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Recoverability for Holevo's just-as-good fidelity

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Abstract—Holevo's just-as-good fidelity is a similarity measure for quantum states that has found several applications. One of its critical properties is that it obeys a data processing inequality: the measure does not decrease under the action of a quantum channel on the underlying states. In this paper, I prove a refinement of this data processing inequality that includes an additional term related to recoverability. That is, if the increase in the measure is small after the action of a partial trace, then one of the states can be nearly recovered by the Petz recovery channel, while the other state is perfectly recovered by the same channel. The refinement is given in terms of the trace distance of one of the states to its recovered version and also depends on the minimum eigenvalue of the other state. As such, the refinement is universal, in the sense that the recovery channel depends only on one of the states, and it is explicit, given by the Petz recovery channel. The appendix contains a generalization of the aforementioned result to arbitrary quantum channels.

I. INTRODUCTION

In Holevo's seminal 1972 work on the quasiequivalence of locally normal states [1], he established the following inequalities for quantum states ρ and σ :

$$1 - \sqrt{F_H(\rho, \sigma)} \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq \sqrt{1 - F_H(\rho, \sigma)}, \quad (1)$$

where $\|\rho - \sigma\|_1$ denotes the well known trace distance and the function F_H is Holevo's "just-as-good fidelity," defined as

$$F_H(\rho, \sigma) \equiv [\text{Tr}\{\sqrt{\sqrt{\rho}\sqrt{\sigma}}\}]^2. \quad (2)$$

After writing it down, he then remarked that "it is evident that F_H is just as good a measure of proximity of the states ρ and σ as $\|\rho - \sigma\|_1$." And so it is that the measure F_H is known as Holevo's just-as-good fidelity.

Some years after this, Uhlmann defined the quantum fidelity as $F(\rho, \sigma) \equiv \|\sqrt{\sqrt{\rho}\sqrt{\sigma}}\|_1^2$ [2]. It is evident that the following relation holds

$$F_H(\rho, \sigma) \leq F(\rho, \sigma), \quad (3)$$

due to the variational characterization of the trace norm of a square operator X as

$$\|X\|_1 = \max_U |\text{Tr}\{XU\}|, \quad (4)$$

where the optimization is with respect to a unitary operator U . Many years after this, at the dawn of quantum computing, with more growing interest in quantum information theory, Fuchs and van de Graaf presented the following widely employed inequalities [3]:

$$1 - \sqrt{F(\rho, \sigma)} \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)}, \quad (5)$$

which bear a striking similarity to (1). Indeed the lower bound on $\frac{1}{2} \|\rho - \sigma\|_1$ in (5) is an immediate consequence of (3) and the lower bound in (1). The upper bound on $\frac{1}{2} \|\rho - \sigma\|_1$ in (5) can be proven by first showing that it is achieved for pure states, employing Uhlmann's "transition probability" characterization of $F(\rho, \sigma)$ [2], and then invoking monotonicity of trace distance with respect to partial trace. The latter inequalities in (5) have been more widely employed in quantum information theory than those in (1) due to Uhlmann's "transition probability" characterization of $F(\rho, \sigma)$ and its many implications.

Nevertheless, Holevo's just-as-good fidelity is clearly a useful measure of similarity for quantum states in light of (1), and it has found several applications in quantum information theory. For example, it serves as an upper bound on the probability of error in discriminating ρ from σ in a hypothesis testing experiment [4], [5], which in some sense is just a rewriting of the lower bound in (1) (see also [6, Lemma 3.2] in this context). In turn, this way of thinking has led to particular decoders for quantum polar codes [7], [8].

The function F_H has also been rediscovered a number of times. For example, it is a particular case of Petz's quasi-entropies [9], [10]. It was studied under the name "quantum affinity" in [11] and shown to be equal to the fidelity of the canonical purifications of quantum states in [12].

One of the most important properties of F_H is that it obeys the following data processing inequality:

$$F_H(\mathcal{N}(\rho), \mathcal{N}(\sigma)) \geq F_H(\rho, \sigma), \quad (6)$$

where \mathcal{N} is a quantum channel (a completely positive and trace preserving map). This inequality is a consequence of data processing for Petz's more general quasi-entropies [9], [10]. This property is one reason that F_H has an interpretation as a similarity measure: the states ρ and σ generally become more similar under the action of a quantum channel.

The main contribution of this paper is the following refinement of the data processing inequality in (6), in the case that ρ is a bipartite density operator, σ is a positive definite bipartite operator, and the channel is a partial trace over the B system:

$$\begin{aligned} \sqrt{F_H(\rho_A, \sigma_A)} &\geq \sqrt{F_H(\rho_{AB}, \sigma_{AB})} \\ &+ \frac{\pi^2}{432} \frac{\lambda_{\min}(\sigma_{AB})}{\text{Tr}\{\sigma_A\}} \|\mathcal{R}_{A \rightarrow AB}^\sigma(\rho_A) - \rho_{AB}\|_1^3, \end{aligned} \quad (7)$$

where $\lambda_{\min}(\sigma_{AB})$ is the minimum eigenvalue of σ_{AB} and

$$\mathcal{R}_{A \rightarrow AB}^\sigma(\cdot) \equiv \sigma_{AB}^{1/2} \left[\sigma_A^{-1/2}(\cdot) \sigma_A^{-1/2} \otimes I_B \right] \sigma_{AB}^{1/2} \quad (8)$$

is a quantum channel known as the Petz recovery channel [13], [14]. The interpretation of this inequality is the same as that given in previous work on this topic of refining data processing inequalities (see, e.g., [15]–[17]). If the difference $\sqrt{F_H}(\rho_A, \sigma_A) - \sqrt{F_H}(\rho_{AB}, \sigma_{AB})$ is small, then one can approximately recover the state ρ_{AB} from its marginal ρ_A . The appendix generalizes the result in (7) to arbitrary quantum channels.

The technique that I use for proving the above data processing refinement closely follows the elegant approach recently put forward by Carlen and Vershynina in [18]. This technique appears to be different from every other approach, given in recent years since [15], that has established refinements of data processing inequalities. It builds on Petz’s approach from [9], [10] for proving data processing for the quantum relative entropy, as well as ideas in [19]. Here, I use this same technique and establish a general lemma regarding remainder terms for data processing with Petz’s quasi-entropies, and then I specialize it to obtain the inequality in (7).

An interesting aspect of (7) is that the recovery channel is explicit, given in the Petz form, and universal, having no dependence on the state ρ_{AB} while depending only on σ_{AB} .

In the rest of the paper, I begin by giving background and establish some notation. After that, I prove a general lemma that refines data processing for Petz’s quasi-entropies. Then I specialize it to arrive at the inequality in (7).

II. BACKGROUND AND NOTATION

I begin by reviewing some background and establish notation. Basic concepts of quantum information theory can be found in [6], [20], [21]. Let f be an operator convex function defined on $[0, \infty)$. Examples include $f(x) = x \ln x$, $f(x) = -x^\alpha$ for $\alpha \in (0, 1)$, $f(x) = x^\alpha$ for $\alpha \in (1, 2]$. According to [22, Section 8], such a function has the following integral representation:

$$f(x) = f(0) + ax + bx^2 + \int_0^\infty d\mu(t) \left(\frac{x}{1+t} - 1 + \frac{t}{x+t} \right), \quad (9)$$

where $a \in \mathbb{R}$, $b \geq 0$, and μ is a non-negative measure on $(0, \infty)$ satisfying $\int_0^\infty d\mu(t)/(1+t)^2 < \infty$.

Define the maximally entangled vector as

$$|\Gamma\rangle_{S\tilde{S}} \equiv \sum_{i=0}^{|S|-1} |i\rangle_S |i\rangle_{\tilde{S}}, \quad (10)$$

for orthonormal bases $\{|i\rangle_S\}_i$ and $\{|i\rangle_{\tilde{S}}\}_i$, and for a positive semi-definite operator σ , define its canonical purification by

$$|\varphi^\sigma\rangle_{S\tilde{S}} \equiv \left(\sigma_S^{1/2} \otimes I_{\tilde{S}} \right) |\Gamma\rangle_{S\tilde{S}}. \quad (11)$$

Then, following Petz [9], [10], [23], [24], as well as what was discussed later in [25], [26], we define the f -quasi-relative entropy $Q_f(\rho\|\sigma)$ of a density operator ρ and a positive definite operator σ as

$$Q_f(\rho\|\sigma) \equiv \langle \varphi^\sigma |_{S\tilde{S}} f \left(\sigma_S^{-1} \otimes \rho_{\tilde{S}}^T \right) | \varphi^\sigma \rangle_{S\tilde{S}}. \quad (12)$$

For example, when $f(x) = x \ln x$, then Q_f reduces to the quantum relative entropy from [27].

Now consider the bipartite case and define

$$|\Gamma\rangle_{A\hat{A}B\hat{B}} \equiv |\Gamma\rangle_{A\hat{A}} \otimes |\Gamma\rangle_{B\hat{B}}. \quad (13)$$

We can also write this as $|\Gamma\rangle_{AB\hat{A}\hat{B}}$ with it being understood that there is a permutation of systems. Then, by the above, we have for a density operator ρ_{AB} and a positive definite operator σ_{AB} that

$$Q_f(\rho_{AB}\|\sigma_{AB}) = \langle \varphi^{\sigma_{AB}} |_{AB\hat{A}\hat{B}} f \left(\sigma_{AB}^{-1} \otimes \rho_{\hat{A}\hat{B}}^T \right) | \varphi^{\sigma_{AB}} \rangle_{AB\hat{A}\hat{B}}. \quad (14)$$

Now define the linear operator V by

$$V_{A\hat{A} \rightarrow AB\hat{A}\hat{B}} \equiv \sigma_{AB}^{1/2} \left(\sigma_A^{-1/2} \otimes I_{\hat{A}} \right) |\Gamma\rangle_{B\hat{B}}. \quad (15)$$

This linear operator is an isometric extension of the Petz recovery channel, as discussed recently in [28]. One can readily verify that V is an isometry and that

$$V_{A\hat{A} \rightarrow AB\hat{A}\hat{B}} |\varphi^{\sigma_A}\rangle_{A\hat{A}} = |\varphi^{\sigma_{AB}}\rangle_{AB\hat{A}\hat{B}}, \quad (16)$$

$$V^\dagger \left(\sigma_{AB}^{-1} \otimes \rho_{\hat{A}\hat{B}}^T \right) V = \sigma_A^{-1} \otimes \rho_{\hat{A}}^T. \quad (17)$$

For simple proofs of these properties, see, e.g., [25] or [28]. With all these notions in place, we can recall Petz’s approach [9], [10], [23], [24] for establishing monotonicity of the f -quasi-relative entropy under partial trace:

$$\begin{aligned} Q_f(\rho_{AB}\|\sigma_{AB}) &= \langle \varphi^{\sigma_{AB}} |_{AB\hat{A}\hat{B}} f \left(\sigma_{AB}^{-1} \otimes \rho_{\hat{A}\hat{B}}^T \right) | \varphi^{\sigma_{AB}} \rangle_{AB\hat{A}\hat{B}} \\ &= \langle \varphi^{\sigma_A} |_{A\hat{A}} V^\dagger f \left(\sigma_{AB}^{-1} \otimes \rho_{\hat{A}\hat{B}}^T \right) V | \varphi^{\sigma_A} \rangle_{A\hat{A}} \\ &\geq \langle \varphi^{\sigma_A} |_{A\hat{A}} f \left(V^\dagger \left[\sigma_{AB}^{-1} \otimes \rho_{\hat{A}\hat{B}}^T \right] V \right) | \varphi^{\sigma_A} \rangle_{A\hat{A}} \\ &= \langle \varphi^{\sigma_A} |_{A\hat{A}} f \left(\sigma_A^{-1} \otimes \rho_{\hat{A}}^T \right) | \varphi^{\sigma_A} \rangle_{A\hat{A}} = Q_f(\rho_A\|\sigma_A) \end{aligned} \quad (18)$$

where we made use of everything above and the operator Jensen inequality [29].

III. GENERAL STATEMENT FOR QUASI-ENTROPIES

I now modify the approach from [18] for lower bounds for relative entropy differences to use an arbitrary operator convex function f instead. So we are considering the following f -quasi-relative entropy difference:

$$Q_f(\rho_{AB}\|\sigma_{AB}) - Q_f(\rho_A\|\sigma_A). \quad (19)$$

Recall the integral representation of f from (9). Let

$$\Delta_{AB\hat{A}\hat{B}} \equiv \sigma_{AB}^{-1} \otimes \rho_{\hat{A}\hat{B}}^T, \quad \Delta_{A\hat{A}} \equiv \sigma_A^{-1} \otimes \rho_{\hat{A}}^T, \quad (20)$$

$$V_{A\hat{A} \rightarrow AB\hat{A}\hat{B}} \equiv \sigma_{AB}^{1/2} \left(\sigma_A^{-1/2} \otimes I_{\hat{A}} \right) |\Gamma\rangle_{B\hat{B}} \quad (21)$$

and recall from (17) that $V^\dagger \Delta_{AB\hat{A}\hat{B}} V = \Delta_{A\hat{A}}$. This implies

$$\begin{aligned} \|\Delta_{A\hat{A}}\|_\infty &= \|V^\dagger \Delta_{AB\hat{A}\hat{B}} V\|_\infty = \|VV^\dagger \Delta_{AB\hat{A}\hat{B}} VV^\dagger\|_\infty \\ &\leq \|\Delta_{AB\hat{A}\hat{B}}\|_\infty \end{aligned} \quad (22)$$

with the last equality following from isometric invariance of the operator norm and the inequality from submultiplicativity of the operator norm and the fact that VV^\dagger is a projection.

Lemma 1: Let μ be a measure. For an operator X , define

$$\nu(X) = \int_0^\infty d\mu(t) t \left(\frac{1}{t} - \frac{1}{t+X} \right), \quad (23)$$

and for $T > 0$, define $\mu([0, T]) \equiv \int_0^T d\mu(t)$. For $c > 0$, define $g(c, T) \equiv \int_T^\infty d\mu(t) \frac{1}{1+t/c}$. Let ρ_{AB} be a density operator and σ_{AB} a positive definite operator. Then for all $T > 0$, the following inequality holds

$$\begin{aligned} & \left\| \left[\sigma_{AB}^{1/2} \sigma_A^{-1/2} \nu(\Delta_{A\hat{A}}) \sigma_A^{1/2} - \nu(\Delta_{AB\hat{A}\hat{B}}) \sigma_{AB}^{1/2} \right] |\Gamma\rangle_{A\hat{A}B\hat{B}} \right\|_2 \\ & \leq \mu([0, T])^{1/2} [Q_f(\rho_{AB} \|\sigma_{AB}) - Q_f(\rho_A \|\sigma_A)]^{1/2} \\ & \quad + 2g(\|\Delta_{AB\hat{A}\hat{B}}\|_\infty, T) \text{Tr}\{\sigma_A\}. \end{aligned} \quad (24)$$

Proof. The proof follows [18] quite closely at times but also features some departures. Since V is an isometry satisfying $V^\dagger V = I_{A\hat{A}}$, it follows that VV^\dagger is a projection, so that $VV^\dagger \leq I_{AB\hat{A}\hat{B}}$. Using the integral representation in (9), we arrive at the chain of inequalities in (25), where we made use of (17) and the fact that VV^\dagger is a projection so that $VV^\dagger \leq I_{AB\hat{A}\hat{B}}$. Similarly, we find that

$$\begin{aligned} Q_f(\rho_A \|\sigma_A) &= \langle \varphi^{\sigma_A} |_{A\hat{A}} [f(\Delta_{A\hat{A}})] | \varphi^{\sigma_A} \rangle_{A\hat{A}} \\ &= f(0) + \langle \varphi^{\sigma_A} |_{A\hat{A}} [a\Delta_{A\hat{A}} + b\Delta_{A\hat{A}}^2] | \varphi^{\sigma_A} \rangle_{A\hat{A}} \\ &+ \int_0^\infty d\mu(t) \langle \varphi^{\sigma_A} |_{A\hat{A}} \left(\frac{\Delta_{A\hat{A}}}{1+t} - 1 + \frac{t}{\Delta_{A\hat{A}}+t} \right) | \varphi^{\sigma_A} \rangle_{A\hat{A}}. \end{aligned} \quad (26)$$

Thus, we find that

$$\begin{aligned} & Q_f(\rho_{AB} \|\sigma_{AB}) - Q_f(\rho_A \|\sigma_A) \geq \\ & \int_0^\infty d\mu(t) t \langle \varphi^{\sigma_A} |_{A\hat{A}} \left[V^\dagger (\Delta_{AB\hat{A}\hat{B}} + t)^{-1} V \right. \\ & \quad \left. - (\Delta_{A\hat{A}} + t)^{-1} \right] | \varphi^{\sigma_A} \rangle_{A\hat{A}}. \end{aligned} \quad (27)$$

Now consider that for $t > 0$

$$\begin{aligned} & t \langle \varphi^{\sigma_A} |_{A\hat{A}} \left[V^\dagger (\Delta_{AB\hat{A}\hat{B}} + t)^{-1} V - (\Delta_{A\hat{A}} + t)^{-1} \right] | \varphi^{\sigma_A} \rangle_{A\hat{A}} \\ &= t \langle \varphi^{w^t} |_{AB\hat{A}\hat{B}} (\Delta_{AB\hat{A}\hat{B}} + t) | \varphi^{w^t} \rangle_{AB\hat{A}\hat{B}} \\ &\geq t^2 \left\| | \varphi^{w^t} \rangle_{AB\hat{A}\hat{B}} \right\|_2^2, \end{aligned} \quad (28)$$

where

$$\begin{aligned} | \varphi^{w^t} \rangle_{AB\hat{A}\hat{B}} &\equiv V (\Delta_{A\hat{A}} + t)^{-1} | \varphi^{\sigma_A} \rangle_{A\hat{A}} \\ &\quad - (\Delta_{AB\hat{A}\hat{B}} + t)^{-1} | \varphi^{\sigma_{AB}} \rangle_{AB\hat{A}\hat{B}}. \end{aligned} \quad (29)$$

Consider that

$$\begin{aligned} | \varphi^{w^t} \rangle_{AB\hat{A}\hat{B}} &= \sigma_{AB}^{1/2} \sigma_A^{-1/2} (\Delta_{A\hat{A}} + t)^{-1} \sigma_A^{1/2} |\Gamma\rangle_{A\hat{A}} |\Gamma\rangle_{B\hat{B}} \\ &\quad - (\Delta_{AB\hat{A}\hat{B}} + t)^{-1} \sigma_{AB}^{1/2} |\Gamma\rangle_{A\hat{A}} |\Gamma\rangle_{B\hat{B}} \end{aligned} \quad (30)$$

$$\begin{aligned} &= \left[\sigma_{AB}^{1/2} \sigma_A^{-1/2} (\Delta_{A\hat{A}} + t)^{-1} \sigma_A^{1/2} \right. \\ &\quad \left. - (\Delta_{AB\hat{A}\hat{B}} + t)^{-1} \sigma_{AB}^{1/2} \right] |\Gamma\rangle_{A\hat{A}} |\Gamma\rangle_{B\hat{B}}. \end{aligned} \quad (31)$$

So we set

$$\begin{aligned} w_{AB\hat{A}\hat{B}}^t &\equiv \sigma_{AB}^{1/2} \sigma_A^{-1/2} (\Delta_{A\hat{A}} + t)^{-1} \sigma_A^{1/2} \\ &\quad - (\Delta_{AB\hat{A}\hat{B}} + t)^{-1} \sigma_{AB}^{1/2}, \end{aligned} \quad (32)$$

so that

$$| \varphi^{w^t} \rangle_{AB\hat{A}\hat{B}} = w_{AB\hat{A}\hat{B}}^t |\Gamma\rangle_{A\hat{A}B\hat{B}}. \quad (33)$$

Now invoking the definition in (23) we find that

$$\begin{aligned} & \sigma_{AB}^{1/2} \sigma_A^{-1/2} \nu(\Delta_{A\hat{A}}) \sigma_A^{1/2} - \nu(\Delta_{AB\hat{A}\hat{B}}) \sigma_{AB}^{1/2} \\ &= \int_0^\infty d\mu(t) t \sigma_{AB}^{1/2} \sigma_A^{-1/2} \left(\frac{1}{t} - \frac{1}{t + \Delta_{A\hat{A}}} \right) \sigma_A^{1/2} \\ &\quad - \int_0^\infty d\mu(t) t \left(\frac{1}{t} - \frac{1}{t + \Delta_{AB\hat{A}\hat{B}}} \right) \sigma_{AB}^{1/2} \end{aligned} \quad (34)$$

$$\begin{aligned} &= \int_0^\infty d\mu(t) t \left[\sigma_{AB}^{1/2} \sigma_A^{-1/2} \left(\frac{1}{t} - \frac{1}{t + \Delta_{A\hat{A}}} \right) \sigma_A^{1/2} \right. \\ &\quad \left. - \left(\frac{1}{t} - \frac{1}{t + \Delta_{AB\hat{A}\hat{B}}} \right) \sigma_{AB}^{1/2} \right] \end{aligned} \quad (35)$$

$$\begin{aligned} &= \int_0^\infty d\mu(t) t \left[-\sigma_{AB}^{1/2} \sigma_A^{-1/2} \frac{1}{t + \Delta_{A\hat{A}}} \sigma_A^{1/2} \right. \\ &\quad \left. + \frac{1}{t + \Delta_{AB\hat{A}\hat{B}}} \sigma_{AB}^{1/2} \right] \end{aligned} \quad (36)$$

$$= - \int_0^\infty d\mu(t) t w_{AB\hat{A}\hat{B}}^t. \quad (37)$$

Thus, for any $T > 0$, we have that

$$\begin{aligned} & \left\| \left[\sigma_{AB}^{1/2} \sigma_A^{-1/2} \nu(\Delta_{A\hat{A}}) \sigma_A^{1/2} - \nu(\Delta_{AB\hat{A}\hat{B}}) \sigma_{AB}^{1/2} \right] |\Gamma\rangle_{A\hat{A}B\hat{B}} \right\|_2 \\ &= \left\| \int_0^\infty d\mu(t) t w_{AB\hat{A}\hat{B}}^t |\Gamma\rangle_{A\hat{A}B\hat{B}} \right\|_2 \end{aligned} \quad (38)$$

$$\begin{aligned} &\leq \int_0^T d\mu(t) t \left\| w_{AB\hat{A}\hat{B}}^t |\Gamma\rangle_{A\hat{A}B\hat{B}} \right\|_2 \\ &\quad + \left\| \int_T^\infty d\mu(t) t w_{AB\hat{A}\hat{B}}^t |\Gamma\rangle_{A\hat{A}B\hat{B}} \right\|_2 \end{aligned} \quad (39)$$

Let us study the two terms separately. For the first term, from Cauchy–Schwarz

$$\left| \int_0^T d\mu(t) f(t) g(t) \right|^2 \leq \left[\int_0^T d\mu(t) f^2(t) \right] \left[\int_0^T d\mu(t) g^2(t) \right],$$

we have that

$$\begin{aligned} & \left[\int_0^T d\mu(t) t \left\| w_{AB\hat{A}\hat{B}}^t |\Gamma\rangle_{A\hat{A}B\hat{B}} \right\|_2 \right]^2 \\ &\leq \mu([0, T]) \int_0^T d\mu(t) t^2 \left\| w_{AB\hat{A}\hat{B}}^t |\Gamma\rangle_{A\hat{A}B\hat{B}} \right\|_2^2 \end{aligned} \quad (40)$$

$$\leq \mu([0, T]) \int_0^\infty d\mu(t) t^2 \left\| w_{AB\hat{A}\hat{B}}^t |\Gamma\rangle_{A\hat{A}B\hat{B}} \right\|_2^2 \quad (41)$$

$$\leq \mu([0, T]) [Q_f(\rho_{AB} \|\sigma_{AB}) - Q_f(\rho_A \|\sigma_A)]. \quad (42)$$

$$\begin{aligned}
Q_f(\rho_{AB} \|\sigma_{AB}) &= \langle \varphi^{\sigma_A} |_{A\hat{A}} V^\dagger [f(\Delta_{AB\hat{A}\hat{B}})] V | \varphi^{\sigma_A} \rangle_{A\hat{A}} \\
&= \langle \varphi^{\sigma_A} |_{A\hat{A}} V^\dagger \left[f(0) + a\Delta_{AB\hat{A}\hat{B}} + b\Delta_{AB\hat{A}\hat{B}}^2 + \int_0^\infty d\mu(t) \left(\frac{\Delta_{AB\hat{A}\hat{B}}}{1+t} - 1 + \frac{t}{\Delta_{AB\hat{A}\hat{B}} + t} \right) \right] V | \varphi^{\sigma_A} \rangle_{A\hat{A}} \\
&= f(0) + \langle \varphi^{\sigma_A} |_{A\hat{A}} [aV^\dagger \Delta_{AB\hat{A}\hat{B}} V + bV^\dagger \Delta_{AB\hat{A}\hat{B}}^2 V] | \varphi^{\sigma_A} \rangle_{A\hat{A}} \\
&\quad + \langle \varphi^{\sigma_A} |_{A\hat{A}} \left[\int_0^\infty d\mu(t) \left(\frac{V^\dagger \Delta_{AB\hat{A}\hat{B}} V}{1+t} - 1 + V^\dagger \frac{t}{\Delta_{AB\hat{A}\hat{B}} + t} V \right) \right] | \varphi^{\sigma_A} \rangle_{A\hat{A}} \\
&\geq f(0) + \langle \varphi^{\sigma_A} |_{A\hat{A}} [aV^\dagger \Delta_{AB\hat{A}\hat{B}} V + bV^\dagger \Delta_{AB\hat{A}\hat{B}} V V^\dagger \Delta_{AB\hat{A}\hat{B}} V] | \varphi^{\sigma_A} \rangle_{A\hat{A}} \\
&\quad + \langle \varphi^{\sigma_A} |_{A\hat{A}} \left[\int_0^\infty d\mu(t) \left(\frac{V^\dagger \Delta_{AB\hat{A}\hat{B}} V}{1+t} - 1 + V^\dagger \frac{t}{\Delta_{AB\hat{A}\hat{B}} + t} V \right) \right] | \varphi^{\sigma_A} \rangle_{A\hat{A}} \\
&= f(0) + \langle \varphi^{\sigma_A} |_{A\hat{A}} \left[a\Delta_{A\hat{A}} + b\Delta_{A\hat{A}}^2 + \int_0^\infty d\mu(t) \left(\frac{\Delta_{A\hat{A}}}{1+t} - 1 + V^\dagger \left(\frac{t}{\Delta_{AB\hat{A}\hat{B}} + t} \right) V \right) \right] | \varphi^{\sigma_A} \rangle_{A\hat{A}}, \quad (25)
\end{aligned}$$

Moving to the second term, from the reasoning in the proof of [18, Theorem 1.7], we find that for any positive operator X

$$t \left(\frac{1}{t} - \frac{1}{t+X} \right) \leq t \left(\frac{1}{t} - \frac{1}{t+\|X\|_\infty} \right) I \quad (43)$$

$$= \frac{1}{1+t/\|X\|_\infty} I \quad (44)$$

so that $\int_T^\infty d\mu(t) t \left(\frac{1}{t} - \frac{1}{t+X} \right) \leq g(\|X\|, T) I$. This leads to the development in (45), and after putting everything together, we get (24). ■

IV. APPLICATION TO HOLEVO'S JUST-AS-GOOD FIDELITY

I now specialize the above analysis to the case of the operator convex function $-x^\alpha$ for $\alpha \in (0, 1)$, and I abbreviate the corresponding quasi-entropy as Q_α . For this case, from [22, Section 8], we have that $d\mu(t) = \frac{\sin(\alpha\pi)}{\pi} t^{\alpha-1} dt$. Plugging into the quantities in Lemma 1, we find that

$$\mu([0, T]) = \frac{\sin(\alpha\pi)}{\pi} \int_0^T t^{\alpha-1} dt = \frac{\sin(\alpha\pi)}{\alpha\pi} T^\alpha. \quad (46)$$

We also find that

$$g(\|\Delta_{AB\hat{A}\hat{B}}\|_\infty, T) = \int_T^\infty d\mu(t) \frac{1}{1+t/\|\Delta_{AB\hat{A}\hat{B}}\|_\infty} \quad (47)$$

$$= \frac{\sin(\alpha\pi)}{\pi} \int_T^\infty dt t^{\alpha-1} \frac{1}{1+t/\|\Delta_{AB\hat{A}\hat{B}}\|_\infty} \quad (48)$$

$$\leq \frac{\sin(\alpha\pi)}{\pi} \int_T^\infty dt t^{\alpha-1} \frac{1}{t/\|\Delta_{AB\hat{A}\hat{B}}\|_\infty} \quad (49)$$

$$= \frac{\sin(\alpha\pi)}{\pi T^{1-\alpha} (1-\alpha)} \|\Delta_{AB\hat{A}\hat{B}}\|_\infty. \quad (50)$$

Furthermore, we have that

$$\nu(X) = \int_0^\infty d\mu(t) t \left(\frac{1}{t} - \frac{1}{t+X} \right) \quad (51)$$

$$= \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty dt t^{\alpha-1} t \left(\frac{1}{t} - \frac{1}{t+X} \right) = X^\alpha. \quad (52)$$

Substituting into (24), we find that

$$\begin{aligned}
&\left\| \left[\sigma_{AB}^{1/2} \sigma_A^{-1/2} \Delta_{AA}^\alpha \sigma_A^{1/2} - \Delta_{AB\hat{A}\hat{B}}^\alpha \sigma_{AB}^{1/2} \right] |\Gamma\rangle_{A\hat{A}\hat{B}\hat{B}} \right\|_2 \leq \\
&\left[\frac{\sin(\alpha\pi)}{\alpha\pi} T^\alpha \right]^{1/2} [Q_\alpha(\rho_{AB} \|\sigma_{AB}) - Q_\alpha(\rho_A \|\sigma_A)]^{1/2} \\
&\quad + \frac{2 \sin(\alpha\pi)}{\pi T^{1-\alpha} (1-\alpha)} \|\Delta_{AB\hat{A}\hat{B}}\|_\infty \text{Tr}\{\sigma_A\}. \quad (53)
\end{aligned}$$

We can consider this for an arbitrary $\alpha \in (0, 1)$, but the most interesting and physically relevant case seems to occur when $\alpha = 1/2$. So I now prove the claim in (7).

For $\alpha = 1/2$, the lower bound in (53) simplifies to

$$\begin{aligned}
&\left\| \left[\sigma_{AB}^{1/2} \sigma_A^{-1/2} \rho_A^{1/2} - \rho_{AB}^{1/2} \right] |\Gamma\rangle_{A\hat{A}\hat{B}\hat{B}} \right\|_2 \\
&= \left\| \sigma_{AB}^{1/2} \sigma_A^{-1/2} \rho_A^{1/2} - \rho_{AB}^{1/2} \right\|_2, \quad (54)
\end{aligned}$$

while the upper bound in (53) becomes

$$\begin{aligned}
&[2/\pi]^{1/2} T^{1/4} [Q_{1/2}(\rho_{AB} \|\sigma_{AB}) - Q_{1/2}(\rho_A \|\sigma_A)]^{1/2} \\
&\quad + \frac{4 \|\Delta_{AB\hat{A}\hat{B}}\|_\infty}{\pi T^{1/2}} \text{Tr}\{\sigma_A\}. \quad (55)
\end{aligned}$$

Now minimizing over $T > 0$ gives the choice

$$T = \left[\frac{8 \|\Delta_{AB\hat{A}\hat{B}}\|_\infty \text{Tr}\{\sigma_A\}}{[2\pi]^{1/2} [Q_{1/2}(\rho_{AB} \|\sigma_{AB}) - Q_{1/2}(\rho_A \|\sigma_A)]^{1/2}} \right]^{4/3}, \quad (56)$$

leading to the upper bound

$$\begin{aligned}
&(3/[2^{2/3}]) \left[(8/\pi^2) [Q_{1/2}(\rho_{AB} \|\sigma_{AB}) \right. \\
&\quad \left. - Q_{1/2}(\rho_A \|\sigma_A)] \|\Delta_{AB\hat{A}\hat{B}}\|_\infty \text{Tr}\{\sigma_A\} \right]^{1/3}. \quad (57)
\end{aligned}$$

Thus, the final inequality is

$$\begin{aligned}
&\frac{\pi^2}{54 \|\Delta_{AB\hat{A}\hat{B}}\|_\infty \text{Tr}\{\sigma_A\}} \left\| \sigma_{AB}^{1/2} \sigma_A^{-1/2} \rho_A^{1/2} - \rho_{AB}^{1/2} \right\|_2^3 \\
&\leq Q_{1/2}(\rho_{AB} \|\sigma_{AB}) - Q_{1/2}(\rho_A \|\sigma_A). \quad (58)
\end{aligned}$$

$$\begin{aligned}
& \left\| \int_T^\infty d\mu(t) t w_{AB\hat{A}\hat{B}}^t |\Gamma\rangle_{A\hat{A}B\hat{B}} \right\|_2 \\
& \leq \left\| \int_T^\infty d\mu(t) t \left[(t^{-1} - (\Delta_{AB\hat{A}\hat{B}} + t)^{-1}) \sigma_{AB}^{1/2} - \sigma_{AB}^{1/2} \sigma_A^{-1/2} (t^{-1} - (\Delta_{A\hat{A}} + t)^{-1}) \sigma_A^{1/2} \right] |\Gamma\rangle_{A\hat{A}B\hat{B}} \right\|_2 \\
& \leq \left\| \int_T^\infty d\mu(t) t (t^{-1} - (\Delta_{AB\hat{A}\hat{B}} + t)^{-1}) \sigma_{AB}^{1/2} |\Gamma\rangle_{A\hat{A}B\hat{B}} \right\|_2 + \left\| \int_T^\infty d\mu(t) t \sigma_{AB}^{1/2} \sigma_A^{-1/2} (t^{-1} - (\Delta_{A\hat{A}} + t)^{-1}) \sigma_A^{1/2} |\Gamma\rangle_{A\hat{A}B\hat{B}} \right\|_2 \\
& = \left\| \int_T^\infty d\mu(t) t (t^{-1} - (\Delta_{AB\hat{A}\hat{B}} + t)^{-1}) \sigma_{AB}^{1/2} |\Gamma\rangle_{A\hat{A}B\hat{B}} \right\|_2 + \left\| \int_T^\infty d\mu(t) t (t^{-1} - (\Delta_{A\hat{A}} + t)^{-1}) \sigma_A^{1/2} |\Gamma\rangle_{A\hat{A}} \right\|_2 \\
& \leq g(\|\Delta_{AB\hat{A}\hat{B}}\|_\infty, T) \left\| \sigma_{AB}^{1/2} |\Gamma\rangle_{A\hat{A}B\hat{B}} \right\|_2 + g(\|\Delta_{A\hat{A}}\|_\infty, T) \left\| \sigma_A^{1/2} |\Gamma\rangle_{A\hat{A}} \right\|_2 \\
& = [g(\|\Delta_{AB\hat{A}\hat{B}}\|_\infty, T) + g(\|\Delta_{A\hat{A}}\|_\infty, T)] \text{Tr}\{\sigma_A\} \leq 2g(\|\Delta_{AB\hat{A}\hat{B}}\|_\infty, T) \text{Tr}\{\sigma_A\}. \tag{45}
\end{aligned}$$

Using definitions, this is then equivalent to

$$\begin{aligned}
& \frac{\pi^2}{54 \|\Delta_{AB\hat{A}\hat{B}}\|_\infty \text{Tr}\{\sigma_A\}} \left\| \sigma_{AB}^{1/2} \sigma_A^{-1/2} \rho_A^{1/2} - \rho_{AB}^{1/2} \right\|_2^3 \\
& \leq \text{Tr}\{\rho_A^{1/2} \sigma_A^{1/2}\} - \text{Tr}\{\rho_{AB}^{1/2} \sigma_{AB}^{1/2}\}. \tag{59}
\end{aligned}$$

The estimate from [18, Lemma 2.2] then gives

$$\begin{aligned}
& \frac{\pi^2/432}{\|\Delta_{AB\hat{A}\hat{B}}\|_\infty \text{Tr}\{\sigma_A\}} \left\| \sigma_{AB}^{1/2} \sigma_A^{-1/2} \rho_A \sigma_A^{-1/2} \sigma_{AB}^{1/2} - \rho_{AB} \right\|_1^3 \\
& \leq \text{Tr}\{\rho_A^{1/2} \sigma_A^{1/2}\} - \text{Tr}\{\rho_{AB}^{1/2} \sigma_{AB}^{1/2}\}. \tag{60}
\end{aligned}$$

Observe that $\|\Delta_{AB\hat{A}\hat{B}}\|_\infty = \|\sigma_{AB}^{-1} \otimes \rho_{\hat{A}\hat{B}}^T\|_\infty \leq \frac{1}{\lambda_{\min}(\sigma_{AB})}$ because ρ_{AB} is a density operator. So we then get

$$\begin{aligned}
& \frac{\pi^2 \lambda_{\min}(\sigma_{AB})}{54 \text{Tr}\{\sigma_A\}} \left\| \sigma_{AB}^{1/2} \sigma_A^{-1/2} \rho_A^{1/2} - \rho_{AB}^{1/2} \right\|_2^3 \\
& \leq \text{Tr}\{\rho_A^{1/2} \sigma_A^{1/2}\} - \text{Tr}\{\rho_{AB}^{1/2} \sigma_{AB}^{1/2}\}, \tag{61}
\end{aligned}$$

$$\begin{aligned}
& \frac{\pi^2 \lambda_{\min}(\sigma_{AB})}{432 \text{Tr}\{\sigma_A\}} \left\| \sigma_{AB}^{1/2} \sigma_A^{-1/2} \rho_A \sigma_A^{-1/2} \sigma_{AB}^{1/2} - \rho_{AB} \right\|_1^3 \\
& \leq \text{Tr}\{\rho_A^{1/2} \sigma_A^{1/2}\} - \text{Tr}\{\rho_{AB}^{1/2} \sigma_{AB}^{1/2}\}, \tag{62}
\end{aligned}$$

the latter of which being what was claimed in (7).

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Note: The results and proofs in the main text of this paper were developed after [18] but independently of arXiv:1710.08080 and were communicated privately by email in mid-October 2017.

APPENDIX

This appendix contains a generalization of the result in (7) to arbitrary quantum channels.

Theorem 2: Let ω be a density operator and τ a positive semi-definite operator such that $\text{supp}(\omega) \subseteq \text{supp}(\tau)$. Let \mathcal{N} be a quantum channel. Then

$$\begin{aligned}
& \sqrt{F_H}(\mathcal{N}(\omega), \mathcal{N}(\tau)) \geq \sqrt{F_H}(\omega, \tau) \\
& + \frac{\pi^2 \lambda_{\min}(\tau)}{432 \text{Tr}\{\tau\}} \|\mathcal{P}(\mathcal{N}(\omega)) - \omega\|_1^3, \tag{63}
\end{aligned}$$

where $\lambda_{\min}(\tau)$ now denotes the minimum non-zero eigenvalue of τ and \mathcal{P} denotes the Petz recovery map for τ and \mathcal{N} , defined as

$$\mathcal{P}(\cdot) = \tau^{1/2} \mathcal{N}^\dagger[(\mathcal{N}(\tau))^{-1/2} (\cdot) (\mathcal{N}(\tau))^{-1/2}] \tau^{1/2}. \tag{64}$$

Proof. Let us start by returning to (7) and reflecting on its statement as well as its proof. If $\text{supp}(\rho_{AB}) \subseteq \text{supp}(\sigma_{AB})$, then without loss of generality, we can restrict the whole space of systems A and B to the support of σ_{AB} and the inequality in (7) holds with $\lambda_{\min}(\sigma_{AB})$ equal to the minimum non-zero eigenvalue of σ_{AB} . Now we can apply this result, as well as the well known Stinespring dilation theorem, in order to arrive at the statement of the theorem. Stinespring's theorem states that for a quantum channel \mathcal{N} acting on a state ω of a system S , there exists an isometry $U_{S \rightarrow AB}$ such that

$$\mathcal{N}(\omega) = \text{Tr}_B\{U_{S \rightarrow AB} \omega (U_{S \rightarrow AB})^\dagger\}. \tag{65}$$

So we pick

$$\rho_{AB} = U_{S \rightarrow AB} \omega (U_{S \rightarrow AB})^\dagger, \tag{66}$$

$$\sigma_{AB} = U_{S \rightarrow AB} \tau (U_{S \rightarrow AB})^\dagger, \tag{67}$$

so that $\rho_A = \mathcal{N}(\omega)$, $\sigma_A = \mathcal{N}(\tau)$, and then find that

$$\begin{aligned}
& \sqrt{F_H}(\mathcal{N}(\omega), \mathcal{N}(\tau)) \geq \\
& \sqrt{F_H}(U_{S \rightarrow AB} \omega (U_{S \rightarrow AB})^\dagger, U_{S \rightarrow AB} \tau (U_{S \rightarrow AB})^\dagger) \\
& + \frac{\pi^2 \lambda_{\min}(U_{S \rightarrow AB} \tau (U_{S \rightarrow AB})^\dagger)}{432 \text{Tr}\{\mathcal{N}(\tau)\}} \times \\
& \|\mathcal{R}_{A \rightarrow AB}^\sigma(\mathcal{N}(\omega)) - U_{S \rightarrow AB} \omega (U_{S \rightarrow AB})^\dagger\|_1^3. \tag{68}
\end{aligned}$$

Due to isometric invariance of Holevo's just-as-good fidelity and the minimum non-zero eigenvalue, and the fact that \mathcal{N} is trace preserving, we find that

$$\begin{aligned}
& F_H(U_{S \rightarrow AB} \omega (U_{S \rightarrow AB})^\dagger, U_{S \rightarrow AB} \omega (U_{S \rightarrow AB})^\dagger) = F_H(\omega, \tau), \\
& \lambda_{\min}(U_{S \rightarrow AB} \tau (U_{S \rightarrow AB})^\dagger) = \lambda_{\min}(\tau), \\
& \text{Tr}\{\mathcal{N}(\tau)\} = \text{Tr}\{\tau\}. \tag{69}
\end{aligned}$$

Also, the Petz map $\mathcal{R}_{A \rightarrow AB}^\sigma$ simplifies for our choices as

$$\begin{aligned}
& \mathcal{R}_{A \rightarrow AB}^\sigma(\cdot) \\
&= \sigma_{AB}^{1/2} [\sigma_A^{-1/2}(\cdot) \sigma_A^{-1/2} \otimes I_B] \sigma_{AB}^{1/2} \\
&= [U_{S \rightarrow AB} \tau (U_{S \rightarrow AB})^\dagger]^{1/2} [\mathcal{N}(\tau)^{-1/2}(\cdot) \mathcal{N}(\tau)^{-1/2} \otimes I_B] \\
&\quad \times [U_{S \rightarrow AB} \tau (U_{S \rightarrow AB})^\dagger]^{1/2} \\
&= U_{S \rightarrow AB} \tau^{1/2} (U_{S \rightarrow AB})^\dagger [\mathcal{N}(\tau)^{-1/2}(\cdot) \mathcal{N}(\tau)^{-1/2} \otimes I_B] \\
&\quad \times U_{S \rightarrow AB} \tau^{1/2} (U_{S \rightarrow AB})^\dagger \\
&= U_{S \rightarrow AB} \tau^{1/2} \mathcal{N}^\dagger [\mathcal{N}(\tau)^{-1/2}(\cdot) \mathcal{N}(\tau)^{-1/2}] \tau^{1/2} (U_{S \rightarrow AB})^\dagger.
\end{aligned} \tag{70}$$

Isometric invariance of the trace norm and the above then gives

$$\begin{aligned}
& \|\mathcal{R}_{A \rightarrow AB}^\sigma(\mathcal{N}(\omega)) - U_{S \rightarrow AB} \omega (U_{S \rightarrow AB})^\dagger\|_1 \\
&= \|\mathcal{P}(\mathcal{N}(\omega)) - \omega\|_1,
\end{aligned} \tag{71}$$

concluding the proof. ■

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