4-3-2022

Unavoidable Structures in Large and Infinite Graphs

Sarah Allred

Louisiana State University and Agricultural and Mechanical College

UNAVOIDABLE STRUCTURES IN LARGE AND INFINITE GRAPHS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Sarah Allred
B.S., Auburn University, 2016
M.S., Louisiana State University, 2017
May 2022

We are always open to instruction, willing to be wiser every day than we were before, and to change whatever we can change for the better.

—John Wesley
A Plain Account of the People Called Methodists

Acknowledgments

This dissertation would not be possible without numerous contributions. I first want to thank my advisor, Professor Bogdan Oporowski, for his time, patience, and guidance throughout this process.

Thank you to Professor James Oxley for the support of my research, travel, and career. Thank you for always being willing to give me advice and encouragement when I needed it most.

I also offer thanks to Professor Guoli Ding for exposing me to unavoidable substructure problems; and Professor Oliver Dasbach for his support and advice throughout my time at LSU.

It is a pleasure to offer my gratitude to my academic siblings Matt and Josh for guiding me through each step of my journey and keeping me focused.

To the members of the UUMC community, particularly: Colleen, Katie, and Lisa thank you for being there for me when I needed it most and helping me to grow as a person throughout my time at LSU.

This dissertation is dedicated to my parents, Martha and Jay, and my brother, Will. Without your support, I could not have finished this dissertation.

Table of Contents

List of Figures

Abstract

In this work, we present results on the unavoidable structures in large connected and large 2-connected graphs. For the relation of induced subgraphs, Ramsey proved that for every positive integer r, every sufficiently large graph contains as an induced subgraph either K_r or $\overline{K}_r$. It is well known that, for every positive integer r, every sufficiently large connected graph contains an induced subgraph isomorphic to one of K_r, $K_{1,r}$, and P_r. We prove an analogous result for 2-connected graphs. Similarly, for infinite graphs, every infinite connected graph contains an induced subgraph isomorphic to one of the following: an infinite complete graph, an infinite star, and a ray. Using some techniques from the finite result, we give the unavoidable induced subgraphs of infinite 2-connected graphs.

We then shift our attention to the relation of bipartite minors defined in 2016 by Chudnovsky, Kalai, Nevo, Novik, and Seymour. For the relation of bipartite minors, we present the unavoidable substructures of both large connected and large 2-connected bipartite graphs.

Chapter 1. Introduction

The topics under study in this dissertation are concerned with fundamental properties of large graphs, principally questions about substructures that a graph must contain if it is sufficiently large. These questions about unavoidable substructures are an outgrowth of Ramsey theory, which is the study of properties guaranteed to be present in large structures. Broadly speaking, Ramsey theory can be considered along two avenues of study. One branch of Ramsey theory focuses on investigating the substructures guaranteed to be present in large structures, while another concentrates on the refinement of the bound on the size of structures with specified properties to guarantee the existence of the substructures. In Ramsey theory for graphs, the former branch considers graphs with particular properties, such as connectivity or chromatic number, and examines what substructures must be present in the large graph based on the relation by which the large graph contains the substructure, such as induced subgraph, minor, or bipartite minor. The latter relation is defined in Chapter 4. The second branch seeks to obtain the best possible bounds on the size of the large graph with some given properties that ensure that it contains, by a particular relation, the unavoidable substructures. This dissertation considers the former aspect of Ramsey theory, namely, what substructures must be present in a sufficiently large graph with certain connectivity properties.

A class of graphs called bipartite graphs, which we focus on in Chapter 4, has a variety of applications, including scheduling problems and matching medical school students with residency programs. However, not all substructure relations guarantee that graphs remain in that class. A key aspect of Ramsey theory is the preservation of particular properties in the unavoidable substructures for a specified relation. For each of the relations of subgraph, induced subgraph, and bipartite minor, the resulting substructures belong to the class of bipartite graphs; however, the minor relation does not have to maintain the property that a graph is bipartite. We explore this idea more in Chapter 4.

Section 1.1 presents basic background definitions and concepts; and Section 1.2 gives background and motivating theorems influenced by Ramsey. In Chapter 2, we prove a Ramsey-like result for the relation of induced subgraphs in large finite 2-connected graphs. The contents of Chapter 2 also appear in [1] and have been submitted to a peer-reviewed academic journal. In Chapter 3, we present a Ramsey-like result for the relation of induced subgraphs in infinite 2-connected graphs. In Chapter 4, we shift our focus from the relation of induced subgraphs to that of bipartite minors, where we prove Ramsey-like results for both large connected and large 2-connected bipartite graphs for the relation of bipartite minors.

1.1. Basic Concepts

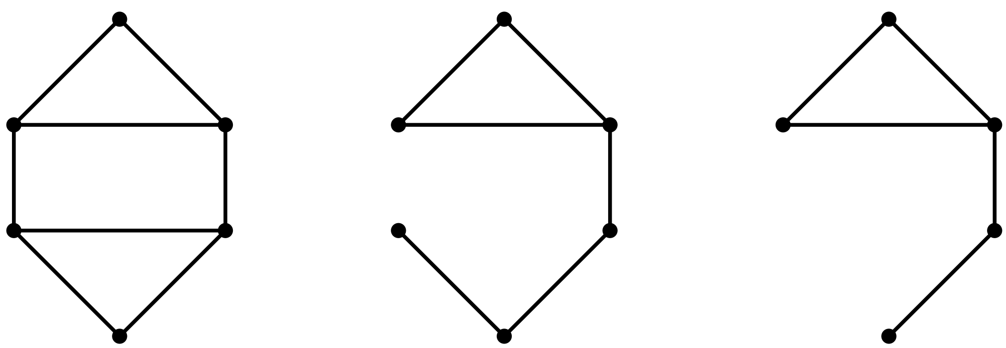
A *graph* G is a triple $(V, E, \mathcal{R})$, where V and E are disjoint sets, called the *vertices* and *edges* of G, respectively. The set $\mathcal{R}$ is a subset of $V \times E$ such that every edge appears in exactly one or two elements of $\mathcal{R}$. The sets V and E may be referred to as $V(G)$ and $E(G)$, respectively, if the identity of the graph is not clear. A vertex v and an edge e of G are called *incident* if the pair (v, e) is an element of $\mathcal{R}$. Two vertices are *adjacent* if they are incident to the same edge. An edge that appears in exactly one element of $\mathcal{R}$ is called a *loop*. Two distinct non-loop edges are *parallel* if they are incident to the same two vertices. A graph is *simple* if it has no loops and no parallel edges. This dissertation will focus almost exclusively on simple graphs.

The *order* of a graph G is the number of vertices in $V(G)$. The *degree* of a vertex v is the number of edges incident to v, where a loop counts twice.

A *complete graph* on n vertices, denoted by K_n, is a simple graph of order n that has an edge between each pair of vertices. A countably infinite complete graph is denoted by K_∞. The *complement* of a graph G with order n, denoted by $\overline{G}$, is a graph with vertex set $V(G)$ and edge set $E(K_n) \setminus E(G)$. A *path* of order n, denoted by P_n, is a graph obtained from a sequence of distinct vertices v_1, v_2, $\ldots$, v_n by having an edge between two consecutive vertices. The vertices v_1 and v_n are the *endpoints* of the path. We often wish to label a path by its endpoints, so a path with endpoints u and v is called a *uv-path*. The *length* of a path is the number of edges on it. The *distance* between two vertices u and v in a graph G, denoted by $\mathrm{dist}_G(u, v)$, is the length of the shortest path between u and v. A path that consists of exactly one vertex and no edges is called *trivial*. A set of two or more (possibly countably infinitely many) uv-paths $\{P^i\}$ for i in a (possibly infinite) subset of natural numbers $\mathcal{I}$ are *pairwise internally-disjoint* if $P^j \cap P^k = \{u, v\}$ for distinct elements j and k in $\mathcal{I}$. A *cycle* C_n is a graph obtained by taking the union of a path v_1, v_2, $\ldots$, v_n and the edge $\{v_1 v_n\}$. Note that in a cycle, the order and length are equal.

A *vertex deletion* is an operation that removes a vertex and all of its incident edges. A graph H is a *subgraph* of G if H can be obtained from G by deleting a set of vertices and a set of edges. We say that H is an *induced subgraph* of G if H can be obtained from G by a sequence of vertex deletions. Notice that the relation of an induced subgraph is more restrictive than that of a subgraph. More specifically, if H is an induced subgraph of G, then H is a subgraph of G, but the converse need not be true. The graph H, shown in Figure 1.1b, is a subgraph of G, shown in Figure 1.1a, but not an induced subgraph of G; whereas the graph H', shown in Figure 1.1c, is an induced subgraph of G. We will often wish in Chapters 2 and 3 to state that a graph G contains an induced subgraph isomorphic to a graph H; in such a case, we will abbreviate this by saying that G *conduces* H.

In all figures of this dissertation, thick segments represent non-trivial paths with an arbitrary number of vertices, while thin lines indicate single edges.



(a) G (b) H (c) H'

Figure 1.1. Difference in subgraph and induced subgraph of G

An *independent set* I of vertices in a graph G is a subset of $V(G)$ such that all members of I are pairwise non-adjacent. A *bipartite graph* is a graph whose vertex set can be partitioned into two sets, A and B, such that every edge is incident with a vertex in A and with a vertex in B. It follows that A and B are independent sets in a bipartite graph. We call $\{A, B\}$ a *bipartition* of $V(G)$, and we call A and B *parts* of G. A *complete bipar-*

tite graph $K_{m,n}$ is a graph with bipartition $\{A, B\}$ such that $|A| = m$, $|B| = n$, and for every pair of vertices $a \in A$ and $b \in B$, the graph has an edge incident to both a and b. Note that m and n may be infinite. Bipartite graphs are characterized by the following theorem:

Theorem 1.1.1 (König [11]). *A graph is bipartite if and only if it has no odd cycles.*

A graph G is *connected* if for each pair of vertices u and v, the graph G has a uv-path. A *component* of G is a maximal connected subgraph of G; a connected graph has one component. A *cut-vertex* v is a vertex such that $G - v$ has more components than G. A graph G of order greater than two is *2-connected* if $G - v$ is connected for each vertex v in $V(G)$. Note that a 2-connected graph has no cut-vertices. In general, a graph G is k-connected, for a natural number k, if the order of G is greater than k and $G - S$ is connected for every subset S of $V(G)$ such that $|S| < k$. In this dissertation, we focus predominantly on the unavoidable substructures of 2-connected graphs. Figure 1.2 shows a graph that is connected, but not 2-connected, and the vertex v, in this figure, is a cut-vertex.

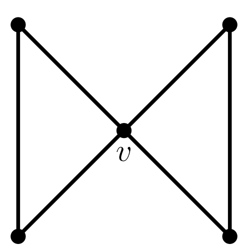


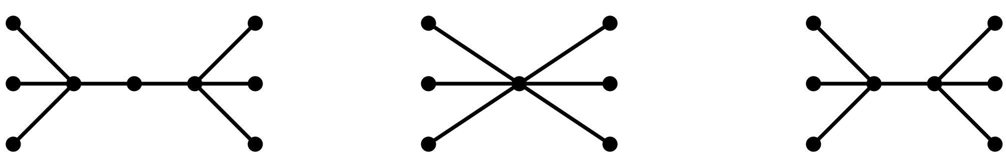
Figure 1.2. Graph that is connected, but not 2-connected

Note that a cycle with n vertices, C_n, shown in Figure 1.3a, and a complete bipartite graph $K_{2,n}$, shown in Figure 1.3b, are some common 2-connected graphs.

(a) A cycle: C_6 (b) $K_{2,4}$

Figure 1.3. Common types of 2-connected graphs

To *subdivide* an edge $e = uv$ of a graph G means to obtain a graph G' by taking the disjoint union of $G - e$ and a non-trivial path P, and identifying one endpoint of P with u and the other endpoint of P with v. A graph H is a *subdivision* of G if H is obtained from G by subdividing edges of G. A *contraction* of a non-loop edge $e = uv$ in a graph G is the operation of deleting e and identifying the vertices u and v. A *minor* H of a graph G is a graph obtained from G by a sequence of operations each, of which is a vertex deletion, an edge deletion, or an edge contraction. A *topological minor* H of a graph G is a graph obtained from G by a sequence of operations, each of which is a vertex deletion, an edge deletion, or a contraction an edge incident with a vertex of degree two. Note that if H is a topological minor of G, then H is a minor of G; however, the converse need not be true. The graph H, shown in Figure 1.4b, is a minor of a graph G, shown in Figure 1.4a, but not a topological minor of G; whereas the graph H', shown in Figure 1.4c, is an topological minor of G.

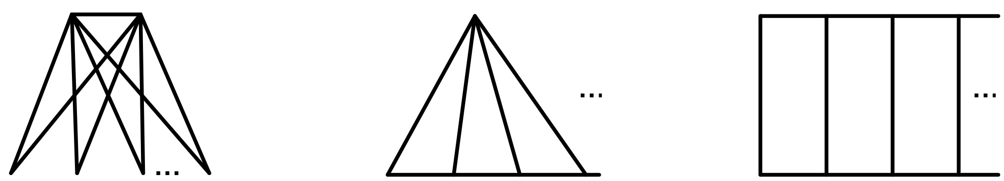


(a) G (b) H (c) H'

Figure 1.4. Difference in topological minor and minor of G

A *parallel minor* H of a graph G is a graph obtained from G by contracting a set of edges of G, then deleting resulting loops and deleting all but one edge in each set of resulting parallel edges. Note that H is a simple graph.

Chapter 3 deals with infinite graphs. In this dissertation, we consider only countably infinite graphs; thus, each use of "infinite" is understood to be countably infinite. Graphs shown in Figure 1.5 below are some examples of infinite 2-connected graphs, which are considered more in Chapter 3.



(a) $K^+_{2,\infty}$ (b) F_∞ (c) L_∞

Figure 1.5. Examples of infinite 2-connected graphs

A *ray* is a graph obtained from an infinite sequence of vertices $v_1, v_2, v_3, \ldots$ by connecting two consecutive vertices with an edge. The vertex v_1 is the *initial vertex* of the ray.

For the definition of terms not found in the above section but appearing in this dissertation, one may consult an introductory textbook in graph theory, such as Diestel [5].

1.2. Motivating Results

The classical result of Ramsey [15], which served as a motivation for this dissertation and many papers, is the following:

Theorem 1.2.1 (Ramsey's Theorem)**.** *For every positive integer r, there is an integer $f_{1.2.1}(r)$ such that every graph on at least $f_{1.2.1}(r)$ vertices conduces K_r (a complete graph* *on r vertices) or $\overline{K}_r$ (an edgeless graph on r vertices).*

There are numerous extensions of Ramsey's Theorem for graphs of various levels of connectivity and different relations on graphs.

For the relation of induced subgraphs of large connected graphs, we have the following:

Theorem 1.2.2 ((5.3) of [6]). *For every positive integer r, there is an integer $f_{1.2.2}(r)$ such that every connected graph on at least $f_{1.2.2}(r)$ vertices conduces one of the following graphs: K_r, $K_{1,r}$, and P_r.*

For 2-connected graphs, we have the following result for the relation of topological minors:

Theorem 1.2.3 ((1.2) of [13]). *For every integer r exceeding two, there is an integer $f_{1.2.3}(r)$ such that every 2-connected graph on at least $f_{1.2.3}(r)$ vertices contains one of the following as a topological minor: $K_{2,r}$ and C_r.*

To present the result for parallel minors, we must define two graphs. First, let $K_{2,n}^+$ be the graph obtained from $K_{2,n}$ by adding an edge between the two vertices of degree n in one side of the bipartition. A *fan* of order s for a natural number s exceeding two, denoted by F_s, is the graph obtained by taking an isolated vertex, called the *apex*, and a path of order $s-1$, called the *rim*, and adding an edge between the apex and every vertex on the rim. C. Chun and Ding proved the following theorem in [4]:

Theorem 1.2.4 ((1.5) of [4]). *There exists a function $f_{1.2.4}(p,q,r,s)$ such that every 2-connected graph on at least $f_{1.2.4}(p,q,r,s)$ vertices contains one of the following as a paral-*

lel minor: K_p*,* $K_{2,q}^+$*,* C_r*, and* F_{s-1}*.*

For 3-connected graphs, Oporowski, Oxley, and Thomas proved in [13] an analogous theorem for topological minors; additionally, C. Chun and Ding proved in [4] a Ramsey-like for parallel minors

Let W_n denote the graph obtained by taking the union of a cycle C of order n and an isolated vertex v and adding an edge from v to each vertex of C. Similar results have been proved for 3-connected binary matroids by Ding, Oporowski, Oxley, and Vertigan in [6]. For more information about matroids, see [14].

Theorem 1.2.5 (1.5 of [6])**.** *For every integer* p *greater than two, there is an integer* $f_{1.2.5}(p)$ *such that every 3-connected binary matroid with more that* $f_{1.2.5}(p)$ *elements contains a minor isomorphic to one of* $M(K_{3,p})$*,* $M^*(K_{3,p})$*,* $M(W_p)$*, and binary* p*-spike.*

For general matroids, Ding, Oporowski, Oxley, and Vertigan proved the following theorem in [7].

Theorem 1.2.6 (1.2 of [7])**.** *For every integer* p *exceeding two, there is an integer* $f_{1.2.6}(p)$ *such that every 3-connected matroid with at least* $f_{1.2.6}(p)$ *has a minor isomorphic to* $U_{p,p+2}$*,* $U_{2,p+2}$*,* $M(K_{3,p})$*,* $M^*(K_{3,p})$*,* $M(W_p)$*, the whirl of rank* p*, or a* p*-spike.*

In Chapter 3 of this dissertation, we consider an analog of Ramsey's Theorem for infinite 2-connected graphs. For infinite graphs, we have similar results to the finite case.

Theorem 1.2.7 (Ramsey's Theorem [15])**.** *Every infinite graph conduces* K_∞ *or* $\overline{K}_\infty$*.*

König's Infinity Lemma [10] implies the following theorem.

Theorem 1.2.8. *Every infinite connected graph has a vertex of infinite degree or a ray.*

The following result for the relation of induced subgraphs of infinite connected graphs follows from Theorems 1.2.7 and 1.2.8.

Theorem 1.2.9. *Every infinite connected graph conduces one of the following: K_∞, $K_{1,\infty}$, and a ray.*

Proof. Let G be an infinite connected graph. By Theorem 1.2.8, either G contains an infinite vertex or a ray. If G contains an infinite vertex v, then let H be the subgraph of G induced by the neighbors of v. By Theorem 1.2.7, H conduces either K_∞ or $\overline{K_\infty}$. If H conduces K_∞, then so does G, and the conclusion follows. If H conduces $\overline{K_\infty}$, then $H \cup \{v\}$ forms an induced $K_{1,\infty}$, and the conclusion follows.

We may therefore assume that every vertex has finite degree. By Theorem 1.2.8, G contains a ray. Let w be a finite vertex of G and let N_k be the set of vertices distance k away from w. Then N_0, N_1, N_2, $\ldots$ form a partition of $V(G)$ into infinitely many finite sets. The edges of G are either between vertices in N_k or between sets N_k and N_{k+1} for $k = 0, 1, 2, \ldots$. Thus each N_i has a vertex x_i for $i = 1, 2, \ldots$ such that x_i has a neighbor in N_{i-1}. Thus the vertices w, x_1, x_2, $\ldots$ and edges wx_1 and $x_i x_{i+1}$, for natural numbers i, form an induced ray, as required. □

For infinite 2-connected graphs, C. Chun and Ding proved the following about subgraphs and parallel minors in [3]:

Theorem 1.2.10. *Let G be a 2-connected infinite graph. Then*

(i) G contains a subdivision of $K_{2,\infty}$, a subdivision of F_∞, see Figure 1.5b, or a subdivision of L_∞ as a subgraph, see Figure 1.5c; and

(ii) G contains K_∞, $K^+_{2,\infty}$, see Figure 1.5a, or F_∞ as a parallel minor.

There are many other results in Ramsey theory; however, the ones presented in this section give some of the unavoidable substructure results for large connected and large 2-connected graphs for relations similar to those discussed in the next chapters.

Chapter 2. Finite Case

2.1. Preliminaries

In this chapter, we present the unavoidable induced subgraphs of large finite 2-connected graphs. Before stating precisely the main result of this chapter, we need to define two families of graphs. Let r be an integer exceeding two. Let $\mathcal{K}_{2,r}$ be the family of graphs obtained from $K_{2,r}$ by subdividing each of the edges of $K_{2,r}$. Let $\mathcal{K}^+_{2,r}$ be the family of graphs obtained from the family $\mathcal{K}_{2,r}$ by adding an edge between the two vertices of degree r to each member of the family $\mathcal{K}_{2,r}$.

Trees and paths will play a significant role in this and the next chapter, so we need some definitions describing their properties. A tree T with a distinguished vertex ρ, called the *root*, is a *rooted tree* and is denoted by (T, ρ). Its *height* is the maximum distance from one of its vertices to the root. The vertices of T have a natural partial ordering: we write $u \leq_T v$ whenever u lies on the ρv-path of T. We write $u <_T v$ whenever u lies on the ρv-path of T and u is distinct from v. If the identity of the tree is clear from the context, we may use $\leq$ or $<$ instead. The vertices v such that $u <_T v$ are called the *descendants* of u. The descendants of u that are also its neighbors are called its *children*. For two vertices a and b of T such that $a \leq b$, the subgraph of T induced by the vertices v such that $a \leq v \leq b$ is denoted by $T[a, b]$. Note that if $a > b$, then $T[a, b]$ is empty. Similarly, the subgraph of T induced by the vertices v such that $a < v < b$ is denoted by $T(a, b)$. The subgraphs $T(a, b]$ and $T[a, b)$ are defined analogously.

This chapter previously appeared, substantially unchanged, in the pre-print framework *arXiv* as: Sarah Allred, Guoli Ding, Bogdan Oporowski, "Unavoidable Induced Subgraphs of Large 2-Connected Graphs," `https://arxiv.org/abs/2009.12503v2`.

A *messy ladder* is a triple (L, X, Y) that consists of a graph L whose vertices all lie on two disjoint induced paths X and Y, called *rails*. Each rail is considered to be a tree rooted at one of its endpoints, which is called the *initial vertex*, and the other endpoint is called the *terminal vertex*. The edges of L that belong to neither X nor Y are called *rungs*. The graph L has an edge between the initial vertices of the rails, called σ, and an edge between the terminal vertices of the rails, called τ. At most one of the rails may be trivial. In some contexts, when we say messy ladder, we mean only the graph L, of which the existence and properties of X and Y are a part. The *order* of a messy ladder is its number of vertices. The following are equal: the order of a messy ladder, the order of the graph L, and the number of vertices in $X \cup Y$.

If e is a rung in a ladder with rails X and Y, then e_X and e_Y denote the endpoints of e on X and Y, respectively. Two rungs in an ordered pair e and f *cross* if $e_X < f_X$ and $f_Y < e_Y$. We also say that (e, f) is a *cross* whose X-*span* is $X[e_X, f_X]$, and whose Y-*span* is $Y[f_Y, e_Y]$. A cross whose X-span and Y-span are both single edges is *degenerate*. A *clean ladder* is a messy ladder whose crosses are all degenerate.

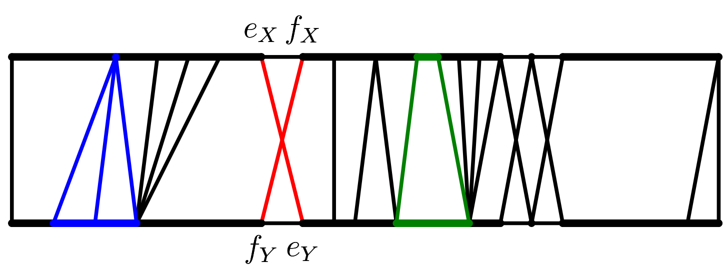


Figure 2.1. A clean ladder

In Figure 2.1, there are a few features to notice: the fan indicated by blue line segments is a clean ladder, and so is the cycle indicated by green line segments. A degenerate cross is depicted by the red line segments. These structures are discussed in detail in Section 2.4.

We may now state the main result of this chapter.

Theorem 2.1.1. *Let r be an integer exceeding two. There is an integer $f_{2.1.1}(r)$ such that every 2-connected graph of order at least $f_{2.1.1}(r)$ conduces one of the following: K_r, a clean ladder of order at least r, a member of $\mathcal{K}_{2,r}$, and a member of $\mathcal{K}^+_{2,r}$.*

REMARK. A clean ladder in the theorem can be replaced by a long cycle, a long fan (where rim edges could be subdivided) and a restricted version of the clean ladder.

The proof uses Ramsey numbers, the known bounds on which are believed to be very far from the best possible. So in the proofs, clarity of the arguments is valued over the tightness of the bounds.

To prove the main theorem, we consider the cases where the large 2-connected graph G either has a long path as a subgraph, or it does not. Section 2.2 discusses the case where G does not have a long path. In that case, we prove that G conduces two of the graphs listed in the conclusion of Theorem 2.1.1. The case where G has a long path is broken into two sections. In Section 2.3, we start with the long path and obtain a large messy ladder. In Section 2.4, we show that if a messy ladder is large enough, then it conduces a sufficiently large clean ladder. Section 2.5 combines the results of Sections 2.2 to 2.4 to prove Theorem 2.1.1.

2.2. Graphs Without a Long Path

In this section, we present that a large 2-connected graph either has a long path or conduces one of the graphs desired in the main result.

A rooted tree (T, ρ) that is a spanning subgraph of a graph G is called *normal* if, for every two adjacent vertices u and v of G, either $u \ \leq_T \ v$ or $v \ \leq_T \ u$. It is well known that every connected graph has a normal spanning tree (Proposition 1.5.6 of [5]). A *rooted sub-tree* (T', ρ') of (T, ρ) has T' as sub-tree of T and (T', ρ') preserves the ordering of (T, ρ).

Lemma 2.2.1. *Let q and r be integers exceeding one. There is an integer $f_{2.2.1}(q, r)$ such that if G is a 2-connected graph on at least $f_{2.2.1}(q, r)$ vertices, then G has either a path of order $q+1$ or an induced subgraph that is a member of one of the following families: $\mathcal{K}^+_{2,r}$ and $\mathcal{K}_{2,r}$.*

Proof. We prove that $f_{2.2.1}(q, r) \ = \ 2 + (d-1) + (d-1)^2 + \ldots + (d-1)^{q-1}$, where $d = 1 + (q-2)(r-1)$, satisfies the conclusion.

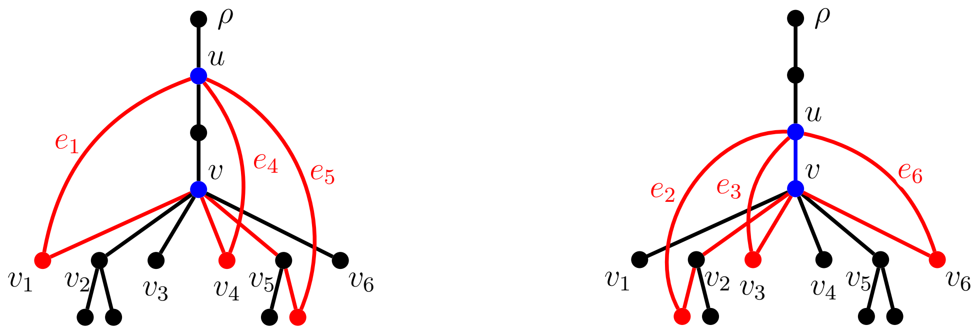
Let (T, ρ) be a normal spanning rooted tree of G. If (T, ρ) has height at least q, then (T, ρ) has a path of order $q+1$, and the conclusion follows.

For the remainder of the proof, we may therefore assume that the height of (T, ρ) is less than q. Since G has order at least $f_{2.2.1}(q, r)$, it follows that the tree (T, ρ) has a vertex v with at least d children.

Let R be the ρv path in (T, ρ), which has order at most $q-1$. For each child v_i of v, let (T_i, v_i) be the rooted sub-tree of (T, ρ) induced by v_i and all of the descendants of v_i.

Since v has at least d children, the tree (T, ρ) has at least d sub-trees rooted at children of v. We need to consider only d of them: (T_1, v_1), (T_2, v_2), $\ldots$, (T_d, v_d). Since (T, ρ) is normal and the rooted sub-trees are distinct, every edge of G with exactly one end in some (T_i, v_i) must have the other end in $R - v$. Since G is 2-connected, it follows that v is not a cut-vertex of G. For each $j \in \{1, 2, \ldots, d\}$, the graph G has an edge e_j incident with both a vertex on (T_j, v_j) and a vertex u_j on $R - v$. By the definition of d, there is a natural number $k \in \{1, 2, \ldots, d\}$ such that u_k is incident to at least r of the edges e_j; let $u = u_k$. Let $\mathcal{I}$ be a set of r indices from $\{1, 2, \ldots, d\}$ of the edges e_i that have u as one endpoint and the other endpoint on (T_i, v_i). Each (T_i, v_i) spans a component G_i of $G - V(R)$. Both vertices u and v have neighbors in G_i. Let G_i' be the subgraph of G that consists of G_i and all the edges between G_i and $\{u, v\}$. Note that G_i' is connected. Let P_i be a shortest uv-path in G_i'.

Let H be the subgraph of G induced by $\bigcup_{i \in \mathcal{I}} P_i$. Since (T, ρ) is normal, G has no edges between internal vertices of distinct paths in $\{P_i\}_{i \in \mathcal{I}}$, and since each P_i is a shortest uv-path in G_i, it follows that H is the union of pairwise internally-disjoint uv-paths. If u is adjacent to v in G, then H is a member of the family $\mathcal{K}_{2,r}^{+}$, and if u is not adjacent to v in G, then H is a member of the family $\mathcal{K}_{2,r}$. The conclusion follows.



(a) member of the family $\mathcal{K}_{2,r}$ in G (b) member of the family $\mathcal{K}^+_{2,r}$ in G

Figure 2.2. Process of obtaining a member of the family $\mathcal{K}_{2,r}$ or $\mathcal{K}^+_{2,r}$

Figure 2.2 shows the paths whose union is either a member of the family $\mathcal{K}_{2,r}$ or a member of the family $\mathcal{K}^+_{2,r}$. The red vertices are the vertices in the bipartition of cardinality r, and the blue vertices are members of the bipartition of cardinality two. The red segments show the edges of a graph in $\mathcal{K}_{2,r}$ and the blue edge in Figure 2.2b illustrates the edge between the two vertices of degree r in a member of the family $\mathcal{K}^+_{2,r}$. $\square$

2.3. From a Long Path to a Messy Ladder

In this section, we prove that if a large 2-connected graph G has a long path as a subgraph, then G conduces one of the following: a large messy ladder, a large complete graph, a large $K_{2,n}$, and a large $K^+_{2,n}$. The goal of this section is to prove the following lemma.

Lemma 2.3.1. *Let p and q be integers exceeding two. There is an integer $f_{2.3.1}(p,q)$ such that every 2-connected graph with a path of order $f_{2.3.1}(p,q)$ conduces one of the following: K_p, $K_{2,p}$, $K^+_{2,p}$, and a messy ladder of order at least q.*

Before proceeding, we need the following result of Galvin, Rival, and Sands [8].

Theorem 2.3.2 (Theorem 4 of [8]). *Let p, q, and r be positive integers. There is an integer $f_{2.3.2}(p,q,r)$ such that every graph with a spanning path of order at least $f_{2.3.2}(p,q,r)$ contains $K_{p,q}$ as a subgraph or conduces a path of order r.*

We use this theorem to prove that a large graph conduces either a graph from the list desired in Theorem 2.1.1 or a long path.

Corollary 2.3.3. *Let q and r be integers exceeding two. There is an integer $f_{2.3.3}(q,r)$ such that every graph with a path of order at least $f_{2.3.3}(q,r)$ conduces one of the following: K_q, $K_{2,q}$, $K^+_{2,q}$, and a path of order r.*

Proof. Let $f_{2.3.3}(q,r) = f_{2.3.2}(2,s,r)$ where $s = f_{1.2.1}(q)$, and $f_{1.2.1}(q)$ and $f_{2.3.2}(2,s,r)$ are the numbers from Ramsey's Theorem (Theorem 1.2.1) and Theorem 2.3.2, respectively. We prove that $f_{2.3.3}(q,r)$ satisfies the conclusion. Suppose G is a graph with a path P of order at least $f_{2.3.3}(q,r)$.

Let H be the graph obtained from G by deleting all vertices except those on the path P. So $V(H) = V(P)$ and H is an induced subgraph of G. Thus, the path P is a spanning path of H of order at least $f_{2.3.2}(2,s,r)$. By Theorem 2.3.2, the graph H conduces a path of order r or contains $K_{2,s}$ as a subgraph. If H conduces a path of order r, then so does G, and the conclusion follows.

Therefore, we may assume that H has a subgraph isomorphic to $K_{2,s}$ whose bipartition is (A,B) with $|A| = 2$ and $|B| = s$. Let $H(B)$ be the subgraph of H induced by B, we apply Ramsey's Theorem (Theorem 1.2.1) to $H(B)$. By Theorem 1.2.1, the graph $H(B)$ conduces either K_q or $\overline{K_q}$. If $H(B)$ conduces K_q, then so does H, and the conclusion follows. If $H(B)$ conduces $\overline{K_q}$, then let I be an independent set of order q in $H(B)$. Since $K_{2,s}$ is a subgraph of H which is not necessarily induced, it follows that H may have an edge between the two vertices of A. The subgraph of H induced by the vertex set $A \cup I$ is isomorphic either to $K_{2,q}$ if the vertices of A are non-adjacent, or to $K_{2,q}^{+}$ otherwise. Since G conduces H, it follows that G conduces one of the following: K_q, $K_{2,q}$, and $K_{2,q}^{+}$, as desired. □

We will use Tutte's notion of a bridge found in [16], see also [9], to build the messy ladder. An *H-bridge* or (a *bridge of H*) is a connected subgraph B of $G \setminus E(H)$ that satisfies either one of the following two conditions:

1. B is a single edge with both endpoints in $V(H)$. In this case, B is called a *degenerate bridge*.

2. $B - V(H)$ is a connected component of $G - V(H)$; and B also includes every edge of G with one end point in $V(B) - V(H)$ and the other end point in H.

Note that every edge of $G \setminus E(H)$ belongs to exactly one H-bridge. Vertices that belong to both B and H are called *vertices of attachment* of B.

Suppose G is a large 2-connected graph that has a long induced uv-path P. Our goal is to use P to form a large induced messy ladder. Since P is an induced path, it has no degenerate bridges. For each bridge B_i of P in G, let u_i and v_i be the two vertices of attachment of B_i such that $P[u_i, v_i]$ includes all vertices of attachment of B_i. We call $P[u_i, v_i]$ the *span* of B_i. A *P-bridge chain* B_1, B_2, …, B_k is a sequence of bridges of an induced uv-path P satisfying the following:

$$u = u_1 < u_2 < v_1 \leq u_3 < v_2 \leq u_4 < v_3 \leq \cdots \leq u_{k-1} < v_{k-2} \leq u_k < v_{k-1} < v_k \leq v$$

The *rank* of a P-bridge chain is the number of bridges that form the P-bridge chain. The *span* of a P-bridge chain is the union of the spans of its elements. Figure 2.3 shows a P-bridge chain of rank six.

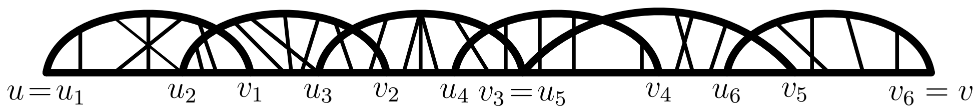


Figure 2.3. P-bridge chain of rank 6

In the next lemma, we prove that if a large 2-connected graph G has a long induced path, then G has either a P-bridge with a long span or a P-bridge chain of large rank.

Lemma 2.3.4. *Let r be an integer exceeding three. There is an integer $f_{2.3.4}(r)$ such that every 2-connected graph with an induced path P of order at least $f_{2.3.4}(r)$ has a P-bridge with span of order at least $r-1$ or a P-bridge chain of rank at least $r-2$.*

Proof. Let $f_{2.3.4}(r) = (r-2) + (r-4)(r-4) + 1$. Suppose G is a 2-connected graph that conduces a path P of order at least $f_{2.3.4}(r)$. Let u and v be the endpoints of P such that $u < v$. If the span of a bridge of P has order at least $r-1$, then the conclusion follows. We may therefore assume that each bridge has span of order at most $r-2$.

We will now show that G has a P-bridge chain of rank at least $r-2$. G has a P-bridge chain $B_1, B_2, \ldots, B_j$ such that the span of each P-bridge B_i is $P[u_i, v_i]$ for $1 \leq i \leq j$ and $u_1 = u$. If $j \geq r-2$, then the conclusion of the lemma follows. We may therefore assume that every P-bridge chain with $u_1 = u$ has rank at most $r-3$.

Select a P-bridge chain $\mathcal{B} = B_1, B_2, \ldots, B_k$ with $u_1 = u$ and maximum span. In order to find an upper bound on the order of the span of $\mathcal{B}$, note that the span of B_1

has at most $r-2$ vertices, $k \leq r-3$, and the span of each of the bridges B_2, B_3, $\ldots$, B_k contributes at most $r-4$ new vertices to the span of $\mathcal{B}$. Since P has order at least $f_{2.3.4}(r)$, it follows that $v_k \neq v$. Moreover, as v_k is not a cut-vertex of G, the path P has a bridge B with a vertex of attachment on $P[u, v_k)$ and another vertex of attachment on $P(v_k, v]$. Let ℓ be minimal subject to B having a vertex of attachment on $P[u, v_\ell)$. The P-bridge chain B_1, B_2, $\ldots$, B_ℓ, B has larger span than $\mathcal{B}$; a contradiction.

Thus, G has a P-bridge chain with rank at least $r-2$, as required. $\square$

The next lemma proves that in either outcome of Lemma 2.3.4, the graph under consideration conduces a large messy ladder.

Lemma 2.3.5. *Let r be an integer exceeding three. There is an integer $f_{2.3.5}(r)$ such that if a 2-connected graph G has an induced path of order $f_{2.3.5}(r)$, then G conduces a messy ladder of order at least r.*

Proof. Let $f_{2.3.5}(r) = (r-2)+(r-4)(r-4)+1$, which is equal to the number $f_{2.3.4}(r)$ from Lemma 2.3.4. Suppose that G has an induced path P of order $f_{2.3.5}(r)$. For each bridge B_i of P in G, let u_i, v_i be the two vertices of attachment of B_i such that $P[u_i, v_i]$ is the span of B_i.

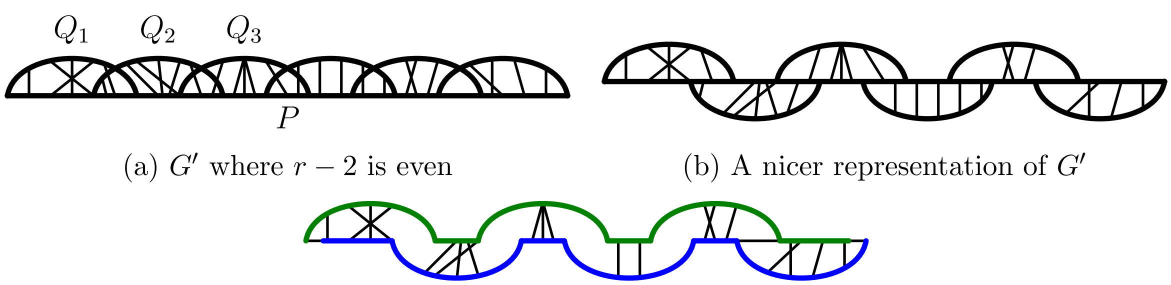
If G has a P-bridge B with span $P[u', v']$ having order at least $r-1$, then let Q be an induced path in B with end-vertices u' and v'. Since P is induced, the path Q has at least one vertex distinct from u' and v'. The subgraph of G induced by $Q \cup P[u', v']$ is a messy ladder of order at least r with rails $P(u', v')$ and Q. The conclusion follows.

Now, we may therefore assume, by Lemma 2.3.4, that G has a P-bridge chain

B_1, B_2, …, B_{r-2}. For each $1 \le i \le r-2$, let Q_i be an induced path in B_i with the endpoints u_i and v_i. Since P is induced, each Q_i contains at least one vertex distinct from u_i and v_i. Define G' to be the subgraph of G induced by $P[u_1, v_{r-2}] \cup \bigcup_{i=1}^{r-2} Q_i$.

In G', we delete vertices on $P(u_{j+1}, v_j)$, if they exist, for $j = \{1, 2, \ldots, r-3\}$ to obtain a graph G''. If $r-2$ is odd, then let $X = Q_1 \cup P(v_1, u_3) \cup Q_3 \cup P(v_3, u_5) \cup Q_5 \cup \cdots \cup P(v_{r-4}, v_{r-2}) \cup Q_{r-2}$ and $Y = P(u_1, u_2) \cup Q_2 \cup P(v_2, u_4) \cup Q_4 \cup P(v_4, u_6) \cup \cdots \cup Q_{r-3} \cup P(v_{r-3}, v_{r-2})$. If $r-2$ is even, then let $X = Q_1 \cup P(v_1, u_3) \cup Q_3 \cup P(v_3, u_5) \cup \cdots \cup Q_{r-3} \cup P(v_{r-3}, v_{r-2})$ and $Y = P(u_1, u_2) \cup Q_2 \cup P(v_2, u_4) \cup Q_4 \cup \cdots \cup P(v_{r-4}, u_{r-2}) \cup Q_{r-2}$. Let the root of X be u_1 and let the root of Y be the neighbor of u_1 on P. Notice that all vertices of G'' lie on $X \cup Y$, the graph G conduces G'', and that (G'', X, Y) is a messy ladder.

Figure 2.4 illustrates this process of obtaining a messy ladder (G'', X, Y) from G'. The rails X and Y of (G'', X, Y) are indicated by the green and blue paths. We remind the reader that thin line segments indicate edges of G' and G'' and thick curves and line segments indicate induced paths of G' and G'', with the straight line segments possibly being trivial.



(a) G' where $r-2$ is even

(b) A nicer representation of G'

(c) (G'', X, Y)

Figure 2.4. Process of obtaining a messy ladder from G'

Since each of the $r-2$ bridges contributes to the messy ladder at least one not on P, it follows that (G'', X, Y) is a messy ladder of order at least r, as required. $\square$

Note that the numbers in the conclusion of two previous lemmas are the same. The process of obtaining a messy ladder from a long induced path has been described in two steps, namely Lemmas 2.3.4 and 2.3.5.

We are now ready to prove Lemma 2.3.1, restated below.

Lemma 2.3.1. *Let p and q be integers exceeding two. There is an integer $f_{2.3.1}(p,q)$ such that every 2-connected graph with a path of order $f_{2.3.1}(p,q)$ conduces one of the following: K_p, $K_{2,p}$, $K_{2,p}^{+}$, and a messy ladder of order at least q.*

Proof. Let $f_{2.3.1}(p,q) = f_{2.3.3}(p,r)$ where $r = f_{2.3.5}(q)$. Since G has a path of order at least $f_{2.3.3}(p,r)$, it follows that G conduces one of the following K_p, $K_{2,p}$, $K_{2,p}^{+}$, and P_r. If G conduces K_p, $K_{2,p}$, or $K_{2,p}^{+}$, then the conclusion follows. We may therefore assume that G conduces P_r. Lemma 2.3.5 implies that G conduces a messy ladder of order at least q, as required. $\square$

2.4. From a Messy Ladder to a Clean Ladder

In this section, we prove that a sufficiently large messy ladder conduces a clean ladder of the desired order.

In order to clean the ladder, we need to define some terms for the crosses. The cross (e,f) is *full* if the messy ladder (L,X,Y) has no other cross whose X-span contains the X-span of (e,f) and whose Y-span contains the Y-span of (e,f). Two crosses are *independent* if their X-spans and Y-spans are edge-disjoint.

In general, crosses may not be ordered in any particular way with respect to the rails X and Y; however, pairwise independent full crosses may be ordered by the position in which their vertices appear on the rails, as explained in Lemma 2.4.1.

Lemma 2.4.1. *Let (e, f) and (g, h) be independent full crosses of a messy ladder (L, X, Y), with the X- and Y-spans being $X[e_X, f_X]$, $Y[f_Y, e_Y]$, $X[g_X, h_X]$, and $Y[h_Y, g_Y]$, respectively. Then $f_X \leq g_X$ if and only if $e_Y \leq h_Y$.*

Proof. Let (e, f) and (g, h) be two independent full crosses. Suppose for a contradiction that $f_X \leq g_X$, however $e_Y \not\leq h_Y$. Then $h_Y \leq f_Y < e_Y$ and $e_X < f_X \leq g_X < h_X$. Thus (e, h) is a cross whose X-span contains the X-spans of (e, f) and of (g, h). This contradicts the fact that the crosses (e, f) and (g, h) are full. Hence $e_Y \leq h_Y$. The other direction of the proof follows an analogous argument. □

For two independent full crosses (e, f) and (g, h), define the relation $(e, f) < (g, h)$ by $f_X \leq g_X$ (or $e_Y \leq h_Y$).

We will be interested in maximal sequences of pairwise independent full crosses, that is, those sequences that do not appear as proper sub-sequences of any other sequence of pairwise independent full crosses.

Let (L, X, Y) be a messy ladder with σ_X, τ_X, σ_Y, and τ_Y as the initial and terminal vertices of X and Y, respectively, and let $\mathcal{X}$ be a maximal sequence of pairwise independent full crosses in (L, X, Y). Our goal is now to use $\mathcal{X}$ to eliminate all non-degenerate crosses in the messy ladder (L, X, Y) to obtain a clean ladder (H, U, W). To do this, we need the following operation on ladders that eliminates non-degenerate pairwise independent full crosses.

Let $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_z)$ be a maximal sequence of pairwise independent full crosses in (L, X, Y). The operation of *resolving the cross* $\mathcal{X}_i = (e^i, f^i)$ results in a triple (L', X', Y') where $L' = L - X(e^i_X, f^i_X) - Y(e^i_Y, f^i_Y)$ and $X' = X[\sigma_X, e^i_X] \cup \{e^i\} \cup Y[e^i_Y, \tau_Y]$ and $Y' = Y[\sigma_Y, f^i_Y] \cup \{f^i\} \cup X[f^i_X, \tau_X]$. Since $\mathcal{X}_i$ is a full cross, the graph L has rungs neither from $X[\sigma_X, e^i_X]$ to $Y[e^i_Y, \tau_Y]$ nor from $Y[\sigma_Y, f^i_Y]$ to $X[f^i_X, \tau_X]$. Thus X' and Y' are induced in L', and (L', X', Y') is a messy ladder. If $\mathcal{X}_i$ is degenerate, then resolving the cross $\mathcal{X}_i$ results in the edges $e^i_X f^i_X$ and $f^i_Y e^i_Y$ becoming rungs of L', and the rungs f^i and e^i becoming edges on the rails X' and Y'. Note that we have not deleted any edges or vertices in this case, so the messy ladders (L', X', Y') and (L, X, Y) are isomorphic.

For a maximal sequence of pairwise independent full crosses $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_z)$ where $\mathcal{X}_i = (e^i, f^i)$ of a messy ladder (L, X, Y), we inductively define the triples that result from resolving consecutive crosses of $\mathcal{X}$. Let (L^1, X^1, Y^1) be the messy ladder obtained by resolving the cross $\mathcal{X}_1$ with $X^1 = X[\sigma_X, e^1_X] \cup \{e^1\} \cup Y[e^1_Y, \tau_Y]$ and $Y^1 = Y[\sigma_Y, f^1_Y] \cup \{f^1\} \cup X[f^1_X, \tau_X]$. Since the crosses in $\mathcal{X}$ are pairwise independent, the operation of resolving $\mathcal{X}_1$ leaves the other crosses in $\mathcal{X}$ unchanged. Since the cross $\mathcal{X}_1$ is full, the operation of resolving $\mathcal{X}_1$ does not create a non-degenerate cross. If $\mathcal{X}_1$ is degenerate, then (L^1, X^1, Y^1) is isomorphic to (L, X, Y). If $\mathcal{X}_1$ is not degenerate, then $(\mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_z)$ is a maximal sequence of pairwise independent full crosses in (L^1, X^1, Y^1).

For the inductive process, the definition of the rails X^i and Y^i depends on the parity of i. Suppose we have defined $(L^{i-1}, X^{i-1}, Y^{i-1})$ for some $2 \leq i \leq z$ where

$(L^{i-1}, X^{i-1}, Y^{i-1})$ is the triple obtained by resolving the crosses $(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_{i-1})$. Since each cross of $\mathfrak{X}$ is full, the operation of resolving the crosses $(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_{i-1})$ does not create non-degenerate crosses. Since the crosses of $\mathfrak{X}$ are pairwise independent, the crosses $(\mathcal{X}_i, \mathcal{X}_{i+1}, \ldots, \mathcal{X}_z)$ are unchanged by the operation of resolving the crosses $(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_{i-1})$. Each cross in $(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_{i-1})$ that was degenerate in (L, X, Y) remains degenerate after resolving the crosses $(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_{i-1})$. So the degenerate crosses from $\mathcal{X}_1$, $\mathcal{X}_2$, $\ldots$, $\mathcal{X}_{i-1}$ together with the crosses $\mathcal{X}_i$, $\mathcal{X}_{i+1}$, $\ldots$, $\mathcal{X}_z$ form a maximal sequence of pairwise independent full crosses in $(L^{i-1}, X^{i-1}, Y^{i-1})$.

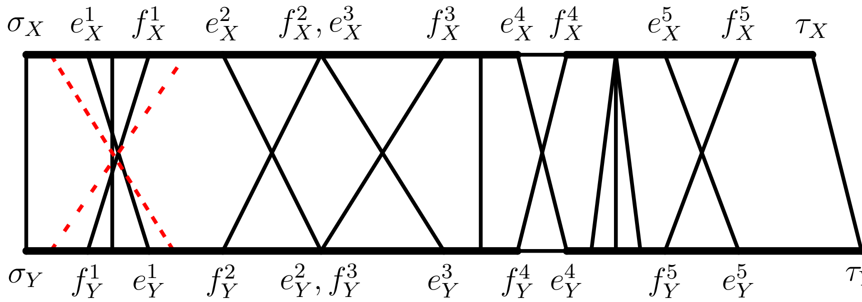
Let (L^i, X^i, Y^i) be the messy ladder obtained from $(L^{i-1}, X^{i-1}, Y^{i-1})$ by resolving $\mathcal{X}_i$. If i is even, then let $X^i = X[\sigma_X, e_X^1] \cup \{e^1\} \cup Y[e_Y^1, f_Y^2] \cup \{f^2\} \cup \cdots \cup \{f^i\} \cup X[f_X^i, \tau_X]$ and $Y^i = Y[\sigma_Y, f_Y^1] \cup \{f^1\} \cup X[f_X^1, e_X^2] \cup \{e^2\} \cup \cdots \cup \{e^i\} \cup Y[e_Y^i, \tau_Y]$. If i is odd, then let $X^i = X[\sigma_X, e_X^1] \cup \{e^1\} \cup Y[e_Y^1, f_Y^2] \cup \{f^2\} \cup \cdots \cup \{e^i\} \cup Y[e_Y^i, \tau_Y]$ and $Y^i = Y[\sigma_Y, f_Y^1] \cup \{f^1\} \cup X[f_X^1, e_X^2] \cup \{e^2\} \cup \cdots \cup \{f^i\} \cup X[f_X^i, \tau_X]$.

Let $(H, U, W) = (L^z, X^z, Y^z)$. Since we have resolved the crosses of $\mathfrak{X}$, every cross from $\mathfrak{X}$ that is in (H, U, W) is degenerate. Thus (H, U, W) is a clean ladder.

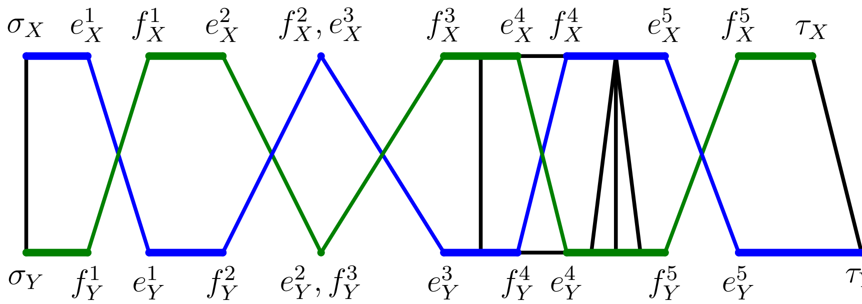
Remark 2.4.2. *The vertices e_X^1, e_Y^1, f_X^1, f_Y^1, $\ldots$, e_X^z, e_Y^z, f_X^z, and f_Y^z of the independent full crosses of $\mathfrak{X}$ are members of the vertex set of (H, U, W).*

This process of resolving the crosses is depicted in Figures 2.5a and 2.5b.

The red dashed lines in Figure 2.5a indicate the locations where rungs that cannot exist due to (e^1, f^1) being a full cross. In Figure 2.5b, the blue path represents the induced path U, the green path represents the induced path W, and the black lines represent rungs



(a) Sequence of independent full crosses



(b) The clean ladder (H, U, W)

Figure 2.5. Resolving the crosses

of (H, U, W). Notice that Figure 2.5a has a rung with an endpoint in the X-span and an endpoint in the Y-span of (e^1, f^1), and this rung is not in (H, U, W).

The next lemma follows from the process described above.

Lemma 2.4.3. *Let $\mathcal{X}$ be a maximal sequence of pairwise independent full crosses of a messy ladder. Resolving the crosses of $\mathcal{X}$ results in a clean ladder.*

In order to prove that a sufficiently large messy ladder conduces a clean ladder of the desired order, we will need the following lemmas to bound from above the order of the spans of crosses and the distance between two consecutive pairwise independent full crosses in a maximal sequence of pairwise independent full crosses. We will combine these lemmas with Lemma 2.4.3 to obtain a clean ladder of the desired order.

The following definitions are essential to creating the bounds. Define a *non-crossing matching* of a messy ladder to be a set of rungs that are pairwise non-adjacent and pairwise non-crossing. Let M be a maximal non-crossing matching in (L, X, Y). The set of edges $M \cup \{\sigma, \tau\}$ is an *augmented matching* of a messy ladder. Note that the rungs of an augmented matching have the following properties: (1) they are pairwise non-crossing; (2) the only vertices that can be endpoints of at most two members of $M \cup \{\sigma, \tau\}$ are σ_X, σ_Y, τ_X, and τ_Y. A *clean cycle* consists of two distinct rungs e and f that do not cross and the sub-paths of X and Y determined by e_X, e_Y, f_X, and f_Y. A ladder is *r-cycle-free* if it conduces no clean cycle of order r or more. A *fan* of order s, denoted by F_s, is the graph obtained by taking an isolated vertex called the *apex* and a path of order $s - 1$ called the *rim* and adding an edge between the apex and every vertex on the rim. Let $\mathcal{F}_s$ be the family of graphs obtained from F_s by subdividing each of the rim edges. A member of the family $\mathcal{F}_s$ that is an induced subgraph of a ladder and has the apex on one rail and the rim entirely on the other rail is called *clean*. A ladder is said to be *s-fan-free* if it conduces no clean member of the family $\mathcal{F}_s$.

In the next three lemmas, we will consider messy ladders that are r-cycle-free and s-fan-free for some values of r and s. Our goal in those lemmas is to bound from above the order of the X-span and the Y-span of every cross by a function of r and s. First, we bound from above the number of vertices on each of the rails of a messy ladder between two consecutive rungs in an augmented matching.

Lemma 2.4.4. *Let r and s be integers exceeding three. Let (L, X, Y) be an r-cycle-free and s-fan-free messy ladder with an augmented matching M. There is an integer $f_{2.4.4}(r, s)$ such that the number of vertices on each X and Y between a pair of consecutive rungs in M, including the endpoints of the rungs, is at most $f_{2.4.4}(r, s)$.*

Proof. Let $f_{2.4.4}(r, s) = 2((s-3)(r-4)+(s-2))+r-5$. Our goal is to show that the number of vertices between a consecutive pair of augmented matching rungs is at most $f_{2.4.4}(r, s)$.

Let $(p^1, p^2, \ldots, p^\ell)$ be an augmented matching M whose rungs are listed in the order of appearance on the rails. Let $X_j = X[p^j_X, p^{j+1}_X]$ and let $Y_j = Y[p^j_Y, p^{j+1}_Y]$ for some $1 \leq j \leq \ell - 1$. Since M is an augmented matching, it follows L has no edge xy such that $x \in X(p^j_X, p^{j+1}_X)$ and $y \in Y(p^j_Y, p^{j+1}_Y)$. Each of the vertices p^j_X, p^j_Y, p^{j+1}_X, and p^{j+1}_Y has at most $s - 2$ incident rungs, including the non-matching rungs, since (L, X, Y) is s-fan-free.

We will bound from above the order of Y_j; the argument for X_j is similar. Let v_1 be the vertex on Y_j such that $p^j_X v_1$ is a rung and the number of vertices on $Y[p^j_Y, v_1]$ is the maximum, and let v_2 be the vertex on Y_j such that $p^{j+1}_X v_2$ is a rung and the number of vertices on $Y[v_2, p^{j+1}_Y]$ is the maximum. We can express $Y[p^j_Y, p^{j+1}_Y]$ as $Y[p^j_Y, v_1] \cup Y(v_1, v_2) \cup Y[v_2, p^{j+1}_U]$, where $Y(v_1, v_2)$ may be empty. We first bound the order of $Y[p^j_Y, v_1]$. Consider the vertices on Y adjacent to p^j_X. Since (L, X, Y) is r-cycle-free and s-fan-free, it follows that Y has at most $s - 2$ such vertices, and the sub-path of Y between every two consecutive neighbors of p^j_X has at most $r - 4$ internal vertices. Since G is s-fan-free, it follows that Y has at most $s - 3$ of the sub-paths determined by the neighbors of p^j_X.

So, $Y[p^j_Y, v_1]$ has at most $s - 2 + (r - 4)(s - 3)$ vertices. Similarly, the number of vertices on $Y[v_2, p_Y^{j+1}]$ is at most $(r-4)(s-3)+s-2$. Since (L, X, Y) is r-cycle-free, the number of vertices on $Y(v_1, v_2)$ is at most $r-5$. So the number of vertices on Y_j is at most $2((s-3)(r-4)+(s-2))+r-5$, as required. □

Next, we bound the number of rungs of an augmented matching that some other rung may cross.

Lemma 2.4.5. *Let r and s be integers exceeding three. Let (L, X, Y) be a messy ladder that is r-cycle-free, s-fan-free, and has an augmented matching M. There is an integer $f_{2.4.5}(r, s)$ such that if a rung crosses two rungs in M, then the number of vertices on the sub-paths of X and Y determined by endpoints of those two rungs in M is at most $f_{2.4.5}(r, s)$.*

Proof. Let $f_{2.4.5}(r, s) = m_1m_2 + (m_1+1)(r-4) - 1$ where $m_1 = (f_{2.4.4}(r, s) - 1)$, $m_2 = (r-4)(s-3)+(s-2)$, and $f_{2.4.4}(r, s)$ is the number from Lemma 2.4.4. Let $(p^1, p^2, \dots, p^\ell)$ be an augmented matching M whose rungs are listed in the order of appearance on the rails. Suppose e is a rung such that e_X is on $X[p_X^j, p_X^{j+1})$ and e_Y is on $Y(p_Y^k, p_Y^{k+1}]$ for some j and k. Without loss of generality, we may assume that $j < k \leq \ell - 1$. Suppose that e crosses two rungs p^m and p^n of M. Since e crosses p^m and p^n, it follows that p_Y^m and p_Y^n are on $Y[p_Y^{j+1}, p_Y^k]$. We bound from above the number of vertices on the sub-path of $Y[p_Y^{j+1}, p_Y^k]$, as the argument for the number of vertices on X is similar.

Lemma 2.4.4 implies the number of vertices on $X[e_X, p_X^{j+1})$ is at most $m_1 = f_{2.4.4}(r, s) - 1$.

Next, we bound from above the number of vertices on $Y[p_Y^{j+1}, e_Y]$. For each vertex

$v \in X[x, p_X^{j+1})$, let E_v be the set of rungs incident with the vertex v. Let D_v be the minimal sub-path of Y that contains the endpoints of all the edges of E_v. Since (L, X, Y) is s-fan-free, we have $|E_v| < s - 1$. Since (L, X, Y) is r-cycle-free, it follows that Y has at most $r - 4$ internal vertices between every two consecutive rungs of E_v. This leads to the following inequality $|V(D_v)| \leq (r-4)(|E_v| - 1) + |E_v| \leq (r-4)(s-3) + (s-2) = m_2$. Let $\mathcal{D} = \bigcup_{v \in X[e_X, p_X^{j+1})} D_v$. Since the union is taken over at most m_1 elements, the graph $\mathcal{D}$ has at most $m_1 m_2$ vertices.

Let $\mathcal{Y}$ be the graph induced by the vertices of $Y[p_Y^{j+1}, e_Y] - \mathcal{D}$. Since $\mathcal{D}$ has at most m_1 components, it follows that $\mathcal{Y}$ has at most $m_1 + 1$ components. Since (L, X, Y) is r-cycle-free, each component of $\mathcal{Y}$ has at most $r - 4$ vertices. So, the number of vertices on $Y[p_Y^{j+1}, e_Y]$ is at most $m_1 m_2 + (m_1 + 1)(r-4)$. Every member of M that crosses e must have an endpoint on $Y[p_Y^{j+1}, e_Y)$. So the number of vertices on $Y[p_Y^{j+1}, p_Y^k]$ is at most $m_1 m_2 + (m_1 + 1)(r-4) - 1 = f_{2.4.5}(r, s)$, as required. $\square$

Now, we bound from above the number of vertices in each the X-span and the Y-span of a cross.

Lemma 2.4.6. *Let r and s be integers exceeding three. There is an integer $f_{2.4.6}(r, s)$ such that if messy ladder (L, X, Y) is a r-cycle-free and s-fan-free, then the X-span and the Y-span of every cross is bounded from above by $f_{2.4.6}(r, s)$.*

Proof. Let $f_{2.4.6}(r, s) = 2(f_{2.4.5}(r, s) + 2f_{2.4.4}(r, s) - 2)$ where $f_{2.4.4}(r, s)$ and $f_{2.4.5}(r, s)$ are the numbers from Lemmas 2.4.4 and 2.4.5, respectively. Let $(p^1, p^2, \ldots, p^m)$ be an augmented matching M whose rungs are listed in the order in which they appear on the rails. Note that $\sigma = p^1$ and $\tau = p^m$. For the cross (e, f) suppose that $e_X \in X[p_X^j, p_X^{j+1})$ and suppose that $f_X \in X(e_X, p_X^\ell]$ where ℓ is minimal subject to $f_X \le p_X^\ell$. Suppose that $e_Y \in Y(\sigma_Y, p_Y^k]$ where k is minimal subject to $e_Y \le p_Y^k$ and suppose also that $f_Y \in Y[p_Y^i, e_Y)$ where i is maximal subject to $p_Y^i \le f_Y$.

We will bound the number of vertices in the Y-span; the argument for the X-span is very similar.

We will define Y_1 depending on the relation of e_Y to p_Y^j. If $p_Y^j \le e_Y$, then let $Y_1 = Y[p_Y^j, p_Y^{j+1}] \cup Y(p_Y^{j+1}, p_Y^{k-1}) \cup Y[p_Y^{k-1}, e_Y]$. By Lemma 2.4.5, the sub-path $Y[p_Y^{j+1}, p_Y^{k-1}]$ has at most $f_{2.4.5}(r, s)$ vertices. So, the sub-path $Y(p_Y^{j+1}, p_Y^{k-1})$ has at most $f_{2.4.5}(r, s) - 2$ vertices. By Lemma 2.4.4, the number of vertices on $Y[p_Y^j, p_Y^{j+1}]$ is at most $f_{2.4.4}(r, s)$. Similarly, the number of vertices on $Y[p_Y^{k-1}, e_Y]$ is at most $f_{2.4.4}(r, s)$. So the number of vertices on Y_1 is at most $2f_{2.4.4}(r, s) + f_{2.4.5}(r, s) - 2$. If $e_Y < p_Y^j$, then let $Y_1 = Y[e_Y, p_Y^k] \cup Y(p_Y^k, p_Y^j) \cup Y[p_Y^j, p_Y^{j+1}]$. By Lemma 2.4.5, the sub-path $Y[p_Y^j, p_Y^k]$ has at most $f_{2.4.5}(r, s)$ vertices. So, the sub-path $Y(p_Y^j, p_Y^k)$ has at most $f_{2.4.5}(r, s) - 2$ vertices. By Lemma 2.4.4, the number of vertices on $Y[p_Y^j, p_Y^{j+1}]$ is at most $f_{2.4.4}(r, s)$. Similarly, the number of vertices on $Y[p_Y^{k-1}, e_Y]$ is at most $f_{2.4.4}(r, s)$. So the number of vertices on Y_1 is at most $2f_{2.4.4}(r, s) + f_{2.4.5}(r, s) - 2$.

Similarly, we will define Y_2 depending on the relation of f_Y to $p_Y^{\ell-1}$. If $p_Y^{\ell-1} \le f_Y$, then let $Y_2 = Y[p_Y^{\ell-1}, p_Y^\ell] \cup Y(p_Y^\ell, p_Y^i) \cup Y[p_Y^i, f_Y]$. The argument for this case is analogous to the argument for $p_Y^j \le e_Y$. If $f_Y < p_Y^{\ell-1}$, then let $Y_2 = Y[f_Y, p_Y^{i+1}] \cup Y(p_Y^{i+1}, p_Y^{\ell-1}) \cup Y[p_Y^{\ell-1}, p_Y^\ell]$. Likewise, this case follows the argument when $e_Y < p_Y^j$; and thus, the number of vertices on Y_2 is at most $2f_{2.4.4}(r, s) + f_{2.4.5}(r, s) - 2$.

Combining the bounds on the number of vertices on Y_1 and Y_2, we get the that $Y_1 \cup Y_2$ has at most $2(f_{2.4.5}(r, s) + 2f_{2.4.4}(r, s) - 2)$ vertices. Since e and f cross, $Y[f_Y, e_Y]$ is a sub-path of $Y_1 \cup Y_2$. Therefore, the number of vertices in the Y-span of a cross is at most $f_{2.4.6}(r, s)$, as required. $\square$

Define a *sub-ladder* (L', X', Y') of (L, X, Y) to be a messy ladder such that X' and Y' are rooted sub-paths of X and Y, respectively, and L' is the subgraph of L induced by the vertices on $X' \cup Y'$. A *cross-free ladder* is a messy ladder such that no pair of its rungs cross. A cross-free ladder is obviously a clean ladder. A messy ladder is *q-cross-crowded* if it does not conduce a cross-free sub-ladder of order q or more. Note that a q-cross-crowded messy ladder is also q-cycle-free and q-fan-free.

In the previous lemmas, we considered messy ladders that were r-cycle-free and s-fan-free for some integers r and s. In the following lemma, we need a stronger assumption, namely that the messy ladder is q-cross-crowded for some integer q. Since a cross-free ladder is a clean ladder, we restrict the order of the largest cross-free sub-ladder and show that a large q-cross-crowded messy ladder has a long maximal sequence of pairwise independent full crosses.

Lemma 2.4.7. *Let q be an integer exceeding three and let w be a positive integer. There is an integer $f_{2.4.7}(q, w)$ such that if a q-cross-crowded messy ladder has order at least $f_{2.4.7}(q, w)$, then the length of every maximal sequence of pairwise independent full crosses is at least w.*

Proof. Let q and w be integers such that $q \geq 4$ and $w \geq 1$, and we prove that

$$f_{2.4.7}(q,w) = 4(f_{2.4.6}(q,q)+1)(q^2+q)+2(w-1)f_{2.4.6}(q,q)+2(2f_{2.4.6}(q,q)+1)(q^2+q)(w-2)$$

and $f_{2.4.6}(q,q)$ is the number from Lemma 2.4.6, satisfies the conclusion. Suppose that (L, X, Y) is a q-cross-crowded messy ladder that has a maximal sequence $\mathcal{X}$ of pairwise independent crosses that has z elements, where $0 \leq z \leq w-1$. We will show that the number of vertices of L is less than $f_{2.4.7}(q,w)$, thereby proving the lemma.

If $z = 0$, then (L, X, Y) is cross-free and so $|V(L)| < q \leq f_{2.4.7}(q,w)$, and the conclusion follows. So, for the remainder of the proof, we may therefore assume that $z \geq 1$. Also, by symmetry, we may assume that $|V(X)| \geq |V(Y)|$, and concentrate on finding an upper bound only for $|V(X)|$.

Let $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_z)$, and, for each i in $\{1, 2, \ldots, z\}$, let S_i be the X-span of $\mathcal{X}_i$. By Lemma 2.4.6, the number of vertices of S_i is at most $f_{2.4.6}(q,q)$, and so the union S of all X-spans of the crosses in $\mathcal{X}$ has order at most $zf_{2.4.6}(q,q)$.

Now, let $T = X - \bigcup_{i=1}^{z} S_i$, and let $\mathcal{X}_i = (e^i, f^i)$ for each i in $\{1, 2, \ldots, z\}$. Every connected component of T is of one of the following forms: $X[\sigma_X, e^1_X)$, $X(e^z_X, \tau_X]$, and $X(f^i_X, e^{i+1}_X)$ for some i in $\{1, 2, \ldots, z-1\}$ in the case $z \geq 2$. Before finding upper bounds on the orders of such segments of X, we will present an upper bound on the number of rungs incident with vertices of those segments.

First, we will bound the number of rungs incident with vertices of $X[\sigma_X, e^1_X)$. Note that the argument for the number of rungs incident with vertices on $X(f^z_X, \tau_X]$ is similar. Since each cross in $\mathcal{X}$ is full, each rung incident with a vertex in $X[\sigma_X, e^1_X)$ has the other endpoint on $Y[\sigma_Y, e_Y^1)$. Since (L, X, Y) has no cross-free sub-ladder of order q, it follows that (L, X, Y) has at most $q - 1$ rungs with one endpoint on $X[\sigma_X, e_X^1)$ and other endpoint on $Y[\sigma_Y, f_Y^1]$. Since (L, X, Y) is q-fan-free, each vertex on $Y(f_Y^1, e_Y^1)$ is incident with at most $q-2$ rungs that have other endpoint on $X[\sigma_X, e_X^1)$. So (L, X, Y) has at most $(q-2)(f_{2.4.6}(q, q) - 2)$ rungs with one endpoint on $X[\sigma_X, e_X^1)$ and a distinct endpoint on $Y(f_Y^1, e_Y^1)$. Thus, (L, X, Y) has at most $(q-2)(f_{2.4.6}(q, q) - 2) + q - 1$ rungs incident with a vertices of $X[\sigma_X, e_X^1)$. Similarly, (L, X, Y) has at most $(q-2)(f_{2.4.6}(q, q) - 2) + q - 1$ rungs incident with a vertices of $X(f_X^z, \tau_X]$.

Next, we assume that $z \geq 2$ and we bound from above the number of rungs incident with an arbitrary segment on X between consecutive crosses of $\mathcal{X}$. Since each cross in $\mathcal{X}$ is full, each rung incident with a vertex in $X(f_X^i, e_X^{i+1})$ for some i in $\{1, 2 \ldots, z-1\}$ have the other endpoint on $Y(f_Y^i, e_Y^{i+1})$. Since (L, X, Y) has no cross-free sub-ladder of order q, it follows that (L, X, Y) has at most $q - 1$ rungs with one endpoint on $X(f_X^i, e_X^{i+1})$ and other endpoint on $Y(e_Y^i, f_Y^{i+1})$. Since (L, X, Y) is q-fan-free, it follows that each vertex on one of $Y(f_Y^i, e_Y^i)$ and $Y(f_Y^{i+1}, e_Y^{i+1})$ for some i in $\{1, 2, \ldots, z-1\}$ is incident with at most $q - 2$ rungs that have a distinct endpoint on $X(f_X^i, e_X^{i+1})$. So (L, X, Y) has at most $2(q-2)(f_{2.4.6}(q, q) - 2)$ rungs with one endpoint on $X(f_X^i, e_X^{i+1})$ and other endpoint on either $Y(f_Y^i, e_Y^i]$ or $Y[f_Y^{i+1}, e_Y^{i+1})$. Thus, (L, X, Y) at most $2(q-2)(f_{2.4.6}(q, q) - 2) + q - 1$ rungs incident with a vertices of $X(f_X^i, e_X^{i+1})$.

Now, we will bound from above the number of vertices on X. Since (L, X, Y) is q-cycle-free, it follows that X has at most $q - 4$ vertices between two consecutive rungs.

The bounds obtained above are cumbersome, and since this dissertation proves an existence result, we relax the number to obtain $f_{2.4.7}(q, w)$. The number of vertices on each of $X[\sigma_X, e_X^1)$ and $X(f_X^z, \tau_X]$ is at most

$$(q-2)(f_{2.4.6}(q,q)-2)+q-1+[(q-2)(f_{2.4.6}(q,q)-2)+q](q-4) < (f_{2.4.6}(q,q)+1)(q^2+q).$$

Similarly, the number of vertices on each $X(f_X^i, e_X^{i+1})$ is at most

$$[2(q-2)(f_{2.4.6}(q,q)-2)+q]\,(q-4)+2(q-2)(f_{2.4.6}(q,q)-2)+q-1 < (2f_{2.4.6}(q,q)+1)(q^2+q).$$

Since $z \le w-1$, the number of vertices on X is at most, $g(q) = 2[(q-2)(f_{2.4.6}(q,q)-1)+q-1+((q-2)(f_{2.4.6}(q,q)-1)+q)(q-4)] + (w-1)f_{2.4.6}(q,q) + (w-2)\,[(2(q-2)(f_{2.4.6}(q,q)-1)+q)\,(q-4)+2(q-2)(f_{2.4.6}(q,q)-1)+q-1]$. Thus the number of of vertices on (L, X, Y) is at most $2g(q)$.

However, $2g(q) < f_{2.4.6}(q, w)$. Hence, every maximal sequence of pairwise independent full crosses has length at least w. $\square$

The following lemma combines the previous lemmas in this section to complete the proof that a clean ladder of desired order is a sub-ladder of every messy ladder that is large enough.

Lemma 2.4.8. *Let t be an integer exceeding two. There is an integer $f_{2.4.8}(t)$ such that every a messy ladder of order at least $f_{2.4.8}(t)$ conduces a clean ladder of order at least t.*

Proof. Without loss of generality, we may assume that t is even. We prove that $f_{2.4.8}(t) = f_{2.4.7}(t, w)$, where $f_{2.4.7}$ is the number from Lemma 2.4.7 and $w = \frac{t}{2} - 1$, satisfies the conclusion. Suppose (L, X, Y) is a messy ladder of order at least $f_{2.4.8}(t)$ and let $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \ \ldots, \ \mathcal{X}_z)$ be a maximal sequence of pairwise independent full crosses in (L, X, Y).

In either of the following three cases, (L, X, Y) has no crosses to resolve. If (L, X, Y) conduces a vertex incident with $t-1$ rungs, then (L, X, Y) conduces a member of the family $\mathcal{F}_t$. Every member of the family $\mathcal{F}_t$ has no crosses and is therefore a clean ladder of order at least t. We may therefore assume that (L, X, Y) is t-fan-free.

If (L, X, Y) conduces a clean cycle of order r for $r \geq t$, this cycle has no crosses and is, therefore, a clean ladder of order at least t. We may therefore assume that (L, X, Y) is t-cycle-free.

If (L, X, Y) conduces a cross-free sub-ladder of order at least t, then the conclusion follows. Therefore, we may therefore assume that (L, X, Y) is t-cross-crowded.

By Lemma 2.4.7, it follows that $z \geq w$.

Let (H, U, W) be the ladder obtained by resolving the crosses of (L, X, Y). By Lemma 2.4.3, (H, U, W) is a clean ladder. It remains to show that (H, U, W) has order at least t. By Remark 2.4.2, each full cross from the sequence has four vertices in (H, U, W). The cross $\mathcal{X}_1$ contributes four vertices to (H, U, W) and each of the subsequent crosses contributes at least two new vertices to (H, U, W). Thus (H, U, W) has at least $4 + 2(w-1) = t$ vertices, as required. □

2.5. Proving Theorem 2.1.1

The main theorem is a straightforward consequence of Lemmas 2.2.1, 2.3.1 and 2.4.8. Specifically, for some integer r exceeding two, if a 2-connected graph G has a sufficiently long path, then it conduces one of the following: K_r, $K_{2,r}$, $K_{2,r}^+$, and a clean ladder of order at least r; and if G fails to have a sufficiently long path, but is large enough, then G conduces a member of one of the families: $\mathcal{K}_{2,r}^+$ and $\mathcal{K}_{2,r}$.

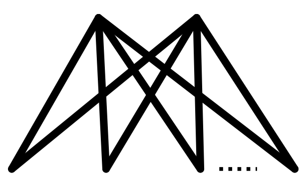
Theorem 2.1.1. *Let r be an integer exceeding two. There is an integer $f_{2.1.1}(r)$ such that every 2-connected graph of order at least $f_{2.1.1}(r)$ conduces one of the following: K_r, a clean ladder of order at least r, a member of $\mathcal{K}_{2,r}$, and a member of $\mathcal{K}_{2,r}^+$.*

Proof. Let $f_{2.1.1}(r) = f_{2.2.1}(q, r)$ where $q = f_{2.3.1}(r, t)$ and $t = f_{2.4.8}(r)$. Since G has at least $f_{2.2.1}(q, r)$ vertices, Lemma 2.2.1 asserts that G either has a path of order q or conduces a member of one of the families: $\mathcal{K}_{2,r}$ and $\mathcal{K}_{2,r}^+$. If G conduces a member of one of the families: $\mathcal{K}_{2,r}$ and $K_{2,r}^+$, then the conclusion follows. We may therefore assume that G has a path of order q. Lemma 2.3.1 implies that G conduces one of the following: K_r, $K_{2,r}$, $K_{2,r}^+$, and a messy ladder of order at least $f_{2.4.8}(r)$. If G conduces K_r, $K_{2,r}$, and $K_{2,r}^+$, then the conclusion follows. We may therefore assume that G conduces a messy ladder of order at least $f_{2.4.8}(r)$. Lemma 2.4.8 implies that G conduces a clean ladder of order at least r. This completes the proof. $\square$

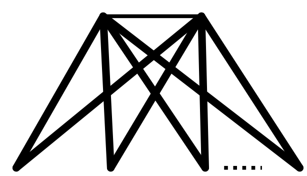
Chapter 3. Infinite Case

3.1. Preliminaries

In this chapter, we will present the unavoidable induced subgraphs of infinite 2-connected graphs. Before giving the main result of this section, we will formally define two families of graphs. Let $K_{2,\infty}$ be the complete bipartite graph where one part of the bipartition contains two infinite vertices, u and v, and the other consists of an infinite number vertices of degree two that are adjacent to both u and v. Let $\mathcal{K}_{2,\infty}$ be the family of graphs obtained from $K_{2,\infty}$ by subdividing each of the edges of $K_{2,\infty}$, shown in Figure 3.1a. Let $\mathcal{K}^+_{2,\infty}$ be the family of graphs obtained from the family $\mathcal{K}_{2,\infty}$ by adding an edge between the two infinite vertices to each member of the family $\mathcal{K}_{2,\infty}$, shown in Figure 3.1b.



(a) $\mathcal{K}_{2,\infty}$



(b) $\mathcal{K}^+_{2,\infty}$

Figure 3.1. Families $\mathcal{K}_{2,\infty}$ and $\mathcal{K}^+_{2,\infty}$

The following theorem is the main result of this chapter.

Theorem 3.1.1. *Let G be a 2-connected infinite graph. Then G conduces one of the following: K_∞, a member of the family $\mathcal{K}^+_{2,\infty}$, a member of the family $\mathcal{K}_{2,\infty}$, a well-described fan-like structure shown in Figure 3.2, and a well-described ladder-like structure shown in Figure 3.3.*

To make a clear distinction between vertices and edges, we will use subscripts for vertices such as v_i and superscripts for edges such as e^i.

(a) $\mathcal{F}_\infty$ (b) $\mathcal{F}^\Delta_\infty$

Figure 3.2. Well-described fan-like structures

(a) $\mathcal{L}_\infty$ (b) $\mathcal{L}^\Delta_\infty$

(c) $\mathcal{L}^{\nabla\Delta}_\infty$ (d) Infinite Clean Ladder

Figure 3.3. Well-described ladder-like structures

To prove Theorem 3.1.1, we consider the cases that an infinite 2-connected graph G either has an infinite vertex or it does not. Section 3.2 discusses the case that G has an infinite vertex. The main challenge in going from finite graphs to infinite graphs is in that section. The case when G is locally-finite is presented into two sections. In Section 3.3, we start with a locally-finite graph and obtain one of the ladder-like structures from Figures 3.3a to 3.3c or an infinite messy ladder. In Section 3.4, we show the process of obtaining an induced infinite clean ladder from the infinite messy ladder. Section 3.5 combines the results of Sections 3.2 to 3.4 to prove Theorem 3.1.1.

3.2. Infinite Vertex

In this section, we present the unavoidable infinite induced subgraphs of a 2-connected graph with an infinite vertex.

Ordered trees will play a significant role in this chapter as well, so we need some additional notations for infinite graphs. The sub-ray of T with initial vertex a is denoted by $T[a, \infty)$. Similarly, the sub-ray of T induced by vertices v such that $a < v$ is denoted by $T(a, \infty)$. A ray is considered to be a tree rooted at its initial vertex.

Next, we formally define the fan-like structures in Figure 3.2. A fan of order s where s in an integer exceeding two, denoted by F_s, is the graph obtained by taking an isolated vertex called the *apex* and a path of order $s - 1$ called the *rim* and adding an edge between the apex and every vertex on the rim. Let $\mathcal{F}_s$ be the family of graphs obtained from F_s by subdividing each of the edges. Let F_∞ be a fan where the apex vertex is infinite and the rim is a ray, and let $\mathcal{F}_\infty$ be the family of graphs obtained from F_∞ by subdividing each of the edges; see Figure 3.2a. An Δ-*fan*, denoted by F_t^Δ of order $3t + 1$ is obtained by taking an apex vertex v of degree t, a collection of t isolated vertices say v_1, v_2, $\ldots$, v_t, and a path of order $2t$ with vertices $p_1, p_2, \ldots, p_{2t}$ and adding an edge from v to v_i for each $1 \leq i \leq t$, an edge from v_i to p_{2i}, and an edge from v_i to p_{2i-1} for each $1 \leq i \leq t$. Let $\mathcal{F}_t^\Delta$ be the family of graphs obtained by subdividing each of the edges vv_i for each $1 \leq i \leq t$ and the edges $p_{2j}p_{2j+1}$ for each $1 \leq j \leq t-1$. Let F_∞^Δ be a Δ-fan such that the apex vertex has infinite degree and the path p_1, p_2 is a ray. Let $\mathcal{F}_\infty^\Delta$ be the family of graphs obtained from F_∞^Δ by subdividing each of the edges vv_i and the edges $p_{2j}p_{2j+1}$

for positive integers i and j; see Figure 3.2b.

For the following Lemmas, we need to define a tree T that controls the edges of an infinite connected graph G that are not in T and whose endpoints are in T.

Definition 3.2.1. *Let $\mathcal{V}$ be an infinite independent set of vertices of a connected graph G. A $\mathcal{V}$-*connecting tree *is a subgraph T of G defined through the following inductive process. Let v_1 be an arbitrary element of $\mathcal{V}$, and let P^1 be a graph consisting of the single vertex $v_1 = t_1$. Let P^2 be the shortest path in G from v_1 to a vertex in $\mathcal{V} \setminus \{v_1\}$. Let v_2 be the endpoint of P^2 in $\mathcal{V} \setminus \{v_1\}$ and $t_2 = v_1$. Suppose now that k is a natural number and that v_i, t_i, and P^i have been defined for all i in $\{1, 2, \ldots, k\}$. Every path Q joining a vertex in $\mathcal{V} \setminus \{v_1, v_2, \ldots, v_k\}$ to a vertex of $P^1 \cup P^2 \cup \cdots \cup P^k$ will receive a triple $(\gamma_1(Q), \gamma_2(Q), \gamma_3(Q))$ as a* grade*. In this triple, let $\gamma_1(Q)$ be the length of Q, let $\gamma_2(Q)$ be the minimum i such that the endpoint u of Q lies on P^i, and let $\gamma_3(Q)$ be $dist_{P^i}(u, t_i)$. Let P^{k+1} be the path with lexicographically minimal grade. Let the endpoint of P^{k+1} on P^i be called t_{k+1} and the endpoint of P^{k+1} in $\mathcal{V} \setminus \{v_1, v_2, \ldots, v_k\}$ be called v_{k+1}. Let T be the union of all the paths P^k that is rooted at v_1.*

We have the following observations about T.

1. At each step, we may select a path because G is connected.

2. Since the graph is countable and $\mathcal{V}$ is countable, the construction process has countably many steps. Thus, the construction process cannot hold for the ordinal ω, and thus terminates. So T exists.

3. T is a tree because at the k-th step for $k > 1$ in Definintion 3.2.1 we added a path P^k whose one endpoint is on $P^1 \cup P^2 \cup \cdots \cup P^{k-1}$ and is otherwise disjoint from $P^1 \cup P^2 \cup \cdots \cup P^{k-1}$.

4. The tree T is a union of induced paths in G, but T is not necessarily induced in G.

For Lemmas 3.2.2 to 3.2.4, we assume that G is an infinite 2-connected graph with an infinite vertex v that does not conduce K_∞, that $\mathcal{V}$ is an infinite independent subset of the neighborhood of v in G, that T is a $\mathcal{V}$-connecting tree of $G - v$, and that the notation used is that from Definintion 3.2.1. Note that G may have an edge from v to any vertex of T. The next three lemmas are of a technical nature, and Lemma 3.2.5 is the main result of this section. Since T is infinite and connected, Theorems 1.2.8 and 1.2.9 imply that T contains an infinite vertex or an induced ray. We address the case that T has an infinite vertex in Lemma 3.2.2 and the case that T contains a ray in Lemmas 3.2.3 and 3.2.4. It is important to keep track of the order that the paths were added in Definintion 3.2.1. In the proofs of Lemmas 3.2.2 and 3.2.4, we will be taking sequences, sub-sequences, and sub-sub-sequences, etc. with respect to the order in which paths were added to T. In particular, one way to take a sub-sequence is to take every other element of the sequence. We call this an *alternating sub-sequence.*

Lemma 3.2.2. *If T has an infinite vertex, then G conduces a member of one of the following families: $\mathcal{K}_{2,\infty}$ and $\mathcal{K}^+_{2,\infty}$.*

Proof. Since T has an infinite vertex, it follows that t_i is the same vertex of G for an infinite sub-sequence $\mathcal{I}$ of natural numbers, and call this vertex x. Since x is an infinite vertex and G does not conduce K_∞, the set of neighbors in T of x denoted by $N_T(x)$ has an infinite subset that induces in G an infinite independent set by Theorem 1.2.7. Let w_i be the neighbor of v on $P^i[x, v_i]$ such that the length of $P^i(x, w_i]$ is minimal. If v does not have a neighbor that is distinct from v_i on $P^i(x, v_i]$, then $w_i = v_i$. Let $\mathcal{P}$ be the sequence of paths $P^i[x, w_i]$ such that the vertices of $N_T(x)$ induce $\overline{K_\infty}$, let x_i be the member of $N_T(x)$ on the path P^i for each path in $\mathcal{P}$, and let $\mathcal{I}_1$ be the sub-sequence of $\mathcal{I}$ that consists of i for which $P^i[x, w_i]$ is in $\mathcal{P}$. Let T^1 be the sub-tree of T consisting of the paths in $\mathcal{P}$. Note that T^1 is a subdivision of $K_{1,\infty}$ and that T^1 is not necessarily induced in G.

Suppose that x_i is a neighbor of v for infinitely many i in $\mathcal{I}_1$. Let $\mathcal{I}_{2a}$ be the sub-sequence of $\mathcal{I}_1$ of indices of paths of $\mathcal{P}$ such that x_i is a neighbor of v. Then the subgraph of G induced by v and vertices of $P^i[x, x_i]$ for $i \in \mathcal{I}_{2a}$ is $K_{2,\infty}$, and the conclusion follows.

We may therefore assume that x_i is not a neighbor of v for infinitely many i in $\mathcal{I}_1$; and thus $P^i[x_i, w_i]$ is non-trivial. Let $\mathcal{I}_{2b}$ be the sub-sequence of $\mathcal{I}_1$ consisting of i for which $P^i[x_i, w_i]$ is not trivial, let $\mathcal{P}^{2b}$ be the sub-sequence of $\mathcal{P}$ that consists of paths $P^j[x, w_j]$ for j in $\mathcal{I}_{2b}$, and let T^{2b} be the sub-tree of T^1 that consists of paths of $\mathcal{P}^{2b}$ with index from $\mathcal{I}_{2b}$.

We will now show that the only edges (if any) of G not in T^{2b} join neighbors of x to vertices of distance two from x in T^{2b}. The red vertices are the neighbors of x, the blue are the vertices of distance two from x in T^{2b}, the purple segments in Figure 3.4 show the only edges (if any) of G that are not in T^{2b}, the thick solid segments represent paths, and the thinner solid segments are edges. For the remainder of the proof, the interval notation for paths will be assumed to be applied to paths P^i in $\mathcal{P}$, so the interval $P^i[x_i, w_i]$ will be assumed to be expressed as $[x_i, w_i]$.

For each $i \in \mathcal{I}_{2b}$, let y_i be the neighbor of x_i on P^i distinct from x. We show that

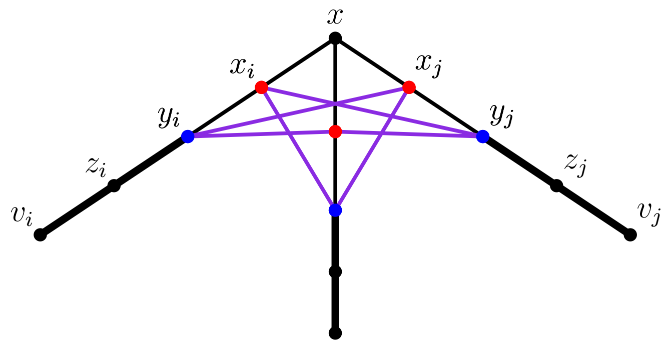


Figure 3.4. Structure of T^{2b}

the only edges of G not on the paths of $\mathcal{P}^{2b}$ are of the form either $x_i y_j$ or $x_j y_i$ for i and j in $\mathcal{I}_{2b}$. Let $\mathcal{X}^{2b}$ be the set of vertices x_i for P^i in $\mathcal{P}^{2b}$. Without loss of generality, for two distinct vertices x_i and x_j in $\mathcal{X}^{2b}$ we may assume that $i < j$. For a nonempty path $(y_i, v_i]$, let z_i an arbitrary vertex on that sub-path. Since the vertices x_i and x_j are members of the independent set $\mathcal{X}^{2b}$, it follows that G has no edge between those two vertices. The graph G has no edge between y_i and y_j; otherwise, the path in G induced by y_i and the vertices of $[y_j, v_j]$ is shorter than the path P^j in G, and has lexicographically smaller grade than P^j, and thus the path P^j would not have been selected in the construction of T; see Definintion 3.2.1. The graph G has no edge from z_i to x_j; otherwise, the path in G induced by x, x_i and the vertices of $[z_i, v_i]$ is shorter than the path P^i in G and has lexicographically smaller grade than P^i, and the path P^i would not have been selected in the construction of T; see Definintion 3.2.1. Additionally, G has no edge from x_i to z_j; otherwise, the path in G induced x, x_j and the vertices of $[z_j, v_j]$ is a shorter path in G from x to v_j, and has lexicographically smaller grade than P^j, and P^j would not have been selected in the construction of T; see Definintion 3.2.1. Thus, edges in G joining internal vertices of members of $\mathcal{P}^{2b}$ must be of the form either x_iy_j or y_ix_j.

Let $\mathcal{Y}^{2b}$ be the set of vertices y_i for i in $\mathcal{I}_{2b}$. Note that $\mathcal{Y}^{2b}$ is infinite. Since we are considering graph T^{2b}, we may assume that every edge of the graph G not in T joins a member of $\mathcal{X}^{2b}$ to a member of $\mathcal{Y}^{2b}$. Let $\mathcal{Y}'$ be the subset of $\mathcal{Y}^{2b}$ consisting of vertices that have more than one neighbor in G in the set $\mathcal{X}$. Note that $\mathcal{Y}'$ is an independent set because it is a subset of $\mathcal{Y}^{2b}$. If $\mathcal{Y}'$ is finite, then there exists an infinite subset $\mathcal{Y}^{3a}$ of $\mathcal{Y}^{2b}$ with only one neighbor in $\mathcal{X}^{2b}$. Let $\mathcal{I}_{3a}$ be the sub-sequence of $\mathcal{I}_{2b}$ that consists of j such that y_j is in $\mathcal{Y}^{3a}$ and let T^{3a} be the sub-tree of T that consists of the paths $P^j[x, w_j]$ for j in $\mathcal{I}_{3a}$. The graph induced in G by v and the vertices of T^{3a} is either a member of the family $\mathcal{K}_{2,\infty}$ if $x \notin N_G(v)$, or a member of the family $\mathcal{K}^+_{2,\infty}$ for $x \in N_G(v)$, and the conclusion follows.

We may therefore assume that $\mathcal{Y}'$ is infinite. Let $\mathcal{I}_{3b}$ be the sub-sequence of $\mathcal{I}_{2b}$ that consists of i for which y_i is in $\mathcal{Y}'$, let $\mathcal{P}^{3b}$ be the sequence of paths P^i for i in $\mathcal{I}_{3b}$, and let T^{3b} be the tree that is the union of paths from $\mathcal{P}^{3b}$. Let $\mathcal{Y}^{3b}$ be the subset of $\mathcal{Y}'$ with indices from the set $\mathcal{I}_{3b}$ and let $\mathcal{X}^{3b}$ be the subset of $\mathcal{X}^{2b}$ with indices from $\mathcal{I}_{3b}$.

We will consider the subgraph H of G induced by $\mathcal{X}^{3b}$ and $\mathcal{Y}^{3b}$. Note that H is infinite and bipartite. We will consider two cases, either H has infinitely many components or H has an infinite component.

First, if H has infinitely many components, we may select one path $[x_i, w_i]$ for each component. Let $\mathcal{I}_{4a}$ be the sub-sequence of $\mathcal{I}_{3b}$ that consists of the indices of these paths and let T^{4a} be the sub-tree of T that consists of paths $P^j[x, w_j]$ for j in $\mathcal{I}_{4a}$. Then H has infinitely many internally-disjoint paths between x and v. Thus T^{4a} is an induced subgraph of G that either is a member of the family $\mathcal{K}_{2,\infty}$ if $x \notin N_G(v)$ or a member of the family $\mathcal{K}^+_{2,\infty}$ otherwise, and the conclusion follows.

We may therefore assume that H has only finitely many components. Thus, H conduces an infinite component H'. Since H' is bipartite and G has no K_∞ subgraph, it does not conduce K_∞. Thus, by Theorem 1.2.9, H' conduces either $K_{1,\infty}$ or a ray. We address each case separately.

First, suppose that H' has an induced $K_{1,\infty}$ and let u denote the infinite vertex in that subgraph. We consider two cases depending on whether u is in $\mathcal{X}^{3b}$ or u is in $\mathcal{Y}^{3b}$.

First, if $u \in \mathcal{X}^{3b}$, then let $\mathcal{I}_{4b}$ be the sub-sequence of $\mathcal{I}_{3b}$ that consists of i for which y_i is a neighbor of u. Let $\mathcal{Y}^{4b}$ be the subset of $\mathcal{Y}^{3b}$ that consists vertices y_i for i in $\mathcal{I}_{4b}$. Note that $\mathcal{Y}^{4b}$ is a subset of $\mathcal{Y}^{3b}$, so $\mathcal{Y}^{4b}$ is an independent set of vertices. The subgraph of G induced by u, v, and the vertices of $[y_i, w_i]$ for i in $\mathcal{I}_{4b}$ is a member of the family $\mathcal{K}_{2,\infty}$, and the conclusion follows.

Next, if $u \in \mathcal{Y}^{3b}$, then let $\mathcal{I}_{4c}$ be the sub-sequence of $\mathcal{I}_{3b}$ that consists of i for which x_i is a neighbor of u. Let $\mathcal{X}^{4c}$ be the subset of $\mathcal{X}^{3b}$ that consists of vertices x_i such that i is in $\mathcal{I}_{4c}$. Note that $\mathcal{X}^{4c}$ is an independent set of vertices. The subgraph of G induced by u, x, and the vertices of $\mathcal{X}^{4c}$ is $K_{2,\infty}$ and thus a member of the family $\mathcal{K}_{2,\infty}$, and the conclusion follows.

We may therefore assume that H' is locally-finite. Thus H' conduces a ray R. Since H' is bipartite, it follows that R alternates vertices from $\mathcal{X}^{3b}$ and $\mathcal{Y}^{3b}$. Let $\mathcal{I}_{4d}$ be the sub-sequence of $\mathcal{I}_{3b}$ that consists of i for which x_i is on R. Let $\mathcal{X}^{4d}$ be the subset of $\mathcal{X}^{3b}$ that consists of vertices x_i for i in $\mathcal{I}_{4d}$. Since R is induced, the paths $[x_i, v_i]$ and $[x_j, v_j]$ are disjoint for non-consecutive i and j in $\mathcal{I}_{4d}$. Let $\mathcal{I}_5$ be an alternating sub-sequence of $\mathcal{I}_{4d}$ and let T^5 be the sub-tree of T that consists of paths $P^j[x, w_j]$ for j in $\mathcal{I}_5$. The subgraph of G induced by v and the vertices of T^5 is either a member of the family $\mathcal{K}_{2,\infty}$ if $x \notin \mathcal{V}$ or a member of the family $\mathcal{K}^+_{2,\infty}$ otherwise, and the conclusion follows. $\square$

We have now considered the case that T has an infinite vertex. As in the above lemma, T does not have to be induced in G. The next two lemmas will address the case that T is locally-finite. Since T is locally-finite, Theorem 1.2.9 implies that T conduces a ray R. Note that since each P^i used to construct T is finite, the ray R has infinitely many distinct vertices t_i for natural numbers i. Let $\mathcal{I}$ be the sub-sequence of the natural numbers that consists of i for which t_i is on R. Let ρ be the sub-ray of R with initial vertex $t_\ell = \sigma_\rho$ such that ℓ is the first element of $\mathcal{I}$.

We will now define a type of edge of G that is not in T but which has both endpoints on T. An edge $\varepsilon = \varepsilon_i\varepsilon_j = f^j$ of $G \setminus E(T)$ where j is in $\mathcal{I}$ is called a *sidestep* of a vertex t_j onto ρ if ε_i is on $P^i(t_j, v_i]$ and ε_j is on $P^j \cap \rho$ where i is the immediately preceding element to j in $\mathcal{I}$; see Figure 3.5. If t_j is a member of $\mathcal{V}$, then G does not have a sidestep of t_j since $\mathcal{V}$ is independent and all vertices of $N_G(v) \cap V(T)$ are in $\mathcal{V}$. We call f^j a *sidestep of type one* if ε_i is a neighbor of t_j on $P^i(t_j, v_i]$, shown in Figure 3.5 as a green edge, and a *sidestep of type two* if ε_i is distance two on $P^i(t_j, v_i]$ from t_j, shown as a gold edge in Figure 3.5.

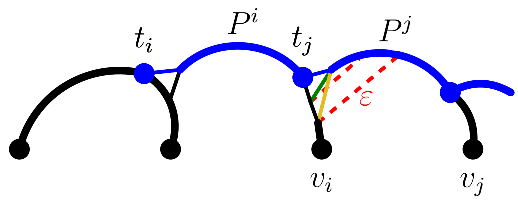


Figure 3.5. Sidestep of t_i onto ρ

In Lemma 3.2.3, we show that the only type of edge of G that is not in T and has both endpoints on T is a sidestep of a vertex t_i onto ρ for i in $\mathcal{I}$.

Lemma 3.2.3. *Suppose that T is locally-finite and has a ray R as a subgraph and suppose ε is an edge of G that is not an edge of T whose endpoints are vertices of T. Then ε is a sidestep of a vertex of T onto R.*

Proof. Let $\mathcal{I}$ be the sub-sequence of the natural numbers that consists of i for which t_i is on R and let ρ be the sub-ray of R with initial vertex $t_k = \sigma_\rho$ such that k is the first element of $\mathcal{I}$.

Suppose that the endpoints of ε are ε_i and ε_j such that ε_i is on P^i and ε_j is on P^j for i and j in $\mathcal{I}$. Note that $i \neq j$ because each path used to construct T in Definintion 3.2.1 is induced in G. Without loss of generality, we may assume that $i < j$. We will consider four cases based on the location of the endpoints of ε and show that only one of those cases can arise. Recall that grades of paths are ordered lexicographically. In Figures 3.6a to 3.6d, the blue segments represent the ray ρ.

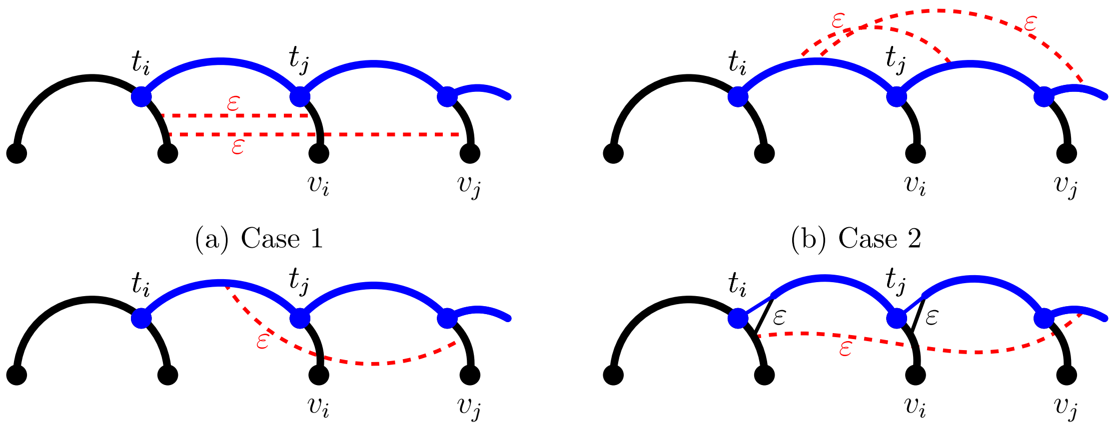
In the first case, both endpoints of ε are on $T \setminus V(\rho)$. Then $\{\varepsilon\} \cup P^j[\varepsilon_j, v_j]$ is shorter than P^j and thus has smaller grade than P^j, and would have been selected by Definintion 3.2.1 instead of P^j; see Figure 3.6a.

In the second case, both endpoints of ε are on ρ, and let $Q = \{\varepsilon\} \cup P^j[\varepsilon_j, v_j]$. If ε_j is not the neighbor of t_j on P^j, then the length of Q is shorter than P^j and so $\gamma_1(Q) < \gamma_1(P^j)$. We may therefore assume that ε_j is the neighbor of t_j on P^j, and so $\gamma_1(Q) = \gamma_1(P^j)$. If i is not immediately preceding j in $\mathcal{I}$, then $\gamma_2(Q) < \gamma_2(P^j)$. We may therefore assume that i is immediately preceding j in $\mathcal{I}$ and that t_j is on P^i so $\gamma_2(Q) = \gamma_2(P^j)$. Then $\text{dist}_{P^i}(\varepsilon_i, t_i) < \text{dist}_{P^i}(t_j, t_i)$ and so $\gamma_3(Q) < \gamma_3(P^j)$. Thus the path Q has a grade smaller than P^j and would have been selected in the construction of T instead of P^j; see Definintion 3.2.1. Thus, G has no edges that are not in T with both endpoints on ρ; see Figure 3.6b.

In the third case, the endpoint ε_i is on $P^i \cap \rho$ and the endpoint ε_j is on $P^j \setminus V(\rho)$. Then the path $Q = \{\varepsilon\} \cup P^j[\varepsilon_j, v_j]$ is shorter than P^j and thus has smaller grade than P^j because $\gamma_1(Q) < \gamma_1(P^j)$. Therefore, the path P^j would not have been selected in the construction of T; see Definintion 3.2.1 and Figure 3.6c.

In the last case, the endpoint ε_i is on $P^i \setminus V(\rho)$ and ε_j is on $P^j \cap \rho$. Suppose first that i and j are not consecutive in $\mathcal{I}$. Then the path $Q = \{\varepsilon\} \cup P^j[\varepsilon_j, v_j]$ is shorter than P^k where k is the element immediately following i in $\mathcal{I}$; see Figure 3.6d. So Q has a smaller grade than P^k, and P^k would not have been selected in the construction of T. We may therefore assume that i and j are consecutive in $\mathcal{I}$. Note that ε_j is the neighbor in T of t_j on P^j; otherwise, the path Q is shorter than P^j and has a smaller grade than P^j; and thus, P^j would not have been selected in the construction process of T; see Definintion 3.2.1. The vertex ε_i is either the neighbor of t_j on $P^i(t_j, v_i]$ or has distance two in T

from t_j on $P^i(t_j, v_i]$, otherwise the path $P^i[v_i, t_j] \cup t_j\varepsilon_j \cup P^i[\varepsilon_i, v_i]$ is shorter than P^i and has a smaller grade than P^i, and thus P^i would not have been selected in the construction of T. Hence, ε is a sidestep of t_j onto ρ where j is an element of $\mathcal{I}$ after the first element, and the conclusion follows.



(a) Case 1 (b) Case 2 (c) Case 3 (d) Case 4

Figure 3.6. Cases of edges of $G \setminus T$ with both endpoints in T.

$\square$

In the next lemma, we use Lemma 3.2.3 to reduce the argument to the case depending on whether G has infinitely many vertices that do not have a sidestep onto ρ or and prove that in either case we obtain a desired graph from the list in Theorem 3.1.1.

Lemma 3.2.4. *If T is locally-finite, then G conduces a member of one of the following families: $\mathcal{F}_\infty$ and $\mathcal{F}_\infty^\Delta$.*

Proof. Since T is locally-finite, Theorem 1.2.9 implies that T conduces a ray R. Note that since each P^i used to construct T is finite, the ray R has infinitely many vertices t_i for natural numbers i. Let $\mathcal{I}$ be the sub-sequence of the natural numbers that consists of i for which t_i is on R. Let ρ be the sub-ray of R with initial vertex $t_k = \sigma_\rho$ such that k is the first element of $\mathcal{I}$. By Lemma 3.2.3, the only type of edges that keep T from being induced in G are sidesteps of distinct vertices t_i onto ρ for i in $\mathcal{I}$.

We will first consider the paths P^i that have a vertex w_i, not in $\mathcal{V}$, that is a neighbor of v. Let $\mathcal{I}_{1a}$ be the sub-sequence of $\mathcal{I}$ that consists of such i. Suppose that $\mathcal{I}_{1a}$ is infinite. Let j be the first element of $\mathcal{I}_{1a}$, and let ρ^{1a} be the sub-ray of ρ with initial vertex w_j. The graph induced by ρ^{1a} and v is a member of the family $\mathcal{F}_\infty$, and the conclusion follows. We may therefore assume that $\mathcal{I}_{1a}$ is finite; and thus has a maximal element k. Let $\mathcal{I}_{1b}$ be the sub-sequence of $\mathcal{I}$ that consists of elements exceeding k.

We will now consider the case that $\mathcal{I}_{1b}$ has infinitely many j such that G does not have a sidestep of t_j onto ρ. Let $\mathcal{I}_{2a}$ be the sub-sequence of $\mathcal{I}_{1b}$ that consists of such j, let $\mathcal{R}^{2a}$ be the union of paths $P^j \setminus V(\rho)$ for j in $\mathcal{I}_{2a}$, and let ρ^{2a} be the sub-ray of ρ with initial vertex t_h where h is the second element of $\mathcal{I}_{2a}$. We choose the second element of $\mathcal{I}_{2a}$ to make it easier to define the paths from ρ to v in a fan. The subgraph of G induced by v, the vertices of ρ^{2a}, and the vertices of $\mathcal{R}^{2a}$ is a member of the family $\mathcal{F}_\infty$.

We may therefore assume that $\mathcal{I}_{1b}$ has only finitely many j such that G does not have a sidestep of t_j onto ρ. Then $\mathcal{I}_{1b}$ has a maximal k such that G does not have a sidestep of t_j onto ρ for $j > k$. Let $\mathcal{I}_{2b}$ be the sub-sequence of $\mathcal{I}_{1b}$ that consists of j for which $j > k$. Let ρ^{2b} be the sub-ray of ρ with initial vertex t_h where h is the second element of $\mathcal{I}_{2b}$. We will now consider two cases depending on whether G has infinitely many sidesteps of type two onto ρ^{2b}.

Suppose that G has infinitely many sidesteps of type two onto ρ^{2b}, let $\mathcal{I}_{3a}$ subsequence of $\mathcal{I}_{2b}$ that consists of j for which $f^j = \varepsilon_i\varepsilon_j$ where i is the element immediately preceding j in $\mathcal{I}_{1b}$ and j is an element of $\mathcal{I}_{3a}$ is a sidestep of type two. Let $\mathcal{R}^{3a}$ be the union of the paths $P^i[v_i, \varepsilon_i] \cup \{f^j\}$ where i the immediately preceding element to j in $\mathcal{I}_{1b}$ and j is an element of $\mathcal{I}_{3a}$ after the first element, and let ρ^{3a} be the sub-ray of ρ^{2b} with initial vertex t_h where h is the second element of $\mathcal{I}_{3a}$. The graph induced by v, vertices of ρ^{3a}, and vertices of $\mathcal{R}^{3a}$ is a member of the family $\mathcal{F}_\infty$, and the conclusion follows.

We may therefore assume that G has only finitely many sidesteps of type two onto ρ^{2b}. Then the sequence $\mathcal{I}_{2b}$ has a k such that each sidestep f^j is a sidestep of type one for $j > k$. Let $\mathcal{I}_{3b}$ sub-sequence of $\mathcal{I}_{3b}$ that consists of j for which $j > k$ and such that f^j is a sidestep of type one, and let ρ^{3b} be the sub-ray of ρ^{2b} with initial vertex t_h where h is the second element of $\mathcal{I}_{3b}$. Note that $\mathcal{I}_{3b}$ is infinite because G has infinitely many sidesteps, but only finitely many of those are of type two.

The definition of $\mathcal{F}_\infty^\Delta$ does not allow for the triangles to share vertices. We now address this situation in two cases.

Suppose $\mathcal{I}_{3b}$ has only finitely many j for which t_j is an endpoint of a sidestep of type one. This means that $t_j = \varepsilon_k$ where ε_k is a vertex on P^k and k is the element immediately preceding j in $\mathcal{I}_{1b}$. Then $\mathcal{I}_{3b}$ has a k such that t_j is not an endpoint of a sidestep of type one for $j > k$. Let $\mathcal{I}_{4a}$ be the sub-sequence of $\mathcal{I}_{3b}$ that consists of j for which $j > k$, and let ρ^{4a} be sub-ray of ρ^{3b} with initial vertex t_h where h is the second element of $\mathcal{I}_{4a}$. Let $\mathcal{R}^{4a}$ be the union of the paths $P^i[v_i, t_j]$ where i is the immediately preceding element to j in $\mathcal{I}_{1b}$ and j is an element of $\mathcal{I}_{4a}$ after the first element. The subgraph of G induced by v, vertices of ρ^{4a}, and the vertices of $\mathcal{R}^{4a}$ is a member of the family $\mathcal{F}^{\Delta}_{\infty}$, and the conclusion follows. The following Figure 3.7 illustrates why i is immediately preceding j in $\mathcal{I}_{1b}$ and not $\mathcal{I}_{4a}$.

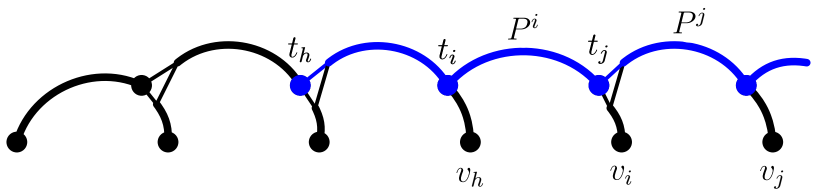


Figure 3.7. Why i immediately precedes j in $\mathcal{I}_{1b}$

We may therefore assume that t_j is the endpoint of a sidestep of type one for infinitely many j in $\mathcal{I}_{3b}$. Let $\mathcal{I}_{4b}$ be the sub-sequence of $\mathcal{I}_{3b}$ such that t_j is the endpoint of a sidestep of type one. Let $\mathcal{I}_5$ be an alternating sub-sequence of $\mathcal{I}_{4b}$, and let ρ^5 be a subray of ρ^{3b} with initial vertex t_h where h is the second element of $\mathcal{I}_5$. Let $\mathcal{R}^5$ be the union of the paths $P^i[v_i, t_j]$ where i is the immediately preceding element to j in $\mathcal{I}_{1b}$ and j is an element of $\mathcal{I}_5$ after the first element. The subgraph of G induced by v, vertices of ρ^5, and the vertices of $\mathcal{R}^5$ is a member of the family $\mathcal{F}^{\Delta}_{\infty}$, and the conclusion follows. $\square$

Now, we will describe the unavoidable infinite induced subgraphs of a 2-connected graph that has an infinite vertex.

Lemma 3.2.5. *Let G be a 2-connected graph with an infinite vertex. Then G conduces either K_∞ or a member of one of the following families: $\mathcal{K}_{2,\infty}$ $\mathcal{K}^+_{2,\infty}$, $\mathcal{F}_\infty$, and $\mathcal{F}^{\Delta}_{\infty}$.*

Proof. Let v be an infinite vertex of G and let $N_G(v)$ be the neighborhood of v. By Theorem 1.2.7, the graph induced by the vertices of $N_G(v)$, contains either K_∞, and the conclusion follows, or $\overline{K_\infty}$. Let $\mathcal{V}$ be the subset of $N_G(v)$ that induces $\overline{K_\infty}$.

Let T be a $\mathcal{V}$-connecting tree of G as described in Definintion 3.2.1. By Theorems 1.2.8 and 1.2.9, T has either an infinite vertex or an induced ray. If T has an infinite vertex, then Lemma 3.2.2 implies that G conduces a member of the one of the families $\mathcal{K}_{2,\infty}$ and $\mathcal{K}^+_{2,\infty}$, and the conclusion follows. We may therefore assume that T is locally-finite. Lemma 3.2.4 implies that G conduces a member of one of the following families: $\mathcal{F}_\infty$ and $\mathcal{F}^\Delta_\infty$, and the conclusion follows. □

3.3. Locally-Finite Graph to Ladder-Like Structure

In this section, we present the unavoidable infinite induced subgraphs of an infinite 2-connected graph that is locally-finite. We will use Tutte's notion of a bridge found in [16], see also [9], and restated below from Chapter 2, to build the well-described ladder-like structures, shown in Figures 3.3a to 3.3c, and formally defined later in this section.

Let H be a non-empty subgraph of a graph G. An *H-bridge* in G (or a *bridge of H in G*) is a connected subgraph B of $G \setminus E(H)$ that satisfies either one of the following two conditions:

1. B is a single edge with both endpoints in $V(H)$. In this case, B is called a *degenerate bridge.*
2. $B - V(H)$ is a connected component of $G - V(H)$; and B also includes every edge of G with one endpoint in $V(B) - V(H)$ and the other endpoint in H.

Note that every edge of $G \setminus E(H)$ belongs to exactly one H-bridge. Vertices that belong to both B and H are called *vertices of attachment* of B.

Suppose that G is an infinite locally-finite 2-connected graph. Then Theorem 1.2.9

implies that G conduces a ray R, which means that G has no degenerate R-bridges. For a finite bridge B of R in G, let u and v be the two vertices of attachment of B such that $R[u, v]$ includes all vertices of attachment of B. We call $R[u, v]$ the *span* of B. An R-bridge has *infinite span* if it has infinitely many vertices of attachment on R; such an R-bridge that is also locally-finite is called *perpetual*. An *infinite R-bridge chain* B_1, B_2, $\ldots$ is a sequence of finite bridges of an induced ray R with initial vertex u and $R[u_i, v_i]$ is the span of B_i that satisfies the following:

$$u = u_1 < u_2 < v_1 \leq u_3 < v_2 \leq u_4 < v_3 \leq \cdots \leq u_{k-1} < v_{k-2} \leq u_k < v_{k-1} \leq u_{k+1} < \cdots .$$

Figure 3.8 shows an infinite R-bridge chain.

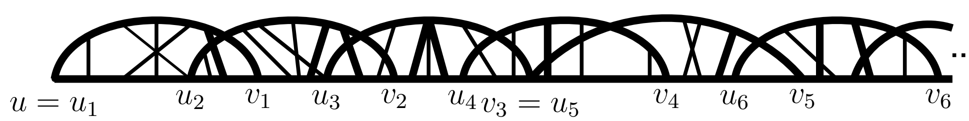


Figure 3.8. An infinite R-bridge chain

Lemma 3.3.1. *Every infinite locally-finite 2-connected graph with an induced ray R has either a perpetual R-bridge or an infinite R-bridge chain.*

Proof. Let G be an infinite locally-finite 2-connected graph with an induced ray R that has initial vertex u. Since G is 2-connected, it follows that R has at least one bridge. If an R-bridge has infinite span, then the conclusion follows. Therefore, we may assume that no R-bridge has infinite span.

Since G is locally-finite, it follows that G has only finitely many bridges of R that have a vertex of attachment at u. Let B_1 be a bridge of R with maximal span of $P[u_1, v_1]$

where $u = u_1$. Since G is a 2-connected locally-finite graph, R is infinite, and every bridge of R has finite span, it follows that G has an R-bridge with one vertex of attachment on $R(u_1, v_1)$ and another vertex of attachment of $R(v_1, \infty)$. Furthermore, G has only finitely many such bridges of R. Let B_2 be such an R-bridge with maximal span of $R[u_2, v_2]$. Since no R-bridge has infinite span, G is a 2-connected locally-finite graph, and R is infinite, it follows that G has at least one, but only finitely many, R-bridges with one vertex of attachment on $R[v_1, v_2]$ and another vertex of attachment on $R(v_2, \infty)$. Suppose that we have obtained the R-bridges B_1, B_2, $\ldots$, B_{i-1} inductively. Let B_i be an R-bridge of G with maximal span $R[u_i, v_i]$ where u_i is a vertex of attachment of B_i on $R[v_{i-2}, v_{i-1})$ and another vertex of attachment v_i on $R(v_{i-1}, \infty)$. Since no R-bridge of G has infinite span and R is infinite, it follows that G has an infinite sequence of such bridges B_1, B_2, B_3, $\ldots$. This sequence of bridges of R forms an infinite R-bridge chain, and the conclusion follows. $\square$

An *end* of an infinite graph G is an equivalence class of rays in G where two rays are *equivalent* if G has by infinitely many pairwise-disjoint paths joining them.

Up until Lemma 3.3.7, we will assume that a 2-connected infinite locally-finite graph G has two disjoint induced rays P and Q in the same end such that G has no edges between them. The existence of such rays will be proved in Lemma 3.3.7.

A *tie* J of $P \cup Q$ is a component of $G \setminus E(P \cup Q)$ such that both $V(J) \cap P$ and $V(J) \cap Q$ are non-empty. The vertices of $V(J) \cap P$ and $V(J) \cap Q$ are called *vertices of attachment* of J. For a tie J, let p_J^ℓ be the vertex of attachment of J on P such that no vertex of attachment p_J of J on P satisfies $p_J < p^\ell_J$, and let p^{r}_J be the vertex of attachment of J on P, if it exists, such that no vertex of attachment p_J of J on P satisfies $p_J > p^{\mathrm{r}}_J$. Note that p^ℓ_J is the left-most vertex of attachment of J on P and p^{r}_J is the right-most vertex of attachment of J on P. The *P-span* is $P[p^\ell_J, p^{\mathrm{r}}_J]$. Similarly, let q^ℓ_J be the vertex of attachment of J on Q, if it exists, such that no vertex of attachment q_J of J on Q satisfies $q_J < q^\ell_J$, and let q^{r}_J be the vertex of attachment of J on Q such that no vertex of attachment q_J of J on Q satisfies $q_J > q^{\mathrm{r}}_J$. The *Q-span* is $Q[q^\ell_J, q^{\mathrm{r}}_J]$.

A tie J of $P \cup Q$ is of *type* I if it has exactly one vertex of attachment, say p_J, on P and one vertex of attachment, say q_J, on Q and its edges form a p_Jq_J-path that is induced in J. A tie J of $P \cup Q$ is of *type* Y if it satisfies the following: the P-span of $P[p^\ell_J, p^{\mathrm{r}}_J]$ is a single edge, the Q-span is a single vertex, say q_J, a vertex u_J of $J \setminus (P \cup Q)$ that is adjacent to both p^ℓ_J and p^{r}_J, and all vertices of J other than p^ℓ_J and p^{r}_J lie on a path from u_J to q_J that is induced in J. A tie J of $P \cup Q$ is of *type* λ if it satisfies the following: the P-span is a single vertex, say p_J, the Q-span of $Q[q^\ell_J, q^{\mathrm{r}}_J]$ is a single edge, a vertex v_J of $J \setminus (P \cup Q)$ that is adjacent to both q^ℓ_J and q^{r}_J, and all vertices of J other than q^ℓ_J and q^{r}_J lie on a path from v_J to p_J that is induced in J. A tie J of $P \cup Q$ is of *type* Y-λ if it satisfies the following: the P-span is a single edge, the Q-span is a single edge, a vertex u_J of $J \setminus (P \cup Q)$ that is adjacent to both p^ℓ_J and p^{r}_J, a vertex v_J of $J \setminus (P \cup Q)$ that is adjacent to both q^ℓ_J and q^{r}_J and all vertices of J other than p^ℓ_J, p^{r}_J, q^ℓ_J, and q^r_J lie on a path from u_J to v_J that is induced in J. A finite tie J of $P \cup Q$ is of *type fork* if it satisfies the following: the P-span is not a single edge or vertex, the Q-span is a single vertex, say q_J, a vertex u_J of

$J \setminus (P \cup Q)$ that is adjacent to each vertex of attachment on P, and all vertices of J that are not vertices of attachment of J on P are on a path from u_J to q_J that is induced in J. A finite tie J of $P \cup Q$ is of *type rake* if it satisfies the following: the P-span is a single vertex, say p_J, the Q-span is not a single edge or vertex, a vertex v of $J \setminus (P \cup Q)$ that is adjacent to each vertex of attachment on Q, and all vertices of J that are not vertices of attachment of J on Q lie on a path from v_J to p_J that is induced in J. A finite tie of $P \cup Q$ is of *type fork-rake* if it satisfies the following: the P-span and Q-span are not a single edge or vertex, a vertex u_J of $J \setminus (P \cup Q)$ that is adjacent to all vertices of attachment on P, a vertex v_J of $J \setminus (P \cup Q)$ that is adjacent to all vertices of attachment of Q, and all other vertices are on a $u_J v_J$-path that is induced in J. Note that u_J and v_J may be equal.

A finite tie J of $P \cup Q$ in G is *full* if G has no other tie of $P \cup Q$ whose P-span contains the P-span of J and whose Q-span contains the Q-span of J. Two finite ties J_1 and J_2 are *independent* if both their P-spans and their Q-spans are disjoint. For finite ties J_i, we will shorten notation for the P-span of J_i from $P[p^{\ell}_{J_i}, p^{\mathrm{r}}_{J_i}]$ to $P[p^{\ell}_i, p^{\mathrm{r}}_i]$ and similarly shorten the notation of the Q-span of J_i to $Q[q^{\ell}_i, q^{\mathrm{r}}_i]$. Two finite ties J_1 and J_2 of $P \cup Q$ with P- and Q-spans being $P[p^{\ell}_1, p^{\mathrm{r}}_1]$, $Q[q^{\ell}_1, q^{\mathrm{r}}_1]$, $P[p^{\ell}_2, p^{\mathrm{r}}_2]$, and $Q[q^{\ell}_2, q^{\mathrm{r}}_2]$, respectively, *cross* if either $q^{\mathrm{r}}_1 < q^{\ell}_2$ and $p^{\mathrm{r}}_2 < p^{\ell}_1$, or $p^{\mathrm{r}}_1 < p^{\ell}_2$ and $q^{\mathrm{r}}_2 < q^{\ell}_1$. Since G is locally-finite, the ray P has only finitely many vertices p such that $p < p^{\ell}_1$ and the ray Q has only finitely many vertices q such that $q < q^{\ell}_1$. So G has only finitely many ties of $P \cup Q$ that can cross J_1. Since G is an infinite locally-finite 2-connected graph, it follows that G has either a maximal infinite sequence of pairwise non-crossing, independent, full ties or an infinite tie. We address the former case in Lemmas 3.3.3 to 3.3.5 and the narrative surround those lemmas, and we address the latter case in the following lemma.

Lemma 3.3.2. *If G has an infinite tie T of $P \cup Q$, then G has an induced subgraph containing $P \cup Q$ that has infinitely many finite ties of $P \cup Q$.*

Proof. Let T be an infinite tie of $P \cup Q$ in G. Since P and Q are in the same end and T is infinite, it follows that T has infinitely many pairwise-disjoint paths with one endpoint on Q and the other endpoint on P such that each of the paths meets P and Q only at its endpoints. Let q_1, q_2, q_3, $\ldots$ be the vertices of Q that are endpoints of such paths from P to Q ordered so that $q_1 <_Q q_2 <_Q \ldots$. Let A_i be such a path in T from q_i to P, for every natural number i, and let p_i be the endpoint of A_i on P. Note that the vertices p_i and p_j for $i < j$ do not have to have the relation $p_i <_P p_j$. Note that each A_i has at least one internal vertex.

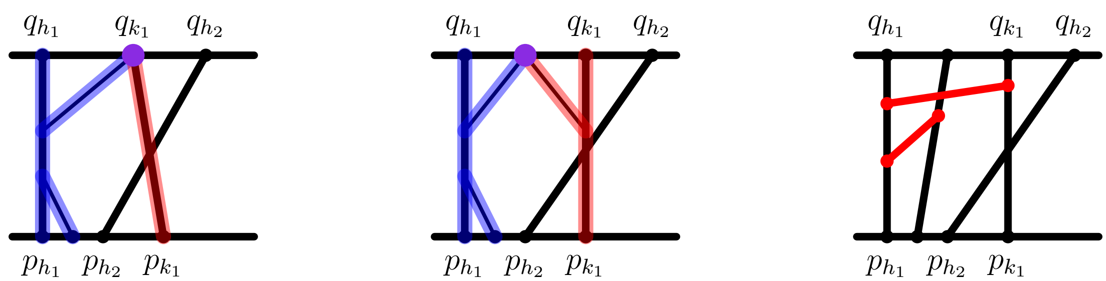
We will inductively construct a sequence of graphs G_{h_1}, G_{h_2}, $\ldots$ and a subsequence $\mathcal{I}_1$ of natural numbers such that the paths A_{h_j} for $h_j \in \mathcal{I}_1$ are parts of pairwise-disjoint finite ties of $P \cup Q$ in the graph G_{h_i}.

Let $1 = h_1$ be the first element of $\mathcal{I}_1$. Since G is locally-finite and T is an infinite tie, there is a maximal natural number k_1 such that one of the following holds:

1. G has two bridges, B_{h_1} and B_{k_1}, of $P \cup Q$ such that the bridge B_{h_1} contains A_{h_1}, the bridge B_{k_1} contains A_{k_1}, and $V(B_{h_1}) \cap V(B_{k_1})$ is not empty. In Figures 3.9a and 3.9b, B_{h_1} is highlighted in blue, B_{k_1} is highlighted in red, and a vertex in $V(B_{h_1}) \cap V(B_{k_1})$ is in purple.

2. T has a path from a vertex on A_{h_1} to a vertex on A_{k_1} that meets no other path A_i for $i \neq h_1, k_1$, see Figure 3.9c. When two segments cross in Figure 3.9c, the tie T does not have a vertex at that crossing.

Let G_{h_1} be the induced subgraph of G obtained by deleting the internal vertices of the paths A_j for $h_1 < j \leq k_1$, and let T_{h_1} be the infinite component of the induced subgraph of T obtained by deleting vertices of A_ℓ for $h_1 \leq \ell \leq k_1$. Let $h_2 = 1 + k_1$, and let h_2 be the second element of $\mathcal{I}_1$. Note that A_{h_1} is a part of a finite tie of $P \cup Q$ in G_{h_1} that is disjoint from T_{h_1}, and that A_{h_1} and A_{h_2} are in separate bridges of $P \cup Q$ in G_{h_1}.



(a) Two bridges where q_{k_1} is in $V(B_{h_1})$

(b) Two bridges with a common neighbor

(c) A_{h_1} and A_{k_1} joined by path in T

Figure 3.9. Selection of k_1

Suppose that we have constructed G_{h_1}, G_{h_2}, ..., G_{h_i} for $h_i \in \mathcal{I}_1$. Note that A_{h_1}, A_{h_2}, ..., A_{h_i} are in i corresponding pairwise-disjoint bridges of $P \cup Q$ in G_{h_i} and also are parts of i corresponding pairwise-disjoint ties of $P \cup Q$ in G_{h_i}, the path $A_{h_{i+1}}$ is a subgraph of T_{h_i}, and at this step $\mathcal{I}_1 = h_1, h_2, \ldots, h_{i+1}$. Since G is locally-finite and T is an infinite tie, there is a maximal natural number k_{i+1} that satisfy one of the following:

1. G_{h_i} has distinct bridges, $B_{h_{i+1}}$ and $B_{k_{i+1}}$, that satisfy the following: the bridge $B_{h_{i+1}}$ contains $A_{h_{i+1}}$, the bridge $B_{k_{i+1}}$ contains $A_{k_{i+1}}$, and $V(B_{h_{i+1}}) \cap V(B_{k_{i+1}})$ is not empty.

2. T_{h_i} has a path from a vertex of $A_{h_{i+1}}$ to a vertex of $A_{k_{i+1}}$ that meets no other path A_j for $j \neq h_{i+1}, k_{i+1}$.

Let $G_{h_{i+1}}$ be the induced subgraph of G_{h_i} obtained by deleting the internal vertices of A_j for $h_{i+1} < j \leq k_{i+1}$, and let $T_{h_{i+1}}$ be the infinite component of the induced subgraph of T_{h_i} obtained by deleting the vertices of A_ℓ for $h_{i+1} \leq \ell \leq k_{i+1}$. Let $h_{i+2} = 1 + k_{i+1}$, and let h_{i+2} be the $i+2$ element of $\mathcal{I}_1$.

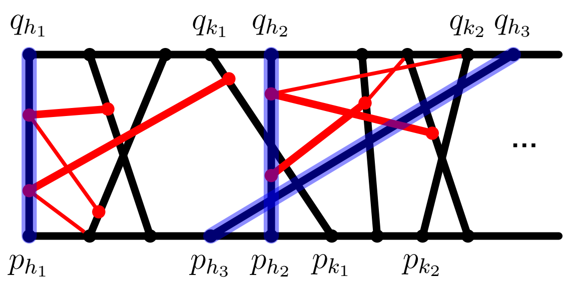


Figure 3.10. Constructing infinitely many finite ties from T

In Figure 3.10, the red segments represent the paths between a vertex on A_i and a vertex on A_j for natural numbers i and j. The blue highlighted segments represent the paths A_{h_1}, A_{h_2}, $\ldots$. When two segments cross in Figure 3.10, the tie T does not have a vertex at that crossing.

It remains to check that the process of constructing $G_{h_{i+1}}$ from G_{h_i} terminates. The process does not delete vertices of $P \cup Q$, and thus the rays do not change throughout the process. The paths A_{h_j} for $j < i$ and $h_j \in \mathcal{I}_1$ are parts of distinct finite ties of $P \cup Q$ in G_{h_i}, and the paths A_ℓ are parts of T_{h_i} for natural numbers ℓ such that $\ell > k_i$. Since G_{h_i} and T_{h_i} are infinite and G_{h_i} has a countably infinite number of paths A_i for $i \in \mathcal{I}_1$, it follows that the process of constructing $G_{h_{i+1}}$ fails for the ordinal ω. Thus, the process of constructing $G_{h_{i+1}}$ from G_{h_i} terminates. Let H be the induced subgraph of G obtained as the result of the above process. Then $\mathcal{I}_1$ is the sub-sequence of natural numbers that consists of i such that A_i is in H. Since P and Q are in the same end, G is locally-finite, and each path A_i is finite for i in $\mathcal{I}_1$, it follows that $\mathcal{I}_1$ is infinite. Each path A_i for i in $\mathcal{I}_1$ is a part of a distinct finite tie. Thus, G has infinitely many finite ties of $P \cup Q$, and the conclusion follows. □

As a consequence of Lemma 3.3.2, up until Lemma 3.3.7, we will assume that G has no infinite ties.

Lemma 3.3.3 shows that pairwise non-crossing, independent, full ties of $P \cup Q$ in this sequence may be ordered by the position in which their vertices appear on P and Q.

Lemma 3.3.3. *Let J_1 and J_2 be non-crossing, independent, and full ties of $P \cup Q$ with the P- and Q-spans being $P[p_1^\ell, p_1^{\mathrm{r}}]$, $Q[q_1^\ell, q_1^{\mathrm{r}}]$, $P[p_2^\ell, p_2^{\mathrm{r}}]$, and $Q[q_2^\ell, q_2^{\mathrm{r}}]$, respectively. Then $p_1^{\mathrm{r}} \leq p_2^\ell$ if and only if $q_1^{\mathrm{r}} \leq q_2^\ell$.*

Proof. Let J_1 and J_2 be non-crossing, independent, and full ties. Suppose for a contradiction that $p_1^{\mathrm{r}} \leq p_2^\ell$, however $q_2^\ell < q_1^{\mathrm{r}}$. Since J_1 and J_2 are independent and $Q(q_2^\ell, q_1^{\mathrm{r}})$ is not empty, it follows that J_1 and J_2 cross, which contradicts that J_1 and J_2 are non-crossing. Hence $q_1^{\mathrm{r}} \leq q_2^\ell$. The other direction of the proof follows an analogous argument. □

For two pairwise non-crossing, independent, full ties J_1 and J_2, define the relation $J_1 < J_2$ by $p_1^{\mathrm{r}} \leq p_2^\ell$. Let $\mathcal{J} = J_1, J_2, \ldots$ be a maximal sequence of pairwise non-crossing, independent, full ties of $P \cup Q$ ordered by the $<$ relation.

We will inductively define the following process of *resolving the ties* for elements of $\mathcal{J}$ by creating a decreasing sequence of induced subgraphs G_1, G_2, …each of which has P and Q as induced rays in the same end. Going from G_i to G_{i+1}, we will resolve the tie J_{i+1} to the tie M_{i+1}.

Let R_1 be the shortest path in J_1 from a vertex of attachment of J_1 on P to a vertex of attachment of J_1 on Q. Let M_1 be the tie obtained from J_1 by deleting the vertices of $J_1 - (R_1 \cup P \cup Q)$, and let G_1 be the subgraph of G induced by the vertices of $P \cup Q$ and of M_1, J_2, J_3, $\ldots$. We say that J_1 is *resolved to* M_1 in G_1. Since R_1 is the shortest path from a vertex of attachment of J_1 on P to a vertex of attachment of J_1 on Q, it follows that R_1 has a vertex u_1 such that u_1 is the only neighbor on R_1 of a vertex of attachment of J_1 on P, and a vertex v_1 such that v_1 is the only neighbor on R_1 of a vertex of attachment of J_1 on Q. Note that u_1 may equal v_1. The edges of M_1 that are not on R_1 are of the form u_1p for a vertex p in the P-span of M_1, or of the form v_1q for a vertex q in the Q-span of M_1. Thus, M_1 is a tie of $P \cup Q$ in G_1 of one of the following types: I, Y, λ, Y-λ, fork, rake, and fork-rake. Since the elements of $\mathcal{J}$ are pairwise non-crossing independent full ties, it follows that resolving J_1 to M_1 does not affect the vertices of $P(p_1^{\mathrm{r}}, \infty)$, the vertices of $Q(q_1^{\mathrm{r}}, \infty)$, or the ties J_2, J_3, $\ldots$. So M_1, J_2, J_3,$\ldots$ is a maximal sequence of pairwise non-crossing independent full ties of $P \cup Q$ in G_1.

Suppose that we have inductively resolved the ties J_1, J_2, $\ldots$, J_{i-1} to M_1, M_2, $\ldots$, M_{i-1} in G_1, G_2, $\ldots$, G_{i-1}, respectively. Let R_i be the shortest path in J_i from a vertex of attachment of J_i on P to a vertex of attachment of J_i on Q. Let M_i be the tie of $P \cup Q$ obtained from J_i by deleting the vertices of $J_i - (R_i \cup P \cup Q)$, and let G_i be the subgraph of G_{i-1} induced by the vertices of $P \cup Q$ and the ties M_1, M_2, $\ldots$, M_i, J_{i+1}, J_{i+2}, $\ldots$. Since R_i is the shortest path from a vertex of attachment of J_i on P to a vertex of attachment of J_i on Q, it follows that R_i has a vertex u_i such that u_i is the only neighbor on R_i of a vertex of attachment of J_i on P, and a vertex v_i such that v_i is the only neighbor on R_i of a vertex of attachment of J_i on Q. Note that u_i may equal v_i. The edges of M_i not on R_i are of the form $u_i p$ for a vertex p in the P-span of M_i, or of the form $v_i q$ for a vertex q in the Q-span of M_i. Thus, M_i is a tie of $P \cup Q$ in G_i of one of the following types: I, Y, λ, Y-λ, fork, rake, and fork-rake. Since the elements of $\mathcal{J}$ are pairwise non-crossing independent full ties, it follows that resolving J_i to M_i does not affect the vertices of $P(p_i^{\mathrm{r}}, \infty)$, the vertices of $Q(q_i^{\mathrm{r}}, \infty)$, or the ties J_{i+1}, J_{i+2}, ... So M_1, M_2, ..., M_i, J_{i+1}, J_{i+2}, ... is a maximal sequence of pairwise independent full ties of $P \cup Q$ in G_i.

It remains to check that the process of resolving the ties terminates. The ties M_1, M_2, ..., M_{i-1} unaffected by resolving J_i, and each is of one of the following types: I, Y, λ, Y-λ, fork, rake, and fork-rake. So the process of resolving the ties of $\mathcal{J}$ has a countably infinite number of steps. Since the graph G is countably infinite and the $\mathcal{J}$ elements of are indexed by natural numbers, it follows that the process of resolving the ties of $\mathcal{J}$ fails at the ordinal ω. Thus, the process of resolving the ties terminates. Let H be the induced subgraph of G obtained by resolving the ties of $\mathcal{J}$ and let $\mathcal{M} = M_1$, M_2, ... Note that P and Q are induced rays in the same end in H, that $\mathcal{M}$ is a maximal sequence of pairwise non-crossing, independent, full ties of H, and that each of the ties of $\mathcal{M}$ are one of the following types: I, Y, λ, Y-λ, fork, rake, and fork-rake.

We now formally define a ladder-like structure that we will use in Section 3.4 to obtain the structure in Figure 3.3d and the ladder-like structures from Theorem 3.1.1, which are shown in Figures 3.3a to 3.3c. An infinite *messy ladder* is a triple (L, W, X) that consists of a locally-finite graph L whose vertices all lie on two disjoint induced rays W and X in the same end of L, called *rails*. The edges of L that belong to neither W nor X are called *rungs*. The graph L has an edge between the initial vertices of the rails. In some contexts, when we say messy ladder, we mean only the graph L, of which the existence and properties of W and X are a part. Figure 3.11 shows an example of an infinite messy ladder. A member of the family $\mathcal{L}_\infty$ is a triple (L, P, Q) that consists of a graph L, and two rays P and Q induced in L such that the only edges and vertices of L neither on P nor on Q form ties of $P \cup Q$ of type I with a tie of type I joining the initial vertices of the rays P and Q, shown in Figure 3.3a. A member of the family $\mathcal{L}_\infty^\Delta$ is a triple (L, P, Q) that consists of a graph L, and two rays P and Q induced in L such that the only edges and vertices of L neither on P nor on Q form ties of $P \cup Q$ where either all are ties of type Y with a tie of type Y joining the initial vertices of the rays P and Q or all form ties of $P \cup Q$ of type λ with a tie of type λ joining the initial vertices of P and Q, shown in Figure 3.3b. A member of the family $\mathcal{L}_\infty^{\nabla\Delta}$ is a triple (L, P, Q) that consists of a graph L, and two rays P and Q induced in L such that the only edges and vertices of L neither on P nor on Q form ties of $P \cup Q$ of type Y-λ with a tie of type Y-λ joining the initial vertices of the rays P and Q, shown in Figure 3.3c.

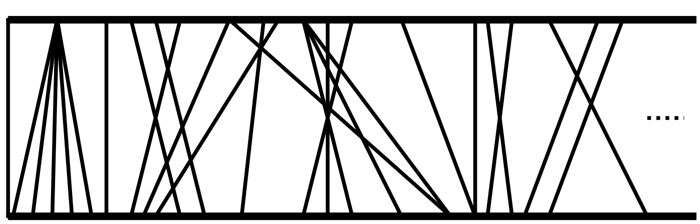


Figure 3.11. An infinite messy ladder

Since $\mathcal{M}$ has infinitely many elements that are ties of $P \cup Q$, it follows that either $\mathcal{M}$ has infinitely many elements that are ties of $P \cup Q$ each of which is of type I, Y, λ, or Y-λ or $\mathcal{M}$ has infinitely many elements that are ties of $P \cup Q$ each of which is of type fork, rake, or fork-rake. We address the first case in Lemma 3.3.4 and the latter in Lemma 3.3.5.

Lemma 3.3.4. *Let H be the induced subgraph of G obtained by resolving the ties of $P \cup Q$ in $\mathcal{J}$ to ties in $\mathcal{M}$. Suppose that $\mathcal{M}$ has infinitely many elements that are ties of $P \cup Q$ each of which is one of the following types: I, Y, λ, and Y-λ. Then H conduces a member of one of the following families: $\mathcal{L}_\infty$, $\mathcal{L}_\infty^{\Delta}$, and $\mathcal{L}_\infty^{\nabla\Delta}$.*

Proof. Since infinitely many elements of $\mathcal{M}$ are ties of $P \cup Q$ each of which is one of the types: I, Y, λ, or Y-λ, it follows that $\mathcal{M}$ has infinitely many elements that are ties of the same type.

Suppose that infinitely many elements of $\mathcal{M}$ are ties of type I, and let $\mathcal{I}_{1a}$ be the sub-sequence natural numbers that consist of i such that M_i is a tie of type I. The subgraph of H induced by the vertices of P, of Q, and of M_i for i in $\mathcal{I}_{1a}$ is a member of the family $\mathcal{L}_\infty$.

We may therefore assume that $\mathcal{M}$ has only finitely many elements that are ties of type I. Suppose that $\mathcal{M}$ has infinitely many elements that are ties of type Y, and let $\mathcal{I}_{1b}$ be the sub-sequence natural numbers that consist of i such that M_i is a tie of type Y. The subgraph of H induced by the vertices of P, of Q, and of M_i for i in $\mathcal{I}_{1b}$ is a member of the family $\mathcal{L}_\infty^{\Delta}$.

We may therefore assume that $\mathcal{M}$ has only finitely many elements that are ties of type Y. Suppose that $\mathcal{M}$ has infinitely many elements that are ties of type Y, and let $\mathcal{I}_{1c}$ be the sub-sequence of natural numbers that consist of i such that M_i is a tie of type λ. The subgraph of H induced by the vertices of P, of Q, and of M_i for i in $\mathcal{I}_{1c}$ is a member of the family $\mathcal{L}_\infty^{\Delta}$.

We may therefore assume that $\mathcal{M}$ has only finitely many elements that are ties of type λ. Thus, $\mathcal{M}$ has infinitely many elements that are ties of type Y-λ, and let $\mathcal{I}_{1d}$ be the sub-sequence of natural numbers that consist of i such that M_i is a tie of type Y-λ. The subgraph of H induced by the vertices of P, of Q, and of M_i for i in $\mathcal{I}_{1d}$ is a member of the family $\mathcal{L}_\infty^{\nabla\Delta}$. $\square$

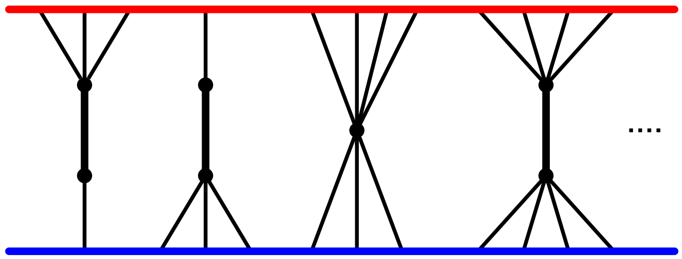
As a result of the above lemma, we may therefore assume that $\mathcal{M}$ has infinitely many elements that are ties of $P \cup Q$ in H, each of which is of type: fork, rake, fork-rake. Let $\mathcal{F}'$ be the family of graphs obtained from F_s for an integer s exceeding two by subdividing each of the rim edges; and the rim is either entirely on P or entirely on Q. Let $\mathcal{M}_1$ be the sub-sequence of $\mathcal{M}$ that consists of M_i such that M_i is a tie of one of the following types: fork, rake, or fork-rake. Let $\mathcal{A}$ be the sub-sequence of natural numbers that consists of i for which M_i is a member of $\mathcal{M}_1$.

We will inductively define the process of *rerouting the rays* by constructing a sequence of ties $\mathcal{M}_2$, a sequence of graphs, and a sub-sequence of natural numbers for which M''_{i_j} is a member of $\mathcal{M}_2$.

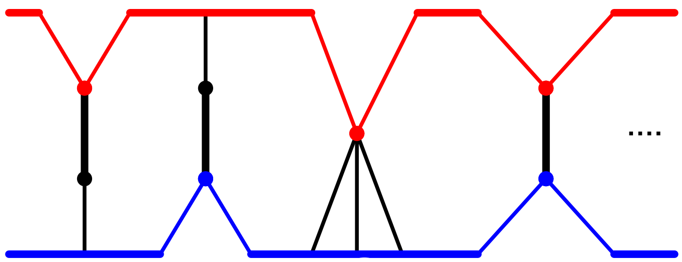
Let i_1 be the first element of $\mathcal{A}$. We will define rerouting the rays of M'_{i_1} depending on the type of tie that it is. If M'_{i_1} is a tie of type fork, we reroute the ray P by letting $P_{i_1} = (P - P(p^{\ell}_{i_1}, p^{\mathrm{r}}_{i_1})) \cup \{p^{\ell}_{i_1}u_{i_1}, u_{i_1}p^{\mathrm{r}}_{i_1}\}$ and $Q_{i_1} = Q$. The resulting induced subgraph M''_{i_1} of M'_{i_1} is a tie of type I of $P_{i_1} \cup Q_{i_1}$. Similarly, if M'_{i_1} is a tie of type rake, we reroute the ray Q by letting $Q_{i_1} = (Q - Q(q^{\ell}_{i_1}, q^{\mathrm{r}}_{i_1})) \cup \{q^{\ell}_{i_1}v_{i_1}, v_{i_1}q^{\mathrm{r}}_{i_1}\}$ and $P_{i_1} = P$. The resulting induced subgraph M''_{i_1} of M'_{i_1} is a tie of type I of $P_{i_1} \cup Q_{i_1}$. If M'_{i_1} is a tie of $P \cup Q$ of type fork-rake, there are two cases to consider: either $u_{i_1} = v_{i_1}$ or not. If $u_{i_1} = v_{i_1}$, we reroute the ray P by letting $P_{i_1} = (P - P(p^{\ell}_{i_1}, p^{\mathrm{r}}_{i_1})) \cup \{p^{\ell}_{i_1}u_{i_1}, u_{i_1}p^{\mathrm{r}}_{i_1}\}$ and $Q_{i_1} = Q$. The resulting induced subgraph M''_{i_1} of M'_{i_1} is either a rung of $P_{i_1} \cup Q_{i_1}$ or a member of the family $\mathcal{F}'_s$ in $P_{i_1} \cup Q_{i_1}$. If $u_{i_1} \neq v_{i_1}$, then we reroute both the rays P and Q by letting $P_{i_1} = (P - P(p^{\ell}_{i_1}, p^{\mathrm{r}}_{i_1})) \cup \{p^{\ell}_{i_1}u_{i_1}, u_{i_1}p^{\mathrm{r}}_{i_1}\}$ and $Q_{i_1} = (Q - Q(q^{\ell}_{i_1}, q^{\mathrm{r}}_{i_1})) \cup \{q^{\ell}_{i_1}v_{i_1}, v_{i_1}q^{\mathrm{r}}_{i_1}\}$. The resulting induced subgraph M''_{i_1} of M'_{i_1} is a tie of type I of $P_{i_1} \cup Q_{i_1}$. Let H_{i_1} be the subgraph of H induced by the vertices of $P_{i_1} \cup Q_{i_1}$, the vertices of M''_{i_1}, and the vertices of M'_k for k in $\mathcal{A}$ such that $k > i_1$. Note that M'_k, for k in $\mathcal{A}$ such that $k > i_1$, is a tie of $P_{i_1} \cup Q_{i_1}$ in H_{i_1} of one of the following types: fork, rake, and fork-rake. Let i_1 be the first element of $\mathcal{A}_1$.

Figure 3.12 shows the process of rerouting the ray for a particular tie M'_{i_j} where the red path represents the ray P, and the blue segment represents the ray Q. The first tie in Figure 3.12a is a tie of type fork, the second tie in Figure 3.12a is a tie of type rake, and the final two ties in Figure 3.12a are ties of type fork-rake. In Figure 3.12b, the red segment shows the resulting P_{i_j}, and the blue segment shows the resulting Q_{i_j}.

Suppose that we have rerouted the rays for M'_{i_1}, M'_{i_2}, …, M'_{i_k} to obtain the follow-



(a) Schematic of cases depending on type of tie M'_{i_j}



(b) Result of rerouting the ray of a tie M'_{i_j}

Figure 3.12. Process of rerouting the rays of a tie M'_{i_j}

ing: the ties M''_{i_1}, M''_{i_2},..., M''_{i_k}, the graphs H_{i_1}, H_{i_2}, ..., H_{i_k}, and a sequence $\mathcal{A}_1 = i_1, i_2, \ldots, i_k$. We will define rerouting the rays of $M'_{i_{k+1}}$ depending on the type of tie that it is. If $M'_{i_{k+1}}$ is a tie of type fork, we reroute the ray P by letting $P_{i_{k+1}} = (P_{i_k} - P(p^{\ell}_{i_{k+1}}, p^{\mathrm{r}}_{i_{k+1}})) \cup \{p^{\ell}_{i_{k+1}} u_{i_{k+1}}, u_{i_{k+1}} p^{\mathrm{r}}_{i_{k+1}}\}$ where and $Q_{i_{k+1}} = Q_{i_k}$. The resulting induced subgraph $M''_{i_{k+1}}$ of $M'_{i_{k+1}}$ is a tie of type I of $P_{i_{k+1}} \cup Q_{i_{k+1}}$. Similarly, if $M'_{i_{k+1}}$ is a tie of type rake, we reroute the ray Q by letting $Q_{i_{k+1}} = (Q_{i_k} - Q(q^{\ell}_{i_{k+1}}, q^{\mathrm{r}}_{i_{k+1}})) \cup \{q^{\ell}_{i_{k+1}} v_{i_{k+1}}, v_{i_{k+1}} q^{\mathrm{r}}_{i_{k+1}}\}$ and $P_{i_{k+1}} = P_{i_k}$. The resulting induced subgraph $M''_{i_{k+1}}$ of $M'_{i_{k+1}}$ is a tie of type I of $P_{i_{k+1}} \cup Q_{i_{k+1}}$. If $M'_{i_{k+1}}$ is a tie of $P \cup Q$ of type fork-rake, there are two cases to consider: either $u_{i_{k+1}} = v_{i_{k+1}}$ or not. If $u_{i_{k+1}} = v_{i_{k+1}}$, we reroute the ray P by letting $P_{i_{k+1}} = (P_{i_k} - P(p^{\ell}_{i_{k+1}}, p^{\mathrm{r}}_{i_{k+1}})) \cup$

$\{p^{\ell}_{i_{k+1}}u_{i_{k+1}}, u_{i_{k+1}}p^{\mathrm{r}}_{i_{k+1}}\}$ and $Q_{i_{k+1}} = Q_{i_k}$. The resulting induced subgraph $M''_{i_{k+1}}$ of $M'_{i_{k+1}}$ is either a rung of $P_{i_{k+1}} \cup Q_{i_{k+1}}$ or a member of the family $\mathcal{F}'_s$ in $P_{i_{k+1}} \cup Q_{i_{k+1}}$. If $u_{i_{k+1}} \neq v_{i_{k+1}}$, then we reroute both the rays P and Q by letting $P_{i_{k+1}} = (P_{i_k} - P(p^{\ell}_{i_{k+1}}, p^{\mathrm{r}}_{i_{k+1}})) \cup \{p^{\ell}_{i_{k+1}}u_{i_{k+1}}, u_{i_{k+1}}p^{\mathrm{r}}_{i_{k+1}}\}$ and $Q_{i_{k+1}} = (Q_{i_k} - Q(q^{\ell}_{i_{k+1}}, q^{\mathrm{r}}_{i_{k+1}})) \cup \{q^{\ell}_{i_{k+1}}v_{i_{k+1}}, v_{i_{k+1}}q^{\mathrm{r}}_{i_{k+1}}\}$. The resulting induced subgraph $M''_{i_{k+1}}$ of $M'_{i_{k+1}}$ is a tie of type I of $P_{i_{k+1}} \cup Q_{i_{k+1}}$. Let $H_{i_{k+1}}$ be the subgraph of H induced by the vertices of $P_{i_{k+1}} \cup Q_{i_{k+1}}$, the vertices of M''_{i_1}, M''_{i_2}, $\ldots$, $M''_{i_{k+1}}$, and the vertices of M'_j for j in $\mathcal{A}$ such that $j > i_{k+1}$. Note that M'_j for j in $\mathcal{A}$ such that $k > i_{k+1}$ is a tie of $P_{i_{k+1}} \cup Q_{i_{k+1}}$ in $H_{i_{k+1}}$ of one of the following types: fork, rake, and fork-rake. Let i_{k+1} be the $k+1$ element of $\mathcal{A}_1$.

It remains to check that the process of rerouting the rays terminates. Note that M''_{i_j} is not affected by rerouting the ray of $M_{i_{k+1}}$ for $i_j < i_{k+1}$ in $\mathcal{A}_1$ and that M'_ℓ is also not affected by rerouting the ray of $M_{i_{k+1}}$ for M'_ℓ for ℓ in $\mathcal{A}$ such that $\ell > i_{k+1}$. Note that the ties M'_{i_j} are indexed by natural numbers. So the process of rerouting the rays for ties in $\mathcal{M}_1$ has a countably infinite number of steps. Since G is countable and the number of ties in $\mathcal{M}_1$ is countable, it follows that the process of rerouting the rays fails for the ordinal ω. Thus, the process of rerouting the rays terminates. Let H' be the induced subgraph of H obtained by rerouting the rays for each M'_i in $\mathcal{M}_1$, and let P' and Q' be the rays obtained by rerouting the rays for the ties of $\mathcal{M}_1$. Let $\mathcal{M}_2$ be the maximal sequence of pairwise non-crossing, independent, full ties of $P' \cup Q'$ in H' obtained by rerouting the rays for each tie in $\mathcal{M}_1$.

Lemma 3.3.5. *Let H' be the induced subgraph of H obtained by rerouting the rays for* *each member of $\mathcal{M}_1$. Then H' is either an infinite messy ladder or a member of the family $\mathcal{L}_\infty$.*

Proof. Suppose that $u_i = v_i$ for infinitely many i such that M_i' in $\mathcal{M}_1$, and let $\mathcal{I}_{1a}$ be the sub-sequence of natural numbers that consists of such i. If M_i' is a tie of type fork or rake, then M_i'' is a single edge. If M_i' is a tie of type fork-rake, then M_i'' is a member of the family $\mathcal{F}_s'$. In either case, H' is a messy ladder, and the conclusion follows.

We may therefore assume that $u_i \neq v_i$ for infinitely many i such that M_i' in $\mathcal{M}_1$, and let $\mathcal{I}_{1b}$ be the sub-sequence of natural numbers that consists of such i. In this case, each tie M_i'' is a tie of type I in H'. Thus, H' is a member of the family $\mathcal{L}_\infty$, and the conclusion follows. $\square$

The previous two lemmas and processes of resolving the ties and rerouting the rays can be summarized in the following lemma. Note that we will make use of the argument below multiple times in the proof of Lemma 3.3.7, so we prove it separately here.

Lemma 3.3.6. *Let G be an infinite 2-connected locally-finite graph with two disjoint induced rays P and Q in the same end such that G has infinitely many finite ties of $P \cup Q$ and has no edges between a vertex on P and a vertex on Q. Then G conduces an infinite messy ladder or a member of one of the following families: $\mathcal{L}_\infty$, $\mathcal{L}_\infty^\Delta$, and $\mathcal{L}_\infty^{\nabla\Delta}$.*

Proof. Let p be the initial vertex of P, and let q be the initial vertex of Q. Let $\mathcal{J} = J_1$, J_2, $\ldots$ be an infinite maximal sequence of pairwise independent full ties of $P \cup Q$ ordered by their appearance on Q.

We will inductively construct an infinite maximal sequence of pairwise non-crossing, independent, full ties of $P \cup Q$ by constructing a sub-sequence $\mathcal{I}$ of natural numbers that consist of i such that J_i is in that maximal sequence.

First, consider J_1 and let $h_1 = 1$ be the first element of $\mathcal{I}$. Since J_{h_1} is the first tie with a vertex of attachment on Q, it follows that ties from $\mathcal{J}$ crossing J_{h_1} have P-span in $P[p, p^\ell_{h_1}]$. Since G is locally-finite and $P[p, p^\ell_{h_1}]$ is finite, it follows that $\mathcal{J}$ has only finitely many elements that are ties that cross J_{h_1}. If $\mathcal{J}$ has no element that crosses J_{h_1}, then let $h_2 = h_1 + 1$. We may therefore assume that $\mathcal{J}$ has an element that crosses J_{h_1}. Thus, there is a maximal natural number k_1 such that J_{k_1} crosses J_{h_1}. Let G_{h_1} be the subgraph of G induced by the vertices of $P \cup Q$, of J_{h_1}, and of J_i for natural numbers $i > k_1$. Let $h_2 = k_1 + 1$ be the second element of $\mathcal{I}$.

Suppose that we have constructed the graphs G_{h_1}, G_{h_2}, $\ldots$, G_{h_i}. Then $\mathcal{I} = h_1, h_2, \ldots, h_{i+1}$. Note that J_{h_1}, J_{h_2}, $\ldots$, $J_{h_{i+1}}$ is a sequence of pairwise non-crossing, independent, full ties of $P \cup Q$ in G_{h_i} and that G has finitely many vertices on each of $P[p, p^\ell_{h_{i+1}}]$ and $Q[q, q^\ell_{h_{i+1}}]$. This, combined with the fact that G is locally-finite, implies that there are finitely many natural numbers i such that J_i crosses $J_{h_{i+1}}$. If $\mathcal{J}$ has no element that crosses $J_{h_{i+1}}$, then let $h_{i+2} = 1 + h_{i+1}$. We may therefore assume that $\mathcal{J}$ has an element that crosses $J_{h_{i+1}}$. Thus, there is a maximal natural number k_{i+1} such that $J_{k_{i+1}}$ crosses $J_{h_{i+1}}$. Let $G_{h_{i+1}}$ be the subgraph of G induced by the vertices of $P \cup Q$, of J_i for $i \in \mathcal{I}$, and of J_j for natural numbers $j > k_{i+2}$. Let $h_{i+2} = 1 + k_{i+2}$ be the $i + 2$ element of $\mathcal{I}$.

It remains to check that the process of resolving the constructing a sequence of pairwise non-crossing independent full ties terminates. Since G is locally-finite, we delete a finite number of vertices at each step. The process of constructing an infinite sequence of pairwise non-crossing independent full ties does not delete vertices of $P \cup Q$. The construction process has a countably infinite number of steps. Since G is countable, the construction process fails for the ordinal ω. Therefore, the inductive process terminates. Let H be the subgraph of G obtained by the above inductive process, and let $\mathcal{J}_1 = J_{h_1}, J_{h_2}, \ldots$ be the infinite sequence of pairwise non-crossing, independent, full ties of $P \cup Q$ in H obtained by the above inductive process.

Note that $\mathcal{J}_1$ is maximal since $\mathcal{J}$ is maximal and by selection of h_i.

Let $\mathcal{M}_1$ be the sequence of ties obtained by resolving the ties of $\mathcal{J}_1$. Let H' be the subgraph of H induced by the vertices of $P \cup Q$ and $\mathcal{M}_1$. If $\mathcal{M}_1$ contains infinitely many elements, each of which is a tie of type I, Y, λ, or Y-λ, then Lemma 3.3.4 implies that H' conduces a member of one of the families: $\mathcal{L}_\infty$, $\mathcal{L}_\infty^{\Delta}$ and $\mathcal{L}_\infty^{\nabla\Delta}$, and the conclusion follows.

We may therefore assume that $\mathcal{M}_1$ has infinitely many elements each of which is a tie of type fork, rake, or fork-rake. Let H'' be the induced subgraph of H' obtained by rerouting the rays for each of those ties. Lemma 3.3.5 implies that H'' conduces an infinite messy ladder or a member of the family $\mathcal{L}_\infty$, and the conclusion follows. $\square$

The following lemma proves that an infinite 2-connected graph that is locally-finite conduces either an infinite messy ladder or one of the structures found in Figures 3.3a to 3.3c.

Lemma 3.3.7. *Let G be a 2-connected infinite locally-finite graph. Then G conduces ei-*

ther an infinite messy ladder or member of one of the following families: $\mathcal{L}_\infty$, $\mathcal{L}_\infty^\Delta$, *and* $\mathcal{L}_\infty^{\nabla\Delta}$.

Proof. Let G be an infinite locally-finite 2-connected graph. Then Theorem 1.2.9 implies that G contains an induced ray P. By Lemma 3.3.1, it follows that G has either a perpetual P-bridge or an infinite P-bridge chain.

Suppose first that G has a perpetual P-bridge B. Let w be the vertex of attachment of B such that B has no vertices of attachment b such that $b <_P w$. Since B is a perpetual bridge of P in G, and G is locally-finite, it follows that B has infinitely many vertices that are neighbors of vertices of attachment on P. Since G is locally-finite, Theorem 1.2.7 implies that B has an infinite independent subset $\mathcal{V}$ of vertices that are neighbors of vertices of attachment on P. Let τ be a $\mathcal{V}$-connecting tree in $B - V(P)$. Then Theorem 1.2.9 implies that τ contains an induced ray R. Note that P and R are disjoint. Since τ is a $\mathcal{V}$-connecting tree, it follows from Lemma 3.2.3 that R is induced in G and that τ has infinitely many pairwise-disjoint paths, each of which is from a vertex on R to a vertex of $\mathcal{V}$. Since G is locally-finite and the vertices of $\mathcal{V}$ are independent, it follows that B has infinitely many pairwise-disjoint paths from R to P. Thus, the rays R and P are in the same end of G. Let $r_1, r_2, \ldots$ be the vertices of R.

The rays R and P need not have the same initial vertex. Now, we will modify R to obtain an induced ray Q in B such that Q meets P at exactly one vertex w, which is the initial vertex of Q, and Q is in the same end as P. Since B is connected, it contains an induced path S from R to w. Let $w = s_1, s_2, \ldots, s_n$ be the consecutive vertices of

S, and let $\mathcal{S} = N(V(S)) \cap R$. Since S has no infinite vertex, it follows that $\mathcal{S}$ is finite. So, R has a vertex r_i in $\mathcal{S}$ that has maximum index i. Let s_k be the neighbor of r_i on S with smallest index k, let $S' = S[w, s_k]$, let R' be the sub-ray of R that begins at r_i, and let $Q = S' \cup \{s_k r_i\} \cup R'$. Note that Q is induced in G and in the same end as P.

If B has infinitely many edges between $P[w, \infty)$ and Q, then the subgraph of G induced by the vertices of $P[w, \infty)$ and Q is an infinite messy ladder, as required. We may therefore assume that B has only finitely many edges between Q and $P[w, \infty)$. Let $e = xy$ be an edge between $P[w, \infty)$ and Q such that $x \in V(Q)$, $y \in V(P)$, and $|V(P[w, y])|$ is maximal. Note that $P' = P(y, \infty)$ and $Q' = Q(x, \infty)$ are both also induced rays in G that are in the same end and that B has no edges from P' to Q'.

Since P' and Q' are in the same end, it follows that either B has infinitely many finite ties of $P' \cup Q'$ or an infinite tie of $P' \cup Q'$. If B has infinitely many finite ties of $P' \cup Q'$, then Lemma 3.3.6 implies that B conduces an infinite messy ladder or a member of the following families: $\mathcal{L}_\infty$, $\mathcal{L}_\infty^\Delta$, and $\mathcal{L}_\infty^{\nabla\Delta}$, and the conclusion follows.

We may therefore assume that $B \cup P$ has an infinite tie T of $P' \cup Q'$. Lemma 3.3.2 implies that $B \cup P$ has an induced subgraph containing $P \cup Q$ that has a sequence of infinitely many finite ties of $P \cup Q$. Then Lemma 3.3.6 implies that B conduces an infinite messy ladder or a member of the following families: $\mathcal{L}_\infty$, $\mathcal{L}_\infty^\Delta$, and $\mathcal{L}_\infty^{\nabla\Delta}$, and the conclusion follows. This completes the case that G has a perpetual P-bridge.

We may therefore assume that G does not have a perpetual P-bridge, and so, it follows from Lemma 3.3.1 that G has an infinite P-bridge chain B_1, B_2, $\ldots$ where $P[u_i, v_i]$

is the span of B_i. Let Q_i be an induced path in B_i with endpoints u_i and v_i. Since P is induced, it follows that each Q_i contains at least one vertex distinct from u_i and v_i. Let G' be the subgraph of G induced by the vertices of $P \cup \bigcup_{i \geq 1} Q_i$. In G', we delete vertices on $P(u_{j+1}, v_j)$, if they exist, for $j \geq 1$ to obtain a graph G''. Let $W = Q_1 \cup P(v_1, u_3) \cup Q_3 \cup P(v_3, u_5) \cup \cdots \cup Q_{2k+1} \cup P(v_{2k+1}, u_{2k+3}) \cup \cdots$, and let $X = P(u_1, u_2) \cup Q_2 \cup P(v_2, u_4) \cup Q_4 \cup \cdots \cup P(v_{2k}, u_{2k+2}) \cup Q_{2k+2} \cup \cdots$.

Let w_1, w_2, $\ldots$ be the vertices of W listed in the order that they appear on W, and similarly, let x_1, x_2, $\ldots$ be the vertices of X listed in the order that they appear on X. If G'' has infinitely many edges with one endpoint on W and the other on X, then we delete the internal vertices, if they exist, of any path from W to X in G''. Thus G'' conduces an infinite messy ladder. We may therefore assume that G'' has only finitely many edges between W and X. Then G'' has an edge $e = w_i x_j$ such that G'' has no edges $w_k x_\ell$ for $k \geq i$ and $\ell \geq j$. Consider the subgraph G''' induced by $W' = W(w_i, \infty)$ and $X' = X(x_j, \infty)$. If G''' has an infinite tie of $W' \cup X'$, then Lemma 3.3.2 implies that G''' has infinitely many finite ties. Lemma 3.3.6 implies G''' conduces an infinite messy ladder or a member of the following families: $\mathcal{L}_\infty$, $\mathcal{L}_\infty^{\Delta}$, and $\mathcal{L}_\infty^{\nabla\Delta}$, and the conclusion follows. We may therefore assume that G''' does not have an infinite tie of $W' \cup X'$. It follows that G''' has infinitely many finite ties of $W' \cup X'$. Lemma 3.3.6 implies that G''' conduces an infinite messy ladder or a member of the following families: $\mathcal{L}_\infty$, $\mathcal{L}_\infty^{\Delta}$, and $\mathcal{L}_\infty^{\nabla\Delta}$, and the conclusion follows. $\square$

3.4. Infinite Messy Ladder

We prove in this section that an infinite messy ladder conduces an infinite clean ladder.

Let (L, W, X) be an infinite messy ladder. If e is a rung in a ladder with rails W and X, then e_W and e_X denote the end-vertices of e on W and X, respectively. Two rungs in an ordered pair e and f *cross* if $e_W < f_W$ and $f_X < e_X$. We also say that (e, f) is a *cross* whose W*-span* is $W[e_W, f_W]$, and whose X*-span* is $X[f_X, e_X]$. Note that the spans of each cross are finite. A cross whose W-span and X-span are both single edges is *degenerate*. An infinite *clean ladder* is an infinite messy ladder whose crosses are all degenerate; see Figure 3.3d. In order to clean the ladder, we need to define some terms for the crosses. The cross (e, f) is *full* if the messy ladder (L, W, X) has no other cross whose W-span contains the W-span of (e, f) and whose X-span contains the X-span of (e, f). Two crosses are *independent* if both their W-spans and X-spans are edge-disjoint.

Lemma 3.4.1. *If an infinite messy ladder has a cross (e, f), then it has a full cross whose W-span contains $W[e_W, f_W]$ and X-span contains $X[f_X, e_X]$.*

Proof. Let (L, W, X) be a messy ladder with σ_W and σ_X as the initial vertices of W and X and let (e, f) be a cross in (L, W, X). Since (L, W, X) is locally-finite and the segment $W[\sigma_W, e_W]$ has a finite number of vertices, it follows that e is crossed by only finitely many edges that have one endpoint on $W[\sigma_W, e_W]$. So, (L, W, X) has a full cross whose W-span contains $W[e_W, f_W]$. Similarly, only finitely many edges cross f that have one endpoint on $X[\sigma_X, f_X]$. So (L, W, X) has a full cross whose X-span contains $X[\sigma_X, f_X]$.

Thus, (L, W, X) has a full cross whose W-span contains $W[e_W, f_w]$ and whose X-span contains $X[f_X, e_x]$, as required. □

In general, crosses may not be ordered in any particular way with respect to the rails W and X; however, pairwise independent full crosses may be ordered by the position in which their vertices appear on the rails, as explained in Lemma 3.4.2.

Lemma 3.4.2. *Let (e, f) and (g, h) be independent full crosses of a messy ladder (L, W, X), with the W- and X-spans being $W[e_W, f_W]$, $X[f_X, e_X]$, $W[g_W, h_W]$, and $X[h_X, g_X]$, respectively. Then $f_W \leq g_W$ if and only if $e_X \leq h_X$.*

Proof. Let (e, f) and (g, h) be independent full crosses. Suppose for a contradiction that $f_W \leq g_W$, however $e_X \not\leq h_X$. Then $h_X \leq f_X < e_X$ and $e_W < f_W \leq g_W < h_W$. Thus (e, h) is a cross whose W-span contains the W-spans of (e, f) and of (g, h). This contradicts the fact that the crosses (e, f) and (g, h) are full. Hence $e_X \leq h_X$. The other direction of the proof follows an analogous argument. □

For two pairwise independent full crosses (e, f) and (g, h), define the relation $(e, f) < (g, h)$ by $f_W \leq g_W$ (or $e_X \leq h_X$).

A *cross-free ladder* is a messy ladder such that no pair of its rungs cross. A cross-free ladder is obviously a clean ladder.

Lemma 3.4.3. *Every infinite messy ladder either has an infinite maximal sequence of pairwise independent full crosses or conduces an infinite cross-free ladder.*

Proof. Let (L, W, X) be an infinite messy ladder. If (L, W, X) has an infinite maximal sequence of pairwise independent full crosses, then the conclusion follows. We may therefore assume that every maximal sequence of pairwise independent full crosses is finite. Let $\mathcal{X} = \mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_z$ be a maximal sequence of pairwise independent full crosses with W-spans $W[e^i_W, f^i_W]$ and X-spans $X[f^i_X, e^i_X]$ for $1 \leq i \leq z$. Let $W' = W[f^z_W, \infty)$, let $X' = X[e^z_X, \infty)$, and let L' be the subgraph of L induced by the vertices of W' and X'. Since (L, W, X) is an infinite messy ladder, $\mathcal{X}$ is finite, and L' is an induced subgraph of L, it follows that (L', W', X') is an infinite cross-free ladder, and the conclusion follows. $\square$

In the second case of the above lemma, we obtain an infinite induced clean ladder. Up until Lemma 3.4.6, we will assume that a messy ladder has an infinite maximal sequence of pairwise independent full crosses, that is, an infinite sequence that does not appear as proper sub-sequence of any other sequence of pairwise independent full crosses.

Let (L, W, X) be a messy ladder with σ_W and σ_X as the initial vertices of W and X, respectively, and let $\mathcal{X}$ be an infinite maximal sequence of pairwise independent full crosses in (L, W, X). Our goal is now to use $\mathcal{X}$ to eliminate all non-degenerate crosses in the messy ladder (L, W, X) to obtain a clean ladder (H, U, Z). To do this, we need the following operation on ladders that eliminates non-degenerate pairwise independent full crosses.

Let $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \ldots)$ be an infinite maximal sequence of pairwise independent full crosses in (L, W, X). The operation of *resolving the cross* $\mathcal{X}_i = (e^i, f^i)$ results in a triple (L', W', X') where $L' = L - W(e^i_W, f^i_W) - X(f^i_X, e^i_X)$ and $W' = W[\sigma_W, e^i_W] \cup \{e^i\} \cup X[e^i_X, \infty)$ and $X' = X[\sigma_X, f^i_X] \cup \{f^i\} \cup W[f^i_W, \infty)$. Since $\mathcal{X}_i$ is a full cross, the graph L has rungs neither from $W[\sigma_W, e^i_W)$ to $X(e^i_X, \infty)$ nor from $X[\sigma_X, f^i_X)$ to $W(f^i_W, \infty)$. Thus W' and X' are induced in L', and (L', W', X') is a messy ladder. If $\mathcal{X}_i$ is degenerate, then resolving the cross $\mathcal{X}_i$ results in the edges $e^i_W f^i_W$ and $f^i_X e^i_X$ becoming rungs of L', and the rungs f^i and e^i becoming edges on the rails W' and X'. Note that we have not deleted any edges or vertices in this case, so the messy ladders (L', W', X') and (L, W, X) are isomorphic.

For a maximal sequence of pairwise pairwise independent full crosses $\mathfrak{X} = (\mathcal{X}_1, \mathcal{X}_2, \dots)$ where $\mathcal{X}_i = (e^i, f^i)$ of a messy ladder (L, W, X), we inductively define the triples that result from resolving consecutive crosses of $\mathfrak{X}$. Let (L^1, W^1, X^1) be the messy ladder obtained by resolving the cross $\mathcal{X}_1$ with $W^1 = W[\sigma_W, e^1_W] \cup \{e^1\} \cup X[e^1_X, \infty)$ and $X^1 = X[\sigma_X, f^1_X] \cup \{f^1\} \cup W[f^1_W, \infty)$. Since the crosses in $\mathfrak{X}$ are pairwise independent, the operation of resolving $\mathcal{X}_1$ is a local and does not affect the other crosses in $\mathfrak{X}$. That is, $(\mathcal{X}_2, \mathcal{X}_3, \dots)$ are crosses of (L^1, X^1, Y^1). Since the cross $\mathcal{X}_1$ is full, the operation of resolving $\mathcal{X}_1$ does not create a non-degenerate cross in (L^1, W^1, X^1). If $\mathcal{X}_1$ is degenerate, then (L^1, W^1, X^1) is isomorphic to (L, W, X). If $\mathcal{X}_1$ is not degenerate, then $(\mathcal{X}_2, \mathcal{X}_3, \dots)$ is a maximal sequence of pairwise independent full crosses in (L^1, W^1, X^1).

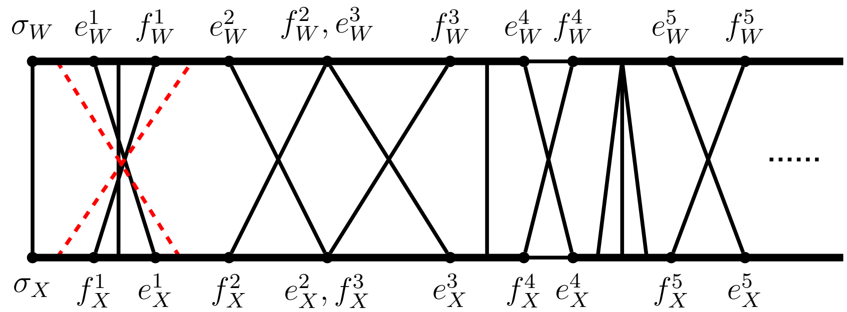
For the inductive process, the definition of the rails W^i and X^i depends on the parity of i. Suppose we have defined $(L^{i-1}, W^{i-1}, X^{i-1})$ for some $i \geq 2$ where $(L^{i-1}, W^{i-1}, X^{i-1})$ is the triple obtained by resolving the crosses $(\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_{i-1})$. Since each cross of $\mathfrak{X}$ is full, the operation of resolving the crosses $(\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_{i-1})$ does not create non-degenerate crosses. Since the crosses of $\mathfrak{X}$ are pairwise independent, the crosses $(\mathcal{X}_i, \mathcal{X}_{i+1}, \dots,)$ are crosses in $(L^{i-1}, W^{i-1}, X^{i-1})$. Each cross in $(\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_{i-1})$ that was degenerate in (L, W, X) is a degenerate cross in $(L^{i-1}, W^{i-1}, X^{i-1})$ after resolving the crosses $(\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_{i-1})$. So the degenerate crosses from $\mathcal{X}_1$, $\mathcal{X}_2$, $\dots$, $\mathcal{X}_{i-1}$ together with the crosses $\mathcal{X}_i$, $\mathcal{X}_{i+1}$, $\ldots$ form a maximal sequence of pairwise independent full crosses in $(L^{i-1}, W^{i-1}, X^{i-1})$.

Let (L^i, W^i, X^i) be the messy ladder obtained from $(L^{i-1}, W^{i-1}, X^{i-1})$ by resolving $\mathcal{X}_i$. If i is even, then let $W^i = W[\sigma_W, e^1_W] \cup \{e^1\} \cup X[e^1_X, f^2_X] \cup \{f^2\} \cup \;\cdots\; \cup \{f^i\} \cup W[f^i_W, \infty)$ and $X^i = X[\sigma_X, f^1_X] \cup \{f^1\} \cup W[f^1_W, e^2_W] \cup \{e^2\} \cup \;\cdots\; \cup \{e^i\} \cup X[e^i_X, \infty)$. If i is odd, then let $W^i = W[\sigma_W, e^1_W] \cup \{e^1\} \cup X[e^1_X, f^2_X] \cup \{f^2\} \cup \;\cdots\; \cup \{e^i\} \cup X[e^i_X, \infty)$ and $X^i = X[\sigma_X, f^1_X] \cup \{f^1\} \cup W[f^1_W, e^2_W] \cup \{e^2\} \cup \;\cdots\; \cup \{f^i\} \cup W[f^i_W, \infty)$.

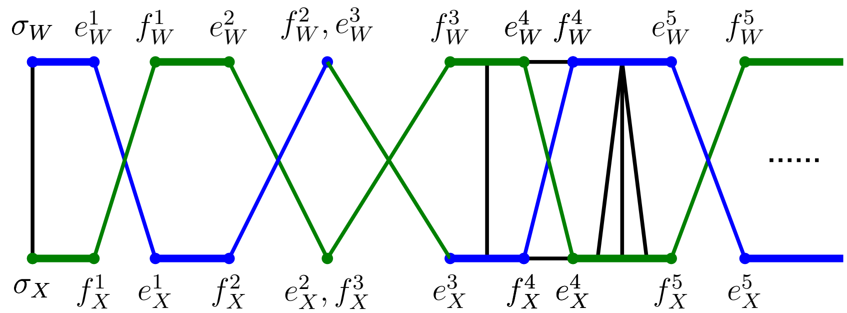
It remains to check that the process of resolving the crosses terminates. Once a cross is resolved, the only crosses in the graph induced by the subpaths of W^i and X^i from the initial vertex to the last vertex in the span of $\mathcal{X}^i$ are degenerate. Since G is countably infinite, it follows that $\mathfrak{X}$ has a countably infinite number of crosses. So, the process of resolving the crosses fails for the ordinal ω. Thus, the process of resolving the crosses of $\mathfrak{X}$ terminates. Let (H, U, Z) be the infinite ladder which is a result of resolving the crosses of $\mathfrak{X}$. Since we have resolved the crosses of $\mathfrak{X}$, every cross from $\mathfrak{X}$ that is in (H, U, Z) is degenerate. Thus (H, U, Z) is an infinite clean ladder.

Remark 3.4.4. *The vertices e^1_W, e^1_X, f^1_W, f^1_X, $\ldots$ of the pairwise independent full crosses of $\mathfrak{X}$ are members of the vertex set of (H, U, Z).*

This process of resolving the crosses is depicted in Figure 3.13a and Figure 3.13b below.



(a) Sequence of Pairwise Independent Full Crosses



(b) The infinite clean ladder (H, U, Z)

Figure 3.13. Resolving the crosses

The red dashed lines in Figure 3.13a indicate the location where rungs cannot exist due to (e^1, f^1) being a full cross. In Figure 3.13b, the blue path represents the induced path U, the green path represents the induced path W, and the black lines represent rungs of (H, U, Z). Notice that Figure 3.13a has a rung that has an endpoint in the W-span and an endpoint in the X-span of (e^1, f^1), and this rung is not in (H, U, Z).

The next lemma follows from the process described above.

Lemma 3.4.5. *Let $\mathcal{X}$ be an infinite maximal sequence of pairwise independent full crosses of a messy ladder. Resolving the crosses of $\mathcal{X}$ results in an infinite clean ladder.*

The following lemma combines the previous lemmas in this section to complete the proof that an infinite clean ladder is a sub-ladder of every infinite messy ladder.

Lemma 3.4.6. *Every infinite messy ladder conduces an infinite clean ladder.*

Proof. Let (L, W, X) be an infinite messy ladder. Lemma 3.4.3 implies that (L, W, X) conduces an infinite cross-free ladder or has an infinite maximal sequence of pairwise independent full crosses. If (L, W, X) conduces an infinite cross-free sub-ladder, then (L, W, X) conduces an infinite clean ladder, as required. We may therefore assume that (L, W, X) has an infinite maximal sequence of pairwise independent full crosses. By Lemma 3.4.5, resolving the crosses of this sequence results in an induced infinite clean ladder, and the conclusion follows. □

3.5. Proving Theorem 3.1.1

We can now prove Theorem 3.1.1 stated below more precisely than in the introduction. Recall that an infinite clean ladder is defined in Section 3.4, and shown in Figure 3.3d; the families $\mathcal{K}_{2,\infty}$ and $\mathcal{K}^+_{2,\infty}$ are defined in Section 3.1, and shown in Figures 3.1a and 3.1b, respectively; the families $\mathcal{F}_\infty$ and $\mathcal{F}^\Delta_\infty$ are defined in Section 3.2, and shown in Figures 3.2a and 3.2b respectively; and lastly the families $\mathcal{L}_\infty$, $\mathcal{L}^\Delta_\infty$, and $\mathcal{L}^{\nabla\Delta}_\infty$ are defined in Section 3.3, and shown in Figures 3.3a to 3.3c respectively.

Theorem 3.1.1. *Let G be a 2-connected infinite graph. Then G conduces one of the following: K_∞, an infinite clean ladder, and a member of one of the families $\mathcal{K}^+_{2,\infty}$, $\mathcal{K}_{2,\infty}$, $\mathcal{F}_\infty$, $\mathcal{F}^\Delta_\infty$, $\mathcal{L}_\infty$, $\mathcal{L}^\Delta_\infty$, and $\mathcal{L}^{\nabla\Delta}_\infty$.*

Proof. Let G be an infinite 2-connected graph. If G has an infinite vertex, then Lemma 3.2.5 implies that G conduces either K_∞ or a member of one the families $\mathcal{F}_\infty$, $\mathcal{F}^\Delta_\infty$, $\mathcal{K}_{2,\infty}$, and

$\mathcal{K}^+_{2,\infty}$, and the conclusion follows.

We may therefore assume that G is locally-finite, and so Theorem 1.2.9 implies that G conduces a ray. Thus, Lemma 3.3.7 implies that G conduces either an infinite messy ladder or a member of the families: $\mathcal{L}_\infty$, $\mathcal{L}^{\Delta}_\infty$, and $\mathcal{L}^{\nabla\Delta}_\infty$. If G conduces a member of one of the following families: $\mathcal{L}_\infty$, $\mathcal{L}^{\Delta}_\infty$, and $\mathcal{L}^{\nabla\Delta}_\infty$, then the conclusion follows.

We may therefore assume that G conduces an infinite messy ladder. Lemma 3.4.6 implies that G conduces an infinite clean ladder, as required. $\square$

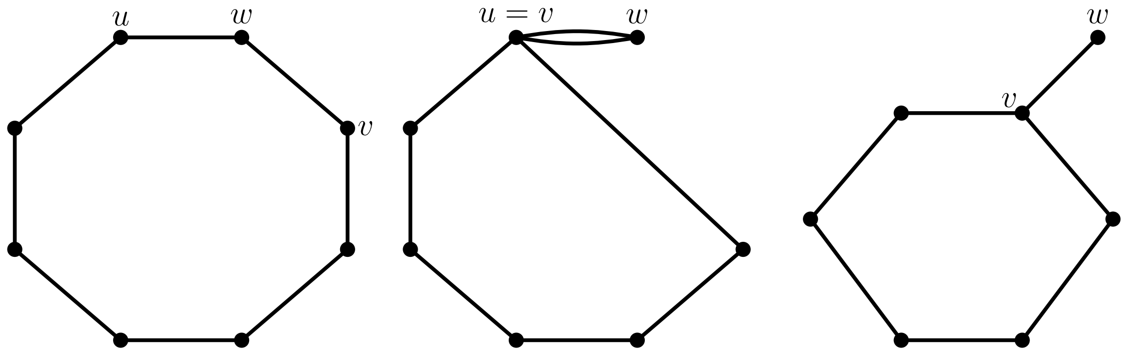
Chapter 4. Bipartite Minors

4.1. Preliminaries

In this chapter, we restrict our considerations to bipartite graphs that are simple and finite.

For a bipartite graph G, a minor of G need not be bipartite. For example, if we contract an edge in a cycle of G to obtain a minor H, then the resulting cycle in H is an odd cycle, thus making H not bipartite. To address this, the authors of [2] created the graph substructure relation of a bipartite minor to guarantee that the graph obtained by taking a bipartite minor of G is bipartite. In this relation, the authors of [2] redefine the operation of a contraction. In order to describe when we are allowed to perform this type of contraction, we first need the following definitions. A *non-separating cycle* C in G is a cycle such that the deletion of the vertices of C does not increase the number of components of G. A *peripheral cycle* of G is an induced, non-separating cycle in G. Let u and v be two vertices in the same part of the bipartition of $V(G)$ that have a common neighbor w such that u, w, v is a sub-path of a peripheral cycle. The *contraction* of u with v is a graph H obtained by identifying u and v then deleting an edge in each pair of resulting parallel edges. We say that the vertex set of H is $V(G) \setminus \{u\}$. Since u and v are in the same side of the bipartition, the graph G does have an edge between u and v. So the contraction of u with v does not create an odd cycle; and thus, H is bipartite. Figure 4.1 illustrates the operation of contraction.

We may now formally define the relation of a bipartite minor. The bipartite graph



(a) C_8 (b) Identifying u with v (c) Result of a contraction

Figure 4.1. The operation of contraction of u with v.

H is a *bipartite minor* of a bipartite graph G if H is obtained from G by a series of the following operations: vertex deletion, edge deletion, and contraction.

In Section 4.2, we will present the unavoidable bipartite minors for a large connected bipartite graph, stated below.

Theorem 4.1.1. *Let q and r be positive integers. There is an integer $f_{4.1.1}(q, r)$ such that every connected bipartite graph of order at least $f_{4.1.1}(q, r)$ contains as a bipartite minor either P_q or $K_{1,r}$.*

Note that this theorem also holds when the relation of bipartite minor is replaced with the relation of subgraph; however, we are only interested in the bipartite minor relation.

In order to state an analogous result for 2-connected bipartite graphs, we need to define two families of graphs. Let u and v be the two vertices of degree n in the same side of the bipartition for each member of the family $\mathcal{K}_{2,n}$, defined in Chapter 2. If a member of $\mathcal{K}_{2,n}$ has two uv-paths that differ in parity of their order, then that member contains an odd cycle and is not bipartite. We may therefore assume that all of the uv-paths in each member of $\mathcal{K}_{2,n}$ have the same parity. So, we will have two families: one where each uv-path has even order, and one where each uv-path has odd order.

Let $\mathcal{K}^{\mathrm{e}}_{2,n}$ be the family of graphs such that each member of the family consists of two distinct vertices u and v such that there are n pairwise internally-disjoint uv-paths that all have even order. Let $\mathcal{K}^{\mathrm{o}}_{2,n}$ be the family of graphs such that each member of the family consists of two distinct vertices u and v such that there are n pairwise internally-disjoint uv-paths that all have odd order. This leads to the following theorem, which will be proved in Section 4.3.

Theorem 4.1.2. *Let p and q be a positive even integers. There is an integer $f_{4.1.2}(p,q)$ such that every 2-connected bipartite graph G of order at least $f_{4.1.2}(p,q)$ contains as a bipartite minor either a cycle of order p or a member of the one of the following families: $\mathcal{K}^{\mathrm{e}}_{2,q}$ and $K^{o}_{2,q}$.*

4.2. Unavoidable Connected Bipartite Minors

In this section, we present the unavoidable bipartite minors for large connected bipartite graphs.

Theorem 4.1.1. *Let q and r be positive integers. There is an integer $f_{4.1.1}(q,r)$ such that every connected bipartite graph of order at least $f_{4.1.1}(q,r)$ contains as a bipartite minor either P_q or $K_{1,r}$.*

Proof. Let $f_{4.1.1}(q,r) = 2 + (r-1) + (r-1)^2 + \cdots + (r-1)^{q-2}$ and let G be a connected bipartite graph of order at least $f_{4.1.1}(q,r)$. If G contains a vertex of degree r, then G contains $K_{1,r}$ as a bipartite minor, and the conclusion follows. We may therefore assume that G has maximum degree at most $r-1$.

Recall the definition of a rooted tree and a normal tree from Sections 2.1 and 2.2, respectively. Let (T,ρ) be a normal tree of G rooted at an arbitrary vertex, ρ. Note that T is a connected bipartite minor of G. Suppose for a contradiction that G does not have a path of order q as a bipartite minor. Since G has maximum degree at most $r-1$, it follows that ρ has at most $r-1$ neighbors in (T,ρ), and thus in G. Let n_s be the number of vertices v that are endpoints of a ρv-path in (T,ρ) of order s. Note that $n_1 = 1$ and $n_2 \leq (r-1)$. In general, since the maximum degree of G is $r-1$, it follows that (T,ρ) has $n_i \leq (r-1)^{i-1}$ vertices that are distance i from ρ in (T,ρ). So $n_s \leq (r-1)^{s-1}$. Since G does not have a path of order q, it follows that $s < q$. So the number of vertices of (T,ρ), and thus G, is $\sum_{i=1}^{q-1} n_i$. This sum is bounded from above by $1+(r-1)+\cdots+(r-1)^{q-2}$. So $|V(G)| \leq 1+(r-1)+\cdots+(r-1)^{q-2}$; a contradiction. Thus, G has a path of order q as a subgraph, and therefore as a bipartite minor, and the conclusion follows. □

4.3. Unavoidable 2-connected Bipartite Minors

In this section, we present the unavoidable bipartite minors for a large 2-connected bipartite graph G. We will accomplish this by splitting the argument into two cases: either G has a long path as a subgraph or not. We address the latter case in the following lemma.

Lemma 4.3.1. *Let m be a positive integer and let n be an integer exceeding one. There is an integer $f_{4.3.1}(m,n)$ such that every 2-connected bipartite graph of order at least*

$f_{4.3.1}(m,n)$ has either a path of length m or has a bipartite minor that is a member of one of the following families: $\mathcal{K}^{\mathrm{e}}_{2,n}$ and $\mathcal{K}^{\mathrm{o}}_{2,n}$.

Proof. We prove that $f_{4.3.1}(m,n) = 1 + d + d^2 + \ldots + d^m$, where $d = (m-1)n$, satisfies the conclusion.

Let G be a 2-connected bipartite graph of order at least $f_{4.3.1}(m,n)$. Let (T,ρ) be a normal spanning tree of G. If (T,ρ) has height at least m, then (T,ρ) has a path of length m, and the conclusion follows.

For the remainder of the proof, we may therefore assume that the height of (T,ρ) is less than m. Since (T,ρ) has at least $f_{4.3.1}(m,n)$ vertices and no vertex w such that a ρw path has length m, it follows that T has a vertex v with more than d children.

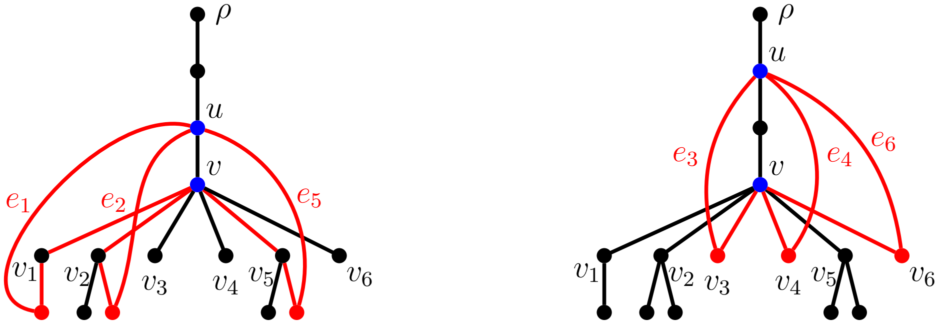
Let R be the ρv path in (T,ρ) that has length at most $m-2$. For each child v_i of v, let (T_i, v_i) be the rooted sub-tree of (T,ρ) induced by v_i and all of its descendants. Since v has at least d children, it follows that (T,ρ) has at least d sub-trees rooted at children of v. We need to consider only d of them, (T_1,v_1), (T_2,v_2), $\ldots$, (T_d,v_d). Since (T,ρ) is normal, it follows that (T_i,v_i) and (T_j,v_j) are disjoint for distinct i and j from $i,j \in \{1,2,\ldots,d\}$. Further, every edge of G with one endpoint (T_i,v_i) for some $1 \le i \le d$ must have other endpoint on $R - v$. Since (T,ρ) is normal, it follows that all of the rooted sub-trees (T_1,v_1), (T_2,v_2), $\ldots$, (T_d,v_d) are disjoint, and that G has no edges between them. Since G is 2-connected, it follows v is not a cut-vertex of G. For each $j \in \{1,2,\ldots,d\}$, the graph G has an edge e_j incident with both a vertex on (T_j,v_j) and a vertex u_j on $R - v$. By definition of d, there is a natural number k in $\{1,2,\ldots,d\}$ such that u_k is incident to at least n of the edges e_j; let $u = u_k$. Let $\mathcal{I}$ be the set of n indices from $\{1, 2, \ldots, d\}$ of the n edges that have u as one endpoint and the other endpoint on (T_j, v_j). Each (T_i, v_i) for $1 \leq i \leq d$ spans a component G_i of $G - V(R)$. The vertices u and v both have at least one neighbor in G_i. Let G_i' be the subgraph of G that consists of G_i and all edges between a vertex of G_i and a vertex of $\{u, v\}$. Note that G_i' is connected. Let P_i be the shortest uv-path in G_i'.

Let H be the subgraph of G induced by $\bigcup_{i \in \mathcal{I}} P_i$. Since (T, ρ) is a normal spanning tree, it follows that G has no edges between internal vertices of distinct paths $\{P_i\}_{i \in \mathcal{I}}$. Since each P_i is a shortest uv-path in G_i', it follows that H is the union of pairwise internally-disjoint paths.

We define a graph H' in the following piecewise manner.

$$H' = \begin{cases} H \setminus \{uv\} & \text{if } uv \in E(G) \\ H & \text{if } uv \notin E(G) \end{cases}$$

Note that H' is a subgraph of H, and thus, a bipartite minor of H. Since G is bipartite, all of the paths P_i for $i \in \mathcal{I}$ have the same parity. If all have odd order, then H' is a member of the family $\mathcal{K}_{2,n}^{\mathrm{o}}$ and is a bipartite minor of G, and the conclusion follows. Otherwise, H' is a member of the family $\mathcal{K}_{2,n}^{\mathrm{e}}$ and is a bipartite minor of G, and the conclusion follows.



(a) member of the family $\mathcal{K}^{\mathrm{e}}_{2,n}$ in G (b) member of the family $\mathcal{K}^{\mathrm{o}}_{2,n}$ in G

Figure 4.2. Process of obtaining a member of the family $\mathcal{K}^{\mathrm{e}}_{2,n}$ or $\mathcal{K}^{\mathrm{o}}_{2,n}$

Figure 4.2 shows the paths whose union is either a member of the family $\mathcal{K}^{\mathrm{e}}_{2,n}$ or a member of the family $\mathcal{K}^{\mathrm{o}}_{2,n}$. The red vertices are the vertices in the bipartition of cardinality$n$, and the blue vertices are members of the bipartition of cardinality two. The red segments show the edges of a graph in $\mathcal{K}^{\mathrm{e}}_{2,n}$ in Figure 4.2a and the edges of $\mathcal{K}^{\mathrm{o}}_{2,n}$ in Figure 4.2b. □

In the case that a large 2-connected bipartite graph has a long path, we will use the following well-known theorem of Menger [12].

Theorem 4.3.2 (3.3.6(i) of [5])**.** *A graph is k-connected if and only if it contains k pairwise internally-disjoint paths between every two vertices.*

Since we consider 2-connected bipartite graphs G, Theorem 4.3.2 implies that G has a cycle that contains two specified vertices u and v for each pair $u, v \in V(G)$. For the following lemma, we need to define a type of vertex. A *pendant vertex* in a graph is a vertex of degree one; the vertex w in Figure 4.1c is an example of a pendant vertex. We will first show that if G has a peripheral cycle of order at least p, where p is an even number exceeding two, then G has a peripheral cycle of order p as a bipartite minor.

Lemma 4.3.3. *Let p be an even integer exceeding two. If a 2-connected bipartite graph G has a cycle of order at least p, then G contains as a bipartite minor a cycle of order p.*

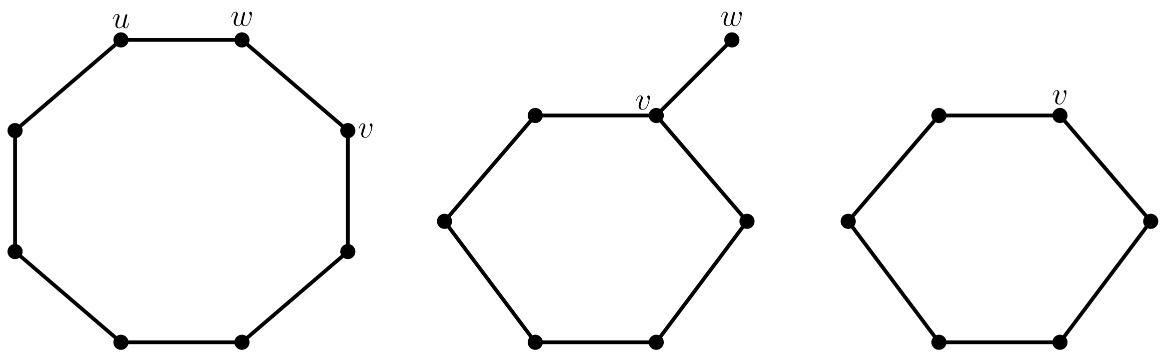
Proof. Let C be a cycle of order $n \geq p$ in G. Since C is an even cycle, it follows that $n = p + 2i$ for some non-negative integer i. If $i = 0$, then G has a cycle of order p as a bipartite minor.

We may therefore assume that $i \geq 1$. Let G' be the subgraph of G with vertex set $V(C)$ and edge set $E(C)$. Note that C is a peripheral cycle of G'. We will inductively obtain, by taking bipartite minors, a sequence of smaller cycles until we get a cycle of length p as a bipartite minor of G'.

Let G'_1 be the graph obtained from G' be performing a contraction and then deleting the resulting pendant vertex. Note that G'_1 has two fewer vertices than G'. So, G'_1 is a cycle of order $p + 2i - 2 = p + 2(i - 1)$. If $i = 1$, then G'_1 is a cycle of order p, and the conclusion follows. We may therefore assume that $i > 1$.

Suppose we have inductively obtained the graphs $G'_1, G'_2, \ldots, G'_j$ for $j < i$ and that G'_j has order greater than p. Note that each graph G'_k for $1 \leq k \leq j$ is a cycle, and thus, a peripheral cycle. So we may perform a contraction on G'_j. Let G'_{j+1} be the graph obtained from G'_j by performing a contraction and then deleting the resulting pendant vertex. Figure 4.3 illustrates this process. Note that G'_{j+1} has $2(j + 1)$ fewer vertices than C. If G'_{j+1} has order p, then G has a cycle of order p as a bipartite minor, and the conclusion follows.

Otherwise, at step i, we have the graph G'_i has $2i$ fewer vertices than C. So G'_i has



(a) G'_i (b) Contraction step (c) G'_{i+1}

Figure 4.3. Process of obtaining G'_{i+1} from G'_i

order $p + 2i - 2i = p$. Therefore the process terminates at step i, the resulting graph G'_i is a cycle of order p as a bipartite minor of C and thus G, and the conclusion follows. □

We will now use Theorem 4.3.2 and Lemma 4.3.3 to address the case that a large 2-connected bipartite graph has a long path.

Lemma 4.3.4. *Let p be a positive even integer. There is a number $f_{4.3.4}(p)$ such that if a 2-connected bipartite graph G has a path of length $f_{4.3.4}(p)$, then G has a cycle of order p as a bipartite minor.*

Proof. Let $f_{4.3.4}(p) = (p-3)(p-2)+1$ and let P be a uv-path of length $f_{4.3.4}(p)$ in G.

Since G is 2-connected, Theorem 4.3.2 implies that G has a cycle C that contains both u and v. Note that the order of C must be even since G is bipartite. We will consider two cases based on whether C has order at least p or not.

Suppose first that C has order n such that $n \geq p$. Let G' be the subgraph of G with vertex set $V(C)$ and edge set $E(C)$. Note that C is a peripheral cycle of G'. Lemma 4.3.3 implies that G' contains as a bipartite minor a cycle of order p, and the conclusion follows.

We may therefore assume that C has order less than p. Since G is bipartite, it follows that C has order at most $p-2$. This implies that the cycle C must intersect P at least once at a vertex u_2 distinct from u and v. Since C has length at most $p-2$, it follows that C intersects P at most $p-2$ times. Let $u = u_1, u_2, \ldots, u_k = v$ where $3 \leq k \leq p-2$ be the vertices of $V(C) \cap V(P)$ appearing on P in that order. The vertices $u_1, u_2, \ldots, u_k$ divide P into $k-1 \leq p-3$ edge disjoint sub-paths of P. Let $C[u_i, u_{i+1}]$ be the sub-path of C such that $C[u_i, u_{i+1}] \cap P[u_i, u_{i+1}] = \{u_i, u_{i+1}\}$ for $1 \leq i \leq k-1 \leq p-3$. Since P has length $f_{4.3.4}(p)$ and C includes both u and v, it follows that P has a sub-path $P[u_i, u_{i+1}]$ that has more than $p-2$ edges. So $C' = C[u_i, u_{i+1}] \cup P[u_i, u_{i+1}]$ forms a cycle of order at least p. Lemma 4.3.3 implies that G contains as a bipartite minor a cycle of order p, and the conclusion follows. $\square$

We now combine Lemmas 4.3.1 and 4.3.4 to prove Theorem 4.1.2.

Theorem 4.1.2. *Let p be an even integer exceeding two and let q be a positive even integer. There is a number $f_{4.1.2}(p, q, r)$ such that every a 2-connected bipartite graph G of order at least $f_{4.1.2}(p, q)$ contains as a bipartite minor either a cycle of order p or a member of the one of the families: $\mathcal{K}^{\mathrm{e}}_{2,q}$ and $\mathcal{K}^{\mathrm{o}}_{2,q}$.*

Proof. Let $f_{4.1.2}(p, q) = f_{4.3.1}(m, q)$ where $m = f_{4.3.4}(p)$ and $f_{4.3.1}$ and $f_{4.3.4}$ are the numbers from Lemmas 4.3.1 and 4.3.4, respectively.

Since G is a 2-connected bipartite graph of order at least $f_{4.1.2}(p, q)$, Lemma 4.3.1 implies that the graph G has either a path of length m or contains as a bipartite minor a member one of the families $\mathcal{K}^{\mathrm{e}}_{2,q}$ and $\mathcal{K}^{o}_{2,q}$. If G contains a a bipartite minor a member of one of the following families: $\mathcal{K}^{\mathrm{e}}_{2,q}$ and $\mathcal{K}^{\mathrm{o}}_{2,q}$, then the conclusion follows.

We may therefore assume that G has a path of length m. Then Lemma 4.3.4 implies that G contains as a bipartite minor a cycle of order p, and the conclusion follows.

$\square$

Chapter 5. Conclusions

In this chapter, we review our results and consider future work in extending them.

5.1. Induced Subgraphs

In Chapters 2 and 3, we showed the unavoidable induced subgraphs for large and infinite 2-connected graphs, respectively. These results fully address the 2-connected analogs of Ramsey's Theorem for the induced subgraph relation.

For the finite case, found in Chapter 2, we considered two cases: either a large 2-connected graph has a long path or not. In the case that the graph does not have a long path, we used the concept of normal trees to find a member of one of the families: $\mathcal{K}_{2,r}$ or $\mathcal{K}^+_{2,r}$ for an integer r exceeding two. In the case that the graph has a long path, there were many instances where we utilized the fact that a cycle, if large enough, is a clean ladder of the target order.

For the infinite case, found in Chapter 3, the large cycle, which was critical to the finite case, does not have an infinite analog. Another barrier to the infinite case was that the division of proof in an analogous way to the finite case, based on whether or not the graph has a ray, did not seem to lead to the desired conclusion. Two of the families of graphs in the infinite result, the fan-like graphs shown in Figures 3.2a and 3.2b, have both a ray and infinite vertex, while four of the families have a ray and are locally-finite, the ladder-like graphs shown in Figures 3.3a to 3.3d. So to prove the result for the unavoidable induced subgraphs of infinite 2-connected graphs, we considered two cases: either the graph has an infinite vertex or not. In the case that the graph has an infinite vertex, we introduced the concept $\mathcal{V}$-connecting tree to control possible edges in the graph that are not in the tree. The latter case used many of the same techniques as the finite case.

An open question that is a natural extension of these results is to find the unavoidable induced sub-matroids for large 2-connected matroids. The intermediate step is to consider the operation on graphs of contracting cycles and then deleting resulting loops and deleting all but one edge in each set of resulting parallel edges. Note that this operation is similar to the operation allowed for producing parallel minors except that it specifies the types of edges that may be contracted.

Another direction is to take a step towards proving a 3-connected analog of Ramsey's Theorem by considering graphs that are "almost" 3-connected; that is, a graph would be 3-connected if not for a large subgraph that is 2-connected but not 3-connected. Professor Geelen of the University of Waterloo suggested this direction at an online seminar where I presented.

5.2. Bipartite Minors

In Chapter 4, we showed the unavoidable bipartite minors of large connected and large 2-connected bipartite graphs. To prove the result on large 2-connected bipartite graphs, we split the proof into two cases, similar to Chapter 2, that either the graph has a long path or not. If the graph has a long path, we proved that it has a large cycle.

Since the relation of bipartite minors has been introduced more recently than other graph relations discussed in this dissertation, many problems that have been solved for those relations are still open for bipartite minors. In [2], the authors present the forbidden bipartite minors for the class of planar graphs and the class of outerplanar graphs. One direction for future work is to prove a forbidden bipartite minors characterization for series-parallel graphs. An intermediate step in this project is to investigate if the tree-width of a bipartite minor H of a bipartite graph G is at most the tree-width of G.

Appendix. Copyright Information

Copyright Information for Chapter 2.

UNAVOIDABLE INDUCED SUBGRAPHS OF LARGE 2-CONNECTED GRAPHS

SARAH ALLRED, GUOLI DING, AND BOGDAN OPOROWSKI

Abstract. Ramsey proved that for every positive integer n, every sufficiently large graph contains an induced K_n or $\overline{K_n}$. Among the many extensions of Ramsey's Theorem there is an analogue for connected graphs: for every positive integer n, every sufficiently large connected graph contains an induced K_n, $K_{1,n}$, or P_n. In this paper, we establish an analogue for 2-connected graphs. In particular, we prove that for every integer exceeding two, every sufficiently large 2-connected graph contains one of the following as an induced subgraph: K_n, a subdivision of $K_{2,n}$, a subdivision of $K_{2,n}$ with an edge between the two vertices of degree n, and a well-defined structure similar to a ladder.

1. Introduction

The terms and symbols that are not defined explicitly in this paper will be understood as defined in Diestel [2]. This paper focuses on the induced subgraph relation, and so we will often wish to state that a graph G contains an induced subgraph isomorphic to a graph H; in such a case we will abbreviate this by saying that G *conduces* H. All graphs we consider are finite, simple, and undirected.

The classical result of Ramsey [8], which served as a motivation for this paper and many others, is the following:

Theorem 1.1 (Ramsey's Theorem). *For every positive integer r, there is an integer $f_{1.1}(r)$ such that every graph on at least $f_{1.1}(r)$ vertices conduces K_r (a complete graph on r vertices) or $\overline{K}_r$ (an edgeless graph on r vertices).*

There are numerous extensions of Ramsey's Theorem for graphs of various levels of connectivity and different relations on graphs. For connected graphs, we have the following:

Theorem 1.2 ((5.3) of [3]). *For every positive integer r, there is an integer $f_{1.2}(r)$ such that every connected graph on at least $f_{1.2}(r)$ vertices conduces one of the following graphs: K_r, $K_{1,r}$, and P_r.*

For 2-connected graphs, we have the following for the relation of topological minors:

Theorem 1.3 ((1.2) of [7]). *For every integer r exceeding two, there is an integer $f_{1.3}(r)$ such that every 2-connected graph on at least $f_{1.3}(r)$ vertices contains a subgraph isomorphic to a subdivision of $K_{2,r}$ or C_r.*

For topological minors, a theorem of this type was proved in [7] for 3- and internally-4-connected graphs. For parallel minors, a theorem of this type was proved in [1] for 1-, 2-, 3-, and internally-4-connected graphs.

Date: September 21, 2021.
2010 *Mathematics Subject Classification.* Primary 05C75; Secondary 05C55.
Key words and phrases. Unavoidable induced subgraphs, Ramsey theory, 2-connected graphs.

Bibliography

[1] S. Allred, G. Ding, and B. Oporowski. Unavoidable induced subgraphs of large 2-connected graphs, 2020, arXiv:2009.12503.

[2] M. Chudnovsky, G. Kalai, E. Nevo, I. Novik, and P. Seymour. Bipartite minors. *J. Combin. Theory Ser. B*, 116:219–228, 2016.

[3] C. Chun and G. Ding. Unavoidable topological minors of infinite graphs. *Discrete Math.*, 310(24):3512–3522, 2010.

[4] C. Chun, G. Ding, B. Oporowski, and D. Vertigan. Unavoidable parallel minors of 4-connected graphs. *J. Graph Theory*, 60(4):313–326, 2009.

[5] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Berlin, fifth edition, 2018. Paperback edition of [MR3644391].

[6] G. Ding, B. Oporowski, J. Oxley, and D. Vertigan. Unavoidable minors of large 3-connected binary matroids. *J. Combin. Theory Ser. B*, 66(2):334–360, 1996.

[7] G. Ding, B. Oporowski, James Oxley, and Dirk Vertigan. Unavoidable minors of large 3-connected matroids. *J. Combin. Theory Ser. B*, 71(2):244–293, 1997.

[8] F. Galvin, I. Rival, and B. Sands. A Ramsey-type theorem for traceable graphs. *J. Combin. Theory Ser. B*, 33(1):7–16, 1982.

[9] R. L. Graham, M. Grötschel, and L. Lovász, editors. *Handbook of combinatorics. Vol. 1, 2*. Elsevier Science B.V., Amsterdam; MIT Press, Cambridge, MA, 1995.

[10] D. König. Über eine schlussweise aus dem endlichen ins unendliche. *Acta Sci. Math.(Szeged)*, 3(2-3):121–130, 1927.

[11] Dénes König. *Theorie der endlichen und unendlichen Graphen: Kombinatorische Topologie der Streckenkomplexe*, volume 16. Akademische Verlagsgesellschaft mbh, 1936.

[12] K. Menger. Zur allgemeinen kurventheorie. *Fund. Math.*, 10:96–115, 1927.

[13] B. Oporowski, J. Oxley, and R. Thomas. Typical subgraphs of 3- and 4-connected graphs. *J. Combin. Theory Ser. B*, 57(2):239–257, 1993.

[14] J. Oxley. *Matroid theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Ox-

ford University Press, Oxford, second edition, 2011.

[15] F. Ramsey. On a Problem of Formal Logic. *Proc. London Math. Soc.*, 30:264–286, 1930.

[16] W. Tutte. *Introduction to the theory of matroids.* Modern Analytic and Computational Methods in Science and Mathematics, No. 37. American Elsevier Publishing Co., Inc., New York, 1971.

Vita

Sarah E. Allred was born in Chattanooga, Tennessee. She finished her undergraduate studies at Auburn University in May 2016. In August 2016, she came to Louisiana State University to begin graduate studies in mathematics. She earned a Master of Science degree in Mathematical Sciences from Louisiana State University in December 2017. She is currently a candidate for the degree of Doctor of Philosophy in mathematics.