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Advanced Gravitational Radiation Transducers With Small Mass and Ultra-Low Temperature SQUIDs.

Ziniu Geng
Louisiana State University and Agricultural & Mechanical College

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ADVANCED GRAVITATIONAL RADIATION TRANSUCERS WITH SMALL MASS AND ULTRA-LOW TEMPERATURE SQUIDS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Physics and Astronomy

by

Ziniu Geng
B.S., Beijing University, 1983
M.S., Institute of Electronics, Academia Sinica, 1986
M.S., Louisiana State University, 1989
December 1994
DEDICATION

To Wei, Danny,

and my parents
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ABSTRACT

The transducer and SQUID are two major components for a resonant-mass gravitational wave antenna. They must always have high sensitivity and low noise. In this work, a two-mode superconducting inductive transducer for a three-mode gravitational wave bar antenna is designed. The construction and tests of small mass resonators that form the transducer are described. The experiments and test results of commercial dc SQUIDs at ultra-low temperature are discussed in detail.

Special problems arise when SQUIDs are tested at ultra-low temperature in vacuum. We address the problems and provide proper solutions. Our experiments show that the dc SQUID made by Quantum Design operates well at temperatures as low as 50 mK. We find that the SQUID has an energy resolution of 2600 $\hbar$. The SQUID's noise did not decrease with temperature as was expected. The possible reasons for this are discussed.

The two-mode inductive transducer is designed based on a noise analysis of the complete three-mode system. A small mass of 8-9 grams is chosen for the transducer's diaphragm. The geometry of both the small mass resonator and the intermediate resonator that form the transducer is designed with the aid of finite element analysis.

One of two aluminum prototype resonators reached a mechanical Q of $8.3 \times 10^6$ at 4.2 K. Among the three fabricated niobium resonators, one achieved mechanical Qs of $2.4 \times 10^7$ at 4.2 K and $3.0 \times 10^7$ at around 7 K. Another one with arms cut by Electrical Discharge Machining has reached mechanical Qs of $7.4 \times 10^6$ at 4.2 K and $1.3 \times 10^7$ at 8.3 K. Its electrical Q has been found to be $5.0 \times 10^5$ with an electromechanical coupling efficiency of 15%. This is equivalent to an average $\beta Q$ of $4.7 \times 10^5$. This resonator is suitable for use in working transducers. An effective
annealing procedure was discovered and applied in the fabrication process of these niobium resonators.
CHAPTER 1

INTRODUCTION

1.1 The Foundations of Gravitational Wave Detection

According to Einstein's theory of general relativity [Einstein, 1916] the gravitational field supports wave propagation. These waves carry energy and momentum [Bondi, 1957; Isaacson, 1968] and should propagate with the speed of light.

There are at least two reasons why we want to detect and study gravitational waves. First, only through the experimental detection can this fundamental law of physics be strongly tested. Although the indirect experimental verification of the existence of gravitational radiation has already been made by observing the orbital period decay rate of the binary pulsar PSR 1913+16 [Hulse and Taylor, 1975; Taylor and Weisberg, 1989], direct detection would confirm the existence of gravitational waves and could measure the speed of the propagation of the waves, thus confirming the predictions of the theory of general relativity. Secondly, because gravitational waves are not easily absorbed nor scattered when they travel through matter [Thorne, 1987], the study of the gravitational waves gives unique information that can complement observations made in the electromagnetic spectrum. As one example, since gravitational waves interact weakly with matter, they can carry out more information from the interior of distant astrophysical objects than electromagnetic waves. Hence we should be able to "see" inside "invisible" regions of supernovae, and might be able to verify the existence of black holes and their behavior in highly dynamic circumstances. As another example, a better upper limit of the neutrino mass
can be set if the neutrinos from a nearby supernova are detected in coincidence with a gravity wave signal. A time delay will be evidence of neutrino rest mass.

In the general theory of relativity, a gravitational wave is described by the dimensionless field $h_{jk}$ (j,k=1,2,3) [Thorne, 1987]. For simplicity, if gravitational waves are the only sources of space-time curvature in a Cartesian coordinates grid of a "free falling observer", a displacement $\delta x^j (x^1, x^2, x^3=x, y, z)$ produced by the gravitational waves can simply be related to $h_{jk}$ by the equation of motion

$$\frac{m}{c^2} \frac{d^2 \delta x^j}{dt^2} = \frac{1}{2} m \frac{\partial^2 h_{jk}}{\partial t^2} x^k.$$  

(1.1)

Since a realistic wave is so weak that it must have $\delta x^j \ll x^k$, $x^k$ on the right-hand side can be considered fundamentally constant when equation (1.1) is integrated to get

$$\delta x^j = \frac{1}{2} h_{jk} x^k$$  

(1.2)

or the dimensionless strain of space $h_{jk}$ can be described as

$$\frac{1}{2} h_{jk} = \frac{\delta x^j}{x^k} = \frac{\text{wave induced displacement}}{\text{original displacement from the origin}}.$$  

(1.3)

The gravitational-wave field $h_{jk}$ is transverse and traceless. Since the gravitational waves come from extremely far away, assuming them to propagate along the z direction of the x, y, z Cartesian axes, the transverse character of the waves means that $h_{xx}$, $h_{xy}$ (=h$_{yx}$), and $h_{yy}$ are the only non-zero components of the field, i.e., they all lie within the x-y plane. The traceless character means that $h_{xx} = -h_{yy}$. This implies that gravitational waves produce a quadrupole force field and the waves have only two independent polarization states: the "plus" wave amplitude

$$h_+ \equiv h_{xx} = -h_{yy}$$  

(1.4)
and the "cross" wave amplitude

\[ h_x \equiv h_{xy} = h_{yx} \]  \hspace{1cm} (1.5)

where \( h_x \) can be induced by rotating \( h_+ \) 45° within the x-y plane.

If a gravitational wave propagating along the z axis strikes a rectangular mass in the x-y plane, the quadrupole deformation caused by the "plus" wave amplitude can be illustrated as in Figure 1.1. The tiny displacement expands distance along the x axis and shrinks distance along the y axis simultaneously, while the center of mass experiences no acceleration at all. Assuming the length of the long side of the plane to be \( 2L \), the displacement \( \Delta x(t) \) at the center of the short side can be derived from (1.2) and (1.4)

\[ \Delta x(t) = \frac{1}{2} h_+(t)L. \]  \hspace{1cm} (1.6)

Figure 1.1  Quadrupole deformation of a rectangular mass in the x-y plane caused by gravitational wave \( h_+ \) striking along the z axis.

Gravitational waves are emitted when a system of masses exhibits oscillating quadrupole moment. Therefore highly active astronomical events, such as the explosions of Type II supernovae, coalescing binary stars, non-axisymmetric pulsars,
colliding black holes and gravitational collapse to form black holes, may all play important roles as sources of gravitational radiation. According to Thorne [Thorne, 1990], the strongest waves are likely to be found at frequencies below 10 kHz. In our galaxy, a core collapse of a star to a neutron star at a distance of 10 kpc would produce a strain amplitude of order from $10^{-20}$ to $10^{-18}$. Thorne [Thorne, 1987] estimated the characteristic gravitational wave amplitude as [Appendix A]

$$h_c = 2.7 \times 10^{-20} \left[ \frac{\Delta E_{GW}}{M_\odot c^2} \right]^{1/2} \left[ \frac{1 \text{ kHz}}{f_c} \right]^{1/2} \left[ \frac{10 \text{ Mpc}}{r} \right]$$

where $\Delta E_{GW}$ is total energy radiated in the form of gravitational waves, $M_\odot$ is the solar mass, $f_c$ is the characteristic frequency, $r$ is the distance to the source, and 10 Mpc is the estimated distance to the center of the Virgo Cluster. According to a recent catalog by Fouqué [Fouqué, 1990], the Virgo Cluster contains some 2096 galaxies. Among these, 105 are Sab-Sdm type spiral galaxies that have high supernova rates based on optical studies [Tammann, 1981].

Theoretically, gravitational waves are hard to detect for two reasons. The first reason is that strong gravitational waves are very rare. For example, the average supernova rate in our galaxy is about once every 30 years. The second reason is that the waves interact very weakly with any matter that they pass through, the same reason that makes them useful for astronomy. The technology to detect gravitational waves did not exist until nearly a half century after the General Theory of Relativity was born.

1.2 A Brief Review of Gravitational Wave Experiments

The effort to detect gravitational waves was pioneered by Weber in the early 1960's [Weber, 1960]. He constructed the first resonant-mass antenna. It was made of
a 1.5 ton aluminum right cylindrical bar working at room temperature. Piezoelectric crystal sensors were glued on the bar to measure the strain that was possibly induced by gravitational waves.

A resonant-mass antenna is an amplifier to signals induced by gravitational waves. The smallest signal energy $E_s$ that an amplifier can detect is limited by its minimum noise energy $E_n = \sqrt{S_v S_l}$, where $S_v$ and $S_l$ are the spectral density of the amplifier's voltage noise and current noise, respectively. Under the condition of signal to noise ratio: $\text{SNR} = E_s/E_n = 1$, the energy sensitivity of an amplifier is usually described by its noise temperature. In terms of $S_v$ and $S_l$, the noise temperature $T_n$ is defined as

$$T_n = \frac{\sqrt{S_v S_l}}{k_B}. \quad (1.8)$$

If the incident gravitational waves are short bursts of duration $\tau_g (=1 \text{ ms}, \text{ say})$, for a bar antenna, the noise temperature $T_n$ is related to its corresponding strain sensitivity $h$ by [Pallottino, 1994]

$$h = \frac{L}{v_s^2 \tau_g} \sqrt{\frac{k_B T_n}{M}} \quad (1.9)$$

where $L$ and $M$ are the length and the mass of the bar cylinder, and $v_s$ is the speed of sound in the bar. In aluminum $v_s=5.4 \text{ km/s}$.

Weber attained, at the best, a noise temperature of $T_n=4 \text{ K}$ with his antenna, which corresponded to a strain amplitude of $h=3\times10^{-16}$ [Thorne, 1987]. His claims to have detected gravity waves were not verified by others.

The second-generation antenna was proposed by Fairbank and Hamilton [Fairbank et al., 1970] in 1969. They attempted to improve the sensitivity not only by
suppressing the thermal Brownian noise of the antenna but also by using superconducting electronics. Some of the details they implemented were: (1) introducing cryogenic techniques to make the bar work at 4.2 K; (2) using material of higher quality factor to build the bar [Suzuki et al., 1978]; (3) employing a more sensitive superconducting inductive transducer [Paik, 1975] and a superconducting quantum interference device (SQUID) as current signal amplifier to the transducer; (4) building a better suspension system to support the bar. Innovations were further employed by groups at the University of Rome and the University of Western Australia.

Antennas of the second-generation have reached energy sensitivities about 1000 times higher than that of the first-generation. The sensitivities reported by three different groups around the world are listed in Table 1.1 [Pallottino, 1994; Solomonson et al., 1994; Linthorne et al., 1994]. Coincidence measurements are on

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<td>$h=8\times10^{-19}$</td>
<td>capacitive, 1-mode</td>
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<tr>
<td>Allegro</td>
<td>Louisiana State University (LSU), Baton Rouge, USA</td>
<td>$T_n=6$ mK</td>
<td>$h=6\times10^{-19}$</td>
<td>inductive, 1-mode</td>
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<td>N/A</td>
<td>University of Western Australia Perth, Australia</td>
<td>$T_n=2$ mK</td>
<td>$h=7\times10^{-19}$</td>
<td>microwave parametric, 1-mode</td>
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going among these groups, which make it highly promising to detect the strongest gravitational radiation from our galaxy or neighboring galaxies.

Third-generation antennas are designed to operate at ultra-low temperatures, which further reduces the Brownian noise of the antenna system. They can operate near 50 mK using a dilution refrigerator. This kind of antennas are currently under development in Italy at Rome and Legnaro [Pizzella, 1994] and under construction at Stanford University [Michelson, 1994]. The tentative goal for these detectors is to reach a noise temperature of a few micro Kelvin or a strain amplitude near $10^{-20}$.

Using a spherical shape for a resonant-mass antenna was first suggested by Forward in the early 70's [Forward, 1971]. Further study was carried out by Wagoner and Paik [Wagoner and Paik, 1977] a few years later. Recently, Johnson and Merkowitz [Johnson and Merkowitz, 1993 and 1994] proposed the TIGA configuration, where TIGA stands for Truncated Icosahedral Gravitational wave Antenna. A TIGA has the shape of the C$_{60}$ cluster or a "buckyball", which closely approximates a sphere. Johnson and Merkowitz have done extensive studies to analyze the noise behavior of the TIGA-transducers system and the arrangement of its six transducers, which are used to measure the direction and polarization of detected waves. Zhou [Zhou, 1994] discussed a different way of arranging five transducers through the mode analysis of an elastic sphere. He also studied its energy cross section and the solution to the inverse problem.

A spherical gravitational wave antenna would offer real hope to gravitational wave astronomy because of the following advantages. (1) A spherical antenna has 40 to 60 times more energy cross section than a bar antenna of the same material and the same frequency, which will directly improve the sensitivity of the antenna. (2) The five lowest order degenerate quadrupole modes can be reconstructed from five or six
transducers mounted at the right places on the spherical surface. This makes it possible to determine the direction as well as the polarization of the wave source. It is also equally sensitive to waves from any direction. (3) A TIGA detector can be made for relatively little cost. It is estimated that a fourth generation antenna made of a sphere may reach a strain sensitivity of $h=10^{-22}$ in the frequency range of 1-2 kHz. This is comparable to that of the first generation LIGO (Laser Interferometer Gravitational-Wave Observatory).

The interferometer detector uses an entirely different approach to measure the displacement caused by gravitational waves. In principle, an interferometer consists of two test masses with reflecting mirrors placed at the ends of two arms to form an "L" shape. Gravitational waves will produce displacements of the two masses with respect to the central beam splitter. The resulting length difference in the two arms is sensed by combining the beams as a Michelson or Fabry-Perot interferometer. Interferometer detectors have been studied for over 20 years [Forward, 1978]. Currently, two full size interferometer projects, the LIGO of the US [Weiss, 1994] and the VIRGO of the Italy-France collaboration [Giazotto, 1994] are expected to make their first operation before the end of this century. The VIRGO project will build only one detector with an arm-length of 3 km, while the LIGO project will build two at different sites separated by 3000 km and each with an arm-length of 4 km. The LIGO project has already started construction. The final goal of its advanced model is to reach strain sensitivity around $10^{-23}$. This sensitivity should allow detection of several gravitational-wave events per year from sources as far as the Virgo Cluster.

However, for the frequency range from 1 to 2 kHz, TIGA is significantly more sensitive in any orientation than is the first generation LIGO detector in its most favorable orientation. While LIGO has lower strain noise from several hundred Hz down to around 100 Hz with wider bandwidth [Abramovici et al., 1992], the TIGA
detector is more sensitive above 1 kHz making the two different type detectors complementary to each other.

1.3 Improving the Sensitivity of Resonant-mass Detectors

The detection of gravitational radiation using cryogenic resonant-mass antennas pushes the limits of precision measurement technology. Such a gravitational wave detector usually consists of three major parts: (1) a resonant mass antenna that weighs a few thousand kilograms and is cooled either to 4.2 K or to an ultra-low temperature of ~50 mK; (2) a transducer that converts very small mechanical vibrations generated in the antenna into electromagnetic signals; and (3) a SQUID that preamplifies the extremely small signals so that conventional electronics working at room temperature can amplify the signals further.

The improvement of gravitational wave detectors is a challenging task. Since astrophysical events producing large strains occur infrequently in our own galaxy, a detector with higher sensitivity is always desired for gravitational wave astronomy. It is clear that any of the three parts forming the detector can be the limiting noise source for the entire system. Considering the antenna solely, its noise temperature $T_{an}$ can be defined from the narrow band Brownian noise energy $E_b$ [Paik, 1974; Solomonson, 1990] as

$$T_{an} = \frac{E_b}{k_B} = 4T \frac{\omega_a \tau_s}{Q_a}.$$  \hspace{1cm} (1.10)

For Allegro with a bar temperature of $T=4.2$ K, a lower limit to the practical sampling time of $\tau_s=1$ ms, an antenna angular frequency of $\omega_a=2\pi(913$ Hz), and the mechanical quality factor of $Q_a=12\times10^6$ [Solomonson et al., 1994], the equivalent Brownian noise


temperature of the antenna is $T_{an}=8 \mu K$. This means that, as long as Allegro is operating at 4.2 K, its overall noise temperature will never get below 8 $\mu K$.

The 6 mK noise temperature of Allegro is far from this limit and is currently dominated by the noise of the commercial dc SQUID. Replacing the current 50,000 $h$ SQUID (the energy sensitivity of a SQUID has the unit of J/Hz, and is usually quoted in units of $h=h/2\pi$) with a 500 $h$ one would reduce the noise temperature of the antenna to 0.8 mK, if there are no other contributions to the noise. To improve the sensitivity of Allegro further, we need to build a multi-mode transducer that will match to the antenna even better. With a 500 $h$ SQUID and a three-mode transducer we can reduce the noise temperature of the antenna to 170 $\mu K$. Beyond this, the bar will have to be cooled to a much lower temperature. If it is cooled from 4.2 K to 50 mK, for instance, not only will the thermal noise of bar be reduced by 80 times, but also that of the transducer and SQUID, since all of them would be operating at the same temperature.

For an ultra-low temperature antenna working at 50 mK, such as the one currently under construction at Stanford University and the ones being tested at Rome and Legnaro, a dc SQUID that works well at 50 mK in vacuum is not automatically guaranteed. It has been found that some SQUIDs cease to operate below 4 K. In Chapter 2 we will discuss the experiments to test a new commercial dc SQUID manufactured by Quantum Design [11578 Sorrento Valley Road, Suite 30, San Diego, CA 92121] at ultra-low temperature. We discuss in detail some of the special problems that arise when a SQUID is used on a gravitational wave antenna at 50 mK in vacuum. We also suggest ways to solve these problems, such as wiring the SQUID leads with good heat sinks and shielding against electromagnetic interference (EMI) on the dilution refrigerator. In the second half of Chapter 2 we show the results of testing a dc SQUID on both conventional probes and the dilution refrigerator. Our
experiments show that: (1) the Quantum Design SQUID has at least an order of magnitude higher energy sensitivity at 4 K than the SQUID currently used on our antenna; (2) this SQUID operates well down to 50 mK with an energy resolution of \( \sim 2600 \, \hbar \). Therefore this SQUID is suitable to work on the Stanford 50 mK antenna. It could also be used on other experiments done at ultra-low temperatures. However the testing showed that the SQUID's noise did not fall with temperature as expected. The possible reasons for this also are discussed.

Among the groups around the world using resonant-mass antennas, capacitive and inductive transducers are the two types of transducers being used. Both types utilize one or more mechanical resonators. A good transducer is mainly defined by its sensitivity and bandwidth. Higher sensitivity enables a weaker signal to be detected. It requires that the transducer must have a high mechanical and electrical quality factor (low losses). A good impedance matching capability between the transducer and SQUID is also necessary. Wider bandwidth ensures more information to reveal the structure of the source. It requires a good mechanical coupling between the antenna mass and the transducer mass. It also demands a good mechanical-electrical coupling between the transducer and its pickup coil.

The superconducting inductive transducer originally developed by Paik [Paik, 1974 and 1976] has the advantage of providing a low output impedance which nearly matches the input impedance of the SQUID. The same type of transducer with a single resonant mode has been developed by Solomonson, and runs currently on Allegro to form a two-mode detector system.

To improve the bandwidth and sensitivity, Richard [Richard, 1979 and 1984] proposed the multimode system. An N-mode system is constructed by an arbitrary number N of coupled harmonic oscillators. These oscillators are made of geometically decreasing masses that have comparably high mechanical Q's.
(and heaviest) oscillator is the antenna. The remaining N-1 lighter oscillators form an (N-1)-mode transducer. There are many discussions about the optimal design of multimode detectors. Through his analysis, Richard has concluded that an N-mode detector would have a bandwidth (single sided) of

$$\Delta f = f_0 \sqrt{\mu} \quad (\mu<1)$$  \hfill (1.11)

where $f_0$ is the mode frequency and $\mu=M_{i+1}/M_i$ (i=1 to N-1) is the successive mass ratio. Through a theoretical analysis, Price [Price, 1987] shows that the bandwidth increases rapidly as the number of modes goes from one to three, but that there is little additional improvement as the number of modes increases further. Thus, in most cases, there is no need to use more than three modes. Bassan [Bassan, 1986] points out that three-mode detectors can perform better than two-mode detectors, while little or no gain is expected by increasing the number N beyond 4.

Previously developed small mass resonators include a 3.6 gram niobium resonator for a three-mode system [Folkner, 1987], a 10 gram resonator for two-mode system [Folkner et al., 1989], and a 0.6 gram (effective mass) first ever niobium-silicon thin-film transducer [Stevenson, 1991].

The simulation done by Solomonson suggested that, for the LSU two-mode detector, a three-mode system appears to be the most practical next step. The reasons are: (1) the noise temperature of the system can be greatly reduced by a factor of 50 to 0.1 mK with a 1000 $\mu$ SQUID; (2) the bandwidth will be increased by a factor of 3 to 70 Hz; (3) within a factor of 2 of its optimal value, the mass variations of the intermediate resonator only increases the noise temperature slightly; (4) within a 30 Hz range, tuning either of the two resonant frequencies of the two-mode transducer in a three-mode system only increases the noise temperature by a few percent; (5) a three-mode system with a final mass of a few grams is practical to make.
In Chapter 3, we discuss the design of the three-mode system. First we will briefly review the principle of an inductive transducer. Then we analyze the relations among the resonant modes of a three-mode system. In the design of the two-mode transducer, we employ the method of Michelson and Taber [Michelson and Taber, 1981] and use the parameters of SQUIDs that are either available now or will be available in the near future. We analyze the noise performance of a two-mode transducer on the LSU gravitational wave antenna and build the mathematical model of the antenna-transducer-SQUID system for noise analysis. This model will be used to optimize the transducer mass. Based on this mass, we apply finite element analysis to aid the design of the uniquely shaped small mass resonator and the intermediate resonator. The main purpose for modeling the resonators using the finite element program is to ensure the decoupling of other mechanical modes from the transducer mode.

In practice, a transducer is usually evaluated by measuring the quality factors of its mechanical resonator(s) and its electrical circuit. In Chapter 4 we discuss the fabrication technologies developed in the process for achieving high quality factors (Qs). Two aluminum and three niobium small mass resonators were fabricated. The results of testing these resonators are reported. Through our tests we show that the geometry we developed permits a larger electro-mechanical coupling factor (0.15) with high electrical Q (5x10^5). The projected performances of the three-mode system made with the best tested resonator are provided. With a 500 \( h \) SQUID at 4.2 K, the three-mode system will have a noise temperature of 170 \( \mu K \) or corresponding strain sensitivity of 1.0x10^{-19}. With a 10 \( h \) SQUID at 50 mK, a noise temperature of 2.4 \( \mu K \) or a strain sensitivity 1.2x10^{-20} is expected.
CHAPTER 2

TEST OF DC SQUIDS AT ULTRA-LOW TEMPERATURE

For an ultra-low temperature antenna which works at 50 mK, a dc SQUID that works well in vacuum is not automatically guaranteed. Some of the previous SQUIDs have been found to cease operation below 4 K due to problems of their junction shunt resistors or junction materials. In this chapter we will discuss the experiments of testing a commercial dc SQUID at ultra-low temperature. This discussion is based on our published paper [Geng et al., 1993] in Review of Scientific Instruments (used by permission – see Appendix B). We first discuss how a dc SQUID operates and what noise properties it has when an input circuit is coupled. Then we discuss in detail some special problems that arise when a SQUID is used on a gravitational wave antenna at 50 mK in vacuum. We also discuss the ways to solve them. In the second half of this chapter we provide the results of testing a dc SQUID with conventional probes and in a dilution refrigerator. Our experiments show that the SQUID made by Quantum Design [Quantum Design, 11578 Sorrento Valley Road, Suite 30, San Diego, CA 92121] is suitable to work on the Stanford 50 mK antenna. It could also be used on other ultra-low temperature experiments. However the SQUID's noise did not fall with temperature as expected. Possible reasons for this are discussed.

2.1 Operation and Noise Properties of a dc SQUID

A Superconducting QUantum Interference Device (SQUID) is a low temperature electromagnetic device used to amplify very small signals. Among the two types of SQUIDs: the rf SQUID and the dc SQUID, the white noise level of a dc
SQUID is typically an order of magnitude lower than that of a rf SQUID. The dc SQUID is the most sensitive low-frequency amplifier for low impedance sources.

A dc SQUID with an input coil $L_i$ is schematically shown in Figure 2.1. The dc SQUID is so named because it is biased with direct current (dc). Structurally, it is a superconducting loop interrupted by two Josephson tunnel junctions, represented by symbol $\times$ in the figure. A Josephson junction consists of two superconductors separated by a thin insulating barrier which allows Cooper pairs of superconducting electrons to tunnel through. The impedance of the dc SQUID depends on the magnetic flux in the superconducting loop.

![Schematic diagram](image1)

**Figure 2.1** Simplified dc SQUID schematic diagram and its input-output characteristic.

To use a dc SQUID with a gravitational wave antenna, the pickup coil $L_0$ of the transducer is electrically connected to the input coil $L_i$ as shown in Figure 2.1 (here we omit the superconducting transformer in between for simplicity). The input coil is inductively coupled to the superconducting loop of the SQUID. The signal current through the input coil produces a flux threading the SQUID loop. The SQUID
converts this flux to a voltage which is further amplified by room temperature electronics.

The basic operation of the device is determined mainly by the ac Josephson effect and ordinary circuit equations. They lead to a set of non-linear equations that usually must be solved numerically. We will discuss this later. The result is that the output voltage is periodic in the external magnetic flux $\Phi$ threading the SQUID loop, with a period given by the flux quantum $\Phi_0 = h/2e = 2.07 \times 10^{-15}$ Weber. With the bias current $I_b$ fixed at an appropriate value, the $V-\Phi$ curve that represents the input-output relation is similar to that shown in Figure 2.1.

The input current $I_i$, generated by the pickup coil $L_0$ of the transducer, flows through $L_i$ and produces a flux $\Phi = M_i I_i$ in the SQUID loop, where $M_i$ is the mutual inductance between the input coil and the SQUID loop. The maximum transfer function $V_{\Phi} = \left| \frac{\partial V}{\partial \Phi} \right|_{I_b}$ is reached by setting the operating point on the steep part of the V-\Phi curve. This is done by adjusting the amount of flux sent through the modulation/feedback coil.

A typical block diagram of the dc SQUID electronics [Clarke et al., 1976] is shown in Figure 2.2. The SQUID is essentially a null detector in this case, since the negative feedback circuit generates a flux $-\delta \Phi$ through the modulation/feedback coil so that the total flux in the SQUID loop remains at a constant value. A change in flux $\delta \Phi$ as small as $10^{-6} \Phi_0$ can be detected. The output voltage $V_{\text{out}}$ is proportional to the external flux applied to the SQUID. The transformer at the input of the low noise amplifier is made to optimize the coupling between the SQUID output impedance of a few ohms and the input impedance of the low noise JFET pre-amplifier of a few kiloohms. While the SQUID is modulated by a constant flux through its modulation/feedback coil, the output voltage from the SQUID is synchronously detected at the modulation frequency.
To analyze the performance of a dc SQUID, the Resistively Shunted Junction (RSJ) model [Stewart, 1968 and McCumber, 1968] is often applied. Figure 2.3 shows a typical configuration of a symmetric dc SQUID in the RSJ model. In that figure, $R$, $C$, $I_c$, are the junction shunt resistance, capacitance, and critical current, respectively. $I_{n1}$ and $I_{n2}$ are statistically independent thermal noise currents of the two shunt

Figure 2.2  Block diagram of a dc SQUID and its electronics.

Figure 2.3  The Resistively Shunted Junction (RSJ) model of a dc SQUID.
resistors, each with a current spectral density $4k_bT/R$. $L$ is the loop inductance of the SQUID.

For each Josephson junction, the current through the junction $I_J$ is related to the phase difference of the junction $\theta_J$ through the current-phase relation

$$I_J = I_c \sin \theta_J$$

(2.1)

where $I_c$ is the critical current, the maximum supercurrent that the junction can sustain. For $I_J > I_c$, a voltage $V_J$ is associated to the time derivative of $\theta_J$ through the voltage-frequency relation:

$$\dot{\theta}_J = \frac{2\pi}{\Phi_0} V_J.$$  

(2.2)

The macroscopic superconducting state of the SQUID loop is characterized by the phase differences $\theta_1(t)$ and $\theta_2(t)$ across each junction. For the two Josephson junctions, the equations of motion based on RSJ model are [Stewart, 1968 and McCumber, 1968]

$$\frac{\Phi_0}{2\pi} \dot{\theta}_1 + \frac{\Phi_0}{2\pi R} \dot{\theta}_1 + I_c \sin \theta_1 + I_n_1 = \frac{I_b}{2} - J$$

(2.3)

$$\frac{\Phi_0}{2\pi} \dot{\theta}_2 + \frac{\Phi_0}{2\pi R} \dot{\theta}_2 + I_c \sin \theta_2 + I_n_2 = \frac{I_b}{2} + J$$

(2.4)

where the terms $I_c \sin \theta_1$ and $I_c \sin \theta_2$ determine the basic properties of the Josephson junction. Equations (2.3) and (2.4) are clearly coupled through the loop circulating current $J$ of the SQUID. Based on the quantization of magnetic flux, the phase difference $\theta_1 - \theta_2$ across the two junction and the circulating current $J$ are related as

$$\frac{\Phi_0}{2\pi} (\theta_1 - \theta_2) = \Phi + LJ.$$  

(2.5)
Using equation (2.2) and the circuit equation, the voltage and the average changing rate of the two phases are related as

\[ V = \frac{\Phi_0}{4\pi} \left( \dot{\theta}_1 + \dot{\theta}_2 \right). \]  

Equations (2.3) through (2.6) determine the performance of a dc SQUID. However, the nonlinear second-order differential-equations (2.3) and (2.4) can only be solved numerically.

By assuming \( C=0 \), the \( \dot{\theta}_1 \) and \( \dot{\theta}_2 \) terms are dropped from equations (2.3) and (2.4), Tesche and Clarke [Tesche and Clarke, 1977 and 1979] solved these equations numerically. They computed the current-voltage (I-V) characteristic as functions of the applied flux \( \Phi \), SQUID loop inductance \( L \), junction critical current \( I_c \), and shunt resistance \( R \). With parameters set at experimentally interesting values, four major conclusions are obtained.

1. The behavior of the dc SQUID is relatively insensitive to quite large asymmetries in the inductance of the two arms, the critical currents, or shunt resistance of the two junctions.

2. The transfer function \( V_\Phi \) peaks smoothly as a function of bias current \( I_b \) for a given \( \Phi \) and exhibits a shallow maximum around \( \Phi=(2n+1)\Phi_0/4 \).

3. Under the following conditions:
   a. the reduced inductance (sometimes it is called modulation parameter [Ketchen, 1981]): \( \beta = 2 LI_c / \Phi_0 = 1 \);
   b. noise parameter: \( \Gamma = 2\pi k_b T / I_c \Phi_0 = 0.05 \);
   c. applied flux: \( \Phi = (2n+1)\Phi_0/4 \);
   d. frequency \( f \) well below (say, \( f < f_J/10 \)) the Josephson frequency:
      \( f_J = V/\Phi_0 = (4.836 \times 10^{14}) V \) Hz;
(e) the optimum current bias \( I_b \) that makes the transfer function a maximum: \( V_{\Phi} = R/L \), the voltage power spectral density \( S_V(f) \), the circulating current power spectral density \( S_J(f) \) and the real part of the voltage-current correlation spectral density \( S_{VJ}(f) \) are all white (all in single sided forms)

\[
S_V(f) \equiv V_n^2(f) = 16k_BT \quad (2.7)
\]

\[
S_J(f) \equiv J_n^2(f) = 11k_BT/R \quad (2.8)
\]

\[
S_{VJ}(f) \equiv \text{Re}(V_n(f)J_n(f)) = 12k_BT. \quad (2.9)
\]

At fixed current bias \( I_b \), the noise voltage \( V_n(t) \) appears across the output of the SQUID as shown in Figure 2.4. Through the transfer function \( V_{\Phi} \), \( V_n(t) \) can be characterized by the equivalent noise flux \( \Phi_n(t) \) in the SQUID loop. The relation between their spectral densities is

\[
S_{\Phi}(f) = \Phi_n^2(f) = \frac{S_V(f)}{V_{\Phi}^2}.
\]

Figure 2.4 A dc SQUID coupled to an input circuit.
The noise current $J_n(t)$ is generated around the SQUID loop. It in turn generates the noise flux $\Phi_n(t)$ shown in Figure 2.4. The flux noise $\Phi_n(f)$ (in unit of $\Phi_0 / \sqrt{\text{Hz}}$) sets a limit for the lowest applied signal flux change $\delta \Phi$ that can be detected by a SQUID. The flux noise due to $J_n$ induces noise into any input circuit coupled to the SQUID loop. Hence an equivalent noise voltage $V_{in}(t) \equiv M_i \frac{dJ_n(t)}{dt}$ is generated by $J_n(t)$ at the input side of the SQUID as depicted in Figure 2.4. This is done through the mutual inductance $M_i \equiv \kappa \sqrt{L_i}$ with coupling coefficient $\kappa$ between the SQUID loop inductance $L$ and SQUID input coil $L_i$. $V_{in}(t)$ has a spectral density as the function of frequency $f$ [Clarke et al., 1979]

$$S_{V_{in}}(f) = (2\pi f)^2 M_i^2 S_f(f). \tag{2.10}$$

The effect of $V_{in}$ is to induce noise in the transducer due to the noise current $J_n$.

(4) With the help of the transfer function and voltage spectral density under the same conditions mentioned at the beginning of (3), the minimum intrinsic noise energy is also white in unit of energy per Hz

$$\varepsilon \approx \frac{9k_B T L}{R} = 16k_B T \sqrt{\frac{L C}{\beta_c}} \tag{2.11}$$

where $\beta_c \equiv 2\pi I_c R^2 C / \Phi_0$ is the hysteresis parameter (or Stewart-McCumber parameter). Usually one makes $\beta_c \leq 1$ for the overdamped case. The intrinsic noise energy is often used to compare different SQUIDs. It is clear that a low noise dc SQUID must have a small junction capacitance $C$, a small loop inductance $L$, proper damping of the junctions, and must operate at the lowest possible temperature $T$. However, further reduction of the junction capacitance is limited practically by the parasitic capacitance effects due to the signal coupling circuits fabricated closely to the SQUID loop.
The most fair indication of SQUID noise is called coupled energy resolution (or sensitivity) $\varepsilon_c$ [Claassen, 1975]. If one assumes that an efficient transformer can be used to match the signal source inductance and SQUID input coil inductance $L_j$ [Clarke et al., 1976], $\varepsilon_c$ is defined by

$$
\varepsilon_c = \frac{\varepsilon}{\kappa^2} = \frac{S_\Phi}{2\kappa^2 L} = \frac{L_i}{2} \left( \frac{\Phi_n}{M_j} \right)^2.
$$

(2.12)

$\kappa$ is the coupling coefficient between the SQUID loop inductance $L$ and SQUID input coil $L_i$. In practice, $V_n(f)$ and the transfer function $V_\Phi$ can be measured directly using the electronics shown in Figure 2.2. $\Phi_n(f) = V_n(f)/V_\Phi$ can be calculated directly. If $L_i$ and $M_j$ are known, $\varepsilon_c$ can also be calculated. $I_{in}(f) = \sqrt{\frac{2\varepsilon_c}{L_i}}$ is an equivalent noise current in parallel with the input coil of the SQUID as illustrated in Figure 2.4. Unlike the equivalent voltage noise $V_n(f) \sim f$, $I_{in}(f)$ appears to be a white noise.

### 2.2 Requirements for a SQUID to Work at Ultra-low Temperature

There are many reports in the literature [Awschalom et al., 1988, Wakai et al., 1988, and Wellstood et al., 1989] about SQUIDs with extremely low noise. Those SQUIDs have energy resolutions much higher than any of the SQUIDs currently used on the gravitational wave antennas around the world. From equation (2.11) it is clear that in principle smaller energy resolution can be achieved by decreasing both the SQUID loop inductance and junction capacitance. For example, Wellstood [Wellstood et al., 1989] designed and built certain types of SQUIDs which had a very small energy resolution at 4 K. As they were cooled, their energy resolution was reduced further, in proportion to temperature, down to at least $\sim 100$ mK, where they were only one order of magnitude away from $\hbar$. $\hbar$
is the quantum limit for a linear amplifier [Koch et al., 1981]. As another example, Awschalom [Awschalom et al., 1988] demonstrated a nearly quantum limited dc SQUID with an energy resolution of $1.7 \hbar$ when it was operated open-loop in a small-signal amplifier mode at $T=290$ mK.

Unfortunately there are many additional requirements on a SQUID for use on a gravitational wave detector. It must have an input coil of 1-2 microhenries that is strongly coupled to the SQUID loop. It must be very robust and reliable when faced with the rigors of thermal cycling, mechanical shock, and humidity. It must have considerable dynamic range, which requires a good flux-locking circuit, in order to handle large interfering inputs at frequencies far below the signal frequency. It must have extensive shielding against electro-magnetic interference (EMI), even though the large size of the system makes that difficult to provide. The SQUID must be operable in a vacuum, with no direct liquid cooling, since it is mounted directly on the end of the gravitational wave antenna, which is in a vacuum. Finally, it must work with rather long (>3 meters) electrical leads to room temperature.

Commercial SQUID manufacturers devote significant effort to several of these requirements. Unfortunately, all previous commercial SQUIDs have been found to cease operation at temperatures somewhere below 4 K, usually because the junction shunt resistors change value or become superconducting, or because the junction material changes its electrical properties.

In the next section we describe a test of a relatively new commercial SQUID that has lower noise and continues working well far below 1 K: a dc SQUID sensor manufactured by Quantum Design. The sensor is a Model 50 dc thin film SQUID along with its Model 500 microPreamp and Model 5000 SQUID controller. Most of the elements of this sensor are thin film structures fabricated
on a single silicon chip. Other experiments done at ultra-low temperatures might also benefit from such a SQUID.

2.3 Experimental Setup for Ultra-low Temperature Measurement

With a conventional probe or mounting configuration, SQUIDs are cooled by immersion in a liquid helium bath, shielded by a tight-fitting superconducting casing, and connected to the room-temperature preamplifier by short lengths of well-shielded twisted pairs of wires.

Use of a SQUID on a dilution refrigerator is, however, complicated by three factors: cooling, shielding, and heat load. We need a cooling method so that the SQUID can operate in vacuum. We must provide efficient electrical shielding of the SQUID wiring, which is extremely sensitive to EMI. The system must also avoid conducting too much heat to the mixing chamber of the dilution refrigerator.

Our strategy to solve these problems was to use low thermal conductivity electrical wiring, with several intermediate heat sinks, all the way down to the lowest temperature heat sink, then use heavy copper wiring to the SQUID terminals, to provide both a cooling path and electrical connections. A more direct thermal contact to the SQUID chip was impossible because it is encapsulated in a fiberglass box. The general arrangement of the system is shown schematically in Figure 2.5, and in more detail in Figure 2.6.

From room temperature to 4 K, we used phosphor-bronze twisted-pair wiring (36 AWG), except for the signal twisted-pair, which was copper for low resistance (3.0 Ω total series resistance) and hence low thermal noise. Between 4
Figure 2.5  Schematic wiring for measuring the dc SQUID on dilution refrigerator.
K and the mixing chamber, all the wiring was phosphor-bronze, which added 6 Ω total series resistance to the signal pair. From the mixing chamber to the LEMO connector [# EGG.1B.310.ZNL from LEMO USA, P. O. Box 11488, Santa Rosa, CA 95406] which mates with the SQUID, all the wiring was 32 AWG copper. The shield tubes
were seamless; above the 4 K plate they were Cu-Ni with a 0.050" OD and 0.037" ID; below the 4 K plate they were Pb with 0.060" OD and .030" ID. Even though the Pb tubing was superconducting, it was found experimentally to conduct sufficient heat to the mixing chamber to raise its lowest temperature from 22 mK to 50 mK.

The heat sink was a thin chip of polycrystalline beryllium oxide obtained from Lake Shore Cryotronics [Lake Shore Cryotronics, Inc., 64 East Walnut St., Westerville, OH 43081-2399]. BeO is a good electrical insulator, but is known to be a good thermal conductor above 4 K. It was metalized on the bottom side, which was soldered into a milled cavity in an OFE (Oxygen free, electronic grade) copper heat sink junction box, shown in Figure 2.6(b). The solder used has the composition of 60%Bi-40%Cd. It remains a normal metal, hence a good thermal conductor, down to at least 0.8 K [Cochran et al., 1956], and so we hoped it would remain normal at even lower temperatures. Each wire was soldered onto one of the two isolated metal solder pads on the top of the chip, using low melting point (155 C) solder Indalloy #2 (80%In-15%Pb-5%Ag) from Indium Corporation of America [Indium Corporation of America, 1676 Lincoln Ave., Utica, NY 13502]. The shield tubes were clamped where they entered the box. Each heat sink box was screwed to a copper attachment plate of the dilution refrigerator, and copper loaded Cry-Con thermal conductive grease also from Lake Shore Cryotronics was used between the pieces to improve thermal contact. The niobium casing holding the SQUID sensor was bolted to the bottom of the mixing chamber, also with the Cry-Con thermal conductive grease between the surfaces.

All tests were done on an Oxford Instruments' Kelvinox dilution refrigerator, which has a cooling power of 20 µW at 100 mK and a no-heat-load (or base) temperature of 22 mK at the mixing chamber.
The temperature of the mixing chamber was measured by a thick film chip resistor [Li et al., 1986], attached with epoxy (Emerson and Cumings 2850FT). In an earlier run, the resistor's calibration curve was checked at 22 mK by comparison with a $^{60}$Co nuclear orientation thermometer. The resistance was measured with a low power 4-terminal bridge. There was no way to confirm the physical temperature of the SQUID chip because the encapsulation made it inaccessible.

For most measurements, the preamplifier and SQUID controller from Quantum Design was used for preamplification, demodulation and flux-locking of the SQUID. All measurements were made in the flux locked mode and the input coil of the SQUID shorted by a superconducting wire. The noise spectral density at the output was measured by using an HP3561A Dynamic Signal Analyzer. The transfer function $V_\Phi$ between flux induced in the SQUID loop and output voltage was measured by (1) unlocking the SQUID and counting flux quanta as the dc current in the modulation coil was varied; (2) locking the SQUID loop and measuring the dc voltage change while the dc current in the modulation coil was varied; (3) dividing measurements (2) by (1). The statistical uncertainty for the measurements was about $0.03 \mu \Phi_0/\sqrt{\text{Hz}}$, where $\Phi_0$ is the flux quantum.

We concentrated our measurements on noise performance near 1 kHz, since that frequency region is most important for gravity wave detection. For selected temperatures between the base temperature of the dilution refrigerator and 4.2 K, the equivalent flux noise of the SQUID was measured. The temperatures below 1 K were controlled by a heater attached to the mixing chamber plate. Temperatures above 1K were measured while the refrigerator was slowly warming up. We found that the bias current needed for lowest noise changed very little (<11%) between the highest and lowest temperatures.
Microphonics is the unwanted electromagnetic response to vibration. It turned out to be a serious problem during the early measurements. During those measurements, cables (serial number: AS 633-2SS) made by Cooner Wire [Cooner Wire, 9186 Independence, Chatsworth, CA 91311] were used for all SQUID leads from the feedthrough to the mixing chamber. Those cables were made of stainless steel stranded conductors and braided shield. Evidently the dilution refrigerator vibrates several orders of magnitude more than an ordinary liquid helium dewar. The large size of the microphonics was a surprise, because we had expected the thick niobium casing to provide sufficient magnetic shielding.

Two measures were taken to reduce the microphonic effects: Teflon and masking tapes were wrapped around the SQUID encapsulation to provide a snug fit between it and the niobium case, so that relative motion would be suppressed; and magnetic shielding material was added to the outside of the vacuum can (in the liquid helium bath) to reduce the ambient magnetic field. The magnetic shield was a spot-welded cylinder of CO-NETIC AA shielding alloy from Perfection Mica Company [Perfection Mica Company, 740 North Thomas Drive, Bensenville, IL 60106], 0.014" thick, and closed at one end. We estimate that the ambient field (mainly the earth’s field) was reduced by roughly a factor of 35. An extra two layers of the same shielding material was wrapped around this cylinder for some additional shielding effect.

2.4 Test Results with a Conventional Probe and in a Dilution Refrigerator

Most of the tests have been performed on a single sensor (serial number A5.6) manufactured by Quantum Design. The parameters specified by the manufacturer are given in Table 2.1.
Table 2.1 Specifications of Quantum Design dc SQUID

<table>
<thead>
<tr>
<th>Specification</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bias current</td>
<td>28 µA</td>
</tr>
<tr>
<td>Signal magnitude</td>
<td>33 µV</td>
</tr>
<tr>
<td>Modulation coil coupling</td>
<td>1.49 µA/Φ₀</td>
</tr>
<tr>
<td>Input coil coupling</td>
<td>0.20 µA/Φ₀</td>
</tr>
<tr>
<td>Input coil inductance</td>
<td>1.85 µH</td>
</tr>
<tr>
<td>1/f noise corner frequency</td>
<td>0.5 Hz</td>
</tr>
<tr>
<td>Noise at 100 Hz without load</td>
<td>3.44 µΦ₀/√Hz</td>
</tr>
<tr>
<td>Noise at 100 Hz with 1.06 µH load</td>
<td>3.49 µΦ₀/√Hz</td>
</tr>
<tr>
<td>Manufacture date</td>
<td>02/08/1992</td>
</tr>
</tbody>
</table>

Based on Table 2.1, the mutual inductance $M_i$ between SQUID loop inductance $L$ and input coil inductance $L_i$ can be estimated from the given input coil coupling of $0.20 \, \mu\text{A}/\Phi_0$

$$M_i = \frac{1}{0.20 \, \mu\text{A} / \Phi_0} \approx 10.3 \, \text{nH}.$$  

This SQUID has survived roughly 30 thermal cycles and several months of exposure to Louisiana humidity without noticeable degradation, which is a reasonable minimum requirement for robustness.

For comparison with the results at lower temperatures, we first made noise tests of the dc SQUID sensor in a conventional probe which is similar to commercial probes, but is of our own design. It was mounted in an ordinary glass dewar, where we could vary the temperature of the liquid helium between 1.9 and 4.2 K. The results for this probe, using the commercial preamplifier and controller made by Quantum...
Design, are shown as curve #1 in Figure 2.7. It shows both flux noise $\Phi_n$ referred to the SQUID loop and the coupled energy resolution that is given by equation (2.12)

$$\varepsilon_c \equiv \frac{S_{\Phi}}{2\kappa^2 L} = \frac{L_j}{2} \left( \frac{\Phi_n}{M_i} \right)^2$$

where $L$ is the SQUID loop inductance, $M_i$ is the mutual inductance between $L$ and $L_i$, and $\kappa$ is the coupling coefficient of $M_i$. The manufacturer has specified that these particular sensors have $L_i=1.85 \mu H$ and we have further calculated $M_i=10.3$ nH. In Figure 2.7, we quote the energy resolution or energy sensitivity in units of $\hbar (=1.055 \times 10^{-34} J Hz^{-1})$.

The final results of the flux-locked noise measurements, as a function of temperature, are shown in Figure 2.7. The conventional probe results, using the manufacturer's electronics, are labeled curve #1. The results on the dilution refrigerator, still using the manufacturer's electronics, are labeled curve #2. A special measurement, using the probe of curve #1, but substituting "Berkeley" electronics [Clarke et al., 1976], is shown in curve #3.

For all the measurements reported, the terminals of the input coil were short-circuited with niobium wire, since in the few cases tested, leaving them open-circuited was found to increase the flux noise by 5 to 20%.

One detail was found to have a significant effect on the noise: the termination of the heater circuit provided by the manufacturer to heat the SQUID and drive off flux trapped in the junctions. The heater is used on some probes, but not ours. Early measurements were made with an open circuit for the heater terminal on the preamplifier, until we discovered that shorting that terminal to ground lowered the flux noise "floor" by 30%, and also eliminated a noise peak that drifted between 400 and 800 Hz.
Figure 2.7 Flux noise and energy resolution of the dc SQUID at 1 kHz vs. temperature.

Curve #1: conventional probe, using the manufacturer's electronics;
Curve #2: dilution refrigerator, using the manufacturer's electronics;
Curve #3: conventional probe, using "Berkeley" electronics.
The flux noise measured in our probe, $3.10 \mu\Phi_0/\sqrt{\text{Hz}}$, was 11% lower than the value listed on the manufacturer's specification sheet for this particular sensor, and was also 18% lower than the noise we measured for this same sensor in one of the manufacturer's probes.

The two measures taken to reduce the microphonic effects mentioned at the end of section 2.3, plus shorting the heater terminal, cleaned up the noise spectrum considerably. Before-and-after flux noise spectra are shown plotted in Figure 2.8. The total reduction in the noise floor is about 50% (about 15% of which was due to acquisition of a better sensor.)

2.5 Discussion of the Results

Our first goal of testing, lower noise at 4 K, was met. Under the same conditions (conventional probe, 4.2 K, shorted input), the energy resolution $\varepsilon_c$ of this Quantum Design SQUID has shown itself to be better by at least a factor of 5 than the BTi SQUID that we are currently using on our antenna.

Our second goal, operation down to 50 mK, was met. This is the first commercial SQUID known to us which continues to operate well at all temperatures down to at least 50 mK. Thus this SQUID can be used on the new generation of ultra-low temperature gravitational wave antennas, such as the antenna under construction at Stanford University. It is worthwhile to mention that during a preliminary run, the SQUID would work well even down to 20 mK, but low noise measurements could not be made probably due to the microphonic noise generated by the braided stainless steel cables and the high resistance in the signal cable.

Our third goal, much lower noise at temperatures far below 4 K, was not met. Possible reasons for this are outlined below.
The noise, expressed as energy resolution $\varepsilon_c$, is well described by a term proportional to temperature, which we quantify with the slope $\alpha$, and by a temperature independent term, quantified by the extrapolation to $T=0$

$$\varepsilon_c(T) = \varepsilon_o + \alpha T.$$ (2.13)

A term proportional to $T$ is to be expected for the thermal noise in the junction shunt resistors.
We conclude that a single slope $\alpha = 470 \; \hbar / \text{K}$ represents the true internal thermal noise of this sensor, because nearly the same slope $\alpha$ was found for curves #1 and #2 in the $\varepsilon_c$-T plot of Figure 2.7. Comparison between the conventional probe result (where the SQUID is liquid cooled) to the dilution refrigerator result (where the SQUID is cooled only by its wiring) convinced us that the wiring had adequate heat sinking, at least to a few tenths of a degree, and perhaps to 50 mK. This also suggests that the question of the true temperature of the SQUID might ultimately be answered by using a separate SQUID fabricated on the same chip as a noise thermometer.

We were surprised that the measured slope $\alpha$ was so large. The Tesche-Clarke calculations [Tesche and Clarke, 1977 and Clarke, 1990] for the thermal noise in a dc SQUID lead to the following approximate formula

$$
\varepsilon = \frac{9k_B T L}{k^2 R} \approx 67 \; \hbar / \text{K} \times \left( \frac{0.7}{k^2} \right) \left( \frac{L}{80 \text{pH}} \right) \left( \frac{2\Omega}{R} \right) T
$$

(2.14)

where $R$ is the single junction shunt resistance, and we have normalized to the manufacturer's estimate for the actual values for this SQUID. This calculated value is a factor of 7 smaller than the measured value. Such a large departure from the Tesche-Clarke result suggests that there is a much larger coupling to thermal (Johnson-Nyquist) noise than expected, perhaps through the input coil.

Nevertheless, at lower temperatures, the noise in every case is dominated by the temperature independent part $\varepsilon_0$. We would like to have determined the causes, but the manufacturer's electronics are arranged for easy operation, and not for detailed diagnostics of the noise. We did make one set of noise measurements with a different set of electronics which would allow those detailed measurements, and they are shown as curve #3 in Figure 2.7. This set of electronics is a version of an electronics design long used by the Berkeley group [Clarke et al., 1976], with an input transformer.
altered to match to the sensor. Unfortunately, these electronics proved to have more temperature independent noise $\varepsilon_0$ than the commercial electronics, presumably due to higher preamplifier noise, so we took only a few measurements with them.

We can speculate about possible causes for the temperature independent noise $\varepsilon_0$.

(1) Perhaps EMI has not been completely eliminated. It is well known that SQUID noise is increased by EMI at almost any frequency below the infra-red. Perhaps small amounts are still leaking through the preamplifier, or perhaps small amounts can penetrate some of the joints in the conventional probe. There are no direct tests to prove the absence of EMI in a SQUID. The indirect test is to systematically improve the shielding, grounding, filtering, etc., and if exhaustive efforts produce no improvement, then declare it negligible. We believe we have approached this condition with the conventional probe, which has had two complete redesigns based on accumulated experience, but are less confident about the dilution refrigerator system.

(2) It could be preamplifier noise, whose elimination has been the motivation for various two stage SQUID systems [Ketchen and Tsuei, 1980 and Wellstood et al., 1987]. This contribution to the total noise would be temperature independent as long as the SQUID's flux-to-voltage transfer coefficient and output impedance were temperature independent.

(3) It could be caused by flux-locking. In a quiet environment, a SQUID can be operated as a small signal amplifier, i.e., with the flux-locking circuitry disconnected. Some of the very good SQUIDs in the literature have not been flux-locked [Aowschalom et al., 1988], because it has sometimes been observed that the noise when flux-locked is significantly larger. This might be due to insufficient filtering of the flux modulation source, or to the traversal of noisy regions as the flux
is modulated [Wellstood, 1988]. Unfortunately, as stated above, on a gravitational wave antenna we require flux-locked operation to achieve sufficient dynamic range.

We also noticed that mounting the SQUID on the dilution refrigerator increased the temperature independent noise $\xi_0$ by 100%, compared to the conventional probe. Most likely that was due to either: (1) reduction in the signal strength at the preamplifier, due to the extra impedance and impedance mismatches in the signal line [Wellstood, 1992], or (2) pickup of EMI through either the heat sink boxes and/or the extra joints in the shielding.
CHAPTER 3

ANALYSIS AND DESIGN OF A THREE-MODE GRAVITATIONAL WAVE DETECTOR WITH AN INDUCTIVE TRANSUDER

In this chapter we discuss the design of a new transducer for resonant gravity wave detectors. First we will briefly review the principles of operation of a superconducting inductive transducer. This is applicable to inductive transducers with any number of modes. Then we analyze the relations of resonant modes of a three-mode system. We analyze signal to noise performance of the LSU gravitational wave antenna and build the mathematical model of the antenna-transducer-SQUID system for noise analysis. This model will be used to optimize the transducer mass. Having chosen an optimal mass, we use finite element analysis to aid the design of the uniquely shaped small mass resonator and the intermediate resonator of the transducer.

3.1 The Superconducting Inductive Transducer Coupled to a dc SQUID

The superconducting inductive transducer has two advantages over other types of transducers that are used for gravitational wave detectors: good electrical impedance match to the SQUID amplifier and ease of operation. The superconducting inductive transducer was first developed by Paik [Paik, 1974 and 1976]. We deal with the Paik transducer first to set the foundation for the detailed design of our two-mode inductive transducer.

A superconducting inductive transducer coupled to a dc SQUID is depicted schematically in Figure 3.1. All of the electrical circuits in the figure are superconducting to minimize electrical noise. The pickup coil \( L_1 \) is usually wound in the shape of a pancake. One side is attached to the transducer body, while the other
side is placed very close to the surface of the transducer's diaphragm which has an effective mass \( m \). A displacement \( x(t) \) of the diaphragm modulates the inductance of the pickup coil.

\[
L_1 = L_0 \left( 1 - \frac{x(t)}{d} \right) = L_0 - \alpha x(t) \tag{3.1}
\]

where \( L_0 \) is the equilibrium value of \( L_1 \) and \( d \) is the equilibrium spacing between the diaphragm and the pickup coil. Since \( x(t) \ll d \), we have

\[
\alpha \equiv \frac{L_0}{d} \approx \frac{L_1}{d}. \tag{3.2}
\]

A dc current \( I_0 \) is trapped in the pickup coil loop, and creates a magnetic field that is pushed by the diaphragm.

There are two reasons to use a transformer between the pickup coil of the transducer and the SQUID. The first is to optimize the impedance match between \( L_1 \) and \( L_i \). The second is to isolate the dc persistent supercurrent \( I_0 \) from disturbing the
normal performance of the dc SQUID. The mutual inductance of the transformer between $L_2$ and $L_3$ is

$$M_{23} = \gamma_{23}\sqrt{L_2 L_3}$$  \hspace{1cm} (3.3)$$

where $\gamma_{23}$ is the coupling coefficient of the transformer. The mutual inductance between the SQUID input coil $L_i$ and SQUID loop inductance $L$ is

$$M_i = \kappa\sqrt{L L_i}$$  \hspace{1cm} (3.4)$$

where $\kappa$ is the coupling coefficient.

In the remaining part of this section, we follow Solomonson's work [Solomonson, 1990 and 1992] closely, using it to build the necessary relations concerning the operational properties and optimal design of an inductive transducer.

Using the quantization property of the magnetic flux in the two superconducting loops, the ac signal currents $I$ and $I_i$ produced by the displacement $x(t)$ are

$$I = \frac{(L_3 + L_i)\alpha I_0 x(t) + M_{23}M_j\bar{J}}{(L_1 + L_2)(L_3 + L_i) - M_{23}^2}$$  \hspace{1cm} (3.5)$$

$$I_i = \frac{-M_{23}\alpha I_0 x(t) + (L_1 + L_2)M_j\bar{J}}{(L_1 + L_2)(L_3 + L_i) - M_{23}^2}$$  \hspace{1cm} (3.6)$$

where $\bar{J}$ is the time-averaged circulating current in the SQUID loop. The total current $I_0 + I$ in the pickup coil $L_1$ creates a magnetic force on the diaphragm $m$

$$F_m = -\frac{\alpha}{2}(I_0 + I)^2 = -\frac{\alpha}{2}I_0^2 - \alpha I_0 I.$$  \hspace{1cm} (3.7)$$

The $I_0^2$ term in equation (3.7) causes two effects. First, it induces a net displacement of the diaphragm from its equilibrium position $d_0$. The new position $d$ is
where $\omega_0 = 2\pi f_0$ is the intrinsic angular resonant frequency of the transducer when $I_0 = 0$. Secondly, that term shifts the resonant frequency from $f_0$ to a tuned frequency $f_t$, which we will discuss later. The $I_0 I$ term in equation (3.7) generates an additional magnetic force on the diaphragm which is useful to use to derive the magnetic spring constant. Using equation (3.5), the magnetic spring constant produced by $I_0$ and $I$ is

$$
\Delta K = -\frac{\partial F_m}{\partial x} = \frac{\alpha^2 I_0 (I_0 + I)}{L_1 + L_2 - \frac{M_{23}^2}{L_3 + L_1}}.
$$

The ac signal current $I$ can be dropped from equation (3.9), since it is much smaller than the trapped dc current $I_0$.

We define the resonant frequency under no magnetic field ($I_0 = 0$) as

$$
f_0 = \frac{1}{2\pi} \sqrt{\frac{K}{m}}
$$

and the shifted resonant frequency due to the magnetic tuning ($I_0 \neq 0$) as

$$
f_t = \frac{1}{2\pi} \sqrt{\frac{K + \Delta K}{m}}.
$$

Equations (3.9)-(3.11) form the tuning equation for the transducer

$$
f_t^2 = f_0^2 + \Gamma I_0^2
$$

where the tuning parameter $\Gamma$ is
The electro-mechanical coupling coefficient describes the conversion efficiency of mechanical motion into electrical signal. It is the ratio of the induced electrical energy to the total energy of motion of transducer. It is defined as [Paik, 1976]

\[
\Gamma = \left( \frac{1}{2\pi} \right)^2 \frac{\alpha^2}{m \left( L_1 + L_2 - \frac{M_{23}^2}{L_3 + L_i} \right)}.
\] (3.13)

\[\beta = \frac{\Delta K}{K + \Delta K} = 1 - \left( \frac{f_0}{f_t} \right)^2 = \frac{\alpha^2 l_0^2}{m\omega_t^2} \left( \frac{1}{L_1 + L_2 - \frac{M_{23}^2}{L_3 + L_i}} \right) \] (3.14)

where \(\omega_t = 2\pi f_t\) is the shifted angular frequency.

The SQUID energy coupling coefficient is defined as the ratio of the ac signal energy in the SQUID input inductor \(L_i\) to the mechanical energy in the oscillator \(m\) [Paik, 1974]

\[
\beta_i = \frac{L_1 l_1^2}{m\omega_t^2 x^2} = \frac{L_1}{m\omega_t^2} \left( \frac{M_{23} \alpha l_0}{(L_1 + L_2)(L_3 + L_i) - M_{23}^2} \right)^2. \] (3.15)

The larger the \(\beta_i\), the more mechanical energy in the oscillator will be converted into signal energy. To maximize \(\beta_i\), the primary and secondary inductions of the transformer, \(L_2\) and \(L_3\), should be optimized for the given values of pickup coil \(L_1\) and SQUID input coil \(L_i\). To do this, Solomonson introduced the dimensionless parameters

\[
\Lambda_1 = \frac{L_2}{L_1}, \quad \Lambda_2 = \frac{L_i}{L_3}
\] (3.16)
If we substitute equations (3.3), (3.12), (3.14), (3.16) and (3.17) into (3.15) and maximize $\beta_i$ with respect to $A_1$ and $A_2$, we obtain the transformer design equations

$$ A_1 = \frac{1 + \Lambda_3}{\sqrt{1 - \gamma_{23}^2}}. \quad (3.18) $$

$$ A_2 = \sqrt{1 - \gamma_{23}^2}. \quad (3.19) $$

Based on the previous results, Solomonson describes the transducer in a more fundamental way. He writes the previous equations by determining how the inductances will vary as the transducer's size is varied.

**Assuming that the area of the pickup coil $S_c$ covers a constant fraction of the diaphragm's area $S_d$, where $S_d = (S_d h)/h = (m/\rho)/h$, (where $\rho$ is the density of the diaphragm and $h$ is the thickness of the diaphragm), then we have**

$$ S_c \propto \frac{m}{h}. \quad (3.20) $$

This implies that, changing the mass of the resonator requires us to change the size of the diaphragm in definite way. Any size change should be done in a manner such that all of the diaphragm's internal resonances remain constant. Since the internal mode resonant frequencies of the diaphragm can be expressed as [Blevins, 1979]: $f_{ij} \propto h/S_d$, to maintain constant frequencies will require

$$ S_c \propto h. \quad (3.21) $$

The thickness $h$ can be canceled by substituting (3.21) into (3.20), thus we have $S_c \propto \sqrt{m}$. Based on this, a transducer constant is defined as
\[ \eta = \frac{S_e}{\sqrt{m}}. \] (3.22)

This means if one varies the mass or surface area of the diaphragm but keeps \( \eta \) constant, the internal mode frequencies will not change.

If the first critical field of the superconductor is not exceeded, we have

\[ B = \mu_0 n I_0 \] (3.23)

\[ L_1 = \alpha d \] (3.24)

\[ \alpha = \mu_0 S_c n^2 \] (3.25)

where \( 1/n \) is the spacing between adjacent wire turns in the pickup coil, \( B \) is the magnetic field in the transducer gap, and \( \mu_0 = 4\pi \times 10^{-7} \) H/m is the permeability constant.

Based on equation (3.23), we express \( I_0 \) in term of \( B \)

\[ I_0 = \frac{B}{\mu_0 n}. \] (3.26)

Using equations (3.3), (3.18), (3.19), (3.22) and (3.26) to substitute variables, equations (3.25), (3.8), and (3.14) can be rewritten as

\[ \alpha = \mu_0 n^2 \eta \sqrt{m} \] (3.27)

\[ d = d_0 + \frac{\eta B^2}{2\mu_0 \omega_0^2 \sqrt{m}} \] (3.28)

\[ \beta = \frac{\eta B^2}{d\mu_0 \omega_0^2 \sqrt{m}} \left[ 1 + \frac{1 + \Lambda_3}{1 - \gamma_{23}^2} - \gamma_{23}^2 \sqrt{1 + \frac{1 + \Lambda_3}{1 - \gamma_{23}^2} (1 + \sqrt{1 - \gamma_{23}^2})^{-1}} \right]^{-1} \] (3.29)

respectively, where from (3.17) we get
Equations (2.27)-(2.29) are now expressed in terms of the transducer mass \( m \), transducer constant \( \eta \), magnetic field \( B \), and the coupling coefficient \( \gamma_{23} \) of the transformer. Thus all of the inductances which vary with the size of the transducer are not involved in the design parameters.

3.2 Analysis of the Resonant Modes of a Three-mode System

The multimode antenna system was first suggested by Richard [Richard, 1979 and 1984]. A simplified model of a three-mode antenna system is illustrated in Figure 3.2, while the detailed model is discussed in the next section. The equations of motion for the three-oscillator system are

\[
\begin{align*}
\Lambda_3 &= \frac{\eta B^2}{\sqrt{\mu_0 m \omega_0^2}}. \\
\text{(3.30)}
\end{align*}
\]

\[
\begin{align*}
M_1 \ddot{x}_1 + K_1 x_1 - K_2 (x_2 - x_1) &= 0 \quad (3.31) \\
M_2 \ddot{x}_2 + K_2 (x_2 - x_1) - K_3 (x_3 - x_2) &= 0 \quad (3.32)
\end{align*}
\]

Figure 3.2  Model of a three-mode antenna system.
\[ M_3 \ddot{x}_3 + K_3 (x_3 - x_2) = 0. \]  

(3.33)

Assuming harmonic solutions of the form

\[ x_i(t) = x_i(0)e^{i\omega t} \quad (i=1,2,3) \]  

(3.34)

the equations of motion (3.31)-(3.33) become

\[ \left(K_1 + K_2 - M_1 \omega^2\right)x_1 - K_2 x_2 = 0 \]  

(3.35)

\[ -K_2 x_1 + \left(K_2 + K_3 - M_2 \omega^2\right)x_2 - K_3 x_3 = 0 \]  

(3.36)

\[ -K_3 x_2 + \left(K_3 - M_3 \omega^2\right)x_3 = 0. \]  

(3.37)

The linear equations (3.35)-(3.37) have non-zero solutions only if the determinant of the coefficients of \( x_1, x_2 \) and \( x_3 \) is zero

\[
\begin{vmatrix}
K_1 + K_2 - M_1 \omega^2 & -K_2 & 0 \\
-K_2 & K_2 + K_3 - M_2 \omega^2 & -K_3 \\
0 & -K_3 & K_3 - M_3 \omega^2
\end{vmatrix} = 0. \tag{3.38}
\]

This gives

\[
\omega^6 - \left(\frac{K_1}{M_1} + \frac{K_2}{M_2} + \frac{K_3}{M_3} + \frac{K_2}{M_1} + \frac{K_3}{M_2}\right) \omega^4 \\
+ \left(\frac{K_1 K_2}{M_1 M_2} + \frac{K_1 K_3}{M_1 M_3} + \frac{K_2 K_3}{M_2 M_3} + \frac{K_1 K_2}{M_1 M_2} + \frac{K_2 K_3}{M_1 M_3} + \frac{K_2 K_3}{M_1 M_3}\right) \omega^2 \\
- \frac{K_1 K_2 K_3}{M_1 M_2 M_3} = 0. \tag{3.39}
\]

The intrinsic angular frequencies of the oscillators are
Each mass is less than the one preceding it and the mass ratios are

$$\mu_1 = \frac{M_2}{M_1} << 1$$ \quad (3.41)

$$\mu_2 = \frac{M_3}{M_2} << 1.$$ \quad (3.42)

If all of the oscillators have the same intrinsic angular frequency \(\omega_0\), i.e.

$$\frac{K_1}{M_1} = \frac{K_2}{M_2} = \frac{K_3}{M_3} = \omega_0^2$$ \quad (3.43)

with the higher order term \(\mu_1\mu_2\) dropped because each of the \(\mu\)'s is small, equation (3.39) can be written as

$$\left(\omega^2 - \omega_0^2\right)\left[\omega^4 - (2 + \mu_1 + \mu_2)\omega_0^2\omega^2 + \omega_0^4\right] = 0.$$ \quad (3.44)

Solving this equation, we get the normal modes of the three oscillator system

$$\omega_- = \omega_0\left(1 - \frac{1}{2}\sqrt{\mu_1 + \mu_2}\right)$$ \quad (3.45)

$$\omega_c = \omega_0$$ \quad (3.46)

$$\omega_+ = \omega_0\left(1 + \frac{1}{2}\sqrt{\mu_1 + \mu_2}\right).$$ \quad (3.47)

With identical mass ratio

$$\frac{M_2}{M_1} = \frac{M_3}{M_2} = \mu$$ \quad (3.48)
we obtain the so-called motion amplification factor (Solomonson suggests that we should use the term "amplitude transformation ratio" instead, since we are dealing with a passive system, using the word "amplification" can be confusing)

$$\frac{|x_3|}{|x_1|} = \frac{1}{\mu} = \sqrt{\frac{M_1}{M_3}} \gg 1$$  \hspace{1cm} (3.49)

which is satisfied for all of the three normal modes. Since

$$\frac{|x_3|}{|x_1|} = \sqrt{\frac{M_1}{M_3}}$$  \hspace{1cm} (3.50)

can be easily derived from the assumption of energy conservation between $M_1$ and $M_3$. Equation (3.49) also implies that almost the entire mechanical energy has been exchanged between $M_1$ and $M_3$.

The two advantages of using a multimode system are clear. The first advantage is the improvement of the transducer's sensitivity, because the use of a lighter final mass $M_3$ can provide greater motion amplification expressed by equation (3.49). The other advantage is the wider bandwidth through a wider splitting of its normal modes. From equations (3.45) and (3.47), the "-" and "+" are split by an amount of $\Delta \omega = \sqrt{\mu_1 + \mu_2}$.

Now we examine the situation when all three oscillators have their own resonant frequencies $f_i=\omega_i/2\pi$ ($i=1,2,3$) in uncoupled cases. We intend to derive relations between these uncoupled frequencies and the coupled frequencies that we can directly measure on an actual three-mode system. Through these relations the uncoupled frequencies ($f_1, f_2, f_3$) can be evaluated. If we substitute equations (3.40)-(3.42) into equation (3.39) and replace $\omega$ with $f=\omega/2\pi$, we have
\[ f^6 - [f_1^2 + (1 + \mu_1)f_2^2 + (1 + \mu_2)f_3^2]f^4 \]
\[ + [f_1^2f_2^2 + (1 + \mu_2)f_1^2f_3^2 + (1 + \mu_1)f_2^2f_3^2]f^2 - f_1^2f_2^2f_3^2 = 0. \quad (3.51) \]

According to the general rules of solving polynomial equations, the three coefficients \((f_1, f_2, f_3)\) of the equation are related to the three solutions \((f_-, f_c, f_+)\) as

\[ f_-^2 + f_c^2 + f_+^2 = f_1^2 + (1 + \mu_1)f_2^2 + (1 + \mu_2)f_3^2 \quad (3.52) \]
\[ f_-^2f_c^2 + f_c^2f_+^2 + f_+^2f_-^2 = f_1^2f_2^2 + (1 + \mu_1)f_2^2f_3^2 + (1 + \mu_2)f_3^2f_1^2 \quad (3.53) \]
\[ f_-^2f_c^2f_+^2 = f_1^2f_2^2f_3^2. \quad (3.54) \]

From equation (3.12), we replace \(f_3\) as \(f_3^2 = f_{30}^2 + \Gamma I_0^2\), then equations (3.52)-(3.54) are expressed as linear functions of \(I_0^2\)

\[ f_-^2 + f_c^2 + f_+^2 = a_1 I_0^2 + b_1 \quad (3.52) \]
\[ f_-^2f_c^2 + f_c^2f_+^2 + f_+^2f_-^2 = a_2 I_0^2 + b_2 \quad (3.53) \]
\[ f_-^2f_c^2f_+^2 = a_3 I_0^2 + b_3 \quad (3.54) \]

where

\[ a_1 = (1 + \mu_2)\Gamma \quad (3.55) \]
\[ b_1 = f_1^2 + (1 + \mu_1)f_2^2 + (1 + \mu_2)f_{30}^2 \quad (3.56) \]
\[ a_2 = [(1 + \mu_2)f_1^2 + (1 + \mu_1)f_2^2]\Gamma \quad (3.57) \]
\[ b_2 = f_1^2f_2^2 + [(1 + \mu_2)f_1^2 + (1 + \mu_1)f_2^2]f_{30}^2 \quad (3.58) \]
\[ a_3 = f_1^2f_2^2\Gamma \quad (3.59) \]
The coupled mode frequencies ($f_-$, $f_c$, and $f_+$) and the stored dc supercurrent ($I_0$) in the transducer are directly measurable in a three-mode system. From the measured data, we can apply the least squares method to obtain a linear fit of the relations: $f_+^2 + f_c^2 + f_+^2$ vs. $I_0^2$, $f_-^2 f_c^2 + f_+^2 f_+^2 + f_-^2 f_+^2$ vs. $I_0^2$, and $f_-^2 f_c^2 f_+^2$ vs. $I_0^2$. $a_1$, $a_2$, $a_3$ and $b_1$, $b_2$, $b_3$ can then be calculated as slopes and intercepts, respectively. Hence, the uncoupled oscillator frequencies, tuning parameter, and mass ratios can be calculated from the relations

\[
b_3 = f_1^2 f_2^2 f_30^2. \tag{3.60}
\]

\[
The coupled mode frequencies ($f_-$, $f_c$, and $f_+$) and the stored dc supercurrent ($I_0$) in the transducer are directly measurable in a three-mode system. From the measured data, we can apply the least squares method to obtain a linear fit of the relations: $f_+^2 + f_c^2 + f_+^2$ vs. $I_0^2$, $f_-^2 f_c^2 + f_+^2 f_+^2 + f_-^2 f_+^2$ vs. $I_0^2$, and $f_-^2 f_c^2 f_+^2$ vs. $I_0^2$. $a_1$, $a_2$, $a_3$ and $b_1$, $b_2$, $b_3$ can then be calculated as slopes and intercepts, respectively. Hence, the uncoupled oscillator frequencies, tuning parameter, and mass ratios can be calculated from the relations

\[
f_1^2 = \frac{a_2(a_3b_2 - a_2b_3) + a_3(a_1b_3 - a_3b_1)}{a_3^2 - a_1(a_3b_2 - a_2b_3)} \tag{3.61}
\]

\[
f_2^2 = \frac{(a_3b_2 - a_2b_3)\left[a_3^2 - a_1(a_3b_2 - a_2b_3)\right]}{a_3[a_2(a_3b_2 - a_2b_3) + a_3(a_1b_3 - a_3b_1)]} \tag{3.62}
\]

\[
f_30^2 = \frac{a_3b_3}{a_3b_2 - a_2b_3} \tag{3.63}
\]

\[
\Gamma = \frac{a_3^2}{a_3b_2 - a_2b_3} \tag{3.64}
\]

\[
1 + \mu_1 = \left[2a_1a_2(a_3b_2 - a_2b_3) - a_1a_3(a_1b_3 - a_3b_1)\right] \times \frac{a_2(a_3b_2 - a_2b_3) + a_3(a_1b_3 - a_3b_1)}{a_3^2 - a_1(a_3b_2 - a_2b_3)^2} \tag{3.65}
\]

\[
1 + \mu_2 = \frac{a_1(a_3b_2 - a_2b_3)}{a_3^2}. \tag{3.66}
\]

To summarize, through the direct measurements of the coupled mode frequencies $f_-$, $f_c$, $f_+$, and stored current $I_0$ from a three-mode system, we get the slopes $a_i$ and the intercepts $b_i$ (i=1,2,3). Then substitute these slopes and intercepts to equations (3.61)-
(3.66), we are able to evaluate the uncoupled oscillator frequencies $f_1$, $f_2$, $f_3$, tuning parameter $\Gamma$, and mass ratios $\mu_1$ and $\mu_2$.

### 3.3 Noise Analysis of the Antenna-Transducer-SQUID System

In this section, we develop the detailed model following closely the work of Solomonson [Solomonson, 1990 and 1992]. The three-mode gravitational wave detector with a two-mode inductive transducer and a dc SQUID amplifier is illustrated schematically in Figure 3.3. The Michelson and Taber model [Michelson and Taber, 1981] of the detector includes the antenna mass, an inductive transducer, a superconducting transformer, and a SQUID amplifier. The transformer is intended to match the impedance between the pickup coil of the transducer and the input coil of the SQUID employed as described in section 3.1.

![Figure 3.3](image.png)  
**Figure 3.3** Model of a three-mode gravitational wave detector with a single coil inductive transducer, a matching transformer, and a dc SQUID.
When all the relevant Gaussian noise sources are included, the equations of motion for the detector are expressed as following

\[ M_1\ddot{x}_1(t) + H_1\dot{x}_1(t) + K_1x_1(t) - H_2\dot{x}_2(t) - K_2x_2(t) = F_1(t) - F_2(t) + F_s(t) \]  
\[ (3.67) \]

\[ M_2\ddot{x}_2(t) + H_2\dot{x}_2(t) + K_2x_2(t) - H_3\dot{x}_3(t) - K_3x_3(t) + M_2\ddot{x}_1(t) - \alpha I_0 I(t) = F_2(t) - F_3(t) \]  
\[ (3.68) \]

\[ M_3\ddot{x}_3(t) + H_3\dot{x}_3(t) + K_3x_3(t) + M_3\left[\ddot{x}_1(t) + \ddot{x}_2(t)\right] + \alpha I_0 I(t) = F_3(t) \]  
\[ (3.69) \]

\[ -\alpha I_0 \dot{x}_3(t) + \left(L_1 + L_2\right)\ddot{I}(t) + rI(t) + M_{23}\dot{I}_i(t) = V_r(t) \]  
\[ (3.70) \]

\[ M_{23}\ddot{I}(t) + (L_3 + L_i)\dot{I}_i(t) = V_n. \]  
\[ (3.71) \]

\( M_1 \) represents the effective mass of the antenna, \( M_2 \) is the effective intermediate mass of the transducer, and \( M_3 \) represents the effective mass of the final mass, which is referred to as the diaphragm. \( K_i = M_i\omega_i^2 \) and \( H_i = M_i\omega_i / Q_i \) (i=1,2,3) are the spring constants and damping coefficients of those effective masses, respectively, where \( \omega_i \) and \( Q_i \) are the intrinsic angular frequencies and mechanical quality factors of each oscillators. \( F_i \) are the Langevin force noise generators associated with the dissipation coefficients \( H_i \). \( F_s \) represents a signal force applied to the antenna. As previously defined in section 3.1, \( \alpha \) is the inductance modulation parameter given by equation (3.2): \( \alpha = L_1/d \), where \( d \) is the equilibrium spacing between the pickup coil and the diaphragm \( M_3 \). \( I_0 \) is the dc supercurrent stored in the \( L_1-L_2 \) loop. This dc supercurrent \( I_0 \) produces an additional magnetic spring constant which acts in parallel with \( K_3 \) as shown in equations (3.68) and (3.69), and changes the resonant frequency of \( M_3 \). \( I(t) \) is the ac signal generated in the \( L_1-L_2 \) loop.
According to equation (3.7), only the ac magnetic force \( a \| \) affects the dynamic motion of the detector. \( I(t) \) is the ac current in the L3-L1 loop. \( L1 \) is the inductance of the pickup coil of the transducer. \( L2 \) and \( L3 \) are the primary and secondary inductance of the transformer, while \( M_{23} \) is the mutual inductance of the transformer. \( r \) represents losses due to ac current \( I(t) \) in the superconductors and \( V_r \) is the associated voltage noise generator.

The dc SQUID is modeled as a linear current amplifier [Clarke et al., 1979] with a SQUID input inductance \( L_i \). The equivalent input-current noise \( I_n \) determines the white current noise level of the SQUID. \( V_n \) is the back acting or voltage noise shown at the input side of the SQUID amplifier. \( I_n \) and \( V_n \) have been extensively discussed in section 2.1. A more detailed SQUID model would be needed only if a nearly quantum noise limited SQUID was assumed [Solomonson, 1994].

After Fourier transforming and dropping the signal force \( F_s(\omega) \), equations (3.67)-(3.71) can be expressed in matrix form as

\[
\begin{bmatrix}
    x_1(\omega) \\
    x_2(\omega) \\
    x_3(\omega) \\
    I(\omega) \\
    I_i(\omega)
\end{bmatrix}
\begin{bmatrix}
    F_1(\omega) \\
    F_2(\omega) \\
    F_3(\omega) \\
    V_f(\omega) \\
    V_n(\omega)
\end{bmatrix}
= \begin{bmatrix}
    I_1(\omega) \\
    I_2(\omega) \\
    I_3(\omega) \\
    I_4(\omega) \\
    I_5(\omega)
\end{bmatrix}
\]

(3.72)

where the matrix elements are

\[
V_{11} = M_1 \left( \omega_1^2 - \omega^2 + j\omega \frac{\omega_1}{Q_1} \right),
\]

\[
V_{12} = -M_2 \left( \omega_2^2 + j\omega \frac{\omega_2}{Q_2} \right),
\]

\[
V_{13} = V_{14} = V_{15} = 0.
\]
\[ V_{21} = -M_2 \omega^2, \]
\[ V_{22} = M_2 \left( \omega_2^2 - \omega^2 + j \omega \frac{\omega_2}{Q_2} \right), \]
\[ V_{23} = -M_3 \left( \omega_3^2 + j \omega \frac{\omega_3}{Q_3} \right), \]
\[ V_{24} = -\alpha I_0, \]
\[ V_{25} = 0, \]
\[ V_{31} = -M_3 \omega^2, \]
\[ V_{32} = -M_2 \omega^2, \]
\[ V_{33} = M_3 \left( \omega_3^2 - \omega^2 + j \omega \frac{\omega_3}{Q_3} \right), \]
\[ V_{34} = \alpha I_0, \]
\[ V_{35} = 0, \]
\[ V_{41} = V_{42} = 0, \]
\[ V_{43} = -j \omega \alpha I_0, \]
\[ V_{44} = r + j \omega (L_1 + L_2), \]
\[ V_{45} = j \omega M_{23}, \]
\[ V_{51} = V_{52} = V_{53} = 0, \]
\[ V_{54} = j \omega M_{23}. \]
\[ V_{55} = j\omega(L_3 + L_i). \]

\[
\mathbf{A} = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

If we define the matrix \( \mathbf{G} = \mathbf{V}^{-1}\mathbf{A} \), and assume that the current and the voltage noise sources of the SQUID are not correlated to each other, then the total current to the SQUID input is

\[ I_i = G_{51}F_1 + G_{52}F_2 + G_{53}F_3 + G_{54}V_r + G_{55}V_n + I_n. \] (3.73)

The total current noise spectral density at the SQUID input is then given as

\[ S_{i} = |G_{51}|^2 S_{F_1} + |G_{52}|^2 S_{F_2} + |G_{53}|^2 S_{F_3} + |G_{54}|^2 S_{V_r} + |G_{55}|^2 S_{V_n} + S_{I_n} \] (3.74)

where \( S_{F_i} \) (i=1,2,3), \( S_{V_r} \), \( S_{V_n} \), and \( S_{I_n} \) are the power spectral densities associated with each of the noise generators. The voltage noise spectral density \( S_{V_r} = 2k_B T r \) is associated with electrical losses. These losses are caused by the equivalent noise resistor \( r \) in the corresponding circuit of Figure 3.3. The noise resistor \( r \) can be defined in terms of the electrical quality factor \( Q_e \) of the electrical circuit

\[ r = \frac{\omega_t}{Q_e} \left( L_1 + L_2 - \frac{M_{23}^2}{L_3 + L_1} \right) \] (3.75)

where \( \omega_t \) is the shifted angular frequency of the transducer. Obviously, a higher electrical quality factor \( Q_e \) is associated with a lower electrical loss measured by the value \( r \) of the system.
Since the signal force $F_s$ applied to the antenna is in parallel with the force $F_1$, the
signal current spectral density is $S = |G_{51}F_s(\omega)|^2$. The optimally filtered signal-to-
noise ratio is therefore

$$\frac{S}{N} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|G_{51}F_s(\omega)|^2}{S_{i1}} d\omega.$$  \hspace{1cm} (3.76)

For burst sources of gravitational radiation, the signal appears as an impulse on
the antenna. The signal can be approximated as a $\delta$ function which deposits an energy
$E$ into the antenna of mass $M_1$. The impulse signal force is therefore

$$F_s(t) = \sqrt{2EM_1}\delta(t)$$  \hspace{1cm} (3.77)

the signal spectrum of this short pulse signal force over the bandwidth of the detector
is white

$$F_s(\omega) = \sqrt{2EM_1}.$$  \hspace{1cm} (3.78)

The pulse detection noise temperature $T_n$ of the system is defined in terms of
the impulse energy $E$, which makes the signal-to-noise ratio equal to 1 [Michelson and
Taber, 1981]. By setting $E=k_B T_n$, $S/N=1$, and solving for $T_n$, we get the noise
temperature of the detector [Solomonson, 1990 and 1992]

$$T_n = \frac{\pi}{M_1 k_B} \left[ \int_{-\infty}^{+\infty} \frac{|G_{51}|^2}{S_{i1}} d\omega \right]^{-1}.$$  \hspace{1cm} (3.79)

To get the strain noise spectrum $\tilde{h}(f)$ of the bar detector, we write the total
equivalent noise force $F_n$ as if it were all applied to $M_1$. According to equation (3.74),
the total equivalent noise spectral density to the antenna is
Using a harmonic-oscillator equivalent [Gibbons and Hawking, 1971] to model the response of a bar antenna to a gravitational waveform \( h(t) \), the equation of motion is given as [Dewey, 1987]

\[
M_j \ddot{x}_j + M_j \omega_j^2 x_j = F_s(t) = -\frac{M_j}{L} h(t) \quad (3.81)
\]

where \( \omega_j \) is the angular frequency of \( M_j \), and \( L \) is the actual length of the cylindrical antenna. After Fourier transforming, equation (3.81) becomes

\[
F_s(\omega) = \frac{2M_1 L}{\pi^2} \omega^2 h(\omega). \quad (3.82)
\]

Replacing the force signal \( F_s(\omega) \) in equation (3.82) with the total equivalent force noise \( F_n(\omega) \) being exerted on the antenna, the strain noise spectrum \( \tilde{h}(f) \) of the bar detector can be written as

\[
\tilde{h}(f) = \frac{1}{8M_1 L f^2} \sqrt{S_{F_n}(f)} \quad (3.83)
\]

where \( S_{F_n}(f) \) (single sided) is given in equation (3.80).

### 3.4 Optimal Design of the Two-mode Transducer

According to the simplified model of Figure 3.2, a two-mode transducer consists of the second and third stage resonators. The second resonator is called the intermediate resonator, which mainly consists of \( K_2 \) and \( M_2 \), but can be affected by
compliance in $M_1$. The third resonator is called the small resonator (or the
diaphragm), which mainly consists of $K_3$ and $M_3$, but also can be affected by
compliance in $M_2$. In this section we first select the transducer mass $M_3$ based on the
noise analysis established in section 3.3. Then we design both the small and the
intermediate resonators with the aid of finite element analysis.

### 3.4.1 Selection of the Small Transducer Mass $M_3$ Based on the Noise Analysis

For a given set of noise sources: $S_{F_i}$ (i=1,2,3), $S_{V_f}$, $S_{V_n}$, $S_{I_n}$, the transducer's
mass $M_3$ should be chosen to minimize the detector's noise temperature. To make the
choice, we modeled the expression for the noise temperature of the detector with
different values for $M_3$. We used MATLAB [The MathWorks, Inc., 24 Prime Park
Way, Natick, MA 01760] to evaluate the equations with the following conditions. (1)
$M_1=1148$ kg and $M_2=2.2$ kg (estimated from $M_2 = \sqrt{M_1 M_3}$ with $M_3$ around 4
grams). (2) The oscillators $M_1$, $M_2$, and $M_3$ are all perfectly tuned to the same
resonant frequency of 913 Hz. (3) The Q's of the oscillators are $Q_1=Q_2=Q_3=8\times10^6$
and the electrical $Q_e=5\times10^5$. (4) The magnetic field at the diaphragm equals 0.1 T.
(5) The coil diaphragm spacing is 80 μm when $I_0=0$. (6) The transformer coupling
coefficient is 0.9 and the thermodynamic temperature is 4.2 K. Using these values,
Figure 3.4 shows how the detector's noise temperature varies as a function of the
transducer's mass $M_3$. Each curve is labeled by the extrinsic energy sensitivity of the
SQUID that is coupled to the transducer. The noise temperature is optimal at $M_3=8$–$9$
g for a 500 $h$ SQUID. Since all the curves are relatively shallow, the noise
temperature with the same $M_3$ for a 4000 $h$ SQUID is not far from its optimal value
either.
To confirm that this choice of $M_3$ is reasonable, we also plot the detector's noise temperature versus $M_3$ with different electrical Q's of the transducer and different extrinsic energy sensitivities of the SQUID in Figure 3.5. This figure shows that for the same $M_3$ the noise temperature can be reduced by improving the electrical Q of the transducer in general. It also shows that $M_3=8\sim9$ g is a good compromise for a 500 $h$ SQUID and any possible electrical Q's that transducer might have. For a 500 $h$ SQUID, if only a lower electrical Q was obtained, choosing a larger $M_3$ would not degrade the noise temperature much from its optimal value. Also if a higher electrical
Figure 3.5 Detector noise temperature vs. transducer mass with extrinsic energy sensitivity of the dc SQUID = 4000 $h$ and 500 $h$. 
Q was reached, the noise temperature would not be far either from the optimal value for which \( M_3 = 8 - 9 \) g, since the curves are shallow for larger values of \( M_3 \). We have set our design goal at \( Q_e = 5 \times 10^5 \) since the detector's noise temperature does not improve rapidly as a function of \( Q_e \) beyond this value. As Figure 3.5 shows, if \( Q_e = 5 \times 10^5 \) is reached for a 4000 \( \hbar \) SQUID, \( M_3 = 8 - 9 \) g is not far from the optimal noise temperature. Obviously we would get closer to the optimal state when a better SQUID is used.

We elected to build a transducer with a diaphragm mass of approximately 8 g, because we already possessed a 4000 \( \hbar \) SQUID and discussed in Chapter 2 and anticipated that a SQUID with lower noise would soon become available.

Now we employ the transducer constant \( \eta \) introduced in section 3.1. Replacing \( m \) in equation (3.22) with transducer's mass \( M_3 \), the transducer constant can be written as

\[
\eta = \frac{S_e}{\sqrt{M_3}}.
\]

(3.84)

This constant depends upon the specific geometry and material constants of the transducer. For a given \( \eta \), any change of the resonator's mass will be compensated by a change in its size in order to maintain all of its internal resonances constant.

The noise temperature as function of \( M_3 \) with different transducer constants is illustrated in Figure 3.6 for a 4000 \( \hbar \) SQUID. It shows that for a given \( M_3 \), larger \( \eta \) corresponds to lower noise temperature of the detector. Therefore, it is a good idea to make the surface area of the pickup coil as large as we can or make the diaphragm as thin as we can. However, the surface area enlargement is limited by the thickness of the diaphragm. A very thin diaphragm is hard to machine. Our design goal is to keep the transducer constant slightly above 0.008.

Table 3.1 summarizes the design parameters for the transducer.
Figure 3.6  Detector noise temperature vs. transducer mass with extrinsic energy sensitivity of the dc SQUID $= 4000 \, \hbar$. 
Table 3.1  Values of the parameters for the design of the 3-mode detector

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Antenna's effective mass</td>
<td>M₁=1148 kg</td>
</tr>
<tr>
<td>Antenna frequency</td>
<td>f₁=913.0 Hz</td>
</tr>
<tr>
<td>Antenna Q</td>
<td>Q₁=12x10⁶</td>
</tr>
<tr>
<td>Intermediate mass</td>
<td>M₂=2.20 kg</td>
</tr>
<tr>
<td>Uncoupled M₂ frequency</td>
<td>f₂=913.0 Hz</td>
</tr>
<tr>
<td>Q of M₂</td>
<td>Q₂=8x10⁶</td>
</tr>
<tr>
<td>Resonator mass</td>
<td>M₃= ~8 g</td>
</tr>
<tr>
<td>Coupled M₃ frequency</td>
<td>f₃=913.0 Hz</td>
</tr>
<tr>
<td>Q of M₃</td>
<td>Q₃=5x10⁶</td>
</tr>
<tr>
<td>Electrical Q</td>
<td>Qₑ=5x10⁵</td>
</tr>
<tr>
<td>Transducer constant</td>
<td>η= ~0.008</td>
</tr>
<tr>
<td>Magnetic field in the gap</td>
<td>B=0.1 Tesla</td>
</tr>
<tr>
<td>Coil-diaphragm gap at I=0</td>
<td>d₀=80 μm</td>
</tr>
<tr>
<td>Pickup coil wire spacing</td>
<td>1/n=100 μm</td>
</tr>
<tr>
<td>Operational temperature</td>
<td>T=4.2 K</td>
</tr>
<tr>
<td>SQUID energy sensitivity</td>
<td>$E_{ext}$=4000 $\mu$</td>
</tr>
<tr>
<td>SQUID input inductance</td>
<td>$L_i$=1.85 $\mu$H</td>
</tr>
<tr>
<td>Mutual inductance</td>
<td>M$_i$=10 nH</td>
</tr>
<tr>
<td>Loop inductance</td>
<td>L=58 pH</td>
</tr>
<tr>
<td>Shunt resistance</td>
<td>R=2 $\Omega$</td>
</tr>
</tbody>
</table>

3.4.2  Design of the Niobium Resonator Core Aided by Finite Element Analysis

The two major parts of the resonator core, the (small) resonator and the coil holder, are made of niobium. As mentioned earlier in this section, the resonator contains the diaphragm (M₃), the base ring (part of M₂), and the arms (K₃) to connect
Figure 3.7 Comparison between (a) the traditional design and (b) our design under the condition of a limited circular space for arranging the diaphragm and the arms of the resonator.

M2 and M3. For the resonator, the major differences between the traditional design and our design are illustrated in Figure 3.7. We moved the arms that function as the springs from within the same plane of the diaphragm to underneath it. The purpose of doing this is to increase the effective area of the pickup coil (i.e., to increase the transducer constant η) based on a given circular area. We use a single coil to increase the dependability of the transducer by reducing the complexity of assembly.

Having made this decision, we needed to decide where exactly to place the arms, how many arms to use, and what dimensions the arms should have. We used
finite element analysis to seek these answers. Before starting, we set a few rules to observe for doing the design-simulation iteration.

(1) Since the uncoupled transducer mode of $f_3=913$ Hz is given under the condition of $T=4$ K and $B=0.1$ Tesla, reducing the magnetic field to $B=0$ should lower the uncoupled mode to about 860 Hz, and warming the transducer to room temperature should further lower the uncoupled mode frequency to $0.968 \times 860 = 832$ Hz (where 0.968 is an empirical factor). We also require all the other modes to be approximately 200 Hz away so that those modes will not disturb the transducer mode within its bandwidth. Solomonson mentioned that the noise temperature of a 3-mode detector is not sensitive to variations in the ratio of the consecutive masses within a factor of 2 of the optimal value, therefore we have some freedom to vary the value of $M_3$ in certain range. We do not have to worry too much about breaking the condition of $M_2/M_3=M_1/M_2$, which maximizes the mechanical transfer function between the first and third oscillator [Solomonson, 1990].

(2) We preferred to have a large base mass to diaphragm mass ratio. This was done by requiring that the base mass not be reduced too much while keeping the overall transducer diameter constant.

(3) We set the "smallest limit" of the arms' cross section to be $0.040''(\text{thickness}) \times 0.0625''(\text{width})$ in order to limit the stress buildup from the machining.

(4) For a given diaphragm mass $M_3$, it is better to have a larger $\eta$ (Figure 3.6). Therefore we should make the diaphragm as thin as possible so long as all internal diaphragm resonances are far from the antenna mode frequency.

(5) The shape of the resonator should not be too hard to machine.

We used the finite element analysis program COSMOS/M developed by Structural Research and Analysis Corporation [2951 28th Street, Suite 1000, Santa
Monica, CA 90405] to do the simulation. We started with all the arms arranged on the outer edge of the diaphragm. The biggest advantage of this is its ease in machining. The biggest disadvantage of this is the reduction of the base mass due to a bigger circular hole underneath the diaphragm for machining purpose. The simulation showed that the transducer mode is the lowest mode in this situation. There are two orthogonal rocking modes less than 100 Hz above it for the three-arm case. Simulations with four and six arms on the same edge split the transducer mode from the rocking mode further apart, but to do that we need to further reduce the cross section of the arms much beyond the "smallest limit" set in the third rule above. We decided to move the arms inward from the edge of the diaphragm and keep the three-arm geometry. We found that as we moved the arms in, the two orthogonal rocking modes appeared at a lower frequency than the transducer mode. They split even more when we moved the arms further inward. After many trials, we found a location for those arms where the transducer mode was split more than 190 Hz away from the rocking modes. This is far enough for the other modes to fall out the possible bandwidth of the 3-mode detector. This choice of the geometry satisfied all the basic rules set above.

Figure 3.8 shows the cross section of the 1/3 meshed resonator. Some characters of the design are clear from this figure. First, the central hole (diameter=0.7") on the base ring needed to machine the structure was kept as small as possible. Since the base ring mass belongs to M2, a small central hole is equivalent to a larger M2. Second, the diaphragm disc (diameter=1.4") is flat on the top. The outer rim (diameter from 0.7" to 1.4") is thicker (0.040") than the inner circle (diameter=0.7") of the disc (thickness=0.025") to stiffen the edge. This thinner inner disc can keep the mass M3 down slightly. Third, a buffer ring is set to connect the diaphragm disc and the small tabs of the three arms going out. It also helps to stiffen
the connection and makes it easier to machine. Finally, the cross section of the arm is rectangular with the thickness (0.0625") shorter than the width (0.072"). For limited arm length and fixed arm cross section, we prefer this arrangement to bring all of the mode frequencies lower[Blevins, 1979]. The arms span about 66° in arc on the constant radius (i.d.=0.7"). Through another buffer ring and a guarding lip the arms were connected to the body of the base rim (diameter=2.75") of the resonator.

We used the finite element analysis program to simulate the modes of this niobium resonator. The simulation results are listed in Table 3.2. The parameters of niobium used in the simulation are: modulus of elasticity (Young's modulus)=15\times10^6 psi, Poisson's ratio=0.3, and mass density=8.03\times10^{-4} lbs sec^2/in^4.
Table 3.2  Simulation results of the five lowest resonator modes using finite element analysis.

<table>
<thead>
<tr>
<th>Mode number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency (Hz)</td>
<td>634.8</td>
<td>634.8</td>
<td>835.0</td>
<td>2665.9</td>
<td>2665.9</td>
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<td>Description of the mode</td>
<td>rocking mode</td>
<td>rocking mode</td>
<td>transducer mode</td>
<td>bending mode</td>
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</table>

The exaggerated deflection of the resonator in its transducer mode and four other adjacent modes are shown in Figure 3.9. The transducer mode (Mode 3 in the figure) offers a large net displacement of the diaphragm's surface and center of mass. All the other modes have much less net movement. Therefore when the pickup coil of the transducer is placed near the surface of the diaphragm of the resonator, the inductance of the pickup coil is primarily modulated by the motion of the diaphragm due to the transducer mode.

Now we describe the design of the coil holder. One of its functions is to hold the pickup coil and to keep the correct gap between the coil and the diaphragm when the holder is bolted to the resonator that is depicted in Figure 3.7. The other function is to shield the enclosed diaphragm and pickup coil from electromagnetic interference (EMI). Figure 3.10 shows how the coil holder mounts on the assembled transducer.

Since the mass of the coil holder belongs to $M_2$, it was designed to be as massive as possible within a limited volume. A larger $M_2/M_3$ ratio would make the actual $M_2$ closer to its optimal value.
Mode 1: 634.8 Hz  
Mode 2: 634.8 Hz  
Mode 3: 835.0 Hz  
Mode 4: 2665.9 Hz  
Mode 5: 2665.9 Hz

Figure 3.9 The motion of the resonator in its five lowest modes, an exaggerated drawing from the software of finite element analysis.
3.4.3 Design of the Intermediate Resonator Aided by Finite Element Analysis

To couple the resonator core to the end of the bar antenna which is made of aluminum alloy 5056, we wish to make an intermediate resonator also of aluminum 5056. This intermediate resonator will structurally consist of an inner ring (part of $M_2$), an outer ring (part of $M_1$), and a spring to connect them with a total spring constant $K_2$. The niobium resonator core will be coupled to the aluminum intermediate resonator both by bolting and by differential thermal contraction. The thermal contraction contact occurs on the boundary of the round surfaces of the coil holder and the intermediate resonator. The reason for not using the round surface of the resonator itself is due to the consideration of assembly and alignment. The design of the coil holder and the sketch of all the rest part of the 3-mode detector system is drawn in Figure 3.10.

The intermediate resonator will be made from a single piece of aluminum. It mainly contains some spring arms on two different physical levels as shown in Figure 3.10 to reach a combined spring constant $K_2$. Based on Table 3.1, the second resonant mass will be $M_2=2.2$ kg. Its resonant frequency should be $f_2=913.0$ Hz, the same as that of the antenna.

As illustrated in Figure 3.10, the intermediate resonator is formed by two rings. Between the rings are the spring arms that are confined to the space given. The inside diameter (i.d.) of the inner smaller ring matches the outside diameter (o.d.) of the coil holder (2.75") of the niobium resonator core. Its o.d.=4.25" and height=2.25" were chosen to give the correct mass, $M_2=2.2$ kg, when combined with the niobium resonator core. The outer ring o.d.=7.5" is limited by the space available on the real antenna system. Its i.d.=5.25" was chosen to ensure that the outer ring is sufficiently stiff, since everything shown in Figure 3.10 is attached to the antenna through this ring.
Figure 3.10 Sketched design of the 3-mode gravitational wave detector with a 2-mode transducer.

with some big bolts. The spring arms are confined within a space of i.d.=4.25", o.d.=5.25" and height=2.25". We arranged 6 "T" shaped arms every 60°, at alternative levels as shown by the shaded parts in Figure 3.11. Each arm is formed by joining two bridges together. Each bridge behaves like a spring with a cross section of 0.15"x0.15" and spanning a length of 12° arc. The bottom end of the "T" shaped arms is connected to the inner ring and the two ends of the two bridges of the "T" shaped arms are connected to the blocks attaching to the outer ring. It is clear from the figure
that this intermediate resonator is designed so that it can be easily milled out of a single aluminum block. The central hole in the figure is for fitting the niobium resonator and coil holder assembly.

Figure 3.11  Arm (shaded) arrangement of the six-arm intermediate resonator.

To simulate the intermediate resonator using the finite element analysis program, we used the parameters for the materials involved as follows, (1) niobium: modulus of elasticity (Young's modulus) = 15×10^6 psi, Poisson's ratio = 0.3, and mass
density=8.03×10^{-4} \text{ lbs sec}^2/\text{in}^4; (2) aluminum: modulus of elasticity (Young's modulus)=10\times10^6 \text{ psi}, Poisson's ratio=0.3, and mass density=2.526\times10^{-4} \text{ lbs sec}^2/\text{in}^4.

As in the case of designing the niobium resonator, we wanted to have the resonator mode \( f_2=913 \text{ Hz} \) at 4 K and any other internal modes at least 180 Hz away from it. At room temperature a resonant frequency of about 850 Hz is required. This frequency then was the simulation goal to be achieved. The results of the simulation are listed in Table 3.3.

<table>
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<tr>
<th>Mode number</th>
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<td>Frequency (Hz)</td>
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<td>1821.8</td>
<td>1821.9</td>
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<tr>
<td>Description of the mode</td>
<td>resonator mode</td>
<td>rocking mode</td>
<td>rocking mode</td>
</tr>
</tbody>
</table>

It is clear from the table that the frequency of the resonator mode is close enough to the design goal. Also, the transducer mode is more than 900 Hz away from its next closest modes. The motion of the intermediate resonator in its designated resonator mode is shown in Figure 3.12. We attached the equivalent mass of the niobium resonator core to the hole of the inner ring. In figure 3.12, this equivalent mass of the whole niobium core is represented inside the dark ring. The inclusion of this equivalent mass is needed to get the correct mode frequencies.
Figure 3.12  The motion of the intermediate resonator in its designated mode in an exaggerated drawing.
CHAPTER 4

DEVELOPMENT OF THE FABRICATION TECHNIQUES TO ACHIEVE HIGH Q'S IN SMALL MASS RESONATORS AND THE TEST RESULTS

4.1 The Facility for Testing the Mechanical and Electrical Q's of the Isolated Small Mass Resonator

To test the mechanical and electrical Qs of an isolated resonator we need an experimental chamber that works at 4 K and a good mechanical isolation system to hang a resonator without degrading its mechanical Q. Our testing facility is illustrated in Figure 4.1. It mainly consists of the dewar insert, the suspension system to hang the isolated resonator, the wiring, and the instruments for making the measurements.

The dewar insert is made of stainless steel. There is a single layer of niobium foil glued to the inside wall of the probe to form a superconducting shield between the resonator and the stainless steel wall. The shield was used because we did not want any normal metal in the neighborhood of the resonator and its superconducting circuits. Three ports from the experimental chamber were assigned to the pumping line, pressure gauges and electrical leads. Pressure gauges include a thermocouple gauge and an ion gauge. The ion gauge was connected to the Model 307 vacuum gauge controller made by Granville-Phillips [Granville-Phillips Company, 5675 Arapahoe Ave., Boulder, CO 80303]. This controller has the capability to dump data to the HP computer through its IEEE-488 interface.

The suspension system was designed to mechanically isolate the resonator and the leads going down to the resonator from outside vibrational noise. As shown in Figure 4.1, the suspension system consists of a four stage Taber type isolation stack [Taber, 1979] bolted to the top of the experimental chamber. Each stage of the stack...
is made of a brass disc. Each lower disc is hung from the disc above it with titanium wire. Titanium is chosen because it has high tensile strength and low creep noise. Finally the tested resonator was hung from the lowest stage of the disc by two pairs of titanium wires (diameter=0.012”).

The feedthrough on top of the probe consists of one hermetic 8-pin connector for the heat switch and power leads, and four hermetic BNC connectors for the sensors. The electrical leads from the feedthrough at room temperature down to the top of the experimental chamber at 4 K were made of: (1) two SS-85 coaxial cables to carry the signals of the piezoelectric strain gauge (PZT) and the thermometer diode, (2) one twisted 32 AWG copper wire for the heat switch, (3) one pair of 14 AWG copper wire for the power leads to carry the high current. All the leads were heat sunk at the top of the experimental chamber mainly using the OFHC made heat sink bobbins potted with GE 7031 varnish. The leads that go down to the bottom stage of the Taber isolator were glued to the top of each stage of the Taber stack using either GE 7031 varnish or 5 minute epoxy. Those leads include: (1) a copper UT34 coaxial cable to carry the signal from the PZT, (2) two pairs of twisted 32 AWG copper wire for the temperature sensor and the heat switch, (3) a set of copper clad niobium wires for bringing the high dc current to the pickup coil. Finally, fine 41 AWG copper wires (diameter=0.003”) were used to connect the leads at the last stage of Taber isolator and the PZT [Keramos, Inc., 5460 W. 84th Street, Indianapolis, IN 46268] as well as the diode sensor glued (5 minute epoxy and GE 7031 varnish, respectively) on the coil holder. The reason to use fine wire was again to avoid degrading the mechanical Q of the resonator.

The cryogenic temperature sensor was a silicon diode with type #DT-470-SD-12A, made by Lake Shore Cryogenic [Lake Shore Cryotronics, Inc., 64 East Walnut St., Westerville, OH 43081-2399]. We calibrated it before we glued it on the
Figure 4.1 Testing facility for the resonator.
resonator. Its leads coming out from the feedthrough were connected to the
temperature controller made by the same company. A HP 9000 series 300 computer
was used for data collection.

To measure the Q of the resonator, we excite its mechanical resonance and
then measure the decay of its amplitude. If a resonator is being measured for the first
time, we do the following to find the mode frequencies. We excite all the modes by
shaking the probe top slightly, in the direction along the axis of the resonator-coil
holder assembly. The shaking caused the back side of the resonator to touch the head
of a nylon screw anchored about 0.05" away from it. The amplitude of the resonance
was sensed by a PZT glued on the back of the resonator and measured by an HP3561A
Dynamic Signal Analyzer. We spot the vibrational modes on the spectrum analyzer,
then decide which one to trace further. To measure the Q of a particular mode of a
resonator, we set the spectrum analyzer at its mode frequency and measure the
amplitude of the PZT as a function of time. The measured data include the amplitude
of a mode, the associated time, and the temperature. The data collection program is
written in BASIC and it runs on the HP computer. Data is stored on the hard disc on
the HP computer. After a unit measurement is over and a set of data collected, this set
of data is transferred to a Macintosh IIci computer for analysis.

4.2 Two Prototype Aluminum Resonators

Niobium was the material used for the resonator because of its excellent
mechanical and electrical Q at 4.2 K. Since niobium is rather expensive and difficult
to machine, we first made a prototype out of aluminum 5056 alloy to verify our design
based on finite element analysis and to test the procedure for machining the resonator.
We chose this material because it is easy to machine and has a high mechanical Q at
4.2 K.
The first aluminum resonator (Al-#1), was machined in a conventional way. Cutting the 3 arms using a 1/16" diameter end mill proved to be difficult. Our initial design did not allow enough clearance for the shank of the end mill which rubbed against one of the solid surface of the resonator and generated vibrations. The cutting speed of the tools was also too high. This produced tremendous stress in the arms.

After using NaOH solution (with 130 g 97% solid NaOH added water to a total 280 ml) to etch the resonator and clean its surface, the arms warped and caused the diaphragm to tilt. Still with hope, we baked the resonator in the air at around 525 °C for 10 hours to relieve the stress. The resonator was chemically brightened for 30 minutes in a mixture of 14:2:35 by volume ratio of 85% H2PO4, 70% HNO3, and H2O. Then it was put in the etching solution of a mixture of 14:1 by volume ratio of 85% H2PO4 and 70% HNO3 for 80 minutes.

We measured the mechanical Q of Al-#1 at 4.2 K, but we could hardly find any mode with a decent Q in this resonator. We suspect that one of the badly warped arms came into contact with resonator body during the cooling.

The second aluminum resonator we made was named Al-#2. We made some small changes in design to allow more clearance this time during fabrication. We cleaned and etched the aluminum 5056, which was in the form of a hockey puck, for 2 hours with the etching solution mentioned above, followed by annealing it in air at around 540 °C for 5 hours. The resonator was machined to the state where we were ready to cut its 3 arms. Then it was immersed in a new batch of the etching solution for 1 hour, followed by 3 hours stress relief annealing at around 550 °C in the same furnace. The machining speed of the whole process was much slower comparing to that of Al-#1 to avoid extra stress buildup in the resonator.

After the resonator was machined, abrasive cords made of silicon carbide were used to deburr and polish the arms. The resonator was immersed in trichloroethane for
45 minutes to degrease. Right after that, the resonator received deep etching in the typical etching bath of H₂PO₄ and HNO₃ for 8 hours and 20 minutes. The diameter of the diaphragm was changed from 1.401" to 1.389", while the edge thickness was changed from 0.040" to 0.035".

Al-#2 was tested at 4.2 K. After excitation, the vibrational amplitude was sensed by a PZT strain gauge glued on the back of the resonator and measured by an HP3561A Dynamic Signal Analyzer. The mechanical Q measured was Q=8.3×10⁶ at 1104.625 Hz, T=4.6~6.0 K and P=(4.2~8.2)×10⁻⁶ Torr.

The two lower rocking modes were found at 826.3 Hz with Q=8.5×10⁶, T~5 K and P=5.1×10⁻⁶ Torr. The nearest higher modes were found at 3726.5 Hz and 3759.0 Hz.

The high Q of Al-#2 proved that sufficient high Qs could be achieved using this geometry. This encouraged us to move on and build the following niobium resonators.

4.3 Construction and Test Results of the Niobium Resonators

The following three Nb resonators were all machined from the same Nb plate purchased from Teledyne Wah Chang [Teledyne Wah Chang Albany, P.O. Box 460, Albany, OR 97321]. Its chemical analysis and 3R rating as supplied from the manufacturer is listed in Table 4.1.

The typical general procedure of making the resonators includes the following steps:

Step 1: cut a niobium cylinder from the plate;

Step 2: immerse the niobium cylinder in the trichloroethane then ultrasonic clean it for 20 to 50 minutes;
Step 3: clean the niobium cylinder using a mixture of 1:1:1 by volume ratio of 37% HCl, 70% HNO₃, and H₂O for 40 to 50 minutes;

Step 4: chemical etch the niobium cylinder using a mixture of 1:2:2 by volume ratio of 49% HF, 70% HNO₃, and H₂O for 7 to 8 minutes;

Step 5: anneal the niobium cylinder for stress relief in an all niobium furnace built at LSU [Solomonson, 1990] with T=760-800 °C at pressure of (2.3~9.9)x10⁻⁷ Torr for 45 to 55 minutes;

Step 6: conventionally machine the niobium to the form before cutting the arms;

Step 7: immerse the niobium in the trichloroethane and ultrasonically clean it for 25 to 120 minutes;

Step 8: clean the niobium using the solution mentioned in step 3 for 25 to 50 minutes;

Step 9: etch the niobium using the etching solution mentioned in step 4 for about 2 minutes;

Step 10: anneal the niobium for stress relief in the vacuum furnace with T=750~900 °C at pressure of (0.3~2.1)x10⁻⁶ Torr for 45-55 minutes;

Step 11: cut the arms of the resonator by means of conventional milling or electrical discharge machining (EDM);

Step 12: deburr and polish the arms using abrasive cords made of silicon carbide;

Step 13: immerse the niobium resonator in the trichloroethane and ultrasonic clean it for 20 to 30 minutes;

Step 14: clean the resonator using the solution mentioned in step 3 for 20 to 40 minutes;
Table 4.1 Chemical analysis of the niobium plate used for making resonators.

1. INGOT analysis with impurities in ppm

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</table>

2. Residual Resistivity Ratio

Spec.: 150 min.

Results: 353 (Room Temp./LHe)

334 (Ice/LHe)

Step 15: tune the mode frequency of the resonator down by etching it using the etching solution mentioned in step 4 for 6 to 13 minutes;
Step 16: anneal the resonator for stress relief in the furnace with $T=930~980 \degree C$ at pressure of $(1.1~1.7) \times 10^{-6}$ Torr for 50 to 70 minutes;

Step 17: cool the resonator to 4.2 K to measure its mechanical Q;

Step 18: anneal further (some of the resonators) with $T=1400~1500 \degree C$ at pressure range of $(0.7~2.3) \times 10^{-6}$ Torr for about 40 minutes.

All the chemical treatments mentioned above were done for the purpose of eliminating any contamination due to the machining and other processing steps.

4.3.1 Niobium Resonator #1

Niobium resonator #1 (Nb-#1) was made following the above general procedure from step 1 to step 16. But instead of using EDM in step 11, the arms of this resonator were cut by conventional milling at relatively high speed. Vibrations at the tip of the mill bit occurred because of the long shank length (>1") and the small diameter (1/16") of the mill. This kind of milling must build up extra stress in the resonator, since all three arms of the resonator were permanently deformed, and one appeared more warped than the others when step 15 was conducted. This brought about a tilted diaphragm for the resonator. In additional, the arms were cut too much due to the vibrating mill tip. This made the mode frequencies of this resonator too low to be used on the antenna. Nevertheless, the resonator was tested the first time as stated in step 17, then it was deep annealed to 1485 \degree C in step 18 and tested the second time.

During the first test of Nb-#1 in step 17, the resonator was cooled to 4.2 K in the dewar insert, a mechanical $Q=2.4 \times 10^6$ was measured at the transducer mode $f_0=705.6$ Hz. The pressure surrounding the resonator was below $P=4.5 \times 10^{-6}$ Torr. The two rocking modes were found at 516 Hz and 533 Hz (the splitting was due to the...
asymmetry of its three arms). The two higher bending modes were found at 2463 Hz and 2574 Hz, respectively.

The comparison between the simulation results of the finite element analysis and the modes actually measured at room temperature (RT) is summarized in Table 4.2.

<table>
<thead>
<tr>
<th></th>
<th>mode 1</th>
<th>mode 2</th>
<th>mode 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>finite element analysis</td>
<td>503.1</td>
<td>503.9</td>
<td>684.8</td>
</tr>
<tr>
<td>results of measurement</td>
<td>499.8</td>
<td>516.6</td>
<td>683.3</td>
</tr>
</tbody>
</table>

The table shows that the finite element analysis gives fairly reliable predictions about where the mode frequencies are going to be. We intended to push down the two lower rocking modes at least 180 Hz away from the transducer mode during our earlier design, and this was well confirmed by our testing.

The second test of Nb-#1 was made after the deep annealing in step 18. A mechanical $Q=2.4\times10^7$ was measured at the transducer mode when the resonator was cooled in the dewar insert to 4.2 K. This result was an order of magnitude higher than that of the case before the deep annealing. The same transducer mode dropped about 12 Hz to $f_0=694.0$ Hz due to the deep annealing. The highest $Q$ of Nb-#1 was reached at temperatures around $T=7$ K as shown in Figure 4.2. It was found that the $Q$ measurement in our setup was independent of pressure once the pressure was below $4\times10^{-5}$ Torr. The two rocking modes were found at 493 Hz and 533 Hz. The nearest higher bending modes were found at 2382 Hz and 2584 Hz.
4.3.2 Niobium Resonator #2

We had to make Nb-#2 because Nb-#1 could not be used on the antenna. As mentioned before, Nb-#1 had neither the right diaphragm orientation due to the permanently deformed arms, nor the correct mode frequencies.

To make Nb-#2, we again followed the general procedure from step 1 to step 17 except in step 11, the arms were cut using the electrical discharge machining (EDM) method. EDM is believed to have much less stress buildup to the machined
part when comparing to that of the other conventional machining methods. We hoped that through EDM we could make resonators with better overall quality factors (Qs) as result.

The EDM machine we used was actually a small spark cutter that we modified. To avoid contamination, we used new EDM oil and a new work piece tank. All of the parts involved in the machining procedure were thoroughly cleaned with trichloroethane. The electrode used to burn through the niobium wall was made of OFHC copper. A niobium electrode was a natural choice to use because it involves no other foreign materials, but it tended to stick to the niobium part that was being cut. Copper electrodes are much easier to make into the requested shape. We believed that the oxides and nitrides or other copper compounds possibly produced from the sparks of machining should be easily etched off by the acid solution used in the general procedure. To ensure that any contamination due to the machining was completely eliminated, we did a much deeper etching in step 15. After over 9 minutes, we removed at least a 0.001” thick contaminated and stressed layer from the surface. This final etching process had another purpose, namely to tune the transducer mode to around 832 Hz at room temperature. We stuck a small accelerometer on the back of the resonator and used a HP3561A signal analyzer to measure the resonant frequencies at the site of etching. Since etching makes the arms thinner which lowers the mode frequencies of the resonator, we usually approach the tuning frequency by iterating the etching-measuring process many times. We stopped the etching when we reached 840 Hz for Nb-#2.

After the right mode frequency was reached, we annealed the resonator following step 16 of the general procedure.

Nb-#2 was tested in the dewar insert. At 4.2 K, a mechanical Q=3.4x10^6 was measured at the transducer mode f_0=861.4 Hz (designed for f_0=857 Hz at 4.2 K). At
4.9 K, Q=3.9x10^6 was measured. The measured Qs were slightly higher than that of the Nb-#1 made with the similar treatment except that the arms of Nb-#2 were cut using EDM. It was encouraging to know that EDM did improve the Q of the resonator. The two rocking modes were found at 649.1 Hz and 679.3 Hz. The higher bending modes were around 2.7 kHz.

Although the Q was improved slightly, it was not high enough to reach the design goal of 5x10^6. The deep annealing in step 18 was not conducted because we wanted to compare the electrical Q (Q_e, defined in section 3.3) before and after the deep annealing to know how it affects Q_e. We did not yet intend to measure the electrical Q at this stage, because we were interested in searching for a way to reach a higher mechanical Q first.

4.3.3 Niobium Resonator #3 and Its Electrical Q Measurement

We made another resonator (Nb-#3) to learn the reproducibility of the high Q resonator under the influences of many factors, particularly the EDM, chemical etching and heat treatment.

Since Nb-#1 offered the highest mechanical Q, Nb-#3 was made following the same general procedure of making Nb-#1 from step 1 to step 16, except in step 11, the EDM was applied to reduce the stress buildup into the niobium. However, Solomonson discovered that EDM is not a completely stress free method for machining niobium. We therefore modified our spark cutter further by cutting its power by a factor of 4, since this reduced the stress buildup and could help improve the Q further.

We deeply etched Nb-#3 in step 15 for over 13 minutes to remove any contaminated and stressed layers from the surface of the resonator and its arms.
During this etching process, we tuned the transducer mode by iterating the etching-measuring process three times. We stopped the etching after reaching a mode frequency of 827 Hz at room temperature.

The resonator was stress relieved at $T=930\sim 940\, ^\circ C$ and at a pressure of $(0.9\sim 1.1)\times 10^{-6}$ Torr for 50 minutes. The mechanical Q of the resonator was measured once at 4.2 K. Then we deep annealed the resonator at $T=1400\sim 1490\, ^\circ C$ with pressure range of $(0.7\sim 2.3)\times 10^{-6}$ Torr for 40 minutes.

Now we introduce the electrical circuit used to measure the electrical Q in our experiment. This circuit consists of a pickup coil, a heat switch and a few superconducting joints as depicted in Figure 4.3.

![Electrical circuit for measuring the electrical Q of Nb-#3.](image)

The electrical Q was measured as follows. The flat pickup coil is placed parallel to the diaphragm of the resonator with a very small gap around 0.003". A persistent dc supercurrent is charged into the coil. It generates a magnetic field in the gap between the coil and the diaphragm. The magnetic field is kept below the first critical field of superconducting niobium of $B_1=0.1\, T$ at 4.2 K, since a relatively large
magnetic field may cause motion of trapped flux in the superconducting diaphragm and pickup coil and induces electrical losses. The field exerts a pressure on the diaphragm and changes the overall spring constant of the resonator. The tuned resonant frequency \( f_t \) of the resonator changes as the magnetic field \( B \) changes. The transducer coupling coefficient \( \beta \) changes with the tuned frequency \( f_t \) according to equation (3.14)

\[
\beta = 1 - \left( \frac{f_0}{f_t} \right)^2
\]  

(4.1)

where \( f_0 \) is the resonator frequency of the transducer at zero magnetic field. Since the loss is proportional to \( 1/Q \), and assuming the mechanical and electrical losses add linearly, we get the expression for the overall \( Q \) (\( Q_{\text{loaded}} \)) of the resonator [Paik, 1976]

\[
\frac{1}{Q_{\text{loaded}}} = \frac{1}{Q_0} + \frac{\beta}{Q_e}
\]

(4.2)

where \( Q_0 \) is the mechanical \( Q \) at \( I_0=0 \) and \( Q_e \) is the electrical \( Q \) of the resonator.

In practice, since \( Q_{\text{loaded}} \) is a function of \( \beta \) in the phenomenological equation (4.2), it is therefore a function of the magnetic field in the gap between the pickup coil and the diaphragm. After each pair \( (\beta, 1/Q_{\text{loaded}}) \) is measured with a different dc current \( I_0 \) in the pickup coil, \( Q_e \) can be calculated from the slope of straight line fit to the \( (\beta, 1/Q_{\text{loaded}}) \) pairs.

The pickup coil we used was wound with 236 turns of niobium wire and had an inside diameter of 0.35" and an outside diameter of 1.35", which is slightly smaller than the size of the diaphragm's outer diameter (=1.40"). The wire was formvar insulated with an outside diameter of 0.004" and a 0.003" diameter niobium core. Emerson and Cuming's Stycast 1266 epoxy [Emerson and Cuming, Inc., 77 Dragon Court, Woborn, MA 01888] was used to glue the wire that was wound into a spiral on
a G-10 fiberglass form. The G-10 form with the pickup coil was held to the niobium coil holder with a 5/16" niobium screw. No glue was used between the G-10 coil form and the coil holder. A 0.032" diameter hole was drilled through the niobium coil holder to allow the twisted coil leads to come outside the coil holder.

The heat switch was made by first wrapping one layer of thin lead foil around a 1.8 kΩ 1/8 W carbon resistor and then winding 7~8 turns of Nb wire non-inductively around the outside. The assembly is potted together using 5 minute epoxy. The 0.008" diameter of the Nb heat switch wire is chosen to be larger than the 0.003" diameter of the pickup coil wire. This is to ensure that the Nb leads at the switch do not burn up in case of an accidental quench under vacuum.

All the four superconducting joints are made using a spot welder. The thin copper cladding at the end of the copper clad niobium wire was etched off prior to the welding by immersing it in a mixture of 1:1:1 by volume ratio of 37% HCl, 70% HNO₃, and H₂O.

We tested the mechanical Q of Nb-#3 before we did the deep annealing at T=1400~1490 °C. At 4.2 K, a mechanical Q=3.4x10⁶ was measured at the transducer mode f₀=851.5 Hz (designed for f₀=857 Hz at 4.2 K). This Q was the same as that of the Nb-#2 made under exactly the same condition of fabrication and treatment. At this stage we knew that the reproducibility was pretty good. But the mechanical Q was still not high enough without the deep annealing.

After we did the deep annealing at above T=1400 °C for 40 minutes, we measured the electrical Q of Nb-#3. We first measured the mechanical Q at I₀=0 to check the efficiency of the deep anneal. We got Q=7.4x10⁶ at 4.2 K with a resonant frequency f₀=845.850 Hz, Q=9x10⁶ at around 7 K and Q=1.3x10⁷ at around 8.3 K.

The HP computer was used to collect the data set including the amplitude and the time for each different dc current I₀. Amplitudes measured at a resonant frequency
$f_0$ were read in units of dBV on the HP signal analyzer. After each set was measured, that data set was transferred through the RS-232C interface to a Macintosh IIci, where data was processed. To get the $Q$ from each data set, we did the calculation based on the relationship among the $Q$, amplitude $x(t)$ and time $t$

$$x(t) = x(0)e^{-\frac{t}{\tau}}$$  \hspace{1cm} (4.3)

where the time constant is

$$\tau = \frac{Q}{\pi f_0}.$$ \hspace{1cm} (4.4)

Since the decibel amplitude $x_{dB} = 20 \log_{10}(x)$ was what we actually measured, equation (4.3) becomes

$$\ln \left[ 10^{x_{dB}(t)/20} \right] = \left( -\frac{\pi f_0}{Q} \right) t + \ln[x(0)].$$ \hspace{1cm} (4.5)

Applying least square method to fit the data $[\ln (10^{x_{dB}(t)/20})$ vs. $t]$ into a straight line, we get the $Q$ from the slope according to equation (4.5).

Before charging dc current $I_0$ into the pickup coil, a few to a few tens of millitorr was usually put into the experimental chamber to prevent the superconducting joints from quenching. We charged the coil with the current we wanted through the power leads to the pickup coil and quickly turned on the heat switch. After we saw the mode frequency shift due to this current, we turned off the heat switch to trap the supercurrent. The source of the dc current could be reduced to zero at this time.

To make the $Q$ measurement, one has to keep the temperature stable, since the $Q$ of the niobium resonator is temperature dependent. Our dewar insert did not hold constant temperature automatically when it was immersed in the helium bath. If one
pumps a good vacuum (<10^{-6} Torr), the temperature at the resonator drifts up slowly.

To avoid the drifting temperature we pumped the LHe bath for an hour to reach T~2.1 K, then we pumped out the helium exchange gas to get P~2.3\times10^{-5} Torr in the experimental chamber. Under this condition the system reached an equilibrium at T~2.3 K in about an hour. We excited the mode that we wanted to measure by connecting the PZT to a signal generator briefly, and then started the program to collect data.

For a mechanical Q measurement at I_0=0, a typical data set and the line fitted to it are plotted in Figure 4.4. Q_0=9.41\times10^6 was measured at T=2.3 K and f_0=845.850 Hz.

![Figure 4.4](image-url)

\[ T = 2.3 \text{ K}, \quad P = 2.5 \times 10^{-5} \text{ Torr} \]

For I_0= 0.00 A:
- \( f = 845.850 \text{ Hz} \)
- \( \beta = 0 \)
- \( Q_{fr} = 9.41 \times 10^6 \)
For a typical loaded Q measurement, we got $Q_{\text{fit}} = 3.22 \times 10^6$ with $I_0 = 5.32$ A at $T = 2.3$ K. The resonant frequency was shifted to 916.682 Hz. This corresponded to a coupling coefficient of $\beta = 15\%$. The typical data set and the line fit are plotted in Figure 4.5 for this case. We observed that the Q displayed an amplitude dependence that was a function of the magnetic field and the resonant amplitude. At sufficiently small amplitudes, Q seemed to be amplitude independent. We fitted for Q where there was no amplitude dependence.

For $I_0 = 5.32$ A:

- $f = 916.682$ Hz
- $\beta = 0.1486$
- $Q_{\text{fit}} = 3.22 \times 10^6$ ($t = 2200 \sim 3000$ sec)

Figure 4.5 $\frac{x_{\text{dB}}}{\ln[10^{20}]}$ vs. time for Nb-#3 at $I_0 = 5.32$ A.
According to equation (3.12), \( f_t^2 \) is a linear function of \( I_0^2 \) with slope \( \Gamma \)

\[
f_t^2 = f_0^2 + \Gamma I_0^2
\]

(4.6)

where \( f_t \) is the tuned frequency of the resonator and \( \Gamma \) is the tuning parameter. A measured tuning curve verifies this linearity and gives the value of \( \Gamma \). The tuning curve for Nb-#3 is shown in Figure 4.6. When the persistent current \( I_0 \) was smaller than about 3.3 amps, a straight line fits the tuning curve well. Good linearity agrees with theory in this range of the magnetic field between the pickup coil and the diaphragm. When the introduced current was larger than this value (3.3 amps) the

![Figure 4.6 The tuning curve for Nb-#3.](image)
tuning curve started to deflect away from the fitted straight line. We suspect that this might be due to the field exceeding 0.1 T, the first critical field of Nb. The coil we wound was not a perfect one layer coil, but instead, its number of turns was almost doubled.

All the loaded Q's measured are listed in Table 4.3 and shown in Figure 4.7 as

<table>
<thead>
<tr>
<th>Q&lt;sub&gt;loaded&lt;/sub&gt;</th>
<th>f&lt;sub&gt;t&lt;/sub&gt; (Hz)</th>
<th>I&lt;sub&gt;0&lt;/sub&gt; (A)</th>
<th>β</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.4x10&lt;sup&gt;6&lt;/sup&gt;</td>
<td>845.850</td>
<td>0.00</td>
<td>0.0000</td>
</tr>
<tr>
<td>5.2x10&lt;sup&gt;6&lt;/sup&gt;</td>
<td>851.460</td>
<td>1.41</td>
<td>0.0131</td>
</tr>
<tr>
<td>2.3x10&lt;sup&gt;6&lt;/sup&gt;</td>
<td>857.060</td>
<td>2.00</td>
<td>0.0260</td>
</tr>
<tr>
<td>1.4x10&lt;sup&gt;6&lt;/sup&gt;</td>
<td>859.880</td>
<td>2.24</td>
<td>0.0324</td>
</tr>
<tr>
<td>3.6x10&lt;sup&gt;6&lt;/sup&gt;</td>
<td>862.590</td>
<td>2.45</td>
<td>0.0384</td>
</tr>
<tr>
<td>1.0x10&lt;sup&gt;6&lt;/sup&gt;</td>
<td>867.880</td>
<td>2.83</td>
<td>0.0501</td>
</tr>
<tr>
<td>2.5x10&lt;sup&gt;6&lt;/sup&gt;</td>
<td>870.480</td>
<td>3.00</td>
<td>0.0558</td>
</tr>
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<td>3.4x10&lt;sup&gt;6&lt;/sup&gt;</td>
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<td>3.16</td>
<td>0.0615</td>
</tr>
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<td>3.46</td>
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<td>880.870</td>
<td>3.61</td>
<td>0.0779</td>
</tr>
<tr>
<td>5.8x10&lt;sup&gt;6&lt;/sup&gt;</td>
<td>883.310</td>
<td>3.74</td>
<td>0.0830</td>
</tr>
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<td>885.610</td>
<td>3.87</td>
<td>0.0878</td>
</tr>
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<td>4.9x10&lt;sup&gt;6&lt;/sup&gt;</td>
<td>888.310</td>
<td>4.00</td>
<td>0.0933</td>
</tr>
<tr>
<td>4.4x10&lt;sup&gt;6&lt;/sup&gt;</td>
<td>892.900</td>
<td>4.24</td>
<td>0.1026</td>
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<td>4.2x10&lt;sup&gt;6&lt;/sup&gt;</td>
<td>897.750</td>
<td>4.47</td>
<td>0.1123</td>
</tr>
<tr>
<td>4.0x10&lt;sup&gt;6&lt;/sup&gt;</td>
<td>902.270</td>
<td>4.69</td>
<td>0.1212</td>
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<td>3.6x10&lt;sup&gt;6&lt;/sup&gt;</td>
<td>907.100</td>
<td>4.90</td>
<td>0.1305</td>
</tr>
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<td>3.7x10&lt;sup&gt;6&lt;/sup&gt;</td>
<td>911.650</td>
<td>5.10</td>
<td>0.1391</td>
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<td>3.3x10&lt;sup&gt;6&lt;/sup&gt;</td>
<td>915.910</td>
<td>5.29</td>
<td>0.1471</td>
</tr>
<tr>
<td>3.2x10&lt;sup&gt;6&lt;/sup&gt;</td>
<td>916.680</td>
<td>5.32</td>
<td>0.1486</td>
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<td>920.290</td>
<td>5.48</td>
<td>0.1552</td>
</tr>
<tr>
<td>1.7x10&lt;sup&gt;6&lt;/sup&gt;</td>
<td>932.270</td>
<td>6.00</td>
<td>0.1768</td>
</tr>
</tbody>
</table>
$1/Q_{\text{loaded}}$ vs. $\beta$. The highest current that we tried to charge into the pickup coil was 6.0 amps. The peak in the data of Figure 4.7 is probably due to a suspension mode that lowered the loaded $Q_s$ in the neighborhood of its resonant frequency.

![Graph](image)

**Figure 4.7** Overall measurements of $1/Q$ vs. $\beta$ for Nb-#3.

The electrical $Q$ was the most important parameter measured for Nb-#3. It was obtained from the slope of a plot of $1/Q_{\text{loaded}}$ vs. $\beta$. A straight line fit to the open circles in Figure 4.7 gave $Q_e=5.0 \times 10^5$, and is shown in close detail in Figure 4.8.
Figure 4.8 Selected $1/Q_{\text{loaded}}$ vs. $\beta$ for Nb-#3.

Figure 4.9 Selected $\beta Q_{\text{loaded}}$ vs. $\beta$ for Nb-#3.
The data at low $\beta$ near the suspension resonance is ignored. Another way to identify the description of the electrical quality factor of Nb-#3 is expressed by a plot of $\beta Q_{\text{loaded}}$ vs. $\beta$ as shown in Figure 4.9. This relationship is easier to establish in practice, because one just needs a single measurement of the $Q_{\text{loaded}}$ at a single $\beta$ to get a value of $\beta Q_{\text{loaded}}$. When $Q_0$ is large, $\beta Q_{\text{loaded}}$ is a good approximation of $Q_e$ as seen in equation (4.2).

### 4.4 Discussion of the Test Results and Suggestions for Future Work

Table 4.4 summarizes all the Q measurements we have done so far.

<table>
<thead>
<tr>
<th>Resonator</th>
<th>$Q_m$ at 4 K</th>
<th>Mode frequency at 4 K</th>
<th>$Q_m$ at 4 K after deep anneal</th>
<th>Mode freq. at 4 K after deep anneal</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al-#1</td>
<td>very low</td>
<td></td>
<td></td>
<td></td>
<td>deformed arms</td>
</tr>
<tr>
<td>Al-#2</td>
<td>$8.3 \times 10^6$</td>
<td>1104.5 Hz</td>
<td></td>
<td></td>
<td>good indication</td>
</tr>
<tr>
<td>Nb-#1</td>
<td>$2.4 \times 10^6$</td>
<td>705.6 Hz</td>
<td>$2.4 \times 10^7$ (3.0$\times 10^7$ at $\sim$7 K)</td>
<td>694.0 Hz</td>
<td>big improvement of $Q_m$ after the deep anneal</td>
</tr>
<tr>
<td>Nb-#2</td>
<td>$3.4 \times 10^6$</td>
<td>861.5 Hz</td>
<td>yet to measure</td>
<td></td>
<td>started to use EDM to cut the arms</td>
</tr>
<tr>
<td>Nb-#3</td>
<td>$3.4 \times 10^6$</td>
<td>851.8 Hz</td>
<td>$7.4 \times 10^6$ (1.3$\times 10^7$ at 8.3 K)</td>
<td>846.0 Hz</td>
<td>$Q_e = 5.0 \times 10^5$ and $\beta = 0.15$ at 2.3 K</td>
</tr>
</tbody>
</table>

Obviously, deep annealing made the decisive contribution in improving the $Q$s of the resonators. The mechanical $Q$ of Nb-#1 was improved by more than an order of
magnitude after deep annealing. The high Q of Nb-#1 also proved that the niobium plate purchased from Teledyne Wah Chang had the required quality. Getting the right material is always the most important step to start with. The EDM method may also receive its part of the credit for getting the high Qs.

A few clues seem to suggest that the actual $Q_0$ and $Q_e$ may be higher than we measured.

1. The dissipation peak shown in Figure 4.7 seems to indicate that all the Qs measured were more or less under the influence of the suspension losses or loose wires to the resonator-coil holder assembly. In the transducer mode there is a significant motion of the center of mass. This energy can be exchanged and even absorbed by the suspension system.

2. Based on figure 4.7, we got $Q_e=5.0 \times 10^5$ and $Q_{0\text{ fit}}=6.9 \times 10^7$ from the slope and the intercept of the line fit, respectively. Although $Q_{0\text{ fit}}=6.9 \times 10^7$ may be too high to be believed, it may imply that the mechanical Q that we measured was still lower than its actual value. Figure 4.7 seems to show that $Q_0$ was measured in the tail of the dissipation peak. The high value of $Q_{0\text{ fit}}$ is based on the fit to the data marked with "o" in Figure 4.7. If the actual $Q_0$ is between $9.4 \times 10^6$ and $Q_{0\text{ fit}}=6.9 \times 10^7$, a shallower slope would result. This means that the electrical Q may be higher than $5.0 \times 10^5$.

3. Amplitude dependence of the loaded Q measurements is clearly seen in Figure 4.5. Its higher Q was obtained in the region of small amplitudes. It is yet to be confirmed that an even higher Q might result when an rf SQUID is used to read even smaller amplitudes. The PZT requires much larger amplitudes than we would use on a transducer operating on the antenna.

4. It is possible that the Q measurements were affected by the pressure. This allows the possibility that the measured Qs were lower than their actual values. Our
study of Nb-#1 showed that once below $4 \times 10^{-5}$ Torr the Q measurement was not
dependent on pressure. But when a pickup coil was added in the measurement of Nb-
#3, there was a very small gap (~ 0.003") between the pickup coil and the diaphragm
of the resonator. This may require a lower pressure for decent Q measurements.
Since most of our measurements were done around $P=2.3 \times 10^{-5}$ Torr, this pressure
might not be low enough for the measurements to be pressure independent. This
pressure effect needs more extensive study.

In the future we intend to do the following. (1) Wind a new pickup coil with
perfect single layer of superconducting wire. This will make the calculation of the
magnetic field between the pickup coil and diaphragm of the resonator more accurate.
The effect of the magnetic field on the loaded Q measurements will be done carefully.
(2) Anchor down the loose wires coming out the resonator more carefully to eliminate
the possibility of having uncontrolled resonances as shown in Figure 4.7. After doing
these two things, we will test Nb-#3 again under the same conditions as previously
done. (3) Use an rf SQUID to amplify the small signal induced in the coil to see
whether this will change the Q measurement. (4) We will also deep anneal Nb-#2 to
test whether the annealing improves its quality factor. (5) Build an intermediate
resonator out of aluminum alloy 5056 and install the Nb-#3 into it to form a 2-mode
transducer. Besides using bolts, the intermediate resonator will hold the inner niobium
resonator by differential thermal contraction. To really use this concept for our
transducer, we need to make sure that the Qs of the niobium resonator are not
worsened by the contraction force that holds the resonator. (6) Design and make some
cavities in a niobium cover plate to hold the dc SQUID amplifier, other
superconducting joints, and heat switches with good shielding against EMI.
Finally, assemble the whole system together by attaching the 2-mode transducer to the aluminum bar of our antenna to form a 3-mode detector of gravitational waves.

4.5 Predicted Future Performance of the Three-mode Detector System

The performance of the 3-mode system with the resonator (Nb-#3) developed above can be predicted by running the program based on the analysis discussed in section 3.3. The noise temperature of the three-mode system at 4.2 K with a $500 \, \hbar$ SQUID is calculated to be $T_n=170 \, \mu K$ which corresponds to a strain sensitivity $h=1.0 \times 10^{-19}$. Judged by the noise performance and bandwidth, the improvement from our current 2-mode system to this 3-mode system can be clearly seen from the strain noise spectrum in Figure 4.10. In addition, if this 3-mode detector is cooled to 50 mK,

![Figure 4.10](image-url)  
Figure 4.10 The strain noise spectrum for the LSU detector at 4 K.
with a 10 $h$ SQUID, a noise temperature of 2.4 $\mu$K or a strain sensitivity $1.2 \times 10^{-20}$ is expected.
CHAPTER 5

CONCLUSION

In this dissertation, we first discussed the test of the best available commercial dc SQUID that can be used from 4.2 K to 50 mK. Then we discussed the design of a two-mode transducer for the LSU 4.2 K antenna. The test results of some fabricated resonators was discussed.

Our experiments showed that the dc SQUID made by Quantum Design operates well down to 50 mK. We found that the SQUID had an energy resolution of 2600 $\hbar$. It is suitable to work on any 50 mK antenna. It could also be used on other experiments done at ultra-low temperatures. The tests showed that the SQUID's noise did not fall with temperature as expected. The possible reasons for this were discussed in detail.

A two-mode inductive transducer was designed to improve the sensitivity and bandwidth of the LSU antenna. Based on the noise analysis model and the parameters of SQUIDs, we chose the mass of the transducer to be 8-9 grams. We designed the geometry of the small mass resonator with large surface area to mass ratio. Both the small mass resonator and the intermediate resonator were designed with the aid of finite element analysis. This helped us to decouple all the other internal resonant modes of the designed resonators by at least 180 Hz from the transducer mode. Subsequent tests verified the design.

We built two aluminum prototype resonators. A high mechanical Q of 8.3x10^6 was measured at 4.2 K for one of the aluminum resonators, which assured us that the design was good. We constructed three more niobium resonators afterwards. During the fabricating process of the resonators, an effective method of annealing the niobium
resonators was discovered and applied to obtain high mechanical Qs. One of the niobium resonators reached mechanical Qs of $2.4 \times 10^7$ at 4.2 K and $3.0 \times 10^7$ at around 7 K. We cannot use this resonator to build the transducer because it does not have the correct resonant frequency. Another niobium resonator that underwent the same procedure but used EDM arm-cutting reached mechanical Qs of $7.4 \times 10^6$ at 4.2 K and $1.3 \times 10^7$ at 8.3 K. Further measurement of this resonator showed an electrical Q ($Q_e$) of $5.0 \times 10^5$ and an electro-mechanical coupling efficiency ($\beta$) of 15%. This is equivalent to an average $\beta Q_{\text{loaded}}$ of $4.7 \times 10^5$.

A three-mode detector system made of this resonator and a 500 $\hbar$ SQUID would reach a noise temperature of 170 $\mu$K or corresponding strain sensitivity of $\hbar=1.0 \times 10^{-20}$. This resonator is also suitable to be used on the spherical antenna of Johnson and Merkowitz [Johnson and Merkowitz, 1993 and 1994].

We suspect that the actual mechanical and electrical Qs may be higher than we measured, due to the mechanical dissipation in the suspension system and other reasons. We need to make further improvements in our experiments and measurements to verify this concern.
REFERENCES


APPENDIX A

THE CHARACTERISTIC GRAVITATIONAL WAVE AMPLITUDE

A gravitational wave source produces a quadrupole force field at the earth. To discuss the detection of this signal, Thorne [Thorne, 1987] defined the characteristic frequency $f_c$ and the characteristic amplitude $h_c$ of a gravitational wave which include information about the noise performance of the detector as

$$f_c = \left[ \int_0^\infty \frac{\left| \tilde{h}_+^2(f) + \tilde{h}_\times^2(f) \right|^2}{S_h(f)} df \right]^{1/2},$$

(A.1)

$$h_c = \left[ 3 \int_0^\infty \frac{S_h(f_c)}{S_h(f)} \left( \left| \tilde{h}_+^2(f) + \tilde{h}_\times^2(f) \right|^2 \right) df \right]^{1/2},$$

(A.2)

where the detector's noise spectral density $S_h(f)$ is related to its noise spectrum $\tilde{h}(f)$ as

$$S_h(f) = \left[ \tilde{h}(f) \right]^2$$

(A.3)

with dimension $Hz^{-1}$, $\tilde{h}_+(f)$ and $\tilde{h}_\times(f)$ are the spectra of the corresponding amplitudes of a plane gravitational wave associated with two independent polarization states, and $\langle ... \rangle$ denotes average over several wavelengths.

It is often useful to rewrite the $f_c$ and $h_c$ in terms of the energy flux per unit frequency $dE_{GW}/dAdf$ carried past the detector by the waves. According to the waves' stress-energy tensor and Parseval's theorem, it has

$$\frac{dE_{GW}}{dAdf} \approx \frac{\pi c^3}{2 G} f^2 \left\langle \left| \tilde{h}_+^2(f) + \tilde{h}_\times^2(f) \right|^2 \right\rangle.$$  

(A.4)
When this quantity is averaged over all directions associated with the source, it gives \( \text{d}E_{GW}/(4\pi r^2) \text{d}f \), where \( r \) is the distance to the source. Consequently, equation (A.2) becomes

\[
h_c = \left[ \int_0^\infty \frac{S_h(f_c)}{S_h(f)} \frac{3}{2\pi^2 r^2} \frac{G \text{d}E_{GW}}{c^3} \frac{\text{d}f}{\ln f} \right]^{1/2}.
\]

(A.5)

For most burst sources (e.g. supernovae) the waveform is so uncertain that a careful calculation of \( h_c \) is unjustified. In such cases it is useful to express the characteristic amplitude \( h_c \), approximately, in terms of the total energy \( \Delta E_{GW} \)

radiated as

\[
h_c \approx \left[ \frac{3}{2\pi^2} \frac{G \Delta E_{GW}}{c^2 f_c r^2} \right]^{1/2} = 2.7 \times 10^{-20} \left[ \frac{\Delta E_{GW}}{M_\odot c^2} \right]^{1/2} \left[ \frac{1 \text{ kHz}}{f_c} \right]^{1/2} \left[ \frac{10 \text{ Mpc}}{r} \right]^{1/2}
\]

(A.6)

where \( M_\odot \) is the solar mass and 10 Mpc is the estimated distance to the center of the Virgo Cluster (assuming a Hubble constant of 100 km s\(^{-1}\) Mpc\(^{-1}\)).
APPENDIX B

LETTER OF PERMISSION

October 17, 1994

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500 Sunnyside Blvd.
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Dear Madam/Sir,


Thank you very much for your timely response.

Sincerely yours,

Ziniu Kenny Geng

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500 Sunnyside Blvd.
Woodbury, NY 11797
VITA

Ziniu Geng was born on April 28, 1961 in Beijing, China, to Jinluan Liu and Shuping Geng. He attended the high school attached to the People's University of China in Beijing.

He attended Beijing University in China, from 1979 to 1983, and graduated with a Bachelor of Science degree in Electrical Engineering.

In 1983 he began his graduate study in the Institute of Electronics of the Chinese Academy of Sciences. He received his Master's degree in Electrical Engineering in 1986.

He attended Louisiana State University in 1987 and received his master's degree in physics in the fall of 1989. He then began research in gravitational wave detection under his major professor, William O. Hamilton and is currently a candidate for the degree of Doctor of Philosophy.
DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate:
Ziniu Geng

Major Field:
Physics

Title of Dissertation:
Advanced Gravitational Radiation Transducers with Small Mass and Ultra-low Temperature SQUIDs

Approved:

[Signatures]

Major Professor and Chairman
Dean of the Graduate School

EXAMINING COMMITTEE:

[Signatures]

Date of Examination:
11-3-94