-minimal non-2-cographs that are critically 2-connected.

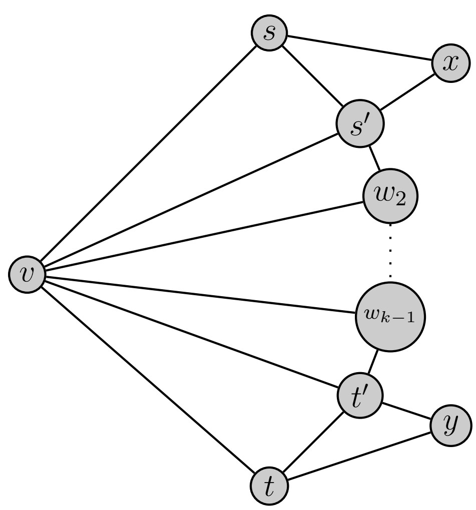


Figure 3.2. For each $k \geq 1$, the complement of the above graph L_k is an induced-minor-minimal non-2-cograph.

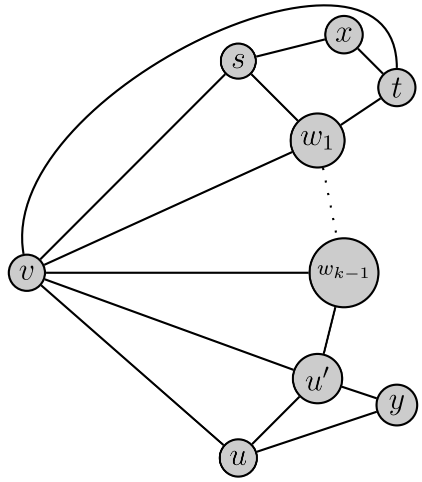


Figure 3.3. M_k, a graph whose complement is an induced-minor-minimal non-2-cograph.

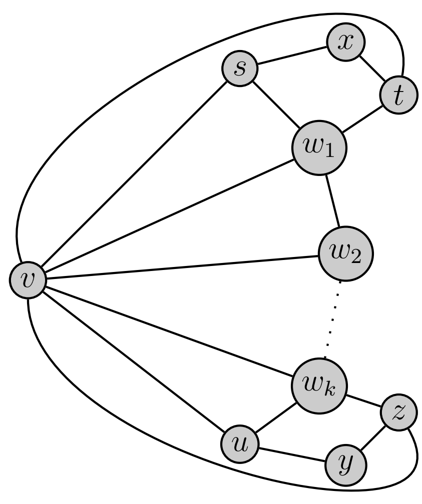


Figure 3.4. N_k, a graph whose complement is an induced-minor-minimal non-2-cograph.

3.2. Preliminaries

Let G be a graph. The *neighbourhood* $N_G(v)$ of v in G is the set of all neighbours of v in G. Viewing G as a subgraph of K_n where $n = |V(G)|$, we colour the edges of G green while assigning the colour red to the non-edges of G. In this chapter, we use the terms *green graph* and *red graph* for G and its complementary graph $\overline{G}$, respectively. An edge of G is called a *green edge* while a *red edge* refers to an edge of $\overline{G}$. The *green degree* of a vertex v of G is the number of *green neighbours* of v, while the *red degree* of v is its number of *red neighbours*.

Let G_1 and G_2 be graphs. If their vertex sets are disjoint, their 0-*sum*, $G_1 \oplus_0 G_2$, is their disjoint union. Now, suppose that $V(G_1) \cap V(G_2) = T$, that $G_1[T] = G_2[T]$, and that $|T| = t$. Then the union of G_1 and G_2, which has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$, is a t-*sum*, $G_1 \oplus_t G_2$, of G_1 and G_2.

For $k \geq 1$, a graph G is a k-*cograph* if, for every induced subgraph H of G, at least one of H and $\overline{H}$ is not k-connected. Thus a 1-cograph is just a cograph. Clearly, every k-cograph is also a $(k+1)$-cograph.

We omit the straightforward proofs of the next three results.

Lemma 3.2.1. *Let G be a k-cograph.*

(i) Every induced subgraph of G is a k-cograph.

(ii) $\overline{G}$ is a k-cograph.

Lemma 3.2.2. *For $0 \leq t < k$, a t-sum of two k-cographs is a k-cograph.*

Lemma 3.2.3. *Let G be a graph and let uv be an edge e of G. Then $\overline{G/e}$ is the graph obtained by adding a vertex w with neighbourhood $N_{\overline{G}}(u) \cap N_{\overline{G}}(v)$ to the graph $\overline{G} - \{u, v\}$.*

Cographs are also called complement-reducible graphs due to the following

recursive-generation result [5]. The operation of taking the complement of a graph is called *complementation.*

Lemma 3.2.4. *A graph G is a cograph if and only if G can be generated from K_1 using complementation and 0-sum.*

Next, we show that, for $k \geq 2$, the class of k-cographs can be generated similarly.

Lemma 3.2.5. *For all positive integers k, a graph G is a k-cograph if and only if G can be generated from K_1 using complementation and the operation of t-sum for all t with $0 \leq t < k$.*

Proof. Let G be a k-cograph. If $|V(G)| \leq 2$, the result holds. We proceed via induction on the number of vertices of G. Assume that the result holds for all k-cographs of order less than $|V(G)|$. Since G is a k-cograph, G or $\overline{G}$ is not k-connected. Without loss of generality, we may assume that G is not k-connected. Therefore, for some $t < k$, we can write G as a t-sum of two induced subgraphs G_1 and G_2 of G. By Lemma 3.2.1, G_1 and G_2 are k-cographs and the result follows by induction.

Conversely, let G be a graph that can be generated from K_1 using complementation and t-sums. Since K_1 is a k-cograph, the result follows by Lemmas 3.2.1 and 3.2.2. □

The following recursive-generation result for cographs is due to Royle [25]. It uses the concept of *join* of two disjoint graphs G and H, which is the graph $G \bigtriangledown H$ that is obtained from the union of G and H by joining every vertex of G to every vertex of H.

Lemma 3.2.6. *Let $\mathcal{C}$ be the class of graphs defined as follows:*

(i) K_1 *is in* $\mathcal{C}$;

(ii) if G *and* H *are in* $\mathcal{C}$*, then so is* $G \oplus_0 H$*; and*

(iii) if G and H are in $\mathcal{C}$, then so is $G \bigtriangledown H$.

Then $\mathcal{C}$ is the class of cographs.

For graphs G and H such that $V(G) \cap V(H) = T$ and $G[T] = H[T]$, suppose that $|T| = t$. We generalize the join operation letting $G \bigtriangledown_t H$ be the graph that is obtained from the union of G and H by joining every vertex of $V(G) - V(H)$ to every vertex of $V(H) - V(G)$. Note that $G \bigtriangledown_t H$ is the graph $\overline{\overline{G} \oplus_t \overline{H}}$.

The next result generalizes Lemma 3.2.6 to k-cographs.

Proposition 3.2.7. *For $k \geq 1$, let $\mathcal{C}$ be the class of graphs defined as follows:*

(i) K_1 is in $\mathcal{C}$;

(ii) if G and H are in $\mathcal{C}$, then so is $G \oplus_t H$ for all t with $0 \leq t < k$; and

(iii) if G and H are in $\mathcal{C}$, then so is $G \bigtriangledown_t H$ for all t with $0 \leq t < k$.

Then $\mathcal{C}$ is the class of k-cographs.

Proof. Since $G \bigtriangledown_t H$ can be written in terms of t-sum and complementation, every graph in $\mathcal{C}$ is a k-cograph. Conversely, let G be a k-cograph. If $|V(G)| = 1$, then $G \in \mathcal{C}$. We proceed by induction on $|V(G)|$. Let $|V(G)| = n \geq 2$ and assume that $H \in \mathcal{C}$ when H is a k-cograph with $|H| < n$. By Lemma 3.2.5, G or $\overline{G}$ is a t-sum of two smaller k-cographs. If G is the graph that can be decomposed as a t-sum, then the result follows by induction. Therefore we may assume that $\overline{G}$ is $G_1 \oplus_t G_2$ for two smaller k-cographs G_1 and G_2. Observe that $G = \overline{G_1} \bigtriangledown_t \overline{G_2}$. By Lemma 3.2.1, $\overline{G_1}$ and $\overline{G_2}$ are k-cographs and so are in $\mathcal{C}$ by induction. Therefore G is in $\mathcal{C}$. $\square$

Next we show that the class of 2-cographs is closed under contractions.

Lemma 3.2.8. *Let G be a 2-cograph and e be an edge of G. Then G/e is a 2-cograph.*

Proof. Assume to the contrary that G/e is not a 2-cograph. Then there is an induced subgraph H of G/e such that both H and $\overline{H}$ are 2-connected. Let $e = uv$ and let w denote the vertex in G/e obtained by identifying u and v. We may assume that w is a vertex of H, otherwise H is an induced subgraph of G, a contradiction. We assert that the subgraph H' of G induced on the vertex set $(V(H) \cup \{u, v\}) - \{w\}$ is 2-connected, as is its complement $\overline{H'}$. To see this, note that, since H is 2-connected, H' is 2-connected unless one of u and v, say u, is a leaf of H'. In the exceptional case, we have $H' - u \cong H$, so G has an induced subgraph for which both it and its complement are 2-connected, a contradiction. We deduce that H' is 2-connected.

By Lemma 3.2.3, the neighbours of w in $\overline{H}$ are the common neighbours of u and v in $\overline{H'}$. Thus the degrees of u and v in $\overline{H'}$ each equal at least the degree of w in $\overline{H}$. Moreover, $\overline{H'} - u$ has a spanning subgraph isomorphic to $\overline{H}$ and is therefore 2-connected. Since u has degree at least two in $\overline{H'}$, it follows that $\overline{H'}$ is 2-connected, a contradiction. □

We show next that, for all $k \geq 3$, a contraction of a k-cograph need not be a k-cograph. We use the following construction for the proof. Start with a graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$ and a copy G' of G with vertex set $\{v'_1, v'_2, \ldots, v'_n\}$. Take the disjoint union of G and G', and add all the edges joining v_i to v'_i. The resulting graph, $G \square K_2$, is the Cartesian product of G and K_2.

Lemma 3.2.9. *For $k \geq 3$, the class of k-cographs is not closed under contraction.*

Proof. Let $G_2 = C_5$. For all $k \geq 3$, let $G_k = G_{k-1} \square K_2$. One can easily check that G_k is a k-connected, k-regular graph whose complement is also k-connected.

Let G'_k be a graph having an edge e that is in no 3-cycles such that $G'_k/e = G_k$ and

Require: Input a simple graph G
  Set $H \leftarrow G$, BlocksList $\leftarrow [G]$
  if $|V(H)| \leq 4$ **then**
    remove H from BlocksList
    if BlocksList is empty **then**
      return G is a 2-cograph and exit the algorithm
    else
      update H to be an element of BlocksList
  if some K in $\{H, \overline{H}\}$ can be decomposed into 2-connected blocks **then**
    remove H from BlocksList
    add all the blocks of K to BlocksList
    update H to be an element of BlocksList
  else
    return G is not a 2-cograph and exit the algorithm

Figure 3.5. Algorithm for recognizing a 2-cograph.

the endpoints of e each have degree less than k. Note that every proper induced subgraph of G'_k has a vertex of degree less than k and so G'_k is a k-cograph. However, G'_k/e is not a k-cograph as it equals G_k. $\square$

By Lemmas 3.2.1 and 3.2.8, the class of 2-cographs is closed under taking induced minors. In the rest of the chapter, we will focus our attention on 2-cographs. The next lemma is straightforward.

Lemma 3.2.10. *All graphs having at most four vertices are* 2*-cographs.*

Since we can compute the blocks of a graph in polynomial time [30, 4.1.23.], the algorithm in Figure 3.5 recognizes 2-cographs in polynomial time. Since 2-cographs do not have induced subgraphs isomorphic to odd cycles of length at least five or their complements, it follows by the Strong Perfect Graph Theorem [4] that all 2-cographs are perfect. However, this inclusion is proper. For example, the graph C_6^+ obtained from a 6-cycle by adding a chord to create two 4-cycles is a perfect graph that is not a 2-cograph.

Akiyama and Harary [1, Corollary 1a] claimed that a 2-connected graph G has a

2-connected complement if and only if the red and green degrees of every vertex of G are at least two and G has no spanning complete bipartite subgraph. However, this result is not true. The graphs in Figure 3.6 are complements of each other. The first graph in the figure satisfies the hypotheses of [1, Corollary 1a] but its complement, $C_4 \oplus_1 C_4$, is not 2-connected.

Figure 3.6. Counterexample to a result of Akiyama and Harary.

We can repair Akiyama and Harary's result as follows.

Proposition 3.2.11. *If G is a 2-connected graph, then $\overline{G}$ is a 2-connected graph if and only if G has no complete bipartite subgraph using at least $|V(G)| - 1$ vertices.*

Proof. Note that if $\overline{G}$ is not 2-connected, then G has a spanning complete bipartite subgraph or a complete bipartite subgraph on $|V(G)| - 1$ vertices. The converse is immediate. $\square$

3.3. Induced-minor-minimal non-2-cographs

We noted in Section 2 that 2-cographs are closed under induced minors. In this section, we consider those non-2-cographs for which every proper induced minor is a 2-cograph. We call these graphs *induced-minor-minimal non-2-cographs*. The goal of this section is to characterize such graphs. We begin by showing that there are infinitely many of them. Theorem 3.1.2, whose proof appears at the end of this section, specifies all of the infinite families of such graphs.

Lemma 3.3.1. *Let G be the complement of a cycle C of length exceeding four. Then G is an induced-minor-minimal non-2-cograph.*

Proof. Certainly G is not a 2-cograph since both G and its complement are 2-connected. Moreover, by Lemma 3.2.1, $G - v$ is a 2-cograph for all vertices v of G because $\overline{G} - v$ is a path and is therefore a 2-cograph. It remains to show that G/e is a 2-cograph for all edges e of G. By Lemma 3.2.3, the complement of G/e is either a 0-sum of two paths and an isolated vertex, or a 0-sum of a path and K_2. This implies that the complement of G/e is a 2-cograph and, by Lemma 3.2.1, the result follows. □

Note that the complements of cycles of length at least five are not the only induced-minor-minimal non-2-cographs. It can be checked that both C_6^+ and its complement are induced-minor-minimal non-2-cographs.

The following lemma is obtained by applying [19, Lemma 2.3] (see also [18, Lemma 4.3.10]) to the bond matroid of a 2-connected graph.

Lemma 3.3.2. *Let G be a 2-connected graph other than K_3 and let v be an arbitrary vertex of G. Then G has at least two edges e incident to v such that G/e is 2-connected.*

An edge e of a 2-connected graph G is *contractible* if G/e is 2-connected. The following observation is immediate.

Lemma 3.3.3. *Let G be an induced-minor-minimal non-2-cograph. Then both G and $\overline{G}$ are 2-connected.*

In the rest of the section, we use the next two theorems of Chan about contractible edges in 2-connected graphs [3, Theorems 3.1, 3.3, and 3.5]. A component of a graph is *trivial* if it has just one vertex. In a 2-connected graph, a 2-*cut* is a 2-element vertex cut.

Theorem 3.3.4. *Let G be a 2-connected graph that is not isomorphic to K_3. Suppose all the contractible edges of G meet a 3-element subset S of $V(G)$. Then either $G - S$ has no edges, or $G - S$ has exactly one non-trivial component and this component has at most three vertices.*

Theorem 3.3.5. *Let G be a 2-connected graph that is not isomorphic to K_3. Suppose all the contractible edges of G meet a subset S of $V(G)$ such that $|S| \geq 4$. Then $G - S$ has at most $|S| - 2$ non-trivial components and, between them, these components have at most $2|S| - 4$ vertices.*

We will also frequently use the following straightforward result.

Lemma 3.3.6. *Let G be a 2-connected graph. If G has a 2-cut $\{g_1, g_2\}$ such that each of g_1 and g_2 has red degree at least two and the components of $G - \{g_1, g_2\}$ can be partitioned into two sets each of which contains at least two vertices, then the red graph $\overline{G}$ is 2-connected.*

Lemma 3.3.7. *Let G be an induced-minor-minimal non-2-cograph such that $|V(G)| \geq 6$ and let $wxyz$ be a path P of G such that both x and y have degree two in G. Then w and z are adjacent.*

Proof. Assume that w and z are not adjacent. By Lemma 3.3.3, G is 2-connected, so there is a path P' joining w and z such that P and P' are internally disjoint. This implies that G has C_5 as a proper induced minor. As C_5 is not a 2-cograph, this is a contradiction. $\square$

Lemma 3.3.8. *Let G be an induced-minor-minimal non-2-cograph. If G has two adjacent vertices of degree two, then $|V(G)| \leq 10$.*

Proof. Assume $|V(G)| \geq 11$. Let a and b be two vertices of G of degree two such that ab is a green edge. Let c be the green neighbour of a distinct from b, and let d be the green neighbour of b distinct from a. Then $c \neq d$, otherwise G is not 2-connected, contradicting Lemma 3.3.3. By Lemma 3.3.7, cd is a green edge. Observe that every vertex of $V(G) - \{a, b, c, d\}$ has red edges joining it to each of a and b. Thus $\overline{G} - \{c, d\}$ is 2-connected.

Suppose that both c and d have red degree at least three. Let w be a red neighbour of d such that $w \neq a$. It follows by Lemma 3.3.2 that w has a contractible green edge incident to it, say e, such that the other endpoint of e is not c. Then $\overline{G/e}$ is 2-connected, a contradiction.

Next suppose that both c and d have red degree two. First, we assume that c and d have the same red neighbour, say v, in $G-\{a, b\}$. Since v has green degree at least two, we have two green neighbours of v, say x and y. Note that x and y are in $V(G) - \{a, b, c, d\}$. Since x and y are adjacent to both c and d in the green graph, both the red and the green graphs induced on $\{a, b, c, d, v, x, y\}$ are 2-connected. This implies $|V(G)| \leq 7$, a contradiction. We may now assume that c and d have distinct red neighbours in $G - \{a, b\}$; call them v and w, respectively. Note that $vdcw$ is a green vw-path.

3.3.8.1. $G - \{a, b\}$ *has no vw-path P internally disjoint from the path $vdcw$.*

Assume that $G - \{a, b\}$ has such a path. Observe that the red graph and the green graph induced on the vertex set $V(P) \cup \{a, b, c, d\}$ are 2-connected and therefore, $V(G) = V(P) \cup \{a, b, c, d\}$. Now $|V(P)| \geq 7$ since $|V(G)| \geq 11$. Let e be an edge in the path P such that neither of the endpoints of e is in $\{v, w\}$. Note that G/e and $\overline{G/e}$ are both 2-connected, a contradiction. Thus 3.3.8.1 holds.

Let P_1 and P_2 be shortest vw-paths in $G - \{a, b, d\}$ and $G - \{a, b, c\}$ respectively. By 3.3.8.1, P_1 contains the vertex c and P_2 contains d. Note that $V(G) = V(P_1) \cup V(P_2) \cup \{a, b\}$. As $|V(G)| \geq 11$, we may assume that $P_1 - w$ has length at least three. Let e be an edge in $P_1 - w$ such that the endpoints of e are not in $\{c, v\}$. Note that G/e and $\overline{G/e}$ are both 2-connected, a contradiction.

Finally, without loss of generality, we may assume that c has red degree two and d has red degree at least three. Let v be the red neighbour of c distinct from b. Suppose that dv is red. Let x and y be two green neighbours of v and let P be a shortest path from d to $\{v, x, y\}$ in $G - \{a, b, c\}$. Then, for $V' = \{a, b, c, d, v, x, y\} \cup V(P)$, the red and green graphs induced by V' are 2-connected, so $V' = V(G)$. As $|V(G)| \geq 11$, we may assume that P has length at least three. Let e be an edge in P such that the endpoints of e are not in $\{d, v, x, y\}$. Note that G/e and $\overline{G/e}$ are both 2-connected, a contradiction. Therefore, dv is green. Let w be a red neighbour of d in $G - \{a, b\}$. Let u be a green neighbour of v distinct from d. Observe that $u \neq w$, otherwise $|V(G)| \leq 6$ since both $G[\{a, b, c, d, v, w\}]$ and $\overline{G}[\{a, b, c, d, v, w\}]$ are 2-connected. Let P be a shortest path from w to $\{d, u, v\}$ in $G - \{a, b, c\}$. Then $V(G) = \{a, b, c, d, u, v, w\} \cup V(P)$, so we may assume that P has length at least three. Then, for an edge e of P having neither endpoint in $\{d, u, v, w\}$, both G/e and $\overline{G/e}$ are 2-connected, a contradiction. $\square$

The next lemma shows that if a path of an induced-minor-minimal non-2-cograph G has three consecutive vertices of degree two, then $G \cong C_5$.

Lemma 3.3.9. *Let G be an induced-minor-minimal non-2-cograph such that G has a path P of length exceeding three and all the internal vertices of P are of degree two, then $G \cong$*

C_5.

Proof. Let u and v be vertices of P such that the subpath P_{uv} of P joining u and v has length four. Since G is 2-connected, there is a uv-path P' such that P_{uv} and P' are internally disjoint. Assume that P' is a shortest such path. Then contracting all but one edge in P' and deleting all the vertices not in $V(P_{uv})$, we obtain C_5. Since G cannot have C_5 as a proper induced minor, $G \cong C_5$. $\square$

A 2-connected graph H is *critically 2-connected* if $H - v$ is not 2-connected for all vertices v of H.

Lemma 3.3.10. *If G is a non-2-cograph such that $G - v$ is a 2-cograph for all vertices v of G, then G or $\overline{G}$ is critically 2-connected, or both G and $\overline{G}$ have vertex connectivity two.*

Proof. Certainly, G and $\overline{G}$ are 2-connected and, for all vertices v of G, either $G - v$ or $\overline{G} - v$ is not 2-connected. Observe that if neither G nor $\overline{G}$ is critically 2-connected, then G has vertices v and v_c such that $G - v$ and $\overline{G} - v_c$ are 2-connected. It follows that $G - v_c$ and $\overline{G} - v$ are not 2-connected so both G and $\overline{G}$ have vertex connectivity two. $\square$

Next we find those induced-minor-minimal non-2-cographs G such that G or $\overline{G}$ is critically 2-connected. We will use the following result of Nebesky [17].

Lemma 3.3.11. *Let G be a critically 2-connected graph such that $|V(G)| \geq 6$. Then G has at least two distinct paths of length exceeding two such that the internal vertices of these paths have degree two in G.*

Proposition 3.3.12. *Let G be an induced-minor-minimal non-2-cograph such that G is critically 2-connected. Then G is isomorphic to C_5 or C_6^+.*

Proof. By Lemmas 3.2.10 and 3.3.1, it follows that C_5 is the unique induced-minor-minimal non-2-cograph with at most five vertices, so we may assume that $|V(G)| \geq 6$. Thus, by Lemma 3.3.11, G has two distinct paths P_1 and P_2 of length exceeding two such that their internal vertices have degree two. Since G is not isomorphic to C_5, by Lemma 3.3.9, we may assume that both P_1 and P_2 have length three. Lemma 3.3.7 implies that, for each i, the endpoints of P_i are adjacent. We deduce that G has C_6^+ as an induced minor. As C_6^+ is an induced-minor-minimal non-2-cograph, we deduce that $G \cong C_6^+$. □

For a graph G, let V_g and V_r be its set of vertices of green-degree two and its set of vertices of red-degree two.

Lemma 3.3.13. *A graph G is an induced-minor-minimal non-2-cograph for which the graph $G[V_r]$ induced on V_r has at least two disjoint red edges if and only if $\overline{G}$ is a cycle with at least five vertices, or $\overline{G}$ is isomorphic to H_1, H_2, or F_k for some $k \geq 0$ where H_1, H_2, and F_k are shown in Figure* 3.1.

Proof. First we observe that if $\overline{G}$ is a cycle with $|V(\overline{G})| \geq 5$ or if $\overline{G}$ is isomorphic to H_1, H_2, or F_k, then $G[V_r]$ has at least two disjoint red edges. Moreover, by Lemma 3.3.1, if $\overline{G}$ is a cycle with $|V(\overline{G})| \geq 5$, then G is an induced-minor-minimal non-2-cograph. It is straightforward to check that if $\overline{G}$ is isomorphic to H_1 or H_2, then G is an induced-minor-minimal non-2-cograph. Finally, we show that, for all $k \geq 0$, the complement of F_k is an induced-minor-minimal non-2-cograph. Since $F_0 \cong C_6^+$ and the complement of the latter is an induced-minor-minimal non-2-cograph, we may assume that $k > 0$. As both F_k and $\overline{F_k}$ are 2-connected, the graph $\overline{F_k}$ is not a 2-cograph. We show that every proper induced

minor H of $\overline{F_k}$ is a 2-cograph. First assume that H is an induced subgraph of $\overline{F_k}$. Deleting the vertex x from F_k leaves a path, which is a 2-cograph. Thus we may assume that x is a vertex of H. Once a vertex distinct from x is deleted from F_k, if we were to find a non-2-cograph, it must be contained in one of the blocks of the vertex deletion. Each block B of a vertex deletion of F_k that has at least three vertices must have x as a vertex. Moreover, B has x adjacent to all but at most one other vertex, so its complement is not 2-connected. It is now straightforward to see that H is a 2-cograph. For an edge uv of $\overline{F_k}$, it follows by Lemma 3.2.3 that the complement of $\overline{F_k}/uv$ is either an induced subgraph of F_k or a 1-sum of an induced subgraph of F_k with K_2 or K_3. Thus $\overline{F_k}/uv$ is a 2-cograph and so $\overline{F_k}$ is an induced-minor-minimal non-2-cograph.

Conversely, assume that G is an induced-minor-minimal non-2-cograph for which $G[V_r]$ has u_1u_2 and v_1v_2 as two disjoint red edges. Since C_5 is the unique induced-minor-minimal non-2-cograph with five vertices, we may assume that $|V(\overline{G})| \geq 6$, that $\overline{G}$ is not a cycle, and that no F_k for $k \geq 0$ is isomorphic to $\overline{G}$. Next we show the following.

3.3.13.1. *In $\overline{G}$, no u_i is adjacent to any v_j.*

Note that if we have a red edge connecting $\{u_1, u_2\}$ to $\{v_1, v_2\}$, then $\overline{G}$ has a path P of length three such that all the vertices of P have red degree two. Let Q be a shortest path in $\overline{G}\backslash E(P)$ joining the endpoints of P. Then $\overline{G}$ has as an induced subgraph a cycle with edge set $E(P) \cup E(Q)$. This cycle has at least five edges, a contradiction. Thus 3.3.13.1 holds.

In $\overline{G}$, let x and y be the neighbours of u_1 and u_2, respectively, other than u_2 and u_1; and let w and z be the neighbours of v_1 and v_2, respectively, other than v_2 and v_1. Be-

cause $\overline{G}$ is 2-connected, it has a cycle C containing u_1u_2 and v_1v_2. We show next that C is Hamiltonian. Assume it is not. Certainly $\overline{G}[V(C)]$ is 2-connected. Consider $G[V(C)]$. In it, u_1 and u_2 are adjacent to every vertex not in $\{x, u_1, u_2, y\}$, and v_1 and v_2 are adjacent to every vertex not in $\{w, v_1, v_2, z\}$. In addition, u_1 and v_1 are adjacent to y and it follows by symmetry that $G[V(C)]$ is 2-connected. The minimality of G implies that $V(G) = V(C)$. Thus C is indeed Hamiltonian.

Assume that C consists of the path xu_1u_2y, a path P_{yz} from y to z, the path zv_2v_1w, and a path P_{wx} from w to x. Now x and y must be distinct. Likewise, w and z are distinct. If $x = w$ and $y = z$, then $\overline{G}$ is either C_6 or C_6^+. As $C_6^+ = F_0$, this is a contradiction. Thus $x \neq w$ or $y \neq z$.

The graph $\overline{G} - \{u_1, u_2\}$ is connected. Take a shortest path P in this graph from x to y. This path P must be a single edge otherwise $\overline{G}$ has an induced cycle of length at least five consisting of the union of P and the path xu_1u_2y. By Lemma 3.3.1, the complement of this induced cycle is an induced-minor-minimal non-2-cograph, so G is this complement, a contradiction.

By symmetry, we may assume that $\overline{G}$ has xy and wz as edges. Assume that $x = w$ but $y \neq z$. Because the only cycles of $\overline{G}$ containing u_1u_2 and v_1v_2 are Hamiltonian, the path P_{yz} in C is a shortest path from y to z in $\overline{G} - x$. Let $P_{yz} = y_0y_1 \dots y_k$ where $y = y_0$ and $z = y_k$. For each i in $\{1, 2, \dots, k-1\}$, the only possible neighbour of y_i in $\overline{G}$ other than y_{i-1} and y_{i+1} is x. We argue by induction on i that y_i is adjacent to x. Suppose y_1 is not adjacent to x. If y_2 is adjacent to x, then $\overline{G}$ has C_6^+ as an induced subgraph, a contradiction. Thus y_2 is not adjacent to x. As y_k is adjacent to x, for some $j \geq 3$, the vertex y_j is adjacent to x, but none of $y_{j-1}, y_{j-2}, \dots, y_2, y_1$ is adjacent to x. Then $\overline{G}$ has a cycle of

length at least five as an induced subgraph, a contradiction. We conclude that y_1 is adjacent to x. Assume that all of $y_1, y_2, \ldots, y_t$ are adjacent to x but y_{t+1} is not. If y_{t+2} is not adjacent to x, then $\overline{G}$ contains an induced cycle of length at least five, a contradiction. Thus y_{t+2} is adjacent to x and $\overline{G}$ has F_t as a proper induced subgraph, a contradiction. We conclude that y_{t+1} is adjacent to x. Hence, by induction, y_i is adjacent to x for all i in $\{1, 2, \ldots, k-1\}$. Thus $\overline{G} \cong F_k$, a contradiction.

It remains to consider the case when $x \neq w$ and $y \neq z$. If xz and wy are both green, then $G - \{v_1, v_2\}$ and its complement are both 2-connected, a contradiction. Suppose both xz and wy are red. Then $\overline{G}$ has a cycle using u_1u_2 and v_1v_2 and having exactly eight vertices. Thus $|V(\overline{G})| = 8$. If both xw and yz are green, then $\overline{G} - \{u_1, u_2\} \cong C_6^+$, a contradiction. Thus $\overline{G}$ is isomorphic to either H_1 or H_2. Now assume that xz is red and wy is green. If both xw and yz are red, then $|V(G)| = 8$ and $\overline{G}$ is isomorphic to H_2. If xw is green, then, using the paths P_{yz} and P_{wz} in $\overline{G}$, we see that $G - \{v_1, v_2\}$ and its complement are both 2-connected. Thus we may assume that xw is red. Likewise, yz is red otherwise $G - \{u_1, u_2\}$ and its complement are both 2-connected, a contradiction. Hence $\overline{G}$ is isomorphic to H_2. $\square$

The following is a straightforward consequence of Lemmas 3.3.11 and 3.3.13.

Proposition 3.3.14. *A graph G is an induced-minor-minimal non-2-cograph for which $\overline{G}$ is critically 2-connected if and only if $\overline{G}$ is a cycle with at least five vertices, or $\overline{G}$ is isomorphic to H_1, H_2, or F_k for some $k \geq 0$.*

The next three lemmas show that the number of vertices of an induced-minor-minimal non 2-cograph is bounded above given some conditions on the sizes of compo-

nents after the removal of a green 2-cut and on the red degrees of the vertices in that cut.

Lemma 3.3.15. *Let $\{g_1, g_2\}$ be a 2-cut of an induced-minor-minimal non-2-cograph G such that each of g_1 and g_2 has red degree exceeding two and the components of $G-\{g_1, g_2\}$ can be partitioned into two subgraphs, A and B, each having at least two vertices. Then $|V(G)| \leq 8$.*

Proof. Assume that $|V(G)| > 8$. Without loss of generality, let $|V(A)| \geq 4$. Suppose A contains no red neighbours of g_1 or g_2. Then all vertices in A are incident to both g_1 and g_2 via a green edge. Let v be any vertex in A. Note that both $G - v$ and $\overline{G} - v$ are 2-connected, a contradiction. Therefore, we may assume that A has a red neighbour, say a_1, of g_1. Lemma 3.3.2 implies that we can find a contractible green edge, say e, of G incident to a_1 such that the other endpoint of e is in A. By Lemma 3.3.6, $\overline{G/e}$ is 2-connected, a contradiction. □

Lemma 3.3.16. *Let $\{g_1, g_2\}$ be a 2-cut of an induced-minor-minimal non-2-cograph G such that the red degree of g_1 is two and the components of $G - \{g_1, g_2\}$ can be partitioned into subgraphs A and B such that $|V(A)| \geq |V(B)| \geq 2$. Suppose that A contains exactly one red neighbour v of g_1, and either g_2 has no red neighbours in $A - v$, or g_2 has red degree greater than two. If all of the contractible edges of G having both endpoints in $V(A) \cup \{g_1, g_2\}$ are incident to a vertex in $\{g_1, g_2, v\}$, then $|V(A)| \leq 4$.*

Proof. Assume that $|V(A)| > 4$. Let G_A be the subgraph of G induced by $V(A) \cup \{g_1, g_2\}$, and let Q denote the vertex set $\{g_1, g_2, v\}$. By colouring the edge g_1g_2 green if necessary, we may assume that G_A is 2-connected. Since the contractible edges of G_A must meet Q, by Theorem 3.3.4, either $G_A - Q$ has no edges, or $G_A - Q$ has one non-trivial component

and this component has at most three vertices. First suppose that $G_A - Q$ is edgeless. Let $\Gamma = V(G_A) - Q$. Next we show the following.

3.3.16.1. *There is no vertex γ in Γ such that $G_A - \gamma$ is 2-connected.*

If such a vertex exists, then $G - \gamma$ is 2-connected. Moreover, by Lemma 3.3.6, $\overline{G} - \gamma$ is 2-connected, a contradiction. Thus 3.3.16.1 holds.

3.3.16.2. *The edge vg_2 is red.*

Suppose vg_2 is green. Let α be a neighbour of v in Γ. Then $g_1 \alpha v g_2 g_1$ is a cycle of G_A. Because $G_A - Q$ is edgeless and G_A is 2-connected, every vertex in $\Gamma - \alpha$ is adjacent to at least two members of $\{g_1, g_2, v\}$. Thus $G_A - \gamma$ is 2-connected for all γ in $\Gamma - \alpha$, a contradiction to 3.3.16.1. Thus 3.3.16.2 holds.

Observe that v and g_2 have a common neighbour β in Γ otherwise, as $G_A - Q$ is edgeless, g_1 is a cut vertex of G_A. By 3.3.16.2, v has a neighbour α in $\Gamma - \beta$. Since $g_1 \alpha v \beta g_2 g_1$ is a cycle and all vertices in $\Gamma - \{\alpha, \beta\}$ are adjacent to at least two vertices in $\{g_1, g_2, v\}$, we deduce that $G_A - \gamma$ is 2-connected for all γ in $\Gamma - \{\alpha, \beta\}$, a contradiction.

We may now assume that $G_A - Q$ has one non-trivial component, say C_A, and a set I_A of isolated vertices. Moreover, $|V(C_A)| \leq 3$. Then I_A is non-empty since $|V(A)| > 4$. Let $\alpha\beta$ be an edge in C_A. Note that $\alpha\beta$ is not contractible in G_A, so $\{\alpha, \beta\}$ is a 2-cut of G_A and, therefore, of G. Since $|V(B)| \geq 2$ and I_A is non-empty, each of α and β has red degree at least three in G. Therefore, by Lemma 3.3.15, as $|V(G)| = |V(A)| + 2 + |V(B)| > 8$, there is a vertex t of G whose only green neighbours are α and β. Since g_1 is adjacent to all vertices in $I_A \cup V(C_A)$, it follows that $t = v$. This implies that all vertices

in I_A are adjacent only to g_1 and g_2. Taking w in I_A, we see that $G_A - w$ is 2-connected, a contradiction to 3.3.16.1 □

Lemma 3.3.17. *Let $\{g_1, g_2\}$ be a 2-cut of an induced-minor-minimal non-2-cograph G such that the components of $G - \{g_1, g_2\}$ can be partitioned into subgraphs, A and B, each having at least two vertices. If the red degree of g_1 is two and that of g_2 is greater than two such that one red neighbour of g_1 is in A and the other is in B, then $|V(G)| \leq 10$.*

Proof. Without loss of generality, assume $|V(A)| \geq |V(B)|$. Let G_A be the subgraph of G induced by $V(A) \cup \{g_1, g_2\}$. Note that G_A is 2-connected since g_1g_2 is green. Denote the red neighbour of g_1 in A by v and let $Q = \{g_1, g_2, v\}$. Observe that if we have a contractible edge e of G having both endpoints in $V(A) \cup \{g_1, g_2\}$ such that neither of the endpoints of e is in Q, then, by Lemma 3.3.6, both G/e and $\overline{G/e}$ are 2-connected, a contradiction. Therefore, we may assume that all contractible edges of G that have both endpoints in $V(A) \cup \{g_1, g_2\}$ meet Q. Thus, by Lemma 3.3.16, $|V(A)| \leq 4$, so $|V(G)| \leq 10$. □

Lemma 3.3.18. *Let $\{g_1, g_2\}$ be a 2-cut of an induced-minor-minimal non-2-cograph G such that the components of $G - \{g_1, g_2\}$ can be partitioned into two subgraphs, A and B, each having at least two vertices. Suppose that, for each i in $\{1, 2\}$, if g_i has red degree two, then g_i has no red neighbour in B. Then $|V(B)| = 2$.*

Proof. Suppose $|V(B)| \geq 3$. If all vertices in B are green neighbours of both g_1 and g_2, then $G - z$ is 2-connected for all z in $V(B)$. But, by Lemma 3.3.6, $\overline{G} - z$ is also 2-connected, a contradiction. Thus B has a red neighbour, say b, of g_1. Note that g_1 has red degree greater than two. Now, by Lemma 3.3.2, we can find a contractible edge, say e,

of G incident to b such that the other endpoint of e is in $V(B)$. By Lemma 3.3.6, $\overline{G/e}$ is 2-connected, a contradiction. □

Lemma 3.3.19. *Let $\{g_1, g_2\}$ be a 2-cut of an induced-minor-minimal non-2-cograph G such that the red degree of g_1 is two and the components of $G - \{g_1, g_2\}$ can be partitioned into subgraphs A and B such that $|V(A)| \geq |V(B)| \geq 2$. Suppose that one of the following holds.*

(i) A contains both the red neighbours $\{x, y\}$ of g_1, and g_2 has no red neighbour in $A - \{x, y\}$ if the red degree of g_2 is two; or

(ii) g_2 has red degree two and A contains exactly one pair $\{x, y\}$ of distinct vertices such that x is a red neighbour of g_1, and y is a red neighbour of g_2.

If all contractible edges of G having both endpoints in $V(A) \cup \{g_1, g_2\}$ are incident to a vertex in $\{g_1, g_2, x, y\}$, then $|V(A)| \leq 6$.

Proof. Assume that $|V(A)| > 6$ and so $|V(G)| > 10$. Let G_A be the graph induced on $V(A) \cup \{g_1, g_2\}$. Let $Q = \{g_1, g_2, x, y\}$. By colouring the edge g_1g_2 green if necessary, we may assume that G_A is 2-connected. Note that all the contractible edges of G_A must meet Q, otherwise we have a contractible edge e of G such that $\overline{G/e}$ is 2-connected, a contradiction. By Theorem 3.3.5, $G_A - Q$ has at most two non-trivial components and, between them, these components have at most four vertices.

Let I_A and N_A be the sets of isolated and non-isolated vertices of $G_A - Q$ respectively. We note the following.

3.3.19.1. *If two vertices i_1 and i_2 in I_A have the same green neighbourhood in G, then $\{i_1, i_2\}$ is a green 2-cut in G.*

As $\overline{G} - \{g_1, g_2, i_1\}$ is a complete bipartite graph with each part having at least two vertices, it is 2-connected. Both g_1 and g_2 have at least two red neighbours in $\overline{G} - \{i_1\}$. Thus $\overline{G} - i_1$ is 2-connected. Therefore $G - i_1$ is not 2-connected. It follows that i_2 is a cut-vertex of $G - i_1$ and so $\{i_1, i_2\}$ is a green 2-cut. Thus 3.3.19.1 holds.

First suppose that N_A is empty. As $|V(A)| \geq 7$, we see that $|I_A| \geq 5$. Suppose that g_2 has red degree two. Then all vertices in I_A are adjacent to both g_1 and g_2 in G. Observe that if a vertex s in I_A has green neighbourhood $\{g_1, g_2\}$, then both $G - s$ and $\overline{G} - s$ are 2-connected, a contradiction. Since g_1 and g_2 have no red neighbours in I_A, the green neighbourhood of a vertex in I_A is $\{g_1, g_2, x\}, \{g_1, g_2, y\}$, or $\{g_1, g_2, x, y\}$. It follows that there are at least two pairs of vertices in I_A such that each vertex in a pair has the same green neighbourhood. Let $\{i_1, i_2\}$ be such a pair. By 3.3.19.1, $\{i_1, i_2\}$ is a green 2-cut. Since the red degrees of both i_1 and i_2 are greater than two, by Lemma 3.3.15, it follows that there is a vertex t of G that has green neighbourhood $\{i_1, i_2\}$. Note that t is either x or y. Since we have at least two such green 2-cuts, it follows that g_2x and g_2y are both red, and there is a red edge connecting $\{x, y\}$ to I_A. Observe that B has no red neighbour of g_1 or g_2. It now follows that, for each b in $V(B)$, both $G - b$ and $\overline{G} - b$ are 2-connected, a contradiction. Therefore g_2 has red degree at least three. By Lemma 3.3.18, $|V(B)| = 2$. Suppose there is no red edge connecting $\{x, y\}$ to I_A. Then the possible green neighbourhoods of the vertices in I_A are $\{x, y\}, \{x, y, g_1\}, \{x, y, g_2\}$, or $\{x, y, g_1, g_2\}$. Thus, by 3.3.19.1, I_A contains a green 2-cut $\{i_1, i_2\}$ of G. Then we get $|V(G)| \leq 8$ by applying Lemma 3.3.15 to the green 2-cut $\{i_1, i_2\}$. Therefore there is a red edge connecting $\{x, y\}$ to I_A. It follows that, for some b in $V(B)$, both $G - b$ and $\overline{G} - b$ are 2-connected, a contradiction.

We may now assume that $G_A - Q$ has at least one non-trivial component. Let C be such a component and let $\alpha\beta$ be an edge in C. Since $\alpha\beta$ is a non-contractible edge of G_A, we see that $\{\alpha, \beta\}$ is a green 2-cut of G_A and thus of G. Then $G_A - Q \neq C$ otherwise, by Theorem 3.3.5, $|V(A)| \leq 6$, a contradiction. Thus both α and β have red degree at least three in G. Therefore, by Lemma 3.3.15, G has a unique vertex t that has green neighbourhood $\{\alpha, \beta\}$. Since all vertices in G_A except x and y are adjacent to g_1 via a green edge, t is either x or y. As $\alpha\beta$ is an arbitrary green edge in $G_A - Q$, it follows that $G_A - Q$ has at most two edges and therefore has either one non-trivial component with at most three vertices, or has two non-trivial components each with two vertices.

Suppose that $G_A - Q$ has only one edge, $\alpha\beta$, and let t be the unique member of $\{x, y\}$ that has green neighbourhood $\{\alpha, \beta\}$. Then $|I_A| \geq 3$ and the green neighbourhood of every vertex in I_A is contained in $\{g_1, g_2, s\}$ where $\{t, s\} = \{x, y\}$. It is clear that if a vertex w in I_A has green neighbourhood $\{g_1, g_2\}$, then $G - w$ and $\overline{G} - w$ are 2-connected. It follows that the green neighbourhood of a vertex in I_A is either $\{g_1, s\}$ or $\{g_1, g_2, s\}$. As $|I_A| \geq 3$, it contains vertices i_1 and i_2 that have the same green neighbourhood. By 3.3.19.1, $\{i_1, i_2\}$ is a green 2-cut in G. As neither t nor s has $\{i_1, i_2\}$ as its green neighbourhood, Lemma 3.3.15 gives the contradiction that $|V(G)| \leq 8$. We now know that $G_A - Q$ has exactly two edges, so $3 \leq |N_A| \leq 4$. Observe that $I_A \neq \emptyset$ and all vertices in I_A have green neighbourhood equal to $\{g_1, g_2\}$ since x and y have their green neighbourhoods contained in N_A. Thus, for $w \in I_A$, both $G - w$ and $\overline{G} - w$ are 2-connected, a contradiction. $\square$

Lemma 3.3.20. *Let $\{g_1, g_2\}$ be a 2-cut of an induced-minor-minimal non-2-cograph G*

such that g_1 and g_2 are not adjacent in G and the components of $G - \{g_1, g_2\}$ can be partitioned into two subgraphs, A and B, each having at least two vertices. Then $|V(G)| \leq 10$.

Proof. First suppose that both g_1 and g_2 have red degree two and that the red neighbour v of g_1 that is distinct from g_2 is in A, and the red neighbour u of g_2 distinct from g_1 is in B. We may assume that $|V(A)| \geq |V(B)|$. Observe that, if we can find a contractible edge e of G having both the endpoints in $V(A) - v$, then, by Lemma 3.3.6, $\overline{G/e}$ is 2-connected, a contradiction. This implies that all the contractible edges of G that have both endpoints in $V(A) \cup \{g_1, g_2\}$ are incident to $\{g_1, g_2, v\}$. By Lemma 3.3.16, $|V(A)| \leq 4$ and so $|V(G)| \leq 10$. Thus we may assume that both u and v are in A and all contractible edges of G that have both endpoints in $V(A) \cup \{g_1, g_2\}$ are incident to $\{g_1, g_2, u, v\}$. Note that $u \neq v$, otherwise $\overline{G}$ has a cut vertex. We get our result now by Lemmas 3.3.18 and 3.3.19. We may now assume that the red degree of g_2 exceeds two. By Lemma 3.3.15, we may further assume that the red degree of g_1 is two.

Let v be the red neighbour of g_1 other than g_2. We may assume that v is in A. By Lemma 3.3.18, $|V(B)| = 2$. Note that all the contractible edges of G that have both endpoints in $V(A) \cup \{g_1, g_2\}$ are incident to $\{g_1, g_2, v\}$. The result now follows by Lemma 3.3.16. □

Lemma 3.3.15 can be modified as follows.

Proposition 3.3.21. *Let $\{g_1, g_2\}$ be a 2-cut of an induced-minor-minimal non-2-cograph G such that the components of $G-\{g_1, g_2\}$ can be partitioned into two subgraphs, A and B, each having at least two vertices. If g_2 has red degree greater than two, then $|V(G)| \leq 10$.*

Proof. Assume that $|V(G)| \geq 11$. Then, by Lemma 3.3.15, the red degree of g_1 is two.

Let x and y be the two red neighbours of g_1. Note that if x is in A and y is in B, then the result follows by Lemma 3.3.17. By Lemma 3.3.20, we may suppose that the edge g_1g_2 is green and both x and y are in A.

The graph G_A induced on $V(A) \cup \{g_1, g_2\}$ is 2-connected. Let $Q = \{g_1, g_2, x, y\}$. Then every contractible edge e of G_A must meet Q otherwise, by Lemma 3.3.6, we obtain the contradiction that both G/e and $\overline{G/e}$ are 2-connected. The result now follows by Lemmas 3.3.18 and 3.3.19. □

We can generalize the above result by removing the condition on the red degrees of the vertices in the 2-cut at the cost of raising the bound on the number of vertices of G to 16.

Proposition 3.3.22. *Let $\{g_1, g_2\}$ be a 2-cut of an induced-minor-minimal non-2-cograph G such that the components of $G - \{g_1, g_2\}$ can be partitioned into two subgraphs, A and B, each having at least two vertices. Then $|V(G)| \leq 16$.*

Proof. Assume that $|V(G)| \geq 17$. By Lemma 3.3.20 and Proposition 3.3.21, we may assume that the red degrees of both g_1 and g_2 are two and g_1g_2 is green. We may further assume that $|V(A)| \geq |V(B)|$. The graph G_A induced on $V(A) \cup \{g_1, g_2\}$ is 2-connected. Let Q be the union of $\{g_1, g_2\}$, the set of red neighbours of g_1 in A, and the set of red neighbours of g_2 in A. Then every contractible edge e of G_A must meet Q, otherwise, by Lemma 3.3.6, we obtain a contradiction. Note that if $|Q| = 2$, then, by Lemma 3.3.18, $|V(A)| = 2$ and so $|V(G)| \leq 6$, a contradiction. By Theorems 3.3.4 and 3.3.5, $G_A - Q$ has at most four non-trivial components and between them, these components have at most eight vertices.

Let I_A and N_A be the sets of isolated and non-isolated vertices of $G_A - Q$, respectively. We note the following.

3.3.22.1. $|N_A| \leq 4$.

Assume that $|N_A| > 4$ and so $G_A - Q$ has at least three edges. Let $\alpha\beta$ be an edge of $G_A - Q$. Because $\alpha\beta$ is not a contractible edge of G_A, it follows that $\{\alpha, \beta\}$ is a green 2-cut of G. Observe that each of α and β has red degree at least three unless $|I_A|$ is empty, and $G_A - Q$ has one non-trivial component, and $|V(B)| = 2$. The exceptional case does not arise since it implies, as $V(G) = (V(A) - Q) \cup Q \cup V(B)$, that $|V(G)| \leq 8 + 6 + 2 = 16$, a contradiction. By Lemma 3.3.15, there is a vertex t that has green neighbourhood $\{\alpha, \beta\}$. Note that the only vertices that could have green neighbourhood $\{\alpha, \beta\}$ are the common red neighbours of g_1 and g_2. Since there are at most two such vertices and at least three edges in $G_A - Q$, each of which must have an associated such vertex, we have a contradiction. Thus 3.3.22.1 holds.

Next we show the following.

3.3.22.2. $|I_A| \leq 4$.

Assume that $|I_A| \geq 5$. Note that all vertices in I_A are adjacent to both g_1 and g_2. Suppose I_A contains a vertex i such that all vertices in $Q - \{g_1, g_2\}$ have degree at least two in $G - i$. Then both $G - i$ and $\overline{G}$–i are 2-connected, a contradiction. It follows that, for every vertex i of I_A, there is a *special* green edge joining i to a vertex q of $Q - \{g_1, g_2\}$ such that q has green degree two. The set Q' of such vertices q is contained in $Q - \{g_1, g_2\}$. If a member q' of Q' is a common red neighbour of g_1 and g_2, then it meets at most two special green edges from I_A. If, instead, q' has a single red neighbour in $\{g_1, g_2\}$, then it

has a single green neighbour in $\{g_1, g_2\}$ and so meets at most one special green edge. Thus the number of red edges from $\{g_1, g_2\}$ to Q' is an upper bound on the number of special green edges from I_A. Hence $|I_A| \leq 4$, a contradiction. Thus 3.3.22.2 holds.

3.3.22.3. $|V(B)| \geq 3$.

Suppose that $|V(B)| = 2$. Then, by 3.3.22.1 and 3.3.22.2, $|V(G)| \leq 4{+}4{+}6{+}2 = 16$, a contradiction. Thus 3.3.22.3 holds.

By 3.3.22.3, since $|V(B)| \neq 2$, Lemma 3.3.18 implies that B contains at least one red neighbour of $\{g_1, g_2\}$. Assume that B contains exactly one such red neighbour v. Let x and y be two green neighbours of v in $V(B) \cup \{g_1, g_2\}$. If $V(B) - \{v, x, y\}$ contains a vertex t, then $G - t$ and $\overline{G} - t$ are both 2-connected. It follows that $|V(B)| \leq 3$. Again by 3.3.22.1 and 3.3.22.2, we get $|V(G)| \leq 4 + 4 + 5 + 3 = 16$, a contradiction. Note that if A contains exactly one of the red neighbours of $\{g_1, g_2\}$, then, by Lemma 3.3.16, $|V(A)| \leq 4$, so $|V(G)| \leq 10$, a contradiction. We may now assume that the red neighbourhood of $\{g_1, g_2\}$ has size four, and each of A and B contains exactly two of those vertices. Then, by Lemma 3.3.19, $|V(A)| \leq 6$, so $|V(G)| \leq 14$, a contradiction. □

The following corollary summarizes our results about the induced-minor-minimal non-2-cographs so far.

Corollary 3.3.23. *Let G be an induced-minor-minimal non-2-cograph. Then*

(i) $|V(G)| \leq 16$; *or*

(ii) $\overline{G}$ *is a cycle of length at least five; or*

(iii) G *has vertex connectivity two, and, for every* 2*-cut* $\{g_1, g_2\}$ *of* G*, the graph* $G - \{g_1, g_2\}$ *has exactly two components and one component contains a single vertex.*

If an induced-minor-minimal non-2-cograph G satisfies (iii) of the above corollary, we say that G is an induced-minor-minimal non-2-cograph of *type* (iii). The next lemma identifies several infinite families of such graphs.

Lemma 3.3.24. *Let G be a graph such that $\overline{G}$ is isomorphic to L_k, M_k, M'_k, N_k, N'_k, or N''_k for some $k \geq 1$ where L_k, M_k, and N_k are shown in Figures 3.2, 3.3, and 3.4, respectively. Then G is an induced-minor-minimal non-2-cograph of type (iii).*

Proof. It is clear that G is not a 2-cograph as both G and $\overline{G}$ are 2-connected. Assume that H is an induced subgraph of G such that both H and $\overline{H}$ are 2-connected. It is clear that $v \in V(H)$ otherwise H or $\overline{H}$ is not 2-connected. Note that $V(H)$ also contains the vertices x and y since x and y are the only green neighbours of v. It now follows that $V(H)$ contains the red neighbours of x and the red neighbours of y. It is now straightforward to see that $H = G$. Therefore every proper induced subgraph of G and of $\overline{G}$ is a 2-cograph. For an edge $\alpha\beta$ of G, it follows by Lemma 3.2.3 that the complement of $G/\alpha\beta$ is either a proper induced subgraph of $\overline{G}$ or a proper induced subgraph of $\overline{G}$ 1-summed with K_2 or K_3. Thus $G/\alpha\beta$ is a 2-cograph and so G is an induced minor-minimal non-2-cograph. Moreover, $\{x, y\}$ is its unique 2-cut and G is an induced-minor-minimal non-2-cograph of type (iii). □

By a similar argument to that just given, we obtain the following.

Lemma 3.3.25. *For a non-negative integer k, the graph $\overline{F_k}$ is an induced-minor-minimal non-2-cograph of type (iii).*

In the rest of the section, we find all the other classes of induced-minor-minimal non-2-cographs of type (iii) thereby proving Theorem 3.1.2.

Lemma 3.3.26. *Let G be an induced-minor-minimal non-2-cograph of type (iii). Then $|V_g| \leq 3$ or $\overline{G}$ is of type (iii).*

Proof. Suppose that $\overline{G}$ is not of type (iii). By Lemma 3.3.10 and Proposition 3.3.12, we may assume that $\overline{G}$ has vertex connectivity two. Take a red 2-cut $\{r_1, r_2\}$ of G such that the components of $\overline{G} - \{r_1, r_2\}$ can be partitioned into subgraphs A and B, and $|V(A)| \geq |V(B)| \geq 2$. If $|V(B)| \geq 3$, then all vertices in $V(G) - \{r_1, r_2\}$ have green degree at least three and so $|V_g| \leq 2$. Now suppose that $V(B) = \{b_1, b_2\}$. Note that there is at most one vertex a in A that has green neighbourhood $\{b_1, b_2\}$ since G is of type (iii). One can now check that all vertices in $V(G) - \{r_1, r_2, a\}$ have green degree at least three, and so $|V_g| \leq 3$. □

Lemma 3.3.27. *Let G be an induced-minor-minimal non-2-cograph such that $|V(G)| > 10$. Suppose that $\overline{G}$ is not isomorphic to a cycle or to F_k for some $k \geq 0$. Then the graph induced on the vertex set V_g is a complete red graph and the graph induced on V_r has at most one red edge.*

Proof. By Lemma 3.3.8, the graph induced on V_g is a complete red graph. Assume that the graph induced on V_r has two red edges $e = u_1u_2$ and $f = v_1v_2$. Note that if e and f are disjoint, then, by Lemma 3.3.13, we obtain a contradiction. Therefore we may assume that $u_2 = v_1$. Let α and β be the respective neighbours of u_1 and v_2 in $\overline{G} - v_1$. Note that α and β are distinct otherwise we have a cut vertex in $\overline{G}$, a contradiction. Let P be a shortest $\alpha\beta$-path distinct from $\alpha u_1 u_2 v_2 \beta$. Then P avoids $\{u_1, u_2, v_2\}$ and the red graph induced on $V(P) \cup \{u_1, u_2, v_2\}$ is a cycle. It follows by the minimality of G that $V(G) = V(P) \cup \{u_1, u_2, v_2\}$ and so $\overline{G}$ is a cycle, a contradiction. □

In the following lemma, we note that either $|V_g|$ or $|V_r|$ is bounded.

Lemma 3.3.28. *Let G be an induced-minor-minimal non-2-cograph. Then either $|V_g|$ or $|V_r|$ is at most three, or $|V_g| = |V_r| = 4$.*

Proof. Note that there are at most $2|V_g|$ green edges and at most $2|V_r|$ red edges joining a vertex in V_g to a vertex in V_r. Since there are $|V_g||V_r|$ edges joining vertices in V_g to vertices in V_r, we have

$$2|V_g| + 2|V_r| \geq |V_g||V_r|.$$

This inequality is symmetric with respect to $|V_g|$ and $|V_r|$, so we may assume that $|V_g| \geq |V_r|$. Then $2 + 2\frac{|V_r|}{V_g} \geq |V_r|$. Thus $|V_r| \leq 4$. Moreover, if $|V_r| = 4$, then $|V_g| = 4$. $\square$

Next we note the following useful observation.

Lemma 3.3.29. *Let G be an induced-minor-minimal non-2-cograph such that $|V(G)| > 10$. If all vertices of a subset S of $V(G) - (V_g \cup V_r)$ either have a red neighbour in V_r or a green neighbour in V_g, then*

3.3.29.1. $|S| \leq 2|V_g \cup V_r| - |V_g||V_r|$.

Moreover, when equality holds here, either each vertex in S has exactly one green neighbour in V_g or has exactly one red neighbour in V_r but not both. In particular, if $S = V(G) - (V_g \cup V_r)$, then

$$11 + |V_g||V_r| \leq 3|V_g| + 3|V_r|.$$

Proof. There are $|V_g||V_r|$ red or green edges joining a vertex in V_g to a vertex in V_r. There are at most $2|V_g|$ green such edges and at most $2|V_r|$ red such edges. Thus among the green edges meeting V_g and the red edges meeting V_r at most $2|V_g \cup V_r| - |V_g||V_r|$ have

an endpoint in $V(G) - (V_g \cup V_r)$. Therefore, $|S| \leq 2|V_g \cup V_r| - |V_g||V_r|$ and it is clear that, when equality holds, each vertex in S satisfies the given condition. If $S = V(G) - (V_g \cup V_r)$, then it is clear that $11 + |V_g||V_r| \leq 3|V_g| + 3|V_r|$ since $|V(G)| \geq 11$. □

Lemma 3.3.26 can be improved in the following way.

Lemma 3.3.30. *Let G be an induced-minor-minimal non-2-cograph of type (iii). Then $|V(G)| \leq 10$ or $|V_g| \leq 3$.*

Proof. By Lemma 3.3.26, it is enough to show that if $\overline{G}$ is of type (iii), then $|V(G)| \leq 10$ or $|V_g| \leq 3$. Suppose that $\overline{G}$ is of type (iii). Since every vertex of $V(G)$ is either in a red 2-cut or a green 2-cut, and both G and $\overline{G}$ are of type (iii), we have the following.

3.3.30.1. *Every vertex in $V(G)$ either has a green neighbour in V_g or a red neighbour in V_r.*

Since a vertex in V_g has no green neighbour in V_g by Lemma 3.3.8, it follows by 3.3.30.1 that $|V_g| \leq 2|V_r|$ since the number o f red-degree-two neighbours of vertices in V_g is at least $|V_g|$ and at most $2|V_r|$. The following is an immediate consequence of Lemma 3.3.29 and 3.3.30.1.

3.3.30.2. $|V(G)| \leq |V_g| + |V_r| + 2|V_r \cup V_g| - |V_g||V_r| = 3|V_g| + 3|V_r| - |V_g||V_r|$.

Note that if $|V_g| = |V_r| = 4$, then $|V(G)| \leq 8$ and the result holds. Therefore, by Lemma 3.3.28, we may assume that $|V_r|$ is at most three. As $|V_g| \leq 2|V_r|$, by 3.3.30.2, checking the possibilities for $|V_r|$, we obtain that $|V(G)| \leq 10$. □

In the next proof, we adopt the convention that, for a 2-cut $\{x, y\}$ of a graph H, the graphs A and B are disjoint subgraphs of $H - \{x, y\}$ with $V(A) \cup V(B) = V(H - \{x, y\})$

such that $|V(A)| \geq |V(B)|$, and $|V(B)|$ is maximal.

Lemma 3.3.31. *Let G be an induced-minor-minimal non-2-cograph such that G is of type (iii). Then $|V(G)| \leq 16$ or $|V_g| \leq 1$.*

Proof. By Lemma 3.3.30, we may assume that $|V_g| \leq 3$. The following observation is immediate.

3.3.31.1. *Let $\{r_1, r_2\}$ be a red 2-cut of G. If $|V(B)| \geq 3$, then $\{r_1, r_2\} \subseteq V_g$.*

Next we show the following.

3.3.31.2. *There are at most two vertices outside of V_g that have neither a red neighbour in V_r nor a green neighbour in V_g.*

Every vertex v of $V(G) - V_g$ is in a green 2-cut or a red 2-cut. In the first case, because G is of type (iii), v has a green neighbour in V_g. In the second case, let $\{v, r\}$ be a red 2-cut. By 3.3.31.1, we may assume that $|V(B)| \leq 2$. If $|V(B)| = 1$, then v has a red neighbour in V_r. Suppose $|V(B)| = 2$. If w is a vertex with green neighbourhood $V(B)$, then $V(B)$ is a green 2-cut. As G is of type (iii), w is unique. If $|V_g| = 3$, it follows that $\{v, r\} \subseteq V_g$, a contradiction, so 3.3.31.2 holds.

If $|V_g| \leq 1$, then the lemma holds, so we may assume $|V_g| = 2$. For the red 2-cut $\{v, r\}$, we know that $|V(B)| = 2$. Now each vertex u of $V(G) - V(B) - \{v, r\}$ has $V(B)$ in its green neighbourhood. Thus $\{u, r\}$ cannot be a red 2-cut with the same $V(B)$. Thus $\{v, r\}$ is the unique red 2-cut with the given $V(B)$. As $v \notin V_g$ and G is of type (iii), the set $V(B)$ is the green neighbourhood of exactly one vertex in V_g. Since $|V_g| = 2$, it follows that we have at most two red 2-cuts for which $|V(B)| = 2$. Moreover, each such red 2-cut

contains a member of V_g. Now 3.3.31.2 follows immediately.

By 3.3.31.2 and Lemma 3.3.29, $|V(G)| \leq 3|V_g| + 3|V_r| - |V_g||V_r| + 2$. If $|V_g| = 3$, then $|V(G)| \leq 11$. Suppose $|V_g| = 2$. Then, by Lemma 3.3.27, $|V_r| \leq 2|V_g| + 2 + 2 = 8$, so $|V(G)| \leq 16$. □

Proof of Theorem 3.1.2. We may assume that G is of type (iii) otherwise we have the result by Corollary 3.3.23. We may also assume that neither G nor $\overline{G}$ is critically 2-connected, otherwise the result follows by Proposition 3.3.12 or Proposition 3.3.14. It is now clear that V_g is non-empty. Therefore, by Lemma 3.3.31, $|V_g| = 1$ or $|V(G)| \leq 16$. If $|V(G)| \leq 16$, then we have our result. Therefore we may assume that $|V_g| = 1$. It now follows that G has a unique green 2-cut $\{x, y\}$. Thus every vertex not in $\{x, y\}$ is in a red 2-cut. As $\overline{G}$ is not critically 2-connected, we may assume that $\overline{G} - \{x\}$ is 2-connected. Note that $G - \{x, y\}$ has a non-trivial component A and a trivial component, say $\{v\}$.

3.3.32.1. *There is no vertex t in A such that $\overline{G} - \{x, v, t\}$ is connected and each of x and y has at least two neighbours in $\overline{G} - \{t\}$.*

Assume that this fails. Since $\overline{G} - \{x, v, t\}$ is connected and v is adjacent to all vertices of $\overline{G} - \{x, t\}$ except y, we conclude that $\overline{G} - \{x, t\}$ is 2-connected as $\overline{G} - x$ is 2-connected and y has at least two neighbours in $\overline{G} - \{x, t\}$. It now follows that $\overline{G} - \{t\}$ is 2-connected since x has at least two neighbours $\overline{G} - \{t\}$. This is a contradiction since t is in a red 2-cut.

3.3.32.2. *$\overline{G}[A]$ is connected.*

To show this, assume $\overline{G}[A]$ is disconnected. Because $\overline{G} - x$ is 2-connected, $\overline{G} -$

$\{x, v\}$ is connected. Since $\overline{G}[A] = \overline{G} - \{x, v, y\}$, it follows that y has a neighbour in each component of $\overline{G}[A]$. As $\overline{G} - \{x, v\}$ is connected, there is a vertex t in A such that $\overline{G} - \{x, v, t\}$ is connected where, if possible, t is chosen from a component of $\overline{G}[A]$ with at least two vertices. By 3.3.32.1, t is a red neighbour of some z in $\{x, y\}$ such that z has degree two in $\overline{G}$. Suppose $z = y$. Then, as $\overline{G} - \{x, v, t\}$ is connected, y is adjacent to t and to each component of $\overline{G}[A]$, we deduce that $\{t\}$ is a component of $\overline{G}[A]$ and $|V(A)| = 2$. Thus $|V(G)| = 5$ and so, as G is a non-2-cograph, G is a 5-cycle, a contradiction. We deduce that $z = x$ and x has red degree two. Thus $\overline{G} - \{x, v\}$ has exactly two vertices t for which $\overline{G} - \{x, v, t\}$ is connected, and each such vertex is a red neighbour of x. It follows that $\overline{G} - \{x, v\}$ is a path and the leaves of this path are the neighbours of x in $\overline{G} - \{v\}$. Therefore $\overline{G} - \{v\}$ is a cycle, a contradiction.

Similar to 3.3.32.1, we have the following.

3.3.32.3. *There is no vertex t in A such that $\overline{G} - \{y, v, t\}$ is connected and each of x and y has at least two neighbours in $\overline{G} - \{t\}$.*

Assume that this fails. If x has at least two neighbours in $\overline{G} - \{y, t\}$, then the proof follows as in 3.3.32.1 by interchanging x and y. Therefore we may assume that x has exactly one neighbour in $\overline{G} - \{y, t\}$. Thus $\overline{G}[A] - \{t\}$ is connected and so $\overline{G} - \{x, y, t\}$ is 2-connected. Since each of x and y has at least two neighbours in $\overline{G} - \{t\}$, we conclude that $\overline{G} - \{t\}$ is 2-connected, a contradiction.

We call a vertex t of $\overline{G}[A]$ *deletable* if $\overline{G}[A] - \{t\}$ is connected. By combining 3.3.32.1 and 3.3.32.3, we obtain the following.

3.3.32.4. *A deletable vertex t of $\overline{G}[A]$ is a neighbour in $\overline{G}$ of some z in $\{x, y\}$ where z has*

degree two in $\overline{G}$.

3.3.32.5. *The number of deletable vertices in $\overline{G}[A]$ is in $\{2, 3, 4\}$.*

To see this, first observe that, since $\overline{G}[A]$ is connected having at least two vertices, it has at least two deletable vertices. Now suppose that $\overline{G}[A]$ has at least five deletable vertices. Then there is such a vertex t so that, in $\overline{G} - \{t\}$, each of x and y has degree at least two. As $\overline{G} - \{x, v, t\}$ is connected, we have a contradiction to 3.3.32.1. Thus 3.3.32.5 holds.

The rest of the proof treats the three possibilities for the number of deletable vertices of $\overline{G}[A]$. First suppose that $\overline{G}[A]$ has exactly two deletable vertices s and t. Then $\overline{G}[A]$ is a path, which we may assume has at least five vertices. Let s' and t' be the respective neighbours of s and t in $\overline{G}[A]$. Note that if either x or y has red neighbourhood $\{s, t\}$, then we have an induced red cycle of size at least six, which is a contradiction. Thus, by 3.3.32.1 and 3.3.32.3, we may assume that both x and y have red degree two, and s is a red neighbour of x, and t is a red neighbour of y. If xy is red, then $\overline{G}$ has an induced cycle of length at least seven, a contradiction. Thus both the red neighbours of x and y are in A. We show next that the respective red neighbourhoods of x and y are $\{s, s'\}$ and $\{t, t'\}$. To see this, let $\{s, w\}$ be the neighbourhood of x in $\overline{G}$ and suppose $w \neq s'$. If s' is not a red neighbour of y, then $\overline{G} - \{y, v, s'\}$ is connected and we get a contradiction to 3.3.32.1. Taking z to be a vertex of A not in $\{s, t, w, s'\}$, we see that $\overline{G} - \{x, v, z\}$ or $\overline{G} - \{y, v, z\}$ is connected and we get a contradiction to 3.3.32.1 or 3.3.32.3. We conclude that $\{s, s'\}$ is the red neighbourhood of x. By symmetry, $\{t, t'\}$ is the red neighbourhood of y. Thus $\overline{G}$ is isomorphic to L_k for some $k \geq 1$.

Next suppose that $\overline{G}[A]$ has exactly three deletable vertices, s, t, and u. Then $\overline{G}[A]$ has a spanning tree T having s, t, and u as its leaves. By 3.3.32.4, each vertex in $\{s, t, u\}$ is adjacent to a red-degree-2 vertex in $\{x, y\}$. Moreover, neither x nor y has red degree exceeding two, and xy is not red. Now $\overline{G}[A]$ is connected, so $\overline{G} - \{x, y\}$ is 2-connected. As xy is red, it follows that $\overline{G} - y$ is 2-connected. Recall that we already know that $\overline{G} - x$ is 2-connected. By symmetry, we may assume that the red neighbourhood of x is $\{s, t\}$, and so u is a red neighbour of y. Let u' be the red neighbour of u in T. Then the red neighbourhood of y is $\{u, u'\}$ otherwise $\overline{G} - \{x, v, u'\}$ is connected and we get a contradiction to 3.3.32.1. Similarly, the distance between s and t in T is two otherwise we get a contradiction to 3.3.32.3. As $\overline{G}[A]$ has exactly three deletable vertices, the only possible edge in $\overline{G}[A]$ that is not in T is st. Thus $\overline{G}$ is isomorphic to M_k or M'_k for some $k \geq 1$.

Finally, suppose that $\overline{G}[A]$ has four deletable vertices, s, t, u, and z. We may assume that the respective red neighbourhoods of x and y are $\{s, t\}$ and $\{u, z\}$. Again let T be a spanning tree of $\overline{G}[A]$ such that s, t, u, and z are leaves of T. Note that the distance between s and t, and u and z in T is two. Thus $\overline{G}$ is isomorphic to N_k, N'_k, or N''_k for some $k \geq 1$. $\square$

3.4. The class $\mathcal{G}$ of induced-minor-minimal non-2-cographs whose complements are also induced-minor-minimal non-2-cographs

In this section, we consider $\mathcal{G}$, the class of induced-minor-minimal non-2-cographs G such that $\overline{G}$ is also an induced-minor-minimal non-2-cograph. We show that all graphs in $\mathcal{G}$ have at most ten vertices. We give an exhaustive list of all these graphs in the last section. We begin the section with the following immediate consequence of Lemma 3.3.8.

Corollary 3.4.1. *Let G be a graph in $\mathcal{G}$ such that $|V(G)| > 10$. Then the graph induced*

on the vertex set V_g is a complete red graph and the graph induced on V_r is a complete green graph.

The next lemma shows that if the number of vertices of a graph G in $\mathcal{G}$ exceeds ten, then $V(G) - (V_g \cup V_r)$ is non-empty.

Lemma 3.4.2. *Let G be a graph in $\mathcal{G}$ such that $|V(G)| > 10$. Then $V(G) \neq V_g \cup V_r$.*

Proof. Assume that $V(G) = V_g \cup V_r$. There are $2|V_g|$ green edges and $2|V_r|$ red edges joining a vertex in V_g to vertex in V_r. Thus

3.4.2.1. $2|V_g| + 2|V_r| = |V_g||V_r|$.

We may assume that $|V_g| \leq |V_r|$. If $|V_g| = |V_r|$, then $4|V_r| = |V_r|^2$, so $|V_r| = 4$, a contradiction. Therefore $|V_g| \leq |V_r| - 1$ so, by 3.4.2.1, $|V_g||V_r| \leq 4|V_r| - 2$. Thus $|V_g| \leq 3$. If $|V_g| = 3$, then, by 3.4.2.1, $|V_r| = 6$, so $|V(G)| = 9$, a contradiction. If $|V_g| \leq 2$, then we contradict 3.4.2.1. $\square$

Next we note a useful observation about the vertices in $V(G) - (V_g \cup V_r)$.

Lemma 3.4.3. *Let G be a graph in $\mathcal{G}$ such that $|V(G)| > 10$. Then every vertex in $V(G) - (V_g \cup V_r)$ either has a green neighbour in V_g or a red neighbour in V_r.*

Proof. Since every vertex of G is in either a red 2-cut or a green 2-cut, the lemma follows by Proposition 3.3.21. $\square$

Lemma 3.4.4. *Let G be a graph in $\mathcal{G}$ such that $|V(G)| > 10$. Then neither V_g nor V_r is empty.*

Proof. It suffices to show that V_r is non-empty. Assume the contrary. By Lemma 3.4.3, every vertex outside V_g has a green neighbour in V_g. Thus, by Lemma 3.3.29, $11 \leq 3|V_g|$,

so $|V_g| \geq 4$. Let $\{r_1, r_2\}$ be a red 2-cut T. Since V_r is empty, applying Proposition 3.3.21 to $\overline{G}$ gives that T is contained in V_g. Let v be a vertex in $V_g - T$ and let α and β be the two green neighbours of v. Consider the graph $\overline{G} - T$. Note that $\overline{G} - T$ is disconnected and v is incident to all the vertices in this graph except α and β. Let X be the component of $\overline{G} - T$ containing v. Since the red graph $\overline{G}$ has no degree-two vertices, $\overline{G} - T$ has exactly two components. The second component must have $\{\alpha, \beta\}$ as its vertex set.

Let w be a vertex in $V_g - T - v$. As w is in a different component of $\overline{G} - T$ from α and β, both $w\alpha$ and $w\beta$ are green edges. Since w has green degree two, it follows that $\{\alpha, \beta\}$ is the green neighbourhood of each vertex in $V_g - T$. By Lemma 3.4.3, each vertex in $V(G) - V_g - \{\alpha, \beta\}$ has a green neighbour in V_g. This neighbour is not in $V_g - T$, so it is in T. Thus $|V(G) - V_g - \{\alpha, \beta\}| \leq 4$. But $|V(G)| > 10$, so $|V_g - T| \geq 3$. Therefore $G - v$ and $\overline{G} - v$ are both 2-connected, a contradiction. We conclude that V_r is non-empty. $\square$

We are now ready to prove the second main result of the chapter.

Proof of Theorem 3.1.3. Assume that $G \in \mathcal{G}$ and $|V(G)| > 10$. Without loss of generality, let $|V_g| \leq |V_r|$. By Lemma 3.4.3, every vertex in $V(G) - (V_g \cup V_r)$ either has a green neighbour in V_g or a red neighbour in V_r. By Lemmas 3.4.4 and 3.3.28, $1 \leq |V_g| \leq 4$. Suppose $|V_g| = 4$. Then, by Lemma 3.3.28, $|V_r| = 4$. Lemma 3.3.29 implies that $V(G) - (V_g \cup V_r)$ is empty. Therefore $|V(G)| = 8$, a contradiction.

Next we assume that $|V_g| = 3$. Then every vertex in V_r is a green neighbour of at least one vertex in V_g. Thus $|V_r| \leq 6$ as there are exactly six green edges incident to vertices in V_g. Then, by Lemma 3.3.29, as $|V_g| = 3$, we deduce that $11 \leq 3|V_g|$, a contradiction.

Now suppose that $|V_g| = 2$. Then, by Lemma 3.3.29, $|V_r| \geq 5$. Let $V_g = \{u, v\}$. Since there are only four green edges meeting V_g, there is a vertex w in V_r whose red neighbours are u and v. Thus $\{u, v\}$ is a red 2-cut. Suppose that $V_r - \{w\}$ contains at least two vertices that are joined to both u and v by red edges. Then one can check that both $G - w$ and $\overline{G} - w$ are 2-connected, a contradiction. Thus V_r has at most two vertices that are joined to both u and v by red edges. Therefore $|V_r| \leq 6$ since V_g meets only four green edges. Assume that $|V_r| = 6$. Then all the green neighbours of u and v are in V_r and are distinct. Since $|V(G)| \geq 11$, we see that $|V(G) - (V_g \cup V_r)| \geq 3$. Let $\{w, x\}$ be the vertices in V_r having both u and v as their red neighbours. All the vertices in $V_r - \{w, x\}$ have one red neighbour in V_g. Since $|V(G) - (V_g \cup V_r)| \geq 3$, Lemma 3.4.3 implies that each vertex in $V(G) - (V_g \cup V_r)$ has at most two red neighbours in $V_r - \{w, x\}$ and thus has at least two green neighbours in $V_r - \{w, x\}$. Thus $G - w$ and $\overline{G} - w$ are 2-connected, a contradiction. We may now assume that $|V_r| = 5$ and $|V(G) - (V_g \cup V_r)| \geq 4$. By Lemma 3.3.29, $|V(G) - (V_g \cup V_r)| = 4$. Thus, as equality holds in 3.3.29.1, every vertex in $V(G) - (V_g \cup V_r)$ has at most one red neighbour in $V_r - w$ and so has at least three green neighbours in $V_r - w$. Therefore we again have that both $G - w$ and $\overline{G} - w$ are 2-connected, a contradiction.

Finally, assume that $|V_g| = 1$. By Lemma 3.3.29, $|V_r| \geq 4$. Let $V_g = \{v\}$ and let $\alpha \in V_r$ be a red neighbour of v. First, we show that V_r does not contain a green 2-cut that contains α. Assume that $\{\alpha, \beta\}$ is a green 2-cut where $\{\alpha, \beta\} \subseteq V_r$. Then $G - \{\alpha, \beta\}$ has a component X that contains $V_r - \{\alpha, \beta\}$ and all but at most two vertices of $V(G) - \{\alpha, \beta\}$. Let Y be a component of $G - \{\alpha, \beta\}$ different from X. Then $|V(Y)| \leq 2$. Suppose $|V(Y)| = 1$. Then the vertex in Y must be in V_g, so it is v. This is a contradic-

tion since αv is red. Thus $|V(Y)| = 2$ and $G - \{\alpha, \beta\}$ has exactly two components. Then $|V(X)| \geq 7$. Let x be a vertex in X such that x is not a red neighbour of α or β, and $X - \{x\}$ contains at least two vertices of $V_r - \{\alpha, \beta\}$. Since each vertex of $V_r - \{\alpha, \beta\}$ has its two red neighbours in Y and so is adjacent in G to every vertex of X, it follows that $G - x$ is 2-connected. Moreover, by Lemma 3.3.6, $\overline{G} - x$ is 2-connected, a contradiction. We conclude that V_r does not have a green 2-cut containing α.

Next, we show that no green 2-cut contains α. Assume that $\{\alpha, z\}$ is a green 2-cut. Then $z \notin V_r$. By Proposition 3.3.21, $G - \{\alpha, z\}$ has a single-vertex component Y. Since the vertex in Y has green degree two, $Y = \{v\}$. Thus αv is green, a contradiction. We conclude that deleting from G any red neighbour of v in V_r leaves a green graph that is still 2-connected.

To complete the proof of the theorem, we show that v has a red neighbour in V_r whose deletion from $\overline{G}$ leaves a 2-connected graph, thus arriving at a contradiction. Let β be a red neighbour of v in $V_r - \{\alpha\}$. If α and β have the same red neighbourhood, say $\{x, v\}$, then $\{x, v\}$ is a red 2-cut and we obtain a contradiction by applying Proposition 3.3.21 to $\overline{G}$. Thus α and β have distinct red neighbourhoods, $\{x, v\}$ and $\{y, v\}$, respectively. Note that if xv is red, then $\overline{G} - \alpha$ is 2-connected. Thus we may assume that both xv and yv are green. This implies γv is red for each γ in $V_r - \{\alpha, \beta\}$ since v has green degree two. Thus, for some fixed γ in $V_r - \{\alpha, \beta\}$, the other red neighbour, z, of γ is distinct from x and y. Since vz is red and γ has red degree two, we see that $\overline{G} - \gamma$ is 2-connected, a contradiction. $\square$

3.5. Graphs in $\mathcal{G}$

We implemented the algorithm given in this section using SageMath [26] and provide a list of all graphs in $\mathcal{G}$ up to complementation. The graphs in this section are drawn using SageMath.

Graphs on six vertices. There are two graphs on six vertices in $\mathcal{G}$, the graph in Figure 3.7 and its complement.

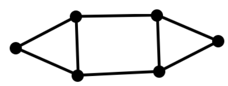


Figure 3.7. A 6-vertex graph in $\mathcal{G}$.

Graphs on seven vertices. There are sixteen graphs on seven vertices in $\mathcal{G}$, the graphs in Figure 3.8 and their complements.

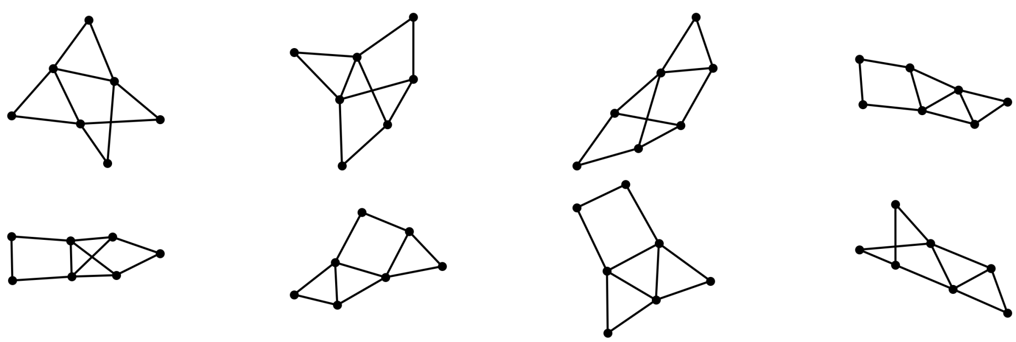


Figure 3.8. Graphs on seven vertices in $\mathcal{G}$.

Graphs on eight vertices. There are 87 graphs on eight vertices in $\mathcal{G}$, of which five are self-complementary. Figure 3.9 shows these self-complementary graphs. Figure 3.10 shows 41 non-self-complementary graphs that, with their complements, are the remaining 8-vertex graphs in $\mathcal{G}$.

Graphs on nine vertices. There are 86 graphs on nine vertices in $\mathcal{G}$. These are the 43 graphs in Figure 3.11 and their complements.

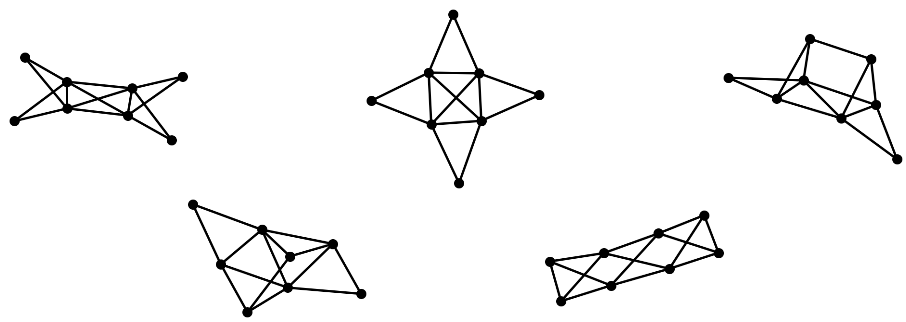


Figure 3.9. Self-complementary graphs on eight vertices in $\mathcal{G}$.

Graphs on ten vertices. There are two graphs on ten vertices in $\mathcal{G}$, the graph in Figure 3.12 and its complement.

Algorithm Finding graphs in $\mathcal{G}$ of order at most ten

---

Require: $n = 6, 7, 8, 9$ or 10.

```
Set FinalList ← ∅, i ← 0, j ← 0
Generate all two connected graphs of order n using nauty geng [15] and store in an
iterator L
for g in L such that vertex connectivity of g and ḡ is 2 do
    for v in V(g) do
        h = g\v
        if h is a 2-cograph then
            i ← i + 1
    for e in E(g) do
        h = g/e
        if h is a 2-cograph then
            j ← j + 1
    if i equals |V(g)| and j equals |E(g)| then
        Add g to FinalList
for g in FinalList do
    if FinalList does not contain ḡ then
        remove g from FinalList
```

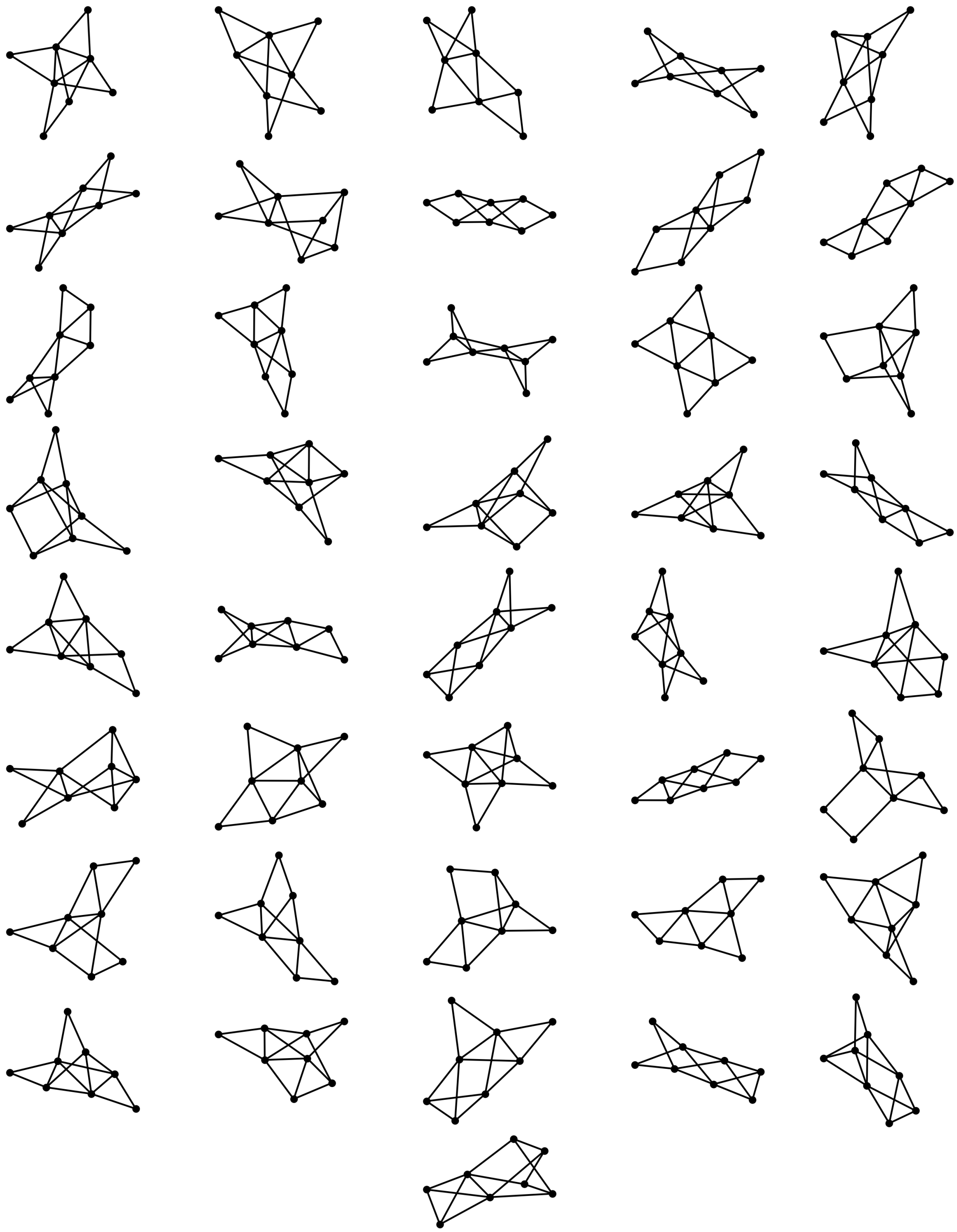


Figure 3.10. Graphs on eight vertices in $\mathcal{G}$.

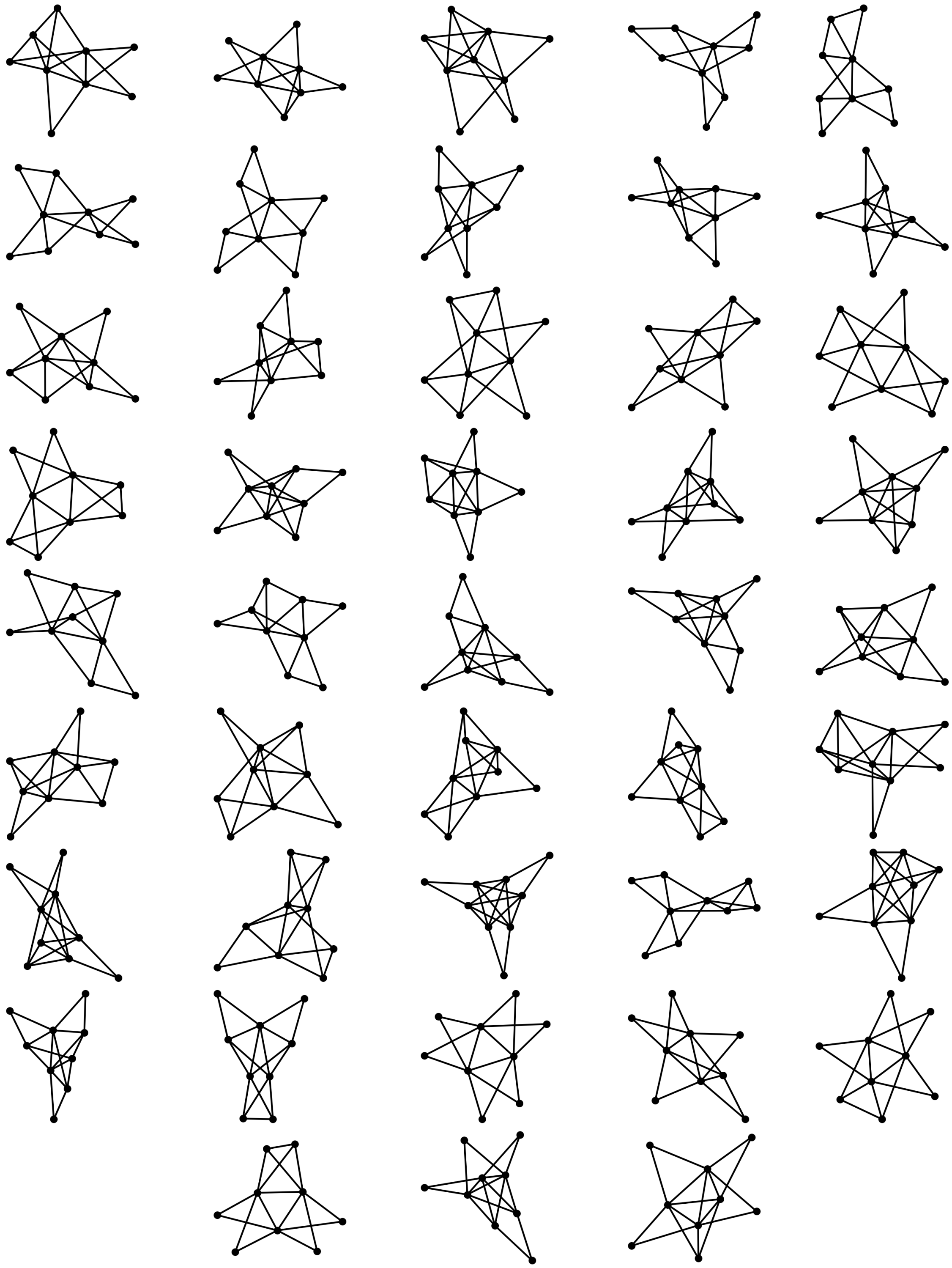


Figure 3.11. Graphs on nine vertices in $\mathcal{G}$.

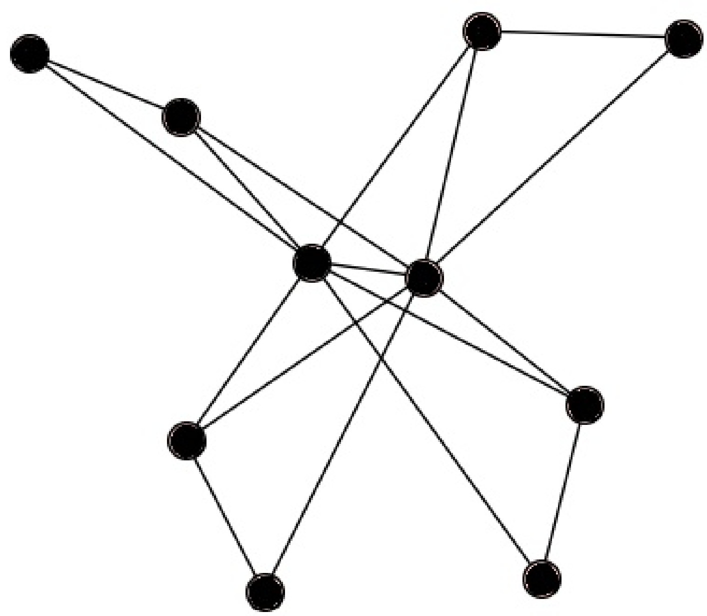


Figure 3.12. A 10-vertex graph in $\mathcal{G}$.

Chapter 4. Comatroids

4.1. Introduction

Recall that a *cograph* is defined recursively as follows:

(i) K_1 is a cograph;

(ii) if G_1 and G_2 are cographs, then so is their disjoint union; and

(iii) if G is a cograph, then so is its complement.

Our goal in this chapter, which is joint work with James Oxley [24], is to give a matroid analogue of cographs by, loosely speaking, considering the smallest class of matroids that is closed under direct sums and complementation. One immediate obstacle to achieving this goal is that matroids in general do not have complements. However, if M is a simple uniquely $GF(q)$-representable matroid and $k \geq r(M)$, the $(GF(q), k)$-*complement* of M is the matroid $PG(k-1, q) \backslash T$ where $M \cong PG(k-1, q)|T$. Brylawski and Lucas [2] (see also [18, Proposition 10.1.7]) showed that this $(GF(q), k)$-complement of M is well-defined. By convention, we write M^c for the $(GF(q), r(M))$-complement of M. Although a $GF(q)$-representable matroid need not be uniquely representable when $q \geq 4$, it is uniquely representable for q in $\{2, 3\}$. Thus we only introduce analogues of cographs for binary and ternary matroids. In particular, for q in $\{2, 3\}$, we define a $GF(q)$-*comatroid* recursively as follows:

(i) $U_{0,0}$ is a $GF(q)$-comatroid;

(ii) if M_1 and M_2 are $GF(q)$-comatroids, then so is their direct sum; and

(iii) if M is a $GF(q)$-comatroid, then so is its $(GF(q), t)$-complement for all $t \geq r(M)$.

As $PG(r-1, q)$ is the $(GF(q), r)$-complement of $U_{0,0}$, every projective geometry

$PG(r-1,q)$ for $r \geq -1$ is a $GF(q)$-comatroid. In particular, as $U_{1,1}$ is $PG(0,q)$, we see that $U_{n,n}$ is a $GF(q)$-comatroid for all $n \geq 0$. We sometimes call $GF(2)$- and $GF(3)$-comatroids, *binary* and *ternary comatroids*, respectively.

The following characterization of $GF(q)$-comatroids is particularly useful.

Theorem 4.1.1. *For q in $\{1,2\}$, a simple $GF(q)$-representable matroid M is a $GF(q)$-comatroid if and only if M is $U_{0,0}$ or, for all flats F of $PG(r(M)-1,q)$ with $r(F \cap E(M)) = r(F - E(M))$, the restriction of $PG(r(M)-1,q)$ to either $F \cap E(M)$ or $F - E(M)$ is disconnected.*

Corneil, Lerchs, and Stewart [5] proved that a graph G is a cograph if and only if G does not have the 4-vertex path as an induced subgraph. The next two theorems, which are our main results, prove matroid analogues of this theorem for binary and ternary comatroids by using the fact that a set X of edges in a graph H is the edge set of an induced subgraph of H if and only if X is a flat of $M(H)$. The matroid $P(U_{3,4}, U_{3,4})$, the parallel connection of two 4-circuits, is the cycle matroid of a 6-cycle with a single chord where this chord lies in two 4-cycles.

Theorem 4.1.2. *A binary matroid M is a binary comatroid if and only if neither M nor M^c has a flat isomorphic to a circuit of size exceeding five, to $P(U_{3,4}, U_{3,4})$, or to the cycle matroid of one of the six 5-vertex graphs shown in Figure 4.1.*

Theorem 4.1.3. *A ternary matroid M is a ternary comatroid if and only if neither M nor M^c has a flat isomorphic to a circuit of size exceeding three, to a matroid that can be obtained from a circuit of size at least three by 2-summing a copy of $U_{2,4}$ to at least one of the elements of the circuit, or to one of the five rank-3 matroids $P(U_{2,3}, U_{2,3}), U_{2,4} \oplus_2 U_{2,4}, P(U_{2,4}, U_{2,3})$, $M(K_4)$, and $\mathcal{W}^3$.*

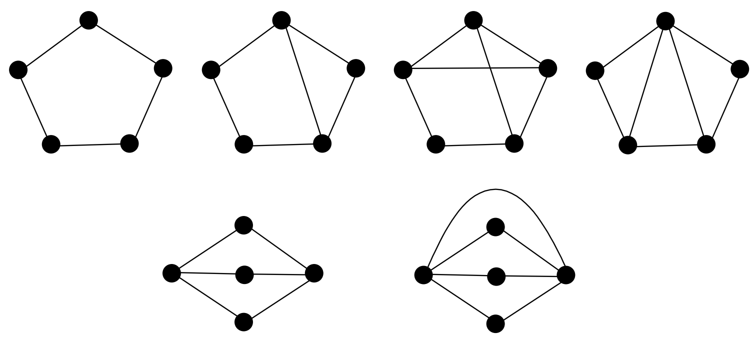


Figure 4.1. The cycle matroids of these graphs are induced-restriction-minimal binary non-comatroids.

The proofs of these theorems are given in Sections 4.4 and 4.5, respectively. In Section 4.2, we prove a number of preliminary results including Theorem 4.1.1. In particular, we show that if we contract an element from a $GF(q)$-comatroid and simplify the resulting matroid, then we get another $GF(q)$-comatroid. A simple matroid N is an *induced minor* of a simple matroid M if N can be obtained from M by a sequence of operations each of which consists of restricting to a flat, or contracting an element and then simplifying. As consequences of Theorems 4.1.2 and 4.1.3, we have the following characterizations of binary and ternary comatroids in terms of forbidden induced minors.

Corollary 4.1.4. *A binary matroid is a binary comatroid if and only if it has no induced minor isomorphic to the complement of a circuit of size exceeding five, to $P(U_{3,4}, U_{3,4})$, to the cycle matroid of one of the six 5-vertex graphs in Figure 4.1, or to the complements of these cycle matroids in $PG(3,2)$.*

Corollary 4.1.5. *A ternary matroid is a ternary comatroid if and only if it has no induced minor isomorphic to any of the following: the complements of all matroids that can be obtained from a circuit of size at least three by 2-summing a copy of $U_{2,4}$ to some, possibly empty, set of elements of the circuit; the matroids $U_{3,4}$, $P(U_{2,3}, U_{2,3})$, $U_{2,4} \oplus_2 U_{2,3}$, $U_{2,4} \oplus_2$*

$U_{2,4}$, $P(U_{2,4}, U_{2,3})$, $M(K_4)$, and $\mathcal{W}^3$; or the complements of these matroids in $PG(2,3)$.

4.2. Preliminaries

Throughout the chapter, we call cocircuits, flats, and hyperplanes of $PG(r-1,q)$ *projective cocircuits*, *projective flats*, and *projective hyperplanes*, respectively. Viewing a $GF(q)$-representable matroid M as a restriction of $PG(r(M)-1,q)$, we color the elements of $E(M)$ green while assigning the color red to the elements of $PG(r(M)-1,q)$ not in $E(M)$. We will frequently use G and R to denote both the sets of green and red elements and the matroids obtained by restricting $PG(r(M)-1,q)$ to these sets of elements. The next lemma is an immediate consequence of the fact that the elements of a projective geometry are not all contained in two hyperplanes.

Lemma 4.2.1. *For an arbitrary prime power q, let (G,R) be a 2-coloring of $PG(r-1,q)$. Then $r(G)=r$ or $r(R)=r$.*

Proposition 4.2.2. *For an arbitrary prime power q, let (G,R) be a 2-coloring of $PG(r-1,q)$. Let j and k be the vertical connectivities of G and R, respectively. Then $j+k \geq r$ unless $(q,r)=(2,3)$ and $\{G,R\}=\{U_{3,3}, U_{2,3}\oplus U_{1,1}\}$.*

Proof. Assume that the result fails. If R is empty, then $k=0$ and $j=r$, a contradiction. Thus we may assume that G and R are both non-empty. Then j and k are both non-zero, so we may assume that $j,k \in \{1,2,\dots,r-2\}$. Observe that if $r(R)<r$, then G contains $AG(r-1,q)$ and hence $j \geq r-1$, a contradiction. Therefore $r(G)=r(R)=r$. Thus G has an exact vertical j-separation (A,B) with $r(A)\geq r(B)$, and R has an exact vertical k-separation (X,Y) with $r(X)\geq r(Y)$. Let $r(A)=a$. Then

$$|G| \leq \frac{q^a-1}{q-1} + \frac{q^a-1}{q-1} - \frac{q^{j-1}-1}{q-1} = \frac{2q^a-q^{j-1}-1}{q-1}. \qquad (4.2.1)$$

By symmetry, with $r(X) = x$, we have

$$|R| \le \frac{2q^x - q^{k-1} - 1}{q-1}. \tag{4.2.2}$$

First suppose that $x = r - 1$. Then $r(Y) = k$. Let H_X be the projective hyperplane spanned by X. Observe that the intersection of H_X with the projective flat F_Y spanned by Y is a projective flat of rank $k - 1$. Thus, as $R - H_X \subseteq F_Y - (F_Y \cap H_X)$, we have

$$|R - H_X| \le \frac{q^k - 1}{q-1} - \frac{q^{k-1} - 1}{q-1} = q^{k-1}. \tag{4.2.3}$$

Suppose that $a = r - 1$. Then $r(B) = j$ and so, as for (4.2.3), we have $|G - H_A| \le q^{j-1}$ where H_A is the projective hyperplane spanned by A. Note that $E(PG(r-1, q)) - (H_A \cup H_X)$ has at least q^{r-2} elements and so it follows by (4.2.3) that $q^{k-1} + q^{j-1} \ge q^{r-2}$. Since j and k are in $\{1, 2, \ldots, r-2\}$, this is a contradiction unless $(q, r) = (2, 3)$ and $k = 1 = j$ and $r = 3$. In the exceptional case, it is straightforward to check that $\{G, R\} = \{U_{3,3}, U_{2,3} \oplus U_{1,1}\}$, and we get the exceptional case noted in the proposition.

We may now assume that $a < r - 1$. Let F_A be the projective flat spanned by A. Observe that $F_A \cap H_X$ is a projective flat of rank a or $a - 1$. Thus

$$\begin{aligned} |(R \cup G) - H_X| &= |R - H_X| + |G - H_X| \\ &\le |R - H_X| + |G| - |F_A \cap H_X| \\ &\le q^{k-1} + \frac{2q^a - q^{j-1} - 1}{q-1} - \frac{q^{a-1} - 1}{q-1} \end{aligned}$$

where the last step follows by (4.2.3) and (4.2.1). As $|(R \cup G) - H_X| = |E(AG(r-1, q)| = q^{r-1}$, we have $q^{k-1} + \frac{2q^a - q^{j-1} - 1}{q-1} - \frac{q^{a-1}-1}{q-1} \ge q^{r-1}$. As the left-hand side of the last inequality is bounded above by $q^{r-3} + \frac{q^{r-3}(2q-1)}{q-1} - \frac{q^{j-1}-1}{q-1}$, we deduce that $1 + \frac{2q-1}{q-1} > q^2$. This is a

contradiction as $q \geq 2$. We conclude that $x < r-1$. By symmetry, $a < r-1$. Then, by (4.2.2) and (4.2.1), $|R| + |G| < \frac{q^r-1}{q-1}$, a contradiction. $\square$

Next, we move towards proving Theorem 4.1.1. We omit the straightforward proof of the following result.

Lemma 4.2.3. *Let M be a simple $GF(q)$-representable matroid.*

(i) If $q = 2$ and $r(M) \leq 3$, then M is a $GF(q)$-comatroid.

(ii) If $q = 3$ and $r(M) \leq 2$, then M is a $GF(q)$-comatroid.

Lemma 4.2.4. *Let M be a $GF(q)$-comatroid and suppose that M is disconnected. Then each of its components is a $GF(q)$-comatroid.*

Proof. By Lemma 4.2.3, we may assume that $r(M) \geq 4$. Let $M = M_1 \oplus M_2$. Take a shortest sequence of direct sums and complements that shows that M is a $GF(q)$-comatroid. Assume that the final step in creating M is not a direct sum. Then this final step involves taking the $(GF(q), t)$-complement of some matroid N_1 where $t \geq r(N_1)$. As M is disconnected, $t = r(N_1) \geq r(M)$, otherwise M has $AG(t-1, q)$ as a restriction and so is connected. Thus $N_1^c = M$. Moreover, as the vertical connectivity of M is one, Proposition 4.2.2 implies that N_1 is connected. Since $N_1^c = M$, the predecessor of N_1 in the construction of M is its $(GF(q), s)$-complement N_2 for some $s \geq r(N_1) + 1$. Then N_2 has $AG(s-1, q)$ as a restriction, so it is connected. The predecessor of N_2 in the production of M must again be a connected matroid N_3 of rank exceeding $r(N_2)$. Tracing back the predecessors of M in its creation as a $GF(q)$-comatroid, we obtain an infinite sequence of matroids of increasing ranks. This contradicts the fact that M is created by a finite process. We conclude that, when M is a disconnected $GF(q)$-comatroid, the final step in con-

structing it is taking the direct sum of two $GF(q)$-comatroids. Thus if M has exactly two components, then each component is a $GF(q)$-comatroid. We now argue by induction on the number of components of M. As the final step in creating M is taking a direct sum of two $GF(q)$-comatroids, it follows by induction that each component of M is a $GF(q)$-comatroid. □

Lemma 4.2.5. *For q in $\{2,3\}$, a $GF(q)$-representable matroid M such that $r(M) = r(M^c)$ and both M and M^c are connected is not a $GF(q)$-comatroid.*

Proof. For M to satisfy the hypotheses of the lemma, we must have $r(M) \geq 3$. Moreover, $r(M) \geq 4$ if $q = 2$. Assume that M is a $GF(q)$-comatroid. Again we take a shortest sequence of direct sums and complements that shows that M is a $GF(q)$-comatroid. Then the final step in creating M must have been taking a complement. As $r(M) = r(M^c)$, for some N_0 in $\{M, M^c\}$, the predecessor of N_0 in the creation of M is the $(GF(q), t)$-complement N_1 of N_0 for some $t > r(M)$. This matroid is also connected. Its predecessor in the construction of M is the $(GF(q), s)$-complement N_2 of N_1 for some $s > r(N_1)$. Again, N_2 is connected and this process must continue indefinitely, a contradiction. □

As an immediate consequence of the last lemma, we have the following.

Corollary 4.2.6. *A k-circuit is a $GF(q)$-comatroid if and only if $q + k \leq 6$.*

Lemma 4.2.7. *The restriction of a $GF(q)$-comatroid to one of its flats is a $GF(q)$-comatroid.*

Proof. We argue by induction on the rank of the $GF(q)$-comatroid M. The result holds by Lemma 4.2.3 if $r(M) \leq 2$. Now assume it holds for every $GF(q)$-comatroid of rank less than n and let M be a $GF(q)$-comatroid of rank n where $n \geq 3$. Then M is obtained by

taking complements and direct sums. Let F be a proper flat of M. Assume first that M is disconnected. Then, by Lemma 4.2.4 and the induction assumption, $M|(F \cap E(M_i))$ is a $GF(q)$-comatroid for each component M_i of M. Thus $M|F$ is a $GF(q)$-comatroid. We may now assume that M is connected. Suppose $N = M^c$. Then N is a $GF(q)$-comatroid. Let F_P be the projective flat of $PG(r(M)-1, q)$ that is spanned by F. Then $F_P - F$ is a flat of N. The complement of $N|(F_P - F)$ in F_P is $M|F$. Assume first that $r(N) < r(M)$. Then, by the induction assumption, $N|(F_P - F)$ is a $GF(q)$-comatroid. Thus $M|F$ is also a $GF(q)$-comatroid. Hence we may assume that $r(N) = r(M)$. By Lemma 4.2.5, N is not connected, so, by the induction assumption, $N|(F_P - F)$ and hence $M|F$ is a $GF(q)$-comatroid. □

Proof of Theorem 4.1.1. Suppose M is a non-empty $GF(q)$-comatroid. By Lemma 4.2.7, if F is a flat of $PG(r(M)-1, q)$, then $M|(F \cap E(M))$ is a $GF(q)$-comatroid. Hence so is its complement in F, namely $PG(r(M)-1, q)|(F - E(M))$. By Lemma 4.2.1, at least one of $r(F \cap E(M))$ and $r(F - E(M))$ is $r(F)$. Thus, by Lemma 4.2.5, if $r(F \cap E(M)) = r(F - E(M))$, then the restriction of $PG(r(M)-1, q)$ to $F \cap E(M)$ or $F - E(M)$ is disconnected.

Conversely, suppose that M is non-empty and that, for all flats F of $PG(r(M)-1, q)$ with $r(F \cap E(M)) = r(F - E(M))$, the restriction of $PG(r(M)-1, q)$ to either $F \cap E(M)$ or $F - E(M)$ is disconnected. We argue by induction on $r(M)$ that M is a $GF(q)$-comatroid. By Lemma 4.2.3, this is true if $r(M) \leq 2$. Assume it is true for $r(M) < n$ and let $r(M) = n$. If M is disconnected, then, by the induction assumption, each component is a $GF(q)$-comatroid. Hence so too is M. Thus M is connected. Sup-

pose $r(M) = r(M^c)$. Then, by the hypothesis, M^c is disconnected. Since M^c obeys the same condition as M, each of its components is a $GF(q)$-comatroid. Thus so is M^c. Hence M is a $GF(q)$-comatroid. We may now assume that $r(M) > r(M^c)$. Take F_0 to be the flat of $PG(r(M)-1,q)$ spanned by M^c. Let F_1 be a flat of F_0 with $r(F_1 \cap E(M^c)) = r(F_1 - E(M^c))$. Then $r(F_1 \cap E(M)) = r(F_1 - E(M))$ so the restriction of $PG(r(M)-1,q)$ to $F_1 \cap E(M)$ or $F_1 - E(M)$ is disconnected. Thus the restriction of F_0 to $F_1 \cap E(M^c)$ or $F_1 - E(M^c)$ is disconnected. We conclude that M^c obeys the same condition as M, so, by the induction assumption, M^c is a $GF(q)$-comatroid. The $(GF(q), r(M))$-complement of M^c is M so it too is a $GF(q)$-comatroid, as required. □

In the following result, we note that if a $GF(q)$-comatroid is connected, it is highly connected.

Proposition 4.2.8. *Let M be a connected $GF(q)$-comatroid of rank r. Then M is vertically $(r-1)$-connected.*

Proof. By Lemma 4.2.5, M^c is either disconnected or has rank less than r. If M^c is disconnected, then, by Proposition 4.2.2, M is vertically $(r-1)$-connected. We may now assume that M^c has rank less than r. Then M is an extension of $AG(r-1,q)$. Since this affine geometry is vertically $(r-1)$-connected, the result follows. □

Next we show that the class of $GF(q)$-comatroids is closed under induced minors. For a subset X of the ground set of a simple $GF(q)$-representable matroid M, we say X is *connected* if $M|X$ is connected. When X is a flat of M, we denote by X^c the matroid $(M|X)^c$ that is obtained from the projective closure of X by deleting X.

Proposition 4.2.9. *Every induced minor of a $GF(q)$-comatroid is a $GF(q)$-comatroid.*

Proof. By Lemma 4.2.7, the restriction of a $GF(q)$-comatroid to one of its flats is a $GF(q)$-comatroid. Now take an element e of M and assume that $\mathrm{si}(M/e)$ is not a $GF(q)$-comatroid. View $\mathrm{si}(M/e)$ as a restriction of $PG(r(M)-2,q)$. Then, by Theorem 4.1.1, there is a flat F of $\mathrm{si}(M/e)$ such that F and F^c are both connected and each has rank k, say. Observe that $\mathrm{cl}_M(F \cup e)$ is a connected flat of M of rank $k+1$ unless e is a coloop of $M|(F \cup e)$. In the exceptional case, F is a flat of M and, therefore, M has a flat F such that both F and F^c are connected of rank k, a contradiction. We deduce that $\mathrm{cl}_M(F \cup e)$ is a connected flat of rank $k+1$. We complete the proof by establishing the contradication that the complement of $\mathrm{cl}_M(F \cup e)$ is also connected of rank $k+1$. To see this, note that, for each element g of F^c, all the elements apart from e that are on the projective line containing $\{e,g\}$ in $PG(r(M)-1,q)$ are in the complement of $\mathrm{cl}_M(F \cup e)$. Thus this complement contains a set of rank $k+1$ that is a union of interlocking 4-circuits. Hence this complement is connected. □

4.3. Connected hyperplanes

Kelmans [13] and Seymour (in [20]) independently established that if M is a simple connected binary matroid that has no cocircuits of size less than three, then M has a connected hyperplane. That theorem was extended in several ways by McNulty and Wu [16]. In this section, we note two of these extensions and prove an analogue for ternary matroids of the result of Kelmans and Seymour. These results on connected hyperplanes will be crucial in proving our characterizations of binary and ternary comatroids.

We begin the section by identifying when there is a free element in a binary or ternary matroid, where an element e is *free* in a matroid M if e is not a coloop of M and

the only circuits that contain e are spanning. Doubtless, the results in the next lemma are known but we include the proofs for completeness. In a rank-zero matroid, every element is free. In a rank-one matroid, the free elements are the non-loops unless the matroid has a coloop in which case there are no free elements. Thus the next result only considers matroids M of rank at least two noting that e is free in M if and only if e is free in $\mathrm{si}(M)$ and e is in no 2-circuits of M.

Lemma 4.3.1. *Let M be a simple $GF(q)$-representable matroid of rank at least two and let e be a free element of M. Then*

(i) M is a circuit when $q = 2$; and

(ii) when $q = 3$, either $M \cong U_{2,4}$, or M can be obtained from a circuit C containing e by, for some subset D of $C - e$, 2-summing a copy of $U_{2,4}$ across each element of D.

Proof. Since e is free in M, there is a spanning circuit C_0 of M containing e. Then $M|C_0$ is represented over $GF(q)$ by $[I_r|\mathbf{1}]$ where $\mathbf{1}$, the column of all ones, is labelled by e. When $q = 2$, we cannot add any further elements without creating either a 2-circuit, or a circuit that contains e and has fewer than $r + 1$ elements. Thus (i) holds.

Now suppose that $q = 3$. If $r(M) = 2$, then M is isomorphic to $U_{2,3}$ or $U_{2,4}$. Assume that $r(M) \geq 3$. Let Z be a matrix representing M over $GF(3)$ and having $[I_r|\mathbf{1}]$ as its first $r + 1$ columns. We will write the elements of $GF(3)$ as $0, 1$, and -1. Let f be a column of Z other than one of the first $r + 1$ columns. As M is simple, f has at least two non-zero entries. If f has two non-zero entries with a common sign, then there is a circuit containing $\{e, f\}$ having at most r elements, a contradiction. It follows that f has exactly two non-zero entries and that these entries have different signs. If columns f and

g have their non-zero entries in, respectively, rows 1 and 2, and rows 1 and 3, then M has an r-element circuit containing $\{e, f, g\}$. We conclude that two distinct columns of Z that are not columns of $[I_r|\mathbf{1}]$ must have disjoint sets of rows containing their non-zero entries. It follows that M can be obtained from a circuit C containing e by, for some subset D of $C - e$, 2-summing a copy of $U_{2,4}$ across each element of D. To see this, for each column of Z with two entries of opposite signs, add an additional column to Z obtained by changing the sign of one of these entries. These added elements form the basepoints of the 2-sums. □

The following technical result will be helpful in proving our results on connected hyperplanes.

Lemma 4.3.2. *In a simple connected matroid M, let e be an element and A be a maximal subset of $E(M)$ that is connected, non-spanning and contains e. Let C be a circuit of M that meets both A and $E(M)-A$ such that $C-A$ is minimal. Then A is a flat, $r(A \cup C) = r(M)$, the set $C - A$ is a series class of $M|(A \cup C)$,*

$$r(M) = r(A) + |C - A| - 1,$$

and one of the following holds:

(i) A is a connected hyperplane of M; or

(ii) $C - A$ is a series class of M with at least three elements; or

(iii) $|C - A| \geq 3$ and $E(M) - (A \cup C)$ is non-empty.

Proof. The minimality of $C-A$ implies that, in $M|(A \cup C)$, a circuit that meets $C-A$ must contain $C-A$, so $C-A$ is a series class. The maximality of A implies that A is a flat of M and that $r(A \cup C) = r(M)$. Thus $r(M) = r(A) + |C-A| - 1$, so $|C-A| \geq 2$. If $|C-A| = 2$,

then A is a connected hyperplane of M. Thus we may assume that $|C - A| \geq 3$. In that case, (ii) or (iii) holds. $\square$

The next result extends the theorem of Kelmans and Seymour, borrowing much from Seymour's proof.

Theorem 4.3.3. *Let e be an element of a simple connected binary matroid M. Then*

(i) M is a circuit; or

(ii) M has a connected hyperplane containing e; or

(iii) M has a series class of size at least three that avoids e.

Proof. Assume that the theorem fails. By Lemma 4.3.1, e is not free in M. Thus M has a subset A that contains a circuit containing e and is maximal with respect to being connected and non-spanning. We take a circuit C_1 that meets both A and its complement such that $|C_1 - A|$ is minimal. By Lemma 4.3.2, $|C_1 - A| \geq 3$ and $E(M) - (C_1 \cup A)$ contains an element, say x. Moreover, for y in $C_1 - A$, the set $A \cup (C_1 - y)$ contains a basis B of M. Let $C_2 = C(x, B)$. Then C_2 meets A otherwise, as M is binary, $C_1 \triangle C_2$ is a circuit that contradicts the choice of C_1. Now $|C_2 - A| \geq |C_1 - A|$. Hence C_2 contains exactly $|C_1 - A| - 1$ elements of $C_1 - A$. Thus $C_1 \triangle C_2$ contains a circuit D that contains x, meets A, and has exactly two elements not in A. Then D contradicts the choice of C_1. $\square$

As an immediate consequence of this theorem, we have the following.

Corollary 4.3.4. *Let M be a simple connected binary matroid M. Then*

(i) M is a circuit; or

(ii) M has a connected hyperplane; or

(iii) M has at least two distinct series classes of size at least three.

The next two results of McNulty and Wu [16, Theorem 1.4 and Lemma 2.10] are much more substantial extensions of the theorem of Kelmans and Seymour. Both of these results will be used in the proof of Theorem 4.1.2.

Theorem 4.3.5. *Let M be a simple connected binary matroid with no cocircuits of size less than three. Then every element of M is in at least two connected hyperplanes; and M has at least four connected hyperplanes.*

Lemma 4.3.6. *Let M be a 3-connected binary matroid with at least four elements. Then, for any two distinct elements e and f of M, there is a connected hyperplane containing e and avoiding f.*

McNulty and Wu [16, Fig. 1] also showed that a simple connected binary matroid with no cocircuits of size less than three may have exactly four connected hyperplanes. In addition, they noted that Joseph Bonin has pointed out that the dual of $PG(2,3)$ is a 3-connected ternary matroid having no connected hyperplanes. Of course, the same is true for the duals of all of the matroids $PG(r-1,3)$ with $r \geq 3$. As another example of a simple connected ternary matroid with no cocircuits of size less than three and no connected hyperplanes, take a circuit with at least three elements and 2-sum a copy of $U_{2,4}$ across each element. Each of these examples has numerous triads. As we shall see, by confining our attention to simple connected ternary matroids having no cocircuits of size less than four, we can establish the existence of at least two connected hyperplanes. The next result is key to proving this.

Theorem 4.3.7. *Let M be a simple connected matroid having no cocircuits of size less than four. Assume that M has no $U_{2,5}$-minor and no $U_{3,5}$-minor. Let e be an element of M that is not free. Then M has a connected hyperplane containing e.*

Proof. Since e is not free, $E(M)$ has a subset A that contains a circuit containing e and is maximal with respect to being connected and non-spanning. Assume that the theorem fails. As M is connected, it has a circuit meeting both A and its complement. Choose such a circuit C_1 for which $|C_1 - A|$ is a minimum. By Lemma 4.3.2, A is a flat of M, while $C_1 - A$ is a series class in $M|(A \cup C_1)$, and $r(A \cup C_1) = r(M)$. Moreover,

4.3.7.1. $r(M) = r(A) + |C_1 - A| - 1.$

As M has no cocircuits of size less than four, $|E(M) - (A \cup C_1)| \geq 2$. Take s in $C_1 - A$. Then M has a basis B that contains $C_1 - s$ and is contained in $A \cup (C_1 - s)$. Choose x in $E(M) - (A \cup C_1)$ and let C_2 be $C(x, B)$.

Next we show the following.

4.3.7.2. *If* $C_2 \cap A = \emptyset$, *then* $|C_2| = 3$.

Let $C_1 \cap C_2 = \{y_1, y_2, \ldots, y_t\}$. By circuit elimination, M has a circuit D_i that contains x and not y_i such that $D_i \subseteq C_1 \cup x$. Then $D_i \supseteq C_1 - C_2$. Moreover, the choice of C_1 implies that $D_i - A = ((C_1 - A) - y_i) \cup x$. Thus $D_i = (C_1 - y_i) \cup x$. From $M|(C_1 \cup x)$, contract $C_1 - \{y_1, y_2, \ldots, y_k, s\}$. The resulting matroid N has ground set $\{y_1, y_2, \ldots, y_k, s, x\}$ and has every subset of size $k+1$ as a circuit except possibly $\{y_1, y_2, \ldots, y_k, x\}$. If a proper subset of $\{y_1, y_2, \ldots, y_k, x\}$ is a circuit of N, then this circuit is a proper subset of a $(k+1)$-element circuit of N, a contradiction. Thus $N \cong U_{k,k+2}$. As M has no $U_{3,5}$-minor and M is simple, we deduce that $k = 2$, so $|C_2| = 3$. Hence (4.3.7.2) holds.

Next we note that

4.3.7.3. $|C_1 - A| \geq 4.$

As the theorem fails, $|C_1 - A| > 2$, by 4.3.7.1. Assume that $|C_1 - A| = 3$. Thus $r(M/A) = 2$. Moreover, $|E(M/A)| \geq 5$ as $|E(M) - (A \cup C_1)| \geq 2$. Since A is a flat of M, the matroid M/A has no loops. Suppose it has a 2-circuit $\{u, v\}$. Then M has a circuit C' such that $\{u, v\} \subseteq C' \subseteq \{u, v\} \cup A$. Thus C' contradicts the choice of C_1. We deduce that M/A is simple, so M/A has $U_{2,5}$ as a restriction, a contradiction. Thus 4.3.7.3 holds.

4.3.7.4. *For x in $E(M) - (A \cup C_1)$, there is an element s of $C_1 - A$ such that M has a triangle that contains x and has its other two elements in $C_1 - (A \cup s)$.*

Assume that M has no such triangle. For s in $C_1 - A$, let B_s be a basis of M containing $C_1 - s$ and let $C_s = C(x, B_s)$. By 4.3.7.2, C_s meets A. By the choice of C_1, we deduce that $C_s - A = (C_1 - A - s) \cup x$. Let $|C_1 - A| = m$ and $N' = M|(A \cup C_1 \cup x)$. Then, for every m-element subset Y of $(C_1 - A) \cup x$, there is a circuit of N' that meets $(C_1 - A) \cup x$ in Y and also meets A. Now $r(N') = r(M) = r(A) + |C_1 - A| - 1$. Contracting A from N' gives a matroid of rank $m - 1$ having $m + 1$ elements. Take an m-element subset Y of $(C_1 - A) \cup x$. Then Y contains a circuit Y' of N'/A. Thus $Y' \cup A$ contains a circuit Y'' of M containing Y'. Then Y'' meets A otherwise $Y'' = Y' \subseteq Y$, a contradiction as Y is independent in M. Thus, by the choice of C_1, we must have that $|Y'' - A| = m$. Hence $m = |Y| \geq |Y'| \geq |Y'' - A| \geq m$, so $Y' = Y$ and Y is a circuit of N'/A. Thus $N'/A \cong U_{m-1,m+1}$. By 4.3.7.3, $m \geq 4$, so M has a $U_{3,5}$-minor, a contradiction. Thus 4.3.7.4 holds.

4.3.7.5. *M does not have a 4-element subset X with exactly two elements in $C_1 - A$ and exactly two elements in $E(M) - (A \cup C_1)$ such that $M|X \cong U_{2,4}$.*

Assume that $\{y_1, y_2, x_1, x_2\}$ is such a 4-element subset X of $E(M)$ where $\{y_1, y_2\} \subseteq$

$C_1 - A$. Then $r(A \cup (C_1 - \{y_1, y_2\})) = r(M) - 1$ and $r(X) = 2$. Thus, in $M|(A \cup C_1 \cup \{x_1, x_2\})$, which is connected, X is 2-separating. Therefore, $M|(A \cup C_1 \cup \{x_1, x_2\})$ is the 2-sum, with basepoint p of connected matroids M_1 and M_2 with ground sets $A \cup (C_1 - \{y_1, y_2\}) \cup p$ and $X \cup p$. Since $|X \cup p| = 5$ and M_2 has rank 2, we must have that p is parallel to some element of X otherwise M has a $U_{2,5}$-minor. Thus a member of $\{y_1, y_2, x_1, x_2\}$ is in the closure of $A \cup (C_1 - \{y_1, y_2\})$. Neither y_1 nor y_2 is in this closure. If x_1 or x_2 is, then there is a circuit D containing x_i for some i in $\{1, 2\}$ such that $D \subseteq A \cup (C_1 - \{y_1, y_2\}) \cup x_i$. The choice of C_1 implies that D does not meet A. Thus $D \subseteq (C_1 - A) \cup x_i$. Then, by circuit elimination, $(D \cup \{x_i, y_1, y_2\}) - x_i$ contains a circuit. But this circuit is properly contained in C_1, a contradiction. We conclude that 4.3.7.5 holds.

4.3.7.6. *M does not have two triangles $\{y_1, y_2, x_2\}$ and $\{y_1, y_3, x_3\}$ where y_1, y_2, and y_3 are distinct elements of $C_1 - A$, and x_2 and x_3 are distinct elements of $E(M) - (A \cup C_1)$.*

Assume that M does have two such triangles. Then M has $(C_1 - y_1) \cup x_2$ as a circuit, C_1' say. Thus $\{y_2, x_2, y_3, x_3\}$ is a circuit of M having exactly three elements in $C_1' - A$. This is the fundamental circuit of x_3 with respect to a basis of M that contains $C_1' - t$ where $t \in C_1' - A - \{x_2, y_2, y_3\}$, the existence of such an element t being a consequence of 4.3.7.3. Thus, using C_1' in place of C_1 in 4.3.7.2, we get a contradiction. Hence 4.3.7.6 holds.

By 4.3.7.4, for each element x in $E(M) - (A \cup C_1)$, there is a triangle of M that contains x and two elements of $C_1 - A$. Moreover, by 4.3.7.5 and 4.3.7.6, if x_1 and x_2 are distinct elements of $E(M) - (A \cup C_1)$, then the corresponding triangles are disjoint.

Suppose that there are exactly k elements, $x_1, x_2, \ldots, x_k$, in $E(M) - (A \cup C_1)$ and

that the corresponding triangles are $\{x_i, y_i, z_i\}$ for $1 \leq i \leq k$ where $\{y_i, z_i\} \subseteq C_1 - A$. Then $E(M) - (A \cup C_1) - \{y_1, z_1\}$ has rank $r(M) - 1$. The closure of this set contains $\{x_2, x_3, \ldots, x_k\}$. The complement of this closure is $\{x_1, y_1, z_1\}$. Therefore M has a triad. This contradiction completes the proof of the theorem. □

Corollary 4.3.8. *Let M be a simple connected ternary matroid having no cocircuits of size less than four. Then M has at least two connected hyperplanes.*

Proof. By Lemma 4.3.1(ii), since M has no cocircuits of size less than four, M has no free elements. Let e be an element of M. Then, by Theorem 4.3.7, M has a connected hyperplane H_e containing e. For f in $E(M) - H_e$, there is a connected hyperplane H_f containing f, so the corollary holds. □

In view of Theorem 4.3.5, it is of interest to specify the minimum number of connected hyperplanes in a simple connected ternary matroid with no cocircuits of size less than four. There are infinitely many examples of such matroids with exactly four connected hyperplanes but we do not know if four is indeed the minimum number of connected hyperplanes. To get a family of examples with exactly four connected hyperplanes, first take a graph G formed from two vertex-disjoint paths $x_1x_2 \ldots x_n$ and $y_1y_2 \ldots y_n$ for some $n \geq 1$ by adding the n edges x_iy_i, the $n-1$ edges of the form x_iy_{i+1} for $1 \leq i \leq n-1$, and the $n-1$ edges of the form $x_{j+1}y_j$ for $1 \leq j \leq n-1$. Then take two copies, N_1 and N_2, of $M(K_4)$, pick a point p_i of N_i and freely add a point q_i to one of the triangles of N_i not containing p_i. Finally, take the parallel connection of N_1 and $M(G)$ with respect to the basepoints p_1 and x_1y_1, and then attach N_2 to the resulting matroid via parallel connection with respect to the basepoints x_ny_n and p_2. The resulting simple connected ternary

matroid has $5n+8$ elements, rank $2n+3$, and has no cocircuits of size less than four. It also has exactly four connected hyperplanes.

4.4. Induced-restriction-minimal non-$GF(2)$-comatroids

An *induced-restriction-minimal non-$GF(q)$-comatroid* is a $GF(q)$-representable matroid M that is not a $GF(q)$-comatroid such that every proper flat of M is a $GF(q)$-comatroid. The collection of such matroids M will be denoted by $\mathcal{M}_q$. Clearly, $M^c \in \mathcal{M}_q$ for every matroid M in $\mathcal{M}_q$. This section begins with some preliminary results that will be used in the proofs of the main theorems. It concludes with proofs of Theorem 4.1.2 and Corollary 4.1.4.

Lemma 4.4.1. *For q in $\{2,3\}$, let X be a subset of $PG(r-1,q)$ having at least $q^{r-1}+1$ elements. Then the matroid $PG(r-1,q)|X$ is connected and has rank r.*

Proof. Observe that X has more elements than a hyperplane of $PG(r-1,q)$, so $PG(r-1,q)|X$ has rank r. Assume that $PG(r-1,q)|X$ is disconnected. Then, for some j with $1 \leq j \leq r-1$, the matroid $PG(r-1,k)|X$ is contained in $PG(j-1,q) \oplus PG(r-j-1,q)$. Thus $|X| \leq \frac{q^j-1}{q-1} + \frac{q^{r-j}-1}{q-1} = \frac{q^j+q^{r-j}-2}{q-1}$. This function of j is maximized when j is 1 or $r-1$, so $|X| \leq q^{r-1}$, a contradiction. □

Lemma 4.4.2. *For q in $\{2,3\}$, let N be the parallel connection of $PG(j-1,q)$ and $PG(k-1,q)$ where $2 \leq j \leq k$ and $k \geq 3$. Then the complement N^c of N has rank equal to $r(N)$.*

Proof. Assume that $r(N^c) < r(N)$. Then N has $AG(r(N)-1,q)$ as a restriction. Now $AG(r(N)-1,q)$ is 3-connected since $r(N) \geq 4$ so N is 3-connected, a contradiction. □

The next result is from [21].

Theorem 4.4.3. *Let n be an integer exceeding one and X and Y be subsets of the ground set of a matroid M. Suppose $M|X$ and $M|Y$ are both vertically n-connected and $r(X) + r(Y) - r(X \cup Y) \geq n - 1$. Then $M|(X \cup Y)$ is vertically n-connected.*

The following is a straightforward consequence of Proposition 4.2.2.

Lemma 4.4.4. *Let $r \geq 4$. For an arbitrary prime power q, color the elements of $PG(r-1, q)$ red or green. Then either $PG(r-1,q)|G$ or $PG(r-1,q)|R$ is connected of rank r.*

Recall that, for a flat F in a simple $GF(q)$-represented matroid M, we write F^c for the matroid $(M|F)^c$.

Lemma 4.4.5. *Let M be a matroid in $\mathcal{M}_2$ such that $r(M) \geq 5$ and M has a 2-cocircuit. Then M is isomorphic to a circuit or to $P(U_{3,4}, U_{3,4})$.*

Proof. Assume that the result fails. Since M has a 2-cocircuit, it has a maximal non-trivial series class S. Thus $M = M_1 \oplus_2 M_2$ where M_2 is a circuit with ground set $S \cup p$, and p is the basepoint of the 2-sum. If p is parallel to an element s in M_1, then we move s into M_2 so that it become parallel to p there.

Suppose that p is free in M_1. Then, by Lemma 4.3.1(i), M_1 is a circuit. As M is not a circuit and $r(M) \geq 5$, we deduce that the element s exists. Thus M is the parallel connection of two circuits. By Corollary 4.2.6, neither of these circuits has more than four elements. Hence $M \cong P(U_{3,4}, U_{3,4})$, a contradiction. We deduce that p is not free in M_1. Thus M_1 has a non-spanning circuit C_p that contains p. If $r((C_p \cup S) - p) \geq 4$, then the closure, F, of $(C_p \cup S) - p$ is a connected proper flat in M, Moreover, by Lemmas 4.2.2 and 4.4.2, F^c is also connected of rank $r(F)$. This contradicts the minimality of M. We deduce that $r((C_p \cup S) - p) = 3$. Hence every non-trivial series class of M has exactly

two elements. Now, by Theorem 4.3.3, as M_1 is not a circuit and does not have a series class of size at least three avoiding p, it has a connected hyperplane H containing p. Then $\mathrm{cl}_M((H \cup S) - p)$ is a connected proper flat, F, of M of rank $r(M) - 1$. As above, F^c is connected of rank $r(F)$, a contradiction. $\square$

Lemma 4.4.6. *Let M be a matroid in $\mathcal{M}_2$ such that $r(M) \geq 5$. Then*

(i) M is a circuit; or

(ii) $M \cong P(U_{3,4}, U_{3,4})$; or

(iii) M is 3-connected.

Proof. Assume that neither (i) nor (ii) holds. Then, by Lemma 4.4.5, we may assume that M is cosimple. Suppose that M is not 3-connected. Then $M = M_1 \oplus_2 M_2$ where $r(M_1) \geq r(M_2)$ and one of M_1 and M_2 may have an element parallel to the basepoint p of the 2-sum. When this occurs, we may assume, by moving an element from M_2 to M_1 if needed, that the element is parallel to p in M_2. Since M is cosimple, neither M_1 nor M_2 is either a circuit or a circuit with an element parallel to p. Hence $r(M_2) \geq 3$. As M_2 is not a circuit, by Lemma 4.3.1, M_2 has a non-spanning circuit C_p containing p. Then the closure F of $(E(M_1) \cup C_p) - p$ is a connected proper flat of M. By Lemmas 4.2.2 and 4.4.2, F^c is connected of rank $r(F)$, a contradiction. $\square$

The next result shows that a matroid M in $\mathcal{M}_2$ such that neither M nor M^c is a circuit has rank at most five.

Theorem 4.4.7. *Let M be a matroid in $\mathcal{M}_2$ such that $r(M) \geq 6$. Then M or M^c is a circuit.*

Proof. Let $P_{r(M)}$ denote the binary projective geometry of rank $r(M)$ such that the set G

of green elements of $P_{r(M)}$ corresponds to M and the set R of red elements of $P_{r(M)}$ corresponds to M^c. Observe that, for each projective flat F of $P_{r(M)}$ with $4 \leq r(F) < r(M)$, it follows by Lemma 4.4.4 and the minimality of M that exactly one of $F|R$ and $F|G$ is connected of rank $r(F)$. We call F *red* or *green* depending on whether $F|R$ or $F|G$ is connected of rank $r(F)$. We may assume that both M and M^c are cosimple otherwise we have our result by Lemma 4.4.5. Let F be a rank-$(r-2)$ flat of $P_{r(M)}$. Note that F is contained in exactly three hyperplanes, say H_1, H_2, and H_3 of $P_{r(M)}$. We note the following.

4.4.7.1. *At least two of H_1, H_2, and H_3 have the same color as F.*

Suppose that F is green and assume that H_1 and H_2 are red. It follows that each of $H_1 - F$ and $H_2 - F$ contains at most one green element and so the green elements in $(H_1 \cup H_2) - F$ form a cocircuit of M with at most two elements, a contradiction. Similarly, if F is red, we get a cocircuit of M^c of size at most two, a contradiction.

Now let G_1 and R_1 be the sets of green and red hyperplanes, respectively, of $P_{r(M)}$. We note the following.

4.4.7.2. *At most one of the rank-$(r-2)$ projective flats contained in a projective hyperplane H has a color different from that of H.*

Observe that if two rank-$(r-2)$ projective flats contained in H have the same color, then, by Theorem 4.4.3, their join is colored the same as the two flats, a contradiction. Thus 4.4.7.2 holds.

Let G_2 and R_2 be the sets of green and red projective flats of $P_{r(M)}$ of rank $r-2$. We consider the bipartite graph B with vertex sets $G_1 \cup R_1$ and $G_2 \cup R_2$ such that a vertex X in $G_1 \cup R_1$ is adjacent to a vertex Y in $G_2 \cup R_2$ if $Y \subseteq X$. We count the number of *cross*

edges of this graph, that is, the G_1R_2-edges and G_2R_1-edges. By 4.4.7.2, the number of G_1R_2-edges is at most $|G_1|$, and the number of G_2R_1-edges is at most $|R_1|$. Observe that each pair $\{H_G, H_R\}$ where $H_G \in G_1$ and $H_R \in R_1$ corresponds to either a G_1R_2- edge or a G_2R_1-edge e depending on whether $H_G \cap H_R$ is red or green. Note that there is a third projective hyperplane H' such that $H' \cap H_G = H' \cap H_R$ and, by 4.4.7.1, has the same color as $H_G \cap H_R$. Observe that either $\{H', H_G\}$ or $\{H', H_R\}$ corresponds to the cross edge e depending on whether $H_G \cap H_R$ is red or green. Hence each cross edge corresponds to exactly two pairs $\{H_G, H_R\}$ where $H_G \in G_1$ and $H_R \in R_1$. As the number of cross edges is bounded above by $|G_1| + |R_1|$ and below by $\frac{1}{2}|G_1||R_1|$, we have $\frac{1}{2}|G_1||R_1| \leq |G_1| + |R_1|$. We may assume that $|R_1| \geq |G_1|$. Then $|G_1| \leq \frac{2|G_1|}{|R_1|} + 2 \leq 3$, a contradiction to Theorem 4.3.5. □

It remains to determine the members of $\mathcal{M}_2$ of rank 4 or 5. The next lemma takes care of the rank-4 case.

Lemma 4.4.8. *A rank-4 binary matroid M is a member of $\mathcal{M}_2$ if and only if M or M^c is the cycle matroid of one of the six graphs in Figure 4.1.*

Proof. First assume that $M \in \mathcal{M}_2$. We may assume that $|E(M)| \leq |E(M^c)|$, so $|E(M)| \leq 7$. If M has a 5-circuit, then M is a 5-circuit or a 1- or 2-element extension thereof. One can now check that M is the cycle matroid of one of the graphs on the first line of Figure 4.1. We may now assume that M has no 5-circuits. Thus $|E(M)|$ is 6 or 7. If $|E(M)| = 6$, then M^* is connected of rank two, so M is the cycle matroid of $K_{2,3}$. Finally, if $|E(M)| = 7$, then M is the cycle matroid of the last graph in Figure 4.1. The proof of the converse is immediate as every rank-3 binary matroid is a binary

comatroid. □

The following result from [14] will be used to simplify the computational task of finding the rank-5 members of $\mathcal{M}_2$. The matroids in this theorem will only appear in the proof of Lemma 4.4.10 and they will be defined there.

Theorem 4.4.9. *An internally* 4*-connected binary matroid has no* $M(K_{3,3})$*- minor if and only if it is*

(i) cographic; or

(ii) isomorphic to a triangular or triadic Möbius matroid; or

(iii) isomorphic to one of 18 *sporadic matroids of rank at most* 11.

Lemma 4.4.10. *Let* M *be a matroid in* $\mathcal{M}_2$ *such that* $r(M) = 5$*. Then* M *or* M^c *is not cosimple.*

Proof. Assume that M and M^c are cosimple. Then, by Lemma 4.4.6, both M and M^c are 3-connected. By Lemma 4.4.4, for every hyperplane H of P_5, exactly one of $H|G$ or $H|R$ is connected. We call the H red or green depending on whether $H|R$ or $H|G$ is connected of rank four. We first show that

4.4.10.1. $E(M)$ *has no set* X *of rank* 3 *such that* $r(E(M) - X) = 4$.

Denote $E(M) - X$ by Y and assume that $r(X) = 3$ and $r(Y) = 4$. Let Y_P and X_P denote the projective flats spanned by Y and X, respectively. Observe that $Y_P \cap X_P$ is a projective line, say, $L = \{x, y, z\}$. As M has no 2-cocircuits, it follows that $E(M) - Y_P$ has at least three elements, including say b_1, b_2, and b_3 such that $\{x, b_1, b_2\}$ is a projective line, say L_1. Let M' be the matroid obtained from $M|Y_P$ by adding x, y, and z if they are not already in $E(M)$. Note that, for k in $\{1, 2\}$, a k-separation of M' induces a k-separation of

M and therefore M' is 3-connected. By Lemma 4.3.6, M' has a connected hyperplane H that contains x but not y or z. Observe that either $P(H, L_1)$ or $P(H, L_1)\backslash x$ is a connected hyperplane H' of M depending on whether or not x is an element of $E(M)$. By Proposition 4.2.2 and Lemma 4.4.2, it follows that the complement of H' is connected of rank four, a contradiction.

4.4.10.2. *A connected hyperplane of M has at least seven elements.*

Suppose such a connected hyperplane has at most six elements. Then its complement in $PG(3, 2)$ has at least nine elements. By Lemma 4.4.1, this complement is connected of rank four, a contradiction. Thus 4.4.10.2 holds.

By 4.4.10.1, it follows that M is internally 4-connected and has no triads. By Theorem 4.3.5, M has a connected hyperplane, so $|E(M)| \geq 11$ by 4.4.10.2.

Suppose that M has no $M(K_{3,3})$-minor. By Theorem 4.4.9, we first suppose that M is cographic and therefore $E(M) \leq 12$ [11]. Since M has no cocircuits of size less than four, it follows that every hyperplane of M has at most eight elements. Therefore, for a connected hyperplane H of M, by 4.4.10.2, $|H|$ is either seven or eight. It follows that H^c has seven or eight elements. As this complement is either disconnected or has rank at most three, it is either F_7 or $F_7 \oplus U_{1,1}$. This implies that H has F_7^* as a restriction. Thus M is not regular, a contradiction.

Next suppose that M is a triangular or a triadic Möbius matroid. Since M has no triads, M is the rank-5 triangular Möbius matroid, Δ_5, and has the reduced representation [14] shown on the left of Figure 4.2. Observe that $\{e, j, k, l, m\}$ is a connected hyperplane of M of size five, a contradiction to 4.4.10.2. We may now assume that M is a rank-5 spo-

radic matroid, so M is isomorphic to a matroid in $\{M_{5,11}, T_{12}/e, M^a_{5,12}, M^b_{5,12}, M_{5,13}\}$ [14]. Since $M_{5,11}$ has a triad, M is not isomorphic to $M_{5,11}$. When M is isomorphic to T_{12}/e, it has the representation shown on the right of Figure 4.2. Then $\{f, g, h, i, j\}$ is a connected hyperplane of M, a contradiction to 4.4.10.2.

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\!\!\begin{array}{c} \begin{array}{cccccccc} f & g & h & i & j & k & l & m \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} f & g & h & i & j & k \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Figure 4.2. Δ_5 and T_{12}/e.

If M is isomorphic to $M^a_{5,12}$, then M has the representation shown on the left of Figure 4.3. Then M has $\{f, g, h, i, j, l\}$ as a connected hyperplane of M, contradicting 4.4.10.2. Similarly, if M is isomorphic to $M^b_{5,12}$, then M has the representation shown on the right of Figure 4.3. Then $\{f, g, h, i, j, l\}$ is a connected hyperplane of M, again contradicting 4.4.10.2.

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\!\!\begin{array}{c} \begin{array}{ccccccc} f & g & h & i & j & k & l \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\!\!\begin{array}{c} \begin{array}{ccccccc} f & g & h & i & j & k & l \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Figure 4.3. $M^a_{5,12}$ and $M^b_{5,12}$.

We may now assume that M is isomorphic to $M_{5,13}$ and therefore has the representation in Figure 4.4. Observe that $\{a, b, d, e, f, i, j\}$ is a connected hyperplane, H, of M such that H^c is also connected of rank 4, a contradiction. We conclude that M is not one

$$
\begin{array}{c}
 \\ a \\ b \\ c \\ d \\ e
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} f & g & h & i & j & k & l & m \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}
\end{array}
$$

Figure 4.4. $M_{5,13}$.

of the five rank-5 sporadic matroids.

We may now assume that M has an $M(K_{3,3})$-minor and so M is an extension of $M(K_{3,3})$. By symmetry, M^c is also an extension of $M(K_{3,3})$. Since P_5 has 31 hyperplanes and $M(K_{3,3})$ has six connected hyperplanes, we deduce that

4.4.10.3. *M has at most 25 connected hyperplanes.*

Figure 4.5 shows the vertex-edge incidence matrix of $K_{3,3}$, which is a binary representation for $M(K_{3,3})$. Although $r(M(K_{3,3})) = 5$, we use this representation because it displays the symmetries of $M(K_{3,3})$ well. The P_5 into which M is embedded is spanned by $\{a, b, c, d, e, f, g, h, i\}$.

For $1 \leq i \leq 6$, let H_i be the connected hyperplane of $M(K_{3,3})$ that is complementary to the vertex bond of v_i in $K_{3,3}$, and let H_i' be the hyperplane of P_5 spanned by

$$
\begin{array}{c}
 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc} a & b & c & d & e & f & g & h & i \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

Figure 4.5. The vertex-edge incidence matrix of $K_{3,3}$.

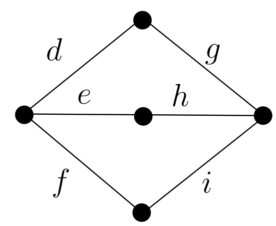


Figure 4.6. The labelled $K_{2,3}$ corresponding to the hyperplane H_1.

H_i. As $H_1'|G$ is an extension of $M(K_{2,3})$, it follows that $H_1'|R$ is a restriction of the complement of $M(K_{2,3})$ in P_4. This complement is isomorphic to $P(F_7, U_{2,3})$, where p is the basepoint of the parallel connection. The $K_{2,3}$ corresponding to H_1 is shown in Figure 4.6. The matroid $P(F_7, U_{2,3})$ that is the complement of this $M(k_{2,3})$ is labelled as in Figure 4.7. Here elements are labelled by the corresponding vectors. Because $H_1'|R$ is not connected of rank 4, it is isomorphic to a restriction of either $U_{2,3} \oplus U_{2,3}$ or $F_7 \oplus U_{1,1}$. Assume the former. Then the red elements of H_1' are contained in the 2-separating triangle in $P(F_7, U_{2,3})$ and one of the four triangles of F_7 that avoid p, where p corresponds to the vector $d + g$. Thus we have the following four cases:

(i) $e + g, e + i$, and $d + i$ are green;

(ii) $e + g, g + i$, and $e + f$ are green;

(iii) $d + e, g + i$, and $e + i$ are green;

(iv) $d + e, d + i$, and $e + f$ are green.

By permuting the vertices v_4, v_5, and v_6, we see that the last three cases are symmetric. Thus M is an extension of one of the two matroids whose representations are shown in Figure 4.8.

Using SageMath [26], we apply the given hyperplane-counting algorithm to the two matroids whose representations are given in Figure 4.8. This shows that, for every exten-

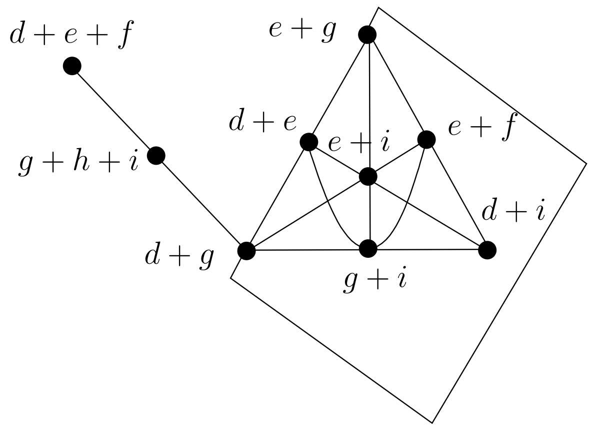


Figure 4.7. The labelled $P(F_7, U_{2,3})$ corresponding to the complement of H_1 in P_4.

$$\begin{array}{c}
\begin{array}{cccccccccccc} a & b & c & d & e & f & g & h & i & g+e & d+i & i+e \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}
\end{array}$$

$$\begin{array}{c}
\begin{array}{cccccccccccc} a & b & c & d & e & f & g & h & i & g+e & g+i & e+f \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}
\end{array}$$

Figure 4.8. Two extensions of $M(K_{3,3})$.

sion of these matroids, either the number of green hyperplanes exceeds 25, a contradiction to 4.4.10.3; or the sum of the number of red and green hyperplanes exceeds 31, the number of hyperplanes of P_5, and again we have a contradiction. Note that, when we run the above algorithm with $|S| = 10$, we do not obtain any matroids. Thus the search can be restricted to sets S with at most ten elements.

Next suppose that $H_1'|R$ is a restriction of a copy of $F_7 \oplus U_{1,1}$. First we assume that these red elements are contained in the union of the 2-separating triangle of $P(F_7, U_{2,3})$

Algorithm Counting hyperplanes of M and M^c

Require: Input a simple binary matroid N of rank five

Set $i \leftarrow 0$, $j \leftarrow 0$

for a subset S of $P_5 - E(N)$ **do**

    Set $i \leftarrow 0$, $j \leftarrow 0$

    Set $M = P_5|(E(N) \cup S)$

    $i \leftarrow$ number of connected hyperplanes of M.

    if $i < 26$ **then**

        $j \leftarrow$ number of connected hyperplanes of M^c.

if $i + j < 32$ **then**

    print M

with another triangle through p and one further point, z. Although there are three such lines through p and four choices for z for each, permuting v_4, v_5, and v_6 reduces these twelve cases to the following two cases:

(i) $e+i, e+f$, and $g+i$ are green;

(ii) $e+i, e+f$ and $d+i$ are green.

Thus M is an extension of one of the two matroids whose representations are shown in Figure 4.9. Again using SageMath [26] and applying the given hyperplane-counting algorithm to these two matroids, we see that, for every extension of these matroids, either the number of green hyperplanes exceeds 25, or the sum of the number of red and green hyperplanes exceeds 31, so we obtain a contradiction. As in the previous case check, we find that we can restrict the search to sets S with at most ten elements.

We may now assume that, for $1 \leq i \leq 6$, each $H'_i|R$ is a restriction of $F_7 \oplus U_{1,1}$ where the coloop in $F_7 \oplus U_{1,1}$ is one of the elements $d+e+f$ or $g+h+i$. Then, for the red elements of each of H'_1, H'_2, H'_3 to behave in this way, we must have at least two points in $\{a+b+c, d+e+f, g+h+i\}$ colored green. Similarly, for H'_4, H'_5, H'_6, we must have at least two points in $\{a+d+g, b+e+h, c+f+i\}$ colored green. Using symmetry, we

$$
\begin{array}{c}
\begin{array}{cccccccccccc} a & b & c & d & e & f & g & h & i & e+i & e+f & g+i \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{cccccccccccc} a & b & c & d & e & f & g & h & i & e+i & e+f & d+i \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}
\end{array}
$$

Figure 4.9. Two more extensions of $M(K_{3,3})$.

may assume that $a+b+c, d+e+f, a+d+g, b+e+h$ are green. Thus M is an extension of the matroid whose representation is shown in Figure 4.10. Using SageMath [26], we see that this matroid has exactly 27 connected hyperplanes, a contradiction. Hence the lemma holds. $\square$

We can now prove our main results for binary comatroids.

Proof of Theorem 4.1.2. Let M be a binary comatroid. Then, by Theorem 4.1.1, every flat of each of M and M^c is a binary comatroid. Thus, by Corollary 4.2.6, none of these flats

$$
\begin{array}{c}
\begin{array}{ccccccccccccc} a & b & c & d & e & f & g & h & i & & & & \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}
\end{array}
$$

Figure 4.10. M is an extension of this matroid whose last four columns are $a+b+c$, $d+e+f$, $a+d+g$, and $b+e+h$.

is a circuit of size exceeding five. By Proposition 4.2.2 and Lemma 4.4.2, none of the flats is isomorphic to $P(U_{3,4}, U_{3,4})$. Finally, by Lemma 4.4.8, none of the flats is isomorphic to the cycle matroid of one of the six graphs in Figure 4.1.

Conversely, suppose that M is a binary matroid that is not a comatroid. Then M has a flat N that is a member of $\mathcal{M}_2$. By Lemma 4.2.3, $r(N) \geq 4$. By Lemma 4.4.8, if $r(N) = 4$, then N or N^c is isomorphic to the cycle matroid of one of the six graphs in Figure 4.1. Thus M or M^c has a flat that is isomorphic to one of these cycle matroids. We may now assume that $r(N) \geq 5$. If $r(N) \geq 6$, then, by Theorem 4.4.7, N or N^c is a circuit, so M or M^c has a circuit as a flat. Finally, if $r(N) = 5$, then, by Lemmas 4.4.6 and 4.4.10, we get that M or M^c has as a flat either a circuit or $P(U_{3,4}, U_{3,4})$. □

Because we are only dealing with simple matroids, in the next proof and from now on, whenever we write M/e, we shall mean $\mathrm{si}(M/e)$.

Proof of Corollary 4.1.4. By Lemma 4.2.3, every binary matroid of rank at most three is a comatroid. Thus, in view of Theorem 4.1.2, it suffices to prove that M is an induced-minor-minimal binary non-comatroid when M^c is either a circuit of size at least six or is isomorphic to $P(U_{2,3}, U_{2,3})$. Consider the first case. Since both M and M^c are connected, M is not a binary comatroid. Observe that, for any proper flat F of M, the matroid $(M|F)^c$ is free and so, by Lemma 4.2.7, $M|F$ is a binary comatroid. Note that, in view of Lemma 4.2.7 and Proposition 4.2.9, it is enough to show that M/e is a binary comatroid for all e in $E(M)$. Because there is at most one red element on any line though e, we see that $(M/e)^c$ has a most one point, so M/e is a comatroid.

Now suppose that $M^c \cong P(U_{2,3}, U_{2,3})$. Again, it is enough to show that M/e is a

binary comatroid for all e in $E(M)$. If e is in the projective closure of one of the 4-circuits of $P(U_{2,3}, U_{2,3})$, then $(M/e)^c$ has a coloop. Thus $(M/e)^c$ is disconnected with each component having rank at most three, so it is a comatroid. If e is not in one of these projective closures, then $(M/e)^c$ has a most one point and, again, M/e is a comatroid. $\square$

4.5. Induced-restriction-minimal non-$GF(3)$-comatroids

In this section, we prove Theorem 4.1.3 and Corollary 4.1.5.

Lemma 4.5.1. *Let M be a matroid in $\mathcal{M}_3$ such that $r(M) \geq 4$ and M has a cocircuit of size less than four. Then M can be obtained from a circuit of size at least three by 2-summing a copy of $U_{2,4}$ to some, possibly empty, set of elements of the circuit.*

Proof. First we show the following.

4.5.1.1. *M has no non-spanning circuit C of rank at least three such that C intersects a cocircuit of M of size less than four.*

Assume that such a circuit C exists and let F be the projective flat spanned by C. Observe that F has a cocircuit A_F that has at most three green elements and so $F|R$ contains A_F minus three points. Since A_F is a ternary affine geometry of rank at least three, $F|R$ is connected of rank $r(F)$, a contradiction to the minimality of M. Thus 4.5.1.1 holds.

Now suppose that M has a 2-cocircuit $\{a, b\}$, say. Then $M = N \oplus_2 U_{2,3}$. If N has an element s parallel to the basepoint p of the 2-sum, then we move s so that it is parallel to p in $U_{2,3}$. Observe that if the basepoint p is contained in a non-spanning circuit D of N, then there is a non-spanning circuit D' of M that contains $\{a, b\}$ and has rank at least three, a contradiction to 4.5.1.1. Therefore p is free in N. Thus, by Lemma 4.3.1(ii),

N is obtained from a circuit of size at least three by 2-summing a copy of $U_{2,4}$ to some of the elements of the circuit. If $\{a, b\}$ is in a triangle of M, then M has a flat isomorphic to N. Thus N is either a circuit of size at least four, or N breaks up as a 2-sum. By Corollary 4.2.6, or by Proposition 4.2.2 and Lemma 4.4.2, N is not a ternary comatroid, contradicting the minimality of M. Thus $\{a, b\}$ is not in a triangle of M, so M satisfies the lemma.

Next suppose that M has a triad $\{a, b, c\}$, say. Observe that if $\{a, b, c\}$ is a triangle, we get our result as above. We may now assume that $\{a, b, c\}$ is independent. Let Π be the projective plane spanned by $\{a, b, c\}$ and let H be the projective hyperplane spanned by $E(M^c) - \{a, b, c\}$. It is clear that $\Pi|R$ is connected of rank three. Suppose that the projective lines spanned by $\{a, b\}, \{a, c\}$, and $\{b, c\}$ meet H at p, q, and s, respectively. Note that at most one of the points in $\{p, q, s\}$ is green otherwise $\Pi|G$ is connected of rank three, a contradiction. Thus we may assume that q and s are red. We may also assume that c is not free in M otherwise we have the result by Lemma 4.3.1(ii). Let D be a non-spanning circuit of M containing c. By orthogonality, D contains either a or b and so D has rank at least three. The result now follows by 4.5.1.1. $\square$

Lemma 4.5.2. *Let M be a matroid in $\mathcal{M}_3$ such that $r(M) = 4$. Then M or M^c has a cocircuit of size less than four.*

Proof. Assume that neither M nor M^c has a cocircuit of size less than four.

4.5.2.1. *A rank-3 simple ternary matroid N that is connected either has a connected rank-3 ternary complement or is $AG(2, 3)\backslash e$ or an extension of it.*

Assume that N^c is not connected or that $r(N^c) < 3$. Then N^c is a restiction of

$U_{2,4} \oplus U_{1,1}$. Thus N is $AG(2,3)\backslash e$ or an extension of it. Hence 4.5.2.1 holds.

The next assertion is an immediate consequence of Corollary 4.3.8.

4.5.2.2. *Both M and M^c have a connected hyperplane.*

By 4.5.2.1 and 4.5.2.2, it follows that we have both a red and a green triangle. In the arguments that follow, we will frequently exploit the symmetry between R and G.

4.5.2.3. *If a red triangle T is contained in exactly t red hyperplanes, then $|R| \geq 5t + 3$.*

By 4.5.2.1, each red hyperplane containing T has at least five red points not in the projective closure of T. The result is immediate.

4.5.2.4. *Every red triangle T is contained in exactly three red hyperplanes. Moreover, $|R| \geq 18$ and $|G| \geq 18$.*

Note that T is in at least two red hyperplanes otherwise we get a red cocircuit of size less than four. Assume that T is in exactly two red hyperplanes, H_1 and H_2. Because each of H_1 and H_2 is $AG(2,3)\backslash e$ or an extension of it, one can check that each element t of T is in a red triangle T_i in H_i that meets T in t. Now consider the projective plane Π that is spanned by T_1 and T_2. There are two green planes that contain T. Each of these has $AG(2,3)\backslash e$ as a restriction and meets Π in a line through t. This line contains at least two green points. Hence Π contains both a red 4-circuit and a green 4-circuit, a contradiction. We conclude that T is in at least three red hyperplanes. Thus, by 4.5.2.3, $|R| \geq 18$. By symmetry, $|G| \geq 18$, so $|R| \leq 22$. Hence, by 4.5.2.3 again, T is not in four red hyperplanes. Therefore 4.5.2.4 holds.

4.5.2.5. *There is a red triangle that is not contained in a red 4-point line.*

Suppose every red triangle is contained in a red 4-point line. As every red hyperplane has $AG(2,3)\backslash e$ as a restriction, one easily checks that every red hyperplane is a $PG(2,3)$. Since every red triangle is in three red hyperplanes, it follows that $|R| \geq 31$, a contradiction to 4.5.2.4.

We now take a red triangle T for which the fourth point, g, on the projective line spanned by T is green. Now T is in exactly three red hyperplanes, R_1, R_2, and R_3, and one green hyperplane, G_0. Because G_0 contains at most one red point not in T, there are three lines, G_1, G_2, and G_3, in G_0 that contain g and at least two other green points.

We may assume that $|R| \leq |G|$, so $|R| \in \{18, 19, 20\}$. We may also assume that $|R_1 - T| \geq |R_2 - T| \geq |R_3 - T| \geq 5$. As $|R_1 - T| + |R_2 - T| + |R_3 - T| \in \{15, 16, 17\}$, we see that $|R_3 - T| = 5$, that $|R_2 - T| \in \{5, 6\}$, and that $|R_1 - T| \in \{5, 6, 7\}$. Thus R_3 contains a green triangle T_1 containing g, so each of the projective planes spanned by $T_1 \cup G_1, T_1 \cup G_2$, and $T_1 \cup G_3$ contains a green 4-circuit. Moreover, each such plane meets each of R_1 and R_2 in a line through g. For each i in $\{1, 2\}$, the plane R_i has at most one line through g that does not contain at least two red points. Hence, for some j in $\{1, 2, 3\}$, the projective plane spanned by $T_1 \cup G_j$ meets both R_1 and R_2 in a line through g containing at least two red points. Then $T_1 \cup G_j$ contains a red 4-circuit. As it contains a green 4-circuit, we have a contradiction. $\square$

Theorem 4.5.3. *Let M be a matroid in $\mathcal{M}_3$ such that $r(M) \geq 4$. Then M or M^c has a cocircuit of size less than four.*

Proof. By Lemma 4.5.2, the result holds when $r(M) = 4$. Therefore we may assume that $r(M) \geq 5$. Further assume that neither M nor M^c has a cocircuit of size less than four.

We now let $P_{r(M)}$ denote the ternary projective geometry of rank $r(M)$. A flat F of $P_{r(M)}$ with $3 \leq r(F) < r(M)$ is *red* or *green* depending on whether $F|R$ or $F|G$ is connected of rank $r(F)$. Let F be a rank-$(r-2)$ flat of $P_{r(M)}$. Then F is contained in exactly four hyperplanes, say H_1, H_2, H_3, and H_4 of $P_{r(M)}$. Moreover, as neither M nor M^c has a cocircuit of size less than four, we deduce the following.

4.5.3.1. *At least two of H_1, H_2, H_3, and H_4 have the same color as F.*

Now let G_1 and R_1 be the sets of green and red hyperplanes, respectively, of $P_{r(M)}$. The following is a straightforward consequence of Theorem 4.4.3.

4.5.3.2. *If $H \in G_1$, then at most one of the rank-$(r-2)$ projective flats contained in H is red.*

Let G_2 and R_2 be the sets of green and red projective flats of $P_{r(M)}$ of rank $r-2$. We consider the bipartite graph B with vertex classes $G_1 \cup R_1$ and $G_2 \cup R_2$ such that a vertex X in $G_1 \cup R_1$ is incident to a vertex Y in $G_2 \cup R_2$ if $Y \subseteq X$. As in the proof of Theorem 4.4.7, by 4.5.3.2, the number of cross edges of this graph is at most $|G_1| + |R_1|$.

Each pair (H_G, H_R), where $H_G \in G_1$ and $H_R \in R_1$, corresponds to a cross edge e. Note that at most three such pairs can correspond to this edge e. Thus the number of cross edges is at least $\frac{1}{3}|G_1||R_1|$, so $\frac{1}{3}|G_1||R_1| \leq |G_1| + |R_1|$. By symmetry, we may suppose that $|G_1| \geq |R_1|$. Thus $|R_1| \leq 3 + \frac{3|R_1|}{|G_1|} \leq 6$. Since $|G_1| + |R_1| = \frac{3^{r(M)}-1}{2}$ and $r(M) \geq 5$, we see that $|G_1| \geq 115$, so $|R_1| \leq 3 + \frac{3|R_1|}{|G_1|}$. Hence $|R_1| \leq 3$ and $|G_1| \geq 118$. Since every red projective flat of rank $r(M) - 2$ is contained in at least two red projective hyperplanes, it follows that $|R_2| \leq 3$. By 4.5.3.1, a flat in R_2 is contained in at most two hyperplanes in G_1 and so the number of G_1R_2-edges is at most six. Thus $\frac{1}{3}|G_1||R_1| \leq |R_1| + 6$, so

$|R_1| \leq \frac{3|R_1|+18}{|G_1|}$. As $|R_1| \leq 3$ and $|G_1| \geq 118$, it follows that $|R_1| = 0$, a contradiction to Lemma 4.3.8. $\square$

Proof of Theorem 4.1.3. A routine check shows that, up to complementation, $U_{3,4}$, $P(U_{2,3}, U_{2,3})$, $U_{2,4} \oplus_2 U_{2,3}$, R_6, $P(U_{2,4}, U_{2,3})$, $M(K_4)$, and $\mathcal{W}^3$ are the only connected ternary matroids of rank three whose complements are also connected of rank three. Theorem 4.1.3 now follows from Lemma 4.2.3, Lemma 4.5.1, Lemma 4.5.2, and Theorem 4.5.3. $\square$

Proof of Corollary 4.1.5. In view of Lemma 4.2.3 and Theorem 4.1.3, it suffices to show that if M^c is obtained from a circuit of size at least three by 2-summing a copy of $U_{2,4}$ to some, possibly empty, set of elements of the circuit, then M is an induced-minor-minimal ternary non-comatroid. But, when M^c is as specified, M/e has at most one point and so is a ternary comatroid. $\square$

Appendix. Permissions

Complementation, local complementation, and switching in binary matroids

Author: James Oxley,Jagdeep Singh

Publication: Advances in Applied Mathematics

Publisher: Elsevier

Date: May 2021

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Vita

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