

11-19-2015

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Recommended Citation

Datta, N., & Wilde, M. (2015). Quantum Markov chains, sufficiency of quantum channels, and Rényi information measures. *Journal of Physics A: Mathematical and Theoretical*, 48 (50) <https://doi.org/10.1088/1751-8113/48/50/505301>

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Quantum Markov chains, sufficiency of quantum channels, and Rényi information measures

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October 6, 2015

Abstract

A short quantum Markov chain is a tripartite state ρ_{ABC} such that system A can be recovered perfectly by acting on system C of the reduced state ρ_{BC} . Such states have conditional mutual information $I(A; B|C)$ equal to zero and are the only states with this property. A quantum channel \mathcal{N} is sufficient for two states ρ and σ if there exists a recovery channel using which one can perfectly recover ρ from $\mathcal{N}(\rho)$ and σ from $\mathcal{N}(\sigma)$. The relative entropy difference $D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$ is equal to zero if and only if \mathcal{N} is sufficient for ρ and σ . In this paper, we show that these properties extend to Rényi generalizations of these information measures which were proposed in [Berta *et al.*, J. Math. Phys. **56**, 022205, (2015) and Seshadreesan *et al.*, J. Phys. A **48**, 395303, 2015], thus providing an alternate characterization of short quantum Markov chains and sufficient quantum channels. These results give further support to these quantities as being legitimate Rényi generalizations of the conditional mutual information and the relative entropy difference. Along the way, we solve some open questions of Ruskai and Zhang, regarding the trace of particular matrices that arise in the study of monotonicity of relative entropy under quantum operations and strong subadditivity of the von Neumann entropy.

1 Introduction

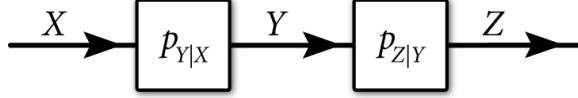
Markov chains and sufficient statistics are two fundamental notions in probability [Fel97, Nor97] and statistics [Ric94]. Three random variables X , Y , and Z constitute a three-step Markov chain (denoted as $X - Y - Z$) if X and Z are independent when conditioned on Y . In particular, if $p_{XYZ}(x, y, z)$ is their joint probability distribution, then

$$\begin{aligned} p_{XYZ}(x, y, z) &= p_X(x) p_{Y|X}(y|x) p_{Z|Y}(z|y) \\ &= p_{X|Y}(x|y) p_{Z|Y}(z|y) p_Y(y). \end{aligned} \tag{1.1}$$

In the information-theoretic framework, such a Markov chain corresponds to a *recoverability condition* in the following sense. Consider X , Y , and Z to be the inputs and outputs of two channels (i.e., stochastic maps) $p_{Y|X}$ and $p_{Z|Y}$, as in the figure below. If $X - Y - Z$ is a three-step Markov

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chain, then the input X , if lost, can be recovered from Y alone (without any knowledge of Z) by the action of the stochastic map $p_{X|Y}$, as is evident from (1.1).

It is well known that such a Markov chain $X - Y - Z$ can be characterized by an information measure [CT91], namely, the *conditional mutual information* $I(X; Z|Y)$. For any three random variables X , Y and Z , it is defined as

$$I(X; Z|Y) \equiv H(XY) + H(ZY) - H(Y) - H(XYZ), \quad (1.2)$$

where $H(W) \equiv -\sum_w p_W(w) \log p_W(w)$ is the Shannon entropy of a random variable $W \sim p_W(w)$. It is non-negative and equal to zero if and only if $X - Y - Z$ is a Markov chain.

In statistics, for a given sample of independent and identically distributed data conditioned on an unknown parameter θ , a *sufficient statistic* is a function of the sample whose value contains all the information needed to compute any estimate of the parameter. One can extend this notion to that of a sufficient channel (or sufficient stochastic map), as discussed in [Pet86b, Pet88, MP04, Mos05]. A channel $T \equiv T_{Y|X}$ is sufficient for two input distributions p_X and q_X if there exists another channel (a *recovery channel*) such that both these inputs can be recovered perfectly by sending the outputs of the channel $T_{Y|X}$ corresponding to them through it. This notion of channel sufficiency is likewise characterized by an information measure, namely, the relative entropy difference

$$D(p_X \| q_X) - D(T(p_X) \| T(q_X)), \quad (1.3)$$

where $T(p_X)$ and $T(q_X)$ are the distributions obtained after the action of the channel, and $D(p_X \| q_X)$ denotes the relative entropy (or Kullback-Leibler divergence) [CT91] between p_X and q_X . It is defined as

$$D(p_X \| q_X) \equiv \sum_x p_X(x) \log \left(\frac{p_X(x)}{q_X(x)} \right), \quad (1.4)$$

if for all x , $q_X(x) \neq 0$ if $p_X(x) \neq 0$. It is equal to $+\infty$ otherwise. The notion of recoverability provides a connection between the notions of Markov chains and sufficient channels.

The generalization of the above ideas to quantum information theory has been a topic of continuing and increasing interest (see, e.g., [HMPB11, BSW15, FR15] and references therein). In the quantum setting, density operators play a role analogous to that of probability distributions in the classical case, and in [HJPW04], a quantum Markov chain $A - C - B$ was defined to be a tripartite density operator ρ_{ABC} with conditional (quantum) mutual information $I(A; B|C)_\rho$ equal to zero, where

$$I(A; B|C)_\rho \equiv H(AC)_\rho + H(BC)_\rho - H(C)_\rho - H(ABC)_\rho, \quad (1.5)$$

and $H(F)_\sigma \equiv -\text{Tr}\{\sigma_F \log \sigma_F\}$ denotes the von Neumann entropy of a density operator σ_F . (We take the convention $A - C - B$ for a quantum Markov chain because we are often interested in quantum correlations between Alice (A) and Bob (B), which are potentially mediated by a third party, here labeled by C .) Strong subadditivity of the von Neumann entropy guarantees that the conditional

mutual information $I(A; B|C)_\rho$ is non-negative for all density operators [LR73a, LR73b], and it is equal to zero if and only if there is a decomposition of \mathcal{H}_C as

$$\mathcal{H}_C = \bigoplus_j \mathcal{H}_{C_{L_j}} \otimes \mathcal{H}_{C_{R_j}} \quad (1.6)$$

such that

$$\rho_{ABC} = \bigoplus_j q(j) \rho_{AC_{L_j}}^j \otimes \rho_{C_{R_j}B}^j, \quad (1.7)$$

for a probability distribution $\{q(j)\}$ and sets of density operators $\{\rho_{AC_{L_j}}^j, \rho_{C_{R_j}B}^j\}$ [HJPW04]. Following [HJPW04], we call such states *short quantum Markov chains* $A - C - B$. In analogy with the classical case, $I(A; B|C)_\rho = 0$ is equivalent to the full state ρ_{ABC} being recoverable after the loss of system A by the action of a quantum recovery channel $\mathcal{R}_{C \rightarrow AC}^P$ on system C alone:

$$I(A; B|C)_\rho = 0 \Leftrightarrow \rho_{ABC} = \mathcal{R}_{C \rightarrow AC}^P(\rho_{BC}), \quad (1.8)$$

where

$$\mathcal{R}_{C \rightarrow AC}^P(\cdot) \equiv \rho_{AC}^{\frac{1}{2}} \rho_C^{-\frac{1}{2}}(\cdot) \rho_C^{-\frac{1}{2}} \rho_{AC}^{\frac{1}{2}} \quad (1.9)$$

is a special case of the so-called *Petz recovery channel* [Pet86b, Pet88]. Note that this channel acts as the identity on system B and the superscript P refers to Petz.

As a generalization of the classical notion of a sufficient channel, several works have discussed and studied the notion of a sufficient quantum channel [Pet86b, Pet88, MP04, Mos05]. A definition of this concept is as follows:

Definition 1 (Sufficiency of a quantum channel) *Let ρ and σ be density operators acting on a Hilbert space \mathcal{H} , and let \mathcal{N} be a quantum channel taking these density operators to density operators acting on a Hilbert space \mathcal{K} . Then the quantum channel \mathcal{N} is sufficient for them if one can perfectly recover ρ from $\mathcal{N}(\rho)$ and σ from $\mathcal{N}(\sigma)$ by the action of a quantum recovery channel \mathcal{R} , i.e., if there exists an \mathcal{R} such that*

$$\rho = (\mathcal{R} \circ \mathcal{N})(\rho), \quad \sigma = (\mathcal{R} \circ \mathcal{N})(\sigma). \quad (1.10)$$

We define sufficiency in the same way even if ρ and σ are arbitrary positive semi-definite operators.

If (1.10) is true for some recovery channel \mathcal{R} , it is known that the following Petz recovery channel $\mathcal{R}_{\sigma, \mathcal{N}}^P$ satisfies (1.10) as well [Pet88]:

$$\mathcal{R}_{\sigma, \mathcal{N}}^P(\omega) \equiv \sigma^{\frac{1}{2}} \mathcal{N}^\dagger \left([\mathcal{N}(\sigma)]^{-\frac{1}{2}} \omega [\mathcal{N}(\sigma)]^{-\frac{1}{2}} \right) \sigma^{\frac{1}{2}}. \quad (1.11)$$

(Note that the Petz recovery channel in (1.9) is a special case of (1.11) with $\sigma = \rho_{AC}$ and $\mathcal{N} = \text{Tr}_A$.) As a generalization of the classical case, the sufficiency of a quantum channel is characterized by the following information measure, the relative entropy difference

$$D(\rho \parallel \sigma) - D(\mathcal{N}(\rho) \parallel \mathcal{N}(\sigma)), \quad (1.12)$$

where $D(\rho \parallel \sigma)$ denotes the *quantum relative entropy* [Ume62]. It is defined as

$$D(\rho \parallel \sigma) \equiv \text{Tr} \{ \rho [\log \rho - \log \sigma] \}, \quad (1.13)$$

whenever the support of ρ is contained in the support of σ and it is equal to $+\infty$ otherwise. The relative entropy difference in (1.12) is non-negative due to the monotonicity of relative entropy under quantum channels [Lin75, Uhl77], and it is equal to zero if and only if \mathcal{N} is sufficient for ρ and σ , i.e., if the Petz recovery channel $\mathcal{R}_{\sigma, \mathcal{N}}^P$ satisfies (1.10) [Pet86b, Pet88]. Further, Mosonyi and Petz have shown that the relative entropy difference in (1.12) is equal to zero if and only if ρ , σ , and \mathcal{N} have the explicit form recalled below in Theorem 7 of Section 3 [MP04, Mos05], which generalizes the result stated in (1.6)-(1.7).

Due to its operational interpretation in the quantum Stein's lemma [HP91], the quantum relative entropy plays a central role in quantum information theory. In particular, fundamental limits on the performance of information-processing tasks in the so-called ‘‘asymptotic, memoryless (or i.i.d.) setting’’ are given in terms of quantities derived from the quantum relative entropy.

There are, however, other generalized relative entropies (or divergences) which are also of operational significance. Important among these are the *Rényi relative entropies* [Rén61, Pet86a] and the more recently defined *sandwiched Rényi relative entropies* [MLDS⁺13, WWY14]. For $\alpha \in (0, 1)$, the Rényi relative entropies arise in the quantum Chernoff bound [ACMnT⁺07], which characterizes the minimum probability of error in discriminating two different quantum states in the setting of asymptotically many copies. Moreover, in analogy with the operational interpretation of their classical counterparts, the Rényi relative entropies can be viewed as generalized cutoff rates in quantum binary state discrimination [MH11]. The sandwiched Rényi relative entropies find application in the strong converse domain of a number of settings dealing with hypothesis testing or channel capacity [WWY14, MO15, GW15, TWW14, CMW14, HT14].

This motivates the introduction of Rényi generalizations of the conditional mutual information. Two of these generalizations, defined in [BSW15], are given as follows:

$$I_\alpha(A; B|C)_\rho \equiv \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho_{ABC}^\alpha \rho_{AC}^{\frac{1-\alpha}{2}} \rho_C^{\frac{\alpha-1}{2}} \rho_{BC}^{1-\alpha} \rho_C^{\frac{\alpha-1}{2}} \rho_{AC}^{\frac{1-\alpha}{2}} \right\}, \quad (1.14)$$

$$\tilde{I}_\alpha(A; B|C)_\rho \equiv \frac{2\alpha}{\alpha - 1} \log \left\| \rho_{ABC}^{\frac{1}{2}} \rho_{AC}^{\frac{1-\alpha}{2\alpha}} \rho_C^{\frac{\alpha-1}{2\alpha}} \rho_{BC}^{\frac{1-\alpha}{2\alpha}} \right\|_{2\alpha}, \quad (1.15)$$

where $\alpha \in (0, 1) \cup (1, \infty)$ denotes the Rényi parameter. (Note that we use the notation $\|A\|_\alpha \equiv \text{Tr}\{(\sqrt{A^\dagger A})^\alpha\}^{1/\alpha}$ even for $\alpha \in (0, 1)$, when it is not a norm.) Both these quantities converge to (1.5) in the limit $\alpha \rightarrow 1$, they are non-negative, and obey several properties of the conditional mutual information defined in (1.5), as shown in [BSW15]. In [SBW15], the authors proposed some definitions for Rényi generalizations of a relative entropy difference, two of which are as follows:

$$\Delta_\alpha(\rho, \sigma, \mathcal{N}) \equiv \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho^\alpha \sigma^{\frac{1-\alpha}{2}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N}(\rho)^{1-\alpha} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right) \sigma^{\frac{1-\alpha}{2}} \right\}, \quad (1.16)$$

$$\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}) \equiv \frac{\alpha}{\alpha - 1} \log \left\| \rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha}{2\alpha}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \mathcal{N}(\rho)^{\frac{1-\alpha}{\alpha}} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \right) \sigma^{\frac{1-\alpha}{2\alpha}} \rho^{\frac{1}{2}} \right\|_\alpha. \quad (1.17)$$

The quantities above converge to (1.12) in the limit $\alpha \rightarrow 1$ [SBW15].

The quantities defined in (1.14)-(1.17) can be expressed in terms of Rényi relative entropies, namely the α -Rényi relative entropy and the α -sandwiched Rényi relative entropy defined in Section 3.1 [BSW15]. The corresponding expressions for the Rényi generalizations of the conditional

mutual information are

$$I_\alpha(A; B|C)_\rho = D_\alpha \left(\rho_{ABC} \left\| \left(\rho_{AC}^{\frac{1-\alpha}{2}} \rho_C^{\frac{\alpha-1}{2}} \rho_{BC}^{1-\alpha} \rho_C^{\frac{\alpha-1}{2}} \rho_{AC}^{\frac{1-\alpha}{2}} \right)^{\frac{1}{1-\alpha}} \right. \right), \quad (1.18)$$

$$\tilde{I}_\alpha(A; B|C)_\rho = \tilde{D}_\alpha \left(\rho_{ABC} \left\| \left(\rho_{AC}^{\frac{1-\alpha}{2\alpha}} \rho_C^{\frac{\alpha-1}{2\alpha}} \rho_{BC}^{\frac{1-\alpha}{\alpha}} \rho_C^{\frac{\alpha-1}{2\alpha}} \rho_{AC}^{\frac{1-\alpha}{2\alpha}} \right)^{\frac{\alpha}{1-\alpha}} \right. \right), \quad (1.19)$$

respectively, and those for the relative entropy difference are given in (3.60) and (3.61), respectively.

Before proceeding, we should note that the conditional mutual information is a special case of a relative entropy difference. Indeed, one can check that

$$I(A; B|C)_\rho = D(\rho||\sigma) - D(\mathcal{N}(\rho)||\mathcal{N}(\sigma)) \quad (1.20)$$

for the choices

$$\rho = \rho_{ABC}, \quad \sigma = \rho_{AC} \otimes I_B, \quad \mathcal{N} = \text{Tr}_A. \quad (1.21)$$

This reduction extends as well to the Rényi quantities for all $\alpha \in (0, 1) \cup (1, \infty)$:

$$I_\alpha(A; B|C)_\rho = \Delta_\alpha(\rho_{ABC}, \rho_{AC} \otimes I_B, \text{Tr}_A), \quad (1.22)$$

$$\tilde{I}_\alpha(A; B|C)_\rho = \tilde{\Delta}_\alpha(\rho_{ABC}, \rho_{AC} \otimes I_B, \text{Tr}_A), \quad (1.23)$$

as pointed out in [SBW15]. This realization is helpful in simplifying some of the arguments in this paper.

2 Summary of results

As highlighted in the Introduction, an important property of the conditional (quantum) mutual information of a tripartite quantum state is that it is always non-negative and vanishes if and only if the state is a short quantum Markov chain. The relative entropy difference of a pair of quantum states and a quantum channel is also non-negative and vanishes if and only if the channel is sufficient for the pair of states. Consequently, it is reasonable to require that Rényi generalizations of these information measures are also non-negative and vanish under the same necessary and sufficient conditions as mentioned above.

In this paper, we prove these properties for the quantities I_α and \tilde{I}_α , and the quantities Δ_α and $\tilde{\Delta}_\alpha$ defined in the Introduction. This contributes further evidence that $I_\alpha(A; B|C)_\rho$ and $\tilde{I}_\alpha(A; B|C)_\rho$ are legitimate Rényi generalizations of the conditional mutual information $I(A; B|C)_\rho$ and that $\Delta_\alpha(\rho, \sigma, \mathcal{N})$ and $\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N})$ are legitimate Rényi generalizations of the relative entropy difference $D(\rho||\sigma) - D(\mathcal{N}(\rho)||\mathcal{N}(\sigma))$. In particular, we prove the following:

1. $I_\alpha(A; B|C)_\rho = 0$ for some $\alpha \in (0, 1) \cup (1, 2)$ if and only if ρ_{ABC} is a short quantum Markov chain with a decomposition as in (1.7).
2. $\tilde{I}_\alpha(A; B|C)_\rho = 0$ for some $\alpha \in (1/2, 1) \cup (1, \infty)$ if and only if ρ_{ABC} is a short quantum Markov chain with a decomposition as in (1.7).
3. $\Delta_\alpha(\rho, \sigma, \mathcal{N})$ is non-negative for $\alpha \in (0, 1) \cup (1, 2)$ and $\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N})$ is non-negative for $\alpha \in (1/2, 1) \cup (1, \infty)$.

4. $\Delta_\alpha(\rho, \sigma, \mathcal{N}) = 0$ for some $\alpha \in (0, 1) \cup (1, 2)$ if and only if the quantum channel \mathcal{N} is sufficient for states ρ and σ , so that \mathcal{N} , ρ , and σ decompose as in (3.20)-(3.21).
5. $\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}) = 0$ for some $\alpha \in (1/2, 1) \cup (1, \infty)$ if and only if the quantum channel \mathcal{N} is sufficient for states ρ and σ , so that \mathcal{N} , ρ , and σ decompose as in (3.20)-(3.21).
6. Generalizations of the conditional mutual information and the relative entropy difference arising from the so-called min- and max-relative entropies [Dat09, DKF⁺12] (which play an important role in one-shot information theory) satisfy identical properties.

Along the way, we resolve some open questions stated in [Rus02, Zha14]. Let ρ_{ABC} be a positive definite density operator. We prove that

$$\mathrm{Tr} \left\{ \left(\rho_{AC}^{\frac{1-\alpha}{2}} \rho_C^{\frac{\alpha-1}{2}} \rho_{BC}^{1-\alpha} \rho_C^{\frac{\alpha-1}{2}} \rho_{AC}^{\frac{1-\alpha}{2}} \right)^{\frac{1}{1-\alpha}} \right\} \leq 1. \quad (2.1)$$

for all $\alpha \in (0, 1) \cup (1, 2)$, and

$$\mathrm{Tr} \left\{ \left(\rho_{AC}^{\frac{1-\alpha}{2\alpha}} \rho_C^{\frac{\alpha-1}{2\alpha}} \rho_{BC}^{\frac{1-\alpha}{\alpha}} \rho_C^{\frac{\alpha-1}{2\alpha}} \rho_{AC}^{\frac{1-\alpha}{2\alpha}} \right)^{\frac{\alpha}{1-\alpha}} \right\} \leq 1, \quad (2.2)$$

for all $\alpha \in (1/2, 1) \cup (1, \infty)$. Let ρ and σ be positive definite density operators and let \mathcal{N} be a strict completely positive trace preserving (CPTP) map (that is, a CPTP map such that $\mathcal{N}(X)$ is positive definite whenever X is positive definite). We prove that

$$\mathrm{Tr} \left\{ \left[\sigma^{\frac{1-\alpha}{2}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N}(\rho)^{1-\alpha} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right) \sigma^{\frac{1-\alpha}{2}} \right]^{\frac{1}{1-\alpha}} \right\} \leq 1, \quad (2.3)$$

for $\alpha \in (0, 1) \cup (1, 2)$, and

$$\mathrm{Tr} \left\{ \left[\sigma^{\frac{1-\alpha}{2\alpha}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \mathcal{N}(\rho)^{\frac{1-\alpha}{\alpha}} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \right) \sigma^{\frac{1-\alpha}{2\alpha}} \right]^{\frac{\alpha}{1-\alpha}} \right\} \leq 1, \quad (2.4)$$

for $\alpha \in (1/2, 1) \cup (1, \infty)$. By taking the limit $\alpha \rightarrow 1$, the inequalities in (2.3) and (2.4) imply that

$$\mathrm{Tr} \left\{ \exp \left\{ \log \sigma + \mathcal{N}^\dagger (\log \mathcal{N}(\rho) - \log \mathcal{N}(\sigma)) \right\} \right\} \leq 1. \quad (2.5)$$

The rest of the paper is devoted to establishing these claims. We begin by recalling some mathematical preliminaries and known results, and follow by establishing the latter claims first and then move on to the former ones.

3 Preliminaries

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators acting on a Hilbert space \mathcal{H} . We restrict to finite-dimensional Hilbert spaces throughout this paper. We denote the support of an operator A by $\mathrm{supp}(A)$. For a Hermitian operator A , by A^{-1} we mean the inverse restricted to $\mathrm{supp}(A)$, so that $AA^{-1} = A^{-1}A$ is the orthogonal projection onto $\mathrm{supp}(A)$. More generally, for a function f and Hermitian operator A with spectral decomposition $A = \sum_i \lambda_i |i\rangle\langle i|$, we define $f(A)$ to be

$\sum_{i:\lambda_i \neq 0} f(\lambda_i)|i\rangle\langle i|$. Let $\mathcal{B}(\mathcal{H})_+$ denote the subset of positive semidefinite operators, and let $\mathcal{B}(\mathcal{H})_{++}$ denote the subset of positive definite operators. We also write $X \geq 0$ if $X \in \mathcal{B}(\mathcal{H})_+$ and $X > 0$ if $X \in \mathcal{B}(\mathcal{H})_{++}$. An operator ρ is in the set $\mathcal{S}(\mathcal{H})$ of density operators (or states) if $\rho \in \mathcal{B}(\mathcal{H})_+$ and $\text{Tr}\{\rho\} = 1$, and an operator ρ is in the set $\mathcal{S}(\mathcal{H})_{++}$ of positive definite density operators if $\rho \in \mathcal{B}(\mathcal{H})_{++}$ and $\text{Tr}\{\rho\} = 1$. Throughout much of the paper, for technical convenience and simplicity, we consider states in $\mathcal{S}(\mathcal{H})_{++}$. For $\alpha \geq 1$, we define the α -norm of an operator X as

$$\|X\|_\alpha \equiv [\text{Tr}\{|X|^\alpha\}]^{1/\alpha}, \quad (3.1)$$

where $|X| \equiv \sqrt{X^\dagger X}$, and we use the same notation even for the case $\alpha \in (0, 1)$, when it is not a norm. The fidelity $F(\rho, \sigma)$ of two states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ is defined as

$$F(\rho, \sigma) \equiv \|\sqrt{\rho}\sqrt{\sigma}\|_1^2. \quad (3.2)$$

A quantum channel is given by a completely positive, trace-preserving (CPTP) map $\mathcal{N} : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{K})$, with \mathcal{H} and \mathcal{K} being the input and output Hilbert spaces of the channel, respectively. Let $\langle C, D \rangle \equiv \text{Tr}\{C^\dagger D\}$ denote the Hilbert-Schmidt inner product of $C, D \in \mathcal{B}(\mathcal{H})$. The adjoint of the quantum channel \mathcal{N} is a completely positive unital map $\mathcal{N}^\dagger : \mathcal{B}(\mathcal{K}) \mapsto \mathcal{B}(\mathcal{H})$ defined through the following relation:

$$\langle B, \mathcal{N}(A) \rangle = \langle \mathcal{N}^\dagger(B), A \rangle, \quad (3.3)$$

for all $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. A linear map is said to be a strict CPTP map if it is CPTP and if $\mathcal{N}(A) \in \mathcal{B}(\mathcal{K})_{++}$ for all $A \in \mathcal{B}(\mathcal{H})_{++}$. Note that a CPTP map is strict if and only if $\mathcal{N}(I) > 0$ [Bha07, Section 2.2]. We denote the identity channel as id but often suppress it for notational simplicity.

The set $\{U^i\}$ of Heisenberg-Weyl unitaries acting on a finite-dimensional Hilbert space \mathcal{H} of dimension d has the property that

$$\frac{1}{d^2} \sum_i U^i X (U^i)^\dagger = \text{Tr}\{X\} \frac{I}{d}, \quad (3.4)$$

for any operator X acting on \mathcal{H} .

3.1 Generalized relative entropies

The following relative entropies of a density operator ρ with respect to a positive semidefinite operator σ play an important role in this paper. In what follows, we restrict the definitions to the case in which ρ and σ satisfy the condition $\text{supp } \rho \subseteq \text{supp } \sigma$, with the understanding that they are equal to $+\infty$ when $\alpha > 1$ and $\text{supp } \rho \not\subseteq \text{supp } \sigma$. The α -Rényi relative entropy is defined for $\alpha \in (0, 1) \cup (1, \infty)$ as follows [Pet86a]:

$$D_\alpha(\rho\|\sigma) \equiv \frac{1}{\alpha - 1} \log \text{Tr}\{\rho^\alpha \sigma^{1-\alpha}\}. \quad (3.5)$$

The α -sandwiched Rényi relative entropy is defined for $\alpha \in (0, 1) \cup (1, \infty)$ as follows [MLDS⁺13, WWY14]:

$$\tilde{D}_\alpha(\rho\|\sigma) \equiv \frac{1}{\alpha - 1} \log \text{Tr}\left\{\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right\}. \quad (3.6)$$

Both quantities above reduce to the quantum relative entropy in (1.13) in the limit $\alpha \rightarrow 1$.

A fundamental property of the quantum relative entropy is that it is monotone with respect to quantum channels (also known as the *data-processing inequality*):

$$D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)), \quad (3.7)$$

where \mathcal{N} is a quantum channel. It is known that the data-processing inequality is satisfied by the α -Rényi relative entropy for $\alpha \in [0, 1) \cup (1, 2]$ [Pet86a], and for the α -sandwiched Rényi relative entropy for $\alpha \in [1/2, 1) \cup (1, \infty]$ [FL13, Bei13, MLDS⁺13, WWY14, MO15].

Two special cases of the α -sandwiched Rényi relative entropy are of particular significance in one-shot information theory [Ren05, Tom12], namely the min-relative entropy [DKF⁺12] and the max-relative entropy [Dat09]. These are defined as follows:

$$D_{\min}(\rho\|\sigma) \equiv \tilde{D}_{1/2}(\rho\|\sigma) = -\log F(\rho, \sigma), \quad (3.8)$$

and

$$D_{\max}(\rho\|\sigma) \equiv \inf\{\lambda : \rho \leq 2^\lambda \sigma\} = \lim_{\alpha \rightarrow \infty} \tilde{D}_\alpha(\rho\|\sigma). \quad (3.9)$$

The relative entropies defined above, satisfy the following lemma [MLDS⁺13]:

Lemma 2 For $\omega \in \mathcal{S}(\mathcal{H})$ and $\tau \in \mathcal{B}(\mathcal{H})_+$, such that $\text{Tr}\{\omega\} \geq \text{Tr}\{\tau\}$,

$$D_\alpha(\omega\|\tau) \geq 0 \text{ for } \alpha \in (0, 1) \cup (1, 2), \quad (3.10)$$

$$\tilde{D}_\alpha(\omega\|\tau) \geq 0 \text{ for } \alpha \in (1/2, 1) \cup (1, \infty), \quad (3.11)$$

$$D_{\min}(\omega\|\tau) \geq 0, \quad (3.12)$$

$$D_{\max}(\omega\|\tau) \geq 0, \quad (3.13)$$

with equalities holding if and only if $\omega = \tau$.

In proving our results, we also employ the notion of a *quantum f -divergence*, first introduced by Petz [Pet86a]. It can be conveniently expressed as follows [TCR09]:

Definition 3 For $A \in \mathcal{B}(\mathcal{H})_+$, $B \in \mathcal{B}(\mathcal{H})_{++}$ and an operator convex function f on $[0, \infty)$, the f -divergence of A with respect to B is given by

$$S_f(A\|B) = \langle \Gamma | \left(\sqrt{B} \otimes I \right) f \left(B^{-1} \otimes A^T \right) \left(\sqrt{B} \otimes I \right) | \Gamma \rangle, \quad (3.14)$$

where $|\Gamma\rangle = \sum_i |i\rangle \otimes |i\rangle$, and $\{|i\rangle\}$ is an orthonormal basis of \mathcal{H} with respect to which the transpose is defined.

Remark 4 Special cases of this are as follows:

1. The trace expression $\text{Tr}\{\rho^\alpha \sigma^{1-\alpha}\}$ of the α -Rényi relative entropy, for the choice $f(x) = x^\alpha$ for $\alpha \in (1, 2)$, and $-\text{Tr}\{\rho^\alpha \sigma^{1-\alpha}\}$ for the choice $f(x) = -x^\alpha$ for $\alpha \in (0, 1)$.
2. The quantum relative entropy for the choice $f(x) = x \log x$.

The equivalence relations given in Lemma 5 below follow directly from [HMPB11, Theorem 5.1].

Lemma 5 For $A, B \in \mathcal{B}(\mathcal{H})_{++}$ and a strict CPTP map \mathcal{N} acting on $\mathcal{B}(\mathcal{H})$, the following conditions are equivalent:

$$S_f(\mathcal{N}(A) \parallel \mathcal{N}(B)) = S_f(A \parallel B) \quad \text{for the functions in Remark 4,} \quad (3.15)$$

$$\mathcal{N}^\dagger [\log \mathcal{N}(A) - \log \mathcal{N}(B)] = \log A - \log B, \quad (3.16)$$

$$\mathcal{N} \text{ is sufficient for } A \text{ and } B. \quad (3.17)$$

Lemma 6 If ρ_{ABC} is a positive definite density operator such that

$$\log \rho_{ABC} = \log \rho_{AC} + \log \rho_{BC} - \log \rho_C, \quad (3.18)$$

then it is a short quantum Markov chain $A - C - B$.

Proof. The identity (3.18) is known from [Rus02] to be a condition for the conditional mutual information $I(A; B|C)_\rho$ of ρ_{ABC} to be equal to zero, which, by [HJPW04], implies that ρ_{ABC} is a short quantum Markov chain $A - C - B$. ■

Theorem 7 ([MP04, Mos05]) Let $\rho \in \mathcal{S}(\mathcal{H})_{++}$, $\sigma \in \mathcal{B}(\mathcal{H})_{++}$, and \mathcal{N} be a strict CPTP map. Then \mathcal{N} is sufficient for ρ and σ (as in (1.10)) if and only if the following conditions hold

1. There exist decompositions of \mathcal{H} and \mathcal{K} as follows:

$$\mathcal{H} = \bigoplus_j \mathcal{H}_{L_j} \otimes \mathcal{H}_{R_j}, \quad \mathcal{K} = \bigoplus_j \mathcal{K}_{L_j} \otimes \mathcal{K}_{R_j}, \quad (3.19)$$

where $\dim(\mathcal{H}_{L_j}) = \dim(\mathcal{K}_{L_j})$ for all j .

2. With respect to the decomposition in (3.19), ρ and σ can be written as follows:

$$\rho = \bigoplus_j p(j) \rho_{L_j}^j \otimes \tau_{R_j}^j, \quad \sigma = \bigoplus_j q(j) \sigma_{L_j}^j \otimes \tau_{R_j}^j, \quad (3.20)$$

for some probability distribution $\{p(j)\}$, positive reals $\{q(j)\}$, sets of states $\{\rho_{L_j}^j\}$ and $\{\tau_{R_j}^j\}$ and set of positive definite operators $\{\sigma_{L_j}^j\}$.

3. With respect to the decomposition in (3.19), the quantum channel \mathcal{N} can be written as

$$\mathcal{N} = \bigoplus_j \mathcal{U}_j \otimes \mathcal{N}_j^R, \quad (3.21)$$

where $\{\mathcal{U}_j : \mathcal{B}(\mathcal{H}_{L_j}) \rightarrow \mathcal{B}(\mathcal{K}_{L_j})\}$ is a set of unitary channels and $\{\mathcal{N}_j^R : \mathcal{B}(\mathcal{H}_{R_j}) \mapsto \mathcal{B}(\mathcal{K}_{R_j})\}$ is a set of quantum channels. Furthermore, with respect to the decomposition in (3.19), the adjoint of \mathcal{N} acts as

$$\mathcal{N}^\dagger = \bigoplus_j \mathcal{U}_j^\dagger \otimes (\mathcal{N}_j^R)^\dagger. \quad (3.22)$$

3.2 Trace inequalities

The following lemma is a consequence of [Eps73] (see also [CL08, Theorem 1.1]):

Lemma 8 For $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{H})_+$, and $p \in (0, 1)$, the map $B \mapsto \text{Tr}\{(AB^p A^\dagger)^{1/p}\}$ is concave. For invertible $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{H})_{++}$, and $p \in (-1, 0)$, the map $B \mapsto \text{Tr}\{(AB^p A^\dagger)^{1/p}\}$ is concave.

Lemma 9 Let $\rho \in \mathcal{S}(\mathcal{H})$, $\sigma \in \mathcal{B}(\mathcal{H})_+$, and let $\mathcal{N} : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{K})$ be a CPTP map. For $\alpha \in (0, 1)$

$$\text{Tr} \left\{ \left[\sigma^{\frac{1-\alpha}{2}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N}(\rho)^{1-\alpha} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right) \sigma^{\frac{1-\alpha}{2}} \right]^{\frac{1}{1-\alpha}} \right\} \leq 1. \quad (3.23)$$

For $\alpha \in (1/2, 1)$

$$\text{Tr} \left\{ \left[\sigma^{\frac{1-\alpha}{2\alpha}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \mathcal{N}(\rho)^{\frac{1-\alpha}{\alpha}} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \right) \sigma^{\frac{1-\alpha}{2\alpha}} \right]^{\frac{\alpha}{1-\alpha}} \right\} \leq 1. \quad (3.24)$$

Let $\rho \in \mathcal{S}(\mathcal{H})_{++}$, $\sigma \in \mathcal{B}(\mathcal{H})_{++}$, and let $\mathcal{N} : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{K})$ be a CPTP map such that $\mathcal{N}(\rho) \in \mathcal{S}(\mathcal{K})_{++}$, $\mathcal{N}(\sigma) \in \mathcal{B}(\mathcal{K})_{++}$. For these choices, the inequality in (3.23) holds if $\alpha \in (1, 2)$, and the inequality in (3.24) holds if $\alpha \in (1, \infty)$.

Proof. We begin by proving (3.23) for $\alpha \in (0, 1)$. By Stinespring's dilation theorem [Sti55], a given quantum channel \mathcal{N} can be realized as

$$\mathcal{N}(\omega) = \text{Tr}_{E'} \left\{ U (\omega \otimes |0\rangle\langle 0|_E) U^\dagger \right\} \quad \forall \omega \in \mathcal{B}(\mathcal{H}), \quad (3.25)$$

for some unitary U taking $\mathcal{H} \otimes \mathcal{H}_E$ to $\mathcal{K} \otimes \mathcal{H}_{E'}$ and fixed state $|0\rangle_E$ in an auxiliary Hilbert space \mathcal{H}_E . Furthermore, it suffices to take $\dim(\mathcal{H}_E) \leq \dim(\mathcal{H}) \dim(\mathcal{K})$ because the number of Kraus operators for the channel \mathcal{N} can always be taken less than or equal to $\dim(\mathcal{H}) \dim(\mathcal{K})$ and this is the dimension needed for an environment system \mathcal{H}_E . The adjoint of \mathcal{N} is given by

$$\mathcal{N}^\dagger(\tau) = \langle 0|_E U^\dagger (\tau \otimes I_{E'}) U |0\rangle_E \quad \forall \tau \in \mathcal{B}(\mathcal{K}), \quad (3.26)$$

Then

$$\begin{aligned} & \text{Tr} \left\{ \left(\sigma^{\frac{1-\alpha}{2}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N}(\rho)^{1-\alpha} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right) \sigma^{\frac{1-\alpha}{2}} \right)^{\frac{1}{1-\alpha}} \right\} \\ &= \text{Tr} \left\{ \left(\sigma^{\frac{1-\alpha}{2}} \langle 0|_E U^\dagger \left(\left[\mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N}(\rho)^{1-\alpha} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right] \otimes I_{E'} \right) U |0\rangle_E \sigma^{\frac{1-\alpha}{2}} \right)^{\frac{1}{1-\alpha}} \right\} \\ &= \text{Tr} \left\{ \left(A A^\dagger \right)^{\frac{1}{1-\alpha}} \right\} \end{aligned} \quad (3.27)$$

where

$$A = \sigma^{\frac{1-\alpha}{2}} \langle 0|_E U^\dagger K_\alpha^\dagger, \quad (3.28)$$

$$K_\alpha \equiv K_\alpha(\rho, \sigma, \mathcal{N}) \equiv \mathcal{N}(\rho)^{\frac{1-\alpha}{2}} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \otimes I_{E'}. \quad (3.29)$$

Then the above is equal to

$$\mathrm{Tr} \left\{ \left(A^\dagger A \right)^{\frac{1}{1-\alpha}} \right\} = \mathrm{Tr} \left\{ \left(K_\alpha U |0\rangle_E \sigma^{\frac{1-\alpha}{2}} \sigma^{\frac{1-\alpha}{2}} \langle 0|_E U^\dagger K_\alpha^\dagger \right)^{\frac{1}{1-\alpha}} \right\}, \quad (3.30)$$

because the eigenvalues of AA^\dagger and $A^\dagger A$ are the same for any operator A . Using that

$$\begin{aligned} U |0\rangle_E \sigma^{\frac{1-\alpha}{2}} \sigma^{\frac{1-\alpha}{2}} \langle 0|_E U^\dagger &= U (\sigma^{1-\alpha} \otimes |0\rangle\langle 0|_E) U^\dagger \\ &= U (\sigma \otimes |0\rangle\langle 0|_E)^{1-\alpha} U^\dagger \\ &= \left[U (\sigma \otimes |0\rangle\langle 0|_E) U^\dagger \right]^{1-\alpha}, \end{aligned} \quad (3.31)$$

the right hand side of (3.30) is equal to

$$\mathrm{Tr} \left\{ \left(K_\alpha \left[U (\sigma \otimes |0\rangle\langle 0|_E) U^\dagger \right]^{1-\alpha} K_\alpha^\dagger \right)^{\frac{1}{1-\alpha}} \right\}. \quad (3.32)$$

Let $\{U_{E'}^i\}$ be a set of Heisenberg-Weyl operators for the E' system and let $\pi_{E'}$ denote the maximally mixed state on system E' . Now we use Lemma 8 to establish the inequality below:

$$\begin{aligned} &\mathrm{Tr} \left\{ \left(K_\alpha \left[U (\sigma \otimes |0\rangle\langle 0|_E) U^\dagger \right]^{1-\alpha} K_\alpha^\dagger \right)^{\frac{1}{1-\alpha}} \right\} \\ &= \frac{1}{d_{E'}^2} \sum_i \mathrm{Tr} \left\{ \left(K_\alpha \left[U_{E'}^i U (\sigma \otimes |0\rangle\langle 0|_E) U^\dagger (U_{E'}^i)^\dagger \right]^{1-\alpha} K_\alpha^\dagger \right)^{\frac{1}{1-\alpha}} \right\} \end{aligned} \quad (3.33)$$

$$\leq \mathrm{Tr} \left\{ \left(K_\alpha \left[\frac{1}{d_{E'}^2} \sum_i U_{E'}^i U (\sigma \otimes |0\rangle\langle 0|_E) U^\dagger (U_{E'}^i)^\dagger \right]^{1-\alpha} K_\alpha^\dagger \right)^{\frac{1}{1-\alpha}} \right\} \quad (3.34)$$

$$= \mathrm{Tr} \left\{ \left(K_\alpha \left[\mathrm{Tr}_{E'} \left\{ U (\sigma \otimes |0\rangle\langle 0|_E) U^\dagger \right\} \otimes \pi_{E'} \right]^{1-\alpha} K_\alpha^\dagger \right)^{\frac{1}{1-\alpha}} \right\}. \quad (3.35)$$

Continuing, the last line above is equal to

$$\begin{aligned} &\mathrm{Tr} \left\{ \left(K_\alpha [\mathcal{N}(\sigma) \otimes \pi_{E'}]^{1-\alpha} K_\alpha^\dagger \right)^{\frac{1}{1-\alpha}} \right\} \\ &= \mathrm{Tr} \left\{ \left(K_\alpha [\mathcal{N}(\sigma)^{1-\alpha} \otimes \pi_{E'}^{1-\alpha}] K_\alpha^\dagger \right)^{\frac{1}{1-\alpha}} \right\} \end{aligned} \quad (3.36)$$

$$= \mathrm{Tr} \left\{ \left(\mathcal{N}(\rho)^{\frac{1-\alpha}{2}} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N}(\sigma)^{1-\alpha} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N}(\rho)^{\frac{1-\alpha}{2}} \otimes \pi_{E'}^{1-\alpha} \right)^{\frac{1}{1-\alpha}} \right\} \quad (3.37)$$

$$= \mathrm{Tr} \left\{ \left(\mathcal{N}(\rho)^{\frac{1-\alpha}{2}} [\mathcal{N}(\sigma)]^0 \mathcal{N}(\rho)^{\frac{1-\alpha}{2}} \otimes \pi_{E'}^{1-\alpha} \right)^{\frac{1}{1-\alpha}} \right\} \quad (3.38)$$

$$\leq \mathrm{Tr} \left\{ \left(\mathcal{N}(\rho)^{1-\alpha} \otimes \pi_{E'}^{1-\alpha} \right)^{\frac{1}{1-\alpha}} \right\} \quad (3.39)$$

$$= 1, \quad (3.40)$$

where the inequality follows because

$$\mathcal{N}(\rho)^{\frac{1-\alpha}{2}} [\mathcal{N}(\sigma)]^0 \mathcal{N}(\rho)^{\frac{1-\alpha}{2}} \leq \mathcal{N}(\rho)^{1-\alpha}, \quad (3.41)$$

and

$$\mathrm{Tr} \{f(A)\} \leq \mathrm{Tr} \{f(B)\} \quad (3.42)$$

when $A \leq B$ and $f(x) \equiv x^{1/(1-\alpha)}$ is a monotone non-decreasing function on $[0, \infty)$. The other inequality in (3.24) follows by recognizing that

$$\frac{1-\alpha}{\alpha} = 1 - \gamma, \quad (3.43)$$

with $\gamma \equiv (2\alpha - 1)/\alpha$, and $\gamma \in (0, 1)$ when $\alpha \in (1/2, 1)$. Thus, we can apply (3.23) to conclude (3.24).

To prove the last two statements, we exploit Choi's inequality [Cho74, Corollary 2.3] (see also [Bha07, Theorem 2.3.6]), which states that

$$\mathcal{M}(A)^{-1} \leq \mathcal{M}(A^{-1}), \quad (3.44)$$

for \mathcal{M} a strictly positive map and $A \in \mathcal{B}(\mathcal{H})_{++}$. Consider that the adjoint \mathcal{N}^\dagger of a channel \mathcal{N} is strictly positive because it is unital (recall that a map \mathcal{M} is strictly positive if and only if $\mathcal{M}(I) > 0$, which is the case if \mathcal{M} is unital). So let $\alpha \in (1, 2)$ and consider that

$$\begin{aligned} & \mathrm{Tr} \left\{ \left[\sigma^{\frac{1-\alpha}{2}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N}(\rho)^{1-\alpha} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right) \sigma^{\frac{1-\alpha}{2}} \right]^{\frac{1}{1-\alpha}} \right\} \\ &= \mathrm{Tr} \left\{ \left[\sigma^{-\frac{1-\beta}{2}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{-\frac{\beta-1}{2}} \mathcal{N}(\rho)^{-(1-\beta)} \mathcal{N}(\sigma)^{-\frac{\beta-1}{2}} \right) \sigma^{-\frac{1-\beta}{2}} \right]^{-\frac{1}{1-\beta}} \right\}, \quad (3.45) \end{aligned}$$

for $\beta = 2 - \alpha \in (0, 1)$. Then

$$\begin{aligned} & \left[\sigma^{-\frac{1-\beta}{2}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{-\frac{\beta-1}{2}} \mathcal{N}(\rho)^{-(1-\beta)} \mathcal{N}(\sigma)^{-\frac{\beta-1}{2}} \right) \sigma^{-\frac{1-\beta}{2}} \right]^{-1} \\ &= \sigma^{\frac{1-\beta}{2}} \left[\mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{-\frac{\beta-1}{2}} \mathcal{N}(\rho)^{-(1-\beta)} \mathcal{N}(\sigma)^{-\frac{\beta-1}{2}} \right) \right]^{-1} \sigma^{\frac{1-\beta}{2}} \quad (3.46) \end{aligned}$$

$$\leq \sigma^{\frac{1-\beta}{2}} \mathcal{N}^\dagger \left(\left[\mathcal{N}(\sigma)^{-\frac{\beta-1}{2}} \mathcal{N}(\rho)^{-(1-\beta)} \mathcal{N}(\sigma)^{-\frac{\beta-1}{2}} \right]^{-1} \right) \sigma^{\frac{1-\beta}{2}} \quad (3.47)$$

$$= \sigma^{\frac{1-\beta}{2}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\beta-1}{2}} \mathcal{N}(\rho)^{1-\beta} \mathcal{N}(\sigma)^{\frac{\beta-1}{2}} \right) \sigma^{\frac{1-\beta}{2}}, \quad (3.48)$$

where the equalities follow from the assumptions that $\rho \in \mathcal{S}(\mathcal{H})_{++}$, $\sigma \in \mathcal{B}(\mathcal{H})_{++}$, $\mathcal{N}(\rho) \in \mathcal{S}(\mathcal{K})_{++}$, and $\mathcal{N}(\sigma) \in \mathcal{B}(\mathcal{K})_{++}$, and the inequality is an application of Choi's inequality. By applying (3.42) and the result above, we find that

$$\begin{aligned} & \mathrm{Tr} \left\{ \left[\left[\sigma^{-\frac{1-\beta}{2}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{-\frac{\beta-1}{2}} \mathcal{N}(\rho)^{-(1-\beta)} \mathcal{N}(\sigma)^{-\frac{\beta-1}{2}} \right) \sigma^{-\frac{1-\beta}{2}} \right]^{-1} \right]^{\frac{1}{1-\beta}} \right\} \\ & \leq \mathrm{Tr} \left\{ \left[\sigma^{\frac{1-\beta}{2}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\beta-1}{2}} \mathcal{N}(\rho)^{1-\beta} \mathcal{N}(\sigma)^{\frac{\beta-1}{2}} \right) \sigma^{\frac{1-\beta}{2}} \right]^{\frac{1}{1-\beta}} \right\} \leq 1, \quad (3.49) \end{aligned}$$

where the last inequality is a consequence of what we have previously shown. The other inequality (3.24) for $\alpha \in (1, \infty)$ follows from (3.23) for $\alpha \in (1, 2)$ by making the same identification as in (3.43): note here that $\gamma \in (1, 2)$ if $\alpha \in (1, \infty)$. ■

As a corollary, we establish the following trace inequality, which was left as an open question in [Rus02]:

Corollary 10 *For $\rho \in \mathcal{S}(\mathcal{H})_{++}$, $\sigma \in \mathcal{B}(\mathcal{H})_{++}$, and a strict CPTP map \mathcal{N} acting on $\mathcal{S}(\mathcal{H})$, the following inequality holds:*

$$\mathrm{Tr} \left\{ \exp \left\{ \log \sigma + \mathcal{N}^\dagger (\log \mathcal{N}(\rho) - \log \mathcal{N}(\sigma)) \right\} \right\} \leq 1. \quad (3.50)$$

Proof. This follows by taking the limit $\alpha \nearrow 1$ in Lemma 9 and using [SBW15, Lemma 24]:

$$\begin{aligned} & \exp \left\{ \log \sigma + \mathcal{N}^\dagger (\log \mathcal{N}(\rho) - \log \mathcal{N}(\sigma)) \right\} \\ &= \lim_{\alpha \nearrow 1} \left[\sigma^{\frac{1-\alpha}{2}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N}(\rho)^{1-\alpha} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right) \sigma^{\frac{1-\alpha}{2}} \right]^{\frac{1}{1-\alpha}}. \end{aligned} \quad (3.51)$$

The inequality in Lemma 9 is preserved in the limit. ■

We can also solve an open question from [Zha14] for some values of α , simply by applying Lemma 9 for the choices $\rho = \rho_{ABC}$, $\sigma = \rho_{AC} \otimes I_B$, and $\mathcal{N} = \mathrm{Tr}_A$:

Corollary 11 *Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$. If $\alpha \in (0, 1)$, then*

$$\mathrm{Tr} \left\{ \left(\rho_{AC}^{\frac{1-\alpha}{2}} \rho_C^{\frac{\alpha-1}{2}} \rho_{BC}^{1-\alpha} \rho_C^{\frac{\alpha-1}{2}} \rho_{AC}^{\frac{1-\alpha}{2}} \right)^{\frac{1}{1-\alpha}} \right\} \leq 1. \quad (3.52)$$

If $\alpha \in (1/2, 1)$, then

$$\mathrm{Tr} \left\{ \left(\rho_{AC}^{\frac{1-\alpha}{2\alpha}} \rho_C^{\frac{\alpha-1}{2\alpha}} \rho_{BC}^{\frac{1-\alpha}{\alpha}} \rho_C^{\frac{\alpha-1}{2\alpha}} \rho_{AC}^{\frac{1-\alpha}{2\alpha}} \right)^{\frac{\alpha}{1-\alpha}} \right\} \leq 1. \quad (3.53)$$

If $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$, then (3.52) holds for $\alpha \in (1, 2)$ and (3.53) holds for $\alpha \in (1, \infty)$.

Remark 12 *Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$. The well-known inequality [LR73b]*

$$\mathrm{Tr} \{ \exp \{ \log \rho_{AC} + \log \rho_{BC} - \log \rho_C \} \} \leq 1 \quad (3.54)$$

follows from Corollary 11 by taking the limit $\alpha \rightarrow 1$ and using the generalized Lie-Trotter product formula [Suz85]:

$$\exp \{ \log \rho_{AC} + \log \rho_{BC} - \log \rho_C \} = \lim_{\alpha \rightarrow 1} \left(\rho_{AC}^{\frac{1-\alpha}{2}} \rho_C^{\frac{\alpha-1}{2}} \rho_{BC}^{1-\alpha} \rho_C^{\frac{\alpha-1}{2}} \rho_{AC}^{\frac{1-\alpha}{2}} \right)^{\frac{1}{1-\alpha}}. \quad (3.55)$$

The following proposition establishes some important properties of the $\Delta_\alpha(\rho, \sigma, \mathcal{N})$ and $\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N})$ quantities, which were left as open questions in [SBW15]:

Proposition 13 *Let $\rho \in \mathcal{S}(\mathcal{H})$, $\sigma \in \mathcal{B}(\mathcal{H})_+$, and let \mathcal{N} be a CPTP map. For all $\alpha \in (0, 1)$*

$$\Delta_\alpha(\rho, \sigma, \mathcal{N}) \geq 0, \quad (3.56)$$

with equality occurring if and only if

$$\rho = \left[\sigma^{\frac{1-\alpha}{2}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N}(\rho)^{1-\alpha} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right) \sigma^{\frac{1-\alpha}{2}} \right]^{\frac{1}{1-\alpha}}. \quad (3.57)$$

For all $\alpha \in (1/2, 1)$

$$\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}) \geq 0, \quad (3.58)$$

with equality occurring if and only if

$$\rho = \left[\sigma^{\frac{1-\alpha}{2\alpha}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \mathcal{N}(\rho)^{\frac{1-\alpha}{\alpha}} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \right) \sigma^{\frac{1-\alpha}{2\alpha}} \right]^{\frac{\alpha}{1-\alpha}}. \quad (3.59)$$

Let $\rho \in \mathcal{S}(\mathcal{H})_{++}$, $\sigma \in \mathcal{B}(\mathcal{H})_{++}$, and let $\mathcal{N} : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{K})$ be a CPTP map such that $\mathcal{N}(\rho) \in \mathcal{S}(\mathcal{K})_{++}$, $\mathcal{N}(\sigma) \in \mathcal{B}(\mathcal{K})_{++}$. For these choices, the inequality in (3.56) and the equality condition in (3.57) hold if $\alpha \in (1, 2)$, and the inequality in (3.58) and equality condition in (3.59) hold if $\alpha \in (1, \infty)$.

Proof. Note that the quantities Δ_α and $\tilde{\Delta}_\alpha$ defined in (1.16) and (1.17), respectively, can be expressed in terms of the α -Rényi relative entropy and the α -sandwiched Rényi relative entropy as follows [SBW15]:

$$\Delta_\alpha(\rho, \sigma, \mathcal{N}) = D_\alpha \left(\rho \left\| \left[\sigma^{\frac{1-\alpha}{2}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N}(\rho)^{1-\alpha} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right) \sigma^{\frac{1-\alpha}{2}} \right]^{\frac{1}{1-\alpha}} \right. \right), \quad (3.60)$$

$$\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}) = \tilde{D}_\alpha \left(\rho \left\| \left[\sigma^{\frac{1-\alpha}{2\alpha}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \mathcal{N}(\rho)^{\frac{1-\alpha}{\alpha}} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \right) \sigma^{\frac{1-\alpha}{2\alpha}} \right]^{\frac{\alpha}{1-\alpha}} \right. \right). \quad (3.61)$$

The non-negativity and equality conditions of $\Delta_\alpha(\rho, \sigma, \mathcal{N})$ for $\alpha \in (0, 1) \cup (1, 2)$, and that of $\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N})$ for $\alpha \in (1/2, 1) \cup (1, \infty)$, then follow by applying Lemmas 2 and 9. ■

Appendix A gives alternate proofs of the inequality in (3.56) for $\alpha \in (1, 2)$ and the inequality in (3.58) for $\alpha \in (1, \infty)$. These were the original proofs appearing in a preprint version of this paper and might be of independent interest. Appendix A gives some other equality conditions.

4 Sufficiency of quantum channels and Rényi generalizations of relative entropy differences

Theorem 14 *Let $\rho \in \mathcal{S}(\mathcal{H})_{++}$, $\sigma \in \mathcal{B}(\mathcal{H})_{++}$, and let \mathcal{N} be a strict CPTP map. Then $\Delta_\alpha(\rho, \sigma, \mathcal{N}) = 0$ and $\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}) = 0$ for all $\alpha \in (0, 1) \cup (1, \infty)$ if \mathcal{N} is sufficient for ρ and σ . Furthermore, \mathcal{N} is sufficient for ρ and σ if $\Delta_\alpha(\rho, \sigma, \mathcal{N}) = 0$ for some $\alpha \in (0, 1) \cup (1, 2)$ or if $\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}) = 0$ for some $\alpha \in (1/2, 1) \cup (1, \infty)$.*

Proof. We begin by proving that $\Delta_\alpha(\rho, \sigma, \mathcal{N}) = 0$ for all $\alpha \in (0, 1) \cup (1, \infty)$ if \mathcal{N} is sufficient for ρ and σ . So suppose that \mathcal{N} is sufficient for ρ and σ . Then according to Theorem 7, this implies that the decompositions in (3.20)-(3.21) hold. To simplify things, we exploit the direct sum structure

in (3.20)-(3.21) and first evaluate the contribution arising from the j^{th} block, for any j . Then for a given block, evaluating the formula

$$\rho^\alpha \sigma^{\frac{1-\alpha}{2}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N}(\rho)^{1-\alpha} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right) \sigma^{\frac{1-\alpha}{2}} \quad (4.1)$$

gives the following (where we suppress the index j , for simplicity)

$$\begin{aligned} & (p\rho_L \otimes \tau_R)^\alpha (q\sigma_L \otimes \tau_R)^{\frac{1-\alpha}{2}} \times \\ & \left[\mathcal{U}^\dagger \otimes (\mathcal{N}^R)^\dagger \right] \left((q\mathcal{U}(\sigma_L) \otimes \mathcal{N}^R(\tau_R))^{\frac{\alpha-1}{2}} (p\mathcal{U}(\rho_L) \otimes \mathcal{N}^R(\tau_R))^{1-\alpha} (q\mathcal{U}(\sigma_L) \otimes \mathcal{N}^R(\tau_R))^{\frac{\alpha-1}{2}} \right) \times \\ & \hspace{20em} (q\sigma_L \otimes \tau_R)^{\frac{1-\alpha}{2}} \\ & = p(\rho_L \otimes \tau_R)^\alpha (\sigma_L \otimes \tau_R)^{\frac{1-\alpha}{2}} \times \\ & \left[\mathcal{U}^\dagger \otimes (\mathcal{N}^R)^\dagger \right] \left((\mathcal{U}(\sigma_L) \otimes \mathcal{N}^R(\tau_R))^{\frac{\alpha-1}{2}} (\mathcal{U}(\rho_L) \otimes \mathcal{N}^R(\tau_R))^{1-\alpha} (\mathcal{U}(\sigma_L) \otimes \mathcal{N}^R(\tau_R))^{\frac{\alpha-1}{2}} \right) \times \\ & \hspace{20em} (\sigma_L \otimes \tau_R)^{\frac{1-\alpha}{2}} \quad (4.2) \end{aligned}$$

$$\begin{aligned} & = p \left((\rho_L)^\alpha (\sigma_L)^{\frac{1-\alpha}{2}} \otimes \tau_R^{(\alpha+1)/2} \right) \times \\ & \left[\mathcal{U}^\dagger \otimes (\mathcal{N}^R)^\dagger \right] \left((\mathcal{U}(\sigma_L)^{\frac{\alpha-1}{2}} \mathcal{U}(\rho_L)^{1-\alpha} \mathcal{U}(\sigma_L)^{\frac{\alpha-1}{2}} \otimes \mathcal{N}^R(\tau_R)^{\frac{\alpha-1}{2}} \mathcal{N}^R(\tau_R)^{1-\alpha} \mathcal{N}^R(\tau_R)^{\frac{\alpha-1}{2}} \right) \times \\ & \hspace{20em} (\sigma_L \otimes \tau_R)^{\frac{1-\alpha}{2}}. \quad (4.3) \end{aligned}$$

Continuing, the last line is equal to

$$\begin{aligned} & p \left((\rho_L)^\alpha (\sigma_L)^{\frac{1-\alpha}{2}} \otimes \tau_R^{(\alpha+1)/2} \right) \left[\mathcal{U}^\dagger \otimes (\mathcal{N}^R)^\dagger \right] \left(\mathcal{U} \left[(\sigma_L)^{\frac{\alpha-1}{2}} (\rho_L)^{1-\alpha} (\sigma_L)^{\frac{\alpha-1}{2}} \right] \otimes I_R \right) (\sigma_L \otimes \tau_R)^{\frac{1-\alpha}{2}} \\ & = p \left((\rho_L)^\alpha (\sigma_L)^{\frac{1-\alpha}{2}} \otimes \tau_R^{(\alpha+1)/2} \right) \left(\left[(\sigma_L)^{\frac{\alpha-1}{2}} (\rho_L)^{1-\alpha} (\sigma_L)^{\frac{\alpha-1}{2}} \right] \otimes I_R \right) (\sigma_L \otimes \tau_R)^{\frac{1-\alpha}{2}} \\ & = p \left((\rho_L)^\alpha (\sigma_L)^{\frac{1-\alpha}{2}} (\sigma_L)^{\frac{\alpha-1}{2}} (\rho_L)^{1-\alpha} (\sigma_L)^{\frac{\alpha-1}{2}} (\sigma_L)^{\frac{1-\alpha}{2}} \otimes \tau_R^{(\alpha+1)/2} \tau_R^{\frac{1-\alpha}{2}} \right) \\ & = p \left((\rho_L)^\alpha (\rho_L)^{1-\alpha} \otimes \tau_R \right) \\ & = p\rho_L \otimes \tau_R. \quad (4.4) \end{aligned}$$

Taking the trace gives p and thus we find that each block has trace equal to $p(j)$, so that

$$\Delta_\alpha(\rho, \sigma, \mathcal{N}) = \frac{1}{\alpha-1} \log \sum_j p(j) = 0. \quad (4.5)$$

So this proves that $\Delta_\alpha(\rho, \sigma, \mathcal{N}) = 0$ for all $\alpha \in (0, 1) \cup (1, \infty)$ if \mathcal{N} is sufficient for ρ and σ .

We now prove that $\Delta_\alpha(\rho, \sigma, \mathcal{N}) = 0$ for all $\alpha \in (0, 1) \cup (1, \infty)$ if \mathcal{N} is sufficient for ρ and σ . As above, the decompositions in (3.20)-(3.21) hold and we exploit this direct sum structure again. As in the previous proof, it suffices to evaluate the contribution arising from each block j , for any j , in the direct-sum decomposition. Evaluating the formula

$$\rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha}{2\alpha}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \mathcal{N}(\rho)^{1-\alpha} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \right) \sigma^{\frac{1-\alpha}{2\alpha}} \rho^{\frac{1}{2}}, \quad (4.6)$$

for the j th block, (where we once again suppress the index j) gives

$$(p\rho_L \otimes \tau_R)^{\frac{1}{2}} (q\sigma_L \otimes \tau_R)^{\frac{1-\alpha}{2\alpha}} \times \\ \left[\mathcal{U}^\dagger \otimes (\mathcal{N}^R)^\dagger \right] \left((q\mathcal{U}(\sigma_L) \otimes \mathcal{N}^R(\tau_R))^{\frac{\alpha-1}{2\alpha}} (p\mathcal{U}(\rho_L) \otimes \mathcal{N}^R(\tau_R))^{\frac{1-\alpha}{\alpha}} (q\mathcal{U}(\sigma_L) \otimes \mathcal{N}^R(\tau_R))^{\frac{\alpha-1}{2\alpha}} \right) \times \\ (q\sigma_L \otimes \tau_R)^{\frac{1-\alpha}{2\alpha}} (p\rho_L \otimes \tau_R)^{\frac{1}{2}}. \quad (4.7)$$

Then proceeding very similarly as before, we conclude that (4.7) is equal to

$$p^{\frac{1}{\alpha}} \rho_L^{\frac{1}{\alpha}} \otimes \tau_R^{\frac{1}{\alpha}}. \quad (4.8)$$

We evaluate evaluate the norm $\|\cdot\|_\alpha^\alpha$ of the j th block to be

$$\left\| p^{\frac{1}{\alpha}} \rho_L^{\frac{1}{\alpha}} \otimes \tau_R^{\frac{1}{\alpha}} \right\|_\alpha^\alpha = p. \quad (4.9)$$

As a consequence, we find that

$$\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}) = \frac{1}{\alpha-1} \log \left(\sum_j p(j) \right) = 0, \quad (4.10)$$

where we used that $\|A \oplus B\|_\alpha^\alpha = \|A\|_\alpha^\alpha + \|B\|_\alpha^\alpha$.

Now suppose that $\Delta_\alpha(\rho, \sigma, \mathcal{N}) = 0$ for some $\alpha \in (0, 1) \cup (1, 2)$. From Proposition 13, we have that

$$\rho = \left[\sigma^{\frac{1-\alpha}{2}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N}(\rho)^{1-\alpha} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right) \sigma^{\frac{1-\alpha}{2}} \right]^{\frac{1}{1-\alpha}}, \quad (4.11)$$

which is equivalent to

$$\sigma^{\frac{\alpha-1}{2}} \rho^{1-\alpha} \sigma^{\frac{\alpha-1}{2}} = \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N}(\rho)^{1-\alpha} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right). \quad (4.12)$$

Multiply both sides by σ and take the trace to get

$$\begin{aligned} \text{Tr} \left\{ \sigma \sigma^{\frac{\alpha-1}{2}} \rho^{1-\alpha} \sigma^{\frac{\alpha-1}{2}} \right\} &= \text{Tr} \left\{ \sigma \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N}(\rho)^{1-\alpha} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right) \right\} \\ &\Leftrightarrow \text{Tr} \left\{ \rho^{1-\alpha} \sigma^\alpha \right\} = \text{Tr} \left\{ \mathcal{N}(\sigma) \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N}(\rho)^{1-\alpha} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right\} \\ &\Leftrightarrow \text{Tr} \left\{ \rho^{1-\alpha} \sigma^\alpha \right\} = \text{Tr} \left\{ \mathcal{N}(\rho)^{1-\alpha} \mathcal{N}(\sigma)^\alpha \right\} \\ &\Leftrightarrow D_\alpha(\sigma \parallel \rho) = D_\alpha(\mathcal{N}(\sigma) \parallel \mathcal{N}(\rho)). \end{aligned} \quad (4.13)$$

The last line is an equality of f -divergences (see Remark 4), which by Lemma 5 allows us to conclude that \mathcal{N} is sufficient for ρ and σ .

Suppose that $\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}) = 0$ for some $\alpha \in (1/2, 1) \cup (1, \infty)$. From Proposition 13, we have that

$$\rho = \left[\sigma^{\frac{1-\alpha}{2\alpha}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \mathcal{N}(\rho)^{\frac{1-\alpha}{\alpha}} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \right) \sigma^{\frac{1-\alpha}{2\alpha}} \right]^{\frac{\alpha}{1-\alpha}}, \quad (4.14)$$

which can be rewritten as

$$\rho = \left[\sigma^{\frac{1-\gamma}{2}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\gamma-1}{2}} \mathcal{N}(\rho)^{1-\gamma} \mathcal{N}(\sigma)^{\frac{\gamma-1}{2}} \right) \sigma^{\frac{1-\gamma}{2}} \right]^{\frac{1}{1-\gamma}}, \quad (4.15)$$

with γ defined in (3.43), so that $\gamma \in (0, 1) \cup (1, 2)$. By the development in (4.11)-(4.13), we can conclude that \mathcal{N} is sufficient for ρ and σ . ■

The following statement is a direct corollary of the correspondence in (1.20)-(1.21) and Theorem 14:

Corollary 15 *Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$. Then $I_\alpha(A; B|C)_\rho = 0$ and $\tilde{I}_\alpha(A; B|C)_\rho = 0$ for all $\alpha \in (0, 1) \cup (1, \infty)$ if ρ_{ABC} is a short quantum Markov chain $A - C - B$. Furthermore, ρ_{ABC} is a short quantum Markov chain $A - C - B$ if $I_\alpha(A; B|C)_\rho = 0$ for some $\alpha \in (0, 1) \cup (1, 2)$ or if $\tilde{I}_\alpha(A; B|C)_\rho = 0$ for some $\alpha \in (1/2, 1) \cup (1, \infty)$.*

5 Quantum Markov chains, sufficiency of quantum channels, and min- and max-information measures

5.1 Min- and max- generalizations of a relative entropy difference

We consider the following generalizations of a relative entropy difference, motivated by the developments in [BSW15, SBW15].

Definition 16 *Let $\rho \in \mathcal{S}(\mathcal{H})_{++}$, $\sigma \in \mathcal{B}(\mathcal{H})_{++}$, and let \mathcal{N} be a strict CPTP map. Then,*

$$\Delta_{\min}(\rho, \sigma, \mathcal{N}) \equiv D_{\min}(\rho \| \mathcal{R}_{\sigma, \mathcal{N}}(\mathcal{N}(\rho))), \quad (5.1)$$

where $\mathcal{R}_{\sigma, \mathcal{N}}$ is the Petz recovery channel defined in (1.11) and

$$\Delta_{\max}(\rho, \sigma, \mathcal{N}) \equiv D_{\max}(\rho \| \mathcal{R}_{\sigma, \mathcal{N}}(\mathcal{N}(\rho))). \quad (5.2)$$

Theorem 17 *For $\rho \in \mathcal{S}(\mathcal{H})_{++}$, $\sigma \in \mathcal{B}(\mathcal{H})_{++}$, and a strict CPTP map \mathcal{N} ,*

$$\Delta_{\min}(\rho, \sigma, \mathcal{N}) \geq 0, \quad \Delta_{\max}(\rho, \sigma, \mathcal{N}) \geq 0, \quad (5.3)$$

with equality holding if and only if \mathcal{N} is sufficient for ρ and σ .

Proof. The non-negativity conditions follow from the fact that $\mathcal{R}_{\sigma, \mathcal{N}}(\mathcal{N}(\rho))$ is a density operator (since $\mathcal{R}_{\sigma, \mathcal{N}}$ is a CPTP map) and Lemma 2. The equality conditions also follow from Lemma 2 and the fact that \mathcal{N} is sufficient for ρ and σ if and only if $\mathcal{R}_{\sigma, \mathcal{N}}(\mathcal{N}(\rho)) = \rho$. ■

5.2 Min- and max- generalizations of conditional mutual information

Definition 18 ([BSW15]) *For a tripartite state $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$ the max-conditional mutual information is defined as follows:*

$$I_{\max}(A; B|C)_\rho \equiv D_{\max} \left(\rho_{ABC} \left\| \rho_{AC}^{\frac{1}{2}} \rho_C^{-\frac{1}{2}} \rho_{BC} \rho_C^{-\frac{1}{2}} \rho_{AC}^{\frac{1}{2}} \right\| \right) \quad (5.4)$$

$$= \inf \left\{ \lambda : \rho_{ABC} \leq \exp(\lambda) \rho_{AC}^{\frac{1}{2}} \rho_C^{-\frac{1}{2}} \rho_{BC} \rho_C^{-\frac{1}{2}} \rho_{AC}^{\frac{1}{2}} \right\} \quad (5.5)$$

$$= 2 \log \left\| \rho_{ABC}^{\frac{1}{2}} \rho_{AC}^{-\frac{1}{2}} \rho_C^{\frac{1}{2}} \rho_{BC}^{-\frac{1}{2}} \right\|_\infty, \quad (5.6)$$

and the min-conditional mutual information is defined as follows:

$$I_{\min}(A; B|C)_\rho \equiv D_{\min} \left(\rho_{ABC} \left\| \rho_{AC}^{\frac{1}{2}} \rho_C^{-\frac{1}{2}} \rho_{BC} \rho_C^{-\frac{1}{2}} \rho_{AC}^{\frac{1}{2}} \right\| \right) \quad (5.7)$$

$$= -\log F \left(\rho_{ABC}, \rho_{AC}^{\frac{1}{2}} \rho_C^{-\frac{1}{2}} \rho_{BC} \rho_C^{-\frac{1}{2}} \rho_{AC}^{\frac{1}{2}} \right) \quad (5.8)$$

$$= -2 \log \left\| \rho_{ABC}^{\frac{1}{2}} \rho_{AC}^{\frac{1}{2}} \rho_C^{-\frac{1}{2}} \rho_{BC}^{\frac{1}{2}} \right\|_1. \quad (5.9)$$

As observed in [BSW15], we have that

$$I_{\max}(A; B|C)_\rho, I_{\min}(A; B|C)_\rho \geq 0, \quad (5.10)$$

due to Lemma 2 and the fact that $\text{Tr}\{\rho_{AC}^{\frac{1}{2}} \rho_C^{-\frac{1}{2}} \rho_{BC} \rho_C^{-\frac{1}{2}} \rho_{AC}^{\frac{1}{2}}\} = 1$. These quantities are special cases of those from Definition 16, by using (1.21). As such, the following is a corollary of Theorem 17:

Corollary 19 *Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})_{++}$. Then each of the following identities hold if and only if ρ_{ABC} is a short quantum Markov chain $A - C - B$:*

$$I_{\max}(A; B|C) = 0, \quad (5.11)$$

$$I_{\min}(A; B|C) = 0. \quad (5.12)$$

6 Conclusion and open questions

We have shown that the α -Rényi quantities $\Delta_\alpha(\rho, \sigma, \mathcal{N})$ and $\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N})$ from [SBW15] are non-negative and equal to zero if and only if \mathcal{N} is sufficient for ρ and σ . As a consequence, we find that the α -Rényi conditional mutual informations $I_\alpha(A; B|C)_\rho$ and $\tilde{I}_\alpha(A; B|C)_\rho$ from [BSW15] are equal to zero if and only if ρ_{ABC} is a short quantum Markov chain $A - C - B$. Moreover, we have solved some open questions from [Rus02, Zha14].

There are some interesting open questions to consider going forward from here. We would like to know the equality conditions for monotonicity of the sandwiched Rényi relative entropies, i.e., for which triples $(\rho, \sigma, \mathcal{N})$ is it true that

$$\tilde{D}_\alpha(\rho||\sigma) = \tilde{D}_\alpha(\mathcal{N}(\rho)||\mathcal{N}(\sigma)) ? \quad (6.1)$$

Apparently we cannot solve this question using the methods of [HMPB11] because we cannot represent the sandwiched Rényi relative entropy as an f -divergence. Presumably, this equality occurs if and only if \mathcal{N} is sufficient for ρ and σ , but it remains to be proved. Next, is there a characterization of states for which $I_\alpha(A; B|C)_\rho$ and $\tilde{I}_\alpha(A; B|C)_\rho$ are nearly equal to zero? Also, is there a characterization of triples $(\rho, \sigma, \mathcal{N})$ for which $\Delta_\alpha(\rho, \sigma, \mathcal{N})$ and $\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N})$ are nearly equal to zero? Presumably the former has to do with ρ_{ABC} being close to the ‘‘Petz recovered’’ ρ_{BC} and the latter has to do with ρ being close to the ‘‘Petz recovered’’ ρ , as recent developments in [FR15, BLW15] might suggest.

Note: After the posting of a preprint of this work to the arXiv, there have been improvements of the results detailed here for values of $\alpha < 1$ [Wil15, DW15, JRS⁺15], which address some

of the open questions mentioned above for these values of α . In particular, the results of [Wil15] provide stronger lower bounds for $\tilde{I}_\alpha(A; B|C)_\rho$ and $\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N})$ when $\alpha \in (1/2, 1)$. This was further improved in [JRS⁺15]. The results of [DW15] provide stronger lower bounds for $I_\alpha(A; B|C)_\rho$ and $\Delta_\alpha(\rho, \sigma, \mathcal{N})$ when $\alpha \in (0, 1)$.

Acknowledgements. We acknowledge helpful discussions with Mario Berta, Kaushik Sethadreesan, and Marco Tomamichel and thank Perla Sousi for a helpful discussion on Markov chains. We are especially grateful to Milan Mosonyi for his very careful reading of our paper, pointing out a problem with our former justification of Lemma 9 for $\alpha \in (1, 2)$, and for communicating many other observations about our paper that go beyond those stated here. We are indebted to an anonymous referee for suggesting to use Choi's inequality to prove some of the statements in Lemma 9. We acknowledge support from the Peter Whittle Fund, which helped to enable this research. MMW is grateful to the Statistical Laboratory in the Center for Mathematical Sciences at the University of Cambridge for hosting him for a research visit during January 2015. MMW acknowledges support from startup funds from the Department of Physics and Astronomy at LSU, the NSF under Award No. CCF-1350397, and the DARPA Quiness Program through US Army Research Office award W31P4Q-12-1-0019.

A Alternative Proof of Non-Negativity of Δ_α and $\tilde{\Delta}_\alpha$ quantities

Here we provide an alternative proof of the non-negativity of the Δ_α and $\tilde{\Delta}_\alpha$ quantities, which appeared in the original preprint version of our work:

Proposition 20 *Let $\rho \in \mathcal{S}(\mathcal{H})_{++}$, $\sigma \in \mathcal{B}(\mathcal{H})_{++}$, and let \mathcal{N} be a strict CPTP map. Then (3.56) holds for $\alpha \in (1, 2)$ and equality occurs if and only if*

$$\left[\mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N} \left(\sigma^{\frac{1-\alpha}{2}} \rho^\alpha \sigma^{\frac{1-\alpha}{2}} \right) \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right]^{\frac{1}{\alpha}} = \mathcal{N}(\rho). \quad (\text{A.1})$$

Furthermore, for the same choices of ρ , σ , and \mathcal{N} , (3.58) holds for $\alpha \in (1, \infty)$.

Proof. We first prove the non-negativity of $\Delta_\alpha(\rho, \sigma, \mathcal{N})$ for $\alpha \in (1, 2)$, $\rho \in \mathcal{S}(\mathcal{H})_{++}$, $\sigma \in \mathcal{B}(\mathcal{H})_{++}$, and \mathcal{N} a strict CPTP map. Using the definition (1.16) of $\Delta_\alpha(\rho, \sigma, \mathcal{N})$, cyclicity of trace, and the definition of the adjoint map, we can express $\Delta_\alpha(\rho, \sigma, \mathcal{N})$ as

$$\Delta_\alpha(\rho, \sigma, \mathcal{N}) = \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N} \left(\sigma^{\frac{1-\alpha}{2}} \rho^\alpha \sigma^{\frac{1-\alpha}{2}} \right) \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N}(\rho)^{1-\alpha} \right\} \quad (\text{A.2})$$

$$= D_\alpha \left(\left[\mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N} \left(\sigma^{\frac{1-\alpha}{2}} \rho^\alpha \sigma^{\frac{1-\alpha}{2}} \right) \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right]^{\frac{1}{\alpha}} \middle\| \middle\| \mathcal{N}(\rho) \right) \quad (\text{A.3})$$

where

$$D_\alpha(P\|Q) \equiv \frac{1}{\alpha - 1} \log \text{Tr} \{ P^\alpha Q^{1-\alpha} \} \quad (\text{A.4})$$

is the Rényi relative entropy between two positive definite operators P and Q . It is known that $D_\alpha(P\|Q) \geq 0$ if P and Q are such that

$$\text{Tr} \{ P \} \geq \text{Tr} \{ Q \} = 1. \quad (\text{A.5})$$

This is because $D_\alpha(P\|Q)$ is monotone under quantum channels for $\alpha \in (1, 2)$, and one such quantum channel is the trace operation:

$$D_\alpha(P\|Q) \geq D_\alpha(\text{Tr}\{P\} \|\text{Tr}\{Q\}) \quad (\text{A.6})$$

$$= \frac{1}{\alpha - 1} \log \left[[\text{Tr}\{P\}]^\alpha [\text{Tr}\{Q\}]^{1-\alpha} \right] \quad (\text{A.7})$$

$$\geq 0. \quad (\text{A.8})$$

If $D_\alpha(P\|Q) = 0$ for some $\alpha \in (1, 2)$ and P and Q such that $\text{Tr}\{P\} \geq \text{Tr}\{Q\}$, then it is known that $P = Q$ (one can deduce this, e.g., from the above and [HMPB11, Theorem 5.1]). Hence, to prove the non-negativity of $\Delta_\alpha(\rho, \sigma, \mathcal{N})$ for $\alpha \in (1, 2)$, it suffices to prove that

$$\text{Tr} \left\{ \left[\mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N} \left(\sigma^{\frac{1-\alpha}{2}} \rho^\alpha \sigma^{\frac{1-\alpha}{2}} \right) \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right]^{\frac{1}{\alpha}} \right\} \geq \text{Tr}\{\mathcal{N}(\rho)\} = 1. \quad (\text{A.9})$$

Theorem 1.1 of [Hia13] establishes that the map

$$(P, Q) \mapsto \text{Tr} \left\{ \left[Q^{\frac{\alpha-1}{2}} P Q^{\frac{\alpha-1}{2}} \right]^{\frac{1}{\alpha}} \right\} \quad (\text{A.10})$$

is jointly concave for positive definite P and Q when $\alpha \in (1, 2)$. A straightforward argument allows to conclude its joint concavity for $\alpha \in (1, 2)$ and positive semidefinite P and Q . Indeed, let $\varepsilon > 0$, $\{P_x\}$ and $\{Q_x\}$ be sets of positive semidefinite operators, let $p_X(x)$ be a probability distribution, and let $\bar{P} \equiv \sum_x p_X(x) P_x$ and $\bar{Q} \equiv \sum_x p_X(x) Q_x$. Consider that

$$\begin{aligned} & \sum_x p_X(x) \text{Tr} \left\{ \left[Q_x^{\frac{\alpha-1}{2}} P_x Q_x^{\frac{\alpha-1}{2}} \right]^{\frac{1}{\alpha}} \right\} \\ & \leq \sum_x p_X(x) \text{Tr} \left\{ \left[Q_x^{\frac{\alpha-1}{2}} (P_x + \varepsilon I) Q_x^{\frac{\alpha-1}{2}} \right]^{\frac{1}{\alpha}} \right\} \\ & = \sum_x p_X(x) \text{Tr} \left\{ \left[(P_x + \varepsilon I)^{\frac{1}{2}} Q_x^{\alpha-1} (P_x + \varepsilon I)^{\frac{1}{2}} \right]^{\frac{1}{\alpha}} \right\} \\ & \leq \sum_x p_X(x) \text{Tr} \left\{ \left[(P_x + \varepsilon I)^{\frac{1}{2}} (Q_x + \varepsilon I)^{\alpha-1} (P_x + \varepsilon I)^{\frac{1}{2}} \right]^{\frac{1}{\alpha}} \right\} \\ & = \sum_x p_X(x) \text{Tr} \left\{ \left[(Q_x + \varepsilon I)^{\frac{\alpha-1}{2}} (P_x + \varepsilon I) (Q_x + \varepsilon I)^{\frac{\alpha-1}{2}} \right]^{\frac{1}{\alpha}} \right\} \\ & \leq \text{Tr} \left\{ \left[(\bar{Q} + \varepsilon I)^{\frac{\alpha-1}{2}} (\bar{P} + \varepsilon I) (\bar{Q} + \varepsilon I)^{\frac{\alpha-1}{2}} \right]^{\frac{1}{\alpha}} \right\}. \end{aligned} \quad (\text{A.11})$$

The first inequality follows because $P_x \leq P_x + \varepsilon I$ and because $\text{Tr}\{f(A)\} \leq \text{Tr}\{f(B)\}$ for $A \leq B$ and $f(x) = x^{1/\alpha}$ a monotone non-decreasing function on $[0, \infty)$. The next inequality follows because $x^{\alpha-1}$ is operator monotone for $\alpha \in (1, 2)$ and for the same reason as above. The final inequality is a consequence of Theorem 1.1 of [Hia13]. By taking the limit $\varepsilon \searrow 0$, we can conclude that

$$\sum_x p_X(x) \text{Tr} \left\{ \left[Q_x^{\frac{\alpha-1}{2}} P_x Q_x^{\frac{\alpha-1}{2}} \right]^{\frac{1}{\alpha}} \right\} \leq \text{Tr} \left\{ \left[\bar{Q}^{(\alpha-1)/2} \bar{P} \bar{Q}^{(\alpha-1)/2} \right]^{\frac{1}{\alpha}} \right\}. \quad (\text{A.12})$$

Now consider the following chain of equalities:

$$\begin{aligned}
1 &= \text{Tr} \left\{ \left[\sigma^{\frac{\alpha-1}{2}} \sigma^{\frac{1-\alpha}{2}} \rho^\alpha \sigma^{\frac{1-\alpha}{2}} \sigma^{\frac{\alpha-1}{2}} \right]^{\frac{1}{\alpha}} \right\} \\
&= \text{Tr} \left\{ \left[\sigma^{\frac{\alpha-1}{2}} \sigma^{\frac{1-\alpha}{2}} \rho^\alpha \sigma^{\frac{1-\alpha}{2}} \sigma^{\frac{\alpha-1}{2}} \otimes |0\rangle\langle 0|_E \right]^{\frac{1}{\alpha}} \right\} \\
&= \text{Tr} \left\{ \left[\left(\sigma^{\frac{\alpha-1}{2}} \otimes |0\rangle\langle 0|_E \right) \left(\sigma^{\frac{1-\alpha}{2}} \rho^\alpha \sigma^{\frac{1-\alpha}{2}} \otimes |0\rangle\langle 0|_E \right) \left(\sigma^{\frac{\alpha-1}{2}} \otimes |0\rangle\langle 0|_E \right) \right]^{\frac{1}{\alpha}} \right\} \\
&= \text{Tr} \left\{ \left[U \left(\sigma^{\frac{\alpha-1}{2}} \otimes |0\rangle\langle 0|_E \right) U^\dagger U \left(\sigma^{\frac{1-\alpha}{2}} \rho^\alpha \sigma^{\frac{1-\alpha}{2}} \otimes |0\rangle\langle 0|_E \right) U^\dagger U \left(\sigma^{\frac{\alpha-1}{2}} \otimes |0\rangle\langle 0|_E \right) U^\dagger \right]^{\frac{1}{\alpha}} \right\} \\
&= \text{Tr} \left\{ \left[\left[U \left(\sigma \otimes |0\rangle\langle 0|_E \right) U^\dagger \right]^{\frac{\alpha-1}{2}} U \left(\sigma^{\frac{1-\alpha}{2}} \rho^\alpha \sigma^{\frac{1-\alpha}{2}} \otimes |0\rangle\langle 0|_E \right) U^\dagger \left[U \left(\sigma \otimes |0\rangle\langle 0|_E \right) U^\dagger \right]^{\frac{\alpha-1}{2}} \right]^{\frac{1}{\alpha}} \right\} \\
&= \frac{1}{d_{E'}^2} \sum_i \text{Tr} \left\{ \left[\left[K^i \right]^{\frac{\alpha-1}{2}} U_{E'}^i U \left(\sigma^{\frac{1-\alpha}{2}} \rho^\alpha \sigma^{\frac{1-\alpha}{2}} \otimes |0\rangle\langle 0|_E \right) U^\dagger U_{E'}^{i\dagger} \left[K^i \right]^{\frac{\alpha-1}{2}} \right]^{\frac{1}{\alpha}} \right\} \tag{A.13}
\end{aligned}$$

where

$$K^i \equiv U_{E'}^i U \left(\sigma \otimes |0\rangle\langle 0|_E \right) U^\dagger U_{E'}^{i\dagger}, \tag{A.14}$$

with $\{U_{E'}^i\}$ a set of Heisenberg-Weyl operators. Then by the concavity result (A.12) above for positive semidefinite P and Q , it follows that the right-hand side of (A.13) is no larger than

$$\begin{aligned}
&\text{Tr} \left\{ \left[\left[\mathcal{N}(\sigma) \otimes \pi_{E'} \right]^{\frac{\alpha-1}{2}} \left[\mathcal{N} \left(\sigma^{\frac{1-\alpha}{2}} \rho^\alpha \sigma^{\frac{1-\alpha}{2}} \right) \otimes \pi_{E'} \right] \left[\mathcal{N}(\sigma) \otimes \pi_{E'} \right]^{\frac{\alpha-1}{2}} \right]^{\frac{1}{\alpha}} \right\} \\
&= \text{Tr} \left\{ \left[\mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N} \left(\sigma^{\frac{1-\alpha}{2}} \rho^\alpha \sigma^{\frac{1-\alpha}{2}} \right) \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right]^{\frac{1}{\alpha}} \right\}. \tag{A.15}
\end{aligned}$$

Thus we obtain the inequality

$$\text{Tr} \left\{ \left[\mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N} \left(\sigma^{\frac{1-\alpha}{2}} \rho^\alpha \sigma^{\frac{1-\alpha}{2}} \right) \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right]^{\frac{1}{\alpha}} \right\} \geq 1. \tag{A.16}$$

This completes the proof of (3.56) for $\alpha \in (1, 2)$. The equality condition in (A.1) follows from the representation in (A.3) and the equality condition stated after (A.8).

Next we prove that $\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}) \geq 0$ for $\alpha \in (1, \infty)$. We start with the definition (1.17), which we repeat here for convenience:

$$\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}) \equiv \frac{\alpha}{\alpha-1} \log \left\| \left\| \rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha}{2\alpha}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \mathcal{N}(\rho)^{\frac{1-\alpha}{\alpha}} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \right) \sigma^{\frac{1-\alpha}{2\alpha}} \rho^{\frac{1}{2}} \right\| \right\|_\alpha. \tag{A.17}$$

From [MLDS⁺13, Lemma 12], it follows that the right-hand side of (A.17) can be written as

$$\begin{aligned}
&\left\| \left\| \rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha}{2\alpha}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \mathcal{N}(\rho)^{\frac{1-\alpha}{\alpha}} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \right) \sigma^{\frac{1-\alpha}{2\alpha}} \rho^{\frac{1}{2}} \right\| \right\|_\alpha \\
&= \sup_{\tau \geq 0, \text{Tr}\{\tau\} \leq 1} \text{Tr} \left\{ \rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha}{2\alpha}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \mathcal{N}(\rho)^{\frac{1-\alpha}{\alpha}} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \right) \sigma^{\frac{1-\alpha}{2\alpha}} \rho^{\frac{1}{2}} \tau^{\frac{\alpha-1}{\alpha}} \right\}. \tag{A.18}
\end{aligned}$$

Now define

$$\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}; \tau) \equiv \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha}{2\alpha}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \mathcal{N}(\rho)^{\frac{1-\alpha}{\alpha}} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \right) \sigma^{\frac{1-\alpha}{2\alpha}} \rho^{\frac{1}{2}} \tau^{\frac{\alpha-1}{\alpha}} \right\}. \quad (\text{A.19})$$

Let us focus on the trace functional in the above equation:

$$\begin{aligned} & \text{Tr} \left\{ \rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha}{2\alpha}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \mathcal{N}(\rho)^{\frac{1-\alpha}{\alpha}} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \right) \sigma^{\frac{1-\alpha}{2\alpha}} \rho^{\frac{1}{2}} \tau^{\frac{\alpha-1}{\alpha}} \right\} \\ &= \text{Tr} \left\{ \sigma^{\frac{1-\alpha}{2\alpha}} \rho^{\frac{1}{2}} \tau^{\frac{\alpha-1}{\alpha}} \rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha}{2\alpha}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \mathcal{N}(\rho)^{\frac{1-\alpha}{\alpha}} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \right) \right\} \\ &= \text{Tr} \left\{ \mathcal{N} \left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho^{\frac{1}{2}} \tau^{\frac{\alpha-1}{\alpha}} \rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha}{2\alpha}} \right) \mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \mathcal{N}(\rho)^{\frac{1-\alpha}{\alpha}} \mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \right\} \\ &= \text{Tr} \left\{ \mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \mathcal{N} \left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho^{\frac{1}{2}} \tau^{\frac{\alpha-1}{\alpha}} \rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha}{2\alpha}} \right) \mathcal{N}(\sigma)^{\frac{\alpha-1}{2\alpha}} \mathcal{N}(\rho)^{\frac{1-\alpha}{\alpha}} \right\}. \end{aligned} \quad (\text{A.20})$$

By making the substitution $\alpha' \equiv (2\alpha - 1)/\alpha$, so that $(1 - \alpha)/\alpha = 1 - \alpha'$ and thus $\alpha' \in (1, 2)$ when $\alpha \in (1, \infty)$, we see that the last line above is equal to

$$\text{Tr} \left\{ \mathcal{N}(\sigma)^{\frac{\alpha'-1}{2}} \mathcal{N} \left(\sigma^{\frac{1-\alpha'}{2}} \rho^{\frac{1}{2}} \tau^{(\alpha'-1)} \rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha'}{2}} \right) \mathcal{N}(\sigma)^{\frac{\alpha'-1}{2}} \mathcal{N}(\rho)^{(1-\alpha')} \right\}. \quad (\text{A.21})$$

Observe that this is similar to (A.2). Hence we can write $\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}; \tau)$ as

$$\begin{aligned} & \frac{1}{\alpha' - 1} \log \text{Tr} \left\{ \mathcal{N}(\sigma)^{\frac{\alpha'-1}{2}} \mathcal{N} \left(\sigma^{\frac{1-\alpha'}{2}} \rho^{\frac{1}{2}} \tau^{(\alpha'-1)} \rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha'}{2}} \right) \mathcal{N}(\sigma)^{\frac{\alpha'-1}{2}} \mathcal{N}(\rho)^{(1-\alpha')} \right\} \\ &= D_{\alpha'} \left(\left[\mathcal{N}(\sigma)^{\frac{\alpha'-1}{2}} \mathcal{N} \left(\sigma^{\frac{1-\alpha'}{2}} \rho^{\frac{1}{2}} \tau^{(\alpha'-1)} \rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha'}{2}} \right) \mathcal{N}(\sigma)^{\frac{\alpha'-1}{2}} \right]^{\frac{1}{\alpha'}} \left\| \mathcal{N}(\rho) \right\| \right). \end{aligned} \quad (\text{A.22})$$

This implies that

$$\begin{aligned} & \tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}) \\ &= \sup_{\tau \geq 0, \text{Tr}\{\tau\} \leq 1} D_{\alpha'} \left(\left[\mathcal{N}(\sigma)^{\frac{\alpha'-1}{2}} \mathcal{N} \left(\sigma^{\frac{1-\alpha'}{2}} \rho^{\frac{1}{2}} \tau^{\alpha'-1} \rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha'}{2}} \right) \mathcal{N}(\sigma)^{\frac{\alpha'-1}{2}} \right]^{\frac{1}{\alpha'}} \left\| \mathcal{N}(\rho) \right\| \right) \end{aligned} \quad (\text{A.23})$$

$$\geq D_{\alpha'} \left(\left[\mathcal{N}(\sigma)^{\frac{\alpha'-1}{2}} \mathcal{N} \left(\sigma^{\frac{1-\alpha'}{2}} \rho^{\frac{1}{2}} \rho^{(\alpha'-1)} \rho^{\frac{1}{2}} \sigma^{\frac{1-\alpha'}{2}} \right) \mathcal{N}(\sigma)^{\frac{\alpha'-1}{2}} \right]^{\frac{1}{\alpha'}} \left\| \mathcal{N}(\rho) \right\| \right) \quad (\text{A.24})$$

$$= D_{\alpha'} \left(\left[\mathcal{N}(\sigma)^{\frac{\alpha'-1}{2}} \mathcal{N} \left(\sigma^{\frac{1-\alpha'}{2}} \rho^{\alpha'} \sigma^{\frac{1-\alpha'}{2}} \right) \mathcal{N}(\sigma)^{\frac{\alpha'-1}{2}} \right]^{\frac{1}{\alpha'}} \left\| \mathcal{N}(\rho) \right\| \right) \quad (\text{A.25})$$

$$= \Delta_{\alpha'}(\rho, \sigma, \mathcal{N}) \quad (\text{A.26})$$

$$\geq 0, \quad (\text{A.27})$$

where the first inequality follows by setting $\tau = \rho$ and the last inequality follows because we have already shown that $\Delta_{\alpha'}(\rho, \sigma, \mathcal{N}) \geq 0$ for $\alpha' \in (1, 2)$. The above inequality also demonstrates that

$$\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}) \geq \Delta_{\alpha'}(\rho, \sigma, \mathcal{N}). \quad (\text{A.28})$$

■

We can use this result to give an alternative proof of the fact that \mathcal{N} is sufficient for ρ and σ if $\Delta_\alpha(\rho, \sigma, \mathcal{N}) = 0$ for some $\alpha \in (1, 2)$ or if $\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}) = 0$ for some $\alpha \in (1, \infty)$:

Proof. If for some $\alpha \in (1, 2)$ we have $\Delta_\alpha(\rho, \sigma, \mathcal{N}) = 0$, then we know that

$$\left[\mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \mathcal{N} \left(\sigma^{\frac{1-\alpha}{2}} \rho^\alpha \sigma^{\frac{1-\alpha}{2}} \right) \mathcal{N}(\sigma)^{\frac{\alpha-1}{2}} \right]^{\frac{1}{\alpha}} = \mathcal{N}(\rho) \quad (\text{A.29})$$

$$\iff \mathcal{N} \left(\sigma^{\frac{1-\alpha}{2}} \rho^\alpha \sigma^{\frac{1-\alpha}{2}} \right) = \mathcal{N}(\sigma)^{\frac{1-\alpha}{2}} \mathcal{N}(\rho)^\alpha \mathcal{N}(\sigma)^{\frac{1-\alpha}{2}} \quad (\text{A.30})$$

This implies that

$$\text{Tr} \left\{ \mathcal{N} \left(\sigma^{\frac{1-\alpha}{2}} \rho^\alpha \sigma^{\frac{1-\alpha}{2}} \right) \right\} = \text{Tr} \left\{ \mathcal{N}(\rho)^\alpha \mathcal{N}(\sigma)^{1-\alpha} \right\} \quad (\text{A.31})$$

which in turn implies that

$$\text{Tr} \left\{ \rho^\alpha \sigma^{1-\alpha} \right\} = \text{Tr} \left\{ \mathcal{N}(\rho)^\alpha \mathcal{N}(\sigma)^{1-\alpha} \right\}, \quad (\text{A.32})$$

because \mathcal{N} is trace preserving. This is then an equality of f -divergences, which we can use to conclude the sufficiency property as in previous proofs.

If $\tilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}) = 0$, for some $\alpha \in (1, \infty)$, then from (A.28) we have that $\Delta_{\alpha'}(\rho, \sigma, \mathcal{N}) = 0$ for some $\alpha' \in (1, 2)$. Then we know from the above analysis that \mathcal{N} is sufficient for ρ and σ . ■

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