8-10-2016

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Masahiro Takeoka
Japan National Institute of Information and Communications Technology

Kaushik P. Seshadreesan
Max Planck Institute for the Science of Light

Mark M. Wilde
Louisiana State University

Recommended Citation

Takeoka, M., Seshadreesan, K., & Wilde, M. (2016). Unconstrained distillation capacities of a pure-loss bosonic broadcast channel. *IEEE International Symposium on Information Theory - Proceedings, 2016-August*, 2484-2488. https://doi.org/10.1109/ISIT.2016.7541746

Unconstrained distillation capacities of a pure-loss bosonic broadcast channel

Masahiro Takeoka*, Kaushik P. Seshadreesan†, and Mark M. Wilde‡
* *Quantum ICT Laboratory, National Institute of Information and Communications Technology, Koganei, Tokyo 184-8795, Japan*
† *Max-Planck-Institut für die Physik des Lichts, 91058 Erlangen, Germany*
‡ *Hearne Institute for Theoretical Physics, Department of Physics and Astronomy, Center for Computation and Technology, Louisiana State University, Baton Rouge, Louisiana 70803, USA*

***Abstract*—Bosonic channels are important in practice as they form a simple model for free-space or fiber-optic communication. Here we consider a single-sender multiple-receiver pure-loss bosonic broadcast channel and determine the unconstrained capacity region for the distillation of bipartite entanglement and secret key between the sender and each receiver, whenever they are allowed arbitrary public classical communication. We show how the state merging protocol leads to achievable rates in this setting, giving an inner bound on the capacity region. We also determine an outer bound on the region and find that the outer bound matches the inner bound in the infinite-energy limit, thereby establishing the unconstrained capacity region for such channels. Our result could provide a useful benchmark for implementing a broadcasting of entanglement and secret key through such channels. An important open question relevant to practice is to determine the capacity region in both this setting and the single-sender single-receiver case when there is an energy constraint on the transmitter.**

I. Introduction

Quantum key distribution (QKD) [1], [2] and entanglement distillation (ED) [3] are two cornerstones of quantum communication. QKD enables two or more cooperating parties to distill and share unconditionally secure random bit sequences, which could then be used for secure classical communication. ED, on the other hand, allows them to distill pure maximal entanglement from a quantum state shared via a noisy communication channel, which could then be used to faithfully transfer quantum states by using quantum teleportation [4]. In both protocols, the parties are allowed to perform (in principle) an unlimited amount of local operations and classical communication (LOCC).

Quantum communication technologies have matured tremendously in recent years. In particular, QKD has been available commercially for a number of years and has now expanded to inter-city networks [5], [6]. Also, efforts are currently underway to accomplish QKD in free space between the earth and satellites [7].

Quantum communication, however, faces an important challenge. Like most other quantum technologies, its performance is affected by noise. Loss is the main source of error in typical optical communication channels and severely limits the rates and distances at which secret key or quantum entanglement can be distilled using the channel. All practical implementations of QKD to date are known to exhibit a rate-loss tradeoff, in which the rate of secret key extraction drops with increasing distance [8]. In the case of the standard optical-fiber communication channel, the drop is exponential with increasing distance.

Some time after these limitations were observed, Refs. [9], [10] provided a mathematical proof, using the notion of squashed entanglement [11], that the tradeoff is indeed a fundamental limitation even with unconstrained input energy. One of the main results of [9], [10] is an upper bound on the two-way LOCC assisted quantum and secret key agreement capacity of a pure-loss bosonic channel, which is solely a function of the channel transmittance η. If the transmitter can consume only a finite amount of energy (as is in some practical cases), then tighter bounds are available [9], [10]. As a consequence, no yet-to-be-discovered protocol could ever surpass the limitations established in [9], [10]. Ref. [12] extended the squashed entanglement technique to obtain upper bounds for a variety of phase-insensitive Gaussian channels. Concurrently with [12], Ref. [13] improved the infinite-energy bound from [9], [10] and conclusively established the unconstrained capacity of the pure-loss bosonic channel as $\mathcal{C}(\eta) = -\log_2(1-\eta)$. It is still an open question to determine the constrained capacity (i.e., when the transmitter is limited to consuming finite energy).

One of the long-term goals of quantum communication is to establish a quantum internet [14]: a large collection of interconnected quantum networks between multiple users that enables secure classical communication and distributed quantum information processing. Apart from point-to-point links, network architectures based on single-sender multiple-receivers (modeled as broadcast channels) and vice versa (multiple access channels) are also important in this context. Even though various network quantum communication scenarios have been examined [15], [16], [17], [18], [19], [20], there has been limited work on the capacity of entanglement and secret key distillation assisted by unlimited LOCC. Only recently in [21] were outer bounds on the achievable rates established for multipartite secret key agreement and entanglement generation between any subset of the users of a general single-sender m-receiver quantum broadcast channel (QBC) (for any $m \geq 1$)

when assisted by unlimited LOCC between all the users. The main idea was to employ multipartite generalizations of the squashed entanglement [22], [23] and the methods of [9], [10].

In this paper, we consider a single-sender multiple-receiver pure-loss bosonic QBC and establish the unconstrained capacity region for the distillation of bipartite entanglement and secret key between the sender and each receiver assisted by unlimited LOCC. To prove the statement, we establish inner bounds on the achievable rate region by employing the quantum state merging protocol [24], [25]. The converse part relies upon several tools. First, we utilize a teleportation simulation argument originally introduced in [26, Section V] and recently generalized in [13] to wider families of channels and continuous-variable systems. Next, it is known that the relative entropy of entanglement is an upper bound on the distillable key of a bipartite state [27], and the recent work in [13] stated how it is possible to combine the relative entropy of entanglement upper bound with the teleportation simulation argument to arrive at upper bounds on the secret-key agreement capacity of certain single-sender single-receiver channels. We find that the outer bounds match the inner bounds in the infinite-energy limit, thereby establishing the unconstrained capacity region. An important open question is to determine the constrained capacity region, i.e., when only finite energy is available at the transmitter.

The paper is organized as follows. In Section II, we describe a general LOCC-assisted distillation protocol for a QBC and the mathematical and physical model of the pure-loss bosonic QBC. The unconstrained capacity region is given in Section III along with a proof for the single-sender two-receiver case. Section IV concludes the paper. The appendix generalizes the main theorem to the single-sender multiple-receiver case.

II. LOCC-assisted distillation protocol and the channel model

In the main text, we consider a single-sender two-receiver QBC $\mathcal{N}_{A'\to BC}$ (Fig. 1(a)) and an (n, E_{AB}, E_{AC}, K_{AB}, K_{AC}, ε) protocol described as follows (the appendix considers the generalization to multiple receivers). The sender, Alice, prepares some quantum systems in an initial quantum state and successively sends some of these systems to the receivers, Bob and Charlie, by interleaving n channel uses of the broadcast channel with rounds of LOCC. The goal of the protocol is to distill bipartite maximally entangled states Φ_{AB} and Φ_{AC} and private states γ_{AB} and γ_{AC} (equivalently secret keys [28]). After each channel use, they can perform an arbitrary number of rounds of LOCC (in any direction with any number of parties). The quantities E_{AB} and E_{AC} denote entanglement rates (i.e., the logarithm of the Schmidt rank of Φ_{AB} and Φ_{AC}, respectively, normalized by the number of channel uses) and K_{AB} and K_{AC} are secret-key rates (i.e., the number of secret-key bits in γ_{AB}, and γ_{AC}, respectively, normalized by the number of channel uses). The protocol considered here is similar to the one described in [21], except that here we do not consider the other possible rates E_{BC}, K_{BC}, E_{ABC}, and K_{ABC}.

A rate tuple (E_{AB}, E_{AC}, K_{AB}, K_{AC}) is achievable if for all $\varepsilon \in (0,1)$ and sufficiently large n, there exists an (n, E_{AB}, E_{AC}, K_{AB}, K_{AC}, ε) protocol of the above form. The capacity region is the closure of the set of all achievable rates.

In the following, we concentrate on a specific channel: a pure-loss bosonic QBC which we denote by $\mathcal{L}_{A'\to BC}$. For this channel, the input state is split into three systems and one system is sent to each of Bob, Charlie, and the environment with transmittance η_B, η_C, and $1-\eta_B-\eta_C$, respectively, where $\eta_B, \eta_C \in [0,1]$, $\eta_B+\eta_C \leq 1$.

Physically the signal splitting is modeled by a pair of two-input two-output beam splitters in which the signal is mixed with the vacuum state. For example, one can construct such a channel by a sequence of two beam splitters with transmittance $\eta_B+\eta_C$ and $\eta_B/(\eta_B+\eta_C)$, respectively, where the first beam splitter splits the signal and the environment, and the second one splits the signal into Bob's and Charlie's parts (see Fig. 1(b)). Mathematically this is characterized by the following input-output relation:

$$\hat{b} = \sqrt{\eta_B}\,\hat{a}' + \sqrt{\frac{\eta_B(1-\eta_B-\eta_C)}{\eta_B+\eta_C}}\,\hat{f} + \sqrt{\frac{\eta_C}{\eta_B+\eta_C}}\,\hat{g}, \quad (1)$$

$$\hat{c} = -\sqrt{\eta_C}\,\hat{a}' - \sqrt{\frac{\eta_C(1-\eta_B-\eta_C)}{\eta_B+\eta_C}}\,\hat{f} + \sqrt{\frac{\eta_B}{\eta_B+\eta_C}}\,\hat{g}, \quad (2)$$

$$\hat{e} = -\sqrt{1-\eta_B-\eta_C}\,\hat{a}' + \sqrt{\eta_B+\eta_C}\,\hat{f}, \quad (3)$$

where $\hat{a}'$, $\hat{b}$, $\hat{c}$, $\hat{e}$ are annihilation operators for Alice's input, Bob's output, and Charlie's output modes, respectively, and $\hat{f}$ and $\hat{g}$ are annihilation operators for vacuum inputs from the environment.

Critical for our analysis is that the physical implementation of $\mathcal{L}_{A'\to BC}$ is not unique. One can model the same channel by a different concatenation of two other beam splitters: for example, we could have a first beam splitter split system B from C and E, and then a second one split C and E. Obviously, it is also possible to split C at the first beam splitter. These physical models are described in Fig. 1(c) and (d), respectively. In the next section, we will use these other physical models to explicitly calculate the rate regions.

III. Unconstrained capacity region

Our main contribution is the following theorem:

Theorem 1: The LOCC-assisted, unconstrained capacity region of the pure-loss bosonic QBC $\mathcal{L}_{A'\to BC}$ is given by

$$E_{AB} + K_{AB} \leq \log_2([1-\eta_C]/[1-\eta_B-\eta_C]), \quad (4)$$

$$E_{AC} + K_{AC} \leq \log_2([1-\eta_B]/[1-\eta_B-\eta_C]), \quad (5)$$

$$E_{AB} + K_{AB} + E_{AC} + K_{AC} \leq -\log_2(1-\eta_B-\eta_C). \quad (6)$$

See the appendix for a generalization of this theorem to the multiple-receiver model from [29].

A. Achievability part

To achieve the rate region (4)–(6), we consider a distillation protocol which employs quantum state merging. State merging was introduced in [24], [25] and provides an operational

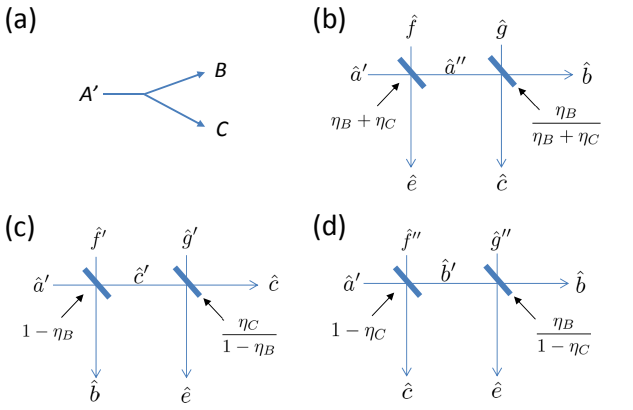

Fig. 1. (a) Single-sender two-receiver quantum broadcast channel. (b)-(d) Various physical implementations of the pure-loss bosonic broadcast channel with transmittances η_B and η_C.

meaning for the conditional quantum entropy. For a state ρ_{AB}, its conditional quantum entropy is defined as $H(A|B)_\rho = H(AB)_\rho - H(B)_\rho$ where $H(AB)_\rho$ and $H(B)_\rho$ are the quantum entropies of ρ_{AB} and its marginal ρ_B, respectively. For many copies of ρ_{AB} shared between Alice and Bob, $H(A|B)_\rho$ is the optimal rate at which maximally entangled two-qubit states need to be consumed to transfer Alice's systems to Bob's side via LOCC. If $H(A|B)_\rho$ is negative, the result is that after transferring Alice's systems, they can gain (i.e., distill) entanglement at rate $-H(A|B)_\rho$. State merging also yields a quantum analog of the Slepian-Wolf theorem in classical distributed compression problem and has been applied to the QBC in [19], [20].

Here we consider the following alternative state merging based protocol. Alice first prepares n copies of a two-mode squeezed vacuum (TMSV)

$$|\Psi(N_S)\rangle_{AA'} = \sum_{m=0}^{\infty} \sqrt{\lambda_m(N_S)}|m\rangle_A|m\rangle_{A'}, \tag{7}$$

where $|m\rangle$ is an m-photon state, $\lambda_m(N_S) = N_S^m/(N_S+1)^{m+1}$, and N_S is the average photon number of the state per mode. She sends system A' to Bob and Charlie through a pure-loss broadcast channel. After n uses of the channel, Alice, Bob, and Charlie share n copies of the state $\phi_{ABC} = \mathcal{L}_{A'\to BC}(|\Psi(N_S)\rangle\langle\Psi(N_S)|_{AA'})$.

Then by using $\phi_{ABC}^{\otimes n}$, they perform state merging to establish entanglement. More precisely, Bob and Charlie transfer their system back to Alice by LOCC (similar to reverse reconciliation in the point-to-point scenario). This could be done by applying the point-to-point state merging protocol successively [24], [25] or alternatively, by applying the multiparty simultaneous decoding state merging [30]. Then we obtain the achievable rate region for E_{AB} and E_{AC} as

$$E_{AB} \le -H(B|AC)_\phi, \tag{8}$$

$$E_{AC} \le -H(C|AB)_\phi, \tag{9}$$

$$E_{AB} + E_{AC} \le -H(BC|A)_\phi. \tag{10}$$

Since one "ebit" of entanglement can generate one private bit of secret key, the left-hand side of the above inequalities can be modified as $E_{AB} \to E_{AB} + K_{AB}$, $E_{AC} \to E_{AC} + K_{AC}$.

The right-hand side of these inequalities can be explicitly calculated. Recall that the marginal of the TMSV $\Psi_{A'}(N_S) = \mathrm{Tr}_A[|\Psi(N_S)\rangle\langle\Psi(N_S)|_{AA'}]$ is a thermal state with mean photon number N_S. Its entropy is equal to $H(A')_\Psi = g(N_S)$, where $g(x) = (x+1)\log_2(x+1) - x\log_2 x$. Also a pure-loss channel with transmittance η maps a thermal state to another thermal state with reduced average photon number. Let $U^{\mathcal{L}}_{A'\to BCE}$ be an isometric extension of $\mathcal{L}_{A'\to BC}$ and let

$$|\phi\rangle_{ABCE} = U^{\mathcal{L}}_{A'\to BCE}|\Psi(N_S)\rangle_{AA'}, \tag{11}$$

be a purification of ϕ_{ABC}. By using $|\phi\rangle_{ABCE}$ and observing the above facts, we have

$$\begin{aligned} -H(B|AC)_\phi &= H(AC)_\phi - H(ABC)_\phi \\ &= H(BE)_\phi - H(E)_\phi \\ &= g((1-\eta_C)N_S) - g((1-\eta_B-\eta_C)N_S). \end{aligned}$$

In the limit as $N_S \to \infty$, this converges to $\log_2\left(\frac{1-\eta_C}{1-\eta_B-\eta_C}\right)$. Similarly, we obtain

$$-H(C|AB)_\phi \to \log_2\left(\frac{1-\eta_B}{1-\eta_B-\eta_C}\right), \tag{12}$$

$$-H(BC|A)_\phi \to \log_2\left(\frac{1}{1-\eta_B-\eta_C}\right), \tag{13}$$

in the limit of infinitely large N_S. Thus (4)–(6) are achievable when there is no energy constraint on the transmitter.

B. Converse part

As stated at the end of Section I, the converse relies upon several tools and is given in terms of the relative entropy of entanglement (REE) [31]. The REE for a quantum state ρ_{AB} is defined by

$$E_R(A;B)_\rho = \inf_{\sigma_{AB}\in \mathrm{SEP}} D(\rho_{AB}\|\sigma_{AB}) \tag{14}$$

where $D(\rho\|\sigma) = \mathrm{Tr}[\rho(\log_2\rho - \log_2\sigma)]$ is the quantum relative entropy and SEP denotes the set of separable states. The original LOCC-assisted communication protocol can equivalently be rewritten by using a teleportation simulation argument [26, Section V] suitably extended to continuous-variable bosonic channels [13]. Teleportation simulation in the case of a point-to-point channel can be understood as the possibility of reducing a sequence of adaptive protocols involving two-way LOCC into a sequence of non-adaptive protocols followed by a final LOCC. For all 'teleportation-simulable channels' (more precisely the channels arising from the action of teleportation on a bipartite state) that allow for such a reduction, an upper bound on the entanglement and secret key agreement capacity can be given by the REE [13], because the REE is an upper bound on the distillable key of any bipartite state [27]. Furthermore, for pure-loss bosonic channels, one can use a concise formula for the REE given in [13]. With these techniques, an upper bound on the unconstrained capacity of a

point-to-point pure-loss channel is equivalent to the REE of the state resulting from sending an infinite-energy TMSV through the channel, explicitly calculated to be equal to $-\log_2(1-\eta)$.

Since the pure-loss bosonic QBC is covariant with respect to displacement operations (which are the teleportation corrections for bosonic channels [32]), it is teleportation-simulable. Then the original broadcasting protocol described in the previous section can be replaced by the distillation of n copies of $\phi_{ABC}=\mathcal{N}_{A'\to BC}(|\Psi(N_S)\rangle\langle\Psi(N_S)|_{AA'})$ via the final LOCC. This LOCC distills entanglement and secret key; i.e., it generates a state ω_{ABC} which is ε-close to $\tilde{\Phi}_{ABC}$:

$$\|\omega_{ABC}-\tilde{\Phi}_{ABC}\|_1\leq\varepsilon \tag{15}$$

with

$$\tilde{\Phi}_{ABC}=\Phi_{A_1B_1}^{\otimes nE_{AB}}\otimes\Phi_{A_2C_1}^{\otimes nE_{AC}}\otimes\gamma_{A_3B_2}^{\otimes nK_{AB}}\otimes\gamma_{A_4C_2}^{\otimes nK_{AC}}, \tag{16}$$

where A_i, B_i, and C_i are subsystems of A, B, and C, respectively. Then by using several well known properties of REE (monotonicity under LOCC, continuity, and subadditivity for product states), we find that

$$\begin{aligned} n(E_{AB}+K_{AB}) &\leq E_R(B;AC)_{\tilde{\Phi}} \\ &\leq E_R(B;AC)_{\omega}+f(n,\varepsilon) \\ &\leq nE_R(B;AC)_{\phi}+f(n,\varepsilon), \end{aligned} \tag{17}$$

where $f(n,\varepsilon)$ is a function such that $\lim_{\varepsilon\to0,n\to\infty}\frac{1}{n}f(n,\varepsilon)=0$. Similar bounds can be obtained in terms of $E_R(C;AB)_\phi$ and $E_R(A;BC)_\phi$, leading to the following outer bound for the capacity region:

$$E_{AB}+K_{AB}\leq E_R(B;AC)_\phi \tag{18}$$

$$E_{AC}+K_{AC}\leq E_R(C;AB)_\phi, \tag{19}$$

$$E_{AB}+K_{AB}+E_{AC}+K_{AC}\leq E_R(A;BC)_\phi. \tag{20}$$

To calculate the right-hand side of each inequality, we use a calculation from [13]: for a point-to-point pure-loss bosonic channel with transmittance η, which we denote by $\mathcal{L}^{\eta}_{A'\to B}$, the REE of $\mathcal{L}^{\eta}_{A'\to B}(|\Psi(N_S)\rangle\langle\Psi(N_S)|_{AA'})$ with $N_S\to\infty$ is given by $-\log_2(1-\eta)$.

For $E_R(A;BC)_\phi$, consider the physical implementation of the channel in Fig. 1(b) and let $\phi'_{AA''}$ be the state such that only the first beam splitter is applied. The second beam splitter is a local unitary in the sense that it operates on B and C whereas our partition is now between A and BC. Thus it does not change the REE. Then we have $E_R(A;A'')_{\phi'}=E_R(A;BC)_\phi$. Also, since $\phi'_{AA''}$ is a TMSV followed by a pure-loss channel with transmittance $\eta_B+\eta_C$, we get

$$\begin{aligned} \lim_{N_S\to\infty}E_R(A;BC)_\phi &= \lim_{N_S\to\infty}E_R(A;A'')_{\phi'} \\ &= -\log_2(1-\eta_B-\eta_C). \end{aligned} \tag{21}$$

To calculate $E_R(C;AB)_\phi$, we employ the physical implementation of the channel in Fig. 1(c) in which the first beam splitter with transmittance $1-\eta_B$ separates B from the others. This beam splitter is followed by the second beam splitter with transmittance $\eta_C/(1-\eta_B)$ which separates C and the environment E.

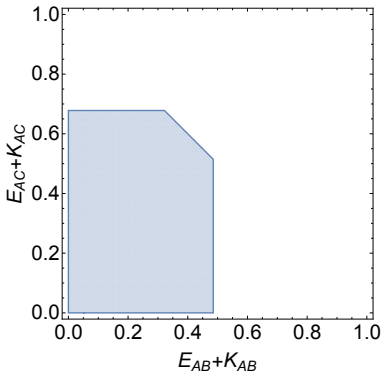

Fig. 2. LOCC-assisted capacity region given by (4)–(6) for the pure-loss bosonic broadcast channel, where $(\eta_B,\,\eta_C)=(0.2,\,0.3)$.

Let $\phi'_{ABC'}$ be the TMSV in which only the first beam splitter is applied. Observe that it is a pure state and its marginal $\phi'_{C'}$ is a thermal state with average photon number $(1-\eta_B)N_S$. Combining these two observations, we can conclude that the state has the following Schmidt decomposition:

$$|\phi''\rangle_{ABC'}=\sum_{m=0}^{\infty}\sqrt{\lambda_m((1-\eta_B)N_S)}|\varphi_m\rangle_{AB}|m\rangle_{C'}, \tag{22}$$

where $\{|\varphi_m\rangle_{AB}\}_m$ is some orthonormal basis. Since $\{|\varphi_m\rangle_{AB}\}$ is an orthonormal set, there exists a local unitary operation acting on systems A and B such that

$$U_{AB}:\,|\varphi_m\rangle_{AB}\to|m\rangle_A|\mathrm{aux}\rangle_B, \tag{23}$$

where $|\mathrm{aux}\rangle$ is some constant auxiliary state. Then we have

$$\begin{aligned} U_{AB}|\phi''\rangle_{ABC'} &= |\mathrm{aux}\rangle_B|\Psi((1-\eta_B)N_S)\rangle_{AC'} \\ &\equiv|\phi'''\rangle_{ABC'}. \end{aligned} \tag{24}$$

Let $\tilde{\phi}_{ABC}=\mathcal{L}^{\bar{\eta}}_{C'\to C}(|\phi'''\rangle\langle\phi'''|_{ABC'})$ where $\bar{\eta}=\eta_C/(1-\eta_B)$. Note that $\phi_{ABC}=\mathcal{L}^{\bar{\eta}}_{C'\to C}(|\phi''\rangle\langle\phi''|_{ABC'})$. Since the local unitary operation U_{AB} does not change the REE between AB and C, we have $E_R(C;AB)_{\tilde{\phi}}=E_R(C;AB)_\phi$. Moreover, $E_R(C;AB)_{\tilde{\phi}}$ is the REE for the TMSV with $(1-\eta_B)N_S$ followed by $\mathcal{L}^{\bar{\eta}}_{C'\to C}$. Then by using the result in [13], we find

$$\begin{aligned} \lim_{N_S\to\infty}E_R(C;AB)_\phi &= \lim_{N_S\to\infty}E_R(C;AB)_{\tilde{\phi}} \\ &= -\log_2(1-\bar{\eta}) \\ &= \log_2([1-\eta_B]/[1-\eta_B-\eta_C]). \end{aligned} \tag{25}$$

Similarly, with the physical implementation picture in Fig. 1(d), we obtain

$$\lim_{N_S\to\infty}E_R(B;AC)_\phi=\log_2([1-\eta_C]/[1-\eta_B-\eta_C]), \tag{26}$$

which completes the proof of the converse part. Figure 2 illustrates an example of the unconstrained capacity region given in (4)–(6).

IV. Conclusion

We have established the unconstrained capacity region of a pure-loss bosonic broadcast channel for LOCC-assisted entanglement and secret key distillation. This result is proved by using the quantum state merging protocol (for the inner bound) and the relative entropy of entanglement (for the outer bound), and it could provide a useful benchmark for the broadcasting of entanglement and secret key through such channels.

There are some interesting problems left open. First, we consider the scenario to share entanglement and a secret key between the sender and each receiver, but we think it is interesting to include other possibilities, i.e., E_{BC} and K_{BC}, and even tripartite entanglement, such as GHZ states, and tripartite common secret key. Such a scenario was discussed in [21], where an outer bound was established by making use of the squashed entanglement. However, these bounds might be further improved, and in addition, it is open to determine how to construct a protocol achieving tight inner bounds.

Second, one might attempt to generalize our scenario to other channels. A consequence of the teleportation simulation argument from [26, Section V] is that the REE upper bound can be applied to all teleportation-simulable channels, as shown in [13]. Thus the approach could work for all broadcast channels for which this is true. Finally, we think it is a pressing open question to determine the constrained capacity region (i.e., with a finite-energy constraint). Since the REE bound using the teleportation reduction technique requires infinite energy to realize an ideal teleportation, one needs an alternative approach here, such as the one given in [21] in order to obtain tighter outer bounds in some cases.

We acknowledge helpful discussions with Sam Braunstein, Saikat Guha, Michal Horodecki, Stefano Pirandola, and John Smolin. This research was supported by the Open Partnership Joint Projects of JSPS Bilateral Joint Research Projects and the ImPACT Program of Council for Science, Technology and Innovation, Japan. MMW is grateful to NICT for hosting and supporting him for a research visit during December 2015, and he acknowledges support from NSF Grant No. CCF-1350397. KPS thanks the Max Planck Society for funding.

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APPENDIX: THE 1-TO-m BROADCAST CHANNEL

In this appendix, we generalize Theorem 1 to the 1-to-m pure-loss broadcast channel for arbitrary positive integer m.

Consider the pure-loss broadcast channel $\mathcal{L}_{A'\to B_1\cdots B_m}$ characterized by a set of transmittances $\{\eta_{B_1},\cdots,\eta_{B_m}\}$ with $\sum_{i=1}^m \eta_{B_i} \leq 1$ [29]. Let $\mathcal{B}=\{B_1,\cdots,B_m\}$, $\mathcal{T}\subseteq\mathcal{B}$, and $\overline{\mathcal{T}}$ be a complement of set $\mathcal{T}$. Then we have the following theorem:

Theorem 2: The LOCC-assisted unconstrained capacity region of the pure-loss bosonic QBC $\mathcal{L}_{A'\to B_1\cdots B_m}$ is given by

$$\sum_{B_i\in\mathcal{T}} E_{AB_i}+K_{AB_i} \leq \log_2\left(\frac{1-\eta_{\overline{\mathcal{T}}}}{1-\eta_{\mathcal{B}}}\right), \tag{27}$$

for all non-empty $\mathcal{T}$, where $\eta_{\mathcal{B}}=\sum_{i=1}^m \eta_{B_i}$ and $\eta_{\overline{\mathcal{T}}}=\sum_{B_i\in\overline{\mathcal{T}}}\eta_{B_i}$.

Proof: The strategy of the proof is quite similar to that of Theorem 1. For the achievability, one can apply the point-to-point state merging protocol successively, which leads to the following achievable rate region:

$$\sum_{B_i\in\mathcal{T}} E_{AB_i} \leq -H(\mathcal{T}|A\overline{\mathcal{T}})_\phi, \tag{28}$$

where

$$\phi_{AB_1\cdots B_m}=\mathcal{L}_{A'\to B_1\cdots B_m}(|\Psi(N_S)\rangle\langle\Psi(N_S)|_{AA'}). \tag{29}$$

The right-hand side of (28) is calculated to be

$$\begin{aligned}-H(\mathcal{T}|A\overline{\mathcal{T}})_\phi &= H(A\overline{\mathcal{T}})_\phi - H(A\mathcal{T}\overline{\mathcal{T}})_\phi \\ &= H(\mathcal{T}E)_\phi - H(E)_\phi \\ &= g((1-\eta_{\overline{\mathcal{T}}})N_S) - g((1-\eta_{\mathcal{B}})N_S),\end{aligned}$$

and taking $N_S\to\infty$, we get $\log_2([1-\eta_{\overline{\mathcal{T}}}]/[1-\eta_{\mathcal{B}}])$. Since one ebit of entanglement can generate one private bit of key, we can replace E_{AB_i} with $E_{AB_i}+K_{AB_i}$ which completes the achievability part.

Remark 3: Since the above rate region reflects the last gain/consumption of entanglement after the sequential operation of the point-to-point state mergings, it could be possible that the protocol is 'catalytic,' meaning that entanglement is consumed at some state merging which is compensated by the following other state mergings. However, this does not happen in our case. We can check it by the following simple observation. Gain/consumption of entanglement at any state merging is given by $-H(\mathcal{S}_1|A\mathcal{S}_2)_\phi$ where $\mathcal{S}_1$ is some nonempty subset of $\mathcal{B}$ and $\mathcal{S}_2$ is other subset (possibly empty) of $\mathcal{B}$ satisfying $\mathcal{S}_1\cap\mathcal{S}_2=\emptyset$. Since $-H(\mathcal{S}_1|A\mathcal{S}_2)_\phi = H(\overline{\mathcal{S}_2})_\phi - H(\overline{\mathcal{S}_1\mathcal{S}_2})_\phi$ and $\mathcal{S}_1$ is non-empty, this quantity always has a positive value meaning that entanglement is generated at all steps of the whole protocol.

For the converse, we need to configure the beam splitter network of the QBC properly. Note that the channel has $m+1$ transmittances $\eta_{B_1},\cdots,\eta_{B_m}$, and $\eta_E\equiv 1-\sum_i\eta_{B_i}$. We can order these transmittances in some sequence and label it as $\eta_1,\eta_2,\cdots,\eta_m,\eta_{m+1}$. Then for any ordering, we can describe the channel by a sequence of m beam splitters where the j-th beam splitter's transmittance is given by

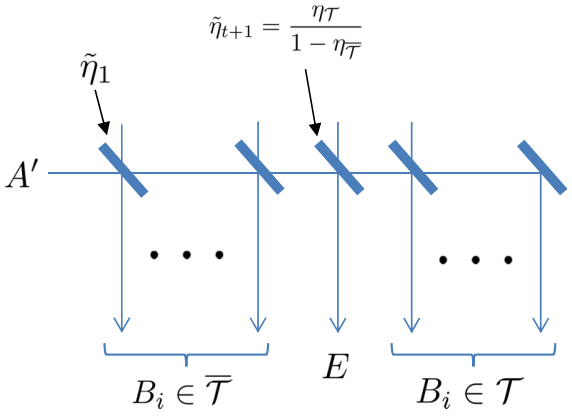

Fig. 3. Implementation of the 1-to-m pure-loss bosonic broadcast channel.

$$\tilde{\eta}_j=\frac{1-\sum_{k=1}^j\eta_k}{1-\sum_{l=1}^{j-1}\eta_l}. \tag{30}$$

Now, for each given $\mathcal{T}$ involving t parties, consider the following specific ordering. For η_i with $1\leq i\leq t$, assign η_{B_j} with $B_j\in\overline{\mathcal{T}}$, $\eta_{t+1}=\eta_E$, and for η_i with $i>t+1$, assign η_{B_j} with $B_j\in\mathcal{T}$. Then the transmittance of the $t+1$ beam splitter is $\eta_{\mathcal{T}}/(1-\eta_{\overline{\mathcal{T}}})$ where $\eta_{\mathcal{T}}=\sum_{B_i\in\mathcal{T}}\eta_{B_i}$ (see Fig. 3). From the main text, we already know that the REE for given partition $\mathcal{T}$ and $A\overline{\mathcal{T}}$ is simply characterized by this transmittance. As a consequence, the REE bound for $N_S\to\infty$ turns out to be

$$\begin{aligned}\sum_{B_i\in\mathcal{T}}(E_{AB_i}+K_{AB_i}) &\leq E_R(\mathcal{T};A\overline{\mathcal{T}})_\phi \\ &= \log_2\left(\frac{1}{1-\frac{\eta_{\mathcal{T}}}{1-\eta_{\overline{\mathcal{T}}}}}\right) \\ &= \log_2\left(\frac{1-\eta_{\overline{\mathcal{T}}}}{1-\eta_{\mathcal{T}}-\eta_{\overline{\mathcal{T}}}}\right), \\ &= \log_2\left(\frac{1-\eta_{\overline{\mathcal{T}}}}{1-\eta_{\mathcal{B}}}\right),\end{aligned} \tag{31}$$

where $\eta_{\mathcal{B}}=\eta_{\mathcal{T}}+\eta_{\overline{\mathcal{T}}}$. ■