Determining Process Target Value and Ordering Policies in Two-Echelon Hierarchy Production System.

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Determining process target value and ordering policies in two-echelon hierarchy production system

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The Louisiana State University and Agricultural and Mechanical Col., 1994
DETERMINING PROCESS TARGET VALUE AND ORDERING POLICIES IN TWO-ECHELON HIERARCHY PRODUCTION SYSTEM

A Dissertation

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Doctor of Philosophy

in

The Interdepartmental Program in Business Administration

by

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ABSTRACT

Selection of the mean (target value) for a production process is a classical problem in quality control. Basically, a process mean is selected based on a balance between production cost and economical consequences associated with conforming items and nonconforming items.

The process mean affects many production decisions. In particular, because it determines the process conforming and yields rates, it affects the production setup policy. These production decisions also affect the raw material requirement and, thus, its procurement policy when the raw material is supplied by outside vendors. Consequently, process mean, production and raw material procurement policies should be jointly determined to minimize the total related costs. Furthermore, in practice, quantity discounts may be available in raw material purchasing. Because of the interaction between the process mean determination and the raw material ordering policy, quantity discounts will affect all of the related decisions.

This dissertation consists of three parts. In the first part, a two-echelon model is formulated to incorporate the issues associated with production setup and raw material procurement into the classical process mean problem for a single-product production process.

In the second part, quantity discounts in raw material purchasing are incorporated into the model. The quantity discounts policy under study is known as all-unit quantity discounts.

In the third part, we consider a situation in which the supply rate of the raw material is finite and constant. Three cases in terms of quantity discounts in the raw material purchasing are considered: no discounts, incremental quantity discounts and all-unit quantity discounts.
Mathematical models are formulated for all the cases discussed above. Analytical properties are derived and efficient solution algorithms are proposed. Examples are used to illustrate the solution procedures and sensitivity analyses are performed to study the effects of model parameters on the optimal solutions.
Selection of the process mean (target value) for a production process is a classical problem in quality control. A typical scenario considered in this problem is as follows. The product of interest has a performance variable with a lower specification limit, and the raw material requirement for producing the product is an increasing function of the performance variable. The amount of raw material used to produce an item is a random normal variable with a known variance, but whose mean depends on a process setting. Each item of the product is inspected after being produced. An item is classified as conforming if its value of performance variable is larger than or equal to the lower limit. Otherwise, the item is a nonconforming item.

Depending on the production and marketing environments, conforming items may be sold at the full price and nonconforming items may be sold at reduced prices, reworked, or scrapped. Since the product is produced by a production process with an adjustable mean and a constant variance, a higher process mean may be used to increase the process conforming rate, but, it will increase the cost of producing the product. Consequently, a process mean should be selected based on a balance between production cost and economical consequences associated with conforming items and nonconforming items.

Typical performance variables under study are weight, volume, count, and concentration. For example, for the producer of a certain night cream, the key ingredient is expensive; therefore the amount of cream filled in a jar becomes a very
important issue. The producer has to consider the material cost associated with overfilling the jar as well as the consequences of underfilling it. Similar examples can be found in the pharmaceutical industry, because many medicines need a minimum dosage of certain ingredients to ensure the effect of the drugs.

The process mean issue is especially important to the producer when material-related costs are a significant portion of production cost. Because the process mean determines the process conforming and yield rates, it affects other important production decisions; in particular, the production setup policy. When the process mean changes and so does the conforming rate, an economical production size has to be determined in order to satisfy the demand and avoid unnecessary setups. These production decisions directly affect the raw material requirement and, thus, its procurement policy if the material has to be ordered from outside vendors. Consequently, process mean, production, and raw material procurement policies should be jointly determined in order to control the total related costs.

Furthermore, the producer may seek a steady supply of the raw material through an arrangement with outside vendors. For the same reasons mentioned above, the process mean may change the production run size as well as the raw material requirement. Therefore, the material supply rate has to be determined simultaneously with the process mean and production setup policy.

When the material procurement policy is studied, it is assumed, most of the time, that the material unit cost is independent of the order quantity. In reality, however, material unit cost (including transportation and other costs) does depend on the order quantity. This may happen when quantity discounts are offered by vendors. Furthermore, a larger order quantity leads to a smaller average material cost per unit, when a fixed setup cost and transportation cost are incurred per order.
Although many different types of quantity discounts policies exist, two are most commonly discussed in the literature; namely, incremental quantity discounts and all-unit quantity discounts. In the incremental quantity discounts model, the discounts apply only to the additional units beyond a certain quantity over which a discount is given. In this case, the total material cost is a continuous function of the order quantity.

In the all-unit quantity discounts model, discounts apply to all the units purchased. As a result, the total material purchasing cost is a discontinuous function of the quantity ordered. This type of discount is very popular in practice. Note that in this model it is possible that, the cost of purchasing a quantity that is below a quantity where a discount is applied may be higher than that of purchasing the larger discount-applicable quantity. In this situation, we may prefer to order a sufficiently large quantity to qualify for a certain discount and then dispose of the excess units to save on inventory costs. This option has not drawn much research interest in the literature.

In this dissertation, we consider three models. In the first model, we incorporate the issues associated with production setup and raw material procurement into the classical process mean problem for a single-product production process. It is assumed that the product of interest requires one major raw material, which is purchased from outside vendors, and that the material unit cost is independent of the procuring quantity. The production cost of an item is a linear function of the amount of the raw material used in producing the item. The product has a lower specification limit, and the items that do not conform to the specification limit are scrapped with no salvage value. A two-echelon model is formulated for jointly determining the process mean, production setup, and raw material ordering policies.

In the second model, we incorporate the all-unit quantity discounts in the raw material cost into the model for joint determination of process mean, production setup,
and raw material ordering policies. The option of ordering an excess amount of the raw material is considered. The results of this model are compared with those of the first model in which no discount is applied.

In the third model, a different raw material ordering policy is considered. It is assumed that the raw material is supplied at a constant rate from outside vendors. Three cases are considered. In the first case, no quantity discounts are available; in cases 2 and 3, the incremental and all-unit quantity discounts policies are considered.

The organization of the paper is as follows. A comprehensive literature review of related areas is given in the next chapter. The three models are discussed in chapters 3, 4, and 5, respectively. Included in the discussion of each model are the assumptions and model formulation, the analytical properties of the optimal solutions, proposed solution algorithms, and numerical examples and sensitivity analyses on the effects of model parameters on the optimal solutions. Chapter 6 includes a brief summary of the results given in this dissertation and a discussion of possible future extensions.
Chapter 2
LITERATURE REVIEW

2.1 Introduction

As stated earlier, it is important to consider production and raw material procurement policies in determining the process mean. Because a change in the process mean affects the process conforming rate, an economical production run size has to be determined not only to satisfy the demand, but also to avoid unnecessary process setups. These decisions directly affect the requirement of the raw material and its procurement policy when the material is supplied by outside vendors. As a result, the process mean, the production run size, and the material procurement policy should be jointly determined.

A simultaneous consideration of production run size and material order policy is a typical two-echelon production and inventory problem in which two product statutes are considered: raw materials and finished products. When the process mean issue is not considered and all the items produced by the process are assumed to be conforming, the problem is very closely related to the single-product, two-echelon inventory problem.

Quantity discounts are a common practice used by vendors to promote their products. Although many types of discounts policies exist, the two most common cases are the all-unit quantity discounts and the incremental quantity discounts. Because quantity discounts affect the raw material procurement cost, quantity discounts therefore should also be incorporated in the decisions of process mean, production run size, and material order quantity.
Although we develop the solution procedures in this study, one-dimensional search methods are required to perform the procedures. Many one-dimensional search techniques have been widely used. The most favorable and commonly used one is the golden-section search.

This chapter presents the literature review on the following four related areas: (a) the process mean determination, (b) single-product, two-echelon inventory models, (c) quantity discounts policies, and (d) the golden-section search method.

2.2 Process Mean Determination

The process mean determination problem is an issue of finding the most economic setting of the mean of a production process. The problem can be illustrated by figure 1, in which the product of interest is sold in cans. A production process is used to fill the cans continuously with an expensive ingredient. Because of uncontrollable variations in the production process, the amount of the ingredient in a can is a random variable with a mean (process mean) determined by the manufacturer. Assume the variance of the process is constant.

![Figure 1. An Illustration of the Process Mean Determination](image-url)
Let $X$ denote the amount of the ingredient in a can. Suppose the product has a lower specification limit, $L$, so that an item is conforming if its $X$ value is larger than or equal to $L$. Otherwise, the item is nonconforming. Consider two possible values for the process mean, $\mu_1$ and $\mu_2$ ($\mu_1 < \mu_2$). The nonconforming rate is represented by the area on the left side of $L$ and under the distribution curve. It is clear that the process nonconforming rate is relatively lower when $\mu_2$ is used. This is at the expense, however, of excess material being put in conforming items. On the other hand, when $\mu_1$ is used, the material use may be lower and the nonconforming rate is higher. The most economical process mean should be determined by considering a balance between the material cost and the economical consequences incurred because of the nonconforming items.

The material cost required to produce an item is usually assumed to be a linear function of the amount of the material used in producing the item. In most of the papers, the nonconforming items may be scrapped, or sold at reduced prices. However, several recent papers have proposed using an artificial upper limit so that nonconforming items, as well as the items that exceed the upper limit, are reprocessed.

Furthermore, it is usually assumed that a screening procedure is used to identify the nonconforming items. Several papers assume that sampling plans or other inspection methods are used instead.

Based on the above discussion, we group the papers in the process mean area into three categories in our discussion. The first comprises the basic models consisting of early models and traditional models that do not consider reprocessing and other inspection methods. The second category contains models that use reprocessing to reduce material cost, and the third contains models that use inspection plans other than screening to inspect the outgoing items.
2.2.1 Traditional Models

Springer's model (1951) is perhaps the first that addresses the issue of process mean setting, although his model assumptions are quite different from those used by others. He considers a production situation where upper and lower specification limits are both presented and the performance variable follows a gamma distribution. The per-item cost associated with the nonconforming items above the upper specification limit (overfilled items) may be different from those below the lower specification limit (underfilled items). These costs, however, are assumed to be constants (independent of the performance variable). The optimal process mean is obtained to minimize the total costs associated with nonconforming items. A nomograph is given by Nelson (1979) for finding solutions to Springer's model.

Hunter and Kartha (1977) considered a product with a lower specification limit, \( L \), and discussed the situation where nonconforming (underfilled) items can be sold at a (constant) reduced price, \( r \) (where \( r < \alpha \), the net selling price of conforming item) and a penalty, \( g(> 0) \) (give-away cost), is incurred for the conforming items with excess quality (the difference of the performance variable and the lower limit).

The net income of a single item is
\[
I = \begin{cases} 
\alpha - g(x - L) & \text{if } x \geq L \\
r & \text{if } x < L,
\end{cases}
\]
and the expected net income per item is
\[
E(I) = \int_{-\infty}^{\infty} f(x)dx - \int_{L}^{\infty} (x - L)f(x)dx + r \int_{L}^{\infty} f(x)dx,
\]
where \( f(x) \) is the probability density function (pdf) of the normal distribution with mean \( \mu \) and variance \( \sigma^2 \).
It was found that the optimal process mean $\mu^*$ is equal to $L + \delta$, where $\delta$ satisfies the following equation:

$$\frac{\phi(\delta/\sigma)}{\Phi(\delta/\sigma)} = \frac{g\sigma}{a - r}.$$ 

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the pdf and the cumulative distribution function (cdf) of the standard normal distribution. Nelson (1978) provided approximate solutions to the same problem, which give at least two-decimal accuracy to this model.

Bisgaard, Hunter, and Pallesen (1984) argued that Hunter and Kartha's assumption concerning the selling price of the underfilled items is not realistic because it implies that empty cans can be sold for the same price as those close to the lower limit. They, therefore, modified Hunter and Kartha's model by assuming that the selling price of the nonconforming items is a linear function of the performance variable. Assuming the production cost is a linear function of $X$, the net income per item is

$$I = rX - cx - c_0 \quad 0 \leq x < L$$

$$= a - cx - c_0 \quad L \leq x,$$

where $rX$ is the per-item selling price of the nonconforming items and $c_0 + cx$ is the per-item production cost.

It was found that the optimal process mean $\mu^*$ is equal to $L + \delta$, where $\delta$ satisfies

$$\frac{r}{a} \Phi(\frac{\delta}{\sigma}) + \phi(\frac{\delta}{\sigma}) \frac{1 - rL/a}{\sigma^2} - \frac{c}{a} = 0.$$ 

A table is also provided for obtaining the most profitable process mean for selected model parameter values.

Carlsson (1984) followed Hunter and Kartha's model except for assuming the net income function is a piecewise linear function of the performance variable. He also
presented a procedure for obtaining the optimal solution and a table for obtaining the most profitable process mean for selected model parameter values.

2.2.2 Reprocessing

All the papers discussed in the last section assume the existence of a secondary market. Golhar (1987) assumed that only the regular market (fixed selling price) is available for the conforming items and that the underfilled items are emptied and reprocessed at a fixed expense (reprocessing cost). The following simple linear expression, which gives the economically optimum setting for the process mean, was proposed:

$$
\mu^* = L + \sigma[0.712 + 0.471 \ln\left(\frac{R}{C\sigma}\right)],
$$

where $R$ is the per-item reprocessing cost, $C$ is unit material cost, and $\sigma$ is the process standard deviation.

To reduce the material cost incurred for overfilled items, artificial limits have been proposed to screen out some overfilled items for reprocessing. Bettes (1962) studied a situation in which an arbitrary upper limit is used. Underfilled (nonconforming) and overfilled items (larger than the artificial limit) are reprocessed at a fixed cost. Optimal process mean and the upper specification limit are determined simultaneously. He reported the results in a tabulate form for selected values of the lower limit, variance, reprocessing cost, and material cost. For any other values not included in the table, however, burdensome computations are required to find the optimal values of the mean and upper limit.

Golhar and Pollock (1988) extended Golhar's (1987) model to include an artificial upper limit so that nonconforming items as well as the items larger than the upper limit are reprocessed. Simple approximate expressions relating optimal process mean and the upper limit to the process parameters are developed. The optimal
process mean and upper limit are given as $\mu^* = L - \sigma t_1$, and $U^* = \mu^* + \sigma t_2$, respectively, where $t_1$ and $t_2$ can be obtained from the following equations:

$$t_1 \approx (0.441 + 0.696 \sqrt{M})^4$$
$$t_2 \approx -0.746 \sqrt{M}$$

and

$$M = \frac{R}{C}$$

Based on this result, they provided a table that can be used to find the optimal process mean and upper specification limit and another table that can be used to evaluate the cost effectiveness of the upper control limit. Golhar (1988) developed a computer program for calculating the optimal process mean and upper control limit based on Golhar and Pollock's work.

Schmidt and Pfeifer (1991) extended Golhar and Pollock's model to a capacitated production process, of which a fixed amount of filling-process capacity is available. Therefore, rather than maximizing the expected profit per can to be filled, the expected profit per fill attempted should be maximized under this capacitated situation. They provided a simple closed-form expression for the optimal upper limit and a table that can be used to determine the optimal mean.

### 2.2.3 Inspection Plan

The models that have been discussed so far assume implicitly or explicitly that a screening (100% inspection) procedure is used to measure the performance variable in order to determine the selling prices and/or the corrective actions. Tang and Lo (1993) discussed a situation in which a surrogate variable, which is correlated with the performance variable, is used as the screening variable. Using a surrogate variable in inspection is attractive when inspection is costly, time-consuming, or destructive. Because the relationship between the performance variable and the surrogate variable is
not exact, some nonconforming items may be accepted as conforming products and vice versa. A model is developed to jointly determine the most economical process mean and the inspection specification limit of the surrogate variable. Carlsson (1989) discussed a situation in which the lots produced by a manufacturing process are subjected to lot-by-lot acceptance sampling by variables. A control plan is used as a tolerance interval with special attention to MIL-STD-414 B, which is a well-known acceptance sampling plan by variable and has been accepted as an ISO standard (numbered ISO 3951). A weight $k$ and a sample size $n$ depending on lot size $N$, choice of acceptable quality level (AQL), and inspection level are chosen from MIL-STD-414 B; if $\bar{x} - ks > L$, where $\bar{x}$ is the sample mean and $s$ is the sample standard deviation, then the lot is accepted. Boucher and Jafari (1991) studied the same problem except that an attributes sampling plan is used to decide whether a lot should be accepted. Let $D$ be the number of nonconforming items found in sample of size $n$ and $d_0$ is the allowable number of nonconforming units. The lot is accepted if $D \leq d_0$ and rejected if $D > d_0$. By determining the process mean that yields maximum profitability per lot, both the cases of destructive testing and nondestructive testing are considered. Melloy (1991) considered the packaged goods that are subject to the regulatory auditing (compliance tests) schemes of the U.S. Department of Commerce and the associated risk of noncompliance. He took into account not only the variability of the amount filled into a package under a fixed mean, but also the variability of the packaging material's weight, and developed a control policy model that determines the economically optimal settings of the mean and the screening limits, subject to an acceptable level of risk.
2.3 Single-Product, Two-Echelon Inventory Models

The concept of formulating the production and inventory structure of the finished product and the raw material is closely related to the single-product, two-echelon inventory model. The focus of the existing literature on this topic, however, is different: the product under consideration requires several raw materials, and the decision to be made is how to optimally group the raw materials in the procurement process.

Goyal (1977) first proposed an integrated model that incorporates the inventory problems of the raw materials and the product for a single-product manufacturing system. It was pointed out in the paper that the production run size and production schedule for the product have to be known before the procurement policy of a raw material can be determined. As a result, determining the economic procurement policy for raw materials cannot be treated in isolation from the problem of determining the economic run size for the products. The inventory model he presented unifies the inventory problem of raw materials and finished product for a single product system. These assumptions are made: (1) demand for the product is uniform and constant over time; (2) the holding costs of the finished product and the raw material are known and constant over time; (3) the setup costs for the finished product and the raw materials are known and constant over time; (4) no lead time is required to procure the raw materials; (5) the production rate for the product is known; (6) no shortage is allowed for the raw materials or the finished product; and (7) no quantity discounts are available for the raw materials. The objective is to minimize the total annual variable cost, which includes setup and holding costs for both the finished product and the raw materials. In order to determine the economic batch quantity for the product and the economic order quantities for all the raw materials, the time between successive manufacturing runs has to be known first. A search procedure for determining the length of production run (T)}
was therefore proposed. The economic batch quantity for the product is determined by multiplying the product demand and T, and the economic order quantity of jth raw material is the product of demand of jth raw material times T and a positive integer (= $T_j/T$, where $T_j$ is the time interval between successive purchase orders).

Based on Goyal's model, Kim and Chandra (1987) and Banerjee, Sylla, and Eiamkanchanalai (1990) proposed heuristic procedures for finding the strategy to group the raw materials in order to simplify inventory control and reduce cost. Kim and Chandra considered the situation in which one order of the raw materials can cover the need of one or multiple production runs and assumed that the unit replenishing cost in a group is a decreasing function of the number of the raw materials in that group and that all the raw materials within a group have the same time interval between successive orders. They developed a dynamic programming algorithm that optimally classifies raw materials into groups and determines the optimum inventory policies for the product and raw materials so that the total inventory cost per unit time was minimized. Banerjee et al. assumed that one order of raw materials can cover the need of, at most, one production run, and multiple orders can be made within one production run. They formulated the model as a nonlinear mixed integer optimization problem and developed an efficient heuristic procedure that simultaneously determines production batch size for the product and order lot sizes for materials.

Hong and Hayya (1992) modified Goyal's model regarding planning horizon and the demand for raw material. They assumed that the planning horizon is finite, in contrast to Goyal's infinite planning horizon assumption. Furthermore, they assumed that the demand for the raw materials is in the production period only, whereas Goyal assumed that the demand for raw materials within a production run is evenly spaced within the time period. The latter assumption implies the unrealistic situation that raw material is also required during the nonmanufacturing period. They found that the
optimal groups of raw materials are consecutive in $\frac{c_{3i}}{c_{1i}r_{i}}$, where $c_{3i}$, $c_{1i}$, and $r_{i}$ are the unit replenishment cost, unit carrying cost, and demand rate of raw material $i$ during the manufacturing period. As a result, the group problem of raw materials can be solved by the shortest route algorithm. Based on this finding, they developed an exact solution procedure for simultaneously finding the optimal production-inventory policy and grouping the raw materials.

2.4 Quantity Discounts

In reality, some forms of discounts (or cost reduction) may be available when placing orders. The discounts may be realized from a saving on paper work or transportation cost when a large enough amount is ordered, which is essentially the concept of "economy of scale." The discounts may also be simply a reduction in the unit cost as a promotion strategy offered by the vendors. In this dissertation, we consider only the discounts applied to the unit cost of the raw material which is offered by the vendors.

Two types of quantity discounts are most commonly discussed in the literature; namely, all-unit quantity discounts and incremental quantity discounts. In all-unit quantity discounts, the discounts sometimes take the form of price breaks in the following pattern: there are given quantities $Q_{0} = 0, Q_{1}, Q_{2}, ..., Q_{k}, (Q_{j} < Q_{j+1})$, and $Q_{k+1} = \infty$ and prices $c_{0}, c_{1}, c_{2}, ..., c_{k}, (c_{j+1} < c_{j})$. If a quantity $Q$ is ordered, then the unit cost is $c_{j}$ if $Q_{j} \leq Q < Q_{j+1}$. In other words, the total purchase cost (TPC) of the order is $c_{j}Q$. The total TPC for $Q$ units is shown in Figure 2.
For incremental quantity discounts, which are different from all-unit quantity discounts (where discounts are applied to all the units), different prices are charged to units in different quantity brackets; i.e., \( c_0 \) per unit is charged to first \( Q_1 \) units, \( c_1 \) per unit is charged to units \( Q_1 + 1, \ldots, Q_2 \), etc. The TPC of incremental quantity discounts is shown in Figure 3.

\[
\text{TPC}(Q) = \text{TPC}(Q_i) + c_i(Q - Q_i), \quad i = 0, 1, \ldots, k,
\]

where \( \text{TPC}(0) = 0 \), \( Q_0 = 0 \) and \( Q_{k+1} = \infty \).
As shown in Figure 2, the TPC in the all-unit quantity discounts policy is a discontinuous function of the order quantity. In fact, using this discount method, the TPC of a larger quantity may be lower than that of a smaller quantity. An obvious approach to take this advantage is to buy a large enough quantity to qualify for a certain discount and then dispose of the excess material at some disposal costs. Of course, the disposal cost should be small enough to justify this practice. In practice, the disposal cost can range from a negative value to positive infinity. A negative disposal cost may imply, for example, that the excess can be sold at a price that is less than the lowest possible unit cost. The disposal cost should not be less than the negative of the lowest price because, otherwise, the producer can make a profit by simply buying a large enough quantity at the cost less than the negative of disposal cost and disposing (reselling) all of it.

Let $c_d$ represent the unit disposal cost. $Q_r$ is the quantity that the costs of buying it at $c_{i-1}$ per unit and buying $Q_i$ at $c_i$ per unit and disposing $(Q_i - Q_r)$ units are the same; that is,

$$c_{i-1}Q_r = c_i Q_i + c_d(Q_i - Q_r),$$

or

$$Q_r = \frac{c_i + c_d}{c_{i-1} + c_d} Q_i, \quad i \in \{1, 2, \ldots, k\}.$$  

Therefore, the new total purchase cost function of Figure 2 will be just like Figure 4, in which $c_d$ is assumed to be larger than 0. The line segments immediately to the left of the break points $Q_i$ will be horizontal if $c_d = 0$ and upward sloping if $c_d < 0$.

Many books and articles contain a discussion of quantity discounts. To our knowledge, Sethi (1984) is the first and the only person who addressed the issue mentioned above and developed a method for obtaining optimal order quantity for an entire range of unit disposal costs.
2.5 Search Procedures

Search procedures have been used extensively by researchers to numerically locate a solution to an equation or search for an optimum point. In this study, one-dimensional search procedures are used in the solution procedures. Wilde (1964) and authors of other general optimization books have included a detailed discussion of several useful search procedures. Wilde's book reviews five one-dimensional search techniques: uniform search, uniform dichotomous search, sequential dichotomous search, Fibonacci search, and golden-section search. Wilde also studied the efficiency of these search techniques and found that the golden-section search and the Fibonacci search are more favorable than the other three methods.

The golden-section search (search by golden sections), which is used throughout our research, is based on the splitting of a line into two segments, which were actually known in ancient times as the "golden section." The ratio of the whole to the larger segment is equal to the ratio of the larger segment to the smaller segment. Two Fibonacci numbers are used:
Note that $F_1 = (F_2)^2$ and $F_1 + F_2 = 1$. It is necessary to start the search in a direction so as to minimize the function, $f(x)$.

An initial range of interval of uncertainty $\Delta^{(0)}$, which is defined as the interval in which the optimum solution is known to exist; i.e., it must contain the minimum of $f(x)$. We always let $\Delta^{(0)} = x^{(0)}_2 - x^{(0)}_1$, where $x^{(0)}_1$ and $x^{(0)}_2$ are the two end points of the interval of uncertainty for the search. For most practical purposes, the search can be terminated when: (1) the functional evaluations of $x^{(k)}_1$ and $x^{(k)}_2$ become arbitrarily close, or/and (2) changes in the objective function, $f(x)$, become negligible. Thereafter, for the $k^{th}$ stage, the next interval of uncertainty can be computed as follows. Determine

$$y^{(k)}_1 = x^{(k)}_1 + F_1 \Delta^{(k)}$$
$$y^{(k)}_2 = x^{(k)}_2 + F_2 \Delta^{(k)} = x^{(k)}_2 - F_1 \Delta^{(k)}$$

If $f(y^{(k)}_1) < f(y^{(k)}_2)$: $\Delta^{(k+1)} = (y^{(k)}_2 - x^{(k)}_1)$, and $x^{(k+1)}_1 = x^{(k)}_1$, $x^{(k+1)}_2 = y^{(k)}_2$

If $f(y^{(k)}_1) > f(y^{(k)}_2)$: $\Delta^{(k+1)} = (x^{(k)}_2 - y^{(k)}_1)$, and $x^{(k+1)}_1 = y^{(k)}_1$, $x^{(k+1)}_2 = x^{(k)}_2$

If $f(y^{(k)}_1) = f(y^{(k)}_2)$: $\Delta^{(k+1)} = (y^{(k)}_2 - x^{(k)}_1)$ = $(x^{(k)}_2 - y^{(k)}_1)$, and
$$x^{(k+1)}_1 = x^{(k)}_1$$, $x^{(k+1)}_2 = y^{(k)}_2$ or
$$x^{(k+1)}_1 = y^{(k)}_1$$, $x^{(k+1)}_2 = x^{(k)}_2$

For better precision, two new points (rather than one new point) are determined each time because the values of $F_1$ and $F_2$ may not be exact, and with only one new point, numerical roundoff can cause the bracket on the minimum to be lost.

To illustrate this method, we consider a search to locate the minimum of the following function, $f(x) = x^2 - 6x + 9$. It is assumed that the search is terminated when $|x^{(k)}_1 - x^{(k)}_2| \leq 0.05$ and $|f(x^{(k)}_1) - f(x^{(k)}_2)| \leq 0.05$, and the initial range of $x$ is $0 \leq x \leq 10$. 

$$F_1 = \frac{3 - \sqrt{5}}{2} \approx 0.38,$$ and 
$$F_2 = \frac{\sqrt{5} - 1}{2} \approx 0.62.$$
The first two points are first placed symmetrically within the interval $0 \leq x \leq 10$. The golden-section ratio places these points at:

$$y_{1}^{(0)} = 0 + 0.38(10 - 0) = 3.8$$

and

$$y_{2}^{(0)} = 0 + 0.62(10 - 0) = 6.2.$$  

Hence,

$$f(y_{1}^{(0)}) = 0.64$$

$$f(y_{2}^{(0)}) = 10.24$$

Because $f(y_{1}^{(0)}) < f(y_{2}^{(0)})$, the region to the right of $x = 6.2$ can be eliminated. Therefore, the interval range of the interval of uncertainty $\Delta^{(1)} = y_{2}^{(0)} - x^{(0)} = 6.2$ and $x_{1}^{(1)} = x^{(0)} = 0$ and $x_{1}^{(1)} = y_{2}^{(0)} = 6.2$. Following the same procedure, Table 2.1 shows the progression of the golden-section search through 12 iterations.

At iteration 12, since

$$x_{1}^{(12)} = 2.982 \text{ and } x_{2}^{(12)} = 3.014, \quad |2.982 - 4.014| = 0.032.$$  

Furthermore, since

$$f(x_{1}^{(12)}) = 0.0003 \text{ and } f(x_{2}^{(12)}) = 0.0002, \quad |0.0003 - 0.0002| = 0.0001.$$  

As a result, the termination criteria are satisfied, and the golden-section search is stopped. The result is

$$x^{*} = \frac{2.982 + 4.014}{2} = 2.998$$

$$f(x^{*}) = 0.$$
Table 2.1: Progression of the Golden-Section Search

<table>
<thead>
<tr>
<th>(n)</th>
<th>$x^{(q)}$</th>
<th>$f(x^{(q)})$</th>
<th>$x^{(q)}$</th>
<th>$f(x^{(q)})$</th>
<th>Interval of Uncertainty</th>
<th>$\Delta^{(n)}$</th>
</tr>
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<tr>
<td>1</td>
<td>0</td>
<td>9</td>
<td>6.2</td>
<td>10.24</td>
<td>$0 \leq x \leq 6.2$</td>
<td>6.2</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>9</td>
<td>3.844</td>
<td>0.7123</td>
<td>$0 \leq x \leq 3.844$</td>
<td>3.844</td>
</tr>
<tr>
<td>3</td>
<td>1.461</td>
<td>2.3685</td>
<td>3.844</td>
<td>0.7123</td>
<td>$1.461 \leq x \leq 3.844$</td>
<td>2.383</td>
</tr>
<tr>
<td>4</td>
<td>2.367</td>
<td>0.4007</td>
<td>3.844</td>
<td>0.7123</td>
<td>$2.367 \leq x \leq 3.844$</td>
<td>1.477</td>
</tr>
<tr>
<td>5</td>
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<td>3.283</td>
<td>0.0801</td>
<td>$2.367 \leq x \leq 3.283$</td>
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</tr>
<tr>
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<td>0.0812</td>
<td>3.283</td>
<td>0.0801</td>
<td>$2.715 \leq x \leq 3.283$</td>
<td>0.568</td>
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<tr>
<td>7</td>
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<td>0.0048</td>
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<td>0.0801</td>
<td>$2.931 \leq x \leq 3.283$</td>
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<tr>
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<td>0.0048</td>
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<td>0.0222</td>
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<td>0.0048</td>
<td>3.066</td>
<td>0.0044</td>
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<td>0.135</td>
</tr>
<tr>
<td>10</td>
<td>2.982</td>
<td>0.0003</td>
<td>3.066</td>
<td>0.0044</td>
<td>$2.982 \leq x \leq 3.066$</td>
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<td>3.034</td>
<td>0.0002</td>
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<td>0.052</td>
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<tr>
<td>12</td>
<td>2.982</td>
<td>0.0003</td>
<td>3.014</td>
<td>0.0002</td>
<td>$2.982 \leq x \leq 3.014$</td>
<td>0.032</td>
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Chapter 3
TWO-ECHelon MODEL FOR PROCESS MEAN DETERMINATION

3.1 Introduction

In this chapter, we incorporate the issues associated with production setup and raw material procurement into the classical process mean problem for a single-product production process. It is assumed that the product of interest requires one major raw material, which is purchased from outside vendors, and that the material unit cost is independent of the procuring quantity. The production cost of an item is a linear function of the amount of the raw material used to produce the item. The product has a lower specification limit, and the items that do not conform to the specification limit are scrapped with no salvage value. A two-echelon model is formulated for jointly determining the process mean, production setup, and raw material ordering policies. This chapter is organized as follows. In the next section, the assumptions of the model are given, and the mathematical model is formulated. In section 3.3, the properties of the optimal solution to the model are discussed and a solution procedure is proposed. Then, a numerical example and sensitivity analysis are given, respectively, in sections 3.4 and 3.5.

3.2 Model Assumptions and Formulation

Consider a product with a constant demand rate of \( D \) items per unit time. A production process with a production rate of \( r \) items per unit time is used to satisfy the demand. Let \( X \) denote the performance variable of interest. As discussed in the last section, \( X \) is a measure of the raw material used in the production, such as weight and
volume. Let \( L \) be the lower specification limit of \( X \), so that an item is conforming if its \( X \) value is larger than or equal to \( L \). Assume that the production process is stable and \( X \) follows a normal distribution with an adjustable mean \( \mu \) and a constant variance \( \sigma^2 \). For given \( \mu \), the conforming rate of the production process is

\[
p = \int_{-\infty}^{L} f(x) \, dx = 1 - \Phi\left(\frac{L-\mu}{\sigma}\right),
\]

where \( f(x) \) is the probability density function of \( X \) and \( \Phi(\cdot) \) is the standard normal distribution function.

Assume that nonconforming items are scrapped with no salvage value. Consequently, for given \( \mu \), the yield rate of the production process is \( \lambda = r \cdot p \). It is assumed that all the demand will be satisfied in such a way that the expected total number of conforming items produced is equal to the total demand and no backlog is allowed. Note that use of the expected conforming items is reasonable, especially in high-speed production, because the production output can be treated approximately constant. Otherwise, it may require considering safety stock or using other inventory models. Note that \( \lambda \) has to be greater than or equal to \( D \) to ensure that the production capacity is large enough to meet the demand.

The expected amount of the raw material required to produce one conforming item is \( \mu/p \). Let \( c \) denote the unit cost of the raw material, and, thus, \( cx \) is the material cost required for producing an item of the finished product. We further assume that the direct cost of producing an item is a linear function of the item's material cost:

\[
pc(x) = b + \alpha cx,
\]

where \( b \) is the fixed production cost, and \( \alpha \) is a constant larger than or equal to 1. This cost function implies that the production cost consists of a fixed cost and a variable cost that is proportional to the raw material used in production. Note that \( \alpha - 1 \) is the
relative value (cost) added on the raw material during the production process. Also, we assume the production cost of nonconforming items are shift to the conforming items. As a result, it can be verified that, for given process mean \( \mu \), the expected cost of yielding a conforming item is \( \frac{b+\alpha \mu}{\mu} \). Let \( h_1 \) be the cost of holding a monetary unit of raw material per unit time. In other word, \( h = c - h_1 \) is the cost of holding each unit of the raw material in the system for a unit time. Assume that the costs of holding a monetary unit of raw material and finished product are the same. Then, the cost of holding a conforming item of finished product for a unit time is

\[
H = \frac{h}{p} (\alpha \mu + \frac{b}{c}).
\]

Let \( q \) be the production run size, which is the number of items (including both conforming and nonconforming items) produced in a production run. The inventory level as a function of time is described in part (a) of Figure 5. Assume that a production run begins at time zero. Until \( q \) items are produced, the finished product inventory increases at a rate of \( \lambda - D \) items per unit time. At time \( q/r \), the production run is complete, and, then, the inventory decreases at a rate of \( D \) items per unit time until time \( q \) when the inventory level reaches zero and the second production run starts. Let \( S \) be the production setup cost. Since the total number of setups required per unit time is \( D/q \), the total setup cost is \( SD/q \). It can verified that the average inventory level for the finished product is \( \frac{q}{2r} (rp-D) \). As a result, the total holding cost for finished products is \( H(q/2r)(rp-D) \) per unit time. Furthermore, because the expected cost of yielding a conforming item is \( \frac{b+\alpha \mu}{\mu} \), the per-unit-time direct production cost is \( D(b+\alpha \mu)/\mu \).
We define the cost associated with the finished product as the sum of the production cost, the process setup cost, and the inventory holding cost:

\[
FPC(\mu, q) = \frac{D(b+c\mu)}{p} + \frac{DS}{pq} + H_2(D(n-D)).
\]

For given process mean \(\mu\) and production run size \(q\), the requirement for the raw material as a function of time is illustrated in part (b) of Figure 5: the requirement is a constant \(r\mu\) during production, and is zero when the production process is idle. We assume instantaneous delivery leadtime and constant order quantity for the raw material procurement. Let \(Q\) denote the raw material order quantity. To determine the setup and holding costs of the raw material, we consider the following two ordering policies:
Case A. Each order quantity of the raw material satisfies the requirement of one or multiple production runs; that is: \(Q = nqp\), where \(n\) is an integer larger than or equal to 1.

In case A, since the periodic raw material requirement is \(D\mu/p\), each order of the raw material will last \(Qp/(D\mu)\) unit time. Let \(K\) denote the setup cost per order of the raw material. Then, the total setup cost associated with the raw material is \(KD\mu/(pQ)\) per unit time. Since \(Q = n\mu q\), the total setup cost can also be expressed as \(KD/(n\mu q)\). The inventory level as a function of time for the raw material is described in part (a) of Figure 6, from which we can find that the average raw material inventory level is \(\frac{(n-1)}{2} \mu q + \frac{\mu q D}{2rp}\). Consequently, the total material setup and holding cost per unit time is

\[
MC_A(\mu,q,n) = \frac{KD}{npq} + h \left[ \frac{n-1}{2} \mu q + \frac{\mu q D}{2rp} \right]
\]

The total expected cost per unit time for case A is the sum of \(FPC(\mu,q)\) and \(MC_A(\mu,q,n)\)

\[
TC_A(\mu,q,n) = H(\frac{q}{2r})(rp-D) + \frac{DS}{pq} + \frac{D(c\mu+b)}{npq} + \frac{KD}{npq} + h \left[ \frac{(n-1)}{2} \mu q + \frac{\mu q D}{2rp} \right] \quad \ldots \quad (3.1)
\]
In case B, because \( m \) orders are made in one production run, the setup cost per unit time is \( \frac{Knmr^2}{pq} \) and the average raw material inventory level is \( \frac{q\mu D}{2rpm} \). The inventory level as a function of time is shown in part (b) of Figure 6. As a result, the total setup and holding costs per unit time for the raw materials procurement is

\[
MC_B(\mu,q,m) = \frac{KmD}{pq} + \frac{q\mu D}{2rpm},
\]

and the total cost per unit time is the sum of \( FPC(\mu,q) \) and \( MC_B(\mu,q,m) \):

\[
TC_B(\mu,q,m) = H\left(\frac{q}{2r}\right)(rp-D) + \frac{DS}{pq} + \frac{D(c\mu+b)}{p} + \frac{KmD}{pq} + \frac{q\mu D}{2rpm} \quad (3.2)
\]
It is easy to verify that expressions (3.1) and (3.2) are equivalent when \( n = m = 1 \). In the next section, we study the properties of the optimal solution. Based on these properties, we propose a solution procedure to find the optimal process mean, production run size, and the raw material ordering policy, which minimize the total cost.

The assumptions used in the model formulation are categorized according to the finished product and raw material and summarized as follows:

**Finished Product.**

1. Demand for the finished product is constant over time.
2. The production rate of the process is uniform and finite.
3. No shortage of the product is permitted.
4. The product requires only one major raw material, and the cost of producing the product is a linear function of the raw material cost.
5. The production process is stable and the performance variable of the product follows a normal distribution with an adjustable process mean and a constant variance.
6. Nonconforming items are scrapped with no salvage value.

**Raw Material.**

1. The raw material is obtained from outside sources and its replenished rate is infinite.
2. No shortage of raw material is allowed.
3. Each order quantity of raw material satisfies the requirement of one or more production runs, or multiple orders are made in a production run.
4. The cost of holding for the finished product and the raw material is known and constant over time.
5. The material unit cost is independent of the size of each order.
Note that the last assumption will be released in the next two chapters when quantity discounts are implied on raw material purchasing.

3.3 Optimal Solution

In this section, we derive several important analytical properties for the optimal solution. On the basis of these properties, we propose an efficient solution algorithm.

3.3.1 Analytical Properties

We first discuss case A. Using (3.1), for given \( \mu \) and \( q \), since the following condition exists, we can verify that \( TCA(\mu, q, n) \) is a convex function of \( n \).

\[
TCA(\mu, q, n-1) - TCA(\mu, q, n) > TCA(\mu, q, n) - TCA(\mu, q, n+1).
\]

Proof. \( TCA(\mu, q, n-1) - TCA(\mu, q, n) = \frac{KD}{pq} \left( \frac{1}{n-1} - \frac{1}{n} \right) - \frac{h u q}{2} \)

\[
TCA(\mu, q, n) - TCA(\mu, q, n+1) = \frac{KD}{pq} \left( \frac{1}{n} - \frac{1}{n+1} \right) - \frac{h u q}{2}
\]

since \( n \geq 1 \) and \( n \) is integer, \( \left( \frac{1}{n-1} - \frac{1}{n} \right) > \left( \frac{1}{n} - \frac{1}{n+1} \right) \). Q.E.D.

As a result, if there is an integer \( n^o \) such that

\[
TCA(\mu, q, n^o) = TCA(\mu, q, n^o) = TCA(\mu, q, n^o+1),
\]

then \( n^o \) is the optimal \( n \) value for given \( \mu \) and \( q \). Using straightforward algebraic manipulation, (3.3) can be translated into the following explicit condition for \( n^o \).

\[
\frac{1}{2} \sqrt{1+4a-1} < n^o < \frac{1}{2} \sqrt{1+4a+1}, \quad (3.4)
\]

where
As a result, we obtain the following results.

**Result 3.1.** For given \( \mu \) and \( q \), the optimal value for \( n \) is given by

\[
 n^* = \left| \frac{1}{2} (\sqrt{1+4a+1}) \right|
\]

where \( \lfloor y \rfloor \) is the largest integer that is less than or equal to \( y \).

Using expression (3.4), we find, for given \( \mu \) and \( n \), that the optimal production run size satisfies the following necessary condition:

\[
\sqrt{\frac{1}{(n+1)n}} \sqrt{\frac{2DK}{h\mu p}} \leq q \leq \sqrt{\frac{1}{(n-1)n}} \sqrt{\frac{2DK}{h\mu p}}.
\]  (3.5)

Note that when \( n = 1 \), the upper bound does not exist.

Furthermore, the optimal \( q \) for given \( \mu \) and \( n \) can be found by solving \( \partial TCA/\partial q = 0 \), resulting in the following explicit expression for the optimal \( q \).

\[
 q = \sqrt{\frac{2rD(S+K/n)}{H(p\mu-D)+h\mu(p(n-1)+D)}}.
\]  (3.6)

However, \( q \) obtained from (3.6) may not satisfy (3.5). The unconditional solution for \( q \) will be given later in result 3.3. Substituting the production run size \( q \) given by (3.6) into (3.5), we obtain the following condition for the optimal \( n \) for given \( \mu \).

**Result 3.2.** For given \( \mu \), the optimal \( n \) satisfies the following condition:

\[
 n_0 < n \leq \frac{1}{2} \left\lfloor \sqrt{1+4\frac{K}{\mu p S \left\{ (\alpha-1)\mu+b \frac{c}{\epsilon} (p-D)+\mu rp \right\} +1}} \right\rfloor
\]

where
or \( n_0 = 0 \) if \( n_0 \) given by (3.7) is not a real number.

Using result 3.2 and expression (3.6), the following result is obtained:

**Result 3.3.** The optimal production run size for given \( \mu \) and its corresponding optimal value for \( n \) obtained from result 3.2 is given by expression (3.6).

We can obtain similar results for case B. First, because

\[
\frac{1}{2}(\sqrt{1+4A-1}) \leq m < \frac{1}{2}(\sqrt{1+4A+1}),
\]

(3.8)

where

\[
A = \frac{h \mu q^2}{2rK},
\]

the following result associated with case B is obtained.

**Result 3.4.** For given \( \mu \) and \( q \), the optimal \( m \) is given by

\[
m^* = \left\lfloor \frac{1}{2}(\sqrt{1+4A+1}) \right\rfloor.
\]

Results 3.5 and 3.6 are obtained using procedures similar to those used in case A.

**Result 3.5.** The optimal \( m \) for given \( \mu \) satisfies the following condition.

\[
m_0 \leq m < \frac{1}{2} \left\{ \sqrt{1+\frac{4\mu D(S+K)}{b}} + \frac{b}{c} \frac{K(\alpha\mu + \frac{1}{c})(rp-D)}{(rp-D)} \right\},
\]

where
or \( m_o = 1 \) if \( m_o \) given by expression (3.9) is not a real number.

**Result 3.6.** The optimal production run size for given \( \mu \) and its corresponding optimal value for \( m \) obtained from result 3.5 is

\[
q = \left\{ \frac{2\sqrt{D(S+km)}}{H\sqrt{(\alpha \mu + b)/c)rp-D}} + h\mu(D/m) \right\}.
\]

From (3.4) we find if \( 2k-1 < \sqrt{1+4\alpha} < 2k+1 \), then \( n = k \). It can be verified that

\[
\begin{align*}
n &= 1 \text{ if } 0 < \alpha < 2; \\
n &= 2 \text{ if } 2 \leq \alpha < 6; \\
n &= 3 \text{ if } 6 \leq \alpha < 12; \\
\end{align*}
\]

... and so on.

We can find a similar situation for case B from (3.8); that is,

\[
\text{if } 2k-1 < \sqrt{1+4A} \leq 2k+1, \text{ then } m = k,
\]

where \( A = \frac{H\mu q^2}{2rK} \). Similarly,

\[
\begin{align*}
m &= 1 \text{ if } 0 < A \leq 2, \\
m &= 2 \text{ if } 2 < A \leq 6, \\
m &= 3 \text{ if } 6 < A \leq 12.
\end{align*}
\]

From the above relationship, it is found that only when \( \frac{DK}{H\mu q^2} < 1 \) and \( \frac{\mu q^2}{rK} \leq 4 \); that is

\[
\frac{\mu q^2}{4r} \leq \frac{K}{h} < \frac{\mu q^2}{D},
\]

then \( n = m = 1 \). Based on (3.10), we obtain the following result.
Result 3.7. For given $\mu$ and $q$,

1. Case A should be used if $\frac{K}{h}$ is higher than or equal to $\frac{\mu q \sigma^2}{D}$, and
2. Case B should be used if $\frac{K}{h}$ is less than $\frac{\mu q \sigma^2}{4r}$.

It should also be noted that the raw material order quantity is directly dependent on the production run size, but does not have an explicit relationship with the value-added factor $\alpha$.

3.3.2 Solution Algorithm

The analytical results presented in the last section provide the basis for developing an efficient solution procedure to find the optimal production run size and the raw material ordering policy for a given process mean. Specifically, if case A is considered, for given $\mu$, the optimal value for $n$ has to satisfy the condition given by result 3.2. Then, for the given $\mu$ and $n$, the optimal production run size is obtained by using result 3.3. If case B is considered, $m$ and $q$ are found, respectively, by using results 3.5 and 3.6. Then, the optimal solutions associated with the two cases are compared, and the one with the smallest cost is selected. The procedure is summarized as follows.

Step 1. Find integer sets $N = \{n: n$ satisfies the condition given in result 3.2} and $M = \{m: m$ satisfies the condition given in result 3.5}.

Step 2. Obtain the production run size, $q_n$, for all $n \in N$, using result 3.3, and $q_m$ for all $m \in M$, using result 3.6.

Step 3. Compute total costs: $TCA(\mu, q_n, n)$ for $n \in N$, and $TCB(\mu, q_m, m)$ for $m \in M$, using expressions (3.1) and (3.2), respectively. Then, find the optimal value for $q$ and the optimal material ordering policy.
(i.e., the optimal value for \( n \) or \( m \)), which give the minimum total cost.

As a result, the total cost becomes a function of single variable \( \mu \). Therefore, a one-dimensional direct search procedure, such as the golden-section search method or any other bi-section search method, can be used to search for the optimal process mean. In this paper, the range for the search is \([L, L + z\sigma]\), where \( z \) is a predetermined real number. The termination criterion is the interval of uncertainty that is less than or equal to 0.0001. In most applications, \( z = 4 \) is large enough to include the possible optimal solution. The reason \( L \) is used as the lower bound is that when \( \mu \) equals \( L \), the process conforming rate is 50%, which is very low in most realistic applications. Note that the golden-section search method provides the optimal solution if the objective function is unimodal, which was found to be true in all the examples that we tested. In general, multiple starting points can always be used in the search procedure to ensure that the global minimum is found.

### 3.4 An Example

In this section, an example is used to illustrate the solution procedure given in the last section. This example will also be used in the sensitivity analysis in the next section.

Consider a product that requires at least 1.1 mg of main content in each item. The item that is less than 1.1 mg is considered nonconforming and is scrapped without salvage value. Because of the variation in the production process, the content of an item produced by the process follows a normal distribution with an adjustable process mean and a constant standard deviation of 0.8 mg. Assume that the product demand rate and the production rate are 5,000 items and 7,500 items per unit time, respectively.
The setup cost per production run is $150, the fixed production cost is $0.16 per item, and $\alpha$ is 4. The raw material is purchased from a vendor. Suppose the material cost is $0.1/mg, and the setup cost per order is $30. Furthermore, the cost for holding $1 of inventory (finished product or raw material) is $0.03 per unit time.

A FORTRAN program has been written for implementing the solution procedure given in the last section. The algorithm is to search the process mean by using the golden-section search method and the optimal process mean is found to be 1.670. For any given process mean $\mu = \mu_0$, for example, $\mu = 1.670$, the following three steps were used to find the optimal material ordering policy associated with the optimal process mean:

Step 1. For $\mu = 1.670$, using results 3.2 and 3.5, the optimal $n$ and $m$ are found to satisfy $0 < n \leq 1.24$ and $1.93 \leq m < 3.45$. Therefore, $N = \{1\}$ and $M = \{2,3\}$.

Step 2. Using result 3.3, $q = 2,709$ for $n=1$; using result 3.6, $q = 22,777$ and $q = 26,220$ for $m = 2, 3$, respectively.

Step 3. Using expressions (3.1) and (3.2), $T_{CA}(\mu,q,1) = 5,879.48$; $T_{CB}(\mu,q,2) = 5,554.43$, $T_{CB}(\mu,q,3) = 5,553.56$.

From step 3, we found that total cost is the minimum when $m = 3$. Consequently, the optimal process mean is 1.670 mg, resulting in a process conforming rate of 76.19%. The optimal production run size is 26,220, and the order for the raw material should be placed three times within each production run. The order quantity for the raw material is 14,621 mg.

For the example used above, if we solve it separately instead of using the model and solution algorithm we proposed, the following results can be obtained. When process mean is 1.724 mg and conforming rate is 78.23%, the producing (material)
cost is minimum. Based on this process mean, solve the two-echelon problem. The cost is minimized when production run size is 16,779 items and raw material order quantity is 28,924 mg. As a result, the total cost under these values is 5,567.25, which is 0.25% higher than what is obtained above. Even though the cost improvement percentage is not high in the case here, however, it can be raised up to about 5% in some situations. For example, if \( r = 6,500 \), \( b = $0.1 \), \( c = $0.05 \), \( h_1 = 0.05 \) and \( S = $350 \), the saving from the joint model is 3.05%. This saving can be increased when the weight of total material cost among the total cost becomes less.

3.5 Sensitivity Analysis

In this section, a sensitivity analysis is performed to study the effects of the following model parameters on the optimal solution: (1) the product demand rate \( D \), (2) the production rate \( r \), (3) the process standard deviation \( \sigma \), (4) the value-added factor \( \alpha \), (5) the production setup cost \( S \), (6) the material ordering setup cost \( K \), and (7) the unit material cost \( c \). The sensitivity analysis is based on the example given in the last section.

In the model formulation given in section 2, the cost components are evaluated in terms of unit time. It is also possible to formulate the model based on per-item costs. Because the demand rate is constant, the results of these two formulation methods are the same. However, in the sensitivity analysis, the demand rate is changed to observe its effects on the optimal solution. For comparing these results, per-item costs may be more appropriate. Consequently, three per-item costs are used to report the results in this section. The first one is the per-item finished product-related cost given by

\[
PFPC = \frac{FPC(\mu, q)}{D}.
\]
The second per-item cost is the total cost of material setup and handling costs given by

$$PMC = MC_A(\mu, q, n)/D, \text{ or } MC_B(\mu, q, n)/D,$$

depending on which policy, A or B, is used for purchasing the raw material. Similarly, the per-item total cost is given by

$$PTC = TC_A(\mu, q, n)/D \text{ or } TC_B(\mu, q, n)/D.$$

The computer program used in the study is written in FORTRAN and run on an IBM3090-600E. The running time for solving each of the problems used in the study was just a fraction of a second.

3.5.1 Effect of Demand Rate

To study the effects of the demand rate, we obtained optimal solutions for selected values of $D$ ranging from 1,500 to 7,000 per unit time with an increment of 500. The results are summarized in Table 3.1.

As the demand rate increases, one would expect that the process mean should be set higher in order to meet the demand. The results show, however, that the process mean actually decreases until the demand rate reaches 5,500 per unit time. The main reason for this result is that when the demand rate is low, the production rate (capacity) is too high. To avoid costs incurred because of frequent production setups and excess finished product inventory, the process mean is set lower to reduce the process yield rate. When the demand rate is closer to the production rate ($D$ is larger than 5,500), we observe more reasonable results in which the production run size, the process mean and the process yield rate increase as the demand rate increases. Furthermore, when the demand rate is close to the production rate, the production run size becomes very
Table 3.1: Effect of Demand Rate for Selected Values of D

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<th>p</th>
<th>λ</th>
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<th>m</th>
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sensitive to the demand rate. In particular, significant changes in the production run size are observed when the demand rate is larger than 5,000. This is because the inventory accumulation rate \((r-p-D)\) during production is so low that a process setup is economical only after a long period of production. For example, when the demand rate is 5,500, the process yield rate is 5,502, resulting an inventory accumulation rate of 2 items per unit time. Consequently, the production run size becomes very large. On the other hand, by comparing \(m\) and \(Q\), when ordering frequency \(m\) keeps the same, material order quantity \(Q\) increases when demand rate increases. If \(m\) increases, \(Q\) decreases. The material-related costs resulting from the ordering policy do not show a clear pattern, however.

Another important observation is that per-item total cost is not a decreasing function of demand rate. It first decreases, then starts to increase when the demand is larger than 6,500. The increase in the total cost suggests that the production capacity is not large enough to effectively satisfy the demand rate. These results suggest that a carefully design production capacity is very important to controlling production cost. In most manufacturing systems, the same facility is used to produce several different products. The result implies that pooling too many products for production in a single, fast machine may not be a good production design, since regardless of the change over cost from one product to the another, the inventory cost for each product may also arise.

3.5.2 Effect of Production Rate

Table 3.2 gives the results for selected values of \(r\). As was argued in the last section, a larger production rate does not necessarily lead to the most economical situation. The per-item total cost has its lowest value when the production rate is 6,500. Although the general pattern of the process mean in response to the change in \(r\)
Table 3.2: Effect of Production Rate for Selected Values of \( r \)

<table>
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<th>( r )</th>
<th>( \mu )</th>
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<th>( \lambda )</th>
<th>( q )</th>
<th>( m )</th>
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is not found, the process mean increases as \( r \) increases when the material ordering frequency is at the same level. Furthermore, the production yield rate increases as \( r \) increases, which is caused by a decreasing production run size in order to reduce the cost of holding finished product inventory.

When the production rate is between 6,000 and 7,000, the production yield rate is just several items larger than the demand rate. As a result, the inventory accumulation rate \((r \cdot p \cdot D)\) is very small, and, thus, the production run size is very sensitive to a change in the production rate.

When the production rate is greater than 7,000, the process mean and the yield rate increase as \( r \) increases. The reason for this is a decreasing production run size will reduce the cost of holding finished product inventory. When \( r \) is less than 7,000, since the demand has to be satisfied, production yield rate is limited to a certain level, as is the process mean. The process mean, therefore, will not follow the pattern (decrease) in response to the decrease of \( r \).

When the material ordering frequency is the same, the material order quantity is affected by an increasing process mean but a smaller production run size. The result shows that when the material ordering frequency is the same, the material order quantity decreases as the production rate increases.

### 3.5.3 Effect of Process Standard Deviation

It is well known that the performance of a process can be improved by reducing its inherent variation (Deming 1986; Taguchi 1978). For a given process mean, a small process standard deviation implies a higher process yield rate. On the other hand, to maintain the same yield rate, the process mean can be set lower when \( \sigma \) is smaller. In this situation, the material requirement is reduced and thus the material ordering policy may be also affected. To study the effect of the process standard deviation on the
optimal solution, the optimal solutions for some selected values of \( \sigma \) ranging from .01 to 1.10 are reported in Table 3.3.

As expected, the per-item total cost decreases as \( \sigma \) decreases. Notice that \( \sigma = 0.01 \) represents a very small process variation. The per-item total cost (0.637), therefore, is near the lowest cost that the current production process can achieve through process variation reduction. When \( \sigma \) increases, the process mean increases until \( \sigma \) is equal to 0.7. Then, the pattern becomes unclear when \( \sigma \) is larger than 0.7. The process conforming rate follows a similar pattern. The decrease in the conforming rate is mainly because of process variation and excess capacity. The conforming rate becomes very stable, however, when the process yield rate is close to the demand rate. The production run size is relatively stable when \( \sigma \) is small, and becomes very sensitive to \( \sigma \) when the production yield rate is close to the demand rate. The material ordering policy is relatively less sensitive to \( \sigma \). It shows, however, that material order quantity increases when \( \sigma \) increases as long as order frequency remains the same. As a result, it was also found that production-related cost is more sensitive to \( \sigma \) than is material-related cost.

3.5.4 Effect of Value-Added Factor

A larger \( \alpha \) implies a larger cost of producing an item. The holding cost \( H \) also becomes larger. Table 3.4 gives the results for selected values of \( \alpha \). From the table, it is clear that the process mean decreases as \( \alpha \) increases. The main reason is that the cost of raw material is reduced relatively in the model because of the increase in the value of the finished product, which, in turn, increases the importance of reducing the holding of finished products. A low process mean can help to achieve this objective by reducing the per unit production cost and cumulative speed of finished product.
### Table 3.3: Effect of Process Variation for Selected Values of $\sigma$

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Table 3.4: Effect of Value-Added Factor for Selected Values of $\alpha$

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A lower process mean reduces the production cost and thus the unit holding cost. In fact, the process mean can be reduced only to a level that a further reduction will result in unsatisfied demand. In the last few cases in the table, the difference between the demand rate and the production yield rate is only 1 item per unit time. When the demand rate is close to the production yield rate, the production run size is very sensitive to the change in $\alpha$. At the same time, the material ordering policy is very stable.

Since the decrease of process mean is in response to the increase of $\alpha$ when $\alpha$ is less than 6, the production run size increases because of the decrease in conforming rate. The material ordering quantity, however, decreases while $\alpha$ increases and order frequency remains the same. Because of the decrease in raw material demand, this will also reduce the production cost when $\alpha$ increases.

### 3.5.5 Effects of Production Setup Cost

When the production setup cost increases, the production run size is expected to increase in order to reduce the number of production setups. This result is observed in Table 3.5. As a result, the inventory holding cost associated with the finished product decreases when the setup cost is less than or equal to 300. When the setup cost is larger than 300, the process mean cannot be lowered because the production yield rate is very close to the demand rate. The production run size shifts to a very high level and the process mean drops significantly to the lowest possible level, resulting in a very low inventory level. It is also observed that the process mean decreases as the production setup cost increases until a further reduction will result in unsatisfied demand. A general pattern is found in material order quantity: as long as the ordering frequency ($m$) remains the same, the material order quantity increases
Table 3.5: Effect of Production Setup Cost for Selected Values of S

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<th>( \lambda )</th>
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when setup cost increases. If the material order frequency increases because of the increase in production run size, the material ordering quantity decreases.

3.5.6 Effect of Material Ordering Setup Cost

In Table 3.6, we find that the material ordering frequency decreases as the setup cost associated with raw material ordering increases. On the other hand, the material order quantity increases. The production run size and the process mean have an interesting relationship with the material ordering frequency: under the same material ordering frequency, the production run size increases, but the process mean decreases as the setup cost increases. For example, when $m$ is 1, the process mean decreases from 1.684 to 1.661, and the production run size increases from 21,814.33 to 34,876.07. The increase of $q$ is due to the increase of material ordering quantity, which is in response to the increase of material ordering setup cost. On the other hand, in order to reduce the holding cost, which is raised by higher $q$, the process mean decreases.

3.5.7 Effect of Material Unit Cost

An increase in $c$ implies larger per-item production cost and holding cost. From Table 3.7, we find that the process mean is very sensitive to the change in $c$. When $c$ is changed to 0.15, the process mean decreases to its lowest possible level. Consequently, the accumulation rate of finished product is kept at its lowest level to reduce the cost of holding finished product inventory. Because of this low accumulation rate, the production run size becomes very sensitive to the changes in $c$. As long as order frequency remains the same, the production run size decreases because of the increase in the holding cost rate. When order frequency increases, the setup cost is reduced by increasing production run size to compensate for the increase in ordering cost.
Table 3.6: Effect of Raw Material Ordering Setup Cost for Selected Values of K

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Table 3.7: Effect of Material Unit Cost for Selected Values of c

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Chapter 4
QUANTITY DISCOUNTS IN MATERIAL COST

4.1 Introduction

In the last chapter, it was assumed that the price (cost) of the raw material is constant. In reality, however, the price of the raw material may depend on the order quantity. For example, quantity discounts may be offered by a vendor in order to promote his or her products (raw materials). It makes intuitive sense therefore, to extend the model given in Chapter 3 to incorporate the quantity discounts issue into the decision on production mean, production lot size, and raw material ordering policy.

Many types of quantity discounts have been studied or applied. The most commonly used discounts, however, are all-unit quantity discounts and incremental quantity discounts. In this chapter, all-unit quantity discounts are considered to develop the model formulation and sensitivity analysis. We also consider the situation in which it may be economical for the producer to order a quantity in excess of what is actually needed in production and to dispose of the excess. Because the model for incremental quantity discounts can be formulated following a similar procedure, the details for that model are not given in this chapter. This chapter is organized as follows. Model formulation is given in the next section. Analytical properties are derived and an efficient solution algorithm is proposed in section 4.3. In section 4.4, two examples are used to illustrate the solution algorithm. A sensitivity analysis is performed in section 4.5, and the results are compared to the model in Chapter 3, without quantity discount consideration.
4.2 Model Formulation

In this section, we extend the model given in Chapter 3 to incorporate the all-unit quantity discounts in the raw material. Let $Q_T$ denote the raw material order quantity and $Q$ denote the raw material use quantity per order; i.e., the raw material quantity actually used in production. The all-unit quantity discounts in the raw material are defined as

$$c = c_{i-1}, \quad \text{for } Q_{i-1} \leq Q < Q_i, \quad i = 1, 2, ..., k.$$ 

Basically, the formulation method of this model is similar to that given in Chapter 3. The cost associated with the finished product -- defined as the sum of the production cost, the process setup cost, and the inventory holding cost -- is identical to that given in Chapter 3, except that the material unit cost depends on $Q_T$, which is a function of $\mu$. Consequently, the cost associated with the finished product, $FPC(\mu,q)$, has to be defined accordingly.

The cost associated with the raw material includes not only ordering and holding costs, but also possible disposal cost. The difference between $Q$ and $Q_T$ is the excess of the raw material. It is assumed that the disposition of the excess raw material is determined when the raw material is received. Based on the two ordering policies discussed in Chapter 3 and on whether excess raw material is ordered, four cases are considered:

Case $A_1$. Each order quantity of the raw material satisfies the requirement of one or multiple production runs and no excess material is ordered; that is: $Q = Q_T = n\mu q$, where $n$ is an integer larger than or equal to 1.
Case A2. Each order quantity of the raw material satisfies the requirement of one or multiple production runs and excess raw material is ordered; that is: for given unit cost $c_i, Q = nμq$ and $Q_T = Q_i$, where $n$ is an integer larger than or equal to 1 and $Q_i$ is the price breakpoint associated with unit cost $c_i$. $Q < Q_T$.

Case B1. Multiple orders are made in a production run and no excess material is ordered; that is $Q = Q_T = \frac{μq}{m}$, where $m$ is an integer larger than or equal to 1.

Case B2. Multiple orders are made in a production run and excess raw material is ordered; that is: for given unit cost $c_i, Q = \frac{μq}{m}$ and $Q_T = Q_i$, where $m$ is an integer larger than or equal to 1 and $Q_i$ is the price breakpoint associated with material unit cost $c_i$. $Q < Q_T$.

These four cases are considered separately for determining the raw material cost as follows.

4.2.1 Case A1

In case A1, in which no excess raw material is ordered, the raw material acquisition cost and inventory cost are the same as that given in Chapter 3, except that different unit material prices are used according to the price brackets; that is, when $Q = Q_T = nμq \in [Q_{i-1}, Q_i)$, the raw material unit cost is $c_{i-1}$. The total cost associated with the finished product, FPC$(μ,q)$, therefore, is

$$FPC_{A1}(μ,q) = \frac{D(b+c_{i-1}μ)}{p} + \frac{DS}{pq} + \frac{h_{i}}{p} (c_{i-1}μ+b)\frac{q}{2l}(rp-D),$$

and the total material ordering and holding cost is
MC_{A_1}(\mu,q,n) = \frac{KD}{npq} + h_1c_{i-1} \left[ \frac{n-1}{2} \mu q + \frac{\mu q D}{2rp} \right].

The total expected cost per unit time for case A_1 is the sum of FPCA_{A_1}(\mu,q) and MC_{A_1}(\mu,q,n):

\begin{align}
TC_{A_1}(\mu,q,n) &= D(b+c_{i-1}c_{\mu}) + DS + h_1(c_{i-1}c_{\mu}+b)q(\mu q - D) \\
&\quad + \frac{KD}{npq} + h_1c_{i-1} \left[ \frac{n-1}{2} \mu q + \frac{\mu q D}{2rp} \right] 
\end{align}

4.4.2 Case A_2

When excess raw material is ordered, the raw material order quantity Q_1 = Q_i is larger than or equal to the raw material use quantity per order, Q. The amount of excess raw material per order is Q_i - Q. The cost associated with the raw material includes setup, holding, and disposal costs. The cost of disposing of a unit of raw material is the sum of c_i and c_d. Since the number of raw material orders per unit time is D/npq, the disposal cost per unit time is given as

DIPC_{A_2}(\mu,q,n) = \frac{D(Q_i - Q)(c_i + c_d)}{npq}.

The total material ordering, holding, and disposal costs can then be expressed as:

MC_{A_2}(\mu,q,n) + DIPC_{A_2}(\mu,q,n) = \frac{KD}{npq} + h_1c_{i-1} \left[ \frac{n-1}{2} \mu q + \frac{\mu q D}{2rp} \right] + \frac{D(Q_i - Q)(c_i + c_d)}{npq}.

On the other hand, since Q_i is ordered, the unit cost is c_i. Consequently, the total cost associated with the finished product becomes

FPCA_{A_2}(\mu,q) = \frac{D(b+c_{\mu}c_{\mu})}{p} + DS + h_1(c_{\mu}c_{\mu}+b)q(\mu q - D).

The total expected cost per unit time for case A_2 when Q_T = Q_i and Q \leq Q_i, therefore, is:
Following a similar process and using equation (3.2), we can accordingly obtain the total costs associated with cases $B_1$ and $B_2$.

### 4.2.3 Case $B_1$

\[
TC_{B_1}(\mu, q, m) = \frac{D(b_1 + c_1 \alpha \mu_1)}{p} + \frac{DS}{pq} + \frac{h_1}{p} (c_1 \alpha \mu_1 + b) \frac{q}{2r} (r - D) \\
+ \frac{KD_{mq}}{npq} + h_1 \left[ \frac{n-1}{2} \mu q + \frac{\mu q D}{2r} \right] + \frac{D(Q_m - Q)(c_1 + c_0)}{npq}
\]  
(4.2)

### 4.2.4 Case $B_2$

\[
TC_{B_2}(\mu, q, m) = \frac{D(b_2 + c_2 \alpha \mu_2)}{p} + \frac{DS}{pq} + \frac{h_1}{p} (c_2 \alpha \mu_2 + b) \frac{q}{2r} (r - D) \\
+ \frac{KD_{mq}}{pq} + h_1 \frac{qmD}{2rpm} + \frac{D_{mq} (Q_m - Q)(c_2 + c_0)}{pq}
\]

for $Q = \frac{\mu q}{m} \leq Q_m (= Q_i)$.  
(4.3)

It is also easy to verify that (4.1) and (4.3) are equivalent, and (4.2) and (4.4) are equivalent, when $n = m = 1$. In the next section, we derive several useful analytical properties, and propose an efficient procedure for finding the optimal process mean, production run size, and the material ordering policy which minimize the total cost.

### 4.3 Optimal Solution

In this section, we derive several important analytical properties for the optimal solution, based on which an efficient solution algorithm is proposed.
4.3.1 Analytical Properties

We first give the analytical results for each of the four cases.

Case A_1

If the relationship between the material unit cost and the order quantity is ignored, Case A_1 is exactly the same as case A presented in Chapter 3. In stating the analytical results, let $TCA_1(\mu, q, n|c_i)$ be the total cost in case $A_1$ evaluated at $\mu$, $q$, and $n$ when the material unit cost is $c_i$. Let $Q^*$ be the raw material use quantity per order resulting from the solution that minimizes $TCA_1(\mu, q, n|c_i)$. $Q^*$ may not be an order quantity that is qualified for the price $c_i$. However, when $Q^*$ is qualified for the price $c_i$ (i.e., $Q_i \leq Q^* < Q_{i+1}$), $Q^*$ is admissible.

It was proved in Chapter 3 that for given $\mu$ and $q$, $TCA_1(\mu, q, n|c_i)$ is a convex function of $n$. As a result, if there is an integer $n^o$ such that

$$TCA_1(\mu, q, n^o-1|c_i) \leq TCA_1(\mu, q, n^o|c_i) < TCA_1(\mu, q, n^o+1|c_i), \quad (4.5)$$

then $n^o$ is the optimal $n$ value for given $\mu$ and $q$. Using straightforward algebraic manipulation, (4.5) can be translated into the following explicit condition for $n^o$:

$$\frac{1}{2} (\sqrt{1+4a-1}) < n^o \leq \frac{1}{2} (\sqrt{1+4a+1}),$$

where

$$a = \frac{2DK}{h_1c_{i}\mu pq^2} > 0.$$

As a result, we obtain the following results.

**Result 4.1.** For given $\mu$ and $q$, the value of $n$ which minimizes $TCA_1(\mu, q, n|c_i)$ is given by

$$n^o = \left\lfloor \frac{1}{2} (\sqrt{1+4a+1}) \right\rfloor.$$
where \(|y|\) is the largest integer that is less than or equal to \(y\).

Result 4.2. For given \(\mu\), the \(n\) value that minimizes \(TCA_1(\mu,q,n|c_i)\) satisfies the following condition:

\[
n_0 < n \leq \frac{1}{2} \left\{ \sqrt{1 + 4 \frac{K}{\mu p S_r} \left\{ \left( \alpha - 1 \right) \mu + b \right\} (rp-D)+\mu rp} + 1 \right\},
\]

where

\[
n_0 = \frac{1}{2} \left\{ \sqrt{1 + 4 \frac{K}{\mu p S_r} \left\{ \left( \alpha - 1 \right) \mu + b \right\} (rp-D)+\mu rp} - 1 \right\}, \tag{4.6}
\]

or \(n_0 = 0\) if \(n_0\) given by (4.6) is not a real number.

Result 4.3. For given \(\mu\) and its corresponding value of \(n\) obtained from result 4.2, the production run size that minimizes \(TCA_1(\mu,q,n|c_i)\) is

\[
q = \sqrt{\frac{2rD(S+K)}{h_1(c_i\mu+b)(rp-D)+h_1c_i\mu(rp(n-1)+D)}}.
\]

Note that the raw material order quantity \(Q_T\) is exactly equal to the raw material use quantity per order \(Q = n\mu q\).

Case A2

Similar to case A1, let \(TCA_2(\mu,q,n|c_i)\) be the total cost in case A2 evaluated at \(\mu\), \(q\), and \(n\) when the material unit cost is \(c_i\). For given \(\mu\) and \(q\), it can be verified that \(TCA_2(\mu,q,n|c_i)\) is a convex function of \(n\). Therefore, the integer \(n\) that minimizes \(TCA_2(\mu,q,n|c_i)\) satisfies the following condition:

\[
TCA_2(\mu,q,n-1|c_i) \geq TCA_2(\mu,q,n|c_i) < TCA_2(\mu,q,n+1|c_i),
\]

which leads to
\[ \frac{1}{2}(\sqrt{1+4a'}-1) < n \leq \frac{1}{2}(\sqrt{1+4a'}+1), \tag{4.7} \]

where \(a' = \frac{2D[K+Q_a(c_i+c_j)]}{h_1c_i\mu q^2}.\)

As a result, the following result is obtained.

**Result 4.4.** For given \(\mu\) and \(q\), the value of \(n\) that minimizes \(TC_{A_2}(\mu,q,n|c_i)\) is given by

\[ n = \left\lfloor \frac{1}{2}(\sqrt{1+4a'}+1) \right\rfloor. \]

On the other hand, if \(\mu\) and \(n\) are given, we can find the value for \(q\) which minimizes \(TC_{A_2}(\mu,q,n|c_i)\) by solving \(\partial TC_{A_2}(\mu,q,n)/\partial q = 0\). As a result, we obtain the following result.

**Result 4.5.** When \(n\) and \(\mu\) are given, the value of \(q\) that minimizes \(TC_{A_2}(\mu,q,n|c_i)\) is

\[ q = \sqrt{\frac{2D[S^{\mu}\frac{KQ_a}{n}(c_i+c_j)]}{h_1[c_i(\alpha+1)\mu+b](\mu\mu-D)+nh_1c_i\mu q}}. \]

Substituting the production run size given in result 4.5 into eq. (4.7) we obtain the following condition for the \(n\) value that minimizes \(TC_{A_2}(\mu,q,n|c_i)\) for given \(\mu\).

**Result 4.6.** For given \(\mu\), the \(n\) value that minimizes \(TC_{A_2}(\mu,q,n|c_i)\) satisfies the following condition

\[ n' \leq n \leq \frac{1}{2}\left\lfloor \sqrt{1+4\frac{K+Q_a(c_i+c_j)}{\mu pSr} \{[(\alpha-1)\mu+b/(c_i+c_j)(\mu\mu-D)+\mu\mu r}\}+1 \right\rfloor, \]

where
\[ n'_i = \frac{1}{2} \left\lfloor \sqrt{1 + 4 \frac{K + Q_i c_i + c_0}{1 + \frac{b}{c_i} \left( (\alpha - 1)\mu + \frac{1}{\mu} \right) (rp-D) - 1} \right\rfloor, \]  

(4.8)

or \( n'_i = 0 \) if \( n'_i \) given by (4.8) is not a real number.

For case \( A_2 \), \( Q_i \) is the upper breakpoint for given \( c_i \). Therefore, if the order quantity \( Q \) resulting from minimizing \( TC(\mu, q, n|c_i) \) is larger than \( Q_i \), then the optimal order quantity \( Q_T \) equals to \( Q_i \). Under this situation, the optimal production run size \( q = \frac{Q_i}{\mu} \), and the total cost function (4.2) can be simplified as

\[
TC_{A_2}(\mu, q, n) = \frac{D(b+c_\alpha \mu)}{p} + \frac{DSn_i q}{p} + \frac{h_i Q_i}{2rp\mu} + \frac{K D\mu}{p Q_i} + \frac{h_i c_i Q_i (n-1)}{2n} + \frac{h_i c_i Q_i D}{2nr}. 
\]  

(4.9)

We can verify that (4.9) is a convex function of \( n \) for given \( \mu \). As a result, if there is an integer \( n' \) such that

\[
TC_{A_2}(\mu, q, n'-1) \geq TC_{A_2}(\mu, q, n') < TC_{A_2}(\mu, q, n'+1),
\]

then \( n' \) is the optimal \( n \) value for given \( \mu \). Using straightforward algebraic manipulation, we found the following result.

**Result 4.7.** When \( Q = Q_i \) for given \( \mu \), the value for \( n \), that minimizes \( TC_{A_2}(\mu, q, n|c_i) \), is determined by the following condition

\[
n' = \left\lfloor \frac{1}{2} \left\lfloor \sqrt{1 + \frac{2h_i Q_i^2}{2DS\mu t} \left\{ (c_i(\alpha - 1)\mu + b)(rp-D) + 1 \right\} \right\rfloor \right\rfloor. 
\]

Similar properties of cases \( B_1 \) and \( B_2 \) can be derived following similar procedures.
Case $B_1$

In presenting the following results, let $TC_{B_1}(\mu,q,m|c_i)$ be the total cost in case $B_1$ evaluated at $\mu$, $q$, and $n$ when the material unit cost is $c_i$.

**Result 4.8.** For given $\mu$ and $q$, the value of $m$ that minimizes $TC_{B_1}(\mu,q,m|c_i)$ is given by

$$m = \left[ \frac{1}{2} \left( 1 + \sqrt{1 + 4A + 1} \right) \right],$$

where $A = \frac{h_1c_1\mu q^2}{2}\frac{1}{2\kappa}$. 

**Result 4.9.** For given $\mu$, the value of $m$ that minimizes $TC_{B_1}(\mu,q,m|c_i)$ satisfies the following condition.

$$m_o \leq m < \frac{1}{2} \left\{ \sqrt{1 + \frac{4\mu D(S+K)}{b\left(\alpha \mu + \frac{b}{c_1}\right)(r_p-D)}} + 1 \right\},$$

where

$$m_o = \frac{1}{2} \left\{ \sqrt{1 + \frac{4\mu D(S+K)}{b\left(\alpha \mu + \frac{b}{c_1}\right)(r_p-D)}} - 1 \right\} \quad (4.10)$$

or $m_o = 1$ if $m_o$ given by expression (4.10) is not a real number.

**Result 4.10.** For given $\mu$ and its corresponding value of $m$ obtained from result 4.9, the production run size that minimizes $TC_{B_1}(\mu,q,m|c_i)$ is given by

$$q = \sqrt{\frac{2\mu D(S+Km)}{h_1(c_1\alpha \mu + b)(r_p-D) + h_1c_1\mu(D/m)}}.$
Case B$_2$

To present the following results, let TC$_{B_2}$(μ,q,m|c$_i$) be the total cost in case B$_2$ evaluated at μ, q, and m when the material unit cost is c$_i$. In this case, the quantity actually used per order, Q = $\frac{μq}{m}$, is less than or equal to the order quantity, Q$_T$, which is the quantity breakpoint for c$_i$.

Result 4.11. For given μ and q, the m that minimizes TC$_{B_2}$(μ,q,m|c$_i$) is given by

$$m = \left[ \frac{1}{2} \left( \sqrt{1 + 4A'} + 1 \right) \right]$$

where $A' = \frac{h_1c_mμq^2}{2r[K+Q_i(c_d+c_i)]}$

Result 4.12. The optimal m for given μ that minimizes TC$_{B_2}$(μ,q,m|c$_i$) satisfies the following condition:

$$m'_1 \leq m < \frac{1}{2} \left\{ \sqrt{1 + \frac{4μD(S+K+Q_i(c_d+c_i))}{[K+Q_i(c_d+c_i)](αμ(c_i+c_i)(rD) + b)}} + 1 \right\}$$

where

$$m'_1 = \frac{1}{2} \left\{ \sqrt{1 + \frac{4μD(S-K-Q_i(c_d+c_i))}{[K+Q_i(c_d+c_i)](αμ(c_i+c_i)(rD) + b)}} - 1 \right\} \quad (4.11)$$

or $m'_1 = 1$ if $m'_1$ given by expression (4.11) is not a real number.

Result 4.13. The optimal production run size for given μ and its corresponding optimal value for m that minimizes TC$_{B_2}$(μ,q,m|c$_i$) obtained from result 4.12 is
If $Q$ resulting from minimizing $TC_{B2}(\mu,q,m|c_i)$ is larger than $Q_1$, then the optimal order quantity $Q_T$ equals $Q_1$. As a result, the optimal production run size, $q = \frac{mQ_1}{\mu}$, and the total cost function (4.4) can be simplified as

$$TC_{B2}(\mu,q,m) = \frac{D(b+c_i\alpha\mu)}{\mu} + \frac{DS}{\mu} \frac{mQ_1}{m} + \frac{h_1}{2r\mu} mQ_1(c_i\alpha\mu+b)(rp-D)$$

$$+ \frac{KD\mu}{pQ_1} + \frac{h_1c_iQ_1D}{2rp} \quad (4.12)$$

and the following result can be obtained.

**Result 4.14.** When $Q = Q_T = Q_1$, for given $\mu$, the value of $m$, which minimizes $TC_{B2}(\mu,q,m|c_i)$, is determined by the following condition:

$$m = \left\lfloor \frac{\sqrt{\frac{8rDS\mu^2}{h_1Q_1(c_i(\alpha-1)\mu+b)(rp-D)}} - 1}{2} \right\rfloor + 1.$$

**Result 4.15.** (a) For given $c_i$, if $Q^*$ resulting from the solution that minimizes $TC_{A1}(\mu,q,n|c_i)$ is admissible, then the feasible solutions in cases $A_1$ and $A_2$ associated with all the material unit costs larger than $c_i$ are not optimal. (b) For given $c_i$, if $Q^*$ resulting from the solution that minimizes $TC_{B1}(\mu,q,m|c_i)$ is admissible, then the feasible solutions in cases $B_1$ and $B_2$ associated with all the unit material prices larger than $c_i$ are not optimal.

**Proof:** We assume, for given $c_i$, $n_i$, $\mu_i$ and $q_i$ are the solution that minimizes the total cost $TC_{A1}(\mu,q,n|c_i)$ and $Q^* = n_i\mu_iq_i$. Considering case $A_1$, for $c_j > c_i$, $n_j$, $\mu_j$ and $q_j$ are the solution that minimizes the total cost $TC_{A1}(\mu,q,n|c_j)$. Then, any
combination of \((\mu, q, n)\) for any \(c_j > c_i\) has a total cost larger than \(TCA_1(\mu, q, n|c_i)\) because of the following obvious relationship:

\[
TCA_1(\mu, q, n|c_i) \leq TCA_1(\mu, q, n|c_i) < TCA_1(\mu, q, n|c_j) \leq TCA_1(\mu, q, n|c_i).
\]

This proves that for all the costs higher than \(c_i\) and no excess raw material is ordered (case \(A_1\)), none of the total costs are lower than \(TCA_1(\mu, q, n|c_i)\). Next, we examine the case when excess material is ordered (case \(A_2\)).

Considering case \(A_2\), given \(c_j \geq c_i\) and \(Q = n\mu q \leq Q_i\),

\[
TCA_2(\mu, q, n|c_i) < TCA_2(\mu, q, n|c_i).
\]

When \(Q \leq Q_i\), for a given unit cost, the total cost of case \(A_1\) is always less than or equal to that of case \(A_2\), because the cost of case \(A_2\) includes the total cost of case \(A_1\) and the disposal cost. They are:

\[
TCA_1(\mu, q, n|c_i) = \frac{D(b+c_i\alpha\mu)}{p} + \frac{DS}{pq} + \frac{h_1}{p}(c_i\alpha\mu+b)\frac{q}{2}\left(rp-D\right) + \frac{KD}{npq} + h_1c_i\left[\frac{n-1}{2}\mu q + \frac{\mu q D^2}{2rp}\right],
\]

and

\[
TCA_2(\mu, q, n|c_i) = \frac{D(b+c_i\alpha\mu)}{p} + \frac{DS}{pq} + \frac{h_1}{p}(c_i\alpha\mu+b)\frac{q}{2}\left(rp-D\right) + \frac{KD}{npq} + h_1c_i\left[\frac{n-1}{2}\mu q + \frac{\mu q D^2}{2rp}\right] + \frac{D(Q_i-Q)(c_i+c_j)}{npq},
\]

that is:

\[
TCA_1(\mu, q, n|c_i) \leq TCA_2(\mu, q, n|c_i).
\]

Since \(n_i\), \(\mu_i\) and \(q_i\) are the solution that minimizes the total cost \(TC(\mu, q, n|c_i)\), so
As a result, 

\[ TC_{A_i}(\mu, q, n|c_i) \leq TC_{A_1}(\mu, q, n|c_i) < TC_{A_2}(\mu, q, n|c_i) \]

This proves that all the solutions associated with a material unit cost larger than \( c_i \) have a total cost larger than \( TC_{A_1}(\mu, q, n|c_i) \). For case \( B_1 \), it can be also proved by the same procedure.

### 4.3.2 Solution Algorithm

From the results presented in the last subsection, we propose a solution algorithm in this section. The algorithm is basically a section-by-section search method based on the material unit cost \( c_j \). Specifically, it starts with the lowest \( c_i \) and the case without excess material ordered (i.e., cases \( A_1 \) and \( B_1 \)). When a feasible material order quantity is found, based on result 4.15, we know that it is not necessary to examine any higher unit cost. Therefore, if this feasible solution happens when the unit cost is the lowest one, then it is the optimal solution; otherwise, we start from the next lower unit cost, examine the cases in which excess material is ordered (cases \( A_2 \) and \( B_2 \)), and find the lowest total costs for all lower unit costs. All these total costs are compared with the total cost of the first feasible solution. The order quantity with the lowest total cost is the optimal order quantity and the corresponding \( \mu \), \( q \) and \( n \) or \( m \) are the optimal process mean, production run size, and order frequency or inverse of order frequency.

Before the detailed steps of the solution algorithm are presented, we discuss how the results given in the last section are used in the algorithm.

First we discuss the procedure to find the best solution in case \( A_1 \) when the material unit cost is \( c_i \) (i.e., \( Q_i < Q < Q_{i+1} \)). For given \( \mu \), the solution for \( n \) and \( q \) that minimizes \( TC_{A_1}(\mu, q, n) \) is obtained by:
Step 1. Find integer sets $N = \{n: n$ satisfies the condition given in result 4.2$\}$.

Step 2. Obtain the production run size, $q_n$, for all $n \in N$, using result 4.3.

Step 3. For each $n$, compute $Q = npq$. If $Q_i \leq Q < Q_{i+1}$, compute total cost $TC_{A_1}(\mu, q_n, n)$. Otherwise, the solution is not feasible.

As a result, $TC_{A_1}(\mu, q, n)$ becomes a function of $\mu$. A search procedure can be used to find the value for $\mu$ that minimizes $TC_{A_1}(\mu, q, n)$ for given $c_i$. In this study, the golden-section search is used to search for $\mu$ in the range $[L, L + 4\sigma]$ and the search terminates when the width of the interval of uncertainty is not more than 0.0001. As discussed in section 3.3, the reason for this range is that when $L$ is used as the lower bound, the process conforming rate is 50%, which is low in most realistic applications; when the upper bound is $4\sigma$ above the lower limit, it is large enough to include the possible optimal solution. This search method and range are also used in the other three cases under the similar situation. Note that when $Q$ is not within the feasible range in step 3, if $Q < Q_i$, the minimum total cost exists when $Q = Q_i$. This situation will be considered in case $A_2$.

The procedure for the three remaining cases is similar: the total cost is first expressed as a function of $\mu$, and then a search procedure is used to find the optimal value for $\mu$.

In case $B_1$, the solution for given $c_i$ and $\mu$ that minimizes $TC_{B_1}(\mu, q, m)$ is obtained by:

Step 1. Find integer sets $M = \{m: m$ satisfies the condition given in result 4.9$\}$. 

Step 2. Obtain the production run size, \( q_m \), for all \( m \in M \), using result 4.10.

Step 3. For each \( m \), compute \( Q = \frac{\mu q}{m} \). If \( Q_i \leq Q < Q_{i+1} \), compute total cost \( TC_B(\mu, q_m, m) \). Otherwise, the solution is not feasible.

In case \( A_2 \), the solution for given \( c_i \) and \( \mu \) that minimizes \( TC_{A_2}(\mu, q, n) \) is obtained by:

Step 1. Find integer sets \( N = \{ n : n \) satisfies the condition given in result 4.6 \}.

Step 2. Obtain the production run size, \( q_n \), for all \( n \in N \), using result 4.5.

Step 3. For each \( n \), compute \( Q = npq \). If \( Q < Q_i \), compute total cost \( TC_{A_2}(\mu, q_n, n) \). If \( Q > Q_i \), then let \( Q = Q_i \), find \( n \) using result 4.7, and compute \( q = \frac{Q_i}{n \mu} \). Use the result to compute total cost \( TC_{A_2}(\mu, q_n, n) \).

In case \( B_2 \), the solution for given \( c_i \) and \( \mu \) that minimizes \( TC_{B_2}(\mu, q, m) \) is obtained by:

Step 1. Find integer sets \( M = \{ m : m \) satisfies the condition given in result 4.12 \}.

Step 2. Obtain the production run size, \( q_m \), for all \( m \in M \), using result 4.13.

Step 3. For each \( m \), compute \( Q = \frac{\mu q}{m} \). If \( Q_i \leq Q < Q_{i+1} \), compute total cost \( TC_{B_2}(\mu, q_m, m) \). If \( Q > Q_i \), then let \( Q = Q_i \), find \( m \) using result 4.14, and compute \( q = \frac{mQ}{\mu} \) and total cost \( TC_{B_2}(\mu, q_m, m) \).

The procedure for finding optimal process mean, production run size, and material order quantity is summarized is the following two steps:
Step 1. Starting from the lowest price, using the procedures presented above for cases A₁ and B₁, find the solution that gives the lowest cost to $T_{C_{A_{1}}}(μ, q_{m}, n|c_{j})$ and $T_{C_{B_{1}}}(μ, q_{m}, m|c_{j})$. This process continues until a feasible solution is found (i.e., $Q^* ∈ [Q_{j}, Q_{j+1})$).

Step 2. If the optimal ordering quantity for the lowest price is feasible, the solution found is optimal. Otherwise, for all the smaller unit costs, $c_{j+1}, c_{j+2}, ...$, using the procedures presented above to find the solutions for cases A₂ and B₂. Then, compare these total costs with the total cost obtained in step 1, and select the one with the lowest total cost as the optimal solution.

A FORTRAN program has been written to implement this solution procedure. In the next section, two numerical examples are used to illustrate the solution procedure.

4.4 Examples

In this section, the example used in Chapter 3 is used to illustrate the solution procedure given in the last section. The model parameters are: $D = 5,000$ items, $r = 7,500$ items, $σ = 0.8$ mg, $L = 1.1$ mg, $S = $150, $K = $30, $b = $0.16 per item, $α = 4$ and $h_{1} = $0.03 per dollar per unit time. The all-unit quantity discounts policy for the raw material considered in the examples is as follows:

<table>
<thead>
<tr>
<th>Order Quantity</th>
<th>Price per mg</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 to 11,999</td>
<td>$0.1</td>
</tr>
<tr>
<td>12,000 to 34,999</td>
<td>$0.099</td>
</tr>
<tr>
<td>35,000 or more</td>
<td>$0.098</td>
</tr>
</tbody>
</table>
We consider two different disposal costs in the following two examples.

**Example 1.** In this example, the disposal cost, $c_d$, per mg is $0.01$. To find the solution in individual cases, the golden-section search method is used. The result of the algorithm is given as follows.

1. **Step 1.** When $c_i = 0.098$, Q associated with the solution in cases $A_1$ and $B_1$ is $Q = 14,702.68$, which is not feasible. When $c_i = 0.099$, the corresponding $Q$ is $14,665$, which is within the range $[12,000, 35,000]$ for $c_i = 0.099$. This solution is feasible, and its total cost is $5,509.19$.

2. **Step 2.** Starting from $c_i = 0.098$, the lowest total cost $TC = 4,724.04$ exists when $Q$ is $169,769.88$. This solution is not feasible because $Q$ is greater than $35,000$. Therefore, we consider $Q = Q_T = 35,000$. The total cost, when $Q = 35,000$, is $5,478.81$. The correspondent $n = m = 1$, $\mu = 1.690$ and $q = 20,710$. Comparing the costs, the lowest total cost is $TC = 5,478.81$ when $Q = Q_T = 35,000$ mgs.

Therefore, the optimal solution for this example is to set the process mean at $1.609$ mgs to produce $20,710$ items per production run and to place an order for $35,000$ mgs of raw material for every production run.

**Example 2.** In this example, the disposal cost per mg is $-0.097$, which means each excess mg of the content can be sold at a reduced price, $0.097$. The solution procedure for this example is as follows.

1. **Step 1.** When $c = 0.098$, the lowest total cost exists when $Q = 14,702.68$, which is not within the range $[35,000, \infty)$. Therefore, it is not feasible. When $c = 0.099$, the $Q$ value associated with the lowest
total cost is 14,665. Because Q is within the range \([12,000,35,000]\) for \(c_i = 0.099\), this solution is feasible and its corresponding total cost is $5,509.19.

**Step 2.** Starting from \(c = 0.098\), for case \(A_2\), the lowest total cost is $5,478.36 when \(n = 1, \mu = 1.694, q = 19,087\) and \(Q = 32,333\) mgs. For case \(B_2\), the lowest total cost is $5,473.23 when \(m = 2, \mu = 1.675, q = 26,394\) and \(Q = 22,102\) mgs. Comparing all the total costs, the lowest one is $5473.23 and its corresponding unit cost is $0.098, \(Q = 22,102\), and \(Q_T = 35,000\).

From the above, we found that the lowest total cost is $5,473.23 in case \(B_2\). Therefore, the optimal solution is to set the process mean at 1.675 mgs, produce 26,394 items per production run, and place two orders, each order with quantity 35,000 mgs, per production run. For each order of the raw materials received, 12,898 (= 35,000 - 22,102) mgs are disposed of right away. The disposal cost for this example is $6,398. The process conforming rate is 76.37%

### 4.5 Sensitivity Analysis

In this section, a sensitivity analysis is performed to study the effects of the following model parameters on the optimal solution: (1) the product demand rate \(D\), (2) the production rate \(r\), (3) the process standard deviation \(\sigma\), (4) the value-added factor \(\alpha\), (5) the production setup cost \(S\), (6) the material ordering setup cost \(K\), and (7) the unit disposal cost \(c_d\). The sensitivity analysis is based on the example 2 given in the last section and compared with what we have in Chapter 3.

For the same reasons given in section 3.4, per-item costs are used to report the results in this section. In addition to what we have in Chapter 3, per-unit disposal cost
is also reported in the results. They are per-item finished product-related cost, PFPC = FPC(μ,q)/D: per-item material setup and handling cost, PMC = MC_A(μ,q,n)/D or MC_B(μ,q,m)/D; per-item disposal cost, PDC = DIPC_A(μ,q,n)/D or DIPC_B(μ,q,m)/D; and per-item total cost, PTC = TC_A(μ,q,n)/D or TC_B(μ,q,m)/D.

The computer program used in the study is written in FORTRAN and run on an IBM3090-600E. The running time for solving each of the problems used in the study was just a few seconds.

4.5.1 Effect of Demand Rate

The demand rate studied here ranges from 1,500 through 7,000 items per unit time with an increment of 500. The results are shown in Table 4.1.

Different from what we may except, the process mean decreases as demand rate increases until the demand rate reaches 5,500 items per unit time. The main reason for this, as we discussed in section 3.5, is mainly caused by "extra" production capacity when D is relatively low. In order to avoid frequently setups and holding excess finished products, the process mean is set lower and the production run size is set higher. When the demand rate is higher than 5,500, we find that the process mean increases as the demand rate increases. This is because the demand must be satisfied. As a result, the process yield rate (λ) is very close to the demand rate and the inventory accumulation rate (λ-D) is very low. The low accumulation rate results in a long production run and, thus, a large production run size. On the other hand, we also observe that the material use quantity per order, Q, increases when the demand rate increases as long as the ordering frequency, n or m, remains the same. When the
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ordering frequency increases (m increases or n decreases), material use quantity decreases if the order quantity remains the same.

Comparing the results in Tables 4.1 and 3.1, although the patterns of changes in the process mean, the production run size and the amount of the raw material used in production are similar, some differences are found because of the quantity discounts. Before the demand rate reaches 5,500, where production yield rate is very close to the demand rate, the process mean is higher when quantity discounts are available. This is mainly because of the decrease in the material unit cost. When the demand rate is not larger than 2,500, in order to reach a higher raw material price breakpoint, 35,000, the order quantity covers the requirement of two production runs. As a result, the production run size decreases. When order frequencies are the same in Tables 4.1 and 3.1, the production run size increases in order to receive the price discount. Consequently, the per-item finished product related cost (PFPC) and per-item total cost (PTC) decreases. The per-item material related (setup and handling) cost (PMC), however, increases because of the higher material inventory level resulted from the high order quantity Q_T. The per-item disposal cost decreases while the demand rate increases as long as the order frequency remains the same. This is because of the increase of the production run size and so the raw material required while material order quantity, Q_T, remains stable at 35,000. An important observation is also found here is the lowest per-item total cost exists when the demand rate is 6,000. This again demonstrates the importance of a careful production capacity design in correspondence to the demand rate.

4.5.2 Effect of Production Rate

Table 4.2 shows the effects of selected production rate ranging from 6,000 to 10,000 items per-unit time. Except when the production rate is close to the demand rate, that is r < 7,000, the process mean increases as r increases. When r is smaller than
Table 4.2: Effect of Production Rate for Selected Values of \( r \)

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<th>( \lambda )</th>
<th>( q )</th>
<th>( m )</th>
<th>( Q )</th>
<th>( Q_T )</th>
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<th>PMC</th>
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</table>
7,000, in order to satisfy the demand, the yield rate has to be maintained at a certain level and so is the process mean. Meanwhile, the accumulation rate is very low, resulting in a very large production run size. When \( r > 7,000 \), under the same ordering frequency, the production run size decreases as \( r \) increases. The decrease is used to compensate the increase in the holding cost caused by the higher process mean.

Compared to the production run size, the amount of the raw material used in production in each order, \( Q \), remains relatively more stable. \( Q \) increases as the ordering frequency becomes smaller and \( Q \) decreases as \( r \) increases when the ordering frequency remains the same. However, the order quantity remains the same, which is 35,000.

The process mean is higher when quantity discounts are applied. The material requirements in Table 4.2 are higher than those in Table 3.2. The main reason for this is to reduce the amount of disposed material when the material order quantity remains 35,000. As a result, the production run size increases in Table 4.2 when material order frequency is same as that in Table 3.2.

One thing needs to be pointed out is that the minimum per-unit total cost occurs when \( r \) is 6,500. Again, a higher production capacity does not mean a lower unit cost. Consequently, a careful allocation on production capacity is an important issue.

### 4.5.3 Effect of Process Variation

From Table 4.3, we find that the per-item total cost increases as the process variation increases. This implies that a better production performance can be obtained from reducing the process variation. In the table, we find that when \( \sigma \) is smaller than 0.2, the production run size increases and the material use quantity decreases as \( \sigma \) decreases. The reason for this is that a lower process mean and a higher conforming rate can be achieved simultaneously by reducing \( \sigma \). As a result, the per-item production
Table 4.3: Effect of Process Variation for Selected Values of $\sigma$

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cost can be reduced. In addition, when \( Q \) is lower and \( q \) is higher, both the holding cost of the raw material and the setup cost of production can then be reduced. Although the disposal and holding costs of the finished product will be higher because of a lower \( Q \) and a higher \( q \), the saving along with the lower process mean, higher \( q \) and lower \( Q \) is more than the increased cost.

The production run size, on the other hand, increases when \( \sigma \) increases from 0.2 to 0.9. This is excepted because of the decrease in the conforming rate. However, a large increase in the production run size at \( \sigma = 0.7 \) is caused by the increase in the ordering frequency of the raw material. When \( \sigma \geq 0.9 \), the process mean stays at a level where the process yield rate is very close to the demand rate. As a result, the production run size becomes very large and sensitive because of the very low accumulation rate of the finished product.

Comparing Table 4.3 to Table 3.3, for the same \( \sigma \) below 0.9, the process mean increases and the ordering frequency, the production run size and the material use quantity, \( Q \), decrease when quantity discounts are available. The main reason for this is the material order quantity remains stable (35,000) when \( \sigma \) changes. In order to reduce the disposal cost (i.e., the difference between \( Q \) and \( Q_{\uparrow} \)), the process mean is set higher and the production run size and ordering frequency are kept lower. However, if the ordering frequency remains the same, e.g. \( \sigma = 0.01 \) and 0.7, the result is totally different. Instead of using a larger process mean and reducing the production run size, a smaller process mean and a larger production run size are used to increase the material use quantity to reduce the disposal cost.

4.5.4 Effect of Value-Added Factor

A larger \( \alpha \) implies higher costs of producing an item and cost of holding inventory of the finished product. One way to compensate this is to lower the process
mean. However, the decrease in the process mean results in a lower production yield rate, and forces the production run size to increase in order to satisfy the demand. Although the process mean can be decreased as $\alpha$ increases, it can be only reduced to a certain level to ensure the demand can be satisfied. As a result, when $\alpha = 6$, the production run size becomes very large and sensitive due to the low inventory accumulation rate of the finished product. The phenomena can be seen from Table 4.4.

Both the process mean and the production run size are slightly higher in Table 4.4 than those in Table 3.4. Note that the material order quantity remains at 35,000 in Table 4.4. Consequently, the price break at $Q_r=35,000$ appears to be very attractive and it is economical to order 35,000 units of the raw material and, at the same time, increase the material usage by using a larger process mean, a larger production run size and a smaller raw material ordering frequency in each production run. By doing so, it actually reduces the production cost and holding cost of the finished product.

4.5.5 Effect of Production Setup Cost

When the production setup cost increases, we can expect an increase in the production run size. However, a larger production size may cause a higher inventory cost. As a result, the process mean decreases to reduce the finished product holding cost and the production yield rate. Moreover, the process mean can only be reduced to a level to ensure the production yield rate satisfies the demand. After the point at which $S = 300$, the process mean remains stable and production run size becomes very large.

On the other hand, $Q$ remains much more stable when $S$ changes. $Q$ increases as $S$ increases when $m$ remains the same. This result is mainly caused by the increase in the production run size and the decrease of the process mean. When $S$ is larger than
Table 4.4: Effect of Value-Added Factor for Selected Values of $\alpha$

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<th>$m$</th>
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Table 4.5: Effect of Production Setup Cost for Selected Values of S

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300, Q remains very stable. These phenomena were also found in Table 3.5. However, the value of Q in Table 4.5 is higher than that in Table 3.5 for the same value of S. As mentioned above, the main reason for this is to decrease the disposal cost which is caused by the constant, high material order quantity.

4.5.6 Effect of Material Ordering Setup Cost

Table 4.6 shows that when the material ordering setup cost increases, the ordering frequency decreases and the material order quantity increases. The increase in the material order quantity results in a larger material holding cost and a higher production run size. On the other hand, in order to reduce the holding cost of the finished products, the process mean is reduced to decrease per-unit production cost and holding cost.

From the table, we found that when K = 30 the material order quantity is larger than the amount of the material used in production. That suggests that a saving in the unit material cost by ordering 35,000 mgs, is higher than the increase of PMC and the disposal cost incurred by the excess material. When K is equal to or larger than 80, the order quantity equals to the material requirement.

Comparing the table to Table 3.6, we can still find the process mean, production run size and material requirement (ordering) quantity in Table 4.6 are higher than those in Table 3.6 for the same K.

4.5.7 Effect of Unit Disposal Cost

As expected, when the unit disposal cost increases, we would be less likely to dispose raw material. However, in order to take the advantage of a lower unit price, we may order up to the lowest quantity of lower price without disposing any material. In other word, \( Q = Q_T \). Table 4.7 shows the results when unit disposal cost \( c_d \) increases from -$0.097 to -$0.055.
Table 4.6: Effect of Raw Material Ordering Setup Cost for Selected Values of K

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Table 4.7 Effect of Unit Disposal Cost for Selected Values of $c_d$

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When the unit disposal cost increases from -$0.097 to -$0.096, the cost of disposing extra materials increases, and both Q and q increase. The increase of the production run size will increase the inventory cost. Therefore, the process mean decreases to reduce the increase in the finished product holding cost. When $c_d$ increases more, the cost of disposing material becomes even higher. As a result, the ordering frequency is reduced and Q is moved to reach the next pricebreak quantity, 35,000 in the case. The decrease of the ordering frequency will also decrease the production run size and increase the process mean. The increase of process mean can increase the process yield rate to compensate the decrease of production run size as well as can increase the material requirement to 35,000. Afterward, the change of $c_d$ will not affect the process mean, production run size and material order quantity which reach stable and so the per-unit total cost.
Chapter 5
CONSTANT MATERIAL SUPPLY

5.1 Introduction

As discussed, the process mean affects the process yield rate and the production run size. If the raw material has to be ordered from outside vendors, the ordering policy should be based on the production run size and the process mean. As a result, in order to minimize total cost, we have to determine simultaneously the process mean, production run size, and material order quantity. In the last chapter, the quantity discounts in raw material purchasing were incorporated in these decisions. It was shown that the process mean and production run size may be affected by the differentials in material unit cost because of the quantity discounts.

In this chapter, we consider a situation in which the raw material is supplied at a constant rate from a reliable source. Note that because of the success of Japanese manufacturing techniques in worldwide competition, the JIT system is adapted by more and more companies in the United States. One of the crucial characteristic in JIT is a single and reliable supply source that delivers small lot material on time. We therefore consider a situation wherein a constant supply rate of raw material is offered by a reliable source. This can be treated as an extreme case of JIT supply. This type of system is especially good for the supplier seeking a stable demand source and capable of providing a good product. The system allows the establishment of a long-term commitment in the supplier-customer relationship.

In addition, we will also consider the situation in which quantity discounts for raw material are offered by vendors. Both the incremental and all-units quantity discounts policies are considered. The organization of this paper is as follows. In the
next section, models are formulated for three cases in quantity discounts: no quantity discounts, incremental discounts, and all-units quantity discounts. In section 5.3, analytical properties associated with the optimal solution are derived and solution procedures are proposed. In section 5.4, numerical examples are given to illustrate the solution procedures, and in section 5.5 a sensitivity analysis is carried out to compare the three cases in terms of quantity discounts.

5.2 Model Formulation

It is assumed that the raw material arrives at a constant rate $\beta$ per unit time, regardless of whether the production process is in operation or is idle. Assume that the production process is operated under the same setup policy as in the previous chapters (shown by Figure 7(a)). Assume all the raw material received by the producer is used in production. The inventory level of the raw material as a function of time is shown in Figure 7(b).

![Inventory Level Graphs](image)

*Figure 7. (a) Inventory Level of Finished Product; (b) Inventory Level of Raw Material*
The requirement for the raw material is $r_1 \mu$ per unit time during production, which should be higher than the raw material supply rate $\beta$. Let $q$ be the production run size. We assume that the initial raw material inventory level at time 0 is $\frac{q}{r}(r_1 \mu - \beta)$, which is sufficient to meet the requirement in the first production run. When production starts, the raw material inventory decreases at a rate of $(-r_1 \mu + \beta)$. At time $\frac{q}{r}$, the raw material inventory is depleted and the first production run is finished. Then, the process is idle, and the raw material inventory level increases at a rate of $\beta$ until the second production run starts at time $\frac{pq}{D}$. Let $q$ be the production run size. At the end of a production run, the inventory level $\beta(\frac{pq}{D} - \frac{q}{r})$ should be same as the level at time 0; that is,

$$\frac{q}{r}(r_1 \mu - \beta) = \beta(\frac{pq}{D} - \frac{q}{r}).$$

For any given process mean $\mu$, therefore, the proper supply rate should be a function of $\mu$; i.e., $\beta = \frac{D \mu}{p}$. On the other hand, $\mu$ corresponding to given $\beta$ is determined by $\mu = \frac{p \beta}{D}$. In this chapter, the material-related cost is expressed as a function of $\beta$ instead of $\mu$. Since the supply rate is constant, the setup cost associated with the raw material is not considered. Let $h_1$ be the holding cost rate per dollar per unit time and $H = h_1(\frac{\alpha C(\beta)}{D} + \frac{b}{p})$ be the cost of holding a conforming item for a unit time. The total holding cost per unit time for the raw material is

$$MC(\beta, q) = h_1 c^2 r(\frac{q}{2r}(r_1 \mu - \beta) = h_1 c \frac{14q}{2rp}(r_1 \mu - \beta).$$

Let $C(\beta)$ denote the raw material cost when the supply rate is $\beta$. In other words, $C(\beta)$ is the product of supply rate $\beta$ and the material unit cost. As a result, it can be verified that (5.1) can be rewritten as
MC(\beta, q) = h \frac{C(\beta)q}{2rD} (rp-D),

and the finished product-related cost -- which is the sum of the production cost, the setup cost and the inventory holding cost -- is

\[ FPC(\beta, q) = \frac{D(b+c\alpha\mu)}{p} + \frac{DS}{pq} + H \frac{q}{2r} (rp-D). \]

where \( H = \frac{h}{p} (c\alpha \mu + b) \). The finished production related cost function can be expressed as a function of \( \beta \) and \( q \) as follows:

\[ FPC(\beta, q) = \alpha C(\beta) + \frac{Db}{p} + \frac{DS}{pq} + H \frac{q}{2r} (rp-D). \]

The total cost is the sum of \( FPC(\beta, q) \) and \( MC(\beta, q) \):

\[ TC(\beta, q) = FPC(\beta, q) + MC(\beta, q) \]

\[ = \alpha C(\beta) + \frac{Db}{p} + \frac{DS}{pq} + h \left( \frac{\alpha C(\beta)}{D} \right) \frac{b}{p} \frac{q}{2r} (rp-D) + h \frac{C(\beta)q}{2rD} (rp-D). \quad (5.2) \]

Three cases, in terms of raw material quantity discounts, are considered. Let \( c_i > c_{i+1} \), and \( 0 \leq \beta_i < \beta_{i+1} \), which are the price breakpoints. The three cases and their corresponding material cost functions are

Case 1. No discounts; that is,

\[ C(\beta) = \beta c_0, \] where \( c_0 \) is the material unit cost.

Case 2. Incremental quantity discounts; that is,

\[ C(\beta) = C(\beta_i) + (\beta - \beta_i)c_p, \] for \( \beta \in [\beta_i, \beta_{i+1}) \), \( i = 0, 1, 2,..., k-1. \)
Case 3. All-units quantity discounts; that is,
\[
C(\beta) = \beta c_i, \text{ for } \beta \in [\beta_i, \beta_{i+1}), i = 0, 1, 2, \ldots, k-1.
\]

5.3 Optimal Solution

In this section, we first derive several important analytical properties for the optimal solutions for the three cases defined in the last section; that is, no discounts, incremental discounts, and all-units quantity discounts policies. Based on these results, efficient solution algorithms are proposed to find the optimal process mean, production run size, and material supply rate.

5.3.1 Analytical Properties

As we mentioned in the last section, the total cost per unit time under constant material supply is the sum of costs associated with finished product, \(\text{FPC}(\beta, q)\), and material holding cost, \(\text{MC}_\beta(\beta, q)\); that is,
\[
\text{TC}(\beta, q) = xC(\beta) + \frac{Db}{p} + \frac{DS}{pq} + h_1\left(\frac{\alpha C(\beta)}{D} + \frac{b}{p}q\right)(rp-D) + h_1\frac{C(\beta)q}{2rD}(rp-D),
\]
where \(C(\beta)\) is the raw material purchasing cost.

In this subsection, we study the properties of three situations: no discounts, incremental quantity discounts, and all-units quantity discounts.

Case 1: No Discount

When no quantity discounts are available, \(C(\beta)\) will be equal to \(c_0\beta\) and the total cost function is
\[
\text{TC}_1(\beta, q) = \alpha c_0\beta + \frac{Db}{p} + \frac{DS}{pq} + h_1\left(\frac{\alpha c_0\beta}{D} + \frac{b}{p}q\right)(rp-D) + h_1\frac{c_0\beta q}{2rD}(rp-D). \tag{5.3}
\]
Because of the function relationship between $\beta$ and $\mu$, $\beta = \frac{D\mu}{p}$, it can be easily verified that $TC_1(\beta,q)$ is a convex function of $q$ when $\mu$ is fixed. The optimal $q$ for given $\beta$ (or, equivalently, $\mu$) can be found, therefore, by solving $\partial TC_1/\partial q = 0$. The result is given as follows.

**Result 5.1.** For given $\beta$, the optimal production run size is

$$q = \sqrt{\frac{2rDS}{h_1(D-1)[Db+p(\alpha+1)C(\beta)]}}. \quad (5.4)$$

Using this result, we can find the optimal production run size and the material supply rate associated with a given process mean.

**Case 2: Incremental Quantity Discounts**

For incremental quantity discounts, the total cost function is

$$TC_2(\beta,q) = \alpha C(\beta) + \frac{Db}{p} + \frac{DS}{pq} + h_1\left(\frac{\alpha C(\beta)}{D} + \frac{b}{p}\right)q + h_1 \frac{C(\beta)q}{2r(D-p)} + h_1 \frac{C(\beta)q}{2rD} (D-p), \quad (5.5)$$

where $C(\beta) = C(\beta_i) + (\beta - \beta_i)c_i$, for $\beta \in [\beta_i, \beta_{i+1})$, is the total material purchasing cost of the constant supply rate, $\beta$ units.

Since disposal is not an economical option under this policy, the material supply rate is determined by $\mu$ by the relationship, $\beta = \frac{D\mu}{p}$. The optimal $q$ corresponding to given $\beta$ can be found, therefore, by solving $\partial TC_2(\beta,q)/\partial q = 0$. The result is given as follows.

**Result 5.2.** For given $\beta$, the optimal production run size is

$$q = \sqrt{\frac{2rDS}{h_1(D-1)[Db+p(\alpha+1)C(\beta)]}}. \quad (5.6)$$
Case 3: All-Units Quantity Discounts

The material purchasing cost $C(P)$ under the all-units quantity discounts is a discontinuous function of $\beta$, and it is possible that the producer orders a larger-than-needed amount of the raw material and disposes of the excess raw material to take advantage of price differentials. We assume that $\beta_u$ is the actual material use rate, where $\beta_u \leq \beta$. Depending on whether excess raw material is ordered, therefore, we consider the following two cases.

Case 3.1: No Excess Raw Material Ordered

In this case, the raw material use rate is equal to the raw material supply rate; i.e., $\beta_u = \beta$. In other words, no excess raw material is ordered. When $\beta \in [\beta_{i-1}, \beta_{i})$, the total cost can be written as

$$TC_{31}(\beta,q) = \alpha \beta c_{i-1} + \frac{Db}{p} + \frac{DS}{pq} + h_i \left( \frac{\alpha \beta c_{i-1}}{D} + \frac{b}{p} \right) q + \frac{\beta q}{2r} (rp-D).$$

(5.7)

It is easy to verify that when $\beta$ (or $\mu$) is fixed, eq. (5.7) is a convex function of $q$. As a result, the value of $q$ that minimizes the total cost function eq. (5.7) can be obtained by setting the first derivative of $TC_{31}(\beta,q)$ respect to $q$ equal to 0. The result is given as follows.

Result 5.3. For given $\beta$, the optimal production run size is

$$q = \sqrt{\frac{2rDS}{h_i (\frac{rP}{D} - 1)[Db+p(\alpha+1)c_{i-1}\beta]}}.$$  

(5.8)
Case 3.2: Excess Raw Material (Disposed)

In this case, \( \beta \) is larger than \( \beta_u \), and the excess material is disposed of. If the supply rate is \( \beta_p \), then the total cost function is:

\[
TC_{32}(\beta_u, q) = \alpha \beta_u c_i + \frac{Db}{p} + \frac{DS}{pq} + h_i \left( \frac{\alpha \beta_u c_i}{D} + \frac{b q}{p} \right) + h_i \frac{\beta_u q}{2r_D} (r_D - D) + (\beta_r - \beta_u)(c_i + c_d). \tag{5.9}
\]

Note that \((\beta_r - \beta_u)(c_i + c_d) = DC(\beta_u, q)\) is the disposal cost per unit time. It is assumed that excess raw material is disposed when it is received, so that no holding cost is associated with the excess raw material. Therefore, the process mean \( \mu \) is \( \frac{p \beta_u}{D} \).

Similar to eq (5.7), eq (5.9) can be verified to be a convex function of \( q \) when \( \beta_u \) (or \( \mu \)) is fixed. Consequently, the minimum point of the total cost for given \( \beta_u \) (or \( \mu \)) can be found by finding the solution to \( \frac{\partial TC_{32}(\beta_u, q)}{\partial q} = 0 \). The following result can be obtained.

**Result 5.4.** For given \( \beta_u \), the optimal production run size is

\[
q = \frac{2r_D S}{h_i (\frac{D p}{D} - 1)[D b + p(\alpha + 1)c_i \beta_u]}. \tag{5.10}
\]

Note that using result 5.3, \( TC_{31}(\beta, q) \) becomes a function of \( \beta \). As a result, a search procedure can be used to find the solution \( \beta^* \) that minimizes \( TC_{31}(\beta, q) \) defined in (5.7). However, for unit cost \( c_o \), the solution may not be feasible because \( \beta^* \) may not be in the range \([\beta_r, \beta_{r+1})\). If \( \beta^* \) is in \([\beta_r, \beta_{r+1})\), we say \( \beta^* \) is admissible.

**Result 5.5.** If \( \beta^* \), which minimizes \( TC_{31}(\beta, q) \), is admissible for given unit cost, \( c_o \), then this solution has a lower total cost than all the solutions associated with the higher unit costs.
Proof. We let $TC_{31}(\beta, q|c_j)$ and $TC_{32}(\beta, q|c_i)$ be the total costs when $c_i$ is used in evaluating total cost functions, eqs. (5.7) and (5.9), respectively. For $c_j > c_i$, we know that $TC_{31}(\beta, q|c_j) > TC_{31}(\beta, q|c_i)$ and $TC_{32}(\beta, q|c_j) > TC_{31}(\beta, q|c_i)$. Let $q^*$ be the production run size corresponding to $P^*$. Since $P^*$ is admissible for $c_i$, $TC_{31}(P^*, q^*|c_i) < TC_{31}(P, q|c_j)$ for all $P \leq P^*$ and $q \leq q^*$. On the other hand, for any $\beta < \beta^*$, $TC_{32}(\beta, q|c_j) > TC_{31}(\beta, q|c_i)$ and $TC_{32}(\beta, q|c_j) > TC_{31}(\beta, q|c_i)$. As a result, $TC_{31}(\beta^*, q^*|c_i) < TC_{31}(\beta, q|c_j)$ and $TC_{31}(\beta^*, q^*|c_i) < TC_{32}(\beta, q|c_j)$.

Q.E.D.

5.3.2 Solution Algorithm

The analytical results presented in the last section provide the basis for developing efficient solution procedures to find the optimal solutions for the three cases. The solution procedures for the three cases are given separately as follows.

Case 1: No Discounts

Because the raw material supply rate is a function of the process mean and result 5.1 gives the optimal production run size corresponding to a given process mean, the total cost becomes a function of the process mean. In this study, the golden-section search is used to locate the optimal process mean. As discussed in Chapter 3, the search range of $\mu$ is $[L, L+4\sigma]$ and the search is terminated when the width of the interval of uncertainty is less than or equal to 0.0001.

Case 2: Incremental Quantity Discounts

Similar to case 1, the optimal production size for given $\beta$ can be obtained by using result 5.2, and the process mean is given by $(p\beta/D)$. The total cost, a function of $\beta$, is evaluated by eq. (5.5). The golden-section search method is used to search for the optimal process mean.
Case 3: All-Units Quantity Discounts

Similar to the procedure proposed in Chapter 4, the solution procedure is a search method based on the unit material cost \( c_i \). Specifically, it starts with the lowest \( c_k \) and case 3.1. The relationship between the material supply rate and the material unit cost is ignored first. The solution that minimizes \( TC_{31}(\beta,q|c_i) \) is feasible and admissible if its resulting \( \beta \) is inside its cost (price) range. The procedure continues until the first feasible \( \beta \) is found. Let \( c' \) be the unit price associated with the first feasible solution (i.e., \( \beta^* \) is admissible). If \( c' \) is the lowest unit cost, then the solution is optimal. Otherwise, based on result 5.5, we can exclude all the solutions associated with the unit cost higher than \( c' \). Therefore, we examine case 3.2 for all the unit costs lower than \( c' \), and compare the results with the first feasible solution (for case 3.1). The one with the lowest total cost is the optimal solution. The procedure is summarized by the following two steps.

Step 1. Starting with the lowest price, find the value of \( \beta^* \) that minimizes \( TC_{31}(\beta,q|c_i) \). \( \beta^* \) is admissible if \( \beta^* \in [\beta_i,\beta_{i+1}) \). This process continues until an admissible \( \beta^* \) is found.

Step 2. If \( \beta^* \) corresponding to the lowest unit material cost is admissible, then the solution is optimal. Otherwise, find the solutions for case 3.2 associated with the unit costs lower than \( c_0 \). Comparing the solution obtained in step 1 and those in this step for case 3.2, the one with the lowest total cost is the optimal solution.

Note that the golden-section search method is used to search for \( \beta^* \) in each problems in the solution procedure.
5.4 An Example

In this section, an example is used to illustrate the solution procedures given in the last section. The following model parameters are the same as those used in chapters 3 and 4: \( D = 5,000 \) items, \( r = 7,500 \) items, \( \sigma = 0.8 \) mg, \( L = 1.1 \) mg, \( S = $150 \), \( b = $0.16 \) per item, \( \alpha = 4 \) and \( h_1 = $0.03 \) per dollar per unit time. The material costs under the incremental quantity discounts are as follows:

\[
\begin{align*}
&c = \begin{cases} 
0.1 & \text{for each of the first 8,000 mgs} \\
0.099 & \text{for each of the next 3,000 mgs} \\
0.098 & \text{for each mg in excess of 11,000.}
\end{cases}
\end{align*}
\]

The all-units quantity discounts are as follows:

<table>
<thead>
<tr>
<th>Order Quantity</th>
<th>Price per mg</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 7,999</td>
<td>$0.100</td>
</tr>
<tr>
<td>8,000 - 10,999</td>
<td>$0.099</td>
</tr>
<tr>
<td>\geq 11,000</td>
<td>$0.098</td>
</tr>
</tbody>
</table>

The raw material disposal cost to the producer is \( c_d = -$0.045/\text{mg} \).

A FORTRAN program has been written for implementing the solution procedures given in the last section. The results are given as follows.

**Case 1: No Discounts**

The optimal process mean is found to be 1.659 mgs. The corresponding raw material supply rate and the production run size are 10,948.32 mgs and 23,391 items, respectively. The total cost is $5,519.25. The process conforming rate is 75.76%.
Case 2: Incremental Quantity Discounts

The optimal process mean is found to be 1.663 mgs and the corresponding process conforming rate is 75.91%. The raw material supply rate is $3 = 10,952 mgs and its corresponding raw material purchasing cost is $C(3) = 1,092.3$. The optimal production run size is $q = 23,391$, and the total cost is $5,510$.

Case 3: All-Units Quantity Discounts

The optimal raw material supply rate and use rate found are 11,000 and 10,986 mgs, respectively. Based on the raw material use rate, the optimal process mean is 1.696 mgs. The following two steps were used to find the optimal supply rate and production run size associated with the optimal process mean:

Step 1. When $c_i = 0.099$, the optimal $3 = 3_u = 10,952.51 mgs$, which is within the range $[8,000, 11,000)$. $TC_{31} = 5,475.09$, and $c' = 0.099$.

Step 2. $c_i = 0.098$ is the only unit cost that is smaller than $c'$. When $c_i = 0.098$, the lowest total cost $TC_{32} = 5,432.56$, and its corresponding $3_u = 10,986.03 mgs$, which is less than $3 = 11,000$. Comparing this result with the result in step 1, we found the lowest total cost is $5,432.56$.

Therefore, the optimal solution is to use the supply rate 11,000 mgs and dispose of 14 mgs. The total disposal cost per unit time is $0.742$. The resulting process mean is 1.696 mgs, and the process conforming rate is 77.17%. The optimal production run size is 21,914.
5.5 Sensitivity Analysis

In this section, a sensitivity analysis is performed to study the effects of the model parameters on the optimal solutions of all three cases: 1. no discounts, 2. incremental quantity discounts, and 3. all-units quantity discounts cases. The model parameters included in the study are the demand rate $D$, the production rate $r$, the process standard deviation $\sigma$, the value-added factor $\alpha$, and the production setup cost $S$. For the all-units quantity discounts policy, the unit disposal cost, $c_d$, is also studied. The sensitivity analyses are based on the example given in the last section.

Similar to what were used in Chapter 4, per-unit costs are used to report the results in this section. They are per-item finished product related cost, $PFPC = FPC(\beta,q)/D$; per-item material handling cost, $PMC = MC(\beta,q)/D$; per-item total cost, $PTC = TC(\beta,q)/D$; and per-item disposal cost, $PDC = DC(\beta,q)/D$.

The computer programs used in the study are written in FORTRAN and run on an IBM3090-600E. The running time for solving each of the problems used in the study was just a fraction of a second.

5.5.1 Effect of Demand Rate

To study the effects of demand rate, we use the results for selected values of $D$ ranging from 1,500 to 7,000 per unit time. As shown in Table 5.1, when $D$ increases, in cases 1 and 2, the process mean decreases and the production run size increases until $D$ reaches 5,500. When $D$ is greater than 5,500, the process mean has to be set at a level so that the yield rate is not smaller than $D$. As a result, the process mean increases as $D$ increases. Furthermore, because of the low inventory accumulation rate, the production run becomes very long and, thus, the production run size becomes very large. In case 3, the process mean and production run size generally follow the same patterns. Different results are observed, however, when $D$ is 3,500 and 5,000. Under
Table 5.1: Effect of Demand Rate for Selected Values of D

<table>
<thead>
<tr>
<th>D</th>
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<th></th>
<th></th>
<th>CASE 2</th>
<th></th>
<th></th>
<th>CASE 3</th>
<th></th>
<th></th>
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</table>
these two demand rates, the process mean in case 3 actually increases. The reason is that both a larger demand and a higher process mean push the material use rate higher. When the material use rate is high enough, it is attractive to use the material supply rate at the next discount-applicable level. Furthermore, the production run size is also reduced because of a higher conforming rate resulting from a higher process mean. Consequently, the per-item production cost becomes lower because of the lower material unit cost and higher conforming rate. The cost saving is larger than the cost of disposing of excess material and holding more raw material.

The values of the process mean, production run size, and material supply rate are the same in the three cases when D is below 3,000. The reason for this is that, when D is below 3,000, the optimal raw material supply rates in all the cases are less than 8,000. The optimal solutions of the three cases are actually the same, therefore, because the material unit costs are the same. When D is between 3,500 and 5,000, the values of μ, q, and β in case 3 are the highest and those in case 1 are the lowest. The differences, however, are small. The reason for this is that the material unit cost in case 3 is the lowest and that in case 1 is the highest. When D is greater than or equal to 5,500, the process means in all the cases have to be maintained at a certain level to guarantee the satisfaction of demand. Also, the material use rate is dependent solely on the process mean. As a result, the values of μ and βu in all the cases are very close to each other. However, the difference among the material unit costs of all cases will affect the production run sizes in all the cases. In addition, because of the low accumulation rate of finished products under the situation when D is greater than or equal to 5,500, the production run size becomes very sensitive and large, corresponding to the change of D in all cases.
5.5.2 Effect of Production Rate

Quite different from what exists in cases 1 and 2, a distinct pattern can be seen in case 3 in Table 5.2, which gives the results under selected values of \( r \) from 6,000 to 10,000.

When the production capacity is less than 7,000 items per unit time, the values of the process mean in all three cases have to be set just high enough, as do the yield rates, to guarantee the satisfaction of demand. As a result, the production run size is very large and is sensitive to the change of \( r \). On the other hand, the values of the process mean and actual material use rate in all three cases are very close to each other. When \( r \) is larger than 7,000, in cases 1 and 2, the values of \( \mu \) and \( \beta \) increase and \( q \) decreases as \( r \) increases. The reason is that, when \( r \) increases, the process mean also becomes larger, which results in a high production yield rate. In order to avoid carrying a very high finished product inventory, the production run size, therefore, becomes smaller to reduce the cost of holding the finished product inventory.

Furthermore, the process mean and the material supply rate in case 3 remain stable when \( r \) is greater than 7,500. This is because, in case 3, the saving from the lower material unit cost at \( \beta = 11,000 \) is larger than the increase in the production and holding costs incurred because of a larger process mean. At the same time, the production run size, which is smaller than that in cases 1 and 2, decreases as the inventory accumulation rate (\( \lambda-D \)) increases. The reason for the lower production run size is that the higher production mean increases the per-item production cost and the holding cost of the finished product. The production run size is thus set lower to decrease the inventory level.
Table 5.2: Effect of Production Rate for Selected Values of r

<table>
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<th>r</th>
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<th></th>
<th></th>
<th></th>
<th>CASE 2</th>
<th></th>
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<th></th>
<th>CASE 3</th>
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<tbody>
<tr>
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</tbody>
</table>
5.5.3 Effect of Process Standard Deviation

As discussed, a smaller process variance implies a better performance of the process and a smaller total cost of operating the process. Table 5.3 shows the results associated with selected values of \( \sigma \) values.

The process mean, production run size, material supply rate, and per-item total cost are all identical for the three cases when \( \sigma \) is below 0.2. On the other hand, the process mean in case 3 is set higher when \( \sigma \) is 0.7 and 0.8. This is because the material supply rate is set at 11,000. In order to increase use of the raw material and to reduce the disposal cost, the process mean has to be set higher. At the same time, the production run size is lowered to reduce the holding cost, which is increased by the larger process mean. Compared to that in cases 1 and 2, the lower production run size in case 3, when \( \sigma \) is 0.7 and 0.8, is different from the values of production run sizes under different \( \sigma \) values; that is, when \( \sigma \) is between 0.2 and 0.7, the production run size in case 3 is the highest among the three cases. When \( \sigma \) is larger than 0.8, however, in order to avoid the higher per-item production cost and holding cost caused by a larger process mean, the process mean is set at a level where process yield rate just satisfies the demand. The material supply rate, however, continues to increase as \( \sigma \) increases.

5.5.4 Effect of Value-Added Factor

As shown in Table 5.4, as \( \alpha \) increases, the process mean first decreases and then remains stable while production run size increases and then becomes very sensitive to the change in \( \alpha \). The main reason for this is that a larger \( \alpha \) implies a higher cost of producing an item and also a higher holding cost. To reduce these costs, a lower process mean is used. On the other hand, the production run size increases to reduce the number of production setups and, thus, the setup cost.
Table 5.3: Effect of Process Variation for Selected Values of $\sigma$

| $\sigma$ | CASE 1 | | CASE 2 | | CASE 3 | |  
|---|---|---|---|---|---|---|---|
| | $\mu$ | $q$ | $\beta$ | $\mu$ | $q$ | $\beta$ |  
| .01 | 1.129 | 14437 | 5655 | 1.129 | 14437 | 5655 | | 1.129 | 14437 | 5655 | 5655 |  
| .10 | 1.295 | 14183 | 6645 | 1.295 | 14183 | 6645 | | 1.295 | 14182 | 6645 | 6645 |  
| .20 | 1.424 | 14296 | 7515 | 1.424 | 14296 | 7515 | | 1.427 | 14291 | 7519 | 8000 |  
| .30 | 1.521 | 14655 | 8268 | 1.521 | 14654 | 8269 | | 1.521 | 14703 | 8269 | 8269 |  
| .40 | 1.594 | 15238 | 8939 | 1.595 | 15224 | 8940 | | 1.594 | 15295 | 8940 | 8940 |  
| .50 | 1.648 | 16076 | 9543 | 1.648 | 16077 | 9543 | | 1.648 | 16148 | 9543 | 9543 |  
| .60 | 1.678 | 17394 | 10080 | 1.679 | 17382 | 10081 | | 1.679 | 17426 | 10082 | 10082 |  
| .70 | 1.685 | 19452 | 10553 | 1.686 | 19454 | 10554 | | 1.708 | 18870 | 10577 | 11000 |  
| .80 | 1.660 | 23534 | 10949 | 1.662 | 23398 | 10952 | | 1.692 | 22029 | 10983 | 11000 |  
| .90 | 1.488 | 2198609 | 11157 | 1.488 | 2201223 | 11157 | | 1.488 | 2216930 | 11157 | 11157 |  
| 1.00 | 1.531 | 1843738 | 11481 | 1.531 | 1846310 | 11481 | | 1.531 | 1859180 | 11481 | 11481 |  
| 1.10 | 1.574 | 1692283 | 11804 | 1.574 | 1694976 | 11804 | | 1.574 | 1706524 | 11804 | 11804 |  


### Table 5.4: Effect of Value-Added Factor for Selected Values of $\alpha$

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<th>CASE 2</th>
<th></th>
<th>CASE 3</th>
<th></th>
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</tbody>
</table>
From Table 5.4, we can see that, when $\alpha$ is 4 and 5, the process mean in case 3 is higher than that in cases 1 and 2. The reason for the increased process mean is to increase the supply rate to 11,000 for the discount. The saving from the lower material unit cost is more than the cost of disposing of excess material and the cost incurred because of higher process mean. As a result, the production run size decreases under these two $\alpha$ values, when compared to that in cases 1 and 2. The reason for this is that the increase in the holding cost caused by a larger process mean is more than the saving from setup cost by increasing production run size. The production run size is decreased, therefore, to reduce the inventory level of the finished product. The disposal of excess material still exists in case 3 when $\alpha$ is larger than 5.

In addition, the process mean and material use rate are the same in all three cases. The difference in the material unit cost, however, will affect the production run size in all three cases. The values of production run size in all cases, therefore, are quite different from each other. The saving of the per-item total cost under the all-units quantity discounts becomes larger, however, as $\alpha$ increases when compared to that of the cases without discount and with the incremental quantity discount.

5.5.5 Effect of Production Setup Cost

The effect of the production setup cost in case 3, shown in Table 5.5, is more similar to that in case 1, except that the process mean is higher and the production run size is lower when $S$ is below 250. The increase in the production run size when $S$ increases is expected. But, in case 3, the lower production run size, which implies more production setups (cost), is due mainly to the lower material unit cost. On one hand, when the process mean is set high, the production run size has to be reduced to ease the increase in the inventory accumulation rate ($\lambda-D$) and thus the inventory
Table 5.5: Effect of Production Setup Cost for Selected Values of S

<table>
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<th>S</th>
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holding cost; on the other hand, a lower material unit cost can be secured by setting the supply rate at 11,000 and by using all the material or disposing of some of it.

In all three cases, when S is large, the production run size increases to reduce the number of setups. In order to reduce the increase in the holding cost resulting from higher inventory levels, the process mean will decrease. Under the limitation of demand satisfaction, however, the process mean can be reduced only to a certain level. As a result, the values of the process mean and material use rate in all three cases when S is larger than 250 are the same. In case 3, when S is 250, the process mean is much higher than that in cases 1 and 2. This implies that the saving from further decreasing the process mean is still greater than the excess setup cost caused by a lower production run size. The main reason for this is that the material unit cost in case 3 is lower than that in cases 1 and 2, even though extra disposal costs are caused by the difference between $\beta$ and $\beta_u$.

5.5.6 Effect of Unit Disposal Cost

Table 5.6 gives the effects of some selected $c_d$ values ranging from $-0.08$ to $0.1$. As expected, when $c_d$ increases, the material use rate will increase in order to reduce possible excess raw material. The table shows that, when $c_d$ is higher than $-0.04$, both the material use rate ($\beta_u$) and the supply rate ($\beta$) are 11,000. This remains the case when we set $c_d$ equal to $1,000$. This implies that, as long as $c_d$ is higher than $-0.04$, it will be cheaper without disposing of any material. The process mean and the production run size will thus remain balanced to keep both $\beta_u$ and $\beta$ equal to 11,000, the highest price breakpoint.
Table 5.6: Effect of Unit Disposal Cost for Selected Values of $c_d$

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<th>$q$</th>
<th>$\beta_u$</th>
<th>$\beta$</th>
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Chapter 6

CONCLUSIONS AND FUTURE RESEARCH

6.1 Conclusions

In this dissertation, two issues are jointly considered. The first issue is selection of the mean (target value) for a production process, and the second is production setup and material ordering policies in a two-echelon inventory system.

A typical scenario of the first issue is as follows. The performance of the product of interest has a lower specification limit, and the items that do not conform to the specification limit are scrapped with no salvage value. The raw material requirement for producing the product is an increasing function of the performance variable. Because of uncontrollable variations in the production process, the amount of raw material used to produce an item is a random variable with a known variance, but its mean depends on a process setting. A higher process mean may be used to increase the process conforming rate. It will increase the cost, however, of producing the product. Also, the production cost of an item is usually assumed to be a linear function of the amount of raw material used in producing the item. Consequently, a process mean should be selected on the basis of a balance between production cost and economical consequences associated with conforming items and nonconforming items.

The process mean issue is especially important to the producer when material-related costs are a significant portion of production cost. Because the process mean determines the process conforming and yield rate, it affects other important production decisions; in particular, the production setup policy. These production decisions also affect the raw material requirement and, thus, procurement policy when the raw
material is supplied by outside vendors. Consequently, process mean, production, and raw material procurement policies should be jointly determined to minimize the total related costs.

The determination of production run size and raw material ordering quantity is a typical two-echelon inventory problem. This dissertation is the first research, however, that combines these two problems, even though each has been studied for a few decades. Furthermore, in practice, quantity discounts may be available in raw material purchasing. Because of the interaction between the process mean determination and the raw material ordering policies, quantity discounts will affect all the related decisions.

This dissertation consists of three parts. In the first part, a two-echelon model is formulated to incorporate the issues associated with production setup and raw material procurement into the classical process mean problem for a single-product production process. A mathematical model is formulated, and analytical properties are derived. An efficient solution procedure is developed to find the optimal process mean, production run size, and raw material ordering policy. A sensitivity analysis is also performed to study the effects of model parameters on the optimal solutions.

In the second part, quantity discounts in raw material purchasing are incorporated into the model. The quantity discounts policy under study is known as all-unit quantity discounts. It is assumed that the producer has the option to order an amount that is more than what is used in production in order to take advantage of the quantity discounts and to dispose of the excess raw material. A mathematical model is formulated, and analytical properties are derived. An efficient solution algorithm is presented to obtain the optimal solutions. Sensitivity analysis on the effects of model parameters is not only presented, but also compared to that in the previous model, which imposes no quantity discounts on raw material unit cost.
In the third part, we consider a special situation in which the supply rate of the raw material is finite and constant. In terms of quantity discounts in the raw material purchasing, three cases are considered: no discounts, incremental quantity discounts and all-unit quantity discounts. Different mathematical models are formulated and analytical properties are derived for all three cases. An efficient solution procedure is presented for each case. A sensitivity analysis is presented which compares the effects of model parameters on the optimal solutions among the three cases.

6.2 Future Research

The model structures presented in this dissertation provide a useful framework for future research on several issues related to this classical problem in quality control. In particular, the following few extensions are possible. The first centers on the time-varying demand pattern. When the demand rate varies with time, we can no longer assume that the best strategy is always to use the same replenishment quantity. The second possible extension is to modify the models to consider the situation in which an upper specification of the performance variable has to be determined and nonconforming product is reprocessed. The third extension is to consider perishable raw material and finished product. The issue becomes very important when the deterioration speeds of the raw material and the finished product are different. The fourth extension is to incorporate production process deterioration into the models. Process mean, instead of remaining stable as assumed in this dissertation, increases or decreases whenever the production process is operating; therefore, not only do process mean setting and production run size determination become more important, but also the material procurement policy becomes more important and unpredictable.
REFERENCES


VITA

Jinshyang Roan, the second son of Yun-Chung Roan and Shan H. Roan, was born on September 29, 1963. He was raised in Guu-Keng, Yun-Lin, Taiwan. He received his bachelor in Industrial Management Science in 1986, from National Cheng Kung University, Tainan, Taiwan. Upon Graduation, he served in the Chinese Army for two years, and then worked for Evergreen Marine Co., Formosa Plastic Co., and Taiwan Randa Co. during the following year. He came to the United States and began his graduate study at Louisiana State University in Quantitative Business Analysis on August 4, 1989. He received his master degree in December 1991 and is going to earn his Doctorate on August 4, 1994, five years after his arrival.
DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Jinshyang Roan

Major Field: Business Administration (Quantitative Business Analysis)

Title of Dissertation: Determining Process Target Value and Ordering Policies in Two-Echelon Hierarchy Production System

Approved:

[Signatures]

Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

[Signatures]

Date of Examination:

June 10, 1994