

7-1-2018

## Rényi relative entropies of quantum Gaussian states

Kaushik P. Seshadreesan  
*Max Planck Institute for the Science of Light*

Ludovico Lami  
*Universitat Autònoma de Barcelona*

Mark M. Wilde  
*Louisiana State University*

Follow this and additional works at: [https://digitalcommons.lsu.edu/physics\\_astronomy\\_pubs](https://digitalcommons.lsu.edu/physics_astronomy_pubs)

---

### Recommended Citation

Seshadreesan, K., Lami, L., & Wilde, M. (2018). Rényi relative entropies of quantum Gaussian states. *Journal of Mathematical Physics*, 59 (7) <https://doi.org/10.1063/1.5007167>

This Article is brought to you for free and open access by the Department of Physics & Astronomy at LSU Digital Commons. It has been accepted for inclusion in Faculty Publications by an authorized administrator of LSU Digital Commons. For more information, please contact [ir@lsu.edu](mailto:ir@lsu.edu).

## Rényi relative entropies of quantum Gaussian states

Kaushik P. Seshadreesan,<sup>1</sup> Ludovico Lami,<sup>2</sup> and Mark M. Wilde<sup>3</sup>

<sup>1</sup>*Max Planck Institute for the Science of Light, Staudtstr. 2, 91058 Erlangen, Germany*

<sup>2</sup>*Física Teòrica: Informació i Fenòmens Quàntics, Departament de Física, Universitat Autònoma de Barcelona, ES-08193 Bellaterra (Barcelona), Spain*

<sup>3</sup>*Hearne Institute for Theoretical Physics, Department of Physics and Astronomy, Center for Computation and Technology, Louisiana State University, Baton Rouge, Louisiana 70803, USA*

The quantum Rényi relative entropies play a prominent role in quantum information theory, finding applications in characterizing error exponents and strong converse exponents for quantum hypothesis testing and quantum communication theory. On a different thread, quantum Gaussian states have been intensely investigated theoretically, motivated by the fact that they are more readily accessible in the laboratory than are other, more exotic quantum states. In this paper, we derive formulas for the quantum Rényi relative entropies of quantum Gaussian states. We consider both the traditional (Petz) Rényi relative entropy as well as the more recent sandwiched Rényi relative entropy, finding formulas that are expressed solely in terms of the mean vectors and covariance matrices of the underlying quantum Gaussian states. Our development handles the hitherto elusive case for the Petz–Rényi relative entropy when the Rényi parameter is larger than one. Finally, we also derive a formula for the max-relative entropy of two quantum Gaussian states, and we discuss some applications of the formulas derived here.

## I. INTRODUCTION

Motivated by the mathematical foundations of entropy, Rényi<sup>1</sup> defined the following  $\alpha$ -dependent relative entropy as a function of two probability distributions  $p$  and  $q$ :

$$D_\alpha(p\|q) \equiv \frac{1}{\alpha - 1} \ln \left( \sum_x p(x)^\alpha q(x)^{1-\alpha} \right), \quad (1)$$

where  $\alpha \in (0, 1) \cup (1, \infty)$ , and for  $\alpha \in \{0, 1, \infty\}$ , the quantity is defined in the limit. An important special case is the limit  $\alpha \rightarrow 1$ , for which the quantity converges to the relative entropy  $D(p\|q)$ <sup>2</sup>:

$$\lim_{\alpha \rightarrow 1} D_\alpha(p\|q) = D(p\|q) \equiv \sum_x p(x) \ln \left( \frac{p(x)}{q(x)} \right). \quad (2)$$

Since Rényi's work<sup>1</sup>, the quantity  $D_\alpha(p\|q)$  has become known as the Rényi relative entropy and has played an important role in hypothesis testing and information theory<sup>3,4</sup>. Most prominently, the Rényi relative entropy has found operational interpretations in these contexts in terms of error exponents or strong converse exponents, which respectively characterize the exponential rate at which error probabilities decay to zero or increase to one for a given information-processing task.

Towards the goal of developing the quantum generalization of the aforementioned fields, several researchers have defined quantum extensions of the Rényi relative entropy<sup>5-7</sup>. Interestingly, in the quantum case, there are several ways to go about this, due to the non-commutativity of quantum states. A first way of generalizing the Rényi relative entropy was put forward in Ref. 5, where the following quantity was defined for two density operators  $\rho$  and  $\sigma$ :

$$D_\alpha(\rho\|\sigma) \equiv \frac{1}{\alpha - 1} \ln \text{Tr}\{\rho^\alpha \sigma^{1-\alpha}\}, \quad (3)$$

with  $\alpha \in (0, 1) \cup (1, \infty)$ . It has since become known as the Petz-Rényi relative entropy and has the following limits:

$$\lim_{\alpha \rightarrow 0} D_\alpha(\rho\|\sigma) = D_0(\rho\|\sigma) = -\ln \text{Tr}\{\Pi_\rho \sigma\}, \quad (4)$$

$$\lim_{\alpha \rightarrow 1} D_\alpha(\rho\|\sigma) = D(\rho\|\sigma) \equiv \text{Tr}\{\rho [\ln \rho - \ln \sigma]\}, \quad (5)$$

where  $\Pi_\rho$  is the projection onto the support of  $\rho$  and  $D(\rho\|\sigma)$  is the quantum relative entropy<sup>8,9</sup>. More recently, a second way of generalizing the Rényi relative entropy was put forward<sup>6,7</sup>:

$$\tilde{D}_\alpha(\rho\|\sigma) \equiv \frac{1}{\alpha - 1} \ln \text{Tr} \left\{ \left( \sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha \right\}. \quad (6)$$

This quantity is known as the sandwiched Rényi relative entropy, due to the operator sandwich in (6), and it has the following limits<sup>6,7</sup>:

$$\lim_{\alpha \rightarrow 1} \tilde{D}_\alpha(\rho\|\sigma) = D(\rho\|\sigma), \quad (7)$$

$$\lim_{\alpha \rightarrow \infty} \tilde{D}_\alpha(\rho\|\sigma) = D_{\max}(\rho\|\sigma) \equiv \inf \{ \lambda \in \mathbb{R} : \rho \leq e^\lambda \sigma \}, \quad (8)$$

where  $D_{\max}$  denotes the max-relative entropy<sup>10</sup>. Both the Petz–Rényi relative entropy and the sandwiched Rényi relative entropy have found widespread application in quantum hypothesis testing and quantum communication theory<sup>7,11–27</sup>. Particular to the quantum case, all evidence to date indicates that the Petz–Rényi relative entropy is the appropriate quantity to employ in the error exponent regime and the sandwiched Rényi relative entropy in the strong converse regime.

Along a different line, the theory of Gaussian quantum information has been intensely investigated and developed<sup>28–30</sup>, the main motivation behind it being that bosonic Gaussian states and evolutions are more accessible in the laboratory than are their non-Gaussian counterparts. These states and evolutions play a prominent role in quantum optics, but they can also describe the physics of particular superconducting degrees of freedom, trapped ions, and atomic ensembles<sup>30</sup>. Similar to the classical case, a quantum Gaussian state of  $n$  modes is uniquely characterized by a mean vector (first moments) and a covariance matrix (second moments). Furthermore, a quantum Gaussian channel is defined to take Gaussian states to Gaussian states, and as such, one can uniquely characterize a quantum Gaussian channel by its action on the mean vector and covariance matrix of an input Gaussian state<sup>31</sup>. These simple characterizations are helpful for theoretical manipulations: even though Gaussian states are density operators acting on infinite-dimensional, separable Hilbert spaces, it often suffices to manipulate their finite-dimensional mean vectors and covariance matrices. A typical goal is to express information-theoretic functions of Gaussian states solely in terms

of their mean vectors and covariance matrices, so that these functions can be easily evaluated numerically or analytically.

With these two threads in mind, the contribution of the present paper lies at the convergence of them. That is, in this paper, we establish formulas for the Petz–Rényi relative entropy and the sandwiched Rényi relative entropy of any two quantum Gaussian states. As desired, these formulas are expressed solely in terms of the mean vectors and covariance matrices of the two states. The most direct consequence of our formulas is in quantum state discrimination, such that it is now possible to characterize error exponents and strong converse exponents in terms of our formulas. We discuss the application to quantum state discrimination in Section VII A. Given the many applications of quantum Rényi relative entropies, we expect there to be further applications of the formulas provided here.

Two special cases of our formulas have already appeared in the literature, and so it is pertinent to recall these developments now. To see the first one, we should note that the following limit holds for the sandwiched Rényi relative entropy:

$$\lim_{\alpha \rightarrow \frac{1}{2}} \tilde{D}_\alpha(\rho \parallel \sigma) = -\log F(\rho, \sigma), \quad (9)$$

where  $F(\rho, \sigma)$  denotes the well known quantum fidelity<sup>32</sup>:

$$F(\rho, \sigma) = \left[ \text{Tr} \left\{ \sqrt{\sqrt{\sigma} \rho \sqrt{\sigma}} \right\} \right]^2. \quad (10)$$

Due to the significance of fidelity in quantum information theory, a number of works have already devised formulas for the fidelity of quantum Gaussian states. The authors of Refs. 33 and 34 determined a general formula for the fidelity of two zero-mean Gaussian states. The authors of Ref. 34 found the first general formula for the fidelity of two zero-mean Gaussian states in terms of their Hamiltonian matrices, using the tools of Ref. 35. In Ref. 33, the determination of the characteristic function of a Gaussian state sandwiched by the square root of another Gaussian state led to a simpler expression involving only the corresponding covariance matrices, again for the zero-mean case. A number of special cases have also been considered in several contributions<sup>36–45</sup>. Some years after these developments, a general formula for the fidelity between two-mode Gaussian states was derived in Ref. 46 (see also the review in Ref. 28). In Ref. 46, an expression for the  $n$ -mode case was also given, which

can be evaluated numerically. More explicit formulas to deal with this latter case were found recently in Ref. 47. See Ref. 48 for further developments.

In addition to the fidelity of Gaussian states, researchers have also investigated the Petz–Rényi relative entropy of Gaussian states exclusively for the case when  $\alpha \in (0, 1)$ . The authors of Ref. 49 contributed a formula for the Petz–Rényi relative entropy for the case of single-mode Gaussian states, and this approach was generalized to the  $n$ -mode case in Ref. 50. It is worthwhile to note that these authors were interested in the symmetric error exponent of quantum hypothesis testing and that the Petz–Rényi relative entropy arises naturally in this context. The particular case of  $\alpha = 1/2$  for the Petz–Rényi relative entropy was considered in Ref. 51, Lemma 2 and Ref. 52, wherein a compact formula was given for this case.

In light of these prior works, the main contribution of our paper can be understood as a general formula for the Petz–Rényi relative entropy for  $\alpha \in (1, \infty)$  and for the sandwiched Rényi relative entropy for  $\alpha \in (0, 1) \cup (1, \infty)$ . Of especial interest is the hitherto elusive case for the Petz–Rényi relative entropy when  $\alpha \in (1, \infty)$ . Additionally, we derive an alternate expression for the Petz–Rényi relative entropy for  $\alpha \in (0, 1)$ . We find that these formulas simplify significantly when  $\alpha = 2$ , and we also devote a section to the derivation of a formula for the max-relative entropy of quantum Gaussian states. Specifically, our main results are as follows:

1. Theorem 18 gives a formula for the Petz–Rényi relative entropy for  $\alpha \in (0, 1)$ .
2. Theorem 19 gives a formula for the Petz–Rényi relative entropy for  $\alpha \in (1, \infty)$ .
3. Theorem 21 gives a formula for the sandwiched Rényi relative entropy for  $\alpha \in (0, 1)$ .
4. Theorem 22 gives a formula for the sandwiched Rényi relative entropy for  $\alpha \in (1, \infty)$ .
5. Theorem 24 gives a formula for the max-relative entropy.

The main tools that we use to derive these formulas are those that were developed to derive the fidelity formula<sup>33–35,46,47</sup>. Given the prominence of both the Rényi relative entropies and quantum Gaussian states in quantum information theory, we expect that the formulas

derived here will find application in a variety of avenues in quantum information and other areas of physics.

We organize our paper as follows. In Section II, we review some basics of quantum Gaussian states that are needed for the remainder of the paper. Section III is devoted to recalling and proving analytic forms for several mappings of quantum Gaussian states. After this preparatory material, Section IV offers a derivation of the Petz–Rényi relative entropy of two quantum Gaussian states, first for  $\alpha \in (0, 1)$  and then for  $\alpha \in (1, \infty)$ . Section V gives a derivation of the sandwiched Rényi relative entropy of two quantum Gaussian states, for  $\alpha \in (0, 1)$  and then for  $\alpha \in (1, \infty)$ . In Section VI, we derive a formula for the max-relative entropy of two quantum Gaussian states. We then discuss applications of our results in Section VII, and we conclude in Section VIII with a summary and some open questions.

## II. PRELIMINARIES ON QUANTUM GAUSSIAN STATES

We begin with a brief review of quantum Gaussian states and point the reader to Ref. 30 for more background. Our development applies to  $n$ -mode Gaussian states, where  $n$  is some fixed positive integer. Let  $\hat{x}_j$  denote each quadrature operator ( $2n$  of them for an  $n$ -mode state), and let

$$\hat{x} \equiv [\hat{q}_1, \dots, \hat{q}_n, \hat{p}_1, \dots, \hat{p}_n] \equiv [\hat{x}_1, \dots, \hat{x}_{2n}] \quad (11)$$

denote the vector of quadrature operators, so that the first  $n$  entries correspond to position-quadrature operators and the last  $n$  to momentum-quadrature operators. The quadrature operators satisfy the following commutation relations:

$$[\hat{x}_j, \hat{x}_k] = i\Omega_{j,k}, \quad \text{where} \quad \Omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes I_n, \quad (12)$$

and  $I_n$  is the  $n \times n$  identity matrix. Note that  $\Omega^T = -\Omega$  and the matrix  $i\Omega$  is involutory, (i.e.,  $(i\Omega)(i\Omega) = I$ ) facts that we use repeatedly in what follows.

A faithful Gaussian state  $\rho$  of  $n$  modes can be written as<sup>30</sup>

$$\rho = \frac{1}{Z_\rho} \exp \left[ -\frac{1}{2} (\hat{x} - s_\rho)^T H_\rho (\hat{x} - s_\rho) \right], \quad (13)$$

$$Z_\rho \equiv \sqrt{\det([V_\rho + i\Omega]/2)}, \quad (14)$$

where  $H_\rho$  is a  $2n \times 2n$  positive-definite real Hamiltonian matrix,  $s_\rho \in \mathbb{R}^{2n}$  is the mean vector, defined as  $s_\rho = \langle \hat{x} \rangle_\rho = \text{Tr}\{\hat{x}\rho\}$ , and  $V_\rho$  is the symmetric covariance matrix, whose entries are defined as

$$[V_\rho]_{j,k} = \langle \{ \hat{x}_j - s_\rho^j, \hat{x}_k - s_\rho^k \} \rangle_\rho. \quad (15)$$

The matrices  $V_\rho$  and  $H_\rho$  are related by<sup>53-57</sup>

$$H_\rho = 2i\Omega \operatorname{arcoth}(V_\rho i\Omega), \quad (16)$$

$$V_\rho = \operatorname{coth}(i\Omega H_\rho/2) i\Omega, \quad (17)$$

where

$$\operatorname{coth}(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1}, \quad (18)$$

$$\operatorname{arcoth}(x) = \frac{1}{2} \ln \left( \frac{x+1}{x-1} \right). \quad (19)$$

These relationships imply for finite  $H$  that

$$H_\rho > 0 \quad \iff \quad V_\rho + i\Omega > 0. \quad (20)$$

Note that the condition  $V_\rho + i\Omega \geq 0$ , if the state is not necessarily faithful, leaves open the possibility that  $H$  diverges. We say that  $V_\rho$  is a legitimate covariance matrix if it satisfies the following uncertainty principle<sup>58</sup>:

$$V_\rho + i\Omega \geq 0, \quad (21)$$

and we note that, by a transpose, this is equivalent to  $V_\rho - i\Omega \geq 0$ .

Alternatively, given a positive-definite real matrix  $H_\rho$ , we have that<sup>59,60</sup>

$$\operatorname{Tr} \left\{ \exp \left[ -\frac{1}{2} \hat{x}^T H_\rho \hat{x} \right] \right\} = \sqrt{\det([V_\rho + i\Omega]/2)}, \quad (22)$$



where  $V_\rho = \coth(i\Omega H_\rho/2)i\Omega$ . We also consider operators of the form  $\exp[-\frac{1}{2}\hat{x}^T H \hat{x}]$ , in which  $H$  is a symmetric matrix with complex entries. In this case, we can still exploit the functional relationships in (16) and (17). Note that  $H$  is symmetric if and only if  $V$  is symmetric, one direction of which can be seen from the following:

$$V^T = [\coth(i\Omega H/2)i\Omega]^T \quad (23)$$

$$= -i\Omega \coth(-Hi\Omega/2) \quad (24)$$

$$= i\Omega \coth(Hi\Omega/2) (i\Omega) (i\Omega) \quad (25)$$

$$= \coth((i\Omega) Hi\Omega (i\Omega) /2) (i\Omega) \quad (26)$$

$$= \coth(i\Omega H/2)i\Omega \quad (27)$$

$$= V, \quad (28)$$

with the other implication following similarly. In the above, we used the fact that  $\coth$  is an odd function and the functional analytic relation  $Mf(L)M^{-1} = f(MLM^{-1})$ .

A  $2n \times 2n$  real matrix  $S$  is symplectic if it preserves the symplectic form:  $S\Omega S^T = \Omega$ . According to Williamson's theorem<sup>61</sup>, there is a diagonalization of the covariance matrix  $V_\rho$  of the form

$$V_\rho = S_\rho (D_\rho \oplus D_\rho) S_\rho^T, \quad (29)$$

where  $S_\rho$  is a symplectic matrix and  $D_\rho \equiv \text{diag}(\nu_1, \dots, \nu_n)$  is a diagonal matrix of symplectic eigenvalues, such that  $\nu_i \geq 1$  for all  $i \in \{1, \dots, n\}$ . A quantum Gaussian state is faithful if all of its symplectic eigenvalues are strictly greater than one (this also means that the state is positive definite). In our paper, we focus exclusively on faithful Gaussian states.

We also define

$$W_\rho = -V_\rho i\Omega, \quad (30)$$

which by the relations in (16) and (17) gives us the following well known Cayley transforms<sup>62-64</sup>:

$$\exp(i\Omega H_\rho) = \frac{W_\rho - I}{W_\rho + I}, \quad (31)$$

$$W_\rho = \frac{I + \exp(i\Omega H_\rho)}{I - \exp(i\Omega H_\rho)}. \quad (32)$$

In the above and in what follows, our convention is that  $\frac{A}{B} = AB^{-1}$  for matrices  $A$  and  $B$ , but observe that the ordering does not matter if  $A$  and  $B$  commute. By substituting (30) into (22), we see that

$$\text{Tr} \left\{ \exp \left[ -\frac{1}{2} \hat{x}^T H_\rho \hat{x} \right] \right\} = \sqrt{\det([I - W] i\Omega/2)}. \quad (33)$$

The mean displacement  $s_\rho \in \mathbb{R}^{2n}$  in (13) can be generated by applying the displacement operator, defined for  $s \in \mathbb{R}^{2n}$  as

$$D(s) = \exp [s^T i\Omega \hat{x}] = \exp [-\hat{x}^T i\Omega s], \quad (34)$$

on a zero-mean state  $\exp [-\frac{1}{2} \hat{x}^T H_\rho \hat{x}] / \sqrt{\det([V_\rho + i\Omega]/2)}$  as follows:

$$D(-s_\rho) \frac{\exp [-\frac{1}{2} \hat{x}^T H_\rho \hat{x}]}{\sqrt{\det([V_\rho + i\Omega]/2)}} D(s_\rho) = \frac{\exp [-\frac{1}{2} (\hat{x} - s_\rho)^T H_\rho (\hat{x} - s_\rho)]}{\sqrt{\det([V_\rho + i\Omega]/2)}}. \quad (35)$$

In our paper, we also consider the operator  $D(s)$  in the more general case when  $s \in \mathbb{C}^{2n}$ , but then we no longer refer to it as a ‘‘displacement operator’’ because it loses its physical interpretation in this more general case.

### III. COMPUTATIONS WITH QUANTUM GAUSSIAN STATES

#### A. Powers of quantum Gaussian states

**Proposition 1** *Given a quantum Gaussian state  $\rho$  expressed as*

$$\rho = \frac{1}{[\det([V + i\Omega]/2)]^{1/2}} \exp \left[ -\frac{1}{2} \hat{x}^T H \hat{x} \right], \quad (36)$$

*for a positive-definite real matrix  $H$  and corresponding covariance matrix  $V$ , the density operator  $\rho^\alpha / \text{Tr}\{\rho^\alpha\}$ , for  $\alpha > 0$ , can be written as*

$$\rho(\alpha) \equiv \frac{\rho^\alpha}{\text{Tr}\{\rho^\alpha\}} = \frac{1}{[\det([V^{(\alpha)} + i\Omega]/2)]^{1/2}} \exp \left[ -\frac{1}{2} \hat{x}^T H^{(\alpha)} \hat{x} \right], \quad (37)$$

*where the positive-definite real matrix  $H^{(\alpha)}$  and the covariance matrix  $V^{(\alpha)}$  are given by*

$$H^{(\alpha)} = \alpha H, \quad (38)$$

$$V^{(\alpha)} \equiv V_{\rho^{(\alpha)}} \equiv \frac{(I + (V_\rho i\Omega)^{-1})^\alpha + (I - (V_\rho i\Omega)^{-1})^\alpha}{(I + (V_\rho i\Omega)^{-1})^\alpha - (I - (V_\rho i\Omega)^{-1})^\alpha} i\Omega. \quad (39)$$

**Proof.** Consider that

$$\rho^\alpha = \left[ \frac{1}{[\det([V + i\Omega]/2)]^{1/2}} \exp \left[ -\frac{1}{2} \hat{x}^T H \hat{x} \right] \right]^\alpha \quad (40)$$

$$= \frac{1}{[\det([V + i\Omega]/2)]^{\alpha/2}} \exp \left[ -\frac{1}{2} \hat{x}^T [\alpha H] \hat{x} \right]. \quad (41)$$

Then

$$\text{Tr}\{\rho^\alpha\} = \frac{1}{[\det([V + i\Omega]/2)]^{\alpha/2}} \text{Tr} \left\{ \exp \left[ -\frac{1}{2} \hat{x}^T [\alpha H] \hat{x} \right] \right\}. \quad (42)$$

To compute the covariance matrix corresponding to  $\alpha H$ , which we call  $V_{\rho(\alpha)}$ , we exploit (16) and (17) and find that

$$V_{\rho(\alpha)} = \coth(i\Omega\alpha H_\rho/2)i\Omega \quad (43)$$

$$= \coth(i\Omega\alpha [2i\Omega \operatorname{arcoth}(V_\rho i\Omega)]/2)i\Omega \quad (44)$$

$$= \coth(\alpha \operatorname{arcoth}(V_\rho i\Omega))i\Omega. \quad (45)$$

To evaluate the last equality, consider for  $|x| > 1$  that

$$\coth(\alpha \operatorname{arcoth}(x)) = \coth \left( \alpha \frac{1}{2} \ln \left( \frac{x+1}{x-1} \right) \right) \quad (46)$$

$$= \coth \left( \alpha \frac{1}{2} \ln \left( \frac{1+1/x}{1-1/x} \right) \right) \quad (47)$$

$$= \coth \left( \frac{1}{2} \ln \left[ \left( \frac{1+1/x}{1-1/x} \right)^\alpha \right] \right) \quad (48)$$

$$= \frac{\exp \left( 2 \left( \frac{1}{2} \ln \left[ \left( \frac{1+1/x}{1-1/x} \right)^\alpha \right] \right) \right) + 1}{\exp \left( 2 \left( \frac{1}{2} \ln \left[ \left( \frac{1+1/x}{1-1/x} \right)^\alpha \right] \right) \right) - 1} \quad (49)$$

$$= \frac{\left( \frac{1+1/x}{1-1/x} \right)^\alpha + 1}{\left( \frac{1+1/x}{1-1/x} \right)^\alpha - 1} \quad (50)$$

$$= \frac{(1+1/x)^\alpha + (1-1/x)^\alpha}{(1+1/x)^\alpha - (1-1/x)^\alpha}. \quad (51)$$

Several of the above manipulations are possible because  $1 \pm 1/x > 0$  for  $|x| > 1$ , so that  $y \rightarrow y^\alpha$  is a well defined function from  $\mathbb{R}^+ \rightarrow \mathbb{R}^+$ . The matrix  $V_\rho i\Omega$  has all of its eigenvalues  $> 1$  or  $< -1$ , so that the above development applies, and we find that

$$V_{\rho(\alpha)} = \frac{(I + (V_\rho i\Omega)^{-1})^\alpha + (I - (V_\rho i\Omega)^{-1})^\alpha}{(I + (V_\rho i\Omega)^{-1})^\alpha - (I - (V_\rho i\Omega)^{-1})^\alpha} i\Omega. \quad (52)$$

The matrix  $V_{\rho(\alpha)}$  is a legitimate covariance matrix as a consequence of (20) and the fact that  $\alpha H > 0$ . So we conclude from (22) that

$$\text{Tr} \left\{ \exp \left[ -\frac{1}{2} \hat{x}^T [\alpha H] \hat{x} \right] \right\} = [\det ([V_{\rho(\alpha)} + i\Omega] / 2)]^{1/2}. \quad (53)$$

Putting together (41), (42), and (53) gives the statement of the proposition. ■

**Remark 2** *The covariance matrix  $V_{\rho(\alpha)}$  is equal to the  $V(\alpha)$  covariance matrix given in Eqs. (54) and (55) of Ref. 50. We give a proof for this equality in Appendix A.*

Now we give alternative proofs of some results from Ref. 33, which can be viewed as consequences of Proposition 1:

**Corollary 3** ([33]) *Given a quantum Gaussian state  $\rho$  expressed as*

$$\rho = \frac{1}{[\det ([V + i\Omega] / 2)]^{1/2}} \exp \left[ -\frac{1}{2} \hat{x}^T H \hat{x} \right], \quad (54)$$

*for a positive-definite real matrix  $H$  and corresponding covariance matrix  $V$ , the density operator  $\rho^2 / \text{Tr}\{\rho^2\}$  can be written as*

$$\frac{\rho^2}{\text{Tr}\{\rho^2\}} = \frac{1}{[\det ([V^{(2)} + i\Omega] / 2)]^{1/2}} \exp \left[ -\frac{1}{2} \hat{x}^T H^{(2)} \hat{x} \right], \quad (55)$$

*where the positive-definite real matrix  $H^{(2)}$  and the covariance matrix  $V^{(2)}$  are given by*

$$H^{(2)} = 2H, \quad (56)$$

$$V^{(2)} = \frac{1}{2} (V + \Omega V^{-1} \Omega^T). \quad (57)$$

**Proof.** Our starting point is the expression for  $V^{(2)}$  in (39). Consider that the matrices in the numerator and denominator are all commuting, so that we can work with the scalar function  $x \rightarrow \frac{(1+1/x)^\alpha + (1-1/x)^\alpha}{(1+1/x)^\alpha - (1-1/x)^\alpha}$  for  $|x| > 1$  and simplify it for  $\alpha = 2$ :

$$\frac{(1 + 1/x)^2 + (1 - 1/x)^2}{(1 + 1/x)^2 - (1 - 1/x)^2} = \frac{1 + 2/x + 1/x^2 + 1 - 2/x + 1/x^2}{1 + 2/x + 1/x^2 - (1 - 2/x + 1/x^2)} \quad (58)$$

$$= \frac{2 + 2/x^2}{4/x} \quad (59)$$

$$= \frac{1}{2} (x + x^{-1}). \quad (60)$$

So we conclude that

$$V^{(2)} = \frac{1}{2} [Vi\Omega + (Vi\Omega)^{-1}] i\Omega \quad (61)$$

$$= \frac{1}{2} [V + i\Omega V^{-1}i\Omega] \quad (62)$$

$$= \frac{1}{2} [V + \Omega V^{-1}\Omega^T]. \quad (63)$$

Proposition 1 already justified that the matrix  $V^{(2)}$  is a legitimate covariance matrix. From Proposition 1, we know that  $H^{(2)} = 2H$ . ■

**Corollary 4** ([33]) *Given a quantum Gaussian state  $\rho$  expressed as*

$$\rho = \frac{1}{[\det([V + i\Omega]/2)]^{1/2}} \exp\left[-\frac{1}{2}\hat{x}^T H \hat{x}\right], \quad (64)$$

for a positive-definite real matrix  $H$  and corresponding covariance matrix  $V$ , the density operator  $\rho^{1/2}/\text{Tr}\{\rho^{1/2}\}$  can be written as

$$\rho^{(1/2)} = \frac{\rho^{1/2}}{\text{Tr}\{\rho^{1/2}\}} = \frac{1}{[\det([V^{(1/2)} + i\Omega]/2)]^{1/2}} \exp\left[-\frac{1}{2}\hat{x}^T H^{(1/2)} \hat{x}\right], \quad (65)$$

where the positive-definite real matrix  $H^{(1/2)}$  and the covariance matrix  $V^{(1/2)}$  are given by

$$H^{(1/2)} = H/2, \quad (66)$$

$$V^{(1/2)} = V_{\rho^{(1/2)}} = \left(\sqrt{I + (V\Omega)^{-2}} + I\right) V. \quad (67)$$

**Proof.** Our starting point is the expression for  $V^{(1/2)}$  in (39). Consider that the matrices in the numerator and denominator are all commuting, so that we can work with the scalar function  $x \rightarrow \frac{(1+1/x)^\alpha + (1-1/x)^\alpha}{(1+1/x)^\alpha - (1-1/x)^\alpha}$  for  $|x| > 1$  and simplify it for  $\alpha = 1/2$ :

$$\begin{aligned} & \frac{(1 + 1/x)^{1/2} + (1 - 1/x)^{1/2}}{(1 + 1/x)^{1/2} - (1 - 1/x)^{1/2}} \\ &= \left[ \frac{(1 + 1/x)^{1/2} + (1 - 1/x)^{1/2}}{(1 + 1/x)^{1/2} - (1 - 1/x)^{1/2}} \right] \left[ \frac{(1 + 1/x)^{1/2} + (1 - 1/x)^{1/2}}{(1 + 1/x)^{1/2} + (1 - 1/x)^{1/2}} \right] \end{aligned} \quad (68)$$

$$= \frac{1 + 1/x + 2\sqrt{(1 + 1/x)(1 - 1/x)} + 1 - 1/x}{1 + 1/x - (1 - 1/x)} \quad (69)$$

$$= \frac{2 + 2\sqrt{1 - 1/x^2}}{2/x} \quad (70)$$

$$= \left(1 + \sqrt{1 - 1/x^2}\right) x. \quad (71)$$

So we conclude that

$$V^{(1/2)} = \left( I + \sqrt{I - (Vi\Omega)^{-2}} \right) (Vi\Omega) i\Omega = \left( \sqrt{I + (V\Omega)^{-2}} + I \right) V. \quad (72)$$

Proposition 1 already justified that the matrix  $V^{(1/2)}$  is a legitimate covariance matrix. From Proposition 1, we know that  $H^{(1/2)} = H/2$ . ■

## B. Traces of compositions of quantum Gaussian states

In what follows, we repeatedly make use of the following well known lemma:

**Lemma 5** ([65]) *Given invertible matrices  $A$  and  $B$ , the following equality holds*

$$(A + B)^{-1} = A^{-1} - A^{-1} (A^{-1} + B^{-1})^{-1} A^{-1}. \quad (73)$$

**Proposition 6** ([35, 33]) *Given symmetric matrices  $H_1$  and  $H_2$ , the following equality holds*

$$\exp \left[ -\frac{1}{2} \hat{x}^T H_1 \hat{x} \right] \exp \left[ -\frac{1}{2} \hat{x}^T H_2 \hat{x} \right] = \exp \left[ -\frac{1}{2} \hat{x}^T H_3 \hat{x} \right], \quad (74)$$

where  $H_3$  is a symmetric matrix such that

$$H_3 = 2i\Omega \operatorname{arcoth}(V_3 i\Omega), \quad (75)$$

$$V_3 = -i\Omega + (V_2 + i\Omega) (V_2 + V_1)^{-1} (V_1 + i\Omega), \quad (76)$$

$$V_1 = \coth(i\Omega H_1/2) i\Omega, \quad (77)$$

$$V_2 = \coth(i\Omega H_2/2) i\Omega. \quad (78)$$

**Proof.** The equality in (74) is one of the main results in Ref. 35, and the particular form of  $V_3$  in (76) was determined in Ref. 33. From Ref. 35, we know that  $H_3$  is a symmetric matrix, and furthermore, the matrix  $H_3$  that satisfies (74) is the same one that satisfies the following equation:

$$\exp [-i\Omega H_1] \exp [-i\Omega H_2] = \exp [-i\Omega H_3]. \quad (79)$$

Note that, by taking inverses, this latter equation is equivalent to

$$\exp [i\Omega H_3] = \exp [i\Omega H_2] \exp [i\Omega H_1]. \quad (80)$$

We use the relations in (16), (17), and (32) to relate  $H_3$  to matrices  $V_3$  and  $W_3$  given by

$$V_3 = -W_3 i\Omega, \quad (81)$$

where

$$W_3 = \frac{I + \exp(i\Omega H_3)}{I - \exp(i\Omega H_3)}. \quad (82)$$

For convenience of the reader, we detail some algebraic manipulations that lead to the form of  $V_3$  in (76), but we note that it is possible to arrive at this form by other means<sup>33</sup>. By (31), we have that

$$W_3 = \frac{I + \exp(i\Omega H_3)}{I - \exp(i\Omega H_3)} \quad (83)$$

$$= (I + \exp[i\Omega H_2] \exp[i\Omega H_1]) (I - \exp[i\Omega H_2] \exp[i\Omega H_1])^{-1} \quad (84)$$

$$= (\exp[i\Omega H_2] + \exp[-i\Omega H_1]) (\exp[-i\Omega H_1] - \exp[i\Omega H_2])^{-1} \quad (85)$$

$$= (\exp[i\Omega H_2] - \exp[-i\Omega H_1] + 2 \exp[-i\Omega H_1]) (\exp[-i\Omega H_1] - \exp[i\Omega H_2])^{-1} \quad (86)$$

$$= -I - 2 \exp[-i\Omega H_1] (\exp[i\Omega H_2] - \exp[-i\Omega H_1])^{-1} \quad (87)$$

$$= -I - 2 \exp[-i\Omega H_1] (\exp[i\Omega H_2] - I - (\exp[-i\Omega H_1] - I))^{-1}. \quad (88)$$

Consider from (31) that

$$\exp[i\Omega H_2] - I = \frac{W_2 - I}{W_2 + I} - \left[ \frac{W_2 + I}{W_2 + I} \right] = -2 [W_2 + I]^{-1}, \quad (89)$$

$$\exp[-i\Omega H_1] - I = \frac{W_1 + I}{W_1 - I} - \left[ \frac{W_1 - I}{W_1 - I} \right] = 2 [W_1 - I]^{-1}. \quad (90)$$

So we find that

$$W_3 = -I - 2 \exp[-i\Omega H_1] (-2 [W_2 + I]^{-1} - 2 [W_1 - I]^{-1})^{-1} \quad (91)$$

$$= -I + \exp[-i\Omega H_1] ([W_2 + I]^{-1} + [W_1 - I]^{-1})^{-1} \quad (92)$$

$$= -I + \frac{W_1 + I}{W_1 - I} ([W_2 + I]^{-1} + [W_1 - I]^{-1})^{-1}. \quad (93)$$

Applying Lemma 5 with  $A = (W_1 - I)^{-1}$  and  $B = (W_2 + I)^{-1}$ , we find that

$$([W_2 + I]^{-1} + [W_1 - I]^{-1})^{-1} = [W_1 - I] - [W_1 - I] (W_1 - I + W_2 + I)^{-1} [W_1 - I] \quad (94)$$

$$= [W_1 - I] - [W_1 - I] (W_1 + W_2)^{-1} [W_1 - I], \quad (95)$$

and this implies that

$$W_3 = -I + \frac{W_1 + I}{W_1 - I} ([W_1 - I] - [W_1 - I] (W_1 + W_2)^{-1} [W_1 - I]) \quad (96)$$

$$= -I + W_1 + I - [W_1 + I] (W_1 + W_2)^{-1} [W_1 - I] \quad (97)$$

$$= W_1 - [W_1 + I] (W_1 + W_2)^{-1} [W_1 - I]. \quad (98)$$

Continuing, we have that

$$W_3 = W_1 - [W_1 + W_2 - W_2 + I] (W_1 + W_2)^{-1} [W_1 - I] \quad (99)$$

$$= W_1 - [W_1 + W_2] (W_1 + W_2)^{-1} [W_1 - I] - [-W_2 + I] (W_1 + W_2)^{-1} [W_1 - I] \quad (100)$$

$$= W_1 - [W_1 - I] + [W_2 - I] (W_1 + W_2)^{-1} [W_1 - I] \quad (101)$$

$$= I + [W_2 - I] (W_1 + W_2)^{-1} [W_1 - I]. \quad (102)$$

So this implies, from (30), that

$$-V_3 i\Omega = I + (-V_2 i\Omega - I) (-V_2 i\Omega - V_1 i\Omega)^{-1} (-V_1 i\Omega - I) \quad (103)$$

$$= I - (V_2 i\Omega + I) ([V_2 + V_1] i\Omega)^{-1} (V_1 i\Omega + I) \quad (104)$$

$$= I - (V_2 i\Omega + I) i\Omega (V_2 + V_1)^{-1} (V_1 i\Omega + I) \quad (105)$$

$$= I - (V_2 + i\Omega) (V_2 + V_1)^{-1} (V_1 i\Omega + I). \quad (106)$$

This finally implies (76). ■

**Lemma 7** *The matrix  $V_3$  from Proposition 6 is symmetric, which follows from the fact that  $H_3$  is symmetric or by inspecting the following identity:*

$$\begin{aligned} & -i\Omega + (V_2 + i\Omega) (V_2 + V_1)^{-1} (V_1 + i\Omega) \\ &= (V_2^{-1} + V_1^{-1})^{-1} - V_1 (V_2 + V_1)^{-1} i\Omega + i\Omega (V_2 + V_1)^{-1} V_1 + \Omega (V_2 + V_1)^{-1} \Omega^T. \end{aligned} \quad (107)$$



**Proof.** Consider that

$$\begin{aligned} & -i\Omega + (V_2 + i\Omega)(V_2 + V_1)^{-1}(V_1 + i\Omega) \\ &= -i\Omega + (V_2 + V_1 - V_1 + i\Omega)(V_2 + V_1)^{-1}(V_1 + i\Omega) \end{aligned} \quad (108)$$

$$= -i\Omega + (V_2 + V_1)(V_2 + V_1)^{-1}(V_1 + i\Omega) + (-V_1 + i\Omega)(V_2 + V_1)^{-1}(V_1 + i\Omega) \quad (109)$$

$$= -i\Omega + (V_1 + i\Omega) + (-V_1 + i\Omega)(V_2 + V_1)^{-1}(V_1 + i\Omega) \quad (110)$$

$$= V_1 + (-V_1 + i\Omega)(V_2 + V_1)^{-1}(V_1 + i\Omega) \quad (111)$$

$$\begin{aligned} &= V_1 - V_1(V_2 + V_1)^{-1}V_1 - V_1(V_2 + V_1)^{-1}i\Omega + i\Omega(V_2 + V_1)^{-1}V_1 \\ &\quad + i\Omega(V_2 + V_1)^{-1}i\Omega \end{aligned} \quad (112)$$

$$= (V_2^{-1} + V_1^{-1})^{-1} - V_1(V_2 + V_1)^{-1}i\Omega + i\Omega(V_2 + V_1)^{-1}V_1 + \Omega(V_2 + V_1)^{-1}\Omega^T. \quad (113)$$

In the last line, we used Lemma 5 with  $A = V_1^{-1}$  and  $B = V_2^{-1}$  and the fact that  $\Omega^T = -\Omega$ .

■

**Proposition 8** ([35, 47, 66]) *Given positive-definite real matrices  $H_4$  and  $H_5$ , we have that*

$$\exp\left[-\frac{1}{2}\hat{x}^T[H_4/2]\hat{x}\right]\exp\left[-\frac{1}{2}\hat{x}^TH_5\hat{x}\right]\exp\left[-\frac{1}{2}\hat{x}^T[H_4/2]\hat{x}\right] = \exp\left[-\frac{1}{2}\hat{x}^TH_6\hat{x}\right], \quad (114)$$

where  $H_6$  is a positive-definite real matrix with corresponding covariance matrix  $V_6$ , given by

$$H_6 = 2i\Omega \operatorname{arcoth}(V_6i\Omega), \quad (115)$$

$$V_6 = V_4 - \left(\sqrt{I + (V_4\Omega)^{-2}}\right)V_4(V_5 + V_4)^{-1}V_4\left(\sqrt{I + (\Omega V_4)^{-2}}\right). \quad (116)$$

**Proof.** Let  $H_7 = H_4/2$  and let  $V_7$  be the covariance matrix defined by

$$V_7 = \operatorname{coth}(i\Omega H_7/2)i\Omega. \quad (117)$$

From Corollary 4, it follows that  $V_7$  can be given in terms of  $V_4$  as

$$V_7 = \left(\sqrt{I + (V_4\Omega)^{-2}} + I\right)V_4, \quad (118)$$

which is equivalent to

$$W_7 = \left(\sqrt{I - W_4^{-2}} + I\right)W_4. \quad (119)$$

Consider from two applications of the composition rule in (80) that

$$\exp(i\Omega H_6) = \exp(i\Omega H_4/2) \exp(i\Omega H_5) \exp(i\Omega H_4/2). \quad (120)$$

This implies from (31) that

$$W_6 = [I + e^{i\Omega H_4/2} e^{i\Omega H_5} e^{i\Omega H_4/2}] [I - e^{i\Omega H_4/2} e^{i\Omega H_5} e^{i\Omega H_4/2}]^{-1} \quad (121)$$

$$= [e^{-i\Omega H_4/2} + e^{i\Omega H_4/2} e^{i\Omega H_5}] [e^{-i\Omega H_4/2} - e^{i\Omega H_4/2} e^{i\Omega H_5}]^{-1} \quad (122)$$

$$= \exp(i\Omega H_4/2) [e^{-i\Omega H_4} + e^{i\Omega H_5}] (\exp(i\Omega H_4/2) [e^{-i\Omega H_4} - e^{i\Omega H_5}])^{-1} \quad (123)$$

$$= \exp(i\Omega H_4/2) [e^{-i\Omega H_4} + e^{i\Omega H_5}] [e^{-i\Omega H_4} - e^{i\Omega H_5}]^{-1} \exp(-i\Omega H_4/2). \quad (124)$$

From the development in (85)–(98), we know that

$$\begin{aligned} & [\exp(-i\Omega H_4) + \exp(i\Omega H_5)] (\exp(-i\Omega H_4) - \exp(i\Omega H_5))^{-1} \\ &= W_4 - [W_4 + I] [W_4 + W_5]^{-1} [W_4 - I]. \end{aligned} \quad (125)$$

This implies that

$$W_6 = \exp(i\Omega H_4/2) [W_4 - [W_4 + I] [W_4 + W_5]^{-1} [W_4 - I]] \exp(-i\Omega H_4/2) \quad (126)$$

$$= \exp(i\Omega H_4/2) W_4 \exp(-i\Omega H_4/2) - \exp(i\Omega H_4/2) [W_4 + I] [W_4 + W_5]^{-1} [W_4 - I] \exp(-i\Omega H_4/2) \quad (127)$$

$$= W_4 - \exp(i\Omega H_4/2) [W_4 + I] [W_4 + W_5]^{-1} [W_4 - I] \exp(-i\Omega H_4/2) \quad (128)$$

$$= W_4 - \frac{\left(\sqrt{I - W_4^{-2}} + I\right) W_4 - I}{\left(\sqrt{I - W_4^{-2}} + I\right) W_4 + I} [W_4 + I] [W_4 + W_5]^{-1} [W_4 - I] \frac{\left(\sqrt{I - W_4^{-2}} + I\right) W_4 + I}{\left(\sqrt{I - W_4^{-2}} + I\right) W_4 - I}, \quad (129)$$

where the last equality follows from (31) and Corollary 4. Considering that the following scalar functions simplify as

$$\frac{(\sqrt{1 - x^{-2}} + 1) x - 1}{(\sqrt{1 - x^{-2}} + 1) x + 1} [x + 1] = \left(\sqrt{1 - x^{-2}}\right) x, \quad (130)$$

$$[x - 1] \frac{(\sqrt{1 - x^{-2}} + 1) x + 1}{(\sqrt{1 - x^{-2}} + 1) x - 1} = x \sqrt{1 - x^{-2}}, \quad (131)$$

we find that

$$W_6 = W_4 - \left( \sqrt{I - W_4^{-2}} \right) W_4 [W_4 + W_5]^{-1} W_4 \left( \sqrt{I - W_4^{-2}} \right). \quad (132)$$

Now substituting, we find that

$$\begin{aligned} & -V_6 i\Omega \\ &= -V_4 i\Omega - \left( \sqrt{I - (-V_4 i\Omega)^{-2}} \right) (-V_4 i\Omega) (-V_5 i\Omega - V_4 i\Omega)^{-1} (-V_4 i\Omega) \left( \sqrt{I - (-V_4 i\Omega)^{-2}} \right) \end{aligned} \quad (133)$$

$$= -V_4 i\Omega + \left( \sqrt{I + (V_4 \Omega)^{-2}} \right) (V_4 i\Omega) ([V_5 + V_4] i\Omega)^{-1} (V_4 i\Omega) \left( \sqrt{I + (V_4 \Omega)^{-2}} \right) \quad (134)$$

$$= -V_4 i\Omega + \left( \sqrt{I + (V_4 \Omega)^{-2}} \right) V_4 (V_5 + V_4)^{-1} (V_4 i\Omega) \left( \sqrt{I + (V_4 \Omega)^{-2}} \right). \quad (135)$$

This then implies that

$$V_6 = V_4 - \left( \sqrt{I + (V_4 \Omega)^{-2}} \right) V_4 (V_5 + V_4)^{-1} (V_4 i\Omega) \left( \sqrt{I + (V_4 \Omega)^{-2}} \right) i\Omega \quad (136)$$

$$= V_4 - \left( \sqrt{I + (V_4 \Omega)^{-2}} \right) V_4 (V_5 + V_4)^{-1} V_4 \left( \sqrt{I + (\Omega V_4)^{-2}} \right), \quad (137)$$

which concludes the proof. ■

Even though it directly follows from the above that  $V_6$  is a legitimate covariance matrix, the following proposition gives an alternative confirmation of this fact:

**Proposition 9** *The matrix  $V_6$  from Proposition 8 is a legitimate covariance matrix, and so we can conclude that*

$$\text{Tr} \left\{ \exp \left[ -\frac{1}{2} \hat{x}^T H_6 \hat{x} \right] \right\} = \sqrt{\det(V_6 + i\Omega/2)}, \quad (138)$$

where  $H_6 = 2i\Omega \text{arcoth}(V_\rho i\Omega)$ .

**Proof.** Since  $V_5$  is a legitimate covariance matrix corresponding to positive-definite real  $H_5$ , we have that  $V_5 - i\Omega > 0$  which implies that  $V_5 + V_4 > V_4 + i\Omega$  and in turn that  $-(V_5 + V_4)^{-1} > -(V_4 + i\Omega)^{-1}$ , by operator monotonicity of the function  $x \rightarrow -x^{-1}$ . Then

we find that

$$V_6 - i\Omega = V_4 - \left( \sqrt{I + (V_4\Omega)^{-2}} \right) V_4 (V_5 + V_4)^{-1} V_4 \left( \sqrt{I + (\Omega V_4)^{-2}} \right) - i\Omega \quad (139)$$

$$> V_4 - \left( \sqrt{I + (V_4\Omega)^{-2}} \right) V_4 (V_4 + i\Omega)^{-1} V_4 \left( \sqrt{I + (\Omega V_4)^{-2}} \right) - i\Omega \quad (140)$$

$$= V_4 - (V_4 - i\Omega) - i\Omega \quad (141)$$

$$= 0. \quad (142)$$

In the above, the second equality follows from Lemma 10 below. ■

**Lemma 10** ([66]) *The following identity holds for a covariance matrix  $V$  such that  $V + i\Omega > 0$ :*

$$\sqrt{I + (V\Omega)^{-2}} V (V + i\Omega)^{-1} V \sqrt{I + (\Omega V)^{-2}} = V - i\Omega. \quad (143)$$

**Proof.** Consider that

$$\begin{aligned} & \sqrt{I + (V\Omega)^{-2}} V (V + i\Omega)^{-1} V \sqrt{I + (\Omega V)^{-2}} \\ &= \sqrt{I + (V\Omega)^{-2}} V i\Omega (V i\Omega + I)^{-1} V \sqrt{I + (\Omega V)^{-2}} \end{aligned} \quad (144)$$

$$= \sqrt{I - (V i\Omega)^{-2}} V i\Omega (V i\Omega + I)^{-1} V i\Omega i\Omega \sqrt{I - (i\Omega V)^{-2}} i\Omega i\Omega \quad (145)$$

$$= \sqrt{I - (V i\Omega)^{-2}} V i\Omega (V i\Omega + I)^{-1} V i\Omega \sqrt{I - (i\Omega i\Omega V i\Omega V i\Omega)^{-1}} i\Omega \quad (146)$$

$$= \left[ \sqrt{I - (V i\Omega)^{-2}} V i\Omega (V i\Omega + I)^{-1} V i\Omega \sqrt{I - (V i\Omega)^{-2}} \right] i\Omega. \quad (147)$$

Now that the expression in square brackets has been reduced to a matrix version of the scalar function  $x \rightarrow \sqrt{1 - x^{-2}} x (x + 1)^{-1} x \sqrt{1 - x^{-2}}$ , we can use the fact that the scalar function collapses as

$$\sqrt{1 - x^{-2}} x (x + 1)^{-1} x \sqrt{1 - x^{-2}} = x - 1, \quad (148)$$

and we find that

$$\begin{aligned} & \sqrt{I - (V i\Omega)^{-2}} V i\Omega (V i\Omega + I)^{-1} V i\Omega \sqrt{I - (V i\Omega)^{-2}} i\Omega \\ &= [V i\Omega - I] i\Omega \end{aligned} \quad (149)$$

$$= V - i\Omega. \quad (150)$$

This concludes the proof. ■

The following proposition is again a consequence of Ref. 35, and the particular form of the determinant in (152) was reported in Ref. 46, Eq. (3.5).

**Proposition 11** ([35, 46]) *Given positive-definite real matrices  $H_1$  and  $H_2$ , it follows that*

$$\exp \left[ -\frac{1}{2} \hat{x}^T H_1 \hat{x} \right] \exp \left[ -\frac{1}{2} \hat{x}^T H_2 \hat{x} \right] = \exp \left[ -\frac{1}{2} \hat{x}^T H_3 \hat{x} \right], \quad (151)$$

where  $H_3$  is such that

$$\text{Tr} \left\{ \exp \left[ -\frac{1}{2} \hat{x}^T H_3 \hat{x} \right] \right\} = \sqrt{\frac{\det([V_1 + i\Omega]/2) \det([V_2 + i\Omega]/2)}{\det([V_1 + V_2]/2)}}, \quad (152)$$

$$V_1 = \coth(i\Omega H_1/2) i\Omega, \quad (153)$$

$$V_2 = \coth(i\Omega H_2/2) i\Omega. \quad (154)$$

**Proof.** Consider that

$$\begin{aligned} & \text{Tr} \left\{ \exp \left[ -\frac{1}{2} \hat{x}^T H_1 \hat{x} \right] \exp \left[ -\frac{1}{2} \hat{x}^T H_2 \hat{x} \right] \right\} \\ &= \text{Tr} \left\{ \exp \left[ -\frac{1}{2} \hat{x}^T [H_1/2] \hat{x} \right] \exp \left[ -\frac{1}{2} \hat{x}^T H_2 \hat{x} \right] \exp \left[ -\frac{1}{2} \hat{x}^T [H_1/2] \hat{x} \right] \right\} \end{aligned} \quad (155)$$

$$= \left[ \det \left( \left[ V_1 - \left( \sqrt{I + (V_1 \Omega)^{-2}} \right) V_1 (V_2 + V_1)^{-1} V_1 \left( \sqrt{I + (\Omega V_1)^{-2}} \right) + i\Omega \right] / 2 \right) \right]^{1/2} \quad (156)$$

$$= \left[ \det \left( \left[ I - W_1 + \left( \sqrt{I - W_1^{-2}} \right) W_1 (W_2 + W_1)^{-1} W_1 \left( \sqrt{I - W_1^{-2}} \right) \right] i\Omega/2 \right) \right]^{1/2} \quad (157)$$

$$= \left[ \det \left( I - W_1 + \left( \sqrt{I - W_1^{-2}} \right) W_1 (W_2 + W_1)^{-1} W_1 \left( \sqrt{I - W_1^{-2}} \right) \right) \det(i\Omega/2) \right]^{1/2}. \quad (158)$$

The first equality follows from cyclicity of trace. The second equality follows from Proposition 9. The third equality follows from (132) and (33). We now prove that the following matrices are similar

$$W' = W_1 - \left( \sqrt{I - W_1^{-2}} \right) W_1 (W_2 + W_1)^{-1} W_1 \left( \sqrt{I - W_1^{-2}} \right), \quad (159)$$

$$W'' = I + (W_2 - I) (W_2 + W_1)^{-1} (W_1 - I), \quad (160)$$

i.e., related as

$$W' = \exp(i\Omega H_1/2)W'' \exp(-i\Omega H_1/2). \quad (161)$$

To this end, consider from (80)–(102) that

$$W'' = \frac{I + \exp(i\Omega H_2) \exp(i\Omega H_1)}{I - \exp(i\Omega H_2) \exp(i\Omega H_1)}, \quad (162)$$

and from applying (80) twice and considering (121)–(132),

$$W' = \frac{I + \exp(i\Omega H_1/2) \exp(i\Omega H_2) \exp(i\Omega H_1/2)}{I - \exp(i\Omega H_1/2) \exp(i\Omega H_2) \exp(i\Omega H_1/2)}. \quad (163)$$

Then we find that

$$\exp(i\Omega H_1/2)W'' \exp(-i\Omega H_1/2) = \exp(i\Omega H_1/2) \frac{I + \exp(i\Omega H_2) \exp(i\Omega H_1)}{I - \exp(i\Omega H_2) \exp(i\Omega H_1)} \exp(-i\Omega H_1/2) \quad (164)$$

$$= \frac{I + \exp(i\Omega H_1/2) \exp(i\Omega H_2) \exp(i\Omega H_1/2)}{I - \exp(i\Omega H_1/2) \exp(i\Omega H_2) \exp(i\Omega H_1/2)} \quad (165)$$

$$= W'. \quad (166)$$

Since these matrices are related by a similarity transformation, we find that

$$\left[ \det \left( I - W_1 + \left( \sqrt{I - W_1^{-2}} \right) W_1 (W_2 + W_1)^{-1} W_1 \left( \sqrt{I - W_1^{-2}} \right) \right) \det(i\Omega/2) \right]^{1/2} \\ = \left[ \det \left( I - [I + (W_2 - I)(W_2 + W_1)^{-1}(W_1 - I)] \right) \det(i\Omega/2) \right]^{1/2} \quad (167)$$

$$= \left[ \det \left( -(W_2 - I)(W_2 + W_1)^{-1}(W_1 - I) \right) \det(i\Omega/2) \right]^{1/2} \quad (168)$$

$$= \left[ \det \left( (I - W_2)(i\Omega/2)(-2i\Omega)(W_2 + W_1)^{-1}(I - W_1)i\Omega/2 \right) \right]^{1/2} \quad (169)$$

$$= \left[ \det \left( (I - W_2)(i\Omega/2) \right) \det \left( (-2i\Omega)(W_2 + W_1)^{-1} \right) \det \left( (I - W_1)i\Omega/2 \right) \right]^{1/2} \quad (170)$$

$$= \left[ \det \left( [V_2 + i\Omega]/2 \right) \det \left( (-2i\Omega)(-V_2i\Omega - V_1i\Omega)^{-1} \right) \det \left( [V_1 + i\Omega]/2 \right) \right]^{1/2} \quad (171)$$

$$= \left[ \det \left( [V_2 + i\Omega]/2 \right) \det \left( ([V_2 + V_1]/2)^{-1} \right) \det \left( [V_1 + i\Omega]/2 \right) \right]^{1/2} \quad (172)$$

$$= \left[ \frac{\det \left( [V_2 + i\Omega]/2 \right) \det \left( [V_1 + i\Omega]/2 \right)}{\det \left( [V_2 + V_1]/2 \right)} \right]^{1/2}. \quad (173)$$

This concludes the proof. ■

**Proposition 12** *Given positive-definite real matrices  $H_4$  and  $H_5$ , we have that*

$$\exp \left[ -\frac{1}{2} \hat{x}^T [H_4/2] \hat{x} \right] \exp \left[ -\frac{1}{2} \hat{x}^T [-H_5] \hat{x} \right] \exp \left[ -\frac{1}{2} \hat{x}^T [H_4/2] \hat{x} \right] = \exp \left[ -\frac{1}{2} \hat{x}^T H_8 \hat{x} \right]. \quad (174)$$

*In the above,  $H_8$  is real and positive definite if  $V_5 > V_4$  and is such that*

$$\text{Tr} \left\{ \exp \left[ -\frac{1}{2} \hat{x}^T H_8 \hat{x} \right] \right\} = \sqrt{\det([V_8 + i\Omega]/2)} \quad (175)$$

$$= \sqrt{\frac{\det([V_4 + i\Omega]/2) \det([V_5 + i\Omega]/2)}{\det([V_5 - V_4]/2)}}, \quad (176)$$

$$V_8 = V_4 + \left( \sqrt{I + (V_4\Omega)^{-2}} \right) V_4 (V_5 - V_4)^{-1} V_4 \left( \sqrt{I + (\Omega V_4)^{-2}} \right). \quad (177)$$

**Proof.** The proof of this proposition amounts to examining again the proofs of Propositions 6 and 8 and instead substituting  $-H_5$  for  $H_5$ . Consider that the product rule from Proposition 6 holds for symmetric  $H_1$  and  $H_2$

$$\exp \left[ -\frac{1}{2} \hat{x}^T H_1 \hat{x} \right] \exp \left[ -\frac{1}{2} \hat{x}^T H_2 \hat{x} \right] = \exp \left[ -\frac{1}{2} \hat{x}^T H_3 \hat{x} \right]. \quad (178)$$

From Ref. 35, we know that the symmetric matrix  $H_3$  that satisfies (74) is the same one that satisfies the following equation:

$$\exp [-i\Omega H_1] \exp [-i\Omega H_2] = \exp [-i\Omega H_3]. \quad (179)$$

Note that, by taking inverses, this latter equation is equivalent to

$$\exp [i\Omega H_3] = \exp [i\Omega H_2] \exp [i\Omega H_1]. \quad (180)$$

Recalling that  $\exp (i\Omega H) = \frac{W-I}{W+I}$ , we find that

$$\exp (-i\Omega H) = [\exp (i\Omega H)]^{-1} = \left[ \frac{W-I}{W+I} \right]^{-1} = \frac{W+I}{W-I} = \frac{-[-W-I]}{W-I} = \frac{-W-I}{-W+I}, \quad (181)$$

which implies that the transformation  $H \rightarrow -H$  induces the transformation  $W \rightarrow -W$ , as observed in Ref. 66. Now propagating this minus sign throughout all of the calculations in the proofs of Propositions 6 and 8, we find that

$$V_8 = V_4 - \left( \sqrt{I + (V_4\Omega)^{-2}} \right) V_4 (-V_5 + V_4)^{-1} V_4 \left( \sqrt{I + (\Omega V_4)^{-2}} \right) \quad (182)$$

$$= V_4 + \left( \sqrt{I + (V_4\Omega)^{-2}} \right) V_4 (V_5 - V_4)^{-1} V_4 \left( \sqrt{I + (\Omega V_4)^{-2}} \right). \quad (183)$$

The latter is a legitimate covariance matrix when  $V_5 - V_4 > 0$  because

$$V_8 + i\Omega = V_4 + i\Omega + \left( \sqrt{I + (V_4\Omega)^{-2}} \right) V_4 (V_5 - V_4)^{-1} V_4 \left( \sqrt{I + (\Omega V_4)^{-2}} \right) \quad (184)$$

$$\geq \left( \sqrt{I + (V_4\Omega)^{-2}} \right) V_4 (V_5 - V_4)^{-1} V_4 \left( \sqrt{I + (\Omega V_4)^{-2}} \right) \quad (185)$$

$$> 0. \quad (186)$$

In the above, we used the fact that  $V_4$  is a legitimate covariance matrix satisfying  $V_4 + i\Omega \geq 0$  and the assumption that  $V_5 - V_4 > 0$ . By (20), this implies that  $H_8 > 0$ . Since  $V_8$  is a legitimate covariance matrix corresponding to  $H_8$ , we conclude (175).

A proof for (176) follows by examining again the proof of Proposition 11 and considering again that the transformation  $H \rightarrow -H$  induces the transformation  $W \rightarrow -W$ . Propagating the minus sign throughout the calculation, we arrive at

$$\begin{aligned} & \left[ \frac{\det([-V_2 + i\Omega]/2) \det([V_1 + i\Omega]/2)}{\det([-V_2 + V_1]/2)} \right]^{1/2} \\ &= \left[ \frac{\det([V_2 - i\Omega]/2) \det([V_1 + i\Omega]/2)}{\det([V_2 - V_1]/2)} \right]^{1/2} \end{aligned} \quad (187)$$

$$= \left[ \frac{\det([V_2 + i\Omega]/2) \det([V_1 + i\Omega]/2)}{\det([V_2 - V_1]/2)} \right]^{1/2}, \quad (188)$$

where the last equality follows because  $[V_2 - i\Omega]^T = V_2 + i\Omega$  and the determinant is invariant with respect to transposition. ■

We close this section by remarking that many of the above calculations can be completed by considering the approach developed in Ref. 66, Appendix A.

### C. Mean vectors and displacement operators

We begin by recalling some standard properties of the operator in (34). For detailed proofs, see, e.g., Ref. 30, but note that they follow from the Baker–Campbell–Hausdorff formula and its corollaries.



**Proposition 13** *The displacement operator in (34) (extended to  $s \in \mathbb{C}^{2n}$ ) satisfies the following properties:*

$$D(s)^{-1} = D(-s), \quad (189)$$

$$D(s)D(t) = D(s+t)e^{-\frac{1}{2}s^T i\Omega t}, \quad (190)$$

$$D(s)\hat{x}D(-s) = \hat{x} + s, \quad (191)$$

$$\exp\left[-\frac{1}{2}\hat{x}^T H \hat{x}\right] \hat{x} = \exp[i\Omega H] \hat{x} \exp\left[-\frac{1}{2}\hat{x}^T H \hat{x}\right], \quad (192)$$

where  $s, t \in \mathbb{C}^{2n}$  and  $H$  is a symmetric matrix with complex entries. If  $s \in \mathbb{R}^{2n}$ , then  $D(s)^{-1} = D(s)^\dagger$ .

The following corollary generalizes some statements from Ref. 47:

**Corollary 14** *The following equalities involving the displacement operator and exponential quadratic forms hold*

$$\exp\left[-\frac{1}{2}\hat{x}^T H \hat{x}\right] D(s) = D(\exp[-i\Omega H] s) \exp\left[-\frac{1}{2}\hat{x}^T H \hat{x}\right], \quad (193)$$

and for  $l = (\exp[-i\Omega H] - I) s$ ,

$$D(l) \exp\left[-\frac{1}{2}\hat{x}^T H \hat{x}\right] = D(-s) \exp\left[-\frac{1}{2}\hat{x}^T H \hat{x}\right] D(s) e^{\frac{1}{4}l^T i\Omega W l}, \quad (194)$$

where  $s \in \mathbb{C}^{2n}$ ,  $H$  is a symmetric matrix with complex entries, and  $W$  is related to  $H$  by (32).

**Proof.** Consider that

$$\exp\left[-\frac{1}{2}\hat{x}^T H \hat{x}\right] \exp[s^T i\Omega \hat{x}] = \exp[s^T i\Omega \exp[i\Omega H] \hat{x}] \exp\left[-\frac{1}{2}\hat{x}^T H \hat{x}\right], \quad (195)$$

which follows from applying (192) of Proposition 13 to  $\exp[s^T i\Omega \hat{x}]$ . This implies that

$$\exp\left[-\frac{1}{2}\hat{x}^T H \hat{x}\right] \exp[s^T i\Omega \hat{x}] = \exp[s^T i\Omega \exp[i\Omega H] \hat{x}] \exp\left[-\frac{1}{2}\hat{x}^T H \hat{x}\right] \quad (196)$$

$$= \exp[s^T \exp[H i\Omega] i\Omega \hat{x}] \exp\left[-\frac{1}{2}\hat{x}^T H \hat{x}\right] \quad (197)$$

$$= \exp\left[(\exp[-i\Omega H] s)^T i\Omega \hat{x}\right] \exp\left[-\frac{1}{2}\hat{x}^T H \hat{x}\right], \quad (198)$$

where we have used the following:

$$i\Omega \exp [i\Omega H] = i\Omega \exp [i\Omega H] i\Omega i\Omega = \exp [i\Omega i\Omega H i\Omega] i\Omega = \exp [Hi\Omega] i\Omega. \quad (199)$$

This establishes (193).

To see (194), consider that

$$D(-s) \exp \left[ -\frac{1}{2} \hat{x}^T H \hat{x} \right] D(s) = D(-s) D(\exp [-i\Omega H] s) \exp \left[ -\frac{1}{2} \hat{x}^T H \hat{x} \right] \quad (200)$$

$$= e^{\frac{1}{2} s^T i\Omega \exp [-i\Omega H] s} D(l) \exp \left[ -\frac{1}{2} \hat{x}^T H \hat{x} \right], \quad (201)$$

where

$$l = (\exp [-i\Omega H] - I) s. \quad (202)$$

In the above, the first equality follows from applying (193), while the second equality results from applying (190) of Proposition 13. Thus,

$$D(l) \exp \left[ -\frac{1}{2} \hat{x}^T H \hat{x} \right] = D(-s) \exp \left[ -\frac{1}{2} \hat{x}^T H \hat{x} \right] D(s) e^{-\frac{1}{2} s^T i\Omega \exp [-i\Omega H] s}. \quad (203)$$

Furthermore, consider that

$$s^T i\Omega \exp [-i\Omega H] s = \frac{1}{2} s^T i\Omega (\exp [-i\Omega H] - \exp [i\Omega H]) s, \quad (204)$$

which follows because a scalar is equal to its transpose. Thus,

$$\begin{aligned} & \frac{1}{2} s^T i\Omega \exp [-i\Omega H] s \\ &= \frac{1}{4} s^T i\Omega (\exp [-i\Omega H] - \exp [i\Omega H]) s \end{aligned} \quad (205)$$

$$= \frac{1}{4} ((\exp [-i\Omega H] - I)^{-1} l)^T i\Omega (\exp [-i\Omega H] - \exp [i\Omega H]) (\exp [-i\Omega H] - I)^{-1} l \quad (206)$$

$$= \frac{1}{4} l^T (\exp [Hi\Omega] - I)^{-1} i\Omega (\exp [-i\Omega H] - \exp [i\Omega H]) (\exp [-i\Omega H] - I)^{-1} l \quad (207)$$

$$= \frac{1}{4} l^T i\Omega (\exp [i\Omega H] - I)^{-1} (\exp [-i\Omega H] - \exp [i\Omega H]) (\exp [-i\Omega H] - I)^{-1} l. \quad (208)$$

Now that we have expressed the middle operator in terms of the scalar function

$$x \rightarrow [x - 1]^{-1} (x^{-1} - x) [x^{-1} - 1]^{-1} = \frac{x + 1}{x - 1}, \quad (209)$$

we can conclude that

$$\frac{1}{2}s^T i\Omega \exp[-i\Omega H] s = \frac{1}{4}l^T i\Omega \frac{(\exp[i\Omega H] + I)}{(\exp[i\Omega H] - I)} l \quad (210)$$

$$= -\frac{1}{4}l^T i\Omega W l. \quad (211)$$

Equations (203) and (211) together establish (194). ■

**Remark 15** *In Corollary 14, if  $H \rightarrow -H$ , (i.e., if the inverse of an exponential quadratic form is considered), then the above statements change as*

$$\exp\left[-\frac{1}{2}\hat{x}^T(-H)\hat{x}\right] D(s) = D(\exp[-i\Omega(-H)]s) \exp\left[-\frac{1}{2}\hat{x}^T(-H)\hat{x}\right], \quad (212)$$

$$D(l) \exp\left[-\frac{1}{2}\hat{x}^T(-H)\hat{x}\right] = D(-s) \exp\left[-\frac{1}{2}\hat{x}^T(-H)\hat{x}\right] D(s) e^{\frac{1}{4}l^T i\Omega(-W)l}, \quad (213)$$

for  $l = (\exp[-i\Omega(-H)] - I)s$ , where  $s \in \mathbb{C}^{2n}$ , and  $W$  is related to  $H$  by (32).

**Lemma 16** ([47]) *For positive-definite real matrices  $H_9$  and  $H_{10}$  such that*

$$\exp[-i\Omega H_9] \exp[-i\Omega H_{10}] = \exp[-i\Omega H_{11}], \quad (214)$$

and

$$l = (\exp[-i\Omega H_9] - I)s, \quad s \in \mathbb{C}^{2n}, \quad (215)$$

the following equality holds

$$-\frac{1}{4}l^T i\Omega W_9 l + \frac{1}{4}l^T i\Omega W_{11} l = -s^T (V_9 + V_{10})^{-1} s, \quad (216)$$

where  $V_9$  and  $V_{10}$  are related to  $H_9$  and  $H_{10}$ , respectively, by (17),  $W_9$  is related to  $H_9$  by (32), and

$$W_{11} = \frac{I + \exp(i\Omega H_{11})}{I - \exp(i\Omega H_{11})} = \frac{I + \exp(i\Omega H_{10}) \exp(i\Omega H_9)}{I - \exp(i\Omega H_{10}) \exp(i\Omega H_9)}. \quad (217)$$

**Proof.** From (205)–(211) of Corollary 14, we have that

$$-\frac{1}{4}l^T i\Omega W_9 l = \frac{1}{4}s^T i\Omega (\exp[-i\Omega H_9] - \exp[i\Omega H_9]) s. \quad (218)$$

We also have that

$$\frac{1}{4}l^T i\Omega W_{11}l = \frac{1}{4}s^T (\exp[-i\Omega H_9] - I)^T i\Omega W_{11} (\exp[-i\Omega H_9] - I) s \quad (219)$$

$$= \frac{1}{4}s^T i\Omega (\exp[i\Omega H_9] - I) W_{11} (\exp[-i\Omega H_9] - I) s, \quad (220)$$

so that the total expression can be written as

$$\begin{aligned} & -\frac{1}{4}l^T i\Omega W_9 l + \frac{1}{4}l^T i\Omega W_{11} l \\ &= \frac{1}{4}s^T i\Omega [(\exp[-i\Omega H_9] - \exp[i\Omega H_9]) + (\exp[i\Omega H_9] - I) W_{11} (\exp[-i\Omega H_9] - I)] s, \end{aligned} \quad (221)$$

The following equalities hold by exploiting (31)

$$\exp[-i\Omega H_9] - \exp[i\Omega H_9] = \frac{W_9 + I}{W_9 - I} - \frac{W_9 - I}{W_9 + I} = \frac{4W_9}{W_9^2 - I}, \quad (222)$$

$$(\exp[i\Omega H_9] - I) W_{11} (\exp[-i\Omega H_9] - I) = -\frac{2}{W_9 + I} W_{11} \frac{2}{W_9 - I}. \quad (223)$$

From (98), we have that

$$W_{11} = W_9 - (W_9 + I) (W_9 + W_{10})^{-1} (W_9 - I), \quad (224)$$

Using (224), we thus have that

$$\begin{aligned} & -\frac{2}{W_9 + I} W_{11} \frac{2}{W_9 - I} \\ &= -\frac{2}{W_9 + I} [W_9 - (W_9 + I) (W_9 + W_{10})^{-1} (W_9 - I)] \frac{2}{W_9 - I} \end{aligned} \quad (225)$$

$$= -\frac{2}{W_9 + I} W_9 \frac{2}{W_9 - I} + \frac{2}{W_9 + I} (W_9 + I) (W_9 + W_{10})^{-1} (W_9 - I) \frac{2}{W_9 - I} \quad (226)$$

$$= -\frac{4W_9}{W_9^2 - I} + 4(W_9 + W_{10})^{-1}. \quad (227)$$

Combining (222) and (227), we obtain that

$$\begin{aligned} & \frac{1}{4}s^T i\Omega [(\exp[-i\Omega H_9] - \exp[i\Omega H_9]) + (\exp[i\Omega H_9] - I) W_{11} (\exp[-i\Omega H_9] - I)] s \\ &= s^T i\Omega (W_9 + W_{10})^{-1} s = -s^T (V_9 + V_{10})^{-1} s, \end{aligned} \quad (228)$$

which is the statement of the lemma. ■

**Lemma 17** *Let  $H_9$  and  $H_{10}$  be positive-definite real matrices such that*

$$\exp[-i\Omega(-H_9)] \exp[-i\Omega H_{10}] = \exp[-i\Omega H_{11}], \quad (229)$$

*with  $H_{11}$  satisfying the above, and let*

$$l = (\exp[-i\Omega(-H_9)] - I) s, \quad s \in \mathbb{C}^{2n}. \quad (230)$$

*If  $V_9 > V_{10}$ , then the following equality holds*

$$-\frac{1}{4} l^T i\Omega(-W_9) l + \frac{1}{4} l^T i\Omega W_{11} l = s^T (V_9 - V_{10})^{-1} s, \quad (231)$$

*where  $V_9$  and  $V_{10}$  are related to  $H_9$  and  $H_{10}$ , respectively, by (17),  $W_9$  is related to  $H_9$  by (32), and*

$$W_{11} = \frac{I + \exp(i\Omega H_{11})}{I - \exp(i\Omega H_{11})} = \frac{I + \exp(i\Omega H_{10}) \exp(-i\Omega H_9)}{I - \exp(i\Omega H_{10}) \exp(-i\Omega H_9)}. \quad (232)$$

**Proof.** This amounts to reexamining the proof of Lemma 16, and noting that, from the discussion around (181),  $W \rightarrow -W$  and  $V \rightarrow -V$  when  $H \rightarrow -H$ , i.e., under the inverse of an exponential quadratic form. This implies that  $-s^T (V_9 + V_{10})^{-1} s \rightarrow -s^T (-V_9 + V_{10})^{-1} s = s^T (V_9 - V_{10})^{-1} s$ . The condition  $V_9 > V_{10}$  suffices to guarantee that the matrix  $V_9 - V_{10}$  is invertible, which is used throughout the calculations in the proof of Lemma 16. ■

#### IV. PETZ-RÉNYI RELATIVE ENTROPY

We now determine a formula for the Petz-Rényi relative entropy<sup>5</sup> of two Gaussian states  $\rho$  and  $\sigma$ . The Petz-Rényi relative entropy is defined for  $\alpha \in (0, 1) \cup (1, \infty)$  as

$$D_\alpha(\rho||\sigma) \equiv \frac{1}{\alpha - 1} \ln Q_\alpha(\rho||\sigma), \quad (233)$$

where  $Q_\alpha(\rho||\sigma)$  denotes the Petz-Rényi relative quasi-entropy:

$$Q_\alpha(\rho||\sigma) \equiv \text{Tr} \{ \rho^\alpha \sigma^{1-\alpha} \}. \quad (234)$$

We first consider the case when  $\alpha \in (0, 1)$ , and then we move on to the case when  $\alpha \in (1, \infty)$ .

A formula for the Petz-Rényi relative entropy was previously given in Ref. 50 for  $\alpha \in (0, 1)$ . The formula given in Theorem 18 below is expressed differently from the one given in Ref. 50 because it depends directly on the covariance matrices of the states involved.

**Theorem 18** Let  $\alpha \in (0, 1)$ , and let  $\rho$  and  $\sigma$  denote two Gaussian states. Then the Petz–Rényi relative quasi-entropy  $Q_\alpha(\rho||\sigma)$  as defined in (234) is given by

$$Q_\alpha(\rho||\sigma) = \frac{Z_{\rho(\alpha)} Z_{\sigma(1-\alpha)}}{Z_\rho^\alpha Z_\sigma^{1-\alpha} [\det([V_{\rho(\alpha)} + V_{\sigma(1-\alpha)}] / 2)]^{1/2}} \exp \left\{ -\delta s^T (V_{\rho(\alpha)} + V_{\sigma(1-\alpha)})^{-1} \delta s \right\}, \quad (235)$$

where

$$V_{\rho(\alpha)} = \frac{(I + (V_\rho i\Omega)^{-1})^\alpha + (I - (V_\rho i\Omega)^{-1})^\alpha}{(I + (V_\rho i\Omega)^{-1})^\alpha - (I - (V_\rho i\Omega)^{-1})^\alpha} i\Omega, \quad (236)$$

$$V_{\sigma(1-\alpha)} = \frac{(I + (V_\sigma i\Omega)^{-1})^{1-\alpha} + (I - (V_\sigma i\Omega)^{-1})^{1-\alpha}}{(I + (V_\sigma i\Omega)^{-1})^{1-\alpha} - (I - (V_\sigma i\Omega)^{-1})^{1-\alpha}} i\Omega, \quad (237)$$

$$Z_{\rho(\alpha)} = \sqrt{\det([V_{\rho(\alpha)} + i\Omega] / 2)}, \quad (238)$$

$$Z_{\sigma(1-\alpha)} = \sqrt{\det([V_{\sigma(1-\alpha)} + i\Omega] / 2)}, \quad (239)$$

$$\delta s = s_\rho - s_\sigma. \quad (240)$$

**Proof.** Let  $\rho_0$  and  $\sigma_0$  denote the following operators:

$$\rho_0 = \exp \left[ -\frac{1}{2} \hat{x}^T H_\rho \hat{x} \right], \quad \sigma_0 = \exp \left[ -\frac{1}{2} \hat{x}^T H_\sigma \hat{x} \right]. \quad (241)$$

Consider that

$$\text{Tr}\{\rho^\alpha \sigma^{1-\alpha}\} = \text{Tr} \left\{ \left[ D(-s_\rho) \left( \frac{\rho_0}{Z_\rho} \right) D(s_\rho) \right]^\alpha \left[ D(-s_\sigma) \left( \frac{\sigma_0}{Z_\sigma} \right) D(s_\sigma) \right]^{1-\alpha} \right\} \quad (242)$$

$$= \frac{1}{Z_\rho^\alpha Z_\sigma^{1-\alpha}} \text{Tr} \left\{ D(-s_\rho) (\rho_0)^\alpha D(s_\rho) D(-s_\sigma) (\sigma_0)^{1-\alpha} D(s_\sigma) \right\} \quad (243)$$

$$= \frac{1}{Z_\rho^\alpha Z_\sigma^{1-\alpha}} \text{Tr} \left\{ D(-\delta s) (\rho_0)^\alpha D(\delta s) (\sigma_0)^{1-\alpha} \right\}, \quad (244)$$

where  $\delta s = s_\rho - s_\sigma$ . Using (194) of Corollary 14 for  $D(-\delta s) (\rho_0)^\alpha D(\delta s)$ , we have that

$$\left( \frac{1}{Z_\rho^\alpha Z_\sigma^{1-\alpha}} \right)^{-1} \text{Tr}\{\rho^\alpha \sigma^{1-\alpha}\} = e^{-\frac{1}{4} l^T i\Omega W_{\rho(\alpha)} l} \text{Tr}\{D(l) (\rho_0)^\alpha (\sigma_0)^{1-\alpha}\}, \quad (245)$$

where  $W_{\rho(\alpha)}$  is related to  $H_{\rho(\alpha)} = \alpha H_\rho$  by (32), and  $l$  is given by

$$l = (\exp[-i\Omega \alpha H_\rho] - I) \delta s. \quad (246)$$

Using (194) of Corollary 14 once again, we get that

$$\left(\frac{1}{Z_\rho^\alpha Z_\sigma^{1-\alpha}}\right)^{-1} \text{Tr}\{\rho^\alpha \sigma^{1-\alpha}\} = e^{-\frac{1}{4}l^T i\Omega W_{\rho(\alpha)} l} e^{\frac{1}{4}l^T i\Omega W_\xi l} \text{Tr}\{D(-t) (\rho_0)^\alpha (\sigma_0)^{1-\alpha} D(t)\} \quad (247)$$

$$= e^{-\frac{1}{4}l^T i\Omega W_{\rho(\alpha)} l} e^{\frac{1}{4}l^T i\Omega W_\xi l} \text{Tr}\{(\rho_0)^\alpha (\sigma_0)^{1-\alpha}\}, \quad (248)$$

where

$$W_\xi = \frac{I + \exp[i\Omega(1-\alpha)H_\sigma] \exp[i\Omega\alpha H_\rho]}{I - \exp[i\Omega(1-\alpha)H_\sigma] \exp[i\Omega\alpha H_\rho]}, \quad (249)$$

and we have used

$$t = (\exp[-i\Omega\alpha H_\rho] \exp[-i\Omega(1-\alpha)H_\sigma] - I)^{-1} (\exp[-i\Omega\alpha H_\rho] - I) \delta s. \quad (250)$$

Note that the particular value of  $t$  is irrelevant because the operators  $D(t)$  and  $D(-t)$  cancel each other in the trace operation. Applying Lemma 16 with  $H_9 = \alpha H_\rho = H_{\rho(\alpha)}$  and  $H_{10} = (1-\alpha)H_\sigma = H_{\sigma(1-\alpha)}$ , we see that

$$e^{-\frac{1}{4}l^T i\Omega W_{\rho(\alpha)} l} e^{\frac{1}{4}l^T i\Omega W_\xi l} = \exp\left[-\delta s^T (V_{\rho(\alpha)} + V_{\sigma(1-\alpha)})^{-1} \delta s\right]. \quad (251)$$

What remains now is to evaluate

$$\text{Tr}\{(\rho_0)^\alpha (\sigma_0)^{1-\alpha}\} = \text{Tr}\left\{\exp\left[-\frac{1}{2}\hat{x}^T (\alpha H_\rho) \hat{x}\right] \exp\left[-\frac{1}{2}\hat{x}^T ((1-\alpha)H_\sigma) \hat{x}\right]\right\}. \quad (252)$$

By Proposition 1, the covariance matrix corresponding to  $\alpha H_\rho$  is given in (236), and the covariance matrix corresponding to  $(1-\alpha)H_\sigma$  is given in (237). We now apply Proposition 11 to conclude that

$$\text{Tr}\left\{\exp\left[-\frac{1}{2}\hat{x}^T (\alpha H_\rho) \hat{x}\right] \exp\left[-\frac{1}{2}\hat{x}^T ((1-\alpha)H_\sigma) \hat{x}\right]\right\} = \frac{Z_{\rho(\alpha)} Z_{\sigma(1-\alpha)}}{[\det([V_{\rho(\alpha)} + V_{\sigma(1-\alpha)}] / 2)]^{1/2}}. \quad (253)$$

This concludes the proof. ■

**Theorem 19** *Let  $\alpha \in (1, \infty)$ , and let  $\rho$  and  $\sigma$  denote two Gaussian states such that  $V_{\sigma(\alpha-1)} > V_{\rho(\alpha)}$ . Then the Petz–Rényi relative quasi-entropy  $Q_\alpha(\rho||\sigma)$  as defined in (234) is given by*

$$Q_\alpha(\rho||\sigma) = \frac{Z_\sigma^{\alpha-1}}{Z_\rho^\alpha} \frac{Z_{\rho(\alpha)} Z_{\sigma(\alpha-1)}}{[\det([V_{\sigma(\alpha-1)} - V_{\rho(\alpha)}] / 2)]^{1/2}} \exp\left[\delta s^T (V_{\sigma(\alpha-1)} - V_{\rho(\alpha)})^{-1} \delta s\right], \quad (254)$$

where

$$V_{\rho^{(\alpha)}} = \frac{(I + (V_{\rho} i \Omega)^{-1})^{\alpha} + (I - (V_{\rho} i \Omega)^{-1})^{\alpha}}{(I + (V_{\rho} i \Omega)^{-1})^{\alpha} - (I - (V_{\rho} i \Omega)^{-1})^{\alpha}} i \Omega, \quad (255)$$

$$V_{\sigma^{(\alpha-1)}} = \frac{(I + (V_{\sigma} i \Omega)^{-1})^{\alpha-1} + (I - (V_{\sigma} i \Omega)^{-1})^{\alpha-1}}{(I + (V_{\sigma} i \Omega)^{-1})^{\alpha-1} - (I - (V_{\sigma} i \Omega)^{-1})^{\alpha-1}} i \Omega, \quad (256)$$

$$Z_{\rho^{(\alpha)}} = \sqrt{\det([V_{\rho^{(\alpha)}} + i \Omega] / 2)}, \quad (257)$$

$$Z_{\sigma^{(\alpha-1)}} = \sqrt{\det([V_{\sigma^{(\alpha-1)}} + i \Omega] / 2)}, \quad (258)$$

$$\delta s = s_{\rho} - s_{\sigma}. \quad (259)$$

**Proof.** Let  $\rho_0$  and  $\sigma_0$  denote the following operators:

$$\rho_0 = \exp \left[ -\frac{1}{2} \hat{x}^T H_{\rho} \hat{x} \right], \quad \sigma_0 = \exp \left[ -\frac{1}{2} \hat{x}^T H_{\sigma} \hat{x} \right]. \quad (260)$$

Consider that

$$\text{Tr}\{\rho^{\alpha} \sigma^{1-\alpha}\} = \text{Tr}\{\rho^{\alpha} [\sigma^{\alpha-1}]^{-1}\} = \text{Tr}\{[\sigma^{\alpha-1}]^{-1} \rho^{\alpha}\} \quad (261)$$

$$= \text{Tr} \left\{ \left[ \left[ D(-s_{\sigma}) \left( \frac{\sigma_0}{Z_{\sigma}} \right) D(s_{\sigma}) \right]^{\alpha-1} \right]^{-1} \left[ D(-s_{\rho}) \left( \frac{\rho_0}{Z_{\rho}} \right) D(s_{\rho}) \right]^{\alpha} \right\} \quad (262)$$

$$= \frac{Z_{\sigma}^{\alpha-1}}{Z_{\rho}^{\alpha}} \text{Tr} \left\{ D(-s_{\sigma}) [(\sigma_0)^{\alpha-1}]^{-1} D(s_{\sigma}) D(-s_{\rho}) (\rho_0)^{\alpha} D(s_{\rho}) \right\} \quad (263)$$

$$= \frac{Z_{\sigma}^{\alpha-1}}{Z_{\rho}^{\alpha}} \text{Tr} \left\{ D(\delta s) [(\sigma_0)^{\alpha-1}]^{-1} D(-\delta s) (\rho_0)^{\alpha} \right\}. \quad (264)$$

Using steps similar to those in the proof of Theorem 18, and based on Remark 15, we arrive at

$$\left( \frac{Z_{\sigma}^{\alpha-1}}{Z_{\rho}^{\alpha}} \right)^{-1} \text{Tr}\{\rho^{\alpha} \sigma^{1-\alpha}\} = e^{\frac{1}{4} l^T i \Omega W_{\sigma^{(\alpha-1)}} l} e^{\frac{1}{4} l^T i \Omega W_{\xi' l}} \text{Tr} \left\{ (\rho_0)^{\alpha} [(\sigma_0)^{\alpha-1}]^{-1} \right\}, \quad (265)$$

where

$$W_{\xi' l} = \frac{I + \exp [i \Omega \alpha H_{\rho}] \exp [-i \Omega (\alpha - 1) H_{\sigma}]}{I - \exp [i \Omega \alpha H_{\rho}] \exp [-i \Omega (\alpha - 1) H_{\sigma}]}, \quad (266)$$

$$l = (\exp [i \Omega (\alpha - 1) H_{\sigma}] - I) (-\delta s), \quad (267)$$

and  $W_{\sigma^{(\alpha-1)}}$  is related to the operator  $H_{\sigma^{(\alpha-1)}} = (\alpha - 1) H_{\sigma}$  by (32). Using Lemma 17 with  $H_9 = (\alpha - 1) H_{\sigma} = H_{\sigma^{(\alpha-1)}}$  and  $H_{10} = \alpha H_{\rho} = H_{\rho^{(\alpha)}}$ , we then have that

$$e^{\frac{1}{4} l^T i \Omega W_{\sigma^{(\alpha-1)}} l} e^{\frac{1}{4} l^T i \Omega W_{\xi' l}} = \exp \left[ \delta s^T (V_{\sigma^{(\alpha-1)}} - V_{\rho^{(\alpha)}})^{-1} \delta s \right]. \quad (268)$$



To finish the proof, consider that

$$\begin{aligned} & \text{Tr}\{\rho_0^\alpha \sigma_0^{1-\alpha}\} \\ &= \text{Tr}\{\rho_0^{\alpha/2} [\sigma_0^{\alpha-1}]^{-1} \rho_0^{\alpha/2}\} \end{aligned} \quad (269)$$

$$= \text{Tr} \left\{ \left[ \exp \left[ -\frac{1}{2} \hat{x}^T H_\rho \hat{x} \right] \right]^{\alpha/2} \left[ \exp \left[ -\frac{1}{2} \hat{x}^T (H_\sigma) \hat{x} \right] \right]^{\alpha-1} \left[ \exp \left[ -\frac{1}{2} \hat{x}^T H_\rho \hat{x} \right] \right]^{\alpha/2} \right\} \quad (270)$$

$$= \text{Tr} \left\{ \left[ \exp \left[ -\frac{1}{2} \hat{x}^T (\alpha H_\rho) \hat{x} \right] \right]^{1/2} \left[ \exp \left[ -\frac{1}{2} \hat{x}^T ((\alpha-1) H_\sigma) \hat{x} \right] \right]^{-1} \left[ \exp \left[ -\frac{1}{2} \hat{x}^T (\alpha H_\rho) \hat{x} \right] \right]^{1/2} \right\}. \quad (271)$$

By Proposition 1, the covariance matrix corresponding to  $\alpha H_\rho$  is given by (255), and the covariance matrix for  $(\alpha-1) H_\sigma$  is given by (256). We can then apply Proposition 12 to find that

$$\begin{aligned} & \text{Tr} \left\{ \left[ \exp \left[ -\frac{1}{2} \hat{x}^T (\alpha H_\rho) \hat{x} \right] \right]^{1/2} \left[ \exp \left[ -\frac{1}{2} \hat{x}^T ((\alpha-1) H_\sigma) \hat{x} \right] \right]^{-1} \left[ \exp \left[ -\frac{1}{2} \hat{x}^T (\alpha H_\rho) \hat{x} \right] \right]^{1/2} \right\} \\ &= \frac{Z_{\rho(\alpha)} Z_{\sigma(\alpha-1)}}{[\det ([V_{\sigma(\alpha-1)} - V_{\rho(\alpha)}] / 2)]^{1/2}}. \end{aligned} \quad (272)$$

For this equality to hold, it suffices that  $V_{\sigma(\alpha-1)} - V_{\rho(\alpha)} > 0$ , as discussed in the proof of Proposition 12. Putting everything together, we arrive at (254). ■

The quasi-entropy  $\text{Tr}\{\rho^2 \sigma^{-1}\}$  has a number of applications that are discussed in Section VII C. As a special case of Theorem 19, we arrive at the following expression for the quasi-entropy  $\text{Tr}\{\rho^2 \sigma^{-1}\}$  after applying Corollary 3:

**Corollary 20** *Let  $\alpha = 2$ , and let  $\rho$  and  $\sigma$  denote two Gaussian states such that  $V_\sigma > V_{\rho(2)}$ . Then the Petz–Rényi relative quasi-entropy as defined in (234) is given by*

$$Q_2(\rho||\sigma) = \text{Tr}\{\rho^2 \sigma^{-1}\} \quad (273)$$

$$= \frac{Z_\sigma^2}{Z_\rho^2} \frac{Z_{\rho(2)}}{[\det ([V_\sigma - V_{\rho(2)}] / 2)]^{1/2}} \exp \left[ \delta s^T (V_\sigma - V_{\rho(2)})^{-1} \delta s \right], \quad (274)$$

where

$$V_{\rho(2)} = \frac{1}{2} (V_\rho + \Omega V_\rho^{-1} \Omega^T), \quad (275)$$

$$Z_{\rho(2)} = \sqrt{\det([V_{\rho(2)} + i\Omega] / 2)}, \quad (276)$$

$$Z_\sigma = \sqrt{\det([V_\sigma + i\Omega] / 2)}. \quad (277)$$

## V. SANDWICHED RÉNYI RELATIVE ENTROPY

We now determine a formula for the sandwiched Rényi relative entropy<sup>6,7</sup> of two quantum Gaussian states  $\rho$  and  $\sigma$ . The sandwiched Rényi relative entropy is defined for  $\alpha \in (0, 1) \cup (1, \infty)$  as

$$\tilde{D}_\alpha(\rho||\sigma) \equiv \frac{1}{\alpha - 1} \ln \tilde{Q}_\alpha(\rho||\sigma), \quad (278)$$

where  $\tilde{Q}_\alpha(\rho||\sigma)$  denotes the sandwiched Rényi relative quasi-entropy:

$$\tilde{Q}_\alpha(\rho||\sigma) \equiv \text{Tr} \left\{ (\sigma^{(1-\alpha)/2\alpha} \rho \sigma^{(1-\alpha)/2\alpha})^\alpha \right\}, \quad (279)$$

$$= \text{Tr} \left\{ (\rho^{1/2} \sigma^{(1-\alpha)/\alpha} \rho^{1/2})^\alpha \right\}. \quad (280)$$

The second equality follows because the eigenvalues of  $\sigma^{(1-\alpha)/2\alpha} \rho \sigma^{(1-\alpha)/2\alpha}$  and  $\rho^{1/2} \sigma^{(1-\alpha)/\alpha} \rho^{1/2}$  are equal, but the latter expression is easier to work with, and thus we do so in what follows.

**Theorem 21** *Let  $\alpha \in (0, 1)$ , and let  $\rho$  and  $\sigma$  denote two Gaussian states. Then the sandwiched Rényi relative quasi-entropy  $\tilde{Q}_\alpha(\rho||\sigma)$  as defined in (280) is given by*

$$\tilde{Q}_\alpha(\rho||\sigma) = \frac{1}{Z_\rho^\alpha Z_\sigma^{1-\alpha}} [\det([V_{\xi(\alpha)} + i\Omega] / 2)]^{1/2} \exp \left\{ -\alpha \delta s^T (V_{\sigma(\beta)} + V_\rho)^{-1} \delta s \right\}, \quad (281)$$

where

$$V_{\xi(\alpha)} = \frac{(I + (V_\xi i\Omega)^{-1})^\alpha + (I - (V_\xi i\Omega)^{-1})^\alpha}{(I + (V_\xi i\Omega)^{-1})^\alpha - (I - (V_\xi i\Omega)^{-1})^\alpha} i\Omega, \quad (282)$$

$$V_\xi = V_\rho - \sqrt{I + (V_\rho \Omega)^{-2}} V_\rho (V_{\sigma(\beta)} + V_\rho)^{-1} V_\rho \sqrt{I + (\Omega V_\rho)^{-2}}, \quad (283)$$

$$V_{\sigma(\beta)} = \frac{(I + (V_\sigma i\Omega)^{-1})^\beta + (I - (V_\sigma i\Omega)^{-1})^\beta}{(I + (V_\sigma i\Omega)^{-1})^\beta - (I - (V_\sigma i\Omega)^{-1})^\beta} i\Omega, \quad (284)$$

$$\beta = (1 - \alpha) / \alpha, \quad (285)$$

$$\delta s = s_\rho - s_\sigma. \quad (286)$$

**Proof.** Let  $\rho_0$  and  $\sigma_0$  denote the following operators:

$$\rho_0 = \exp \left[ -\frac{1}{2} \hat{x}^T H_\rho \hat{x} \right], \quad \sigma_0 = \exp \left[ -\frac{1}{2} \hat{x}^T H_\sigma \hat{x} \right]. \quad (287)$$

To evaluate the expression for the sandwiched Rényi relative quasi-entropy, consider that

$$\begin{aligned} & \text{Tr} \left\{ (\rho^{1/2} \sigma^{(1-\alpha)/\alpha} \rho^{1/2})^\alpha \right\} \\ &= \text{Tr} \left\{ (\rho^{1/2} \sigma^\beta \rho^{1/2})^\alpha \right\} \end{aligned} \quad (288)$$

$$= \text{Tr} \left\{ \left( \left[ D(-s_\rho) \left( \frac{\rho_0}{Z_\rho} \right) D(s_\rho) \right]^{\frac{1}{2}} \left[ D(-s_\sigma) \left( \frac{\sigma_0}{Z_\sigma} \right) D(s_\sigma) \right]^\beta \left[ D(-s_\rho) \left( \frac{\rho_0}{Z_\rho} \right) D(s_\rho) \right]^{\frac{1}{2}} \right)^\alpha \right\} \quad (289)$$

$$= \text{Tr} \left\{ \left( D(-s_\rho) \left( \frac{\rho_0}{Z_\rho} \right)^{\frac{1}{2}} D(\delta s) \left( \frac{\sigma_0}{Z_\sigma} \right)^\beta D(-\delta s) \left( \frac{\rho_0}{Z_\rho} \right)^{\frac{1}{2}} D(s_\rho) \right)^\alpha \right\} \quad (290)$$

$$= \frac{1}{Z_\rho^\alpha Z_\sigma^{1-\alpha}} \text{Tr} \left\{ D(-s_\rho) \left( (\rho_0)^{\frac{1}{2}} D(\delta s) (\sigma_0)^\beta D(-\delta s) (\rho_0)^{\frac{1}{2}} \right)^\alpha D(s_\rho) \right\} \quad (291)$$

$$= \frac{1}{Z_\rho^\alpha Z_\sigma^{1-\alpha}} \text{Tr} \left\{ \left( (\rho_0)^{\frac{1}{2}} D(\delta s) (\sigma_0)^\beta D(-\delta s) (\rho_0)^{\frac{1}{2}} \right)^\alpha \right\}. \quad (292)$$

Using (194) of Corollary 14 for  $D(\delta s) (\sigma_0)^\beta D(-\delta s)$ , we obtain

$$\left( \frac{1}{Z_\rho^\alpha Z_\sigma^{1-\alpha}} \right)^{-1} \text{Tr} \left\{ (\rho^{1/2} \sigma^{(1-\alpha)/\alpha} \rho^{1/2})^\alpha \right\} = e^{-\frac{\alpha}{4} l^T i \Omega W_{\sigma(\beta)} l} \text{Tr} \left\{ \left( (\rho_0)^{\frac{1}{2}} D(l) (\sigma_0)^\beta (\rho_0)^{\frac{1}{2}} \right)^\alpha \right\}, \quad (293)$$

where  $W_{\sigma(\beta)}$  is related to  $H_{\sigma(\beta)} = \beta H_\sigma$  by (32), and  $l$  is given by

$$l = (\exp [-i \Omega H_{\sigma(\beta)}] - I) (-\delta s). \quad (294)$$

Continuing further, using (193) of Corollary 14, we get

$$\begin{aligned} & \left( \frac{1}{Z_\rho^\alpha Z_\sigma^{1-\alpha}} \right)^{-1} \text{Tr} \left\{ (\rho^{1/2} \sigma^{(1-\alpha)/\alpha} \rho^{1/2})^\alpha \right\} \\ &= e^{-\frac{\alpha}{4} l^T i \Omega W_{\sigma(\beta)} l} \text{Tr} \left\{ \left( D(\exp [-i \Omega H_\rho / 2] l) (\rho_0)^{\frac{1}{2}} (\sigma_0)^\beta (\rho_0)^{\frac{1}{2}} \right)^\alpha \right\}. \end{aligned} \quad (295)$$

By applying (194) of Corollary 14 once again, we obtain

$$\begin{aligned} & \left( \frac{1}{Z_\rho^\alpha Z_\sigma^{1-\alpha}} \right)^{-1} \text{Tr} \left\{ \left( \rho^{1/2} \sigma^{(1-\alpha)/\alpha} \rho^{1/2} \right)^\alpha \right\} \\ &= \left( e^{-\frac{1}{4} l^T i \Omega W_{\sigma(\beta)} l} e^{\frac{1}{4} (\exp[-i \Omega H_\rho / 2] l)^T i \Omega W_\xi \exp[-i \Omega H_\rho / 2] l} \right)^\alpha \text{Tr} \left\{ \left( D(-t) (\rho_0)^{\frac{1}{2}} (\sigma_0)^\beta (\rho_0)^{\frac{1}{2}} D(t) \right)^\alpha \right\} \end{aligned} \quad (296)$$

$$= \left( e^{-\frac{1}{4} l^T i \Omega W_{\sigma(\beta)} l} e^{\frac{1}{4} (\exp[-i \Omega H_\rho / 2] l)^T i \Omega W_\xi \exp[-i \Omega H_\rho / 2] l} \right)^\alpha \text{Tr} \left\{ \left( (\rho_0)^{\frac{1}{2}} (\sigma_0)^\beta (\rho_0)^{\frac{1}{2}} \right)^\alpha \right\}, \quad (297)$$

where

$$W_\xi = \frac{I + \exp[i \Omega H_\rho / 2] \exp[i \Omega \beta H_\sigma] \exp[i \Omega H_\rho / 2]}{I - \exp[i \Omega H_\rho / 2] \exp[i \Omega \beta H_\sigma] \exp[i \Omega H_\rho / 2]}, \quad (298)$$

and we have used

$$t = (\exp[-i \Omega H_\rho / 2] \exp[-i \Omega \beta H_\sigma] \exp[-i \Omega H_\rho / 2] - I)^{-1} \exp[-i \Omega H_\rho / 2] (\exp[-i \Omega H_{\sigma(\beta)}] - I) (-\delta s). \quad (299)$$

Note that the particular value of  $t$  is irrelevant because the operators  $D(t)$  and  $D(-t)$  cancel each other in the trace operation. We now simplify the expression in (297) term by term.

First, consider the exponent in the first prefactor:

$$\begin{aligned} & -\frac{1}{4} l^T i \Omega W_{\sigma(\beta)} l \\ &= -\frac{1}{4} \delta s^T i \Omega (\exp[i \Omega H_{\sigma(\beta)}] - I) \left( \frac{\exp[-i \Omega H_{\sigma(\beta)}] + I}{\exp[-i \Omega H_{\sigma(\beta)}] - I} \right) (\exp[-i \Omega H_{\sigma(\beta)}] - I) \delta s \end{aligned} \quad (300)$$

$$= \frac{1}{4} \delta s^T i \Omega (\exp[-i \Omega H_{\sigma(\beta)}] - \exp[i \Omega H_{\sigma(\beta)}]) \delta s. \quad (301)$$

Second, consider the exponent in the second prefactor in (297):

$$\begin{aligned} & \frac{1}{4} (\exp[-i\Omega H_\rho/2] l)^T i\Omega W_\xi \exp[-i\Omega H_\rho/2] l \\ &= \frac{1}{4} l^T i\Omega \exp[i\Omega H_\rho/2] W_\xi \exp[-i\Omega H_\rho/2] l \end{aligned} \quad (302)$$

$$= ((\exp[-i\Omega H_{\sigma(\beta)}] - I) \delta s)^T i\Omega \exp[i\Omega H_\rho/2] W_\xi \exp[-i\Omega H_\rho/2] (\exp[-i\Omega H_{\sigma(\beta)}] - I) \delta s \quad (303)$$

$$= \delta s^T (\exp[H_{\sigma(\beta)} i\Omega] - I) i\Omega \exp[i\Omega H_\rho/2] W_\xi \exp[-i\Omega H_\rho/2] (\exp[-i\Omega H_{\sigma(\beta)}] - I) \delta s \quad (304)$$

$$= \delta s^T i\Omega (\exp[i\Omega H_{\sigma(\beta)}] - I) \exp[i\Omega H_\rho/2] W_\xi \exp[-i\Omega H_\rho/2] (\exp[-i\Omega H_{\sigma(\beta)}] - I) \delta s \quad (305)$$

$$= \delta s^T i\Omega (\exp[i\Omega H_{\sigma(\beta)}] - I) W_{\xi'} (\exp[-i\Omega H_{\sigma(\beta)}] - I) \delta s, \quad (306)$$

where

$$W_{\xi'} = \frac{I + \exp[i\Omega H_\rho] \exp[i\Omega H_{\sigma(\beta)}]}{I - \exp[i\Omega H_\rho] \exp[i\Omega H_{\sigma(\beta)}]}. \quad (307)$$

Based on (301) and (306), and applying Lemma 16 with  $H_9 = H_{\sigma(\beta)}$  and  $H_{10} = H_\rho$ , we arrive at

$$\left( e^{-\frac{1}{4} l^T i\Omega W_{\sigma(\beta)} l} e^{\frac{1}{4} (\exp[-i\Omega H_\rho/2] l)^T i\Omega W_\xi \exp[-i\Omega H_\rho/2] l} \right)^\alpha = \exp \left\{ -\alpha \delta s^T (V_{\sigma(\beta)} + V_\rho)^{-1} \delta s \right\}. \quad (308)$$

Finally, we evaluate the term

$$\text{Tr} \left\{ \left( (\rho_0)^{\frac{1}{2}} (\sigma_0)^\beta (\rho_0)^{\frac{1}{2}} \right)^\alpha \right\}. \quad (309)$$

By Proposition 1, the covariance matrix  $V_{\sigma(\beta)}$  corresponding to  $\beta H_\sigma$  is as given in (284). By Proposition 8, we can write

$$(\rho_0)^{\frac{1}{2}} (\sigma_0)^\beta (\rho_0)^{\frac{1}{2}} = \left[ \exp \left[ -\frac{1}{2} \hat{x}^T H_\rho \hat{x} \right] \right]^{\frac{1}{2}} \exp \left[ -\frac{1}{2} \hat{x}^T \beta H_\sigma \hat{x} \right] \left[ \exp \left[ -\frac{1}{2} \hat{x}^T H_\rho \hat{x} \right] \right]^{\frac{1}{2}} \quad (310)$$

$$= \exp \left[ -\frac{1}{2} \hat{x}^T H_\xi \hat{x} \right], \quad (311)$$

where the covariance matrix corresponding to  $H_\xi$  is  $V_\xi$ , given in (283). Then we have that

$$\begin{aligned} \text{Tr} \left\{ \left( \left[ \exp \left[ -\frac{1}{2} \hat{x}^T H_\rho \hat{x} \right] \right]^{\frac{1}{2}} \exp \left[ -\frac{1}{2} \hat{x}^T \beta H_\sigma \hat{x} \right] \left[ \exp \left[ -\frac{1}{2} \hat{x}^T H_\rho \hat{x} \right] \right]^{\frac{1}{2}} \right)^\alpha \right\} \\ = \text{Tr} \left\{ \left[ \exp \left[ -\frac{1}{2} \hat{x}^T H_\xi \hat{x} \right] \right]^\alpha \right\} = \text{Tr} \left\{ \exp \left[ -\frac{1}{2} \hat{x}^T \alpha H_\xi \hat{x} \right] \right\}. \end{aligned} \quad (312)$$

By Proposition 1, the covariance matrix corresponding to  $\alpha H_\xi$  is  $V_{\xi(\alpha)}$ , given in (282). This finally implies that

$$\text{Tr} \left\{ \exp \left[ -\frac{1}{2} \hat{x}^T \alpha H_\xi \hat{x} \right] \right\} = [\det (V_{\xi(\alpha)} + i\Omega/2)]^{1/2}. \quad (313)$$

Combining the different terms, we arrive at the statement of the theorem. ■

**Theorem 22** *Let  $\alpha \in (1, \infty)$ , and let  $\rho$  and  $\sigma$  denote two Gaussian states such that  $V_{\sigma(\gamma)} > V_\rho$ . Then the sandwiched Rényi relative quasi-entropy  $\tilde{Q}_\alpha(\rho||\sigma)$  as defined in (280) is given by*

$$\tilde{Q}_\alpha(\rho||\sigma) = \frac{Z_\sigma^{\alpha-1}}{Z_\rho^\alpha} [\det ([V_{\xi(\alpha)} + i\Omega] / 2)]^{1/2} \exp \left\{ \alpha \delta s^T (V_{\sigma(\gamma)} - V_\rho)^{-1} \delta s \right\}, \quad (314)$$

where

$$V_{\xi(\alpha)} = \frac{(I + (V_\xi i\Omega)^{-1})^\alpha + (I - (V_\xi i\Omega)^{-1})^\alpha}{(I + (V_\xi i\Omega)^{-1})^\alpha - (I - (V_\xi i\Omega)^{-1})^\alpha} i\Omega, \quad (315)$$

$$V_\xi = V_\rho + \sqrt{I + (V_\rho \Omega)^{-2} V_\rho (V_{\sigma(\gamma)} - V_\rho)^{-1} V_\rho} \sqrt{I + (\Omega V_\rho)^{-2}}, \quad (316)$$

$$V_{\sigma(\gamma)} = \frac{(I + (V_\sigma i\Omega)^{-1})^\gamma + (I - (V_\sigma i\Omega)^{-1})^\gamma}{(I + (V_\sigma i\Omega)^{-1})^\gamma - (I - (V_\sigma i\Omega)^{-1})^\gamma} i\Omega, \quad (317)$$

$$\gamma = (\alpha - 1) / \alpha, \quad (318)$$

$$\delta s = s_\rho - s_\sigma. \quad (319)$$

**Proof.** Let  $\rho_0$  and  $\sigma_0$  denote the following operators:

$$\rho_0 = \exp \left[ -\frac{1}{2} \hat{x}^T H_\rho \hat{x} \right], \quad \sigma_0 = \exp \left[ -\frac{1}{2} \hat{x}^T H_\sigma \hat{x} \right]. \quad (320)$$

Consider that

$$\begin{aligned} & \text{Tr} \left\{ (\rho^{1/2} \sigma^{(1-\alpha)/\alpha} \rho^{1/2})^\alpha \right\} \\ &= \text{Tr} \left\{ \left( \rho^{1/2} [\sigma^{(\alpha-1)/\alpha}]^{-1} \rho^{1/2} \right)^\alpha \right\} = \text{Tr} \left\{ (\rho^{1/2} [\sigma^\gamma]^{-1} \rho^{1/2})^\alpha \right\} \end{aligned} \quad (321)$$

$$= \text{Tr} \left\{ \left( \left[ D(-s_\rho) \left( \frac{\rho_0}{Z_\rho} \right) D(s_\rho) \right]^{\frac{1}{2}} \left[ \left[ D(-s_\sigma) \left( \frac{\sigma_0}{Z_\sigma} \right) D(s_\sigma) \right]^\gamma \right]^{-1} \left[ D(-s_\rho) \left( \frac{\rho_0}{Z_\rho} \right) D(s_\rho) \right]^{\frac{1}{2}} \right)^\alpha \right\} \quad (322)$$

$$= \text{Tr} \left\{ \left( D(-s_\rho) \left( \frac{\rho_0}{Z_\rho} \right)^{\frac{1}{2}} D(\delta s) \left( \left[ \frac{\sigma_0}{Z_\sigma} \right]^\gamma \right)^{-1} D(-\delta s) \left( \frac{\rho_0}{Z_\rho} \right)^{\frac{1}{2}} D(s_\rho) \right)^\alpha \right\} \quad (323)$$

$$= \frac{Z_\sigma^{\alpha-1}}{Z_\rho^\alpha} \text{Tr} \left\{ D(-s_\rho) \left( (\rho_0)^{\frac{1}{2}} D(\delta s) [\sigma_0^\gamma]^{-1} D(-\delta s) (\rho_0)^{\frac{1}{2}} \right)^\alpha D(s_\rho) \right\} \quad (324)$$

$$= \frac{Z_\sigma^{\alpha-1}}{Z_\rho^\alpha} \text{Tr} \left\{ \left( (\rho_0)^{\frac{1}{2}} D(\delta s) [\sigma_0^\gamma]^{-1} D(-\delta s) (\rho_0)^{\frac{1}{2}} \right)^\alpha \right\}. \quad (325)$$

Using steps similar to those in Theorem 21, and based on Remark 15, we arrive at

$$\begin{aligned} & \left( \frac{Z_\sigma^{\alpha-1}}{Z_\rho^\alpha} \right)^{-1} \text{Tr} \left\{ (\rho^{1/2} [\sigma^\gamma]^{-1} \rho^{1/2})^\alpha \right\} \\ &= \left( e^{\frac{1}{4}l^T i\Omega W_{\sigma(\gamma)} l} e^{\frac{1}{4}(\exp[-i\Omega H_\rho/2]l)^T i\Omega W_\xi \exp[-i\Omega H_\rho/2]l} \right)^\alpha \text{Tr} \left\{ \left( (\rho_0)^{\frac{1}{2}} [\sigma^\gamma]^{-1} (\rho_0)^{\frac{1}{2}} \right)^\alpha \right\} \end{aligned} \quad (326)$$

where

$$W_\xi = \frac{I + \exp[i\Omega H_\rho/2] \exp[-i\Omega \gamma H_\sigma] \exp[i\Omega H_\rho/2]}{I - \exp[i\Omega H_\rho/2] \exp[-i\Omega \gamma H_\sigma] \exp[i\Omega H_\rho/2]}, \quad (327)$$

$$l = (\exp[i\Omega H_{\sigma(\gamma)}] - I)(-\delta s), \quad (328)$$

and where  $W_{\sigma(\gamma)}$  is related to  $H_{\sigma(\gamma)} = \gamma H_\sigma$  by (32). Following steps similar to those in (302)–(306) of Theorem 21, we get

$$\begin{aligned} & \left( \frac{Z_\sigma^{\alpha-1}}{Z_\rho^\alpha} \right)^{-1} \text{Tr} \left\{ (\rho^{1/2} [\sigma^\gamma]^{-1} \rho^{1/2})^\alpha \right\} \\ &= \left( e^{\frac{1}{4}l^T i\Omega W_{\sigma(\gamma)} l} e^{\frac{1}{4}l^T i\Omega W_{\xi'} l} \right)^\alpha \text{Tr} \left\{ \left( \rho_0^{1/2} [\sigma^\gamma]^{-1} \rho_0^{1/2} \right)^\alpha \right\}, \end{aligned} \quad (329)$$

where

$$W_{\xi'} = \frac{I + \exp[i\Omega H_\rho] \exp[-i\Omega H_{\sigma(\gamma)}]}{I - \exp[i\Omega H_\rho] \exp[-i\Omega H_{\sigma(\gamma)}]}. \quad (330)$$

Applying Lemma 17 with  $H_9 = H_{\sigma(\gamma)}$  and  $H_{10} = H_\rho$ , we arrive at

$$\left( e^{\frac{1}{4}l^T i\Omega W_{\sigma(\gamma)} l} e^{\frac{1}{4}l^T i\Omega W_{\xi} l} \right)^\alpha = \exp \left\{ \alpha \delta s^T (V_{\sigma(\gamma)} - V_\rho)^{-1} \delta s \right\}. \quad (331)$$

Now consider that

$$\begin{aligned} & \text{Tr} \left\{ \left( \rho_0^{1/2} [\sigma_0^\gamma]^{-1} \rho_0^{1/2} \right)^\alpha \right\} \\ &= \text{Tr} \left\{ \left( \left[ \exp \left[ -\frac{1}{2} \hat{x}^T H_\rho \hat{x} \right] \right]^{\frac{1}{2}} \left[ \left[ \exp \left[ -\frac{1}{2} \hat{x}^T H_\sigma \hat{x} \right] \right]^\gamma \right]^{-1} \left[ \exp \left[ -\frac{1}{2} \hat{x}^T H_\rho \hat{x} \right] \right]^{\frac{1}{2}} \right)^\alpha \right\} \end{aligned} \quad (332)$$

$$= \text{Tr} \left\{ \left( \exp \left[ -\frac{1}{2} \hat{x}^T [H_\rho/2] \hat{x} \right] \exp \left[ -\frac{1}{2} \hat{x}^T (\gamma H_\sigma) \hat{x} \right]^{-1} \exp \left[ -\frac{1}{2} \hat{x}^T [H_\rho/2] \hat{x} \right] \right)^\alpha \right\}. \quad (333)$$

Let  $V_{\sigma(\gamma)}$  denote the covariance matrix corresponding to  $\gamma H_\sigma$ . From Proposition 1, we know that it is given by (317). By applying Proposition 12, we find that

$$\exp \left[ -\frac{1}{2} \hat{x}^T [H_\rho/2] \hat{x} \right] \exp \left[ -\frac{1}{2} \hat{x}^T (\gamma H_\sigma) \hat{x} \right]^{-1} \exp \left[ -\frac{1}{2} \hat{x}^T [H_\rho/2] \hat{x} \right] = \exp \left[ -\frac{1}{2} \hat{x}^T H_\xi \hat{x} \right], \quad (334)$$

where the covariance matrix  $V_\xi$  corresponding to  $H_\xi$  is given by (316). Furthermore, the covariance matrix  $V_\xi$  is legitimate because  $H_\xi > 0$ , which in turn follows from the assumption  $V_{\sigma(\gamma)} - V_\rho > 0$  and the discussion in the proof of Proposition 12. Then we find that

$$\begin{aligned} & \text{Tr} \left\{ \left( \exp \left[ -\frac{1}{2} \hat{x}^T [H_\rho/2] \hat{x} \right] \exp \left[ -\frac{1}{2} \hat{x}^T (\gamma H_\sigma) \hat{x} \right]^{-1} \exp \left[ -\frac{1}{2} \hat{x}^T [H_\rho/2] \hat{x} \right] \right)^\alpha \right\} \\ &= \text{Tr} \left\{ \left( \exp \left[ -\frac{1}{2} \hat{x}^T H_\xi \hat{x} \right] \right)^\alpha \right\} = \text{Tr} \left\{ \left( \exp \left[ -\frac{1}{2} \hat{x}^T [\alpha H_\xi] \hat{x} \right] \right) \right\}. \end{aligned} \quad (335)$$

By Proposition 1, the covariance matrix  $V_{\xi(\alpha)}$  corresponding to  $\alpha H_\xi$  is given by (315). We can then conclude that

$$\text{Tr} \left\{ \left( \exp \left[ -\frac{1}{2} \hat{x}^T [\alpha H_\xi] \hat{x} \right] \right) \right\} = [\det(V_{\xi(\alpha)} + i\Omega/2)]^{1/2}. \quad (336)$$

This, along with (331) implies (314). ■

The collision relative entropy is a special case of the sandwiched Rényi relative entropy, introduced in Ref. 67, Definition 5.3.1 and applied in subsequent work<sup>68-71</sup>. As a special case of Theorem 22, we arrive at the following expression for the collision relative quasi-entropy  $\tilde{Q}_2(\rho||\sigma)$  after applying Corollaries 3 and 4:



**Corollary 23** *The collision relative quasi-entropy  $\tilde{Q}_2(\rho\|\sigma)$  of two Gaussian states  $\rho$  and  $\sigma$  such that  $V_{\sigma(1/2)} - V_\rho > 0$  is given by*

$$\tilde{Q}_2(\rho\|\sigma) = \text{Tr} \left\{ (\rho^{1/2} \sigma^{-1/2} \rho^{1/2})^2 \right\} \quad (337)$$

$$= \frac{Z_\sigma}{Z_\rho^2} [\det ([V_{\xi(2)} + i\Omega] / 2)]^{1/2} \exp \left\{ 2 \delta s^T (V_{\sigma(1/2)} - V_\rho)^{-1} \delta s \right\}, \quad (338)$$

where

$$V_{\xi(2)} = \frac{1}{2}(V_\xi + \Omega V_\xi^{-1} \Omega^T), \quad (339)$$

$$V_\xi = V_\rho + \sqrt{I + (V_\rho \Omega)^{-2}} V_\rho (V_{\sigma(1/2)} - V_\rho)^{-1} V_\rho \sqrt{I + (\Omega V_\rho)^{-2}}, \quad (340)$$

$$V_{\sigma(1/2)} = \left( \sqrt{I + (V_\sigma \Omega)^{-2}} + I \right) V_\sigma. \quad (341)$$

## VI. MAX-RELATIVE ENTROPY

Now we derive a formula for the max-relative entropy<sup>10</sup>, which is defined as

$$D_{\max}(\rho\|\sigma) \equiv \ln \|\rho^{1/2} \sigma^{-1} \rho^{1/2}\|_\infty. \quad (342)$$

**Theorem 24** *For the case in which  $V_\sigma - V_\rho > 0$ , the max-relative entropy  $D_{\max}(\rho\|\sigma)$  of two Gaussian states  $\rho$  and  $\sigma$  is given by*

$$D_{\max}(\rho\|\sigma) = \ln \left( \frac{Z_\sigma}{Z_\rho} \right) - \sum_{j=1}^n \text{arccoth}(\nu'_j) + \delta s^T (V_\sigma - V_\rho)^{-1} \delta s, \quad (343)$$

where  $\delta s = s_\rho - s_\sigma$  and  $\nu'_j$  is the  $j$ th symplectic eigenvalue of the following covariance matrix:

$$V' = V_\rho + \sqrt{I + (V_\rho \Omega)^{-2}} V_\rho (V_\sigma - V_\rho)^{-1} V_\rho \sqrt{I + (\Omega V_\rho)^{-2}}. \quad (344)$$

Alternatively, we have that

$$D_{\max}(\rho\|\sigma) = \ln \left( \frac{Z_\sigma}{Z_\rho} \right) - \frac{1}{2} \text{Tr} \left\{ \text{arccoth} \left( \sqrt{-V' \Omega V' \Omega} \right) \right\} + \delta s^T (V_\sigma - V_\rho)^{-1} \delta s. \quad (345)$$

**Proof.** To begin with, we discuss how to calculate the maximum eigenvalue of a Gaussian state  $\omega$  (i.e.,  $\|\omega\|_\infty$ ). One can also find a discussion of this calculation in Ref. 55. Consider that a thermal state  $\theta(N)$  with mean photon number  $N \geq 0$  is of the form

$\sum_{n=0}^{\infty} N^n / (N+1)^{n+1} |n\rangle\langle n|$ , so that its maximum eigenvalue is equal to  $[N+1]^{-1}$  (corresponding to the weight of the vacuum). From the Williamson theorem, we know that any  $n$ -mode Gaussian state can be written as a unitary operator acting on a tensor product of  $n$  thermal states, and the mean photon number  $N$  of each thermal state is related to the symplectic eigenvalue  $\nu$  as  $\nu = 2N+1$ . In terms of the symplectic eigenvalue  $\nu = 2N+1$ , the maximum eigenvalue of  $\theta(N)$  is equal to  $[N+1]^{-1} = 2/(\nu+1)$ . So, for a general Gaussian state  $\omega$ , if we have the covariance matrix, we simply perform a Williamson decomposition, and then we find that

$$\|\omega\|_{\infty} = \prod_{j=1}^n 2/(\nu_j + 1). \quad (346)$$

Let  $\rho_0$  and  $\sigma_0$  denote the following operators:

$$\rho_0 = \exp\left[-\frac{1}{2}\hat{x}^T H_{\rho}\hat{x}\right], \quad \sigma_0 = \exp\left[-\frac{1}{2}\hat{x}^T H_{\sigma}\hat{x}\right]. \quad (347)$$

Consider that

$$\begin{aligned} & \|\rho^{1/2}\sigma^{-1}\rho^{1/2}\|_{\infty} \\ &= \left\| \left( D(-s_{\rho}) \begin{bmatrix} \rho_0 \\ Z_{\rho} \end{bmatrix} D(s_{\rho}) \right)^{\frac{1}{2}} \left( D(-s_{\sigma}) \begin{bmatrix} \sigma_0 \\ Z_{\sigma} \end{bmatrix} D(s_{\sigma}) \right)^{-1} \left( D(-s_{\rho}) \begin{bmatrix} \rho_0 \\ Z_{\rho} \end{bmatrix} D(s_{\rho}) \right)^{\frac{1}{2}} \right\|_{\infty} \end{aligned} \quad (348)$$

$$= \left\| D(-s_{\rho}) \begin{bmatrix} \rho_0 \\ Z_{\rho} \end{bmatrix}^{\frac{1}{2}} D(s_{\rho}) D(-s_{\sigma}) \begin{bmatrix} \sigma_0 \\ Z_{\sigma} \end{bmatrix}^{-1} D(s_{\sigma}) D(-s_{\rho}) \begin{bmatrix} \rho_0 \\ Z_{\rho} \end{bmatrix}^{\frac{1}{2}} D(s_{\rho}) \right\|_{\infty} \quad (349)$$

$$= \frac{Z_{\sigma}}{Z_{\rho}} \left\| [\rho_0]^{\frac{1}{2}} D(\delta s) [\sigma_0]^{-1} D(-\delta s) [\rho_0]^{\frac{1}{2}} \right\|_{\infty}, \quad (350)$$

where we have used the fact that the infinity norm of an operator is invariant with respect to unitaries. Note that the operator

$$[\rho_0]^{\frac{1}{2}} D(\delta s) [\sigma_0]^{-1} D(-\delta s) [\rho_0]^{\frac{1}{2}} \quad (351)$$

is identical to the operator whose trace is evaluated in (325) of Theorem 22 when  $\gamma$  and  $\alpha$  are independently set to 1 in that expression. Thus, based on the mean-vector-dependent factor that is derived in (331) and the fact that  $\|A\|_{\infty} = \lim_{p \rightarrow \infty} \|A\|_p$ , we have that

$$\|\rho^{1/2}\sigma^{-1}\rho^{1/2}\|_{\infty} = \left( \frac{Z_{\sigma}}{Z_{\rho}} \right) \exp\{\delta s^T (V_{\sigma} - V_{\rho})^{-1} \delta s\} \left\| \rho_0^{1/2} \sigma_0^{-1} \rho_0^{1/2} \right\|_{\infty}. \quad (352)$$

Now consider that

$$\begin{aligned} & \left\| \rho_0^{1/2} \sigma_0^{-1} \rho_0^{1/2} \right\|_{\infty} \\ &= \left\| \left[ \exp \left[ -\frac{1}{2} \hat{x}^T H_{\rho} \hat{x} \right] \right]^{\frac{1}{2}} \left[ \exp \left[ -\frac{1}{2} \hat{x}^T H_{\sigma} \hat{x} \right] \right]^{-1} \left[ \exp \left[ -\frac{1}{2} \hat{x}^T H_{\rho} \hat{x} \right] \right]^{\frac{1}{2}} \right\|_{\infty} \end{aligned} \quad (353)$$

$$= \left\| \exp \left[ -\frac{1}{2} \hat{x}^T [H_{\rho}/2] \hat{x} \right] \exp \left[ -\frac{1}{2} \hat{x}^T [-H_{\sigma}] \hat{x} \right] \exp \left[ -\frac{1}{2} \hat{x}^T [H_{\rho}/2] \hat{x} \right] \right\|_{\infty}. \quad (354)$$

From Proposition 12, we conclude that there exists an  $H'$  such that

$$\exp \left[ -\frac{1}{2} \hat{x}^T [H_{\rho}/2] \hat{x} \right] \exp \left[ -\frac{1}{2} \hat{x}^T [-H_{\sigma}] \hat{x} \right] \exp \left[ -\frac{1}{2} \hat{x}^T [H_{\rho}/2] \hat{x} \right] = \exp \left[ -\frac{1}{2} \hat{x}^T H' \hat{x} \right]. \quad (355)$$

with corresponding covariance matrix  $V'$  given by

$$V' = V_{\rho} + \sqrt{I + (V_{\rho}\Omega)^{-2}} V_{\rho} (V_{\sigma} - V_{\rho})^{-1} V_{\rho} \sqrt{I + (\Omega V_{\rho})^{-2}}. \quad (356)$$

Again applying Proposition 12, we find that

$$\text{Tr} \left\{ \exp \left[ -\frac{1}{2} \hat{x}^T H' \hat{x} \right] \right\} = [\det ([V' + i\Omega] / 2)]^{1/2}. \quad (357)$$

Continuing, we find that

$$\begin{aligned} \left\| \rho_0^{1/2} \sigma_0^{-1} \rho_0^{1/2} \right\|_{\infty} &= [\det ([V' + i\Omega] / 2)]^{1/2} \times \\ & \left\| \frac{\exp \left[ -\frac{1}{2} \hat{x}^T [H_{\rho}/2] \hat{x} \right] \exp \left[ -\frac{1}{2} \hat{x}^T [-H_{\sigma}] \hat{x} \right] \exp \left[ -\frac{1}{2} \hat{x}^T [H_{\rho}/2] \hat{x} \right]}{[\det ([V' + i\Omega] / 2)]^{1/2}} \right\|_{\infty}. \end{aligned} \quad (358)$$

The term inside the infinity norm is a state because  $V'$  is a legitimate covariance matrix.

Using the expression in (346) for the infinity norm of a Gaussian state, we find that

$$\left\| \frac{\exp \left[ -\frac{1}{2} \hat{x}^T [H_{\rho}/2] \hat{x} \right] \exp \left[ -\frac{1}{2} \hat{x}^T [-H_{\sigma}] \hat{x} \right] \exp \left[ -\frac{1}{2} \hat{x}^T [H_{\rho}/2] \hat{x} \right]}{[\det ([V' + i\Omega] / 2)]^{1/2}} \right\|_{\infty} = \prod_{j=1}^n 2/(\nu'_j + 1), \quad (359)$$

where  $\nu'_j$  is the  $j$ th symplectic eigenvalue of  $V'$ . Using the fact that<sup>46</sup> (Eq. (2.14))

$$[\det ([V' + i\Omega] / 2)]^{1/2} = \prod_{j=1}^n \frac{1}{2} \sqrt{(\nu'_j + 1)(\nu'_j - 1)}, \quad (360)$$

we find that

$$\left\| \rho_0^{1/2} \sigma_0^{-1} \rho_0^{1/2} \right\|_{\infty} = [\det ([V' + i\Omega] / 2)]^{1/2} \prod_{j=1}^n 2/(\nu'_j + 1) = \prod_{j=1}^n \sqrt{\frac{\nu'_j - 1}{\nu'_j + 1}}. \quad (361)$$

Taking a logarithm, we see that

$$\ln \left\| \rho_0^{1/2} \sigma_0^{-1} \rho_0^{1/2} \right\|_\infty = \sum_{j=1}^n \frac{1}{2} \ln \left( \frac{\nu'_j - 1}{\nu'_j + 1} \right) = - \sum_{j=1}^n \frac{1}{2} \ln \left( \frac{\nu'_j + 1}{\nu'_j - 1} \right) = - \sum_{j=1}^n \operatorname{arccoth}(\nu'_j). \quad (362)$$

Combining with (352) gives (343).

To arrive at the formula in (345), consider for a covariance matrix  $V$  with symplectic diagonalization  $S(D \oplus D)S^T$ , where  $S$  is a symplectic matrix and  $D$  is a diagonal matrix of symplectic eigenvalues, we have that (see Ref. 72, Appendix A)

$$Vi\Omega = S(U \otimes I_n) ([-D] \oplus D) [S(U \otimes I_n)]^{-1}, \quad (363)$$

where  $U$  is the following unitary matrix:

$$U \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}. \quad (364)$$

From this, we see that

$$-V\Omega V\Omega = (Vi\Omega)(Vi\Omega) = S(U \otimes I_n) ([D^2 \oplus D^2]) [S(U \otimes I_n)]^{-1}, \quad (365)$$

which implies that

$$\sqrt{-V\Omega V\Omega} = S(U \otimes I_n) (D \oplus D) [S(U \otimes I_n)]^{-1}, \quad (366)$$

and in turn that

$$\sum_{j=1}^n \operatorname{arccoth}(\nu'_j) = \frac{1}{2} \operatorname{Tr} \{ \operatorname{arccoth}([D \oplus D]) \} \quad (367)$$

$$= \frac{1}{2} \operatorname{Tr} \{ S(U \otimes I_n) \operatorname{arccoth}([D \oplus D]) [S(U \otimes I_n)]^{-1} \} \quad (368)$$

$$= \frac{1}{2} \operatorname{Tr} \{ \operatorname{arccoth}(S(U \otimes I_n) [D \oplus D] [S(U \otimes I_n)]^{-1}) \} \quad (369)$$

$$= \frac{1}{2} \operatorname{Tr} \{ \operatorname{arccoth}(\sqrt{-V\Omega V\Omega}) \}. \quad (370)$$

This concludes the proof. ■

In the above theorem, we provided a formula for the max-relative entropy that holds whenever  $V_\sigma - V_\rho > 0$ . The proposition below states that the condition  $V_\sigma - V_\rho \geq 0$  is necessary for the max-relative entropy to be finite (it still remains open to determine whether the condition  $V_\sigma - V_\rho > 0$  is necessary and sufficient.)

**Proposition 25** *A necessary condition for the max-relative entropy  $D_{\max}(\rho\|\sigma)$  of two Gaussian states  $\rho$  and  $\sigma$  to be finite is that  $V_\sigma - V_\rho \geq 0$ .*

**Proof.** Suppose that  $D_{\max}(\rho\|\sigma) < +\infty$ , which implies that there exists a constant  $M$  such that  $\rho \leq M\sigma$ . Then for all  $u \in \mathbb{R}^{2n}$ , we can test the inequality on the displaced vacuum state  $|u\rangle = D(u)|0\rangle$ , giving that

$$\langle u|\rho|u\rangle \leq M\langle u|\sigma|u\rangle. \quad (371)$$

These expectation values on displaced vacuum states form what is known as the Husimi Q-function, which for Gaussian states is given by the following Gaussian form<sup>30</sup>:

$$\langle u|\rho|u\rangle = \frac{2^n}{\sqrt{\det(V_\rho + I)}} \exp \left\{ - (u - s_\rho)^T (V_\rho + I)^{-1} (u - s_\rho) \right\}. \quad (372)$$

The constraint in (371) is then equivalent to

$$\exp \left\{ - (u - s_\rho)^T (V_\rho + I)^{-1} (u - s_\rho) + (u - s_\sigma)^T (V_\sigma + I)^{-1} (u - s_\sigma) \right\} \leq M \sqrt{\frac{\det(V_\rho + I)}{\det(V_\sigma + I)}}, \quad (373)$$

which should be obeyed for all  $u \in \mathbb{R}^{2n}$ . This is possible only if  $(V_\rho + I)^{-1} \geq (V_\sigma + I)^{-1}$ , which implies that we should have  $V_\sigma \geq V_\rho$ . ■

**Remark 26** *The development at the end of the proof of Theorem 24 extends more generally to any function  $f : [1, \infty) \rightarrow [0, \infty)$ . Given a covariance matrix  $V$  with symplectic eigenvalues  $\{\nu_j\}_{j=1}^n$ , then*

$$\sum_{j=1}^n f(\nu_j) = \frac{1}{2} \text{Tr} \{ f(\sqrt{-V\Omega V\Omega}) \}, \quad (374)$$

$$\prod_{j=1}^n f(\nu_j) = \sqrt{\det \left( f(\sqrt{-V\Omega V\Omega}) \right)}. \quad (375)$$

*The first equality follows from a development identical to that in (363)–(370). The second*

equality follows because

$$\prod_{j=1}^n f(\nu_j) = \sqrt{\det(f(D \oplus D))} \quad (376)$$

$$= \sqrt{\det(S(U \otimes I_n)) \det(f(D \oplus D)) \det([S(U \otimes I_n)]^{-1})} \quad (377)$$

$$= \sqrt{\det(S(U \otimes I_n) f(D \oplus D) [S(U \otimes I_n)]^{-1})} \quad (378)$$

$$= \sqrt{\det(f[S(U \otimes I_n)(D \oplus D)[S(U \otimes I_n)]^{-1}])} \quad (379)$$

$$= \sqrt{\det\left(f(\sqrt{-V\Omega V\Omega})\right)}, \quad (380)$$

with some of the steps following from the development in (363)–(370).

## VII. APPLICATIONS

### A. Quantum state discrimination and hypothesis testing

Quantum state discrimination is one of the central problems in quantum information theory<sup>73</sup>. It represents the quantum generalization of the classical statistical decision-theoretic problem of deciding the probability distribution corresponding to a random variable, given some candidate distributions. There is an inherent probability of error associated with the task, which in the classical case is due to the overlap between the candidate distributions, and in the quantum case is additionally due to the non-commutativity of the candidate states. The goal in part is to determine fundamental bounds on the error probability associated with the discrimination as dictated by the laws of quantum mechanics. Quantum state discrimination is important in several areas of quantum information, particularly in quantum communication and cryptography, where information is encoded in nonorthogonal quantum states, and optimal decoding requires minimum error discrimination at the quantum limit. Since continuous-variable physical systems such as the bosonic field modes of electromagnetic radiation form particularly good carriers of information in communication scenarios, the discrimination of Gaussian states is especially important, and has been extensively studied in the past (see, e.g., Ref. 74).

Binary quantum state discrimination is largely studied in two flavors, namely with symmetric and asymmetric goals in minimizing the two possible types of error probabilities in decision. In the symmetric case, the goal is to minimize the average probability of error in discriminating two quantum states. The optimal measurement achieving the smallest average error probability was determined in Refs. 75 and 76 and is known as the Helstrom limit. The Helstrom limit is a function of the trace distance between the candidate states, which, at least in the finite-dimensional case, becomes more difficult to calculate as the dimension of the Hilbert space grows larger<sup>77</sup>. Furthermore, as far as we are aware, there is no known simple formula for the trace distance between two Gaussian states. Thus, upper bounds on the Helstrom limit that are easier to calculate have been developed. In this regard, the quantum Chernoff bound<sup>49,78,79</sup> serves as a good substitute, and it actually gives an exact characterization of the exponential decay of the average error probability in the limit when many copies of the state are available. The quantum Chernoff bound can be expressed as an optimized Petz–Rényi relative entropy for  $\alpha \in (0, 1)$ .

In one variant of asymmetric hypothesis testing, the error probability corresponding to one of the types of errors is constrained to decay at a rate  $e^{-nr}$ , for some  $r > 0$  and where  $n$  is the number of copies of the state, while the goal is to determine the behavior of the other kind of error probability. If  $r$  is less than the quantum relative entropy, the quantum Hoeffding bound<sup>14,15</sup> applies and states that the other kind of error probability decays exponentially fast to zero, and the optimal error exponent can be expressed in terms of the Petz–Rényi relative entropy<sup>5</sup>. If  $r$  exceeds the quantum relative entropy, the strong converse bound from Ref. 20 applies and states that the other kind of error probability converges exponentially fast to one, and the optimal strong converse exponent can be expressed in terms of the sandwiched Rényi relative entropy<sup>6,7</sup>.

Gaussian state discrimination has been studied in the contexts of both symmetric and asymmetric cost of errors. The quantum Chernoff bound<sup>49,50</sup> and the quantum Hoeffding bound<sup>80</sup> for Gaussian states have been considered. However, the expressions given in these earlier works were in terms of the symplectic decomposition of the Gaussian states. A quest for more compact and elegant expressions for the quantities that solely depend on the covariance matrices of the candidate states has prompted the development of other less

tight bounds for these quantities<sup>50</sup>.

The formulas derived in our paper readily apply to the settings of the quantum Chernoff bound, the quantum Hoeffding bound, and the strong converse regime and lead to expressions for the exponents of Gaussian state discrimination in these contexts. We do not give details here, but instead we simply note that the results can be obtained by direct substitution of our formulas into the general expressions for the various bounds. We note that our formulas depend only on the mean vectors and covariance matrices of the candidate states.

## B. Quantum communication theory

There is an intimate link between hypothesis testing and communication theory, first realized in the classical case in Ref. 81. This approach has since been successfully explored in the context of quantum communication theory<sup>7,11,15,17,19,21,22,24,25,27</sup>, in order to establish strong converse bounds for a variety of information-processing tasks. In all of the aforementioned works, the strong converse bounds are expressed in terms of the Rényi relative entropies. As such, one would expect the formulas derived here to apply in these contexts, and we now comment on the most direct application of our results in the context of quantum and private communication.

To begin with, let us recall that a quantum channel has a capacity for quantum and private communication when assisted by classical communication between the sender and receiver (see, e.g., Refs. 82–84 for these notions). These capacities are roughly and respectively defined to be the maximum rates at which these communication resources can be used to establish entanglement or secret key reliably between a sender and a receiver, when using the channel many times. It is of interest to understand these capacities in the context of quantum key distribution<sup>85</sup>, in order to understand the limitations on practical protocols. For channels that are teleportation simulable<sup>82</sup>, meaning that they can be realized by the action of local operations and classical communication (LOCC) on a resource state<sup>86</sup>, a general protocol of the above form can be significantly simplified<sup>82,87</sup>, such that it consists of a single round of LOCC acting on a given number of copies of the resource state. As observed in Ref. 82 for the case of quantum communication, one can then bound the assisted quantum capacity of the



channel in terms of the distillable entanglement of the resource state, and the same reasoning trivially extends to the case of assisted private communication. These observations apply as well to Gaussian channels that are teleportation simulable, as identified and discussed in Refs. 88 and 89.

One of the main contributions of Refs. 24 and 27 is that bounds on the strong converse exponent for assisted quantum and private communication over teleportation-simulable channels, respectively, can be expressed in terms of the sandwiched Rényi relative entropy of the underlying resource state. After these developments, a recent work<sup>90</sup>, following the approach of Ref. 91, found finite-energy Gaussian resource states that can be used for the teleportation simulation of thermal Gaussian channels, and as such, they were used to establish bounds on the assisted quantum and private capacities of these channels. Avoiding details, we simply note here that one can evaluate the sandwiched Rényi relative entropy of the finite-energy Gaussian resource states from Ref. 90 in order to determine bounds on the strong converse exponent for communication over these channels.

### C. Mixing times of Markov processes and covert communication

We finally briefly mention some applications of the Petz–Rényi relative entropy of order two. One particular quantum  $\chi^2$  divergence from Ref. 92 can be related to the Petz–Rényi relative entropy of order two. Therein, the authors used the quantum  $\chi^2$  divergence to bound mixing times of quantum Markov processes. As a result, we suspect that the formulas derived in our paper will be useful in the context of bounding mixing times of quantum Gauss–Markov processes, such as the processes considered in Refs. 93 and 94.

Additionally, the Petz–Rényi relative entropy of order two has been employed in the context of bounding error probabilities for covert communication over quantum channels<sup>95</sup>. In covert communication, the goal is for two parties to communicate information over a quantum channel, such that someone else (typically called a warden), who is allowed to observe the channel, is effectively not able to realize that they are in fact communicating. In light of this previous work, we expect that the formula derived in our paper will be useful in the context of covert communication when using a quantum Gaussian channel for the task.

## VIII. CONCLUSION

The main contribution of our paper is the derivation of formulas for the Petz–Rényi relative entropy and the sandwiched Rényi relative entropy of quantum Gaussian states for  $\alpha \in (0, 1) \cup (1, \infty)$ . Interestingly, our approach handles the previously elusive case for the Petz–Rényi relative entropy when  $\alpha \in (1, \infty)$ . We also derived a formula for the max-relative entropy of two quantum Gaussian states. Given the wide applicability of the Rényi relative entropies and quantum Gaussian states in quantum information theory and beyond, we suspect that the formulas derived here will be useful in a number of future applications.

For future work, it remains open to determine whether the sufficient conditions given in Theorems 19 and 22 are in fact necessary for the quantities to be finite. The similarity of the sufficient conditions with the necessary and sufficient conditions from the classical case<sup>96,97</sup> suggest that this might be the case. At the least, Proposition 25 establishes significant progress on this question for the max-relative entropy. Additionally, the approach given in our paper can be used to determine expressions for the  $\alpha$ - $z$  relative entropies<sup>98</sup> and the generalized Rényi quantities from Refs. 99–102 (in the latter case, we would need expressions for the adjoint of a quantum Gaussian channel, as given in Ref. 103).

### Appendix A: Covariance matrix for $\rho(\alpha)$

Recall that the covariance matrix  $V_\rho$  for an  $n$ -mode state has a symplectic (Williamson) decomposition as

$$S_\rho (D_\rho \oplus D_\rho) S_\rho^T = S_\rho (I_2 \otimes D_\rho) S_\rho^T, \quad (\text{A1})$$

where  $S_\rho$  is a  $2n \times 2n$  symplectic matrix satisfying  $S_\rho \Omega S_\rho^T = \Omega$  and  $D_\rho$  is a diagonal matrix of symplectic eigenvalues (each entry being  $> 1$  for a faithful state).

**Proposition 27** *The following equality holds*

$$V_{\rho(\alpha)} = \frac{(I + (V_\rho i\Omega)^{-1})^\alpha + (I - (V_\rho i\Omega)^{-1})^\alpha}{(I + (V_\rho i\Omega)^{-1})^\alpha - (I - (V_\rho i\Omega)^{-1})^\alpha} i\Omega \quad (\text{A2})$$

$$= S_\rho \left( I_2 \otimes \frac{(D_\rho + I)^\alpha + (D_\rho - I)^\alpha}{(D_\rho + I)^\alpha - (D_\rho - I)^\alpha} \right) S_\rho^T, \quad (\text{A3})$$

which demonstrates the equivalence of  $V_{\rho(\alpha)}$  with Eqs. (54) and (55) of<sup>50</sup>.

**Proof.** By definition,

$$V_{\rho(\alpha)} = \frac{(I + (V_{\rho}i\Omega)^{-1})^{\alpha} + (I - (V_{\rho}i\Omega)^{-1})^{\alpha}}{(I + (V_{\rho}i\Omega)^{-1})^{\alpha} - (I - (V_{\rho}i\Omega)^{-1})^{\alpha}}i\Omega. \quad (\text{A4})$$

Consider the following reasoning along the lines from Ref. 72, Appendix A. The covariance matrix  $V_{\rho}$  for an  $n$ -mode state has a symplectic decomposition as

$$S_{\rho}(D_{\rho} \oplus D_{\rho})S_{\rho}^T = S_{\rho}(I_2 \otimes D_{\rho})S_{\rho}^T, \quad (\text{A5})$$

where  $S_{\rho}$  is a  $2n \times 2n$  symplectic matrix and  $D_{\rho}$  is a diagonal matrix of symplectic eigenvalues. After some steps, this implies that

$$V_{\rho}i\Omega = S_{\rho}(U \otimes I_n)([-D_{\rho}] \oplus D_{\rho})(U^{\dagger} \otimes I_n)S_{\rho}^{-1} \quad (\text{A6})$$

$$= S_{\rho}(U \otimes I_n)(-\sigma_Z \otimes D_{\rho})(U^{\dagger} \otimes I_n)S_{\rho}^{-1}, \quad (\text{A7})$$

where

$$U \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}. \quad (\text{A8})$$

So we see that the eigenvalues of  $V_{\rho}i\Omega$  correspond to the symplectic eigenvalues of  $V_{\rho}$  and the eigenvectors of  $V_{\rho}i\Omega$  correspond to the symplectic eigenvectors of  $V_{\rho}$ . Let us abbreviate this as

$$V_{\rho}i\Omega = M\bar{D}M^{-1}, \quad (\text{A9})$$

$$M = S_{\rho}(U \otimes I_n), \quad (\text{A10})$$

$$\bar{D} = [-D_{\rho}] \oplus D_{\rho}. \quad (\text{A11})$$

Note that for a positive definite state  $\rho$ , each entry of  $D$  is  $> 1$ . So this means that both

$Vi\Omega + I$  and  $Vi\Omega - I$  are invertible matrices. Consider that

$$V_{\rho^{(\alpha)}} = \frac{(I + (V_{\rho}i\Omega)^{-1})^{\alpha} + (I - (V_{\rho}i\Omega)^{-1})^{\alpha}}{(I + (V_{\rho}i\Omega)^{-1})^{\alpha} - (I - (V_{\rho}i\Omega)^{-1})^{\alpha}} i\Omega \quad (\text{A12})$$

$$= \frac{(I + (M\bar{D}M^{-1})^{-1})^{\alpha} + (I - (M\bar{D}M^{-1})^{-1})^{\alpha}}{(I + (M\bar{D}M^{-1})^{-1})^{\alpha} - (I - (M\bar{D}M^{-1})^{-1})^{\alpha}} i\Omega \quad (\text{A13})$$

$$= \frac{(M(I + \bar{D}^{-1})M^{-1})^{\alpha} + (M(I - \bar{D}^{-1})M^{-1})^{\alpha}}{(M(I + \bar{D}^{-1})M^{-1})^{\alpha} - (M(I - \bar{D}^{-1})M^{-1})^{\alpha}} i\Omega \quad (\text{A14})$$

$$= M \frac{(I + \bar{D}^{-1})^{\alpha} + (I - \bar{D}^{-1})^{\alpha}}{(I + \bar{D}^{-1})^{\alpha} - (I - \bar{D}^{-1})^{\alpha}} M^{-1} i\Omega. \quad (\text{A15})$$

Finally consider that

$$M^{-1}i\Omega = (U^{\dagger} \otimes I_n) S_{\rho}^{-1} i\Omega = (U^{\dagger} \otimes I_n) i\Omega S_{\rho}^T. \quad (\text{A16})$$

This then implies that

$$V_{\rho^{(\alpha)}} = S_{\rho} (U \otimes I_n) \frac{(I + \bar{D}^{-1})^{\alpha} + (I - \bar{D}^{-1})^{\alpha}}{(I + \bar{D}^{-1})^{\alpha} - (I - \bar{D}^{-1})^{\alpha}} (U^{\dagger} \otimes I_n) i\Omega S_{\rho}^T. \quad (\text{A17})$$

Since the function  $x \rightarrow \frac{(1+1/x)^{\alpha} + (1-1/x)^{\alpha}}{(1+1/x)^{\alpha} - (1-1/x)^{\alpha}}$  is an odd function (it is a composition of three odd functions: arcoth, scaling by  $\alpha$ , and then coth) and  $\frac{(1+1/x)^{\alpha} + (1-1/x)^{\alpha}}{(1+1/x)^{\alpha} - (1-1/x)^{\alpha}} = \frac{(x+1)^{\alpha} + (x-1)^{\alpha}}{(x+1)^{\alpha} - (x-1)^{\alpha}}$  for  $x > 1$ , we can rewrite

$$\frac{(I + \bar{D}^{-1})^{\alpha} + (I - \bar{D}^{-1})^{\alpha}}{(I + \bar{D}^{-1})^{\alpha} - (I - \bar{D}^{-1})^{\alpha}} = \frac{(I + D^{-1})^{\alpha} + (I - D^{-1})^{\alpha}}{(I + D^{-1})^{\alpha} - (I - D^{-1})^{\alpha}} \quad (\text{A18})$$

$$= -\sigma_Z \otimes \frac{(I + D^{-1})^{\alpha} + (I - D^{-1})^{\alpha}}{(I + D^{-1})^{\alpha} - (I - D^{-1})^{\alpha}} \quad (\text{A19})$$

$$= -\sigma_Z \otimes \frac{(D + I)^{\alpha} + (D - I)^{\alpha}}{(D + I)^{\alpha} - (D - I)^{\alpha}}, \quad (\text{A20})$$

which means that

$$S_\rho(U \otimes I_n) \frac{(I + \bar{D}^{-1})^\alpha + (I - \bar{D}^{-1})^\alpha}{(I + \bar{D}^{-1})^\alpha - (I - \bar{D}^{-1})^\alpha} (U^\dagger \otimes I_n) i\Omega S_\rho^T$$

$$= S_\rho(U \otimes I_n) \left( -\sigma_Z \otimes \frac{(D+I)^\alpha + (D-I)^\alpha}{(D+I)^\alpha - (D-I)^\alpha} \right) (U^\dagger \otimes I_n) i\Omega S_\rho^T \quad (\text{A21})$$

$$= S_\rho \left( -U\sigma_Z U^\dagger \otimes \frac{(D+I)^\alpha + (D-I)^\alpha}{(D+I)^\alpha - (D-I)^\alpha} \right) i\Omega S_\rho^T \quad (\text{A22})$$

$$= S_\rho \left( -\sigma_Y \otimes \frac{(D+I)^\alpha + (D-I)^\alpha}{(D+I)^\alpha - (D-I)^\alpha} \right) i\Omega S_\rho^T \quad (\text{A23})$$

$$= S_\rho \left( -\sigma_Y \otimes \frac{(D+I)^\alpha + (D-I)^\alpha}{(D+I)^\alpha - (D-I)^\alpha} \right) (-\sigma_Y \otimes I_n) S_\rho^T \quad (\text{A24})$$

$$= S_\rho \left( I_2 \otimes \frac{(D+I)^\alpha + (D-I)^\alpha}{(D+I)^\alpha - (D-I)^\alpha} \right) S_\rho^T. \quad (\text{A25})$$

So we conclude that

$$V_{\rho(\alpha)} = S_\rho \left( I_2 \otimes \frac{(D+I)^\alpha + (D-I)^\alpha}{(D+I)^\alpha - (D-I)^\alpha} \right) S_\rho^T. \quad (\text{A26})$$

This completes the proof of the equivalence of  $V_{\rho(\alpha)}$  with Eqs. (54) and (55) of Ref. 50. ■

**Acknowledgements.** We are grateful to Leonardo Bianchi, Saikat Guha, Felix Leditzky, Ty Volkoff, and Haoyu Qi for discussions. KPS thanks the Max Planck Society for support. LL acknowledges financial support from the European Research Council (AdG IRQUAT No. 267386), the Spanish MINECO (project no. FIS2013-40627-P and no. FIS2016-86681-P), and the Generalitat de Catalunya (CIRIT Project no. 2014 SGR 966). MMW acknowledges support from the National Science Foundation and the Office of Naval Research. He is also grateful to the Max Planck Institute for the Science of Light, especially Christoph Marquardt, for hosting him for a research visit in June 2016, when this project was initiated.

## REFERENCES

- <sup>1</sup>Alfréd Rényi. On measures of entropy and information. *Proceedings of the 4th Berkeley Symposium on Mathematics, Statistics and Probability*, 1:547–561, 1961. held at the Statistical Laboratory, University of California, 1960, edited by J. Neyman (University of California Press, Berkeley).

- <sup>2</sup>Solomon Kullback and Richard A. Leibler. On information and sufficiency. *The Annals of Mathematical Statistics*, 22(1):79–86, March 1951.
- <sup>3</sup>Imre Csiszár. Generalized cutoff rates and Rényi’s information measures. *IEEE Transactions on Information Theory*, 41(1):26–34, January 1995.
- <sup>4</sup>Tim van Erven and Peter Harremoës. Rényi divergence and Kullback-Leibler divergence. *IEEE Transactions on Information Theory*, 60(7):3797–3820, July 2014. arXiv:1206.2459.
- <sup>5</sup>Dénes Petz. Quasi-entropies for finite quantum systems. *Reports in Mathematical Physics*, 23:57–65, 1986.
- <sup>6</sup>Martin Müller-Lennert, Frédéric Dupuis, Oleg Szehr, Serge Fehr, and Marco Tomamichel. On quantum Rényi entropies: a new definition and some properties. *Journal of Mathematical Physics*, 54(12):122203, December 2013. arXiv:1306.3142.
- <sup>7</sup>Mark M. Wilde, Andreas Winter, and Dong Yang. Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy. *Communications in Mathematical Physics*, 331(2):593–622, October 2014. arXiv:1306.1586.
- <sup>8</sup>Hisaharu Umegaki. Conditional expectations in an operator algebra IV (entropy and information). *Kodai Mathematical Seminar Reports*, 14(2):59–85, 1962.
- <sup>9</sup>Göran Lindblad. Entropy, information and quantum measurements. *Communications in Mathematical Physics*, 33(4):305–322, December 1973.
- <sup>10</sup>Nilanjana Datta. Min- and max-relative entropies and a new entanglement monotone. *IEEE Transactions on Information Theory*, 55(6):2816–2826, June 2009. arXiv:0803.2770.
- <sup>11</sup>Tomohiro Ogawa and Hiroshi Nagaoka. Strong converse to the quantum channel coding theorem. *IEEE Transactions on Information Theory*, 45(7):2486–2489, November 1999. arXiv:quant-ph/9808063.
- <sup>12</sup>Tomohiro Ogawa and Hiroshi Nagaoka. Strong converse and Stein’s lemma in quantum hypothesis testing. *IEEE Transactions on Information Theory*, 46(7):2428–2433, November 2000. arXiv:quant-ph/9906090.
- <sup>13</sup>Tomohiro Ogawa and Masahito Hayashi. On error exponents in quantum hypothesis testing. *IEEE Transactions on Information Theory*, 50(6):1368–1372, June 2004. arXiv:quant-ph/0206151.

- <sup>14</sup>Hiroshi Nagaoka. The converse part of the theorem for quantum Hoeffding bound. November 2006. arXiv:quant-ph/0611289.
- <sup>15</sup>Masahito Hayashi. Error exponent in asymmetric quantum hypothesis testing and its application to classical-quantum channel coding. *Physical Review A*, 76(6):062301, December 2007. arXiv:quant-ph/0611013.
- <sup>16</sup>Koenraad M. R. Audenaert, Michael Nussbaum, Arleta Szkola, and Frank Verstraete. Asymptotic error rates in quantum hypothesis testing. *Communications in Mathematical Physics*, 279:251–283, 2008. arXiv:0708.4282.
- <sup>17</sup>Robert Koenig and Stephanie Wehner. A strong converse for classical channel coding using entangled inputs. *Physical Review Letters*, 103(7):070504, August 2009. arXiv:0903.2838.
- <sup>18</sup>Milán Mosonyi and Fumio Hiai. On the quantum Rényi relative entropies and related capacity formulas. *IEEE Transactions on Information Theory*, 57(4):2474–2487, April 2011. arXiv:0912.1286.
- <sup>19</sup>Naresh Sharma and Naqeeb Ahmad Warsi. On the strong converses for the quantum channel capacity theorems. June 2012. arXiv:1205.1712.
- <sup>20</sup>Milan Mosonyi and Tomohiro Ogawa. Quantum hypothesis testing and the operational interpretation of the quantum Rényi relative entropies. *Communications in Mathematical Physics*, 334(3):1617–1648, March 2015. arXiv:1309.3228.
- <sup>21</sup>Manish K. Gupta and Mark M. Wilde. Multiplicativity of completely bounded  $p$ -norms implies a strong converse for entanglement-assisted capacity. *Communications in Mathematical Physics*, 334(2):867–887, March 2015. arXiv:1310.7028.
- <sup>22</sup>Tom Cooney, Milan Mosonyi, and Mark M. Wilde. Strong converse exponents for a quantum channel discrimination problem and quantum-feedback-assisted communication. *Communications in Mathematical Physics*, 344(3):797–829, June 2016. arXiv:1408.3373.
- <sup>23</sup>Masahito Hayashi and Marco Tomamichel. Correlation detection and an operational interpretation of the Rényi mutual information. *Journal of Mathematical Physics*, 57(10):102201, October 2016. arXiv:1408.6894.
- <sup>24</sup>Marco Tomamichel, Mark M. Wilde, and Andreas Winter. Strong converse rates for quantum communication. *IEEE Transactions on Information Theory*, 63(1):715–727, January 2017. arXiv:1406.2946.

- <sup>25</sup>Dawei Ding and Mark M. Wilde. Strong converse exponents for the feedback-assisted classical capacity of entanglement-breaking channels. *Problems of Information Transmission*, 54(1):1–19, January 2018. arXiv:1506.02228.
- <sup>26</sup>Felix Leditzky, Mark M. Wilde, and Nilanjana Datta. Strong converse theorems using Rényi entropies. *Journal of Mathematical Physics*, 57(8):082202, August 2016. arXiv:1506.02635.
- <sup>27</sup>Mark M. Wilde, Marco Tomamichel, and Mario Berta. Converse bounds for private communication over quantum channels. *IEEE Transactions on Information Theory*, 63(3):1792–1817, March 2017. arXiv:1602.08898.
- <sup>28</sup>Stefano Olivares. Quantum optics in the phase space. *The European Physical Journal Special Topics*, 203(1):3–24, April 2012. arXiv:1111.0786.
- <sup>29</sup>Gerardo Adesso, Sammy Ragy, and Antony R. Lee. Continuous variable quantum information: Gaussian states and beyond. *Open Systems and Information Dynamics*, 21(01–02):1440001, June 2014. arXiv:1401.4679.
- <sup>30</sup>Alessio Serafini. *Quantum Continuous Variables*. CRC Press, 2017.
- <sup>31</sup>Filippo Caruso, Jens Eisert, Vittorio Giovannetti, and Alexander S. Holevo. Multi-mode bosonic Gaussian channels. *New Journal of Physics*, 10:083030, August 2008. arXiv:0804.0511.
- <sup>32</sup>Armin Uhlmann. The “transition probability” in the state space of a  $*$ -algebra. *Reports on Mathematical Physics*, 9(2):273–279, 1976.
- <sup>33</sup>Gh.-S. Paraoanu and Horia Scutaru. Fidelity for multimode thermal squeezed states. *Physical Review A*, 61(2):022306, January 2000. arXiv:quant-ph/9907068.
- <sup>34</sup>Xiang-Bin Wang, L. C. Kwek, and C. H. Oh. Bures fidelity for diagonalizable quadratic Hamiltonians in multi-mode systems. *Journal of Physics A: Mathematical and General*, 33(27):4925, July 2000.
- <sup>35</sup>R. Balian and E. Brezin. Nonunitary Bogoliubov transformations and extension of Wick’s theorem. *Il Nuovo Cimento B (1965-1970)*, 64(1):37–55, November 1969.
- <sup>36</sup>Jason Twamley. Bures and statistical distance for squeezed thermal states. *Journal of Physics A: Mathematical and General*, 29(13):3723, July 1996. arXiv:quant-ph/9603019.
- <sup>37</sup>Horia Scutaru. Fidelity for displaced squeezed thermal states and the oscillator semigroup.



*Journal of Physics A: Mathematical and General*, 31(15):3659, April 1998. arXiv:quant-ph/9708013.

<sup>38</sup>Gh.-S. Paraoanu and Horia Scutaru. Bures distance between two displaced thermal states. *Physical Review A*, 58(2):869–871, August 1998. arXiv:quant-ph/9703051.

<sup>39</sup>Paulina Marian, Tudor A. Marian, and Horia Scutaru. Bures distance as a measure of entanglement for two-mode squeezed thermal states. *Physical Review A*, 68(6):062309, December 2003.

<sup>40</sup>Hyunchul Nha and Howard J. Carmichael. Distinguishing two single-mode Gaussian states by homodyne detection: An information-theoretic approach. *Physical Review A*, 71(3):032336, March 2005. arXiv:quant-ph/0410111.

<sup>41</sup>Stefano Olivares, Matteo G. A. Paris, and Ulrik L. Andersen. Cloning of Gaussian states by linear optics. *Physical Review A*, 73(6):062330, June 2006. arXiv:quant-ph/0601164.

<sup>42</sup>Paulina Marian and Tudor A. Marian. Bures distance as a measure of entanglement for symmetric two-mode Gaussian states. *Physical Review A*, 77(6):062319, June 2008. arXiv:0705.1138.

<sup>43</sup>Paulina Marian and Tudor A. Marian. Gaussian entanglement of symmetric two-mode Gaussian states. *The European Physical Journal Special Topics*, 160(1):281–289, July 2008. arXiv:0711.3477.

<sup>44</sup>Gunnar Björk, Jonas Söderholm, Luis L. Sanchez-Soto, Andrei B. Klimov, Iulia Ghiu, Paulina Marian, and Tudor A. Marian. Quantum degrees of polarization. *Optics Communications*, 283(22):4440–4447, November 2010. arXiv:1005.3935.

<sup>45</sup>Iulia Ghiu, Gunnar Björk, Paulina Marian, and Tudor A. Marian. Probing light polarization with the quantum Chernoff bound. *Physical Review A*, 82(2):023803, August 2010. arXiv:1008.3858.

<sup>46</sup>Paulina Marian and Tudor A. Marian. Uhlmann fidelity between two-mode Gaussian states. *Physical Review A*, 86(2):022340, August 2012. arXiv:1111.7067.

<sup>47</sup>Leonardo Banchi, Samuel L. Braunstein, and Stefano Pirandola. Quantum fidelity for arbitrary Gaussian states. *Physical Review Letters*, 115(26):260501, December 2015. arXiv:1507.01941.

<sup>48</sup>Paulina Marian and Tudor A. Marian. Quantum Fisher information on two manifolds of

- two-mode Gaussian states. *Physical Review A*, 93(5):052330, May 2016. arXiv:1605.06524.
- <sup>49</sup>J. Calsamiglia, R. Muñoz Tapia, Ll. Masanes, A. Acín, and E. Bagan. Quantum Chernoff bound as a measure of distinguishability between density matrices: Application to qubit and Gaussian states. *Physical Review A*, 77(3):032311, March 2008. arXiv:0708.2343.
- <sup>50</sup>Stefano Pirandola and Seth Lloyd. Computable bounds for the discrimination of Gaussian states. *Physical Review A*, 78(1):012331, July 2008. arXiv:0806.1625.
- <sup>51</sup>Alexander S. Holevo. On quasiequivalence of locally normal states. *Theoretical and Mathematical Physics*, 13(2):1071–1082, November 1972.
- <sup>52</sup>Paulina Marian and Tudor A. Marian. Hellinger distance as a measure of Gaussian discord. *Journal of Physics A: Mathematical and Theoretical*, 48(11):115301, March 2015. arXiv:1408.4477.
- <sup>53</sup>Xiao-yu Chen. Gaussian relative entropy of entanglement. *Physical Review A*, 71(6):062320, June 2005. arXiv:quant-ph/0402109.
- <sup>54</sup>Ole Krueger. *Quantum Information Theory with Gaussian Systems*. PhD thesis, Technische Universität Braunschweig, April 2006. Available at [https://publikationsserver.tu-braunschweig.de/receive/dbbs\\_mods\\_00020741](https://publikationsserver.tu-braunschweig.de/receive/dbbs_mods_00020741).
- <sup>55</sup>Alexander S. Holevo. Entropy gain and the Choi–Jamiolkowski correspondence for infinite-dimensional quantum evolutions. *Theoretical and Mathematical Physics*, 166(1):123–138, January 2011.
- <sup>56</sup>Alexander S. Holevo. The Choi–Jamiolkowski forms of quantum Gaussian channels. *Journal of Mathematical Physics*, 52(4):042202, April 2011. arXiv:1004.0196.
- <sup>57</sup>Alexander S. Holevo. *Quantum Systems, Channels, Information*. de Gruyter Studies in Mathematical Physics (Book 16). de Gruyter, November 2012.
- <sup>58</sup>R. Simon, N. Mukunda, and Biswadeb Dutta. Quantum-noise matrix for multimode systems:  $U(n)$  invariance, squeezing, and normal forms. *Physical Review A*, 49(3):1567–1583, March 1994.
- <sup>59</sup>R. W. Munn and R. Silbey. Ensemble averages of exponential quadratic phonon operators. *Journal of Physics A: Mathematical and General*, 11(5):939, 1978 1978.
- <sup>60</sup>Richard Friesner, Montgomery Pettitt, and John M. Jean. Calculation of temperature-dependent multimode resonance Raman line shapes for harmonic potential surfaces. *The*

*Journal of Chemical Physics*, 82(7):2918, April 1985.

- <sup>61</sup>John Williamson. On the algebraic problem concerning the normal forms of linear dynamical systems. *American Journal of Mathematics*, 58(1):141–163, January 1936.
- <sup>62</sup>Arthur Cayley. Sur quelques propriétés des déterminants gauches. *Journal für die reine und angewandte Mathematik*, 32:119–123, 1846.
- <sup>63</sup>Arthur Cayley. *The collected mathematical papers of Arthur Cayley (1841–1853)*. Cambridge University Press, 1889. article 52 (pages 332–336).
- <sup>64</sup>Gene H. Golub and Charles F. Van Loan. *Matrix Computations*. Johns Hopkins University Press, Baltimore, Maryland, USA, third edition, 1996.
- <sup>65</sup>Max A. Woodbury. Inverting modified matrices. Technical Report 42, Statistical Research Group, Princeton University, Princeton, New Jersey, USA, June 1950.
- <sup>66</sup>Ludovico Lami, Siddhartha Das, and Mark M. Wilde. Approximate reversal of quantum Gaussian dynamics. *Journal of Physics A*, 51(12):125301, March 2018. arXiv:1702.04737.
- <sup>67</sup>Renato Renner. *Security of Quantum Key Distribution*. PhD thesis, ETH Zurich, September 2005. arXiv:quant-ph/0512258.
- <sup>68</sup>Mario Berta, Patrick J. Coles, and Stephanie Wehner. Entanglement-assisted guessing of complementary measurement outcomes. *Physical Review A*, 90(6):062127, December 2014. arXiv:1302.5902.
- <sup>69</sup>Frederic Dupuis, Omar Fawzi, and Stephanie Wehner. Entanglement sampling and applications. *IEEE Transactions on Information Theory*, 61(2):1093–1112, February 2015. arXiv:1305.1316.
- <sup>70</sup>Frédéric Dupuis, Mario Berta, Jürg Wullschleger, and Renato Renner. One-shot decoupling. *Communications in Mathematical Physics*, 328(1):251–284, May 2014. arXiv:1012.6044.
- <sup>71</sup>Salman Beigi and Amin Gohari. Quantum achievability proof via collision relative entropy. *IEEE Transactions on Information Theory*, 60(12):7980–7986, December 2014. arXiv:1312.3822.
- <sup>72</sup>Mark M. Wilde, Marco Tomamichel, Seth Lloyd, and Mario Berta. Gaussian hypothesis testing and quantum illumination. *Physical Review Letters*, 119(12):120501, September 2017. arXiv:1608.06991.

- <sup>73</sup>Joonwoo Bae and Leong-Chuan Kwek. Quantum state discrimination and its applications. *Journal of Physics A: Mathematical and Theoretical*, 48(8):083001, February 2015. arXiv:1707.02571.
- <sup>74</sup>Carmen Invernizzi. *Quantum Estimation and Discrimination in Continuous Variable and Fermionic Systems*. PhD thesis, University of Milan, 2011.
- <sup>75</sup>Carl W. Helstrom. Quantum detection and estimation theory. *Journal of Statistical Physics*, 1:231–252, 1969.
- <sup>76</sup>Carl W. Helstrom. *Quantum Detection and Estimation Theory*. Academic, New York, 1976.
- <sup>77</sup>John Watrous. Limits on the power of quantum statistical zero-knowledge. *Proceedings of the 43rd Annual IEEE Symposium on Foundations of Computer Science*, pages 459–468, November 2002. arXiv:quant-ph/0202111.
- <sup>78</sup>Koenraad M. R. Audenaert, John Calsamiglia, Ramon Muñoz Tapia, Emilio Bagan, Lluís Masanes, Antonio Acín, and Frank Verstraete. Discriminating states: The quantum Chernoff bound. *Physical Review Letters*, 98(16):160501, April 2007. arXiv:quant-ph/0610027.
- <sup>79</sup>Michael Nussbaum and Arleta Szkola. The Chernoff lower bound for symmetric quantum hypothesis testing. *Annals of Statistics*, 37(2):1040–1057, 2009. arXiv:quant-ph/0607216.
- <sup>80</sup>Gaetana Spedalieri and Samuel L. Braunstein. Asymmetric quantum hypothesis testing with Gaussian states. *Physical Review A*, 90(5):052307, November 2014. arXiv:1407.0884.
- <sup>81</sup>Richard Blahut. Hypothesis testing and information theory. *IEEE Transactions on Information Theory*, 20(4):405–417, July 1974.
- <sup>82</sup>Charles H. Bennett, David P. DiVincenzo, John A. Smolin, and William K. Wootters. Mixed-state entanglement and quantum error correction. *Physical Review A*, 54(5):3824–3851, November 1996. arXiv:quant-ph/9604024.
- <sup>83</sup>Charles H. Bennett, Igor Devetak, Peter W. Shor, and John A. Smolin. Inequalities and separations among assisted capacities of quantum channels. *Physical Review Letters*, 96(15):150502, April 2006. arXiv:quant-ph/0406086.
- <sup>84</sup>Masahiro Takeoka, Saikat Guha, and Mark M. Wilde. Fundamental rate-loss tradeoff for optical quantum key distribution. *Nature Communications*, 5:5235, October 2014. arXiv:1504.06390.

- <sup>85</sup>Valerio Scarani, Helle Bechmann-Pasquinucci, Nicolas J. Cerf, Miloslav Dušek, Norbert Lütkenhaus, and Momtchil Peev. The security of practical quantum key distribution. *Reviews of Modern Physics*, 81(3):1301–1350, September 2009. arXiv:0802.4155.
- <sup>86</sup>Michał Horodecki, Paweł Horodecki, and Ryszard Horodecki. General teleportation channel, singlet fraction, and quasidistillation. *Physical Review A*, 60(3):1888–1898, September 1999. arXiv:quant-ph/9807091.
- <sup>87</sup>Alexander Müller-Hermes. Transposition in quantum information theory. Master’s thesis, Technical University of Munich, September 2012.
- <sup>88</sup>Michael M. Wolf, David Pérez-García, and Geza Giedke. Quantum capacities of bosonic channels. *Physical Review Letters*, 98(13):130501, March 2007. arXiv:quant-ph/0606132.
- <sup>89</sup>Julien Niset, Jaromír Fiurasek, and Nicolas J. Cerf. No-go theorem for Gaussian quantum error correction. *Physical Review Letters*, 102(12):120501, March 2009. arXiv:0811.3128.
- <sup>90</sup>Eneet Kaur and Mark M. Wilde. Upper bounds on secret key agreement over lossy thermal bosonic channels. *Physical Review A*, 96(6):062318, December 2017. arXiv:1706.04590.
- <sup>91</sup>Pietro Liuzzo-Scorpo, Andrea Mari, Vittorio Giovannetti, and Gerardo Adesso. Optimal continuous variable quantum teleportation with limited resources. *Physical Review Letters*, 119(12):120503, September 2017. arXiv:1705.03017.
- <sup>92</sup>K. Temme, M. J. Kastoryano, M. B. Ruskai, M. M. Wolf, and F. Verstraete. The  $\chi^2$ -divergence and mixing times of quantum Markov processes. *Journal of Mathematical Physics*, 51(12):122201, 2010.
- <sup>93</sup>Teiko Heinosaari, Alexander S. Holevo, and Michael M. Wolf. The semigroup structure of Gaussian channels. *Quantum Information and Computation*, 10(7):619–635, July 2010. arXiv:0909.0408.
- <sup>94</sup>Vittorio Giovannetti, Alexander S. Holevo, Seth Lloyd, and Lorenzo Maccone. Generalized minimal output entropy conjecture for one-mode Gaussian channels: definitions and some exact results. *Journal of Physics A: Mathematical and Theoretical*, 43(41):415305, October 2010. arXiv:1004.4787.
- <sup>95</sup>A. Sheikholeslami, B. A. Bash, D. Towsley, D. Goeckel, and S. Guha. Covert communication over classical-quantum channels. In *2016 IEEE International Symposium on Information Theory (ISIT)*, pages 2064–2068, July 2016. arXiv:1601.06826.

- <sup>96</sup>Manuel Gil. On Rényi divergence measures for continuous alphabet sources. Master's thesis, Queen's University, Kingston, Ontario, Canada, August 2011. Available at <http://qspace.library.queensu.ca/handle/1974/6680>.
- <sup>97</sup>M. Gil, F. Alajaji, and T. Linder. Rényi divergence measures for commonly used univariate continuous distributions. *Information Sciences*, 249:124–131, November 2013.
- <sup>98</sup>Koenraad M. R. Audenaert and Nilanjana Datta.  $\alpha$ - $z$ -Rényi relative entropies. *Journal of Mathematical Physics*, 56(2):022202, February 2015. arXiv:1310.7178.
- <sup>99</sup>Mario Berta, Kaushik Seshadreesan, and Mark M. Wilde. Rényi generalizations of the conditional quantum mutual information. *Journal of Mathematical Physics*, 56(2):022205, February 2015. arXiv:1403.6102.
- <sup>100</sup>Kaushik P. Seshadreesan, Mario Berta, and Mark M. Wilde. Rényi squashed entanglement, discord, and relative entropy differences. *Journal of Physics A: Mathematical and Theoretical*, 48(39):395303, September 2015. arXiv:1410.1443.
- <sup>101</sup>Mario Berta, Kaushik P. Seshadreesan, and Mark M. Wilde. Rényi generalizations of quantum information measures. *Physical Review A*, 91(2):022333, February 2015. arXiv:1502.07977.
- <sup>102</sup>Nilanjana Datta and Mark M. Wilde. Quantum Markov chains, sufficiency of quantum channels, and Rényi information measures. *Journal of Physics A: Mathematical and Theoretical*, 48(50):505301, November 2015. arXiv:1501.05636.
- <sup>103</sup>Marco G. Genoni, Ludovico Lami, and Alessio Serafini. Conditional and unconditional Gaussian quantum dynamics. *Contemporary Physics*, 57(3):331–349, January 2016. arXiv:1607.02619.