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Polar Codes in Network Quantum Information Theory

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Abstract—**Polar coding is a method for communication over noisy classical channels, which is provably capacity achieving and has an efficient encoding and decoding. Recently, this method has been generalized to the realm of quantum information processing, for tasks such as classical communication, private classical communication, and quantum communication. In this paper, we apply the polar coding method to network classical-quantum information theory, by making use of recent advances for related classical tasks. In particular, we consider problems such as the compound multiple access channel and the quantum interference channel. The main result of our work is that it is possible to achieve the best known inner bounds on the achievable rate regions for these tasks, without requiring a so-called quantum simultaneous decoder. Thus, this paper paves the way for developing network classical-quantum information theory further without requiring a quantum simultaneous decoder.**

Index Terms—**Polar codes, channel capacity, classical-quantum networks, successive cancellation decoder, quantum simultaneous decoder.**

I. INTRODUCTION

ONE of the key tasks distinguishing the theory of quantum physics from its classical counterpart is the simultaneous measurement of non-commuting observables. Indeed, the uncertainly principle captures one of the most profound characteristics of quantum mechanics, that is, the impossibility of simultaneously measuring non-commuting operators to arbitrary accuracy. The principle itself is considered a cornerstone of modern physics.

In quantum communication theory, the problem of simultaneous measurement arises in multi-user communication models when one needs to simultaneously learn about two or more non-commuting output states of a quantum channel in order to achieve optimal rates of communication. In particular, the problem of simultaneous decoding in the quantum setting manifests itself in the difficulty in constructing a measurement operator achieving this task. Although this problem is well understood classically, the existence of a general quantum simultaneous decoder has remained a conjecture.

In contrast to simultaneous measurement, building decoders based on measuring outputs of two or more users of a channel successively, i.e., successive decoding, has been successfully realized in the quantum setting [1]. Moreover, a coding strategy based on successive decoding has been shown to achieve the optimal communication rate region for the classical-quantum multiple access channel (cq-MAC) [1], with the help of the gentle measurement lemma [2] bounding the measurement disturbance of quantum states.

Despite this success with successive decoding and the multiple access channel, achieving the rate region for other multi-user quantum channels has remained elusive. An important example of such a channel is provided by the quantum interference channel [3]. In this model two or more senders wish to communicate information simultaneously, and solely with their intended corresponding receiver by means of a noisy channel modeled by cross-talk or interference. In the classical setting, the capacity of the interference channel is known exactly only in the case of very strong interference [4] or strong interference [5].

The best known achievable rate region for the two-user classical interference channel is given by the Han-Kobayashi region [5]. The coding strategy which achieves this region relies on the simultaneous decoding of two three-sender multiple access channels. A quantum version of the Han-Kobayashi rate region for the classical-quantum interference channel has also been conjectured to be achievable, based on the conjectured existence of a three-sender simultaneous decoder [3], [6], and it was in fact proven to be achievable in [7] using a specialized three-sender quantum simultaneous decoder. This result raised the question of whether the quantum Han-Kobayashi region can be achieved using a successive decoder [8].

In this article, we address this question by exploiting Arikan's polar coding technique for classical channels [9].

Indeed, polar codes have attracted a great deal of attention as the first *constructive* capacity-achieving codes with an efficient encoder and decoder. Recently the polar coding technique has been applied to a variety of multi-user classical channels including multiple access channels [10]–[13], broadcast channels [14], [15], interference networks [16] and for the task of source coding [17] and universal coding for compound channels [16], [18], [19]. We make use of some of these advances in our work. Polar coding has also been generalized for the task of sending classical [20], [21] and quantum information [22], [23] over single-user quantum channels. In the quantum setting, efficiency has only been shown in general for the encoder and left open for the decoder [20]–[22] (see [24] for recent progress on the efficient decoder question). However an efficient encoding and decoding scheme has been shown for certain quantum channels in the case of quantum communication [25].

In this work we show that polar coding can also be applied to the cq-MAC to achieve every point in the known achievable rate region [1] and also that an approach for universal polar codes from [18] can be used to obtain achievable rates for compound cq-MACs. We also apply the results obtained for compound cq-MACs in a way similar to [16] in order to achieve the Han-Kobayashi rate region for the two-user classical-quantum interference channel using a successive cancellation decoder.

The paper is organized as follows. In Section II we introduce the necessary mathematical preliminaries and the classical-quantum multiple access and interference channels, along with the technique for polar coding for classical-quantum channels in the case of a single sender and receiver. In Section III we discuss the two-user classical-quantum MAC and generalize in Section IV to multiple users and in Section V to compound MACs. This is applied to interference channels in Section VI, and we conclude in Section VII with a summary and some open questions.

II. PRELIMINARIES

A. Notation and Definitions

A discrete classical-quantum channel W takes realizations $x \in \mathcal{X}$ of a random variable X to a quantum state, denoted ρ_x^B, on a finite-dimensional Hilbert space $\mathcal{H}^B$,

$$W : x \to \rho_x^B, \tag{1}$$

where each quantum state ρ_x is described by a positive semi-definite operator with unit trace. We will take the input alphabet $\mathcal{X} = \{0, 1\}$ unless otherwise stated, and the tensor product $W^{\otimes N}$ of N channels is denoted by W^N.

To characterize the behavior of the channels, we will make use of the symmetric Holevo capacity, defined as follows:

$$I(W) \equiv I(X; B)_\rho, \tag{2}$$

where the quantum mutual information with respect to a classical-quantum state ρ^{XB} is given by

$$I(X; B) \equiv H(X)_\rho + H(B)_\rho - H(XB)_\rho, \tag{3}$$

with $\rho^{XB} = \frac{1}{2}|0\rangle\langle 0| \otimes \rho_0^B + \frac{1}{2}|1\rangle\langle 1| \otimes \rho_1^B$. In the above, the von Neumann entropy $H(\rho)$ is defined as

$$H(\rho) \equiv - \operatorname{Tr}\{\rho \log_2 \rho\}. \tag{4}$$

We will also make use of the quantum conditional mutual information defined for a tripartite state ρ^{XYB} as

$$\begin{aligned} I(X; B|Y)_\rho &\equiv H(XY)_\rho + H(YB)_\rho - H(Y)_\rho \\ &\quad - H(XYB)_\rho. \end{aligned} \tag{5}$$

We characterize the reliability of a channel W as the fidelity between the output states

$$F(W) \equiv F(\rho_0, \rho_1), \tag{6}$$

with

$$F(\rho_0, \rho_1) \equiv \| \sqrt{\rho_0} \sqrt{\rho_1} \|_1^2 \tag{7}$$

and

$$\|A\|_1 \equiv \operatorname{Tr} \sqrt{A^\dagger A}. \tag{8}$$

The Holevo capacity and the fidelity can be seen as quantum generalizations of the mutual information and the Bhattacharya parameter from the classical setting, respectively (see, e.g., [9]).

B. Classical-Quantum Multi-User Channels

In the following sections, we will focus on two particular kinds of multi-user channels: classical-quantum multiple access channels (cq-MACs) and classical-quantum interference channels.

We begin with the classical-quantum interference channel, and for simplicity we focus on the case of two senders and two receivers. The interference channel can be modeled mathematically as the following triple:

$$\left(\mathcal{X}_1 \times \mathcal{X}_2, W, \mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2} \right), \tag{9}$$

with

$$W : x_1, x_2 \to \rho_{x_1, x_2}^{B_1 B_2}. \tag{10}$$

The information processing task for the classical-quantum interference channel [3], [6] is as follows. The kth sender would like to communicate a message to the kth receiver, where $k \in \{1, 2\}$. Sender k chooses a message m_k from a message set $\mathcal{M}_k = \{1, \cdots, 2^{NR_k}\}$, and encodes her message as a codeword $x_k^N(m_k) \in \mathcal{X}_k^N$. The encoding for each sender is given by $\{x_1^N(m_1)\}_{m_1 \in \mathcal{M}_1}$ and $\{x_2^N(m_2)\}_{m_2 \in \mathcal{M}_2}$, respectively, with the corresponding receivers' decoding POVMs denoted by $\{\Lambda_{m_1}\}$ and $\{\Gamma_{m_2}\}$. The code is said to be an (N, R_1, R_2, ϵ)-code, if the average probability of error is bounded as follows

$$\bar{p}_e = \frac{1}{|\mathcal{M}_1||\mathcal{M}_2|} \sum_{m_1, m_2} p_e(m_1, m_2) \leq \epsilon, \tag{11}$$

where the probability of error $p_e(m_1, m_2)$ for a pair of messages (m_1, m_2) is given by

$$p_e(m_1, m_2) = \operatorname{Tr} \left\{ (I - \Lambda_{m_1} \otimes \Gamma_{m_2}) \rho_{x_1^N(m_1), x_2^N(m_2)}^{B_1^N B_2^N} \right\}, \tag{12}$$

with $\rho_{x_1^N(m_1),x_2^N(m_2)}^{B_1^N B_2^N}$ the state resulting when senders 1 and 2 transmit the codewords $x_1^N(m_1)$ and $x_2^N(m_2)$ through N instances of the channel, respectively.

A rate pair (R_1, R_2) is said to be *achievable* for the two-user classical-quantum interference channel described above if there exists an (N, R_1, R_2, ϵ)-code $\forall \epsilon > 0$ and sufficiently large N.

The two-user classical-quantum interference channel induces two c-q MACs which can be modeled as

$$\left(\mathcal{X}_1 \times \mathcal{X}_2, \rho_{x_1,x_2}^{B_1} = \mathrm{Tr}_{B_2}\{\rho_{x_1,x_2}^{B_1 B_2}\}, \mathcal{H}^{B_1} \right), \tag{13}$$

and

$$\left(\mathcal{X}_1 \times \mathcal{X}_2, \rho_{x_1,x_2}^{B_2} = \mathrm{Tr}_{B_1}\{\rho_{x_1,x_2}^{B_1 B_2}\}, \mathcal{H}^{B_2} \right). \tag{14}$$

The *rate region* for a channel is given by the closure of all achievable rates for that channel. We will be particularly interested in the Han-Kobayashi rate region for the two-user interference channel. This region was achieved in the classical setting by exploiting a coding strategy for the interference channel which induces two three-user MACs, together with a simultaneous decoder [5].

The two-user c-q MAC is defined by the following triple, corresponding to the input alphabets, channel output state, and output system:

$$\left(\mathcal{X}_1 \times \mathcal{X}_2, \rho_{x_1,x_2}^{B}, \mathcal{H}^{B} \right). \tag{15}$$

The coding task is for two senders to communicate individual messages to a single receiver. The detailed description of the information processing task is somewhat similar to the above, so we omit it for brevity's sake.

Later, we will be particularly interested in *compound* cq-MACs. Indeed, compound channels form a class of channels with so-called "channel uncertainty." In this model, a channel is chosen from a set of possible channels, and used to transmit the information, thus generalizing the traditional setting in which both sender and receiver have full knowledge of the channel before choosing their code. The classical and quantum capacities of compound quantum channels have been studied in [26] and [27], respectively.

A compound cq-MAC is defined by a set $\mathcal{W} = \{W_i\}$ of cq-MAC channels where each W_i can be written as

$$W_i : x_1, x_2 \to \rho_{x_1,x_2,i}^{B}, \tag{16}$$

and characterized by its output state $\rho_{x_1,x_2,i}^{B}$, taken with respect to the input pairs $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$.

Note that we will look at the case in which the receiver knows the particular channel W_i which has been chosen. However, the sender does not have this knowledge. This assumption can be easily justified in the case of taking many uses of the channel, since channel tomography can be performed in order to give the receiver knowledge of the channel which has been chosen. Moreover, this requires a small number of channel uses when compared to the overall number of channel uses, thereby not affecting the communication rate.

C. Polar Codes for Classical-Quantum Channels

In [20] a polar coding scheme for single-user classical-quantum channels was introduced. We review the scheme briefly below before applying the technique to the c-q MAC and interference channel in the following sections.

Polar codes exploit the effect of channel polarization, which is achieved in two steps, namely, by so-called channel combining and channel splitting. In *channel combining* the input sequence u^N is transformed by a linear transformation given by $x^N = u^N G_N$ where

$$G_N = B_N F^{\otimes n} \tag{17}$$

with $N = 2^n$,

$$F \equiv \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \tag{18}$$

and B_N is a permutation matrix known as a "bit reversal" operation [9]. This induces a channel W_N from N single copies of the channel W. For the *channel splitting* step the combined channel W_N from the previous step is used to define new channels $W_N^{(i)}$ as follows:

$$W_N^{(i)} : u_i \to \rho_{(i),u_i}^{U_1^{i-1} B^N}, \tag{19}$$

where

$$\rho_{(i),u_i}^{U_1^{i-1} B^N} = \sum_{u_1^{i-1}} \frac{1}{2^{i-1}} |u_1^{i-1}\rangle\langle u_1^{i-1}| \otimes \sum_{u_{i+1}^N} \frac{1}{2^{N-i}} \rho_{u^N}^{B^N} \tag{20}$$

and u_1^N denotes a row vector $(u_1, \ldots, u_N)$ and correspondingly u_i^j denotes, for $1 \le i, j \le N$, a subvector $(u_i, \ldots, u_j)$. Note that if $j < i$ then u_i^j is empty. Similarly for a vector u_1^N and a set $A \subset \{1, \ldots, N\}$ we write u_A to denote the subvector $(u_i : i \in A)$. The above can be seen as a "genie-aided" successive cancellation decoder, where the i-th measurement estimates the bit u_i, with the assumptions that the entire output is available to the decoder, the previous bits u_1^{i-1} are correctly decoded and the distribution over the bits u_{i+1}^N is uniform.

The channel polarization effect ensures that the fraction of channels $W_N^{(i)}$ which have the property $I(W_N^{(i)}) \in (1 - \delta, 1]$ goes to the symmetric Holevo information $I(W)$ and the fraction with $I(W_N^{(i)}) \in [0, \delta)$ goes to $1 - I(W)$ for any $\delta \in (0, 1)$, as N goes to infinity through powers of two [9], [20] (see [20] for a precise statement). Hence we choose a polar code as a "G_N-coset code" [9]; that is, we choose a subset $A \subset \{1, \ldots, N\}$ and re-write the input transformation

$$x^N = u^N G_N \tag{21}$$

as

$$x^N = u_A G_N(A) \oplus u_{A^c} G_N(A^c), \tag{22}$$

where $G_N(A)$ denotes the submatrix of G_N constructed from the rows of G_N with indices in A. Now we can fix a code (N, K, A, u_{A^c}) where N is the length of the code, $K = |A|$ is the number of information bits, A fixes the indices for the information bits and u_{A^c} is the vector of so-called frozen bits.

A polar code has the above properties and is such that it obeys the polar coding rule, which is that the set of information indices A is chosen such that the following inequality holds between fidelities

$$F(W_N^{(i)}) \leq F(W_N^{(j)}) \tag{23}$$

for all $i \in A$ and $j \in A^c$.

Lastly, a bound on the block error probability $P_e(N, R)$ for blocklength N and rate R was derived for a fixed $R < I(W)$ and $\beta < \frac{1}{2}$, with the result [20]

$$P_e(N, R) = o(2^{-\frac{1}{2}N^{\beta}}). \tag{24}$$

The measurement achieving this error bound was called a quantum successive cancellation decoder [21], and the error analysis exploited Sen's non-commutative union bound [7].

The technique described in this section can also be extended to non-binary alphabets using the approach in [28], where an additional permutation is introduced to the coding process, which ensures polarization. Since these permutations only act on the inputs of the channel, which are also in our setting classical, this approach can be directly translated to classical-quantum channels. For further details we refer to [28].

III. POLAR CODES FOR THE TWO-USER BINARY-INPUT cq-MAC

The achievable rate region for the classical-quantum MAC is described by the following bounds [1]:

$$R_x \leq I(X; B|Y)_{\rho}, \tag{25}$$
$$R_y \leq I(Y; B|X)_{\rho}, \tag{26}$$
$$R_x + R_y \leq I(XY; B)_{\rho}, \tag{27}$$

with respect to a ccq-state

$$\rho^{XYB} = \sum_{x,y} p_X(x) p_Y(y) |x\rangle\langle x|_X \otimes |y\rangle\langle y|_Y \otimes \rho_{x,y}^B. \tag{28}$$

The case in which the last inequality above is saturated is of particular interest to us and the resulting line, which interpolates between the points $(I(X; B)_{\rho}, I(Y; B|X)_{\rho})$ and $(I(X; B|Y)_{\rho}, I(Y; B)_{\rho})$, is called the *dominant face* of the rate region. It is clear that if every point on the dominant face can be achieved then we can also achieve every other point within the rate region by resource wasting.

Recently, Arikan introduced the technique of "monotone chain rules" for handling the Slepian-Wolf problem [17] with the polar coding technique, and Önay applied this approach to the binary-input MAC [12]. The advantage of this approach is that with each monotone chain rule, we can achieve a rate pair lying on the dominant face of the rate region. Furthermore, the achievable points form a dense subset of all points on the dominant face, so that we can approximate every point on the dominant face to arbitrarily good accuracy. Note that it will be crucial when coding for the interference channel that the compound MAC described later can achieve specific points on the dominant face of the rate region, which is ensured by the approach described in this section. Here we apply the technique to the classical-quantum MAC with two senders, each with binary alphabet.

The main difficulty in generalizing Arikan's technique to the classical-quantum setting lies in adjusting the decoder for the quantum outputs of the channel, such that they properly handle messages from two different senders while keeping the exponentially decreasing error. In addition we generalize the proofs for the continuity of the achievable rates and the path scaling to the classical-quantum setting.

We now recall the idea of a monotone chain rule, but with our discussion here being for the classical-quantum MAC. Let X^N and Y^N each denote a sequence of N uniformly random bits. Let U^N be the result of sender 1 processing the sequence X^N with the polar encoder, and let V^N be the result of sender 2 processing the sequence Y^N with the polar encoder. Let $(S_1, \ldots, S_{2N})$ be a permutation of the input sequence $U^N V^N$ such that the relative order of the elements constituting U^N is preserved. A chain-rule expansion for mutual information is said to be monotone with respect to U^N if it is of the following form:

$$N \cdot I(XY; B) = I(U^N V^N; B^N) \tag{29}$$

$$= \sum_{i=1}^{2N} I(S_i; B^N|S^{i-1}), \tag{30}$$

with the first equality following from the reversibility of the encoders and the second from the chain rule for mutual information. Based on the above permutation, we let b^{2N} denote a binary sequence which we can think of as a "path," where b_k is equal to zero if the kth channel use is transmitting an information bit from the input sequence U^N of the first sender and equal to one if the kth channel use is transmitting an information bit from the input sequence V^N of the second sender. This gives rise to the following rates:

$$R_x = \frac{1}{N} \sum_{k:b_k=0} I(S_k; B^N|S^{k-1})$$

$$\leq \frac{1}{N} I(U^N; B^N|V^N) = I(X; B|Y), \tag{31}$$

$$R_y = \frac{1}{N} \sum_{k:b_k=1} I(S_k; B^N|S^{k-1})$$

$$\leq \frac{1}{N} I(V^N; B^N|U^N) = I(Y; B|X), \tag{32}$$

$$R_x + R_y = I(XY; B), \tag{33}$$

where the inequalitites hold because of the structure of the monotone chain rules in (30), the statistical independence of U^N and V^N, and the one-to-one correspondence between U^N, V^N and X^N, Y^N, respectively.

To illustrate the chain rule approach, consider for example the case where we limit the form of the chain rule to splitting the first sender's messages into two parts corresponding to the chain $b^{2N} = 0^j 1^N 0^{N-j}$. Here we would get the explicit rates

$$R_x = \frac{1}{N} \sum_{i=1}^{j} I(U_i; B^N|U^{i-1})$$

$$+ \frac{1}{N} \sum_{i=j+1}^{N} I(U_i; B^N|U^{i-1} V^N), \tag{34}$$

$$R_y = \frac{1}{N} \sum_{i=1}^{N} I(V_i; B^N | U^j V^{i-1}), \tag{35}$$

which achieve points on the dominant face corresponding to the particular choice of the parameter j.

Using the standard polar coding channel-combining technique outlined in the previous section, we get a combined channel W_N from W^N by transforming both input sequences as

$$x^N = u^N G_N, \quad y^N = v^N G_N. \tag{36}$$

Now for the channel splitting step we have to distinguish whether we want to decode a bit from sender 1 or from sender 2 as follows

$$W_N^{(b_k,i,j)} = \begin{cases} W_N^{(0,i,j)} : u_i \to \rho_{(0,i,j),u_i}^{U_1^{i-1} V_1^j B^N} & \text{if } b_k = 0 \\ W_N^{(1,i,j)} : v_j \to \rho_{(1,i,j),v_j}^{U_1^i V_1^{j-1} B^N} & \text{if } b_k = 1 \end{cases} \tag{37}$$

with output states

$$\rho_{(0,i,j),u_i}^{U_1^{i-1} V_1^j B^N} = \sum_{u_1^{i-1}, v_1^j} \frac{1}{2^{k-1}} |u_1^{i-1}\rangle\langle u_1^{i-1}| \\ \otimes |v_1^j\rangle\langle v_1^j| \otimes \bar{\rho}_{u_1^i, v_1^j}^{B^N}, \tag{38}$$

$$\rho_{(1,i,j),v_j}^{U_1^i V_1^{j-1} B^N} = \sum_{u_1^i, v_1^{j-1}} \frac{1}{2^{k-1}} |u_1^i\rangle\langle u_1^i| \\ \otimes |v_1^{j-1}\rangle\langle v_1^{j-1}| \otimes \bar{\rho}_{u_1^i, v_1^j}^{B^N}, \tag{39}$$

where

$$\bar{\rho}_{u_1^i, v_1^j}^{B^N} = \sum_{u_{i+1}^N, v_{j+1}^N} \frac{1}{2^{2N-k}} \rho_{u^N, v^N}^{B^N}. \tag{40}$$

Similar to the case of classical-quantum polar coding for a single sender, we now discuss how a *quantum successive cancellation decoder* operates for the cq-MAC. As in [20] we can build projectors to decide whether the kth input, corresponding to the split channel $W_N^{(b_k,i,j)}$, is equal to zero or one:

$$\Pi_{(b_k,i,j),0} = \begin{cases} \Pi_{(0,i,j),0}^{U_1^{i-1} V_1^j B^N} & \text{if } b_k = 0 \\ \Pi_{(1,i,j),0}^{U_1^i V_1^{j-1} B^N} & \text{if } b_k = 1 \end{cases} \tag{41}$$

with

$$\Pi_{(0,i,j),0}^{U_1^{i-1} V_1^j B^N} = \left\{ \sqrt{\rho_{(0,i,j),0}^{U_1^{i-1} V_1^j B^N}} - \sqrt{\rho_{(0,i,j),1}^{U_1^{i-1} V_1^j B^N}} \geq 0 \right\} \tag{42}$$

$$\Pi_{(1,i,j),0}^{U_1^i V_1^{j-1} B^N} = \left\{ \sqrt{\rho_{(1,i,j),0}^{U_1^i V_1^{j-1} B^N}} - \sqrt{\rho_{(1,i,j),1}^{U_1^i V_1^{j-1} B^N}} \geq 0 \right\} \tag{43}$$

and

$$\Pi_{(b_k,i,j),1} = \mathbb{1} - \Pi_{(b_k,i,j),0}. \tag{44}$$

The notation $\{A \geq 0\}$ denotes the projector onto the positive eigenspace of A, and $\{A < 0\}$ denotes the projector onto its negative eigenspace.

Note that, similar to [20], we can write

$$\Pi_{(0,i,j),0}^{U_1^{i-1} V_1^j B^N} = \sum_{u_1^{i-1}, v_1^j} |u_1^{i-1}\rangle\langle u_1^{i-1}| \otimes |v_1^j\rangle\langle v_1^j| \\ \otimes \Pi_{(0,i,j),u_1^{i-1}0, v_1^j}^{B^N}, \tag{45}$$

$$\Pi_{(1,i,j),0}^{U_1^i V_1^{j-1} B^N} = \sum_{u_1^i, v_1^{j-1}} |u_1^i\rangle\langle u_1^i| \otimes |v_1^{j-1}\rangle\langle v_1^{j-1}| \\ \otimes \Pi_{(1,i,j),u_1^i, v_1^{j-1}0}^{B^N}, \tag{46}$$

where

$$\Pi_{(0,i,j),u_1^{i-1}0, v_1^j}^{B^N} = \left\{ \sqrt{\bar{\rho}_{u_1^{i-1}0, v_1^j}^{B^N}} - \sqrt{\bar{\rho}_{u_1^{i-1}1, v_1^j}^{B^N}} \geq 0 \right\}, \tag{47}$$

$$\Pi_{(1,i,j),u_1^i, v_1^{j-1}0}^{B^N} = \left\{ \sqrt{\bar{\rho}_{u_1^i, v_1^{j-1}0}^{B^N}} - \sqrt{\bar{\rho}_{u_1^i, v_1^{j-1}1}^{B^N}} \geq 0 \right\}. \tag{48}$$

Now again with arguments presented in [20] we get a POVM with elements

$$\Lambda_{u^N, v^N} = \Pi_{(b_1,i_1,j_1),\{u_1,v_1\}} \\ \cdots \Pi_{(b_{2N},i_{2N},j_{2N}),\{u_1^{N-1} u_N v_1^N, u_1^N v_1^{N-1} v_N\}} \\ \cdots \Pi_{(b_1,i_1,j_1),\{u_1,v_1\}}, \tag{49}$$

where the exact values of i and j depend on the monotone chain rule chosen for decoding, as well as whether a projector attempts to decode u_i or v_j. As required for a POVM we also have that

$$\sum_{u_A, v_A} \Lambda_{u^N, v^N} = \mathbb{1}^{B^N}, \tag{50}$$

by noting that we can set $\Pi_{(b_k,i,j),\{u_i,v_j\}} = \mathbb{1}$ when $\{u_i, v_j\}$ is a frozen bit.

Using the bitwise projections we can build the successive cancellation decoder with the decoding rules:

$$\hat{u}_i = \begin{cases} u_i & \text{if } i \in A^c \\ h(\hat{u}_1^{i-1}, \hat{v}_1^j) & \text{if } i \in A, \end{cases} \tag{51}$$

$$\hat{v}_j = \begin{cases} v_j & \text{if } j \in A^c \\ g(\hat{u}_1^i, \hat{v}_1^{j-1}) & \text{if } j \in A, \end{cases} \tag{52}$$

where $h(\hat{u}_1^{i-1}, \hat{v}_1^j)$ is the outcome of the kth measurement when $b_k = 0$ based on

$$\{\Pi_{(0,i,j),u_1^{i-1}0, v_1^j}^{B^N}, \Pi_{(0,i,j),u_1^{i-1}1, v_1^j}^{B^N}\}, \tag{53}$$

and $g(\hat{u}_1^i, \hat{v}_1^{j-1})$ is the outcome when $b_k = 1$ based on

$$\{\Pi_{(1,i,j),u_1^i, v_1^{j-1}0}^{B^N}, \Pi_{(1,i,j),u_1^i, v_1^{j-1}1}^{B^N}\}. \tag{54}$$

Due to the structure of the decoder and the polarization effect, the block error probability decays exponentially with the number of channel uses as in the single-sender case described in the previous section.

A. Continuity of Rates and Approximations

We now argue that the above approach can be used to achieve the entire dominant face of the rate region. This task was achieved for the classical Slepian-Wolf source coding problem [17] involving rates based on the conditional Shannon entropy, and then extended to the complementary problem of channel coding over the classical MAC [12]. We now extend that technique by applying it to channel coding for the classical-quantum MAC.

We start by defining a distance measure. Let $b^{2N}, \tilde{b}^{2N}$ denote two paths and $(R_u, R_v), (\tilde{R}_u, \tilde{R}_v)$ their corresponding rate pairs. Then we define the distance between the two paths b^{2N} and $\tilde{b}^{2N}$ as follows

$$d(b^{2N}, \tilde{b}^{2N}) \equiv |R_u - \tilde{R}_u| = |R_v - \tilde{R}_v|, \qquad (55)$$

where the last equality holds since $R_u + R_v = \tilde{R}_u + \tilde{R}_v = I(XY; B)$.

We now define two paths $b^{2N}, \tilde{b}^{2N}$ to be neighbors if $\tilde{b}^{2N}$ can be obtained from b^{2N} by transposing b_i with b_j for some $i < j$ such that $b_i \neq b_j$ and b_{i-1}^{j+1} is either all 0 or all 1. We state the following proposition, which bounds the distance between two neighboring paths. This generalizes [17, Proposition 3] to the quantum setting considered here.

Proposition 3.1: If paths b^{2N} and $\tilde{b}^{2N}$ are neighboring, then the following holds

$$d(b^{2N}, \tilde{b}^{2N}) \leq \frac{1}{N}. \qquad (56)$$

Proof: Let b^{2N} be a path with edge variables S^{2N} and let $\tilde{b}^{2N}$ differ from b^{2N} by a transposition in the coordinates $i < j$.

First we check the case, where $b_i = 0$ and $b_j = 1$ and the string b_{i+1}^{j-1} contains only 1's. The rate difference can be written as

$$R_u - \tilde{R}_u = \frac{1}{N}[I(S_i; B^N|S^{i-1}) \\ - I(S_i; B^N|S^{i-1}, S_j, S_{i+1}^{j-1})], \qquad (57)$$

and we can see directly that

$$R_u - \tilde{R}_u \leq \frac{1}{N}, \qquad (58)$$

where the inequality follows from the non-negativity of the conditional mutual information and a dimension bound when one unconditioned system is classical.

Since $R_u - \tilde{R}_u \geq 0$ we can conclude

$$|R_u - \tilde{R}_u| \leq \frac{1}{N}. \qquad (59)$$

This holds for $b_i^j = 01^{j-i}$. The three other options ($b_i^j \in \{0^{j-i}1, 1^{j-i}0, 10^{j-i}\}$) can be proven similarly, by exchanging the roles of b^{2N} and $\tilde{b}^{2N}$ or by considering $R_v - \tilde{R}_v$ or both. ∎

Using this result we can show that the distribution of achievable points on the dominant face of the rate region is dense, and we state the following theorem generalizing [17, Th. 1] to the quantum case considered here.

Theorem 3.2: Let (R_x, R_y) be a given rate pair on the dominant face. For any given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ and a chain rule b^{2N} on $U^N V^N$ such that b^{2N} belongs to the class $v_{2N} = \{0^i 1^N 0^{N-i} : 0 \leq i \leq N\}$ and has a rate pair (R_1, R_2) satisfying

$$|R_1 - R_x| \leq \epsilon \text{ and } |R_2 - R_y| \leq \epsilon. \qquad (60)$$

The proof follows from the fact that in v_{2N} two paths $0^i 1^N 0^{N-i}$ and $0^{i+1} 1^N 0^{N-i-1}$ are neighbors, thus we simply have to fix $N > \frac{1}{\epsilon}$ and Theorem 3.2 follows from Proposition 3.1.

Note that this result is only concerned with permutations from the class v_{2N}. This is sufficient for our purposes, but we note that it may be interesting to consider a more general class when choosing the decoding path.

B. Path Scaling and Polarization

It was shown in the previous section that one can find polar codes that approximate points on the dominant face of the rate region for a cq-MAC. Here we show that these approximations are stable under scaling of the chosen path, implying that performing a step of the polar coding recursion does not change the achievable rates so long as polarization still holds. We generalize ideas from [17] to the classical-quantum MAC.

We will look at paths kb^{2N} which denote the scaling of a path b^{2N} as

$$\underbrace{b_1 \ldots b_1}_{k} \underbrace{b_2 \ldots b_2}_{k} \ldots \ldots \underbrace{b_{2N} \ldots b_{2N}}_{k} \qquad (61)$$

and therefore represent a monotone chain rule for $U^{kN} V^{kN}$. Note that we can write a step of the polar-code transformation as

$$T_{2i-1} = S_i \oplus \tilde{S}_i, \quad T_{2i} = S_i, \qquad (62)$$

and thus we can show that an additional step of polarization does not affect the rate

$$\begin{aligned} I(T_{2i-1}; B^N|T^{2i-2}) &+ I(T_{2i}; B^N|T^{2i-1}) \\ &= I(T_{2i-1}T_{2i}; B^N|T^{2i-2}) \\ &= I(S_i \oplus \tilde{S}_i, S_i; B^N|S^{i-1} \oplus \tilde{S}^{i-1}, S^{i-1}) \\ &= I(\tilde{S}_i, S_i; B^N|\tilde{S}^{i-1}, S^{i-1}) \\ &= 2I(S_i; B^N|S^{i-1}), \end{aligned} \qquad (63)$$

where the last step follows if T^{4N} follows the path $2b^{2N}$. From this we can conclude that if a path b^{2N} achieves a rate (R_1, R_2) then the path $2b^{2N}$ achieves the same rate pair.

Now the polarization argument follows directly using arguments from the single-sender case in [20].

C. Polar Code Performance

Since we use a POVM with the same basic structure as the single-sender case, the analysis of the error probability follows the same arguments. That is, by applying the non-commutative union bound from [7] to the probability of error $P_e(M, b^{2N}, (K_u, K_v), (A_u, A_v), (u_{A_u^c}, v_{A_v^c}))$ for code length M with $M = 2^m N$ and m a positive integer, a chosen
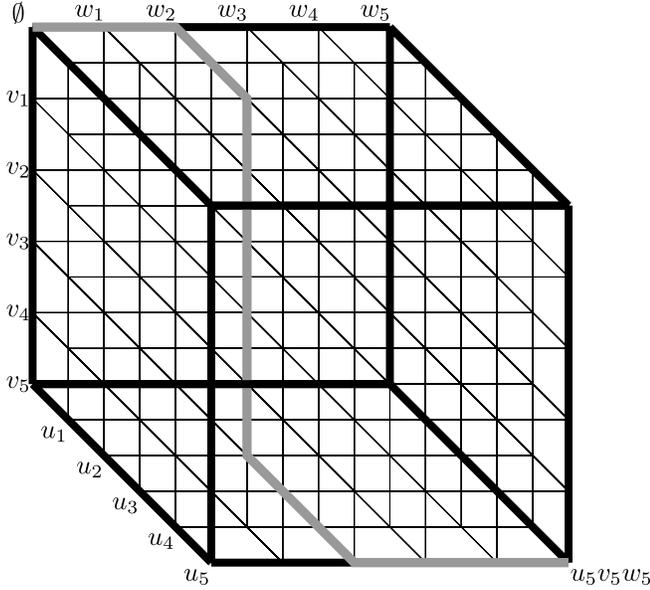
Fig. 1. Order for decoding the 3-user MAC, with example path in grey. Allowed paths are only those which go only to the right, front or bottom at each step, to imply a monotone chain rule. This is important to ensure the correct decoding order for the successive cancellation decoder.

path b^{2N}, the number of information bits (K_u, K_v) and sets of information bits (A_u, A_v) for each sender and the choice for the frozen bits $(u_{A_u^c}, v_{A_v^c})$, we get

$$P_e(M, b^{2N}, (K_u, K_v), (A_u, A_v), (u_{A_u^c}, v_{A_v^c}))$$

$$\leq 2 \sqrt{\sum_{i \in A_u, j \in A_v} \frac{1}{2} \sqrt{F(W_N^{(b_k, i, j)})}}. \tag{64}$$

Therefore, we can state that the error probability in [20] (see Section II-C) also holds for multiple-user settings

$$P_e(M, R) = o(2^{-\frac{1}{2} M^\beta}), \tag{65}$$

with $M = 2^m N$.

IV. THREE SENDER MAC

We can easily extend the approach for two senders discussed above to the case of many senders. For our purposes, we are particularly interested in the three-sender setting. Therefore, similar to the two-dimensional case, we simply follow a path through a three-dimensional cube (see Figure 1 for example), in order to choose a path $b_k \in \{0, 1, 2\}$ giving rise to the following achievable rates

$$R_x = \frac{1}{N} \sum_{k:b_k=0} I(S_k; B^N | S^{k-1}) \leq I(X; B|YZ),$$

$$R_y = \frac{1}{N} \sum_{k:b_k=1} I(S_k; B^N | S^{k-1}) \leq I(Y; B|XZ),$$

$$R_z = \frac{1}{N} \sum_{k:b_k=2} I(S_k; B^N | S^{k-1}) \leq I(Z; B|XY),$$

$$R_x + R_y \leq I(XY; B|Z),$$
$$R_x + R_z \leq I(XZ; B|Y),$$
$$R_y + R_z \leq I(ZY; B|X),$$
$$R_x + R_y + R_z = I(XYZ; B). \tag{66}$$

Note that for the m-sender case we can generalize the above coding method by simply following a path in an m-dimensional structure while making sure that the entropy equations remain monotonic.

V. UNIVERSAL POLAR CODES FOR THE COMPOUND MAC

Next we will describe how the so-called "universal polar codes" introduced in [18] can be applied to the cq-MAC to achieve rates for compound channels. In particular we will make use of the second scheme described in [18] and the generalizations of this scheme to MACs in [16].

For now we will look at compound MACs based on sets of two different MACs. The essential approach is to "align" polarized indices as follows. Note that we assume that the selected channel is known to the receiver but not to the sender. For simplicity we consider two 2-sender MACs with equal sum rate. It is clear that a standard polar code which is good for one of the channels is not necessarily good for the other one. To get around this issue we align the two senders independently.

Conceptually the approach is similar to the classical setting, but needs to be applied to the sets of good and bad classical-quantum channels, defined by upper bounding the fidelity of the polarized channels.

Recall that in Section II-C we reviewed the channel splitting step for classical-quantum channel polarization, first introduced in [20]. Similarly, here we define the partial split channels $P_i : U_i \rightarrow B^N U_1^{i-1} V_1^j$ and $Q_i : U_i \rightarrow B^N U_1^{i-1} V_1^j$, each corresponding to the first sender of one of the two MACs in the set comprising the compound MAC. In the previous section, b_k served as a label indicating which sender should be decoded in the kth step. Moreover, the channels P_i and Q_i can be considered to be equivalent to looking at only the channel uses of the corresponding MACs for which $b_k = 0$. Let

$$\mathcal{G}_{(1)} = \left\{ i \in [1:N] : \sqrt{F(P_i)} < 2^{-N^\beta} \right\},$$

$$\mathcal{G}_{(2)} = \left\{ i \in [1:N] : \sqrt{F(Q_i)} < 2^{-N^\beta} \right\},$$

$$\mathcal{B}_{(1)} = \left\{ i \in [1:N] : \sqrt{F(P_i)} \geq 2^{-N^\beta} \right\},$$

$$\mathcal{B}_{(2)} = \left\{ i \in [1:N] : \sqrt{F(Q_i)} \geq 2^{-N^\beta} \right\}. \tag{67}$$

denote the sets of indices corresponding to whether a bit is good or bad for a channel. These sets tell us whether the attempt of sending an information bit through one of the MACs would be successful with high probability for the ith channel use of the first sender. Due to the polarization effect, all bits will be in one of the following sets:

$$\mathcal{A}_{\mathrm{I}} = \mathcal{G}_{(1)} \cap \mathcal{G}_{(2)}, \tag{68}$$
$$\mathcal{A}_{\mathrm{II}} = \mathcal{G}_{(1)} \cap \mathcal{B}_{(2)}, \tag{69}$$
$$\mathcal{A}_{\mathrm{III}} = \mathcal{B}_{(1)} \cap \mathcal{G}_{(2)}, \tag{70}$$
$$\mathcal{A}_{\mathrm{IV}} = \mathcal{B}_{(1)} \cap \mathcal{B}_{(2)}. \tag{71}$$

Bits belonging to the sets $\mathcal{A}_{\mathrm{I}}$ will also be decoded with high probability in the compound setting and bits in $\mathcal{A}_{\mathrm{IV}}$ will have to be set as frozen bits. Due to Theorem 3.2 in Section III-A we can find monotone chain rules for each MAC which
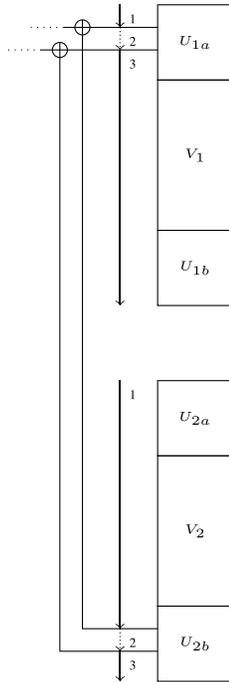
Fig. 2. Coding for compound MAC: first iteration step. The arrows indicate the decoding order and must be followed in ascending order of the attached numbers.

approximate every point on the dominant face of the rate region of the corresponding MAC.

The main idea from here is to align the sets $\mathcal{A}_{\mathrm{II}}$ and $\mathcal{A}_{\mathrm{III}}$ within a recursion to achieve the capacity of the compound MAC. We will do so alternating in each step of the recursion either for the first or the second sender. Here, just as in the classical case, we have to ensure that we align bits such that the successive cancellation decoder can still be applied.

We take two polar coding blocks which have both already been polarized independently of each other. Since both blocks have been built from the same channel, the sets of indices are identical for both blocks. We then combine the first index from $\mathcal{A}_{\mathrm{II}}$ in the first block with the first index of $\mathcal{A}_{\mathrm{III}}$ in the second block by an additional CNOT gate, and similarly for the second indices and so on. With such a scheme, we can halve the fraction of incompatible indices, those from the sets $\mathcal{A}_{\mathrm{II}}$ and $\mathcal{A}_{\mathrm{III}}$, for the first sender.

Intuitively this can be seen as sending the same information bits via both of the aligned channels, so that the reciever will be able to decode one of them independently of which MAC is actually used. Since we assume that the reciever knows the used channel, this works well with the successive cancellation decoder, because the reciever can just decode the channel which is good for the used MAC first and then decode the aligned channel as if it is a frozen bit.

In the next step we take two of the blocks after the first iteration step and repeat the process for the second sender. Hence, we again halve the fraction of incompatible indices for this sender. In the following we repeat this process until the fraction of incompatible indices tends to zero.

To generalize the above scheme to the k-user MAC we simply decode by alternating over the different senders in each

step of the recursion. Therefore each sender becomes aligned in every k-th step of the recursion. In the recursion mK steps reduce the fraction of incompatible indices for the K-user MAC to $(|\mathcal{A}_{\mathrm{II}}| + |\mathcal{A}_{\mathrm{III}}|)/2^m N$. For example for the case of two users this means that every two steps we halve the fraction of incompatible indices. This is due to the fact that we need one recursion step for each sender to halve the fraction of incompatible indices for that particular sender. Then we can reorder our decoding in a way that the successive cancellation decoder still holds. Figure 2 illustrates the process.

In order to use the compound MAC for the interference channel we need to generalize the described approach to the setting of unequal sum rates. Therefore assume that we want to code for a rate pair (R_x, R_y) on the dominant face of the achievable rate region for the compound MAC consisting of a set of two MACs. Now we can find a rate pair (R'_x, R'_y) for the first MAC in that set and a rate pair (R''_x, R''_y) for the second MAC such that

$$R_x \leq \min(R'_x, R''_x) \qquad (72)$$
$$R_y \leq \min(R'_y, R''_y). \qquad (73)$$

We can use the corresponding monotone chain rules for these two MACs to code for the targeted point on the rate region of the compound MAC. Note that in the setting of unequal sum-rate the sets $\mathcal{A}_{\mathrm{II}}$ and $\mathcal{A}_{\mathrm{III}}$ are not necessarily of equal size. This is not a problem for the aligning process, because we can simply align until one of the sets has no unaligned indices left and then handle the remaining indices in the larger set as frozen indices. It is easy to see that this is sufficient to code at rates $\min(R'_x, R''_x)$ and $\min(R'_y, R''_y)$ and therefore achieve the dominant face for the achievable rate region of the compound MAC.

It was previously unknown how to code for the k-user compound MAC in a quantum setting. Having this result will allow us to also code for classical-quantum interference channels, as discussed in the next section.

VI. INTERFERENCE CHANNEL

The two-user classical-quantum interference channel [3], [6], as discussed in Section II-B, can be represented by its set of output states as follows:

$$\{\rho^{B_1 B_2}_{x_1, x_2}\}_{x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2}. \qquad (74)$$

In the classical setting the best known achievable rate region for the interference channel is given by the Han-Kobayashi rate region [5]. We now show that this region can be achieved for the two-user classical-quantum interference channel by using polar codes. The scheme is a direct generalization of that presented in [16]. Indeed note that the conceptual approach is independent of whether we apply it to classical or classical-quantum channels. Therefore the Han-Kobayashi rate region can be achieved by splitting the message of the first sender m_1 into two parts labelled (l_1, l_2) and similarly for the second sender the message m_2 is split into (l_3, l_4). Now we get channel inputs represented by the random variables X_1^N and X_2^N via symbol-to-symbol encoding maps $x_1(v_1, v_2)$ and $x_2(v_3, v_4)$ corresponding to the codewords $v_j^N(L_j)$. Then receiver 1

decodes from $\rho_{x_1,x_2}^{B_1}$ the messages (l_1, l_2, l_3) and receiver 2 decodes $\rho_{x_1,x_2}^{B_2}$ to get the message triple (l_2, l_3, l_4). The Han-Kobayashi rate region is defined for rates (S_1, S_2, T_1, T_2) as follows:

$$S_1 \leq I(V_1; B_1|V_3 V_4),$$
$$T_1 \leq I(V_3; B_1|V_1 V_4),$$
$$T_2 \leq I(V_4; B_1|V_1 V_3),$$
$$S_1 + T_1 \leq I(V_1 V_3; B_1|V_4), \qquad (75)$$
$$S_1 + T_2 \leq I(V_1 V_4; B_1|V_3),$$
$$T_1 + T_2 \leq I(V_3 V_4; B_1|V_1),$$
$$S_1 + T_1 + T_2 \leq I(V_1 V_3 V_4; B_1),$$
$$S_2 \leq I(V_2; B_2|V_3 V_4),$$
$$T_1 \leq I(V_3; B_2|V_2 V_4),$$
$$T_2 \leq I(V_4; B_2|V_2 V_4),$$
$$S_2 + T_1 \leq I(V_2 V_3; B_2|V_4),$$
$$S_2 + T_2 \leq I(V_2 V_4; B_2|V_3),$$
$$T_1 + T_2 \leq I(V_3 V_4; B_2|V_2),$$
$$S_2 + T_1 + T_2 \leq I(V_2 V_3 V_4; B_2), \qquad (76)$$

The achievable rate region for the interference channel is the set of all rates $(S_1 + T_1, S_2 + T_2)$.

This can be seen as an interference network with four senders $x_1(v_1, v_2)$ and $x_2(v_3, v_4)$ and two recievers. The rate tuples (S_1, T_1, T_2) and (S_2, T_1, T_2) coincide with two 3-user MAC regions with two common senders and the intersection of both gives the rate region (S_1, S_2, T_1, T_2). We can find monotone chain rules for every point on the dominant face of each MAC. Now we use the approach for the compound MAC to align the two common senders along with the results of the previous section in order to achieve the rate region for the two-user interference channel using polar coding. Note that it is necessary to use the approach presented for compound MACs in order to ensure that the decoding used for the polar codes is good for both MACs. With this approach we can achieve the Han-Kobayashi rate region by a successive cancellation decoder for each receiver. There are known techniques for handling arbitrary input distributions. Rather than go into detail, we point to the work in [16] which elaborates on this point.

VII. CONCLUSION

We have applied the recently introduced polar coding technique to achieve known rates for a variety of classical-quantum multi-user channels, with our main result being that the Han-Kobayashi rate region for the two-user interference channel can be achieved by a successive cancellation decoder via polar coding. In particular, we emphasize that this was achieved without the use of a quantum simultaneous decoder. The interference channel model forms a basis from which other multi-user channels can be built. This result and the wide range of problems for which polar coding has been applied in classical information theory suggest that it might be possible to generalize a wide range of problems to the classical-quantum setting using a successive decoder and in particular without the need of a quantum simultaneous decoder.

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