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# Classical codes for quantum broadcast channels

Ivan Savov and Mark M. Wilde

## Abstract

We present two approaches for transmitting classical information over quantum broadcast channels. The first technique is a quantum generalization of the superposition coding scheme for the classical broadcast channel. We use a quantum simultaneous nonunique decoder and obtain a proof of the rate region stated in [Yard *et al.*, IEEE Trans. Inf. Theory 57 (10), 2011]. Our second result is a quantum generalization of the Marton coding scheme. The error analysis for the quantum Marton region makes use of ideas in our earlier work and an idea recently presented by Radhakrishnan *et al.* in arXiv:1410.3248. Both results exploit recent advances in quantum simultaneous decoding developed in the context of quantum interference channels.

## I. INTRODUCTION

How can a broadcast station communicate separate messages to two receivers using a single antenna? Two well known strategies [1] for transmitting information over broadcast channels are superposition coding [2], [3] and Marton multicoding using correlated auxiliary random variables [4]. In this paper, we prove that these strategies can be adapted to the quantum setting by constructing random codebooks and matching decoding measurements that have asymptotically vanishing error in the limit of many uses of the channel.

Sending classical data over a quantum channel is one of the fundamental problems of quantum information theory [5]. Single-letter formulas are known for classical-quantum point-to-point channels [6], [7] and multiple access channels [8]. Classical-quantum channels are a useful abstraction for studying general quantum channels and correspond to the transmitters being restricted to classical encodings. Codes for classical-quantum channels (c-q channels), when augmented with an extra optimization over the possible input states, directly generalize to codes for quantum channels. Furthermore, it is known that classical encoding (coherent-state encoding using classical Gaussian codebooks) is sufficient to achieve the capacity of phase-insensitive quantum Gaussian channels, which is a realistic model for optical communication links [9], [10], [11].

Previous work on quantum broadcast channels includes [12], [13], [14]. Yard *et al.* consider both classical and quantum communication over quantum broadcast channels and state a superposition coding inner bound in their Theorem 1 similar to that stated in our Theorem 1 [12]. However, it is unclear to us whether the proof given for their Theorem 1 is complete (we elaborate on this point in what follows). Relying on Theorem 1 of [12], Ref. [13] discusses classical communication over a bosonic broadcast channel. Ref. [14] establishes a Marton rate region for quantum communication.

In this paper, we derive two achievable rate regions for classical-quantum broadcast channels by exploiting error analysis techniques developed in the context of quantum interference channels [15], [16]. In Section III, we prove achievability of the superposition coding inner bound (Theorem 1), by using a quantum simultaneous nonunique decoder at one of the receivers. In Section IV we prove that the quantum Marton rate region with no common message is achievable (Theorem 3). In the Marton coding scheme, the sub-channels to each receiver are essentially point-to-point, but it turns out that two techniques which we call the “projector trick” and “overcounting” [17] seem to be necessary in the proof. We discuss open problems and give an outlook for the future in Section V.

I.S. and M.M.W. were with the School of Computer Science, McGill University, Montréal, Québec, Canada when conducting parts of this research. I.S. is now with Minireference Publishing. M.M.W. is now with the Hearne Institute for Theoretical Physics, the Department of Physics and Astronomy, and the Center for Computation and Technology at Louisiana State University, Baton Rouge, LA 70803. This work was presented in part at the 2012 IEEE International Symposium on Information Theory.

*Note:* The original justification for the quantum Marton region given in our earlier work [18] contained a gap, which was identified by Pranab Sen and relayed to us by Andreas Winter. This gap was addressed in the related paper [17], where an achievable region in the “one-shot” Marton coding setting was established. Here we show how to apply the overcounting method in order to close the aforementioned gap in our earlier work.

## II. PRELIMINARIES

1) *Notation:* We denote classical random variables as  $X, U, W$ , whose realizations are elements of the respective finite alphabets  $\mathcal{X}, \mathcal{U}, \mathcal{W}$ . Let  $p_X, p_U, p_W$  denote their corresponding probability distributions. We denote quantum systems as  $A, B$ , and  $C$  and their corresponding Hilbert spaces as  $\mathcal{H}^A, \mathcal{H}^B$ , and  $\mathcal{H}^C$ . We represent quantum states of a system  $A$  with a density operator  $\rho^A$ , which is a positive semi-definite operator with unit trace. Let  $H(A)_\rho \equiv -\text{Tr}\{\rho^A \log \rho^A\}$  denote the von Neumann entropy of the state  $\rho^A$ . A classical-quantum channel,  $\mathcal{N}^{X \rightarrow B}$ , is represented by the set of  $|\mathcal{X}|$  possible output states  $\{\rho_x^B \equiv \mathcal{N}^{X \rightarrow B}(x)\}$ , meaning that a classical input of  $x$  leads to a quantum output  $\rho_x^B$ . In a communication scenario, the decoding operations performed by the receivers correspond to quantum measurements on the outputs of the channel. A quantum measurement is a positive operator-valued measure (POVM)  $\{\Lambda_m\}_{m \in \{1, \dots, |\mathcal{M}|\}}$  on the system  $B^n$ , the output of which we denote  $M'$ . To be a valid POVM, the set of  $|\mathcal{M}|$  operators  $\Lambda_m$  must all be positive semi-definite and sum to the identity:  $\Lambda_m \geq 0$ ,  $\sum_m \Lambda_m = I$ .

2) *Definitions and useful lemmas:* We define a classical-quantum-quantum broadcast channel as the following map:

$$x \rightarrow \rho_x^{B_1 B_2}, \quad (1)$$

where  $x$  is a classical letter in an alphabet  $\mathcal{X}$  and  $\rho_x^{B_1 B_2}$  is a density operator on the tensor product Hilbert space for systems  $B_1$  and  $B_2$ . The model is such that when the sender inputs a classical letter  $x$ , Receiver 1 obtains system  $B_1$ , and Receiver 2 obtains system  $B_2$ . Since Receiver 1 does not have access to the  $B_2$  part of the state  $\rho_x^{B_1 B_2}$ , we model his state as  $\rho_x^{B_1} = \text{Tr}_{B_2} [\rho_x^{B_1 B_2}]$ , where  $\text{Tr}_{B_2}$  denotes the partial trace over Receiver 2's system.

**Lemma 1** (Gentle Operator Lemma for Ensembles [19]). *Given an ensemble  $\{p_X(x), \rho_x\}$  with expected density operator  $\rho \equiv \sum_x p_X(x) \rho_x$ , suppose that an operator  $\Lambda$  such that  $I \geq \Lambda \geq 0$  succeeds with high probability on the state  $\rho$ :*

$$\text{Tr}\{\Lambda \rho\} \geq 1 - \varepsilon. \quad (2)$$

*Then the subnormalized state  $\sqrt{\Lambda} \rho \sqrt{\Lambda}$  is close in expected trace distance to the original state  $\rho$ :*

$$\mathbb{E}_X \left\{ \left\| \sqrt{\Lambda} \rho_X \sqrt{\Lambda} - \rho_X \right\|_1 \right\} \leq 2\sqrt{\varepsilon}. \quad (3)$$

The following lemma appears in [20, Lemma 2]. When using it for the square-root measurement in (8), we choose  $S = \Pi'_m$  and  $T = \sum_{k \neq m} \Pi'_k$ .

**Lemma 2** (Hayashi-Nagaoka). *The Hayashi-Nagaoka operator inequality applies to a positive operator  $T$  and an operator  $S$  where  $0 \leq S \leq I$ :*

$$I - (S + T)^{-\frac{1}{2}} S (S + T)^{-\frac{1}{2}} \leq 2(I - S) + 4T. \quad (4)$$

3) *Information processing task:* The task of communication over a broadcast channel is to use  $n$  independent instances of the channel in order to communicate with Receiver 1 at a rate  $R_1$  and to Receiver 2 at a rate  $R_2$ . More specifically, the sender chooses a pair of messages  $(m_1, m_2)$  from message sets  $\mathcal{M}_i \equiv \{1, 2, \dots, |\mathcal{M}_i|\}$ , where  $|\mathcal{M}_i| = 2^{nR_i}$ , and encodes these messages into an  $n$ -symbol codeword  $x^n(m_1, m_2) \in \mathcal{X}^n$  suitable as input for the  $n$  channel uses.

The output of the channel is a quantum state of the form:

$$\mathcal{N}^{\otimes n}(x^n(m_1, m_2)) \equiv \rho_{x^n(m_1, m_2)}^{B_1^n B_2^n}. \quad (5)$$

where  $\rho_{x^n}^{B_1^n B_2^n} \equiv \rho_{x_1}^{B_{11} B_{21}} \otimes \dots \otimes \rho_{x_n}^{B_{1n} B_{2n}}$ . To decode the message  $m_1$  intended for him, Receiver 1 performs a POVM  $\{\Lambda_{m_1}\}_{m_1 \in \{1, \dots, |\mathcal{M}_1|\}}$  on the system  $B_1^n$ , the output of which we denote  $M'_1$ . Receiver 2 similarly performs a POVM  $\{\Gamma_{m_2}\}_{m_2 \in \{1, \dots, |\mathcal{M}_2|\}}$  on the system  $B_2^n$ , and the random variable associated with the outcome is denoted  $M'_2$ .

An error occurs whenever either of the receivers decodes the message incorrectly. The probability of error for a particular message pair  $(m_1, m_2)$  is

$$p_e(m_1, m_2) \equiv \text{Tr} \left\{ (I - \Lambda_{m_1} \otimes \Gamma_{m_2}) \rho_{x^n(m_1, m_2)}^{B_1^n B_2^n} \right\}, \quad (6)$$

where the operator  $(I - \Lambda_{m_1} \otimes \Gamma_{m_2})$  represents the complement of the correct decoding outcome.

**Definition 1.** An  $(n, R_1, R_2, \varepsilon)$  broadcast channel code consists of a codebook  $\{x^n(m_1, m_2)\}_{m_1 \in \mathcal{M}_1, m_2 \in \mathcal{M}_2}$  and two decoding POVMs  $\{\Lambda_{m_1}\}_{m_1 \in \mathcal{M}_1}$  and  $\{\Gamma_{m_2}\}_{m_2 \in \mathcal{M}_2}$  such that the average probability of error  $\bar{p}_e$  is bounded from above as

$$\bar{p}_e \equiv \frac{1}{|\mathcal{M}_1||\mathcal{M}_2|} \sum_{m_1, m_2} p_e(m_1, m_2) \leq \varepsilon. \quad (7)$$

A rate pair  $(R_1, R_2)$  is *achievable* if there exists an  $(n, R_1 - \delta, R_2 - \delta, \varepsilon)$  quantum broadcast channel code for all  $\varepsilon, \delta > 0$  and sufficiently large  $n$ .

When devising coding strategies for c-q channels, the main obstacle to overcome is the construction of a decoding POVM that correctly decodes the messages. Given a set of positive operators  $\{\Pi'_m\}$  which are suitable for detecting each message, we can construct a POVM by *normalizing* them using the square-root measurement [6], [7]:

$$\Lambda_m \equiv \left( \sum_k \Pi'_k \right)^{-\frac{1}{2}} \Pi'_m \left( \sum_k \Pi'_k \right)^{-\frac{1}{2}}. \quad (8)$$

Thus, the search for a decoding POVM is reduced to the problem of finding positive operators  $\Pi'_m$  apt at detecting and distinguishing the output states produced by each of the possible input messages ( $\text{Tr}[\Pi'_m \rho_m] \geq 1 - \varepsilon'$  and  $\text{Tr}[\Pi'_m \rho_{m' \neq m}] \leq \varepsilon'$  for some small  $\varepsilon' > 0$ ).

### III. SUPERPOSITION CODING INNER BOUND

One possible strategy for the broadcast channel is to send a message at a rate that is low enough so that both receivers are able to decode. Furthermore, if we assume that Receiver 1 has a better reception signal, then the sender can encode a further message *superimposed* on top of the common message that Receiver 1 will be able to decode *given* the common message. The sender encodes the common message at rate  $R_2$  using a codebook generated from a probability distribution  $p_W(w)$ , and the additional message for Receiver 1 at rate  $R_1$  using a conditional codebook with distribution  $p_{X|W}(x|w)$ .

**Theorem 1** (Superposition coding inner bound). *A rate pair  $(R_1, R_2)$  is achievable for the quantum broadcast channel in (1) if it satisfies the following inequalities:*

$$R_1 \leq I(X; B_1|W)_\theta, \quad (9)$$

$$R_2 \leq I(W; B_2)_\theta, \quad (10)$$

$$R_1 + R_2 \leq I(X; B_1)_\theta, \quad (11)$$

where the above information quantities are with respect to a state  $\theta^{WXB_1B_2}$  of the form

$$\sum_{w,x} p_W(w) p_{X|W}(x|w) |w\rangle\langle w|^W \otimes |x\rangle\langle x|^X \otimes \rho_x^{B_1 B_2}. \quad (12)$$

It suffices to take the cardinality of the alphabet  $\mathcal{W}$  for  $W$  to be no larger than  $\min\{|\mathcal{X}|, |B_1|^2 + |B_2|^2 - 1\}$ , where  $\mathcal{X}$  is the input alphabet of the channel.

*Proof:* The idea of the proof given below is to exploit superposition encoding and a quantum simultaneous nonunique decoder for the decoding of the first receiver [2], [3]. We use a standard HSW decoder for the second receiver [6], [7]. The cardinality bound follows directly from Appendix A of [12].

**Codebook generation.** The sender randomly and independently generates  $M_2$  sequences  $w^n(m_2)$  according to the product distribution

$$p_{W^n}(w^n) \equiv \prod_{i=1}^n p_W(w_i). \quad (13)$$

For each sequence  $w^n(m_2)$ , the sender then randomly and conditionally independently generates  $M_1$  sequences  $x^n(m_1, m_2)$  according to the product distribution:

$$p_{X^n|W^n}(x^n|w^n(m_2)) \equiv \prod_{i=1}^n p_{X|W}(x_i|w_i(m_2)). \quad (14)$$

The sender then transmits the codeword  $x^n(m_1, m_2)$  if she wishes to send  $(m_1, m_2)$ .

**POVM Construction.** We now describe the POVMs that the receivers employ in order to decode the transmitted messages. First consider the state we obtain from (12) by tracing over the  $B_2$  system:

$$\rho^{WXB_1} = \sum_{w,x} p_W(w) p_{X|W}(x|w) |w\rangle\langle w|^W \otimes |x\rangle\langle x|^X \otimes \rho_x^{B_1}. \quad (15)$$

Further tracing over the  $X$  system gives

$$\rho^{WB_1} = \sum_w p_W(w) |w\rangle\langle w|^W \otimes \sigma_w^{B_1}, \quad (16)$$

where  $\sigma_w^{B_1} \equiv \sum_x p_{X|W}(x|w) \rho_x^{B_1}$ . For the first receiver, we exploit a square-root decoding POVM as in (8) based on the following positive operators:

$$\Pi'_{m_1, m_2} \equiv \Pi \Pi_{W^n(m_2)} \Pi_{X^n(m_1, m_2)} \Pi_{W^n(m_2)} \Pi, \quad (17)$$

where we have made the abbreviations

$$\Pi \equiv \Pi_{\rho, \delta}^{B_1^n}, \quad \Pi_{W^n(m_2)} \equiv \Pi_{\sigma_{W^n(m_2)}, \delta}^{B_1^n}, \quad \Pi_{X^n(m_1, m_2)} \equiv \Pi_{\rho_{X^n(m_1, m_2)}, \delta}^{B_1^n}. \quad (18)$$

The above projectors are weakly typical projectors [5, Section 14.2.1] defined with respect to the states  $\rho^{\otimes n}$ ,  $\sigma_{W^n(m_2)}^{B_1^n}$ , and  $\rho_{X^n(m_1, m_2)}^{B_1^n}$ .

Consider now the state in (12) as it looks from the point of view of Receiver 2. If we trace over the  $X$  and  $B_1$  systems, we obtain the following state:

$$\rho^{WB_2} = \sum_w p_W(w) |w\rangle\langle w|^W \otimes \sigma_w^{B_2}, \quad (19)$$

where  $\sigma_w^{B_2} \equiv \sum_x p_{X|W}(x|w) \rho_x^{B_2}$ . For the second receiver, we exploit a standard HSW decoding POVM that is with respect to the above state—it is a square-root measurement as in (8), based on the following positive operators:

$$\Pi'^{B_2^n}_{m_2} = \Pi_{\rho, \delta}^{B_2^n} \Pi_{\sigma_{W^n(m_2)}, \delta}^{B_2^n} \Pi_{\rho, \delta}^{B_2^n}, \quad (20)$$

where the above projectors are weakly typical projectors defined with respect to  $\rho^{\otimes n}$  and  $\sigma_{W^n(m_2)}^{B_2^n}$ .

**Error analysis.** We now analyze the expectation of the average error probability for the first receiver with the POVM defined by (8) and (17):

$$\begin{aligned} \mathbb{E}_{X^n, W^n} \left\{ \frac{1}{M_1 M_2} \sum_{m_1, m_2} \text{Tr} \left\{ (I - \Gamma_{m_1, m_2}^{B_1^n}) \rho_{X^n(m_1, m_2)}^{B_1} \right\} \right\} \\ = \frac{1}{M_1 M_2} \sum_{m_1, m_2} \mathbb{E}_{X^n, W^n} \left\{ \text{Tr} \left\{ (I - \Gamma_{m_1, m_2}^{B_1^n}) \rho_{X^n(m_1, m_2)}^{B_1} \right\} \right\}. \quad (21) \end{aligned}$$

Due to the above exchange between the expectation and the average and the symmetry of the code construction (each codeword is selected randomly and independently), it suffices to analyze the expectation of the average error probability for the first message pair ( $m_1 = 1, m_2 = 1$ ), i.e., the last line above is equal to

$$\mathbb{E}_{X^n, W^n} \left\{ \text{Tr} \left\{ \left( I - \Gamma_{1,1}^{B_1^n} \right) \rho_{X^n(1,1)}^{B_1} \right\} \right\}. \quad (22)$$

Using the Hayashi-Nagaoka operator inequality (Lemma 2 in the appendix), we obtain the following upper bound on this term:

$$2 \mathbb{E}_{X^n, W^n} \left\{ \text{Tr} \left\{ \left( I - \Pi'_{1,1} \right) \rho_{X^n(1,1)}^{B_1} \right\} \right\} + 4 \sum_{(m_1, m_2) \neq (1,1)} \mathbb{E}_{X^n, W^n} \left\{ \text{Tr} \left\{ \Pi'_{m_1, m_2} \rho_{X^n(1,1)}^{B_1} \right\} \right\}. \quad (23)$$

We begin by bounding the first term above. Consider the following chain of inequalities:

$$\mathbb{E}_{X^n, W^n} \left\{ \text{Tr} \left\{ \Pi'_{1,1} \rho_{X^n(1,1)}^{B_1} \right\} \right\} = \mathbb{E}_{X^n, W^n} \left\{ \text{Tr} \left\{ \Pi \Pi_{W^n(1)} \Pi_{X^n(1,1)} \Pi_{W^n(1)} \Pi \rho_{X^n(1,1)}^{B_1} \right\} \right\} \quad (24)$$

$$\begin{aligned} &\geq \mathbb{E}_{X^n, W^n} \left\{ \text{Tr} \left\{ \Pi_{X^n(1,1)} \rho_{X^n(1,1)}^{B_1} \right\} \right\} - \mathbb{E}_{X^n, W^n} \left\{ \left\| \rho_{X^n(1,1)}^{B_1} - \Pi \rho_{X^n(1,1)}^{B_1} \Pi \right\|_1 \right\} \\ &\quad - \mathbb{E}_{X^n, W^n} \left\{ \left\| \rho_{X^n(1,1)}^{B_1} - \Pi_{W^n(1)} \rho_{X^n(1,1)}^{B_1} \Pi_{W^n(1)} \right\|_1 \right\} \end{aligned} \quad (25)$$

$$\geq 1 - \varepsilon - 4\sqrt{\varepsilon}, \quad (26)$$

where the first inequality follows from the inequality

$$\text{Tr} \{ \Lambda \rho \} \leq \text{Tr} \{ \Lambda \sigma \} + \|\rho - \sigma\|_1, \quad (27)$$

which holds for all subnormalized states  $\rho$  and  $\sigma$ , and  $\Lambda$  such that  $0 \leq \Lambda \leq I$ . The second inequality follows from the Gentle Operator Lemma for ensembles (see Lemma 1 in the appendix) and the properties of typical projectors for sufficiently large  $n$ .

We now focus on bounding the second term in (23). We can expand this term as follows:

$$\sum_{m_1 \neq 1} \mathbb{E}_{X^n, W^n} \left\{ \text{Tr} \left\{ \Pi'_{m_1, 1} \rho_{X^n(1,1)}^{B_1} \right\} \right\} + \sum_{\substack{m_1, \\ m_2 \neq 1}} \mathbb{E}_{X^n, W^n} \left\{ \text{Tr} \left\{ \Pi'_{m_1, m_2} \rho_{X^n(1,1)}^{B_1} \right\} \right\}. \quad (28)$$

Consider the first term in (28):

$$\sum_{m_1 \neq 1} \mathbb{E}_{X^n, W^n} \left\{ \text{Tr} \left\{ \Pi'_{m_1, 1} \rho_{X^n(1,1)}^{B_1} \right\} \right\} \quad (29)$$

$$= \sum_{m_1 \neq 1} \mathbb{E}_{X^n, W^n} \text{Tr} \left\{ \Pi \Pi_{W^n(1)} \Pi_{X^n(m_1,1)} \Pi_{W^n(1)} \Pi \rho_{X^n(1,1)}^{B_1} \right\} \quad (30)$$

$$\leq 2^{n[H(B_1|WX)+\delta]} \sum_{m_1 \neq 1} \mathbb{E}_{X^n, W^n} \left\{ \text{Tr} \left[ \Pi \Pi_{W^n(1)} \rho_{X^n(m_1,1)} \Pi_{W^n(1)} \Pi \rho_{X^n(1,1)}^{B_1} \right] \right\} \quad (31)$$

$$= 2^{n[H(B_1|WX)+\delta]} \sum_{m_1 \neq 1} \mathbb{E}_{W^n} \left\{ \text{Tr} \left[ \Pi_{W^n(1)} \mathbb{E}_{X^n|W^n} \left\{ \rho_{X^n(m_1,1)} \right\} \Pi_{W^n(1)} \Pi \mathbb{E}_{X^n|W^n} \left\{ \rho_{X^n(1,1)}^{B_1} \right\} \Pi \right] \right\} \quad (32)$$

$$= 2^{n[H(B_1|WX)+\delta]} \sum_{m_1 \neq 1} \mathbb{E}_{W^n} \left\{ \text{Tr} \left\{ \Pi \Pi_{W^n(1)} \sigma_{W^n(1)} \Pi_{W^n(1)} \Pi \sigma_{W^n(1)} \right\} \right\} \quad (33)$$

$$\leq 2^{n[H(B_1|WX)+\delta]} 2^{-n[H(B_1|W)-\delta]} \sum_{m_1 \neq 1} \mathbb{E}_{W^n} \left\{ \text{Tr} \left\{ \Pi \Pi_{W^n(1)} \Pi \sigma_{W^n(1)} \right\} \right\} \quad (34)$$

$$\leq 2^{n[H(B_1|WX)+\delta]} 2^{-n[H(B_1|W)-\delta]} \sum_{m_1 \neq 1} \mathbb{E}_{W^n} \left\{ \text{Tr} \left\{ \sigma_{W^n(1)} \right\} \right\} \quad (35)$$

$$\leq 2^{-n[I(X;B_1|W)-2\delta]} M_1. \quad (36)$$

The first inequality is due to the *projector trick* inequality [21], [16], [15] which states that

$$\Pi_{X^n(m_1,1)} \leq 2^{n[H(B_1|WX)+\delta]} \rho_{X^n(m_1,1)}^{B_1}. \quad (37)$$

Note that this inequality is a straightforward consequence of the following standard typicality operator inequality and the fact that  $\Pi_{X^n(m_1,1)}$  and  $\rho_{X^n(m_1,1)}^{B_1}$  commute:

$$2^{-n[H(B_1|WX)+\delta]} \Pi_{X^n(m_1,1)} \leq \Pi_{X^n(m_1,1)} \rho_{X^n(m_1,1)}^{B_1} \Pi_{X^n(m_1,1)}. \quad (38)$$

The second inequality follows from the properties of typical projectors:

$$\Pi_{W^n(1)} \sigma_{W^n(1)} \Pi_{W^n(1)} \leq 2^{-n[H(B_1|W)-\delta]} \Pi_{W^n(1)}. \quad (39)$$

Now consider the second term in (28):

$$\sum_{\substack{m_1, \\ m_2 \neq 1}} \mathbb{E}_{X^n, W^n} \left\{ \text{Tr} \left\{ \Pi'_{m_1, m_2} \rho_{X^n(1,1)}^{B_1} \right\} \right\} \quad (40)$$

$$= \sum_{\substack{m_1, \\ m_2 \neq 1}} \mathbb{E}_{X^n, W^n} \left\{ \text{Tr} \left[ \Pi \Pi_{W^n(m_2)} \Pi_{X^n(m_1, m_2)} \Pi_{W^n(m_2)} \Pi \rho_{X^n(1,1)}^{B_1} \right] \right\} \quad (41)$$

$$= \sum_{\substack{m_1, \\ m_2 \neq 1}} \text{Tr} \left[ \mathbb{E}_{X^n, W^n} \left\{ \Pi_{W^n(m_2)} \Pi_{X^n(m_1, m_2)} \Pi_{W^n(m_2)} \right\} \Pi \mathbb{E}_{X^n, W^n} \left\{ \rho_{X^n(1,1)}^{B_1} \right\} \Pi \right] \quad (42)$$

$$= \sum_{\substack{m_1, \\ m_2 \neq 1}} \text{Tr} \left\{ \mathbb{E}_{X^n, W^n} \left\{ \Pi_{W^n(m_2)} \Pi_{X^n(m_1, m_2)} \Pi_{W^n(m_2)} \right\} \Pi \rho^{\otimes n} \Pi \right\} \quad (43)$$

$$\leq 2^{-n[H(B_1)-\delta]} \sum_{\substack{m_1, \\ m_2 \neq 1}} \text{Tr} \left[ \mathbb{E}_{X^n, W^n} \left\{ \Pi_{W^n(m_2)} \Pi_{X^n(m_1, m_2)} \Pi_{W^n(m_2)} \right\} \Pi \right] \quad (44)$$

$$= 2^{-n[H(B_1)-\delta]} \sum_{\substack{m_1, \\ m_2 \neq 1}} \mathbb{E}_{X^n, W^n} \text{Tr} \left[ \Pi_{X^n(m_1, m_2)} \Pi_{W^n(m_2)} \Pi \Pi_{W^n(m_2)} \right] \quad (45)$$

$$\leq 2^{-n[H(B_1)-\delta]} \sum_{m_2 \neq 1, m_1} \mathbb{E}_{X^n, W^n} \left\{ \text{Tr} \left\{ \Pi_{X^n(m_1, m_2)} \right\} \right\} \quad (46)$$

$$\leq 2^{-n[H(B_1)-\delta]} 2^{n[H(B_1|WX)+\delta]} M_1 M_2 \quad (47)$$

$$= 2^{-n[I(WX; B_1)-2\delta]} M_1 M_2 \quad (48)$$

$$= 2^{-n[I(X; B_1)-2\delta]} M_1 M_2. \quad (49)$$

The equality  $I(WX; B_1) = I(X; B_1)$  follows from the way the codebook is constructed (i.e., the Markov chain  $W - X - B$ ), as discussed also in [16]. This completes the error analysis for the first receiver.

For the second receiver, the decoding error analysis follows from the HSW coding theorem. We now present this for completeness and tie the coding theorem together so that the sender and two receivers can agree on a strategy that has asymptotically vanishing error probability in the large  $n$  limit. The following bound holds for the expectation of the average error probability for the second receiver if  $n$  is sufficiently

large:

$$\mathbb{E}_{X^n, W^n} \left\{ \frac{1}{M_2} \sum_{m_2} \text{Tr} \left\{ (I - \Lambda_{m_2}^{B_2^n}) \rho_{X^n(m_1, m_2)}^{B_2^n} \right\} \right\} \quad (50)$$

$$= \mathbb{E}_{W^n} \left\{ \frac{1}{M_2} \sum_{m_2} \text{Tr} \left\{ (I - \Lambda_{m_2}^{B_2^n}) \mathbb{E}_{X^n|W^n} \left\{ \rho_{X^n(m_1, m_2)}^{B_2^n} \right\} \right\} \right\} \quad (51)$$

$$= \mathbb{E}_{W^n} \left\{ \frac{1}{M_2} \sum_{m_2} \text{Tr} \left\{ (I - \Lambda_{m_2}^{B_2^n}) \sigma_{W^n(m_2)}^{B_2^n} \right\} \right\} \quad (52)$$

$$\leq 2(\varepsilon + 2\sqrt{\varepsilon}) + 4 \left[ 2^{-n[I(W; B_2) - 2\delta]} M_2 \right], \quad (53)$$

where the last line follows from an analysis similar to that given above.

Putting everything together, the joint POVM performed by both receivers is of the form:

$$\Gamma_{m_1, m_2}^{B_1^n} \otimes \Lambda_{m_2'}^{B_2^n}, \quad (54)$$

and the expectation of the average error probability for both receivers is bounded from above as

$$\mathbb{E}_{X^n, W^n} \frac{1}{M_1 M_2} \sum_{m_1, m_2} \text{Tr} \left\{ (I - \Gamma_{m_1, m_2}^{B_1^n} \otimes \Lambda_{m_2}^{B_2^n}) \rho_{X^n(m_1, m_2)}^{B_1^n B_2^n} \right\} \quad (55)$$

$$\leq \mathbb{E}_{X^n, W^n} \left\{ \frac{1}{M_1 M_2} \sum_{m_1, m_2} \text{Tr} \left\{ (I - \Gamma_{m_1, m_2}^{B_1^n}) \rho_{X^n(m_1, m_2)}^{B_1^n} \right\} \right\} \quad (56)$$

$$+ \mathbb{E}_{X^n, W^n} \left\{ \frac{1}{M_1 M_2} \sum_{m_1, m_2} \text{Tr} \left\{ (I - \Lambda_{m_2}^{B_2^n}) \rho_{X^n(m_1, m_2)}^{B_2^n} \right\} \right\} \quad (57)$$

$$\leq 4\varepsilon + 12\sqrt{\varepsilon} + 4 \left[ 2^{-n[I(W; B_2) - 2\delta]} M_2 \right] \quad (58)$$

$$+ 4 \left[ 2^{-n[I(X; B_1|W) - 2\delta]} M_1 + 2^{-n[I(X; B_1) - 2\delta]} M_1 M_2 \right], \quad (59)$$

where the first inequality follows from the following ‘‘union bound’’ operator inequality:

$$I^{B_1^n B_2^n} - \Gamma_{m_1, m_2}^{B_1^n} \otimes \Lambda_{m_2}^{B_2^n} \leq (I^{B_1^n B_2^n} - \Gamma_{m_1, m_2}^{B_1^n} \otimes I^{B_2^n}) + (I^{B_1^n B_2^n} - I^{B_1^n} \otimes \Lambda_{m_2}^{B_2^n}), \quad (60)$$

and the second inequality follows from our previous estimates. Thus, as long as the sender chooses the message sizes  $M_1$  and  $M_2$  such that  $M_1 \leq 2^{n[I(X; B_1|W) - 3\delta]}$ ,  $M_2 \leq 2^{n[I(W; B_2) - 3\delta]}$ , and  $M_1 M_2 \leq 2^{n[I(X; B_1) - 3\delta]}$ , then there exists a particular code with asymptotically vanishing average error probability in the large  $n$  limit. ■

**Remark 2.** *It is unclear to us whether the proof of [12, Theorem 1] is complete. These authors begin their proof by claiming that the region in Theorem 1 is equivalent to the following region:*

$$R_1 \leq I(X; B_1|W)_\theta, \quad (61)$$

$$R_2 \leq I(W; B_2)_\theta, \quad (62)$$

$$R_2 \leq I(W; B_1)_\theta. \quad (63)$$

*The regions certainly intersect at the corner point associated with their successive decoding strategy, but the full regions for a fixed distribution do not coincide in general. The proof of [12, Theorem 1] demonstrates achievability of all rates in the rectangular part of Receiver 1’s  $(R_1, R_2)$  region given in our Theorem 1. With our simultaneous decoding non-unique decoding strategy, we can achieve any rate in the triangular part of this region as well, which could be useful if the first constraint above on Receiver 2 is looser than the second constraint above on Receiver 2. In such a case, the successive decoding strategy from [12, Theorem 1] would not be able to achieve the rate  $R_2$  if  $R_2 > I(W; B_1)$ , but the simultaneous*



decoding strategy can. It might be the case that the proof of [12, Theorem 1] could be completed by choosing particular coding distributions and taking unions over the resulting regions, but this is not discussed there.

#### IV. MARTON CODING SCHEME

We now prove that the Marton inner bound is achievable for quantum broadcast channels. The Marton scheme depends on auxiliary random variables  $U_1$  and  $U_2$ , *multicoding*, and the properties of strongly typical sequences and projectors. The proof depends on some ideas originally presented in [18] and critically on the ‘‘overcounting’’ technique recently presented in [17].

**Theorem 3** (Marton inner bound). *Let  $\{\rho_x^{B_1 B_2}\}$  be a classical-quantum broadcast channel and  $x = f(u_1, u_2)$  be a deterministic function. The following rate region is achievable:*

$$R_1 \leq I(U_1; B_1)_\theta, \quad (64)$$

$$R_2 \leq I(U_2; B_2)_\theta, \quad (65)$$

$$R_1 + R_2 \leq I(U_1; B_1)_\theta + I(U_2; B_2)_\theta - I(U_1; U_2)_\theta, \quad (66)$$

where the information quantities are with respect to the state:

$$\theta^{U_1 U_2 B_1 B_2} = \sum_{u_1, u_2} p(u_1, u_2) |u_1\rangle\langle u_1|^{U_1} \otimes |u_2\rangle\langle u_2|^{U_2} \otimes \rho_{f(u_1, u_2)}^{B_1 B_2}. \quad (67)$$

It suffices to take the cardinalities  $\mathcal{U}_1$  and  $\mathcal{U}_2$  of  $U_1$  and  $U_2$  to be no larger than the cardinality of the channel’s input alphabet  $\mathcal{X}$ : i.e.,  $|\mathcal{U}_1|, |\mathcal{U}_2| \leq |\mathcal{X}|$ .

Define the following states:

$$\rho_{f(u_1, u_2)}^{B_1} \equiv \text{Tr}_{B_2} \left\{ \rho_{f(u_1, u_2)}^{B_1 B_2} \right\}, \quad (68)$$

$$\omega_{u_1}^{B_1} \equiv \sum_{u_2} p_{U_2|U_1}(u_2|u_1) \rho_{f(u_1, u_2)}^{B_1}, \quad (69)$$

$$\bar{\rho}^{B_1} \equiv \sum_{u_1} p_{U_1}(u_1) \omega_{u_1}^{B_1}. \quad (70)$$

**Codebook construction.** Define two auxiliary indices  $l_1 \in \{1, \dots, L_1\}$  and  $l_2 \in \{1, \dots, L_2\}$ , and let  $\tilde{R}_1 = (\log L_1)/n$  and  $\tilde{R}_2 = (\log L_2)/n$ . For each  $l_1$ , generate a sequence  $u_1^n(l_1)$  independently and randomly according to the product distribution

$$p_{U_1^n}(u_1^n) \equiv \prod_{i=1}^n p_{U_1}(u_{1,i}). \quad (71)$$

Similarly, for each  $l_2$ , generate a sequence  $u_2^n(l_2)$  independently and randomly according to the product distribution

$$p_{U_2^n}(u_2^n) \equiv \prod_{i=1}^n p_{U_2}(u_{2,i}). \quad (72)$$

Partition the sequences  $u_1^n(l_1)$  into  $2^{nR_1}$  different bins, each of which we label as  $B_{m_1}$ . Partition the sequences  $u_2^n(l_2)$  into  $2^{nR_2}$  different bins, each of which we label as  $C_{m_2}$ . For each message pair, the sender selects a sequence pair  $(u_1^n(l_1), u_2^n(l_2)) \in (B_{m_1} \times C_{m_2}) \cap \mathcal{A}_{p_{U_1, U_2}, \delta}^n$ , where  $\mathcal{A}_{p_{U_1, U_2}, \delta}^n$  is the strongly typical set for  $p_{U_1, U_2}$ . The scheme is such that each sequence is taken from the appropriate bin and the sender demands that they are strongly jointly-typical (otherwise admitting failure by just sending the first sequence pair in the bin). The codebook  $x^n(m_1, m_2)$  is deterministically constructed from  $(u_1^n(l_1), u_2^n(l_2))$ , by applying the function  $x_i = f(u_{1,i}, u_{2,i})$ .

**Transmission.** Let  $\ell_1$  and  $\ell_2$  denote the pair of indices of the joint sequence  $(u_1^n(\ell_1), u_2^n(\ell_2))$  which are chosen as the codewords for the message pair  $(m_1, m_2)$ . Expressed in terms of these indices, the output of the channel is

$$\rho_{\ell_1, \ell_2} \equiv \bigotimes_{i=1}^n \rho_{f(u_{1,i}(\ell_1), u_{2,i}(\ell_2))}^{B_{1,i} B_{2,i}}. \quad (73)$$

We will also make the abbreviation

$$\rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_1^n B_2^n} \equiv \rho_{\ell_1, \ell_2}, \quad (74)$$

and furthermore define  $\rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_1^n}$  in the obvious way by taking the partial trace over  $B_2^n$ .

**Decoding.** The decoding POVM  $\{\Lambda_{l_1}\}_{l_1 \in \{1, \dots, L_1\}}$  for Receiver 1 is a square-root measurement as in (8) and based on the following positive operators:

$$\Gamma_{l_1} \equiv \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{u_1^n(l_1)}, \delta'} \Pi_{\bar{\rho}, \delta''}^n, \quad (75)$$

where  $\Pi_{\bar{\rho}, \delta''}^n$  is a strongly typical projector for the state  $\bar{\rho}^{\otimes n}$  and  $\Pi_{\omega_{u_1^n(l_1)}, \delta'}$  is a strong conditionally typical projector for the state  $\omega_{u_1^n(l_1)}$  (cf. [5, Chapter 14]). Having decoded  $l_1$  correctly and knowing the binning scheme, Receiver 1 can deduce the message  $m_1$  from the bin index. The decoding is essentially the same for Receiver 2 but using the appropriate states and induced conditionally typical projectors. Let  $\Lambda_{l_2}^{B_2^n}$  denote the resulting decoding POVM for Receiver 2.

**Error Analysis.** We begin by analyzing the case when  $(m_1, m_2) = (1, 1)$  and a fixed subcodebook. Let  $\ell_1$  and  $\ell_2$  denote the pair of indices of the joint sequence  $(u_1^n(\ell_1), u_2^n(\ell_2))$  which was chosen as the codeword for the message pair  $(1, 1)$ . If there is none, let  $\ell_1$  and  $\ell_2$  be the first pair in the bin. An error occurs if one or more of the following occurs:

- 1) Let  $\mathcal{E}_0$  be the event that  $((u_1^n(l_1), u_2^n(l_2)) \notin \mathcal{A}_{p_{U_1}, U_2, \delta}^n)$  for all  $(u_1^n(l_1), u_2^n(l_2)) \in B_{m_1} \times C_{m_2}$ .
- 2) The event  $\mathcal{E}_0^c$  occurs, and Receiver 1 decodes some other  $l'_1$  besides the transmitted  $\ell_1$  or Receiver 2 decodes some other  $l'_2$  besides the transmitted  $\ell_2$ .

We can write out an exact expression for the error probability of a fixed subcodebook explicitly as

$$\mathcal{I}(\mathcal{E}_0) + \mathcal{I}(\mathcal{E}_0^c) \text{Tr} \left\{ \left( I^{B_1^n B_2^n} - \Lambda_{\ell_1}^{B_1^n} \otimes \Lambda_{\ell_2}^{B_2^n} \right) \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_1^n B_2^n} \right\}. \quad (76)$$

The interpretation is that:

- 1) If there are no jointly typical pairs in the subcodebook, then an error occurs with probability one. So  $\mathcal{I}(\mathcal{E}_0)$  gives this contribution.
- 2) If there is at least one pair that is jointly typical (event  $\mathcal{E}_0^c$  occurs), let  $(\ell_1, \ell_2)$  denote the first one found when scanning in lexicographic order. This one is sent. The expression for the decoding error probability is exactly equal to

$$\text{Tr} \left\{ \left( I^{B_1^n B_2^n} - \Lambda_{\ell_1}^{B_1^n} \otimes \Lambda_{\ell_2}^{B_2^n} \right) \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_1^n B_2^n} \right\}. \quad (77)$$

Note from the ‘‘union bound’’ stated in (60), we get that

$$\text{Tr} \left\{ \left( I^{B_1^n B_2^n} - \Lambda_{\ell_1}^{B_1^n} \otimes \Lambda_{\ell_2}^{B_2^n} \right) \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_1^n B_2^n} \right\} \leq \text{Tr} \left\{ \left( I^{B_1^n} - \Lambda_{\ell_1}^{B_1^n} \right) \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_1^n} \right\} + \text{Tr} \left\{ \left( I^{B_2^n} - \Lambda_{\ell_2}^{B_2^n} \right) \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_2^n} \right\}. \quad (78)$$

So this means that we can bound (76) from above by

$$\mathcal{I}(\mathcal{E}_0) + \mathcal{I}(\mathcal{E}_0^c) \left[ \text{Tr} \left\{ \left( I^{B_1^n} - \Lambda_{\ell_1}^{B_1^n} \right) \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_1^n} \right\} + \text{Tr} \left\{ \left( I^{B_2^n} - \Lambda_{\ell_2}^{B_2^n} \right) \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_2^n} \right\} \right] \quad (79)$$

We now focus on the term

$$\text{Tr} \left\{ \left( I^{B_1^n} - \Lambda_{\ell_1}^{B_1^n} \right) \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_1^n} \right\}. \quad (80)$$

By applying the Hayashi-Nagaoka operator inequality, we can bound (80) from above by

$$\begin{aligned} & 2\text{Tr} \left\{ \left( I - \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{u_1^n(\ell_1), \delta'} \Pi_{\bar{\rho}, \delta''}^n} \right) \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_1^n} \right\} \\ & \quad + 4 \sum_{l'_1 \in B_{m_1} \wedge l'_1 \neq \ell_1} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{u_1^n(l'_1), \delta'} \Pi_{\bar{\rho}, \delta''}^n} \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_1^n} \right\} \\ & \quad 4 \sum_{l'_1 \notin B_{m_1}} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{u_1^n(l'_1), \delta'} \Pi_{\bar{\rho}, \delta''}^n} \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_1^n} \right\}. \end{aligned} \quad (81)$$

Consider that

$$\begin{aligned} & \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{u_1^n(\ell_1), \delta'} \Pi_{\bar{\rho}, \delta''}^n} \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_1^n} \right\} \\ & \geq \text{Tr} \left\{ \Pi_{\omega_{u_1^n(\ell_1), \delta'} \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_1^n} \right\} - \left\| \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_1^n} - \Pi_{\bar{\rho}, \delta''}^n \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_1^n} \Pi_{\bar{\rho}, \delta''}^n \right\|_1 \\ & \geq 1 - \varepsilon - 2\sqrt{\varepsilon'} \end{aligned} \quad (82)$$

$$\geq 1 - \varepsilon - 2\sqrt{\varepsilon'} \quad (83)$$

where these inequalities follow from Lemma 3 in the appendix, whenever  $u_1^n(\ell_1), u_2^n(\ell_2)$  are strongly jointly typical. We would like to remove the dependence of the second term in (81) on the chosen indices  $\ell_1$  and  $\ell_2$ . In order to do so, we bound the relevant expression using the ‘‘overcounting’’ idea from [17]:

$$\sum_{l'_1 \in B_{m_1} \wedge l'_1 \neq \ell_1} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{u_1^n(l'_1), \delta'} \Pi_{\bar{\rho}, \delta''}^n} \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_1^n} \right\} \quad (84)$$

$$= \sum_{l_1 \in B_{m_1}, l_2 \in C_{m_2}} \mathcal{I}(l_1 = \ell_1, l_2 = \ell_2) \sum_{l'_1 \in B_{m_1} \wedge l'_1 \neq \ell_1} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{u_1^n(l'_1), \delta'} \Pi_{\bar{\rho}, \delta''}^n} \rho_{u_1^n(l_1), u_2^n(l_2)}^{B_1^n} \right\} \quad (85)$$

$$= \sum_{l_1 \in B_{m_1}, l_2 \in C_{m_2}} \mathcal{I}(l_1 = \ell_1, l_2 = \ell_2) \sum_{l'_1 \in B_{m_1} \wedge l'_1 \neq \ell_1} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{u_1^n(l'_1), \delta'} \Pi_{\bar{\rho}, \delta''}^n} \rho_{u_1^n(l_1), u_2^n(l_2)}^{B_1^n} \right\} \quad (86)$$

$$\leq \sum_{l_1 \in B_{m_1}, l_2 \in C_{m_2}} \mathcal{I}((l_1, l_2) \in \mathcal{A}) \sum_{l'_1 \in B_{m_1} \wedge l'_1 \neq \ell_1} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{u_1^n(l'_1), \delta'} \Pi_{\bar{\rho}, \delta''}^n} \rho_{u_1^n(l_1), u_2^n(l_2)}^{B_1^n} \right\} \quad (87)$$

$$= \sum_{l_1 \in B_{m_1}, l_2 \in C_{m_2}} \sum_{l'_1 \in B_{m_1} \wedge l'_1 \neq \ell_1} \mathcal{I}((l_1, l_2) \in \mathcal{A}) \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{u_1^n(l'_1), \delta'} \Pi_{\bar{\rho}, \delta''}^n} \rho_{u_1^n(l_1), u_2^n(l_2)}^{B_1^n} \right\} \quad (88)$$

where  $\mathcal{I}((l_1, l_2) \in \mathcal{A})$  is a shorthand for  $\mathcal{I}((u_1^n(l_1), u_2^n(l_2)) \in \mathcal{A}_{P_{U_1, U_2}, \delta}^n)$ . So the final bound on (80) is

$$\begin{aligned} & 2 \left( \varepsilon + 2\sqrt{\varepsilon'} \right) + 4 \sum_{l_1 \in B_{m_1}, l_2 \in C_{m_2}} \sum_{l'_1 \in B_{m_1} \wedge l'_1 \neq \ell_1} \mathcal{I}((l_1, l_2) \in \mathcal{A}) \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{u_1^n(l'_1), \delta'} \Pi_{\bar{\rho}, \delta''}^n} \rho_{u_1^n(l_1), u_2^n(l_2)}^{B_1^n} \right\} \\ & \quad + 4 \sum_{l'_1 \notin B_{m_1}} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{u_1^n(l'_1), \delta'} \Pi_{\bar{\rho}, \delta''}^n} \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_1^n} \right\}. \end{aligned} \quad (89)$$

In a similar way, we can write a bound on the right hand term in (79) as follows:

$$\begin{aligned} & \text{Tr} \left\{ \left( I^{B_2^n} - \Lambda_{\ell_2}^{B_2^n} \right) \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_2^n} \right\} \leq 2 \left( \varepsilon + 2\sqrt{\varepsilon'} \right) \\ & \quad + 4 \sum_{l_1 \in B_{m_1}, l_2 \in C_{m_2}} \sum_{l'_2 \in C_{m_2} \wedge l'_2 \neq \ell_2} \mathcal{I}((l_1, l_2) \in \mathcal{A}) \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{u_2^n(l'_2), \delta'} \Pi_{\bar{\rho}, \delta''}^n} \rho_{u_1^n(l_1), u_2^n(l_2)}^{B_2^n} \right\} \\ & \quad + 4 \sum_{l'_2 \notin C_{m_2}} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{u_2^n(l'_2), \delta'} \Pi_{\bar{\rho}, \delta''}^n} \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_2^n} \right\}, \end{aligned} \quad (90)$$

where the typical projectors  $\Pi_{\bar{\rho}, \delta''}^n$  and  $\Pi_{\omega_{u_2^n(\ell_2)}, \delta'}$  acting on system  $B_2^n$  are defined from the states

$$\omega_{u_2}^{B_2} \equiv \sum_{u_1} p_{U_1|U_2}(u_1|u_2) \rho_{f(u_1, u_2)}^{B_2}, \quad (91)$$

$$\bar{\rho}^{B_2} \equiv \sum_{u_2} p_{U_2}(u_2) \omega_{u_2}^{B_2}. \quad (92)$$

Putting everything together, we can write a bound on (76) as follows:

$$\begin{aligned} \mathcal{I}(\mathcal{E}_0) + \mathcal{I}(\mathcal{E}_0^c) \text{Tr} \left\{ \left( I^{B_1^n B_2^n} - \Lambda_{\ell_1}^{B_1^n} \otimes \Lambda_{\ell_2}^{B_2^n} \right) \rho_{u_1^n(\ell_1), u_2^n(\ell_2)}^{B_1^n B_2^n} \right\} &\leq \mathcal{I}(\mathcal{E}_0) + 4 \left( \varepsilon + 2\sqrt{\varepsilon'} \right) \\ &+ 4 \sum_{l_1 \in B_{m_1}, l_2 \in C_{m_2}} \sum_{l'_1 \in B_{m_1} \wedge l'_1 \neq l_1} \mathcal{I}((l_1, l_2) \in \mathcal{A}) \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{u_1^n(l'_1)}, \delta'} \Pi_{\bar{\rho}, \delta''}^n \rho_{u_1^n(l_1), u_2^n(l_2)}^{B_1^n} \right\} \\ &\quad + 4 \sum_{l'_1 \notin B_{m_1}} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{u_1^n(l'_1)}, \delta'} \Pi_{\bar{\rho}, \delta''}^n \rho_{u_1^n(l_1), u_2^n(l_2)}^{B_1^n} \right\} \\ &+ 4 \sum_{l_1 \in B_{m_1}, l_2 \in C_{m_2}} \sum_{l'_2 \in C_{m_2} \wedge l'_2 \neq l_2} \mathcal{I}((l_1, l_2) \in \mathcal{A}) \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{u_2^n(l'_2)}, \delta'} \Pi_{\bar{\rho}, \delta''}^n \rho_{u_1^n(l_1), u_2^n(l_2)}^{B_2^n} \right\} \\ &\quad + 4 \sum_{l'_2 \notin C_{m_2}} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{u_2^n(l'_2)}, \delta'} \Pi_{\bar{\rho}, \delta''}^n \rho_{u_1^n(l_1), u_2^n(l_2)}^{B_2^n} \right\}. \quad (93) \end{aligned}$$

At this point, we follow Shannon and recognize that the analysis of a particular subcode can be difficult (the last four terms above), so we instead analyze the expectation of the error probability, where the expectation is with respect to a randomly chosen code. That is, we consider the following quantity instead

$$\mathbb{E}_{\mathcal{C}} \left\{ \mathcal{I}(\mathcal{E}_0) + \mathcal{I}(\mathcal{E}_0^c) \text{Tr} \left\{ \left( I^{B_1^n B_2^n} - \Lambda_{\ell_1}^{B_1^n} \otimes \Lambda_{\ell_2}^{B_2^n} \right) \rho_{U_1^n(\ell_1), U_2^n(\ell_2)}^{B_1^n B_2^n} \right\} \right\}, \quad (94)$$

where

$$\mathbb{E}_{\mathcal{C}} \{ \cdot \} \equiv \mathbb{E}_{U_1^n(1)} \cdots \mathbb{E}_{U_1^n(L_1)} \mathbb{E}_{U_2^n(1)} \cdots \mathbb{E}_{U_2^n(L_2)} \{ \cdot \}, \quad (95)$$

$$\mathbb{E}_{U_1^n(1)} \{ \cdot \} = \sum_{u_1^n(1)} p_{U_1^n}(u_1^n(1)) \{ \cdot \}, \quad (96)$$

with the other expectations defined similarly. Then using the bound from (93), we find the following bound on (94)

$$\begin{aligned} \mathbb{E}_{\mathcal{C}} \{ \mathcal{I}(\mathcal{E}_0) \} + 4 \left( \varepsilon + 2\sqrt{\varepsilon'} \right) \\ &+ 4 \mathbb{E}_{\mathcal{C}} \left\{ \sum_{l_1 \in B_{m_1}, l_2 \in C_{m_2}} \sum_{l'_1 \in B_{m_1} \wedge l'_1 \neq l_1} \mathcal{I}((l_1, l_2) \in \mathcal{A}) \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{U_1^n(l'_1)}, \delta'} \Pi_{\bar{\rho}, \delta''}^n \rho_{U_1^n(l_1), U_2^n(l_2)}^{B_1^n} \right\} \right\} \\ &\quad + 4 \mathbb{E}_{\mathcal{C}} \left\{ \sum_{l'_1 \notin B_{m_1}} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{U_1^n(l'_1)}, \delta'} \Pi_{\bar{\rho}, \delta''}^n \rho_{U_1^n(l_1), U_2^n(l_2)}^{B_1^n} \right\} \right\} \\ &+ 4 \mathbb{E}_{\mathcal{C}} \left\{ \sum_{l_1 \in B_{m_1}, l_2 \in C_{m_2}} \sum_{l'_2 \in C_{m_2} \wedge l'_2 \neq l_2} \mathcal{I}((l_1, l_2) \in \mathcal{A}) \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{U_2^n(l'_2)}, \delta'} \Pi_{\bar{\rho}, \delta''}^n \rho_{U_1^n(l_1), U_2^n(l_2)}^{B_2^n} \right\} \right\} \\ &\quad + 4 \mathbb{E}_{\mathcal{C}} \left\{ \sum_{l'_2 \notin C_{m_2}} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{U_2^n(l'_2)}, \delta'} \Pi_{\bar{\rho}, \delta''}^n \rho_{U_1^n(l_1), U_2^n(l_2)}^{B_2^n} \right\} \right\}. \quad (97) \end{aligned}$$

We focus on the second expectation above and can write

$$\begin{aligned} & \mathbb{E}_{\mathcal{C}} \left\{ \sum_{l_1 \in B_{m_1}, l_2 \in C_{m_2}} \sum_{l'_1 \in B_{m_1} \wedge l'_1 \neq l_1} \mathcal{I}((l_1, l_2) \in \mathcal{A}) \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{U_1^n}(l'_1), \delta'} \Pi_{\bar{\rho}, \delta''}^n \rho_{U_1^n(l_1), U_2^n(l_2)}^{B_1^n} \right\} \right\} \\ &= \sum_{\substack{l_1 \in B_{m_1}, l'_1 \in B_{m_1}, \\ l_2 \in C_{m_2}, l'_1 \neq l_1}} \mathbb{E}_{U_1^n(l'_1)} \mathbb{E}_{U_1^n(l_1)} \mathbb{E}_{U_2^n(l_2)} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{U_1^n}(l'_1), \delta'} \Pi_{\bar{\rho}, \delta''}^n \rho_{U_1^n(l_1), U_2^n(l_2)}^{B_1^n} \mathcal{I}((l_1, l_2) \in \mathcal{A}) \right\} \end{aligned} \quad (98)$$

We focus on bounding the expression inside the sum, keeping in mind that  $l'_1 \neq l_1$ :

$$\mathbb{E}_{U_1^n(l'_1)} \mathbb{E}_{U_1^n(l_1)} \mathbb{E}_{U_2^n(l_2)} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{U_1^n}(l'_1), \delta'} \Pi_{\bar{\rho}, \delta''}^n \rho_{U_1^n(l_1), U_2^n(l_2)}^{B_1^n} \mathcal{I}((l_1, l_2) \in \mathcal{A}) \right\} \quad (99)$$

$$\leq 2^{n[H(B_1|U_1)_\theta + c\delta']} \mathbb{E}_{U_1^n(l'_1)} \mathbb{E}_{U_1^n(l_1)} \mathbb{E}_{U_2^n(l_2)} \left\{ \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \omega_{U_1^n}(l'_1) \Pi_{\bar{\rho}, \delta''}^n \rho_{U_1^n(l_1), U_2^n(l_2)}^{B_1^n} \mathcal{I}((l_1, l_2) \in \mathcal{A}) \right\} \right\} \quad (100)$$

$$= 2^{n[H(B_1|U_1)_\theta + c\delta']} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \mathbb{E}_{U_1^n(l'_1)} \left\{ \omega_{U_1^n}(l'_1) \right\} \Pi_{\bar{\rho}, \delta''}^n \mathbb{E}_{U_1^n(l_1)} \mathbb{E}_{U_2^n(l_2)} \left\{ \rho_{U_1^n(l_1), U_2^n(l_2)}^{B_1^n} \mathcal{I}((l_1, l_2) \in \mathcal{A}) \right\} \right\} \quad (101)$$

$$= 2^{n[H(B_1|U_1)_\theta + c\delta']} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \bar{\rho}^{\otimes n} \Pi_{\bar{\rho}, \delta''}^n \mathbb{E}_{U_1^n(l_1)} \mathbb{E}_{U_2^n(l_2)} \left\{ \rho_{U_1^n(l_1), U_2^n(l_2)}^{B_1^n} \mathcal{I}((l_1, l_2) \in \mathcal{A}) \right\} \right\} \quad (102)$$

$$\leq 2^{n[H(B_1|U_1)_\theta + c\delta']} 2^{-n[H(B_1)_\theta - c'\delta'']} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \mathbb{E}_{U_1^n(l_1)} \mathbb{E}_{U_2^n(l_2)} \left\{ \rho_{U_1^n(l_1), U_2^n(l_2)}^{B_1^n} \mathcal{I}((l_1, l_2) \in \mathcal{A}) \right\} \right\} \quad (103)$$

$$= 2^{-n[I(U_1; B_1)_\theta - c\delta' - c'\delta'']} \mathbb{E}_{U_1^n(l_1)} \mathbb{E}_{U_2^n(l_2)} \left\{ \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \rho_{U_1^n(l_1), U_2^n(l_2)}^{B_1^n} \mathcal{I}((l_1, l_2) \in \mathcal{A}) \right\} \right\} \quad (104)$$

$$\leq 2^{-n[I(U_1; B_1)_\theta - c\delta' - c'\delta'']} \mathbb{E}_{U_1^n(l_1)} \mathbb{E}_{U_2^n(l_2)} \left\{ \mathcal{I}((l_1, l_2) \in \mathcal{A}) \right\} \quad (105)$$

$$\leq 2^{-n[I(U_1; B_1)_\theta - c\delta' - c'\delta'']} 2^{-n[I(U_1; U_2)_\theta - c''\delta''']} \quad (106)$$

$$= 2^{-n[I(U_1; B_1)_\theta + I(U_1; U_2)_\theta - c''\delta'' - c'\delta'' - c\delta']} \quad (107)$$

We then find that (98) is bounded from above by

$$|B_{m_1}|^2 |C_{m_2}| 2^{-n[I(U_1; B_1)_\theta + I(U_1; U_2)_\theta - c''\delta'' - c'\delta'' - c\delta']} \quad (108)$$

A similar analysis for Receiver 2 gives the following bound:

$$\begin{aligned} & \mathbb{E}_{\mathcal{C}} \left\{ \sum_{l_1 \in B_{m_1}, l_2 \in C_{m_2}} \sum_{l'_2 \in C_{m_2} \wedge l'_2 \neq l_2} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{U_2^n}(l'_2), \delta'} \Pi_{\bar{\rho}, \delta''}^n \rho_{U_1^n(l_1), U_2^n(l_2)}^{B_2^n} \right\} \right\} \\ & \leq |B_{m_1}| |C_{m_2}|^2 2^{-n[I(U_2; B_2)_\theta + I(U_1; U_2)_\theta - c''\delta'' - c'\delta'' - c\delta']} \end{aligned} \quad (109)$$

We can again use the same analysis to recover the following bounds (however observing that the joint random variable  $(U_1^n(l_1), U_2^n(l_2))$  is independent of both  $U_1^n(l'_1)$  and  $U_2^n(l'_2)$  for  $l'_1 \notin B_{m_1}$  and  $l'_2 \notin C_{m_2}$ ):

$$\mathbb{E}_{\mathcal{C}} \left\{ \sum_{l'_1 \notin B_{m_1}} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{U_1^n}(l'_1), \delta'} \Pi_{\bar{\rho}, \delta''}^n \rho_{U_1^n(l_1), U_2^n(l_2)}^{B_1^n} \right\} \right\} \leq L_1 2^{-n[I(U_1; B_1)_\theta - c'\delta'' - c\delta']} \quad (110)$$

$$\mathbb{E}_{\mathcal{C}} \left\{ \sum_{l'_2 \notin C_{m_2}} \text{Tr} \left\{ \Pi_{\bar{\rho}, \delta''}^n \Pi_{\omega_{U_2^n}(l'_2), \delta'} \Pi_{\bar{\rho}, \delta''}^n \rho_{U_1^n(l_1), U_2^n(l_2)}^{B_2^n} \right\} \right\} \leq L_2 2^{-n[I(U_2; B_2)_\theta - c'\delta'' - c\delta']} \quad (111)$$

The term  $\mathbb{E}_{\mathcal{C}} \{\mathcal{I}(\mathcal{E}_0)\}$  can be bounded from above by  $\varepsilon$  by employing the mutual covering lemma [22] (see also [1, Lemma 8.1]). Indeed if we choose

$$\left( \tilde{R}_1 - R_1 \right) + \left( \tilde{R}_2 - R_2 \right) \geq I(U_1; U_2)_\theta + \delta'''' \quad (112)$$

then this error can be made arbitrarily small (i.e., less than  $\varepsilon$ ) by increasing  $n$ .

So we finally get that (94) is bounded from above by

$$\begin{aligned} \varepsilon + 4 \left( \varepsilon + 2\sqrt{\varepsilon'} \right) + |B_{m_1}|^2 |C_{m_2}|^2 2^{-n} [I(U_1; B_1)_\theta + I(U_1; U_2)_\theta - c''\delta''' - c'\delta'' - c\delta'] + L_1 2^{-n} [I(U_1; B_1)_\theta - c'\delta'' - c\delta'] \\ + |B_{m_1}| |C_{m_2}|^2 2^{-n} [I(U_2; B_2)_\theta + I(U_1; U_2)_\theta - c''\delta''' - c'\delta'' - c\delta'] + L_2 2^{-n} [I(U_2; B_2)_\theta - c'\delta'' - c\delta']. \end{aligned} \quad (113)$$

Then for any  $\varepsilon'' > 0$ , we can pick

$$2 \left( \tilde{R}_1 - R_1 \right) + \left( \tilde{R}_2 - R_2 \right) < I(U_1; B_1)_\theta + I(U_1; U_2)_\theta - c''\delta''' - c'\delta'' - c\delta', \quad (114)$$

$$\tilde{R}_1 < I(U_1; B_1)_\theta - c'\delta'' - c\delta', \quad (115)$$

$$\left( \tilde{R}_1 - R_1 \right) + 2 \left( \tilde{R}_2 - R_2 \right) < I(U_2; B_2)_\theta + I(U_1; U_2)_\theta - c''\delta''' - c'\delta'' - c\delta', \quad (116)$$

$$\tilde{R}_2 < I(U_2; B_2)_\theta - c'\delta'' - c\delta', \quad (117)$$

and  $n$  sufficiently large so that the quantity in (113) is less than  $\varepsilon''$ . Indeed this estimate can be made for the expected error probability for each of the subcodebooks, so by linearity of the expectation we can finally conclude the existence of a coding scheme for which

$$\frac{1}{M_1 M_2} \sum_{m_1, m_2} \text{Tr} \left\{ (I - \Upsilon_{m_1}^{B_1^n} \otimes \Upsilon_{m_2}^{B_2^n}) \rho_{m_1, m_2} \right\} \leq \varepsilon'' \quad (118)$$

as long as

$$\left( \tilde{R}_1 - R_1 \right) + \left( \tilde{R}_2 - R_2 \right) \geq I(U_1; U_2)_\theta + \delta'''' , \quad (119)$$

$$2 \left( \tilde{R}_1 - R_1 \right) + \left( \tilde{R}_2 - R_2 \right) < I(U_1; B_1)_\theta + I(U_1; U_2)_\theta - c''\delta''' - c'\delta'' - c\delta', \quad (120)$$

$$\tilde{R}_1 < I(U_1; B_1)_\theta - c'\delta'' - c\delta', \quad (121)$$

$$\left( \tilde{R}_1 - R_1 \right) + 2 \left( \tilde{R}_2 - R_2 \right) < I(U_2; B_2)_\theta + I(U_1; U_2)_\theta - c''\delta''' - c'\delta'' - c\delta', \quad (122)$$

$$\tilde{R}_2 < I(U_2; B_2)_\theta - c'\delta'' - c\delta', \quad (123)$$

where  $\Upsilon_{m_1}^{B_1^n}$  and  $\Upsilon_{m_2}^{B_2^n}$  represent the overall decoding POVMs of Bob and  $\rho_{m_1, m_2}$  represents the channel output when sending messages  $m_1$  and  $m_2$ . Thus, since  $\delta', \delta'', \delta''', \delta'''' > 0$  are arbitrary, the following rate region is achievable:

$$R_1 + R_2 + I(U_1; U_2)_\theta \leq \tilde{R}_1 + \tilde{R}_2, \quad (124)$$

$$2\tilde{R}_1 + \tilde{R}_2 \leq I(U_1; B_1)_\theta + I(U_1; U_2)_\theta + 2R_1 + R_2, \quad (125)$$

$$\tilde{R}_1 \leq I(U_1; B_1)_\theta, \quad (126)$$

$$2\tilde{R}_2 + \tilde{R}_1 \leq I(U_2; B_2)_\theta + I(U_1; U_2)_\theta + 2R_2 + R_1, \quad (127)$$

$$\tilde{R}_2 \leq I(U_2; B_2)_\theta. \quad (128)$$

By exploiting the additional constraints  $R_1 \leq \tilde{R}_1$  and  $R_2 \leq \tilde{R}_2$  and applying Fourier-Motzkin elimination (see Appendix A), we find that the following quantum Marton rate region is achievable:

$$R_1 \leq I(U_1; B_1)_\theta, \quad (129)$$

$$R_2 \leq I(U_2; B_2)_\theta, \quad (130)$$

$$R_1 + R_2 \leq I(U_1; B_1)_\theta + I(U_2; B_2)_\theta - I(U_1; U_2)_\theta. \quad (131)$$

As we are dealing with a channel having a classical input, the cardinality bounds given in the statement in the theorem follow directly from what is known in the classical case. Here, one can apply the perturbation method introduced in [23], discussed also in [24], and reviewed in [1].

## V. CONCLUSION

We have proved quantum generalizations of the superposition coding inner bound [2], [3] and the Marton rate region with no common message [4]. A key ingredient in both proofs was the use of the projector trick. A natural followup question would be to combine the two strategies to obtain the Marton coding scheme with a common message.

A much broader goal would be to extend all of network information theory to the study of quantum channels. To accomplish this goal, it would be helpful to have a tool that generalizes El Gamal and Kim's classical packing lemma [1] to the quantum domain. The packing lemma is sufficient to prove all of the known coding theorems in network information theory. At the moment, it is not clear to us whether such a tool exists for the quantum case, but evidence in favor of its existence is that 1) one can prove the HSW coding theorem by using conditionally typical projectors only [5, Exercise 19.3.5], 2) we have solved the quantum simultaneous decoding conjecture for the case of two senders [15], [16], and 3) we have generalized two important coding theorems in the current paper (with proofs somewhat similar to the classical proofs). Ideally, such a "quantum packing lemma" would allow quantum information theorists to prove quantum network coding theorems by appealing to it, rather than having to analyze each coding scheme in detail on a case by case basis.

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### APPENDIX A FOURIER-MOTZKIN ELIMINATION

In the proof of Theorem 3, we conclude that the following rate region is achievable:

$$R_1 + R_2 + I(U_1; U_2)_\theta \leq \tilde{R}_1 + \tilde{R}_2, \quad (124)$$

$$2\tilde{R}_1 + \tilde{R}_2 \leq I(U_1; B_1)_\theta + 2R_1 + R_2 + I(U_1; U_2), \quad (125)$$

$$\tilde{R}_1 \leq I(U_1; B_1)_\theta, \quad (126)$$

$$2\tilde{R}_2 + \tilde{R}_1 \leq I(U_2; B_2)_\theta + 2R_2 + R_1 + I(U_1; U_2), \quad (127)$$

$$\tilde{R}_2 \leq I(U_2; B_2)_\theta. \quad (128)$$

There are additional constraints  $R_1 \leq \tilde{R}_1$  and  $R_2 \leq \tilde{R}_2$ . To eliminate  $\tilde{R}_1$ , we split this system of seven equations into three groups: those that provide lower bounds on  $\tilde{R}_1$ , those that provide upper bounds on  $\tilde{R}_1$ , and equations that do not involve  $\tilde{R}_1$ :

$$I(U_1; U_2) + R_1 + R_2 - \tilde{R}_2 \leq \tilde{R}_1 \quad (132a)$$

$$R_1 \leq \tilde{R}_1 \quad (132b)$$

$$\tilde{R}_1 \leq \frac{I(U_1; B_1)}{2} + R_1 + \frac{R_2}{2} - \frac{\tilde{R}_2}{2} + \frac{I(U_1; U_2)}{2} \quad (132c)$$

$$\tilde{R}_1 \leq I(U_1; B_1) \quad (132d)$$

$$\tilde{R}_1 \leq I(U_2; B_2) + R_1 + 2R_2 - 2\tilde{R}_2 + I(U_1; U_2) \quad (132e)$$

$$\tilde{R}_2 \leq I(U_2; B_2) \quad (132f)$$

$$R_2 \leq \tilde{R}_2 \quad (132g)$$

Now we combine each possible lower bound (a,b) with each possible upper bound (c,d,e) and copy over the others:

$$I(U_1; U_2) + R_1 + R_2 - \tilde{R}_2 \leq \frac{I(U_1; B_1)}{2} + R_1 + \frac{R_2}{2} - \frac{\tilde{R}_2}{2} + \frac{I(U_1; U_2)}{2} \quad (133a)$$

$$I(U_1; U_2) + R_1 + R_2 - \tilde{R}_2 \leq I(U_1; B_1) \quad (133b)$$

$$I(U_1; U_2) + R_1 + R_2 - \tilde{R}_2 \leq I(U_2; B_2) + R_1 + 2R_2 - 2\tilde{R}_2 + I(U_1; U_2) \quad (133c)$$

$$R_1 \leq \frac{I(U_1; B_1)}{2} + R_1 + \frac{R_2}{2} - \frac{\tilde{R}_2}{2} + \frac{I(U_1; U_2)}{2} \quad (133d)$$

$$R_1 \leq I(U_1; B_1) \quad (133e)$$

$$R_1 \leq I(U_2; B_2) + R_1 + 2R_2 - 2\tilde{R}_2 + I(U_1; U_2) \quad (133f)$$

$$\tilde{R}_2 \leq I(U_2; B_2) \quad (133g)$$

$$R_2 \leq \tilde{R}_2 \quad (133h)$$

Cancelling terms and simplifying, we get

$$\frac{I(U_1; U_2)}{2} + \frac{R_2}{2} - \frac{\tilde{R}_2}{2} \leq \frac{I(U_1; B_1)}{2} \quad (134a)$$

$$I(U_1; U_2) + R_1 + R_2 - \tilde{R}_2 \leq I(U_1; B_1) \quad (134b)$$

$$0 \leq I(U_2; B_2) + R_2 - \tilde{R}_2 \quad (134c)$$

$$0 \leq \frac{I(U_1; B_1)}{2} + \frac{R_2}{2} - \frac{\tilde{R}_2}{2} + \frac{I(U_1; U_2)}{2} \quad (134d)$$

$$R_1 \leq I(U_1; B_1) \quad (134e)$$

$$0 \leq I(U_2; B_2) + 2R_2 - 2\tilde{R}_2 + I(U_1; U_2) \quad (134f)$$

$$\tilde{R}_2 \leq I(U_2; B_2) \quad (134g)$$

$$R_2 \leq \tilde{R}_2 \quad (134h)$$

This completes the steps required to eliminate  $\tilde{R}_1$ .

Observe that (134a) is redundant because it is implied by (134b) and the implicit constraint  $R_1 \geq 0$ , and (134c) is redundant because it is implied by (134g) and the implicit constraint  $R_2 \geq 0$ . After removing the redundant inequalities, to eliminate  $\tilde{R}_2$ , we rearrange the equations (134) into lower bounds, upper bounds, and those not containing  $\tilde{R}_2$ :

$$-I(U_1; B_1) + I(U_1; U_2) + R_1 + R_2 \leq \tilde{R}_2 \quad (135a)$$

$$R_2 \leq \tilde{R}_2 \quad (135b)$$

$$\tilde{R}_2 \leq I(U_1; B_1) + R_2 + I(U_1; U_2) \quad (135c)$$

$$\tilde{R}_2 \leq \frac{I(U_2; B_2)}{2} + R_2 + \frac{I(U_1; U_2)}{2} \quad (135d)$$

$$\tilde{R}_2 \leq I(U_2; B_2) \quad (135e)$$

$$R_1 \leq I(U_1; B_1) \quad (135f)$$

Combining each of the lower bounds on  $\tilde{R}_2$  with each of the upper bounds results in the following



equations:

$$-I(U_1; B_1) + I(U_1; U_2) + R_1 + R_2 \leq I(U_1; B_1) + R_2 + I(U_1; U_2) \quad (136a)$$

$$-I(U_1; B_1) + I(U_1; U_2) + R_1 + R_2 \leq \frac{I(U_2; B_2)}{2} + R_2 + \frac{I(U_1; U_2)}{2} \quad (136b)$$

$$-I(U_1; B_1) + I(U_1; U_2) + R_1 + R_2 \leq I(U_2; B_2) \quad (136c)$$

$$R_2 \leq I(U_1; B_1) + R_2 + I(U_1; U_2) \quad (136d)$$

$$R_2 \leq \frac{I(U_2; B_2)}{2} + R_2 + \frac{I(U_1; U_2)}{2} \quad (136e)$$

$$R_2 \leq I(U_2; B_2) \quad (136f)$$

$$R_1 \leq I(U_1; B_1) \quad (136g)$$

After simplification, this system of equations becomes

$$R_1 \leq 2I(U_1; B_1) \quad (137a)$$

$$R_1 \leq \frac{I(U_2; B_2)}{2} + I(U_1; B_1) - \frac{I(U_1; U_2)}{2} \quad (137b)$$

$$R_1 + R_2 \leq I(U_2; B_2) + I(U_1; B_1) - I(U_1; U_2) \quad (137c)$$

$$0 \leq I(U_1; B_1) + I(U_1; U_2) \quad (137d)$$

$$0 \leq \frac{I(U_2; B_2)}{2} + \frac{I(U_1; U_2)}{2} \quad (137e)$$

$$R_2 \leq I(U_2; B_2) \quad (137f)$$

$$R_1 \leq I(U_1; B_1) \quad (137g)$$

Observe that the first inequality is implied by the last and the fourth and fifth inequalities are trivially true. Consider dividing (137c) by two, dividing (137g) by two, and adding the result:

$$R_1 + \frac{R_2}{2} \leq \frac{I(U_2; B_2)}{2} + I(U_1; B_1) - \frac{I(U_1; U_2)}{2}. \quad (138)$$

Using the fact that  $R_2 \geq 0$ , we see that (137b) is redundant and we are left with the three inequalities that specify the Marton region.

## APPENDIX B TYPICALITY LEMMA

The following lemma is an extension of [5, Property 14.2.7].

**Lemma 3.** *The state  $\rho_{\ell_1, \ell_2}$  is well supported by both the averaged state projector:*

$$\text{Tr} [\Pi_{\bar{\rho}, \delta}^n \rho_{\ell_1, \ell_2}] \geq 1 - \epsilon, \quad \forall \ell_1, \ell_2, \quad (139)$$

*and the  $\omega_{u_1}^{B_1}$  conditionally typical projector:*

$$\text{Tr} [\Pi_{u_1^n(\ell_1)} \rho_{\ell_1, \ell_2}] \geq 1 - \epsilon, \quad \forall \ell_2, \quad (140)$$

*when  $u_1^n(\ell_1)$  and  $u_2^n(\ell_2)$  are strongly jointly typical. (Both of these projectors are defined in the main text just after (75).)*

Consider the following sets of all jointly-typical and marginally-typical sequences for the probability distribution  $p_{U_1 U_2}(u_1, u_2)$ :

$$\begin{aligned}\mathcal{A}_{p_{U_1}, \delta}^n &\equiv \left\{ u_1^n \in \mathcal{U}_1^n : \left| \frac{N(u_a | u_1^n)}{n} - p_{U_1}(u_a) \right| \leq \delta \right\}, \\ \mathcal{A}_{p_{U_2}, \delta}^n &\equiv \left\{ u_2^n \in \mathcal{U}_2^n : \left| \frac{N(u_a | u_2^n)}{n} - p_{U_2}(u_a) \right| \leq \delta \right\}, \\ \mathcal{A}_{p_{U_1 U_2}, \delta}^n &\equiv \left\{ u^n \in \mathcal{U}_1^n \times \mathcal{U}_2^n : \left| \frac{N(u_a | u^n)}{n} - p_{U_1 U_2}(u_a) \right| \leq \delta \right\}.\end{aligned}$$

Note that the notion of strong typicality implies that if  $u^n = (u_1^n, u_2^n) \in \mathcal{A}_{p_{U_1 U_2}, \delta}^n$ , then both of its substrings are marginally typical:  $u_1^n \in \mathcal{A}_{p_{U_1}, \delta}^n$  and  $u_2^n \in \mathcal{A}_{p_{U_2}, \delta}^n$ .

*Proof:* Consider the eigen-decomposition of the average state at Receiver 1:

$$\bar{\rho} = \sum_z p_Z(z) |z\rangle\langle z|, \quad (141)$$

and the associated *pinching* operator:

$$\Delta(\psi) = \sum_z |z\rangle\langle z| \psi |z\rangle\langle z|, \quad (142)$$

which turns any quantum state on the output system of Receiver 1 into a classical probability distribution.

In particular, when  $\Delta$  is applied to the state  $\rho_{u_1, u_2}$ , (the channel output when codewords  $u_1$  and  $u_2$  are sent) is given by:

$$\rho'_{u_1, u_2} = \sum_z |z\rangle\langle z| \rho_{u_1, u_2} |z\rangle\langle z| \quad (143)$$

$$= \sum_z p_{Z_p | U_1 U_2}(z_p | u_1, u_2) |z\rangle\langle z|, \quad (144)$$

where  $p_{Z_p | U_1 U_2}(z_p | u_1, u_2)$  is a classical probability distribution.

The statement of the lemma can be expressed in terms of  $n$  copies of this product distribution:

$$\text{Tr} [\Pi_{\bar{\rho}, \delta}^n \rho_{\ell_1, \ell_2}] = \text{Tr} \left[ \sum_{z^n \in \mathcal{A}_{\bar{\rho}}} |z^n\rangle\langle z^n| \rho_{\ell_1, \ell_2} \right] \quad (145)$$

$$= \text{Tr} \left[ \sum_{z^n \in \mathcal{A}_{\bar{\rho}}} |z^n\rangle\langle z^n| |z^n\rangle\langle z^n| \rho_{\ell_1, \ell_2} \right] \quad (146)$$

$$= \text{Tr} \left[ \sum_{z^n \in \mathcal{A}_{\bar{\rho}}} \langle z^n | \rho_{\ell_1, \ell_2} | z^n \rangle |z^n\rangle\langle z^n| \right] \quad (147)$$

$$= \sum_{z^n \in \mathcal{A}_{\bar{\rho}}} p_{Z_p^n | U_1^n U_2^n}(z_p^n | u_1^n, u_2^n). \quad (148)$$

Thus we see that the value of the trace expression is equivalent to the probability of a conditionally typical sequence  $Z_p^n | u_1^n u_2^n$  being in the typical set  $\mathcal{A}_{\bar{\rho}}$ :

$$\text{Pr} \{ Z_p^n | u_1^n u_2^n \in \mathcal{A}_{\bar{\rho}} \}. \quad (149)$$

To evaluate the above expression we start from the following facts: (1) For  $n$  large enough the state classical distribution that corresponds to the pinched state  $\rho'_{u_1^n, u_2^n}$  is going to be conditionally typical on the input sequence:

$$\text{Pr} \left\{ Z_p^n | u_1^n u_2^n \in \mathcal{A}_{Z_p^n | U_1^n U_2^n, \delta'} \right\} \geq 1 - \varepsilon', \quad (150)$$

and (2) the input sequence is jointly typical  $(u_1^n, u_2^n) \in \mathcal{A}_{U_1^n U_2^n, \delta''}$ . It then follows that, with high probability the input-output sequence will be  $(\delta' + \delta'')$ -jointly-typical:

$$Z_p^n \in \mathcal{A}_{Z_p^n U_1^n U_2^n, \delta' + \delta''}, \quad (151)$$

which in turn implies that:

$$Z_p^n \in \mathcal{A}_{Z_p, |U_1| |U_2| (\delta' + \delta'')} = \mathcal{A}_{\bar{p}, |U_1| |U_2| (\delta' + \delta'')}. \quad (152)$$

By a suitable choice of  $n$ ,  $\delta' = \frac{\delta}{2|U_1| |U_2|}$ , and  $\delta'' = \frac{\delta}{2|U_1| |U_2|}$  we have that

$$\text{Tr} [\Pi_{\bar{p}, \delta}^n \rho_{\ell_1, \ell_2}] = \Pr \{Z_p^n | u_1^n u_2^n \in \mathcal{A}_{\bar{p}}\} \quad (153)$$

$$\geq 1 - \varepsilon''. \quad (154)$$

To prove the other inequality, consider the eigen-decompositions of  $\omega_{u_a}^{B_1}$  states at Receiver 1:

$$\omega_{u_a} = \sum_{z_1} p_{Z_1|U_1}(z_1|u_a) |z_1^{(u_a)}\rangle \langle z_1^{(u_a)}|, \quad \forall u_a \in \mathcal{U}_1. \quad (155)$$

The associated *pinching* operator is:

$$\Delta_{u_a}(\psi) = \sum_{z_1} |z_1^{(u_a)}\rangle \langle z_1^{(u_a)}| \psi |z_1^{(u_a)}\rangle \langle z_1^{(u_a)}|, \quad (156)$$

which turns any quantum state on the output system of Receiver 1 into a classical probability distribution expressed in terms of the basis for  $\omega_{u_a}$ :  $|z_1^{(u_a)}\rangle$ . When the symbol  $u_a$  is obvious from the context, we will sometimes refer to the basis elements simply as  $|z_1\rangle \equiv |z_1^{(u_a)}\rangle$ .

When  $\Delta_{u_1}$  is applied to the state  $\rho_{u_1, u_2}$ , (the channel output when codewords  $u_1$  and  $u_2$  are sent) is given by:

$$\rho'_{u_a, u_b} = \Delta_{u_a}(\rho_{u_a, u_b}) \quad (157)$$

$$= \sum_{z_1} |z_1^{(u_a)}\rangle \langle z_1^{(u_a)}| \rho_{u_a, u_b} |z_1^{(u_a)}\rangle \langle z_1^{(u_a)}| \quad (158)$$

$$= \sum_{z_1} p_{Z_1|U_1 U_2}(z_1|u_a, u_b) |z_1^{(u_a)}\rangle \langle z_1^{(u_a)}|, \quad (159)$$

where  $p_{Z_1|U_1 U_2}(z_1|u_a, u_b)$  is a classical probability distribution.

If we take the conditional marginal of this distribution we get

$$p_{Z_1|U_1}(z_1|u_a) = \sum_{u_b} p_{U_2|U_1}(u_b|u_a) p_{Z_1|U_1 U_2}(z_1|u_a, u_b), \quad (160)$$

which is the probability distribution of the  $u_a$ -basis eigenvalues given that  $U_2$  is unknown. This distribution can also be obtained from the pinching of the state  $\omega_{u_1}^{B_1}$ :

$$\begin{aligned} \Delta_{u_a}(\omega_{u_1}^{B_1}) &= \sum_{z_1} |z_1^{(u_a)}\rangle \langle z_1^{(u_a)}| \omega_{u_1}^{B_1} |z_1^{(u_a)}\rangle \langle z_1^{(u_a)}| \\ &= \sum_{z_1} |z_1^{(u_a)}\rangle \langle z_1^{(u_a)}| \sum_{u_b} p_{U_2|U_1}(u_b|u_a) \rho_{u_a, u_b} |z_1^{(u_a)}\rangle \langle z_1^{(u_a)}| \end{aligned} \quad (161)$$

$$= \sum_{z_1} \sum_{u_b} p_{U_2|U_1}(u_b|u_a) \underbrace{\langle z_1^{(u_a)}| \rho_{u_a, u_b} |z_1^{(u_a)}\rangle}_{p_{Z_1|U_1 U_2}(z_1|u_a, u_b)} |z_1^{(u_a)}\rangle \langle z_1^{(u_a)}| \quad (162)$$

$$= \sum_{z_1} p_{Z_1|U_1}(z_1|u_a) |z_1^{(u_a)}\rangle \langle z_1^{(u_a)}| \quad (163)$$

Define the classical conditionally typical sets on sequences of  $m$  symbols drawn from the above probability distributions:

$$p_{Z_{1p}|U_1U_2}(z_1|u_a, u_b) \Rightarrow \mathcal{A}_{Z_{1p}|u_a u_2^n, \delta}^{(m)} \quad (164)$$

$$p_{Z_{1p}|U_1}(z_1|u_a) \Rightarrow \mathcal{A}_{Z_{1p}|u_a, \delta}^{(m)} \quad (165)$$

Applied to the  $n$ -symbols of the channel output we get:

$$\rho'_{u_1^n, u_2^n} = \sum_{z_1^n} |z_1^n\rangle \langle z_1^n | \rho_{u_1^n, u_2^n} | z_1^n\rangle \langle z_1^n | \quad (166)$$

$$= \sum_{z_1^n} p_{Z_{1p}|U_1^n U_2^n}(z_1^n | u_1^n, u_2^n) | z_1^n\rangle \langle z_1^n |, \quad (167)$$

where  $p_{Z_{1p}|U_1^n U_2^n}(z_1^n | u_1^n, u_2^n)$  is a product distribution built from the individual distributions  $p_{Z_{1p}|U_1U_2}(z_1 | u_a, u_b)$  depending on the value of  $u_{1i}$  and  $u_{2i}$ . Note also that the basis  $|z_1^n\rangle$  is built from the different bases  $|z_{1i}^{(u_{1i})}\rangle$  according to whichever input symbol  $u_{1i}$  is used. To make the above statements more explicit, we can permute order of the symbols in the codeword so that they form contiguous blocks where the same input  $u_a$  is used.

$$\rho'_{u_1^n, u_2^n} = \sum_{z_1^n} p_{Z_{1p}|U_1^n U_2^n}(z_1^n | u_1^n, u_2^n) | z_1^n\rangle \langle z_1^n | \quad (168)$$

$$= \bigotimes_{u_a \in \mathcal{U}_1} \left( \prod_{j=1}^{m_1} p_{Z_{1p_j}|U_1U_2}(z_{1j} | u_a, u_{2j}) | z_{1j}\rangle \langle z_{1j} | \right), \quad (169)$$

where  $m_a = N(u_a | u_1^n, u_2^n)$  and  $\sum_a m_a = n$ . In each  $m_a$ -dimensional block, the same basis is used for all symbols:  $|z_1^{(m_1)}\rangle = |z_1^{(u_a)}\rangle$ .

Using the pinching operator, we can reduce the lemma to a question involving only classical probability distribution. Let  $m_a = N(u_a | u_1^n)$  and decompose  $\Pi_{u_1^n(\ell_1)}$  into different blocks:

$$\text{Tr} \left[ \Pi_{u_1^n(\ell_1)} \rho_{\ell_1, \ell_2} \right] = \text{Tr} \left[ \bigotimes_{u_a} \left( \sum_{z_1^{(m_a)} \in \mathcal{A}_{Z_{1p}|u_a, \delta}^{(m_a)}} |z_1^{(m_a)}\rangle \langle z_1^{(m_a)}| \right) \rho_{\ell_1, \ell_2} \right] \quad (170)$$

$$= \prod_{u_a} \sum_{z_1^{(m_a)} \in \mathcal{A}_{Z_{1p}|u_a, \delta}^{(m_a)}} p_{Z_{1p}^{(m_a)}|U_1^{(m_a)}U_2^{(m_a)}}(z_{1p}^{(m_a)} | u_a, u_2^{(m_a)}) \quad (171)$$

$$= \prod_{u_a} \text{Pr} \left\{ Z_{1p}^{(m_a)} | u_a, u_2^{(m_a)} \in \mathcal{A}_{Z_{1p}|u_a, \delta}^{(m_a)} \right\} \quad (172)$$

$$\geq \prod_{u_a} (1 - \varepsilon'') \quad (173)$$

$$= 1 - |\mathcal{U}_1| \varepsilon'' = 1 - \varepsilon''' \quad (174)$$

We use a similar argument as in the previous lemma. In each block we know that w.h.p.  $Z_{1p}^{(m_a)} | u_a, u_2^{(m_a)} \in \mathcal{A}_{Z_{1p}|u_a, \delta}^{(m_a)}$  and  $u_2^{(m_a)} \in \mathcal{A}_{U_2|u_a}^{(m_a)}$  therefore it must be that  $Z_{1p}^{(m_a)} u_2^{(m_a)} | u_a \in \mathcal{A}_{Z_{U_2}|u_a}^{(m_a)}$ . This in turn implies that  $Z_{1p}^{(m_a)} | u_a \in \mathcal{A}_{Z_{1p}|u_a, \delta}^{(m_a)}$ . ■

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