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Applications of position-based coding to classical communication over quantum channels

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Abstract

Recently, a coding technique called *position-based coding* has been used to establish achievability statements for various kinds of classical communication protocols that use quantum channels. In the present paper, we apply this technique in the entanglement-assisted setting in order to establish lower bounds for error exponents, lower bounds on the second-order coding rate, and one-shot lower bounds. We also demonstrate that position-based coding can be a powerful tool for analyzing other communication settings. In particular, we reduce the quantum simultaneous decoding conjecture for entanglement-assisted or unassisted communication over a quantum multiple access channel to open questions in multiple quantum hypothesis testing. We then determine achievable rate regions for entanglement-assisted or unassisted classical communication over a quantum multiple-access channel, when using a particular quantum simultaneous decoder. The achievable rate regions given in this latter case are generally suboptimal, involving differences of Rényi-2 entropies and conditional quantum entropies.

1 Introduction

Understanding optimal rates for classical communication over both point-to-point quantum channels and quantum network channels are fundamental tasks in quantum Shannon theory (see, e.g., [Hol12, Wat16, Wil16, Wil17b]). Early developments of quantum Shannon theory are based on the assumption that the information is transmitted over an arbitrarily large number of independent and identically distributed (i.i.d.) uses of a given quantum channel. By taking advantage of this assumption, general formulas have been established for capacities of various communication protocols, with or without preshared entanglement. When a sender and receiver do not share entanglement before communication begins, it is known that the Holevo information of a quantum channel is an achievable rate for classical communication [Hol98, SW97]. Regularizing the Holevo information leads to a multi-letter formula that characterizes the capacity for this task. Regarding communication over quantum network channels, an achievable rate region for classical communication over quantum multiple-access channels was given in [Win01] and regularizing it leads to a characterization of the

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capacity region for this task. However, only inner bounds on the capacity region for general broadcast channels are known [YHD11, RSW16, SW15], except when the quantum broadcast channel is a particular kind of degraded channel [WDW17]. When there is entanglement shared between the communicating parties, many scenarios have also been studied, including classical communication over point-to-point quantum channels [BSST02, Hol02], quantum multiple-access channels [HDW08, XW13], and quantum broadcast channels [DHL10, WDW17].

Although channel capacity gives a fundamental characterization of the communication capabilities of a quantum channel, many practically important properties of quantum channels are not captured by this quantity. To close this gap, several works have focused on the study of refined notions of capacity, including error exponents [BH98, Win99b, Hol00, Hay07, Dal13, DW14] and second-order asymptotics [TT15, WRG16, DTW16]. The latter works build upon strong connections between hypothesis testing and coding, as considered in [Hay09a, PPV10, WR12]. The refined characterizations of capacity are of importance for regimes of practical interest, in which a limited number of uses of a quantum channel are available. Complementary to these developments, to go beyond the i.i.d. assumption, many works have been dedicated to the one-shot formalism [WR12, DRRW13, DH13, MW14] and the information-spectrum approach [HN03, Hay06, BD06], with very few assumptions made on the structure of quantum channels.

In a recent work [AJW17b], a technique called position-based coding was developed in order to give one-shot achievability bounds for various classical communication protocols that use entanglement assistance. This was then extended to the cases of unassisted classical communication and private classical communication [Wil17a]. The method of position-based coding is a derivative of the well known and long studied coding technique called pulse position modulation (see, e.g., [Ver90, CE03]). In pulse position modulation, a sender encodes a message by placing a pulse in one slot and having no pulse in the other available slots. Each slot is then communicated over the channel, one by one. The receiver can decode well if he can distinguish "pulse" from "no pulse." Position-based coding borrows this idea: a sender and receiver are allowed to share many copies of a bipartite quantum state. The sender encodes a message by sending a share of one of the bipartite states through a channel to the receiver. From the receiver's perspective, only one share of his systems will be correlated with the channel output (the pulse in one slot), while the others will have no correlation (no pulse in the other slots). So if the receiver can distinguish "pulse" from "no pulse" in this context, then he will be able to decode well, just as is the case in pulse position modulation. The authors of [AJW17b] applied the position-based coding technique to a number of problems that have already been addressed in the literature, including point-to-point entanglement-assisted communication [BSST02, Hol02], entanglement-assisted coding with side information at the transmitter [Dup09], and entanglement-assisted communication over broadcast channels [DHL10].

In the present paper, we use position-based coding to establish several new results. First, we establish lower bounds on the entanglement-assisted error exponent and on the one-shot entanglement-assisted capacity. The latter improves slightly upon the result from [AJW17b] and in turn gives a simpler proof of one of the main results of [DTW16], i.e., a lower bound on the second-order coding rate for entanglement-assisted classical communication. We then turn to communication over quantum multiple-access channels when using a quantum simultaneous decoder, considering both cases of entanglement assistance and no assistance. The quantum simultaneous decoding conjecture from [FHS$^+$12, Wil11] stands as one of the most important open problems in network quantum information theory. Here we report progress on this conjecture and connect it to some open questions from [AM14, BHOS15] in multiple quantum hypothesis testing. At the same time, we give new achievable rate regions for entanglement-assisted classical communication over multiple-access channels, where the bounds on achievable rates are expressed as a difference of a Rényi entropy of order two and a conditional quantum entropy.

This paper is organized as follows. We first summarize relevant definitions and lemmas in Section 2. In this section, we also prove Proposition 3, which relates the hypothesis testing relative entropy to the quantum Rényi relative entropy and is an interesting counterpart to [CMW16, Lemma 5]. In Section 3, we consider entanglement-assisted point-to-point classical communication. By using position-based coding, we establish a lower bound on the entanglement-assisted error exponent. We also establish a lower bound on the one-shot entanglement-assisted capacity in terms of hypothesis-testing mutual information and state how it is close to a previously known upper bound from [MW14]. At the same time, we provide a simpler proof of the upper bound from [MW14], which follows from a lemma regarding generalized mutual information. Based upon this one-shot lower bound, we then rederive a lower bound on the second-order coding rate for entanglement-assisted communication with a proof that is arguably simpler than that given in [DTW16]. In Section 4, we apply position-based coding to entanglement-assisted classical communication over multiple-access channels and establish an explicit link to multiple quantum hypothesis testing. We give an achievable rate region for i.i.d. channels by using techniques from the theory of quantum typicality. We demonstrate the power of position-based coding technique in unassisted classical communication in Section 4.2, by considering classical communication over multiple-access channel. We explicitly show how to derandomize a randomness-assisted protocol. In Section 5, we tie open questions in multiple quantum hypothesis testing to quantum simultaneous decoding for the quantum multiple-access channel. Finally, we summarize our main results and discuss open questions in Section 6.

2 Preliminaries

Trace distance, fidelity, and gentle measurement. Let $\mathcal{D}(\mathcal{H})$ denote the set of density operators (positive semi-definite operators with unit trace) acting on a Hilbert space $\mathcal{H}$. The *trace distance* between two density operators $\rho, \sigma \in \mathcal{D}(\mathcal{H})$ is equal to $\|\rho - \sigma\|_1$, where $\|A\|_1 \equiv \operatorname{Tr}\{\sqrt{A^\dagger A}\}$. Another quantity to measure the closeness between two quantum states is the *fidelity*, defined as $F(\rho, \sigma) \equiv \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$ [Uhl76]. Two inequalities relating trace distance and quantum measurement operators are as follows:

Lemma 1 *(Gentle operator [Win99a, ON07]) Consider a density operator $\rho \in \mathcal{D}(\mathcal{H})$ and a measurement operator Λ where $0 \leq \Lambda \leq I$. Suppose that the measurement operator Λ detects the state ρ with high probability $\operatorname{Tr}\{\Lambda\rho\} \geq 1 - \varepsilon$, where $\varepsilon \in [0, 1]$. Then*

$$\left\|\rho - \sqrt{\Lambda}\rho\sqrt{\Lambda}\right\|_1 \leq 2\sqrt{\varepsilon}\ . \tag{2.1}$$

Lemma 2 *Consider two quantum states $\rho, \sigma \in \mathcal{D}(\mathcal{H})$ and a measurement operator Λ where $0 \leq \Lambda \leq I$. Then we have*

$$\operatorname{Tr}\{\Lambda\rho\} \geq \operatorname{Tr}\{\Lambda\sigma\} - \|\rho - \sigma\|_1\ . \tag{2.2}$$

More generally, the same bound holds when ρ and σ are subnormalized, i.e., $\operatorname{Tr}\{\rho\}, \operatorname{Tr}\{\sigma\} \leq 1$.

Information spectrum. The information spectrum approach [Han03, NH07, HN03, DR09] gives one-shot bounds for operational tasks in quantum Shannon theory, with very few assumptions made about the source or channel [NH07, Hay03, HN03, Hay07]. What plays an important role in the information spectrum method is the positive spectral projection of an operator. For a Hermitian operator X with spectral decomposition $X = \sum_i \lambda_i |i\rangle\langle i|$, the associated *positive spectral projection* is denoted and defined as

$$\{X \geq 0\} \equiv \sum_{i:\lambda_i \geq 0} |i\rangle\langle i| \ . \tag{2.3}$$

Relative entropies and mutual informations. For a state $\rho \in \mathcal{D}(\mathcal{H})$ and a positive semi-definite operator σ, the *quantum Rényi relative entropy* of order α, where $\alpha \in [0,1) \cup (1,+\infty)$ is defined as [Pet86, TCR09]

$$D_\alpha(\rho\|\sigma) \equiv \frac{1}{\alpha-1} \log_2 \operatorname{Tr}\{\rho^\alpha \sigma^{1-\alpha}\} \ . \tag{2.4}$$

If $\alpha > 1$ and $\operatorname{supp}(\rho) \nsubseteq \operatorname{supp}(\sigma)$, it is set to $+\infty$. In the limit as $\alpha \to 1$, the above definition reduces to the *quantum relative entropy* [Ume62]

$$D(\rho\|\sigma) \equiv \operatorname{Tr}\{\rho[\log_2 \rho - \log_2 \sigma]\} \ , \tag{2.5}$$

which is defined as above when $\operatorname{supp}(\rho) \subseteq \operatorname{supp}(\sigma)$ and it is set to $+\infty$ otherwise. Using the above definition, we can define the *Rényi mutual information* for a bipartite state θ_{RB} as

$$I_\alpha(R;B)_\theta \equiv D_\alpha(\theta_{RB}\|\theta_R \otimes \theta_B) \ . \tag{2.6}$$

The ε*-hypothesis testing relative entropy* for a state ρ and a positive semi-definite σ is defined for $\varepsilon \in [0,1]$ as [BD10, WR12]

$$D_H^\varepsilon(\rho\|\sigma) \equiv -\log_2 \inf_\Lambda \{\operatorname{Tr}\{\Lambda\sigma\} : 0 \leq \Lambda \leq I \wedge \operatorname{Tr}\{\Lambda\rho\} \geq 1-\varepsilon\} \ . \tag{2.7}$$

Similarly, we define the ε*-hypothesis testing mutual information* of a bipartite state θ_{RB} as

$$I_H^\varepsilon(R;B)_\theta \equiv D_H^\varepsilon(\theta_{RB}\|\theta_R \otimes \theta_B) \ . \tag{2.8}$$

Note that there are alternative definitions [MW14] of ε-hypothesis testing mutual information that involve an optimization with respect to the marginal state on system B:

$$\widetilde{I}_H^\varepsilon(R;B)_\theta \equiv \min_{\sigma_B} D_H^\varepsilon(\theta_{RB}\|\theta_R \otimes \sigma_B). \tag{2.9}$$

The following proposition establishes an inequality relating hypothesis testing relative entropy and the quantum Rényi relative entropy, and it represents a counterpart to [CMW16, Lemma 5]. We give its proof in the appendix, where we also mention how [CMW16, Lemma 5] and the following proposition lead to a transparent proof of the quantum Stein's lemma [HP91, ON00].

Proposition 3 *Let ρ be a density operator and let σ be a positive semi-definite operator. Let $\alpha \in (0,1)$ and $\varepsilon \in (0,1)$. Then the following inequality holds*

$$D_H^\varepsilon(\rho\|\sigma) \geq \frac{\alpha}{\alpha-1} \log_2\left(\frac{1}{\varepsilon}\right) + D_\alpha(\rho\|\sigma). \tag{2.10}$$

The hypothesis testing relative entropy has the following second-order expansion [TH13, Li14, DPR16]:

$$D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) = nD(\rho\|\sigma) + \sqrt{nV(\rho\|\sigma)}\Phi^{-1}(\varepsilon) + O(\log n) \ , \tag{2.11}$$

where $V(\rho\|\sigma) = \mathrm{Tr}\{\rho[\log_2\rho - \log_2\sigma]^2\} - [D(\rho\|\sigma)]^2$ is the *quantum relative entropy variance* and the function $\Phi(a)$ is the cumulative distribution function for a standard normal distribution:

$$\Phi(a) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} dx\ e^{-x^2/2} \ . \tag{2.12}$$

Let σ be a quantum state now. The hypothesis testing relative entropy is relevant for asymmetric hypothesis testing, in which the goal is to minimize the error probability $\mathrm{Tr}\{\Lambda\sigma\}$ subject to a constraint on the other kind of error probability $\mathrm{Tr}\{(I-\Lambda)\rho\} \leq \varepsilon$. We could also consider symmetric hypothesis testing, in which the goal is to minimize both kinds of error probabilities simultaneously. It is useful for us here to take the approach of [AM14] and consider general positive semi-definite operators A and B rather than states ρ and σ. As in [AM14], we can define the error "probability" in identifying the operators A and B as follows:

$$\begin{aligned} P_e^*(A,B) &\equiv \inf_{T\,:\,0\leq T\leq I} \mathrm{Tr}\{(I-T)A\} + \mathrm{Tr}\{TB\} &(2.13)\\ &= \mathrm{Tr}\{A\} - \sup_{T\,:\,0\leq T\leq I} \mathrm{Tr}\{T(A-B)\} &(2.14)\\ &= \mathrm{Tr}\{A\} - \mathrm{Tr}\{\{A-B\geq 0\}(A-B)\} &(2.15)\\ &= \frac{1}{2}\left(\mathrm{Tr}\{A+B\} - \|A-B\|_1\right). &(2.16)\end{aligned}$$

The following lemma allows for bounding $P_e^*(A,B)$ from above, and we use it to establish bounds on the error exponent for entanglement-assisted communication.

Lemma 4 ([ACMnT$^+$07]) *Let A and B be positive semi-definite operators and $s \in [0,1]$. Then the following inequality holds*

$$P_e^*(A,B) = \frac{1}{2}\left(\mathrm{Tr}\{A+B\} - \|A-B\|_1\right) \leq \mathrm{Tr}\{A^sB^{1-s}\}. \tag{2.17}$$

The above lemma was first proved in [ACMnT$^+$07], but the reader should note that a much simpler proof due to N. Ozawa is presented in [JOPS12, Proposition 1.1] and [Aud14, Theorem 1].

Weak typicality. We will use results from the theory of weak typicality in some of our achievability proofs (see, e.g., [Wil16, Wil17b] for a review). Consider a density operator ρ_A with spectral decomposition: $\rho_A = \sum_x p_X(x)|x\rangle\langle x|_A$. The weakly *$\delta$-typical subspace* $T_{A^n}^{\rho,\delta}$ is defined as the span of all unit vectors $|x^n\rangle \equiv |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle$ such that the sample entropy $\overline{H}(x^n)$ of their classical label is close to the true entropy $H(X) = H(A)_\rho$ of the distribution $p_X(x)$:

$$T_{A^n}^{\rho,\delta} \equiv \mathrm{span}\left\{|x^n\rangle : \left|\overline{H}(x^n) - H(X)\right| \leq \delta\right\}, \tag{2.18}$$

where $\overline{H}(x^n) \equiv -\frac{1}{n}\log_2(p_{X^n}(x^n))$ and $H(X) \equiv -\sum_x p_X(x)\log_2 p_X(x)$. The *$\delta$-typical projector* $\Pi_{\rho,\delta}^n$ onto the typical subspace of ρ is defined as

$$\Pi_{A^n}^{\rho,\delta} \equiv \sum_{x^n\in T_\delta^{X^n}} |x^n\rangle\langle x^n|, \tag{2.19}$$

where we have used the symbol $T^\delta_{X^n}$ to refer to the set of δ-typical sequences:

$$T^\delta_{X^n} \equiv \left\{x^n : \left|\overline{H}(x^n) - H(X)\right| \leq \delta\right\}. \tag{2.20}$$

Three important properties of the typical projector are as follows:

$$\mathrm{Tr}\{\Pi^{\rho,\delta}_{A^n}\rho^{\otimes n}\} \geq 1-\varepsilon, \tag{2.21}$$

$$\mathrm{Tr}\{\Pi^{\rho,\delta}_{A^n}\} \leq 2^{n[H(A)+\delta]}, \tag{2.22}$$

$$2^{-n[H(A)+\delta]}\Pi^{\rho,\delta}_{A^n} \leq \Pi^{\rho,\delta}_{A^n}\rho^{\otimes n}\Pi^{\rho,\delta}_{A^n} \leq 2^{-n[H(A)-\delta]}\Pi^{\rho,\delta}_{A^n}, \tag{2.23}$$

where the first property holds for arbitrary $\varepsilon \in (0,1)$, $\delta > 0$, and sufficiently large n. We will also need the following 'projector trick' inequality [GLM12, FHS$^+$12]:

$$\Pi^{\rho,\delta}_{A^n} \leq 2^{n[H(A)+\delta]}\rho^{\otimes n}_A \ , \tag{2.24}$$

which follows as a consequence of the leftmost inequality in (2.23) and the fact that $\Pi^{\rho,\delta}_{A^n}\rho^{\otimes n}\Pi^{\rho,\delta}_{A^n} = \sqrt{\rho^{\otimes n}}\Pi^{\rho,\delta}_{A^n}\sqrt{\rho^{\otimes n}} \leq \rho^{\otimes n}$. A final inequality we make use of is the following one

$$\Pi^{\rho,\delta}_{A^n} \leq 2^{-n[H(A)-\delta]/2}\left[\rho^{\otimes n}\right]^{-1/2}, \tag{2.25}$$

which is a consequence of sandwiching the rightmost inequality of (2.23) by $[\rho^{\otimes n}]^{-1/2}$, applying $\Pi^{\rho,\delta}_{A^n} \leq I^{\otimes n}$, and operator monotonicity of the square root function.

Hayashi-Nagaoka operator inequality. We repeatedly use the following operator inequality from [HN03] when analyzing error probability:

Lemma 5 *Given operators S and T such that $0 \leq S \leq I$ and $T \geq 0$, the following inequality holds for all $c > 0$:*

$$I - (S+T)^{-1/2}S(S+T)^{-1/2} \leq c_{\mathrm{I}}\,(I-S) + c_{\mathrm{II}}\,T \ , \tag{2.26}$$

where $c_{\mathrm{I}} \equiv 1 + c$ and $c_{\mathrm{II}} \equiv 2 + c + c^{-1}$.

3 Entanglement-assisted point-to-point classical communication

We begin by defining the information-processing task of point-to-point entanglement-assisted classical communication, originally considered in [BSST02, Hol02] and studied further in [DH13, MW14, DTW16]. Before communication begins, the sender Alice and the receiver Bob share entanglement in whatever form they wish, and we denote their shared state as Ψ_{RA}. Suppose Alice would like to communicate some classical message m from a set $\mathcal{M} \equiv \{1,\ldots,M\}$ over a quantum channel $\mathcal{N}_{A'\to B}$, where $M \in \mathbb{N}$ denotes the cardinality of the set $\mathcal{M}$. An (M,ε) entanglement-assisted classical code, for $\varepsilon \in [0,1]$, consists of a collection $\{\mathcal{E}^m_{A\to A'}\}_m$ of encoders and a decoding POVM $\{\Lambda^m_{RB}\}_m$, such that the average error probability is bounded from above by ε:

$$\frac{1}{M}\sum_{m=1}^{M}\mathrm{Tr}\{(I-\Lambda^m_{RB})\mathcal{N}_{A'\to B}(\mathcal{E}^m_{A\to A'}(\Psi_{RA}))\} \leq \varepsilon, \tag{3.1}$$

For fixed ε, let $M^*(\mathcal{N},\varepsilon)$ denote the largest M for which there exists an (M,ε) entanglement-assisted classical communication code for the channel $\mathcal{N}$. Then we define the ε-one-shot entanglement-assisted classical capacity as $\log_2 M^*(\mathcal{N},\varepsilon)$. We note that one could alternatively consider maximum error probability when defining this capacity. The entanglement-assisted capacity of a channel $\mathcal{N}$ is then defined as

$$C_{\mathrm{EA}}(\mathcal{N}) \equiv \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log_2 M^*(\mathcal{N}^{\otimes n},\varepsilon). \tag{3.2}$$

For fixed M, the one-shot entanglement-assisted error exponent $-\log_2 \varepsilon^*(\mathcal{N},M)$ is such that ε^* is equal to the smallest ε for which there exists an (M,ε) entanglement-assisted classical communication code. In the i.i.d. setting, the entanglement-assisted error exponent is defined for a fixed rate $R \geq 0$ as

$$E_{\mathrm{EA}}(\mathcal{N},R) \equiv \limsup_{n\to\infty} \left[-\frac{1}{n} \log_2 \varepsilon^*(\mathcal{N}^{\otimes n}, 2^{nR}) \right]. \tag{3.3}$$

3.1 One-shot position-based coding

We now review the method of position-based coding [AJW17b], as applied to point-to-point entanglement-assisted communication, and follow the review by showing how the approach leads to a lower bound on the error exponent for entanglement-assisted communication, a lower bound for one-shot entanglement-assisted capacity, and a simple proof for a lower bound on the second-order coding rate for entanglement-assisted communication. We note that a lower bound for one-shot entanglement-assisted capacity using position-based coding was already given in [AJW17b], but the lower bound given here leads to a lower bound on the entanglement-assisted second-order coding rate that is optimal for covariant channels [DTW16].

The position-based entanglement-assisted communication protocol consists of two steps, encoding and decoding, and we follow that discussion with an error analysis of its performance.

Encoding: Before communication begins, Alice and Bob share the following state:

$$\theta_{RA}^{\otimes M} \equiv \theta_{R_1A_1} \otimes \cdots \otimes \theta_{R_MA_M}, \tag{3.4}$$

where Alice possesses the A systems and Bob has the R systems. To send message m, Alice simply sends the mth A system through the channel. So this leads to the following state for Bob:

$$\rho^m_{R^MB} \equiv \theta_R^{\otimes m-1} \otimes \mathcal{N}_{A\to B}(\theta_{R_mA_m}) \otimes \theta_R^{\otimes M-m}. \tag{3.5}$$

Decoding: Define the following measurement:

$$\Gamma^m_{R^MB} \equiv I_{R^{m-1}} \otimes T_{R_mB_m} \otimes I_{R^{M-m}}, \tag{3.6}$$

where $T_{R_mB_m} = T_{RB}$ is a "test" or measurement operator satisfying $0 \leq T_{RB} \leq I_{RB}$, which we will specify later. For now, just think of it as corresponding to a measurement that should distinguish well between $\mathcal{N}_{A\to B}(\theta_{RA})$ and $\theta_R \otimes \mathcal{N}_{A\to B}(\theta_A)$. This is important for the following reason: If message m is transmitted and the test is performed on the mth R system and the channel output system B, then the probability of it accepting is

$$\mathrm{Tr}\{\Gamma^m_{R^MB}\rho^m_{R^MB}\} = \mathrm{Tr}\{T_{RB}\mathcal{N}_{A\to B}(\theta_{RA})\}. \tag{3.7}$$

If however the test is performed on the m'th R system and B, where $m' \neq m$, then the probability of it accepting is

$$\mathrm{Tr}\{\Gamma^{m'}_{R^MB}\rho^m_{R^MB}\} = \mathrm{Tr}\{T_{RB}[\theta_R \otimes \mathcal{N}_{A\to B}(\theta_A)]\}. \tag{3.8}$$

We use these facts in the forthcoming error analysis.

We use a square-root measurement to form a decoding POVM for Bob as follows:

$$\Lambda^m_{R^M B} \equiv \left(\sum_{m'=1}^{M} \Gamma^{m'}_{R^M B}\right)^{-1/2} \Gamma^m_{R^M B} \left(\sum_{m'=1}^{M} \Gamma^{m'}_{R^M B}\right)^{-1/2}. \tag{3.9}$$

This is called the position-based decoder.

Error analysis: The error probability under this coding scheme is the same for each message m (see, e.g., [AJW17b, Wil17a]) and is as follows:

$$p_e(m) \equiv \mathrm{Tr}\{(I_{R^M B} - \Lambda^m_{R^M B})\rho^m_{R^M B}\}. \tag{3.10}$$

Applying Lemma 5 with $S = \Gamma^m_{R^M B}$ and $T = \sum_{m' \neq m} \Gamma^{m'}_{R^M B}$, we find that this error probability can be bounded from above as

$$\begin{aligned}
&\mathrm{Tr}\{(I_{R^M B} - \Lambda^m_{R^M B})\rho^m_{R^M B}\} \\
&\leq c_{\mathrm{I}} \, \mathrm{Tr}\{(I_{R^M B} - \Gamma^m_{R^M B})\rho^m_{R^M B}\} + c_{\mathrm{II}} \sum_{m' \neq m} \mathrm{Tr}\{\Gamma^{m'}_{R^M B}\rho^m_{R^M B}\} &(3.11)\\
&= c_{\mathrm{I}} \, \mathrm{Tr}\{(I_{RB} - T_{RB})\mathcal{N}_{A\to B}(\theta_{RA})\} + c_{\mathrm{II}} \sum_{m' \neq m} \mathrm{Tr}\{T_{RB}\,[\theta_R \otimes \mathcal{N}_{A\to B}(\theta_A)]\} &(3.12)\\
&= c_{\mathrm{I}} \, \mathrm{Tr}\{(I_{RB} - T_{RB})\mathcal{N}_{A\to B}(\theta_{RA})\} + c_{\mathrm{II}}(M-1) \, \mathrm{Tr}\{T_{RB}\,[\theta_R \otimes \mathcal{N}_{A\to B}(\theta_A)]\}. &(3.13)
\end{aligned}$$

The same bound applies for both the average and the maximum error probability, due to the symmetric construction of the code.

Our bound for a test operator T_{RB} is thus as follows and highlights, as in [AJW17b], an important connection between quantum hypothesis testing (i.e., the ability to distinguish the states $\mathcal{N}_{A\to B}(\theta_{RA})$ and $\theta_R \otimes \mathcal{N}_{A\to B}(\theta_A)$) and entanglement-assisted communication:

$$p_e(m) \leq c_{\mathrm{I}} \, \mathrm{Tr}\{(I_{RB} - T_{RB})\mathcal{N}_{A\to B}(\theta_{RA})\} + c_{\mathrm{II}}(M-1) \, \mathrm{Tr}\{T_{RB}\,[\theta_R \otimes \mathcal{N}_{A\to B}(\theta_A)]\}. \tag{3.14}$$

3.2 Lower bounds on one-shot and i.i.d. entanglement-assisted error exponents

We first prove a lower bound on the one-shot error exponent, and then a lower bound for the entanglement-assisted error exponent in the i.i.d. case directly follows.

Theorem 6 *For a quantum channel $\mathcal{N}_{A\to B}$, a lower bound on the one-shot entanglement-assisted error exponent for fixed message size M is as follows:*

$$-\log_2 \varepsilon^*(\mathcal{N}, M) \geq \sup_{s\in[0,1]} (1-s) \left[\sup_{\theta_{RA}} I_s(R;B)_{\mathcal{N}(\theta)} - \log_2 M\right] - 2 \;, \tag{3.15}$$

where θ_{RA} is a pure bipartite entangled state and $I_s(R;B)_{\mathcal{N}(\theta)}$ is the Rényi mutual information defined in (2.6).

Proof. Following the position-based encoding and decoding procedure described in Section 3.1 and setting $c = 1$ in (3.14), the error probability for each message can be bounded as

$$p_e(m) = \operatorname{Tr}\{(I_{R^M B} - \Lambda^m_{R^M B})\rho^m_{R^M B}\} \tag{3.16}$$
$$\leq 4\left[\operatorname{Tr}\{(I_{RB} - T_{RB})\mathcal{N}_{A\to B}(\theta_{RA})\} + M \operatorname{Tr}\{T_{RB}\left[\theta_R \otimes \mathcal{N}_{A\to B}(\theta_A)\right]\}\right] \tag{3.17}$$
$$= 4\left[\operatorname{Tr}\{\mathcal{N}_{A\to B}(\theta_{RA})\} - \operatorname{Tr}\{T_{RB}\left(\mathcal{N}_{A\to B}(\theta_{RA}) - M\left[\theta_R \otimes \mathcal{N}_{A\to B}(\theta_A)\right]\right)\}\right]. \tag{3.18}$$

To minimize the term in the last line above, it is well known that one should take the test operator T_{RB} as follows:

$$T_{RB} = \{\mathcal{N}_{A\to B}(\theta_{RA}) - M\left[\theta_R \otimes \mathcal{N}_{A\to B}(\theta_A)\right] \geq 0\}. \tag{3.19}$$

The statement for quantum states is due to [Hel69, Hol73, Hel76] and the extension (relevant for us) to the more general case of positive semi-definite operators appears in [AM14, Eq. (22)] (see also (2.13)–(2.16)). This then leads to the following upper bound on the error probability:

$$p_e(m) \leq 4\left[\operatorname{Tr}\{\mathcal{N}_{A\to B}(\theta_{RA})\} - \operatorname{Tr}\{T_{RB}\left(\mathcal{N}_{A\to B}(\theta_{RA}) - M\left[\theta_R \otimes \mathcal{N}_{A\to B}(\theta_A)\right]\right)\}\right] \tag{3.20}$$
$$= 2\Big[\operatorname{Tr}\{\mathcal{N}_{A\to B}(\theta_{RA}) + M\left[\theta_R \otimes \mathcal{N}_{A\to B}(\theta_A)\right]\} - \left\|\mathcal{N}_{A\to B}(\theta_{RA}) - M\left[\theta_R \otimes \mathcal{N}_{A\to B}(\theta_A)\right]\right\|_1\Big] \tag{3.21}$$
$$\leq 4 \operatorname{Tr}\{\mathcal{N}_{A\to B}(\theta_{RA})^s \left(M\theta_R \otimes \mathcal{N}_{A\to B}(\theta_A)\right)^{1-s}\} \tag{3.22}$$
$$= 4M^{1-s} \operatorname{Tr}\{\mathcal{N}_{A\to B}(\theta_{RA})^s \left[\theta_R \otimes \mathcal{N}_{A\to B}(\theta_A)\right]^{1-s}\} \tag{3.23}$$
$$= 4\left(2^{-(1-s)\left[I_s(R;B)_{\mathcal{N}(\theta)} - \log_2 M\right]}\right). \tag{3.24}$$

The first equality is standard, using the relation of the positive part of an operator to its modulus (see, e.g., [AM14, Eq. (23)]). The second inequality is a consequence of [ACMnT+07, Theorem 1], recalled as Lemma 4 in Section 2, and holds for all $s \in [0, 1]$ (see [JOPS12, Proposition 1.1] and [Aud14, Theorem 1] for a simpler proof of [ACMnT+07, Theorem 1] due to N. Ozawa). The last equality follows from the definition of Rényi mutual information in (2.6). Since this bound holds for an arbitrary $s \in [0, 1]$ and an arbitrary input state θ_{RA}, we can conclude the following bound:

$$p_e(m) \leq 4\left(2^{-\sup_{s\in[0,1]}(1-s)\left[\sup_{\theta_{RA}} I_s(R;B)_{\mathcal{N}(\theta)} - \log_2 M\right]}\right). \tag{3.25}$$

Note that it suffices to take θ_{RA} as a pure bipartite state, due to the ability to purify a mixed θ_{RA} and the data-processing inequality for $I_s(R;B)_{\mathcal{N}(\theta)}$, holding for all $s \in [0, 1]$ [Pet86]. Finally taking a negative binary logarithm of both sides of (3.25) gives (3.15). ■

We remark that the above proof bears some similarities to that given in [Hay07] (one can find a related result in the later work [MD09, Lemma 3.1]). One of the results in [Hay07] concerns a bound on the error exponent for classical communication over classical-input quantum-output channels. The fundamental tool used in the proof of this result in [Hay07] is Lemma 4, attributed above to [ACMnT+07]. Our proof above clearly follows the same approach.

Applying the above result in the i.i.d. case for a memoryless channel $\mathcal{N}^{\otimes n}_{A\to B}$ leads to the following:

Proposition 7 *For a quantum channel $\mathcal{N}_{A\to B}$, a lower bound on the entanglement-assisted error exponent $E_{\rm EA}(\mathcal{N},R)$ (defined in (3.3)) for fixed rate $R \geq 0$ is as follows:*

$$E_{\rm EA}(\mathcal{N},R) \geq \sup_{s\in[0,1]} (1-s)\left[\sup_{\theta_{RA}} I_s(R;B)_{\mathcal{N}(\theta)} - R\right], \tag{3.26}$$

where θ_{RA} is a pure bipartite entangled state and $I_s(R;B)_{\mathcal{N}(\theta)}$ is the Rényi mutual information defined in (2.6).

Proof. A proof follows by plugging in the memoryless channel $\mathcal{N}^{\otimes n}_{A\to B}$ into (3.15), setting the number of messages to be $M = 2^{nR}$, and considering that

$$\sup_{\theta^{(n)}_{R^nA^n}} I_s(R^n;B^n)_{\mathcal{N}^{\otimes n}(\theta^{(n)})} \geq \sup_{\theta^{\otimes n}_{RA}} I_s(R^n;B^n)_{[\mathcal{N}(\theta)]^{\otimes n}} \tag{3.27}$$

$$= n \sup_{\theta_{RA}} I_s(R;B)_{\mathcal{N}(\theta)}, \tag{3.28}$$

leading to the following bound:

$$-\frac{1}{n}\log_2 \varepsilon^*(\mathcal{N}^{\otimes n}, 2^{nR}) \geq \sup_{s\in[0,1]} (1-s)\left[\sup_{\theta_{RA}} I_s(R;B)_{\mathcal{N}(\theta)} - R\right] - \frac{2}{n}. \tag{3.29}$$

The equality in (3.28) follows from the additivity of the Rényi mutual information for tensor-power states. Taking the large n limit then gives (3.26). Alternatively, plugging the memoryless channel $\mathcal{N}^{\otimes n}_{A\to B}$ in to (3.25), we find that the bound on the error probability becomes

$$p_e(m) \leq 4\left(2^{-(1-s)n\left[I_s(R;B)_{\mathcal{N}(\theta)}-R\right]}\right), \tag{3.30}$$

holding for all $s\in[0,1]$ and states θ_{RA}. After taking a negative logarithm, normalizing by n, and taking the limit as $n\to\infty$, we arrive at (3.26). ■

3.3 Lower bounds on one-shot entanglement-assisted capacity and entanglement-assisted second-order coding rate

By using position-based coding, here we establish a lower bound on the one-shot entanglement-assisted capacity. Note that a similar lower bound was established in [AJW17b], but the theorem below allows for an additional parameter $\eta\in(0,\varepsilon)$, which is helpful for giving a lower bound on the entanglement-assisted second-order coding rate.

Theorem 8 *Given a quantum channel $\mathcal{N}_{A\to B}$ and fixed $\varepsilon\in(0,1)$, the ε-one-shot entanglement-assisted capacity of $\mathcal{N}_{A\to B}$ is bounded as*

$$\log_2 M^*(\mathcal{N},\varepsilon) \geq \max_{\theta_{RA}} I_H^{\varepsilon-\eta}(R;B)_{\mathcal{N}(\theta)} - \log_2(4\varepsilon/\eta^2), \tag{3.31}$$

where $\eta\in(0,\varepsilon)$ and the hypothesis testing mutual information is defined in (2.8).

Proof. The idea is to use the same coding scheme described in Section 3.1 and take the test operator T_{RB} in Bob's decoder to be Υ^*_{RB}, where Υ^*_{RB} is the optimal measurement operator for $I_H^{\varepsilon-\eta}(R;B)_{\mathcal{N}(\theta)}$, with $\eta \in (0,\varepsilon)$. Then, starting from the upper bound on the error probability in (3.14), the error analysis reduces to

$$\begin{aligned}
&\operatorname{Tr}\{(I_{R^MB} - \Lambda_{R^MB})\rho^m_{R^MB}\} \\
&\leq c_{\mathrm{I}} \operatorname{Tr}\{(I_{RB} - \Upsilon^*_{RB})\,[\mathcal{N}_{A\to B}(\theta_{RA})]\} + c_{\mathrm{II}} M \operatorname{Tr}\{\Upsilon^*_{RB}\,[\theta_R \otimes \mathcal{N}_{A\to B}(\theta_A)]\} &(3.32)\\
&\leq c_{\mathrm{I}}\,(\varepsilon - \eta) + c_{\mathrm{II}} M 2^{-I_H^{\varepsilon-\eta}(R;B)_{\mathcal{N}(\theta)}}. &(3.33)
\end{aligned}$$

The second inequality follows from the definition of quantum hypothesis testing relative entropy, which gives that

$$\operatorname{Tr}\{\Upsilon^*_{RB}\,[\mathcal{N}_{A\to B}(\theta_{RA})]\} \geq 1 - (\varepsilon - \eta), \tag{3.34}$$

$$\operatorname{Tr}\{\Upsilon^*_{RB}\,[\theta_R \otimes \mathcal{N}_{A\to B}(\theta_A)]\} = 2^{-I_H^{\varepsilon-\eta}(R;B)_{\mathcal{N}(\theta)}}. \tag{3.35}$$

To make the error $p_e(m) \leq \varepsilon$, we set $c = \eta/(2\varepsilon - \eta)$ for $\eta \in (0,\varepsilon)$, and this leads to

$$\log_2 M = I_H^{\varepsilon-\eta}(R;B)_{\mathcal{N}(\theta)} - \log_2(4\varepsilon/\eta^2). \tag{3.36}$$

The inequality in the theorem follows after maximizing $I_H^{\varepsilon-\eta}(R;B)_{\mathcal{N}(\theta)}$ with respect to all input states θ_{RA}. ■

Comparison to upper bound. The authors of [MW14] established the following upper bound on one-shot entanglement-assisted capacity:

$$\begin{aligned}
\max_{\theta_{RA}} I_H^{\varepsilon}(R;B)_{\mathcal{N}(\theta)} &\geq \max_{\theta_{RA}} \min_{\sigma_B} D_H^{\varepsilon}(\mathcal{N}_{A\to B}(\theta_{RA}) \| \theta_R \otimes \sigma_B) &(3.37)\\
&\geq \log_2 M^*(\mathcal{N}, \varepsilon). &(3.38)
\end{aligned}$$

Thus, there is a sense in which the upper bound from [MW14] is close to the lower bound in (3.31). In particular, we could pick $\eta = \delta\varepsilon$ for any constant $\delta \in (0,1)$, and the lower bound becomes $\max_{\theta_{RA}} I_H^{\varepsilon(1-\delta)}(R;B)_{\mathcal{N}(\theta)} - \log_2(4/\varepsilon\delta^2)$. Thus the information term $\max_{\theta_{RA}} I_H^{\varepsilon(1-\delta)}(R;B)_{\mathcal{N}(\theta)}$ can become arbitrarily close to $I_H^{\varepsilon}(R;B)_{\mathcal{N}(\theta)}$ by picking δ smaller, but at the cost of the term $-\log_2(4/\varepsilon\delta^2)$ becoming more negative with decreasing δ.

Lower bound on second-order coding rate. To get a lower bound on the entanglement-assisted second-order coding rate for an i.i.d. channel $\mathcal{N}^{\otimes n}$, evaluate the formula $I_H^{\varepsilon-\eta}(R;B)_{\mathcal{N}(\theta)}$ for an i.i.d. state $\mathcal{N}(\theta)^{\otimes n}$, pick $\eta = 1/\sqrt{n}$ and n large enough such that $\varepsilon - \eta > 0$, and use the second-order expansions for D_H^{ε} in (2.11). We then recover one of the main results of [DTW16]:

$$\log_2 M^*(\mathcal{N}^{\otimes n}, \varepsilon) \geq nI(R;B)_{\mathcal{N}(\theta)} + \sqrt{nV(R;B)_{\mathcal{N}(\theta)}}\,\Phi^{-1}(\varepsilon) + O(\log n). \tag{3.39}$$

Interestingly, this is achievable for maximal error in addition to average error due to the above analysis. Additionally, it does seem that this approach for arriving at a lower bound on the entanglement-assisted second-order coding rate is much simpler than the previous approach developed in [DTW16].

3.4 Alternative proof of an upper bound on one-shot entanglement-assisted capacity

In this section, we provide a proof for an upper bound on the one-shot entanglement-assisted classical capacity, which is arguably simpler than the approach taken in [MW14]. A proof along these lines was found recently and independently in [AJW18].

Before doing so, we recall the definition of generalized divergence $\mathbf{D}(\rho\|\sigma)$ [SW12] of two states ρ and σ as any function from two density operators to the reals that is monotone under the action of a quantum channel $\mathcal{N}$, in the sense that

$$\mathbf{D}(\rho\|\sigma) \geq \mathbf{D}(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)). \tag{3.40}$$

From this, we can define the generalized mutual information of a bipartite state ρ_{AB} as

$$I_{\mathbf{D}}(A;B) \equiv \inf_{\sigma_B} \mathbf{D}(\rho_{AB}\|\rho_A \otimes \sigma_B), \tag{3.41}$$

where the optimization is with respect to a density operator σ_B acting on system B. We then have the following lemma:

Lemma 9 *Let ρ_{ABC} be such that the marginal state ρ_{AC} is product (i.e., $\rho_{AC} = \rho_A \otimes \rho_C$). Then*

$$I_{\mathbf{D}}(A;BC)_\rho \leq I_{\mathbf{D}}(AC;B)_\rho. \tag{3.42}$$

Proof. This follows because

$$I_{\mathbf{D}}(A;BC)_\rho = \inf_{\sigma_{BC}} \mathbf{D}(\rho_{ABC}\|\rho_A \otimes \sigma_{BC}) \tag{3.43}$$

$$\leq \inf_{\sigma_B} \mathbf{D}(\rho_{ABC}\|\rho_A \otimes \sigma_B \otimes \rho_C) \tag{3.44}$$

$$= \inf_{\sigma_B} \mathbf{D}(\rho_{ABC}\|\rho_{AC} \otimes \sigma_B) \tag{3.45}$$

$$= I_{\mathbf{D}}(AC;B)_\rho. \tag{3.46}$$

This concludes the proof. ■

We now apply Lemma 9 in the context of entanglement-assisted communication, to establish an alternate proof of the following upper bound (again emphasizing that a proof along these lines was found recently and independently in [AJW18]):

Proposition 10 ([MW14]) *Let $\mathcal{N}_{A\to B}$ be a quantum channel. For an $(|M|, \varepsilon)$ entanglement-assisted classical communication protocol over the channel $\mathcal{N}_{A\to B}$, the following bound holds*

$$\log_2 |M| \leq I^\varepsilon(\mathcal{N}), \tag{3.47}$$

where the ε-mutual information of $\mathcal{N}_{A\to B}$ is defined as

$$I^\varepsilon(\mathcal{N}) \equiv \max_{\psi_{RA}} \min_{\sigma_B} D^\varepsilon_H(\mathcal{N}_{A\to B}(\psi_{RA})\|\psi_R \otimes \sigma_B), \tag{3.48}$$

with ψ_{RA} a pure bipartite state such that system R isomorphic to system A.

Proof. An entanglement-assisted classical communication protocol begins with the sender preparing the maximally classically correlated state $\overline{\Phi}_{MM'}$, defined as

$$\overline{\Phi}_{MM'} \equiv \frac{1}{|M|} \sum_{m=1}^{|M|} |m\rangle\langle m|_M \otimes |m\rangle\langle m|_{M'}. \tag{3.49}$$

Also, the sender and receiver share an arbitrary entangled state $\Psi_{A'B'}$ before communication begins. The sender then performs an encoding channel $\mathcal{E}_{M'A'\to A}$ on systems M' and A', and the resulting state is

$$\mathcal{E}_{M'A'\to A}(\overline{\Phi}_{MM'} \otimes \Psi_{A'B'}) = \frac{1}{|M|} \sum_{m=1}^{|M|} |m\rangle\langle m|_M \otimes \mathcal{E}_{M'A'\to A}(|m\rangle\langle m|_{M'} \otimes \Psi_{A'B'}). \tag{3.50}$$

Defining the quantum channels $\mathcal{E}^m_{A'\to A}$ by $\mathcal{E}^m_{A'\to A}(\tau_{A'}) \equiv \mathcal{E}_{M'A'\to A}(|m\rangle\langle m|_{M'} \otimes \tau_{A'})$, we can write the above state as

$$\mathcal{E}_{M'A'\to A}(\overline{\Phi}_{MM'} \otimes \Psi_{A'B'}) = \frac{1}{|M|} \sum_{m=1}^{|M|} |m\rangle\langle m|_M \otimes \mathcal{E}^m_{A'\to A}(\Psi_{A'B'}). \tag{3.51}$$

The sender transmits the A system through the channel $\mathcal{N}_{A\to B}$, leading to

$$(\mathcal{N}_{A\to B} \circ \mathcal{E}_{M'A'\to A})(\overline{\Phi}_{MM'} \otimes \Psi_{A'B'}) = \frac{1}{|M|} \sum_{m=1}^{|M|} |m\rangle\langle m|_M \otimes (\mathcal{N}_{A\to B} \circ \mathcal{E}^m_{A'\to A})(\Psi_{A'B'}). \tag{3.52}$$

The receiver's goal is then to determine which message m was transmitted. To do so, he performs a quantum-to-classical or measurement channel $\mathcal{D}_{BB'\to\hat{M}}$, defined by

$$\mathcal{D}_{BB'\to\hat{M}}(\tau_{BB'}) := \sum_m \mathrm{Tr}[\Lambda^m_{BB'}\tau_{BB'}]|m\rangle\langle m|_{\hat{M}}, \tag{3.53}$$

for a POVM $\{\Lambda^m_B\}_{m=1}^{|M|}$, and the state becomes

$$\omega_{M\hat{M}} := (\mathcal{D}_{BB'\to\hat{M}} \circ \mathcal{N}_{A\to B} \circ \mathcal{E}_{M'A'\to A})(\overline{\Phi}_{MM'} \otimes \Psi_{A'B'}) \tag{3.54}$$

$$= \frac{1}{|M|} \sum_{m,m'=1}^{|M|} |m\rangle\langle m|_M \otimes \mathrm{Tr}[\Lambda^{m'}_{BB'}\mathcal{N}_{A\to B}(\mathcal{E}^m_{A'\to A}(\Psi_{A'B'}))]|m'\rangle\langle m'|_{\hat{M}}. \tag{3.55}$$

The protocol is an $(|M|, \varepsilon)$ protocol by definition if the following condition holds

$$1 - \frac{1}{|M|} \sum_m p(m|m) \leq \varepsilon, \tag{3.56}$$

where

$$p(m'|m) := \mathrm{Tr}[\Lambda^{m'}_{BB'}\mathcal{N}_{A\to B}(\mathcal{E}^m_{A'\to A}(\Psi_{A'B'}))]. \tag{3.57}$$

The following equality holds by direct calculation:

$$1 - \frac{1}{|M|} \sum_m p(m|m) = 1 - \mathrm{Tr}[\Pi_{M\hat{M}}\omega_{M\hat{M}}], \tag{3.58}$$

where the comparator test or projection $\Pi_{M\hat{M}}$ is defined as

$$\Pi_{M\hat{M}} := \sum_{m=1}^{|M|} |m\rangle\langle m|_M \otimes |m\rangle\langle m|_{M'}. \tag{3.59}$$

Now, let us apply the error condition in (3.56), the equality in (3.58), and the definition of hypothesis testing relative entropy to conclude that

$$\log_2 |M| \leq \widetilde{I}^{\varepsilon}_H(M; \hat{M})_\omega, \tag{3.60}$$

where $\omega_{M\hat{M}}$ is defined in (3.54). From data processing under the action of the decoding channel $\mathcal{D}_{BB'\to\hat{M}}$, we find that

$$\widetilde{I}^{\varepsilon}_H(M; \hat{M})_\omega \leq \widetilde{I}^{\varepsilon}_H(M; BB')_\theta, \tag{3.61}$$

where the state $\theta_{MBB'}$ is the same as that in (3.52):

$$\theta_{MBB'} := (\mathcal{N}_{A\to B} \circ \mathcal{E}_{M'A'\to A})(\overline{\Phi}_{MM'} \otimes \Psi_{A'B'}). \tag{3.62}$$

Observe that the reduced state $\theta_{MB'}$ is a product state because the channel $\mathcal{N}_{A\to B}$ and encoding $\mathcal{E}_{M'A'\to A}$ are trace preserving:

$$\theta_{MB'} = \mathrm{Tr}_B[(\mathcal{N}_{A\to B} \circ \mathcal{E}_{M'A'\to A})(\overline{\Phi}_{MM'} \otimes \Psi_{A'B'})] \tag{3.63}$$
$$= \mathrm{Tr}_{M'A'}[\overline{\Phi}_{MM'} \otimes \Psi_{A'B'}] \tag{3.64}$$
$$= \overline{\Phi}_M \otimes \Psi_{B'} \tag{3.65}$$
$$= \theta_M \otimes \theta_{B'}. \tag{3.66}$$

Thus, from Lemma 9, we have that

$$\widetilde{I}^{\varepsilon}_H(M; BB')_\theta \leq \widetilde{I}^{\varepsilon}_H(MB'; B)_\theta. \tag{3.67}$$

Consider that the state $\theta_{MB'B}$ has the form $\mathcal{N}_{A\to B}(\rho_{SA})$ for some mixed state ρ_{SA}, by identifying S with MB' and ρ_{SA} with $\mathcal{E}_{M'A'\to A}(\overline{\Phi}_{MM'} \otimes \Psi_{A'B'})$. So we find that

$$\widetilde{I}^{\varepsilon}_H(MB'; B)_\theta \leq \max_{\rho_{SA}} \widetilde{I}^{\varepsilon}_H(S; B)_\xi, \tag{3.68}$$

where

$$\xi_{SB} := \mathcal{N}_{A\to B}(\rho_{SA}). \tag{3.69}$$

Now employing the fact that ρ_{SA} can be purified to $\psi_{S'SA}$, the data processing inequality for the ε-mutual information, and the Schmidt decomposition theorem with respect to the bipartite cut $S'S|A$ to conclude that the Schmidt rank of $\psi_{S'SA}$ is no larger than $|A|$, we conclude the statement of the proposition. ■

4 Classical communication over quantum multiple-access channels

We now establish a link between communication over a quantum multiple-access channel and multiple quantum hypothesis testing. One advantage of this development is the reduction of the communication problem to a hypothesis testing problem, which is perhaps simpler to state and could also be considered a more fundamental problem than the communication problem. Later, in Section 5, we discuss the relation of the quantum simultaneous decoding conjecture from [FHS+12, Wil11] to open questions in multiple quantum hypothesis testing from [AM14, BHOS15] (here we note that the solution of the multiple Chernoff distance conjecture from [Li16] does not evidently allow for the solution of the quantum simultaneous decoding conjecture). We point the reader to [Dut11, DF13] for further discussions and variations of the quantum simultaneous decoding conjecture.

We begin by considering the case of entanglement-assisted communication and later consider unassisted communication. We first define the information-processing task of entanglement-assisted classical communication over quantum multiple-access channels (see also [HDW08, XW13]). Consider the scenario in which two senders Alice and Bob would like to transmit classical messages to a receiver Charlie over a two-sender single-receiver quantum multiple-access channel $\mathcal{N}_{AB\to C}$. Alice and Bob choose their messages from message sets $\mathcal{L}$ and $\mathcal{M}$. The cardinality of the sets $\mathcal{L}$ and $\mathcal{M}$ are denoted as L and M, respectively. Suppose that Alice and Bob each share an arbitrary entangled state with Charlie before communication begins. Let $\Phi_{RA'}$ denote the state shared between Charlie and Alice, and let $\Psi_{SB'}$ denote the state shared between Charlie and Bob.

Let $L, M \in \mathbb{N}$ and $\varepsilon \in [0,1]$. An (L, M, ε) entanglement-assisted multiple-access classical communication code consists of a set $\{\mathcal{E}^l_{A'\to A}, \mathcal{F}^m_{B'\to B}, \Lambda^{l,m}_{RSC}\}_{l,m}$ of encoders and a decoding POVM, such that the average error probability is bounded from above by ε:

$$\frac{1}{LM}\sum_{l=1}^{L}\sum_{m=1}^{M} p_e(l,m) \le \varepsilon \ , \tag{4.1}$$

where the error probability for each message pair is given by

$$p_e(l,m) \equiv \operatorname{Tr}\{(I_{RSC} - \Lambda^{l,m}_{RSC})\mathcal{N}_{AB\to C}(\mathcal{E}^l_{A'\to A}(\Phi_{RA'}) \otimes \mathcal{F}^m_{B'\to B}(\Psi_{SB'}))\} \ . \tag{4.2}$$

Figure 1 depicts the coding task.

4.1 One-shot position-based coding scheme

We now describe and analyze a position-based coding scheme for entanglement-assisted communication over a quantum multiple-access channel, in which the decoding POVM is a quantum simultaneous decoder.

Encoding: Before communication begins, suppose that Alice and Charlie share L copies of the same bipartite state: $\theta_{RA}^{\otimes L} \equiv \theta_{R_1A_1} \otimes \cdots \otimes \theta_{R_LA_L}$. Similarly, suppose that Bob and Charlie share M copies of the same bipartite state: $\gamma_{SB}^{\otimes M} \equiv \gamma_{S_1B_1} \otimes \cdots \otimes \gamma_{S_MB_M}$. To send message $(l,m) \in \mathcal{L}\times\mathcal{M}$, Alice sends the lth A system of $\theta_{RA}^{\otimes L}$ and Bob sends the mth B system of $\theta_{SB}^{\otimes M}$ over the quantum multiple-access channel $\mathcal{N}_{AB\to C}$. So this leads to the following state for Charlie:

$$\rho^{l,m}_{R^LS^MC} = \theta_R^{\otimes(l-1)} \otimes \gamma_S^{\otimes(m-1)} \otimes \mathcal{N}_{AB\to C}(\theta_{R_lA_l} \otimes \gamma_{S_mB_m}) \otimes \theta_R^{\otimes(L-l)} \otimes \gamma_S^{\otimes(M-m)} \ . \tag{4.3}$$

Decoding: To decode the message transmitted, Charlie performs a measurement on the systems R^L, S^M, and the channel output C to determine the message pair (l,m) that Alice and Bob

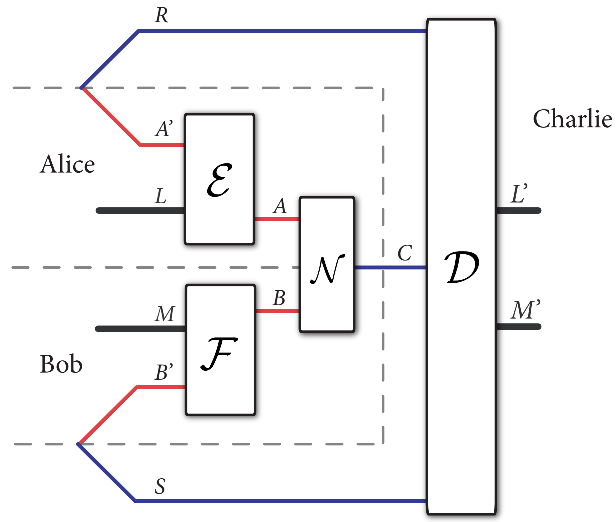

Figure 1: The coding task for entanglement-assisted classical communication over a quantum multiple-access channel. Alice and Bob each share entanglement with the receiver Charlie, and Charlie employs a decoding channel $\mathcal{D}$ to figure out which messages Alice and Bob transmitted.

transmitted. Consider the following measurement operator:

$$\Gamma^{l,m}_{R^L S^M C} \equiv T_{R_l S_m C} \ , \tag{4.4}$$

where identity operators are implicit for all of the R and S systems besides R_l and S_m and T_{RSC} is a measurement operator satisfying $0 \leq T_{RSC} \leq I_{RSC}$. Let us call the action of performing the measurement $\{\Gamma^{l,m}_{R^L S^M C}, I_{R^L S^M C} - \Gamma^{l,m}_{R^L S^M C}\}$ "checking for the message pair (l, m)." If Charlie checks for message pair (l, m) when indeed message pair (l, m) is transmitted, then the probability of incorrectly decoding is

$$\text{Tr}\{(I - \Gamma^{l,m}_{R^L S^M C})\rho^{l,m}_{R^L S^M C}\} = \text{Tr}\{(I - T_{RSC})\mathcal{N}_{AB\to C}(\theta_{RA} \otimes \gamma_{SB})\}. \tag{4.5}$$

The equality above follows by combining (4.3) and (4.4) and applying partial traces. There are three other kinds of error probabilities to consider. If message pair (l, m) was transmitted and Charlie checks for message pair (l', m) for $l' \neq l$, then the probability of decoding as (l', m) is

$$\text{Tr}\{\Gamma^{l',m}_{R^L S^M C}\rho^{l,m}_{R^L S^M C}\} = \text{Tr}\{T_{RSC}\mathcal{N}_{AB\to C}(\theta_R \otimes \theta_A \otimes \gamma_{SB})\}. \tag{4.6}$$

If message pair (l, m) was transmitted and Charlie checks for message pair (l, m') for $m' \neq m$, then the probability of decoding as (l, m') is

$$\text{Tr}\{\Gamma^{l,m'}_{R^L S^M C}\rho^{l,m}_{R^L S^M C}\} = \text{Tr}\{T_{RSC}\mathcal{N}_{AB\to C}(\theta_{RA} \otimes \gamma_S \otimes \gamma_B)\}. \tag{4.7}$$

If message pair (l, m) was transmitted and Charlie checks for message pair (l', m') for $l' \neq l$ and $m' \neq m$, then the probability of decoding as (l', m') is

$$\text{Tr}\{\Gamma^{l',m'}_{R^L S^M C}\rho^{l,m}_{R^L S^M C}\} = \text{Tr}\{T_{RSC}\mathcal{N}_{AB\to C}(\theta_R \otimes \theta_A \otimes \gamma_S \otimes \gamma_B)\}. \tag{4.8}$$

The above observations are helpful in the forthcoming error analysis.

We now take Charlie's position-based quantum simultaneous decoder to be the following square-root measurement:

$$\Lambda^{l,m}_{R^L S^M C} \equiv \left(\sum_{l'=1}^{L}\sum_{m'=1}^{M}\Gamma^{l',m'}_{R^L S^M C}\right)^{-1/2} \Gamma^{l,m}_{R^L S^M C} \left(\sum_{l'=1}^{L}\sum_{m'=1}^{M}\Gamma^{l',m'}_{R^L S^M C}\right)^{-1/2} . \tag{4.9}$$

Error analysis: Due to the code construction, the error probability under the position-based coding scheme is the same for each message pair (l, m):

$$p_e(l, m) = \mathrm{Tr}\{(I - \Lambda^{l,m}_{R^L S^M C})\rho^{l,m}_{R^L S^M C}\} . \tag{4.10}$$

Applying Lemma 5 and (4.5)–(4.8), we arrive at the following upper bound on the decoding error probability:

$$\begin{aligned} p_e(l, m) &\leq c_{\mathrm{I}}\, \mathrm{Tr}\{(I - \Gamma^{l,m}_{R^L S^M C})\rho^{l,m}_{R^L S^M C}\} + c_{\mathrm{II}} \sum_{(l',m')\neq(l,m)} \mathrm{Tr}\{\Gamma^{l',m'}_{R^L S^M C}\rho^{l,m}_{R^L S^M C}\} && (4.11)\\ &= c_{\mathrm{I}}\, \mathrm{Tr}\{(I - \Gamma^{l,m}_{R^L S^M C})\rho^{l,m}_{R^L S^M C}\} + c_{\mathrm{II}} \sum_{l'\neq l} \mathrm{Tr}\{\Gamma^{l',m}_{R^L S^M C}\rho^{l,m}_{R^L S^M C}\} \\ &\quad + c_{\mathrm{II}} \sum_{m'\neq m} \mathrm{Tr}\{\Gamma^{l,m'}_{R^L S^M C}\rho^{l,m}_{R^L S^M C}\} + c_{\mathrm{II}} \sum_{l'\neq l, m'\neq m} \mathrm{Tr}\{\Gamma^{l',m'}_{R^L S^M C}\rho^{l,m}_{R^L S^M C}\} && (4.12)\\ &= c_{\mathrm{I}}\, \mathrm{Tr}\{(I - T_{RSC})\mathcal{N}_{AB\to C}(\theta_{RA}\otimes\gamma_{SB})\} + c_{\mathrm{II}}\big[(L-1)\,\mathrm{Tr}\{T_{RSC}\mathcal{N}_{AB\to C}(\theta_R\otimes\theta_A\otimes\gamma_{SB})\} \\ &\quad + (M-1)\,\mathrm{Tr}\{T_{RSC}\mathcal{N}_{AB\to C}(\theta_{RA}\otimes\gamma_S\otimes\gamma_B)\} \\ &\quad + (L-1)(M-1)\,\mathrm{Tr}\{T_{RSC}\mathcal{N}_{AB\to C}(\theta_R\otimes\theta_A\otimes\gamma_S\otimes\gamma_B)\}\big] . && (4.13) \end{aligned}$$

The error probability associated with c_{I} is the probability of incorrectly decoding when Charlie checks for message pair (l, m). The error probabilities associated with c_{II} are the probabilities of decoding as some other message pair when message pair (l, m) is transmitted. There are $L - 1$ possibilities for Charlie to erroneously decode Alice's message and correctly decode Bob's message, $M - 1$ possibilities to erroneously decode Bob's message and correctly decode Alice's message, and $(M - 1)(L - 1)$ possibilities of incorrectly decoding both Alice and Bob's messages.

One-shot bound for quantum simultaneous decoding. Thus, our bound on the decoding error probability for a position-based entanglement-assisted coding scheme is as follows:

$$\begin{aligned} p_e(l, m) \leq c_{\mathrm{I}}\, \mathrm{Tr}\{(I - T_{RSC})\mathcal{N}_{AB\to C}(\theta_{RA}\otimes\gamma_{SB})\} + c_{\mathrm{II}}\big[(L-1)\,\mathrm{Tr}\{T_{RSC}\mathcal{N}_{AB\to C}(\theta_R\otimes\theta_A\otimes\gamma_{SB})\} \\ + (M-1)\,\mathrm{Tr}\{T_{RSC}\mathcal{N}_{AB\to C}(\theta_{RA}\otimes\gamma_S\otimes\gamma_B)\} \\ + (L-1)(M-1)\,\mathrm{Tr}\{T_{RSC}\mathcal{N}_{AB\to C}(\theta_R\otimes\theta_A\otimes\gamma_S\otimes\gamma_B)\}\big] . \end{aligned} \tag{4.14}$$

Interestingly, this bound is the same for all message pairs, and thus holds for maximal or average error probability. We also remark that the above inequality forges a transparent link between communication over multiple-access channels and multiple quantum hypothesis testing, a point that we will return to in Section 5.

Generalization to multiple senders. The above bound can be extended as follows for an entanglement-assisted quantum multiple-access channel $\mathcal{N}_{A_1\cdots A_K\to C}$ with K senders and a single receiver:

$$p_e(m_1,\ldots,m_K) \leq c_{\rm I} \operatorname{Tr}\left\{(I - T_{R_1\cdots R_K C})\mathcal{N}_{A_1\cdots A_K\to C}\left(\bigotimes_{k=1}^{K}\theta_{R_k A_k}\right)\right\}$$
$$+ c_{\rm II}\sum_{\mathcal{J}\subseteq[K]}\left[\prod_{j\in\mathcal{J}}(M_j-1)\right]\operatorname{Tr}\left\{T_{R_1\cdots R_K C}\mathcal{N}_{A_1\cdots A_K\to C}\left(\bigotimes_{j\in\mathcal{J}}\theta_{R_j}\otimes\theta_{A_j}\otimes\bigotimes_{l\in\mathcal{J}^c}\theta_{R_l A_l}\right)\right\}. \quad (4.15)$$

In the above, $m_1,\ldots,m_K$ are the messages for senders 1 through K, respectively, chosen from respective message sets of size $M_1,\ldots,M_K$. The states $\theta_{R_1A_1}$, $\ldots$, $\theta_{R_KA_K}$ are entangled states shared between the receiver and senders 1 through K, with the receiver possessing all of the R systems. Finally, $T_{R_1\cdots R_K C}$ is a test operator satisfying $0 \leq T_{R_1\cdots R_K C} \leq I_{R_1\cdots R_K C}$ and $\mathcal{J}$ is a non-empty subset of $[K] \equiv \{1,\ldots,K\}$. The above bound is derived by using position-based coding as described above and a square-root measurement that generalizes (4.9). We omit the straightforward proof.

4.2 Unassisted classical communication over multiple-access channels

The position-based coding technique is not only a powerful tool for entanglement-assisted classical communication protocols, but also for those that do not employ entanglement assistance or any other kind of assistance. This was shown explicitly for the single-sender, single-receiver case in [Wil17a]. We now demonstrate this point further by considering unassisted classical communication over a classical-input quantum-output multiple-access channel $\mathcal{N}_{XY\to C}$. We do so by first considering classical communication assisted by shared randomness, such that we can employ a position-based coding scheme, and then we derandomize the protocol to obtain a codebook for unassisted communication.

The classical-classical-quantum channel that we consider can be written in fully quantum form as

$$\mathcal{N}_{X'Y'\to C}(\omega_{X'Y'}) = \sum_{x,y}\langle x|_{X'}\langle y|_{Y'}\omega_{X'Y'}|x\rangle_{X'}|y\rangle_{Y'}\ \rho_C^{x,y}\ . \quad (4.16)$$

Before communication begins, Alice and Charlie share randomness in the form of the following classical–classical state:

$$\rho_{XX'} \equiv \sum_x p_X(x)|x\rangle\langle x|_X \otimes |x\rangle\langle x|_{X'}\ . \quad (4.17)$$

Similarly Bob and Charlie also share randomness represented by the following classical–classical state:

$$\sigma_{YY'} \equiv \sum_y p_Y(y)|y\rangle\langle y|_Y \otimes |y\rangle\langle y|_{Y'}\ . \quad (4.18)$$

We demonstrate the procedure of derandomization by proving the following theorem:

Theorem 11 *There exists an unassisted, simultaneous decoding protocol for classical communication over a classical-input quantum-output quantum multiple-access channel with the following* *upper bound on its average error probability, holding for all T_{XYC} such that $0 \leq T_{XYC} \leq I_{XYC}$:*

$$\frac{1}{LM}\sum_{l,m} p_e(l,m) \leq c_{\mathrm{I}} \operatorname{Tr}\{(I - T_{XYC})\mathcal{N}_{X'Y'\to C}(\rho_{X'X} \otimes \sigma_{YY'})\}$$
$$+ c_{\mathrm{II}}\big[(L-1)\operatorname{Tr}\{T_{XYC}\mathcal{N}_{X'Y'\to C}(\rho_X \otimes \rho_{X'} \otimes \sigma_{YY'})\}$$
$$+ (M-1)\operatorname{Tr}\{T_{XYC}\mathcal{N}_{X'Y'\to C}(\rho_{XX'} \otimes \sigma_Y \otimes \sigma_{Y'})\}$$
$$+ (L-1)(M-1)\operatorname{Tr}\{T_{XYC}\mathcal{N}_{X'Y'\to C}(\rho_X \otimes \rho_{X'} \otimes \sigma_Y \otimes \sigma_{Y'})\}\big] , \quad (4.19)$$

where L is the number of messages for the first sender and M is the number of messages for the second sender. A generalization of this statement holds for multiple senders, with an upper bound on the average error probability given by the right-hand side of (4.15), but with all of the R and A systems being classical.

Proof. The position-based coding scheme operates exactly as specified in Section 4.1, with the states θ_{RA} and γ_{SB} replaced by $\rho_{XX'}$ and $\sigma_{YY'}$, respectively, the channel $\mathcal{N}_{AB\to C}$ replaced by $\mathcal{N}_{X'Y'\to C}$, and the test operator T_{RSC} replaced by T_{XYC}. The same error analysis then leads to the following bound on the error probability when decoding the message pair (l,m):

$$p_e(l,m) \leq c_{\mathrm{I}} \operatorname{Tr}\{(I - T_{XYC})\mathcal{N}_{X'Y'\to C}(\rho_{XX'} \otimes \sigma_{YY'})\}$$
$$+ c_{\mathrm{II}}\big[(L-1)\operatorname{Tr}\{T_{XYC}\mathcal{N}_{X'Y'\to C}(\rho_X \otimes \rho_{X'} \otimes \sigma_{YY'})\}$$
$$+ (M-1)\operatorname{Tr}\{T_{XYC}\mathcal{N}_{X'Y'\to C}(\rho_{XX'} \otimes \sigma_Y \otimes \sigma_{Y'})\}$$
$$+ (L-1)(M-1)\operatorname{Tr}\{T_{XYC}\mathcal{N}_{X'Y'\to C}(\rho_X \otimes \rho_{X'} \otimes \sigma_Y \otimes \sigma_{Y'})\}\big] . \quad (4.20)$$

Derandomization: Extending the development in [Wil17a], first notice that we can rewrite the four trace terms in (4.20) as follows:

$$\operatorname{Tr}\{T_{XYC}\mathcal{N}_{X'Y'\to C}(\rho_{XX'} \otimes \sigma_{YY'})\} = \operatorname{Tr}\{T_{XYC}\sum_{x,y} p_X(x)p_Y(y)|xy\rangle\langle xy| \otimes \rho_C^{x,y}\} \quad (4.21)$$
$$= \sum_{x,y} p_X(x)p_Y(y)\operatorname{Tr}\{Q_C^{x,y}\rho_C^{x,y}\} , \quad (4.22)$$
$$\operatorname{Tr}\{T_{XYC}\mathcal{N}_{X'Y'\to C}(\rho_X \otimes \rho_{X'} \otimes \sigma_{YY'})\} = \sum_{x,y} p_X(x)p_Y(y)\operatorname{Tr}\{Q_C^{x,y}\bar{\rho}_C^{y}\} , \quad (4.23)$$
$$\operatorname{Tr}\{T_{XYC}\mathcal{N}_{X'Y'\to C}(\rho_{XX'} \otimes \sigma_Y \otimes \sigma_{Y'})\} = \sum_{x,y} p_X(x)p_Y(y)\operatorname{Tr}\{Q_C^{x,y}\bar{\rho}_C^{x}\} , \quad (4.24)$$
$$\operatorname{Tr}\{T_{XYC}\mathcal{N}_{X'Y'\to C}(\rho_X \otimes \rho_{X'} \otimes \sigma_Y \otimes \sigma_{Y'})\} = \sum_{x,y} p_X(x)p_Y(y)\operatorname{Tr}\{Q_C^{x,y}\bar{\rho}_C\} , \quad (4.25)$$

where we define the following averaged output states:

$$\bar{\rho}_C \equiv \sum_{x,y} p_X(x)p_Y(y)\rho_C^{x,y}, \quad (4.26)$$
$$\bar{\rho}_C^{y} \equiv \sum_{x} p_X(x)\rho_C^{x,y}, \quad (4.27)$$
$$\bar{\rho}_C^{x} \equiv \sum_{y} p_Y(y)\rho_C^{x,y}, \quad (4.28)$$

and the measurement operator

$$Q_C^{x,y} \equiv \langle x, y|_{XY} T_{XYC} |x, y\rangle_{XY} \ . \tag{4.29}$$

Thus, in the case that the code is randomness-assisted, it suffices to take the test operator T_{XYC} to have the following form:

$$T_{XYC} = \sum_{x,y} |x\rangle\langle x|_X \otimes |y\rangle\langle y|_Y \otimes Q_C^{x,y}, \tag{4.30}$$

because, as we will show, the upper bound on the average error probability does not change when doing so. Then we can rewrite the decoding POVM in (4.9) as follows:

$$\Gamma^{l,m}_{X^L Y^M C} \equiv T_{X_l Y_m C} \tag{4.31}$$

$$= \sum_{x^L, y^M} |x^L\rangle\langle x^L|_{X^L} \otimes |y^M\rangle\langle y^M|_{Y^M} \otimes Q_C^{x_l, y_m} \ , \tag{4.32}$$

where we use the resolution of the identity $I_X = \sum_x |x\rangle\langle x|_X$ to expand the implicit identity operators and we employ the notation $x^L \equiv x_1 \cdots x_L$ and $y^M \equiv y_1 \cdots y_M$. Then this implies that

$$\left(\sum_{l'=1}^{L} \sum_{m'=1}^{M} \Gamma^{l',m'}_{X^L Y^M C} \right)^{-1/2} = \left(\sum_{l'=1}^{L} \sum_{m'=1}^{M} \sum_{x^L, y^M} |x^L\rangle\langle x^L|_{X^L} \otimes |y^M\rangle\langle y^M|_{Y^M} \otimes Q_C^{x_{l'}, y_{m'}} \right)^{-1/2} \tag{4.33}$$

$$= \left(\sum_{x^L, y^M} |x^L\rangle\langle x^L|_{X^L} \otimes |y^M\rangle\langle y^M|_{Y^M} \otimes \sum_{l'=1}^{L} \sum_{m'=1}^{M} Q_C^{x_{l'}, y_{m'}} \right)^{-1/2} \tag{4.34}$$

$$= \sum_{x^L, y^M} |x^L\rangle\langle x^L|_{X^L} \otimes |y^M\rangle\langle y^M|_{Y^M} \otimes \left(\sum_{l'=1}^{L} \sum_{m'=1}^{M} Q_C^{x_{l'}, y_{m'}} \right)^{-1/2} . \tag{4.35}$$

The last step follows from the fact that $\{|x\rangle\}_x$ and $\{|y\rangle\}_y$ form orthonormal bases. Therefore, the decoding POVM for the randomness-assisted protocol can be decomposed as

$$\Lambda^{l,m}_{X^L Y^M C} \equiv \left(\sum_{l'=1}^{L} \sum_{m'=1}^{M} \Gamma^{l',m'}_{X^L Y^M C} \right)^{-1/2} \Gamma^{l,m}_{X^L Y^M C} \left(\sum_{l'=1}^{L} \sum_{m'=1}^{M} \Gamma^{l',m'}_{X^L Y^M C} \right)^{-1/2} , \tag{4.36}$$

$$= \sum_{x^L, y^M} |x^L\rangle\langle x^L|_{X^L} \otimes |y^M\rangle\langle y^M|_{Y^M} \otimes \Omega_C^{x_l, y_m} \ , \tag{4.37}$$

where

$$\Omega_C^{x_l, y_m} \equiv \left(\sum_{l'=1}^{L} \sum_{m'=1}^{M} Q_C^{x_{l'}, y_{m'}} \right)^{-1/2} Q_C^{x_l, y_m} \left(\sum_{l'=1}^{L} \sum_{m'=1}^{M} Q_C^{x_{l'}, y_{m'}} \right)^{-1/2} . \tag{4.38}$$

By definition, the output state of Charlie in (4.3), for our case of interest, can be written as

$$\rho^{l,m}_{X^L Y^M C} = \sum_{x^L, y^M} p_{X^L}(x^L) p_{Y^M}(y^M) |x^L\rangle\langle x^L|_{X^L} \otimes |y^M\rangle\langle y^M|_{Y^M} \otimes \rho_C^{x_l, y_m} \ . \tag{4.39}$$

By combining (4.37) and (4.39), we find that the average error probability for the code can be rewritten as

$$\frac{1}{LM}\sum_{l=1}^{L}\sum_{m=1}^{M}\text{Tr}\{(I_{X^LY^MC}-\Lambda^{l,m}_{X^LY^MC})\rho^{l,m}_{X^LY^MC}\}$$

$$=\frac{1}{LM}\sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{x^L,y^M}p_{X^L}(x^L)p_{Y^M}(y^M)\,\text{Tr}\{(I_C-\Omega^{x_l,y_m}_C)\rho^{x_l,y_m}_C\} \tag{4.40}$$

$$=\sum_{x^L,y^M}p_{X^L}(x^L)p_{Y^M}(y^M)\left[\frac{1}{LM}\sum_{l=1}^{L}\sum_{m=1}^{M}\text{Tr}\{(I_C-\Omega^{x_l,y_m}_C)\rho^{x_l,y_m}_C\}\right]. \tag{4.41}$$

Suppose now that there exists a randomness-assisted position-based code that has an average error probability $\leq\varepsilon$. By the above equalities and since the average can never exceed the maximum, we know there must exist a particular choice of x^L,y^L such that

$$\frac{1}{LM}\sum_{l=1}^{L}\sum_{m=1}^{M}\text{Tr}\{(I-\Omega^{x_l,y_m}_C)\rho^{x_l,y_m}_C\}\leq\varepsilon\ . \tag{4.42}$$

Thus for an unassisted multiple-access classical communication protocol, if we choose $\{x_l\}_{l=1}^L$ as Alice's codebook and $\{y_m\}_{m=1}^M$ as Bob's codebook, and the POVM $\{\Omega^{x_l,y_m}_c\}$ as Charlie's decoder, an upper bound on the average probability error is given by

$$\frac{1}{LM}\sum_{l=1}^{L}\sum_{m=1}^{M}p_e(l,m)=\frac{1}{LM}\sum_{l=1}^{L}\sum_{m=1}^{M}\text{Tr}\{(I-\Omega^{x_l,y_m}_C)\rho^{x_l,y_m}_C\}\leq\varepsilon\ . \tag{4.43}$$

This proves the statement of the theorem after considering the upper bound in (4.20). ■

4.3 Achievable rate region for i.i.d. case

We now demonstrate rates that are achievable when using a particular quantum simultaneous decoder. Interestingly, we show how the same quantum simultaneous decoder leads to two generally different bounds for the achievable rate region. In the first one, the rates are bounded by terms which consist of the difference of a Rényi entropy of order two and a conditional quantum entropy. In the second one, the rates are bounded by terms which consist of the difference of a collision conditional entropy and a conditional quantum entropy. Although these rates are suboptimal when compared to what is achievable by using successive decoding [Win01, HDW08], we nevertheless think that the following coding theorems represent progress toward finding a quantum simultaneous decoder.

Before we state the theorems, we require the following definition: A rate pair (R_1,R_2) is achievable for communication over a quantum multiple access channel if there exists a $(2^{n[R_1-\delta]},2^{n[R_2-\delta]},\varepsilon)$ code for communication over $\mathcal{N}^{\otimes n}_{A'B'\to C}$ for all $\varepsilon\in(0,1)$, $\delta>0$, and sufficiently large n.

Theorem 12 *An achievable rate region for entanglement-assisted classical communication over a two-sender quantum multiple-access channel $\mathcal{N}_{AB\to C}$, by employing a quantum simultaneous de-*

coder, is as follows:

$$R_1 \leq \tilde{I}(S;CR)_\omega \ , \tag{4.44}$$

$$R_2 \leq \tilde{I}(R;CS)_\omega \ , \tag{4.45}$$

$$R_1 + R_2 \leq \tilde{I}(RC;S)_\omega \ , \tag{4.46}$$

where $\omega_{RSC} \equiv \mathcal{N}_{AB\to C}(\theta_{RA} \otimes \gamma_{SB})$ *and* θ_{RA} *and* γ_{SB} *are quantum states. Here we define the following mutual-information-like quantities:*

$$\tilde{I}(R;CS)_\omega \equiv H_2(SC)_\omega - H(SC|R)_\omega \ , \tag{4.47}$$

$$\tilde{I}(S;CR)_\omega \equiv H_2(RC)_\omega - H(RC|S)_\omega \ , \tag{4.48}$$

$$\tilde{I}(RS;C)_\omega \equiv H_2(C)_\omega - H(C|RS)_\omega \ , \tag{4.49}$$

where $H_2(A)_\rho \equiv -\log_2 \operatorname{Tr}\{\rho_A^2\}$ *is the Rényi entropy of order two.*

Proof. In our setting, Alice and Bob use an i.i.d. channel $\mathcal{N}_{A'B'\to C}^{\otimes n}$. In order to bound the error probability, we replace each state in (4.14) by its n-copy version. Defining $\omega_{RSC} \equiv \mathcal{N}_{AB\to C}(\theta_{RA} \otimes \gamma_{SB})$, we choose the test operator T to be the following 'coated' typical projector:

$$T_{R^nS^nC^n} \equiv (\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta})\Pi_{R^nS^nC^n}^{\omega,\delta}(\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta}) \ , \tag{4.50}$$

where $\Pi_{R^n}^{\omega,\delta}$, $\Pi_{S^n}^{\omega,\delta}$, and $\Pi_{R^nS^nC^n}^{\omega,\delta}$ are the typical projectors corresponding to the respective states ω_R, ω_S, and ω_{RSC}. Applying (4.14), we find the following upper bound on the error probability when decoding the message pair (l,m):

$$\begin{aligned} p_e(l,m) \leq c_{\mathrm{I}} \operatorname{Tr}\{(I - T_{R^nS^nC^n})\omega_{RSC}^{\otimes n}\} + c_{\mathrm{II}}\big[L \operatorname{Tr}\{T_{R^nS^nC^n}[\mathcal{N}_{AB\to C}(\theta_R \otimes \theta_A \otimes \gamma_{SB})]^{\otimes n})\} \\ + M \operatorname{Tr}\{T_{R^nS^nC^n}[\mathcal{N}_{AB\to C}(\theta_{RA} \otimes \gamma_S \otimes \gamma_B)]^{\otimes n}\} \\ + LM \operatorname{Tr}\{T_{R^nS^nC^n}[\mathcal{N}_{AB\to C}(\theta_R \otimes \theta_A \otimes \gamma_S \otimes \gamma_B)]^{\otimes n}\}\big] \ . \end{aligned} \tag{4.51}$$

We evaluate each term sequentially. To give an upper bound on the first term, consider the following chain of inequalities, which holds for sufficiently large n:

$$\begin{aligned} &\operatorname{Tr}\{(\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta})\Pi_{R^nS^nC^n}^{\omega,\delta}(\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta})\omega_{RSC}^{\otimes n}\} \\ &\geq \operatorname{Tr}\{\Pi_{R^nS^nC^n}^{\omega,\delta}\omega_{RSC}^{\otimes n}\} - \left\|(\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta})\omega_{RSC}^{\otimes n}(\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta}) - \omega_{RSC}^{\otimes n}\right\|_1 && (4.52) \\ &\geq 1 - \varepsilon - 2\sqrt{2\varepsilon} \ . && (4.53) \end{aligned}$$

The first inequality follows from Lemma 2. The second inequality follows from (2.21), [HDW08, Eq. (81)], and the application of Lemma 1. We then obtain an upper bound on the first term:

$$\operatorname{Tr}\{(I - T_{R^nS^nC^n})\omega_{RSC}^{\otimes n}\} \leq \varepsilon + 2\sqrt{2\varepsilon} \ . \tag{4.54}$$

Now we consider the second term in (4.51):

$$
\begin{aligned}
& L \operatorname{Tr}\{(\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta}) \Pi_{R^n S^n C^n}^{\omega,\delta} (\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta}) [\mathcal{N}_{AB\to C}(\theta_R \otimes \theta_A \otimes \gamma_{SB})]^{\otimes n})\} \\
& \leq L\ 2^{n[H(RSC)_\omega+\delta]} \operatorname{Tr}\{(\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta}) \omega_{RSC}^{\otimes n} (\Pi_{R^n}^{\omega,\delta} \otimes \Pi_{S^n}^{\omega,\delta}) \theta_R^{\otimes n} \otimes [\mathcal{N}_{AB\to C}(\theta_A \otimes \gamma_{SB})]^{\otimes n})\} && (4.55) \\
& = L\ 2^{n[H(RSC)_\omega+\delta]} \operatorname{Tr}\{\Pi_{S^n}^{\omega,\delta} \omega_{RSC}^{\otimes n} \Pi_{S^n}^{\omega,\delta} \left[(\Pi_{R^n}^{\omega,\delta} \theta_R^{\otimes n} \Pi_{R^n}^{\omega,\delta}) \otimes [\mathcal{N}_{AB\to C}(\theta_A \otimes \gamma_{SB})]^{\otimes n}\right]\} && (4.56) \\
& \leq L\ 2^{n[H(RSC)_\omega+\delta]} 2^{-n[H(R)_\omega-\delta]} \operatorname{Tr}\{\Pi_{S^n}^{\omega,\delta} \omega_{RSC}^{\otimes n} \Pi_{S^n}^{\omega,\delta} \left[I_{R^n} \otimes [\mathcal{N}_{AB\to C}(\theta_A \otimes \gamma_{SB})]^{\otimes n}\right]\} && (4.57) \\
& \leq L\ 2^{n[H(SC|R)_\omega+2\delta]} \operatorname{Tr}\{\Pi_{S^n}^{\omega,\delta} \omega_{SC}^{\otimes n} \Pi_{S^n}^{\omega,\delta} \omega_{SC}^{\otimes n}\} && (4.58) \\
& \leq L\ 2^{n[H(SC|R)_\omega+2\delta]} \operatorname{Tr}\{(\omega_{SC}^{\otimes n})^2\} && (4.59) \\
& = 2^{nR_1} 2^{n[H(SC|R)_\omega+2\delta]} 2^{-nH_2(SC)_\omega} && (4.60) \\
& = 2^{-n[H_2(SC)_\omega - H(SC|R)_\omega - R_1 - 2\delta]}\ . && (4.61)
\end{aligned}
$$

The first inequality follows from the application of the projector trick inequality from (2.24) to the state $\omega_{RSC}^{\otimes n}$. The first equality follows from cyclicity of trace. The second inequality follows from the right-hand side of (2.23), the fact that $\theta_R = \omega_R$, and the inequality $\Pi_{R^n}^{\omega,\delta} \leq I_{R^n}$. The third inequality follows from a partial trace over the R^n systems. The fourth inequality follows because

$$
\operatorname{Tr}\{\Pi_{S^n}^{\omega,\delta} \omega_{SC}^{\otimes n} \Pi_{S^n}^{\omega,\delta} \omega_{SC}^{\otimes n}\} \leq \operatorname{Tr}\{\omega_{SC}^{\otimes n} \Pi_{S^n}^{\omega,\delta} \omega_{SC}^{\otimes n}\} = \operatorname{Tr}\{(\omega_{SC}^{\otimes n})^2 \Pi_{S^n}^{\omega,\delta}\} \leq \operatorname{Tr}\{(\omega_{SC}^{\otimes n})^2\}, \qquad (4.62)
$$

which is a consequence of the facts that $\Pi_{S^n}^{\omega,\delta} \leq I_{S^n}$ and $\omega_{SC}^{\otimes n} \Pi_{S^n}^{\omega,\delta} \omega_{SC}^{\otimes n}$ and $(\omega_{SC}^{\otimes n})^2$ are positive semi-definite. Finally, the second equality follows from the fact $L = 2^{nR_1}$ and the definition of Rényi entropy of order two.

Following a similar analysis, we obtain the following upper bounds for the other two terms:

$$
M \operatorname{Tr}\{T_{R^n S^n C^n} [\mathcal{N}_{AB\to C}(\theta_{RA} \otimes \gamma_S \otimes \gamma_B)]^{\otimes n}\} \leq 2^{-n[H_2(RC)_\omega - H(RC|S)_\omega - R_2 - 2\delta]}\ , \qquad (4.63)
$$

$$
LM \operatorname{Tr}\{T_{R^n S^n C^n} [\mathcal{N}_{AB\to C}(\theta_R \otimes \theta_A \otimes \gamma_S \otimes \gamma_B)]^{\otimes n}\} \leq 2^{-n[H_2(C)_\omega - H(C|RS)_\omega - (R_1+R_2) - 3\delta]}\ . \qquad (4.64)
$$

Taking the sum of the upper bounds for the above four terms, we find the following upper bound on the error probability when decoding the message pair (l, m):

$$
\begin{aligned}
p_e(l,m) \leq \varepsilon + 2\sqrt{2\varepsilon} + 2^{-n[H_2(SC)_\omega - H(SC|R)_\omega - R_1 - 2\delta]} \\
+ 2^{-n[H_2(RC)_\omega - H(RC|S)_\omega - R_2 - 2\delta]} + 2^{-n[H_2(C)_\omega - H(C|RS)_\omega - (R_1+R_2) - 3\delta]}\ . \qquad (4.65)
\end{aligned}
$$

Thus, if the rate pair (R_1, R_2) satisfies the following inequalities (related to those in the statement of the theorem)

$$
\begin{aligned}
R_1 + 3\delta &\leq \tilde{I}(S; CR)_\omega\ , && (4.66) \\
R_2 + 3\delta &\leq \tilde{I}(R; CS)_\omega\ , && (4.67) \\
R_1 + R_2 + 4\delta &\leq \tilde{I}(RC; S)_\omega\ , && (4.68)
\end{aligned}
$$

then the error probability can be made arbitrarily small with increasing n. However, since δ can be taken arbitrarily small after the limit of large n, we conclude that the rate region given in the statement of the theorem is achievable. ■

Our results can be easily extended to the multiple-sender scenario, which we state below without explicitly writing down a proof (note that the proof is a straightforward generalization of the above analysis for two senders).

Theorem 13 *An achievable rate region for entanglement-assisted classical communication over a K-sender multiple-access quantum channel $\mathcal{N}_{A_1A_2,\ldots A_K\to C}$, by employing a quantum simultaneous decoder, is given by the following:*

$$\sum_{j\in\mathcal{J}} R_j \leq \tilde{I}(S(\mathcal{J});CS(\mathcal{J}^c))_\omega, \qquad \textit{for every } \mathcal{J}\subseteq[K]\ , \tag{4.69}$$

where $\omega_{S_1\cdots S_K C}\equiv\mathcal{N}_{A_1\cdots A_K\to C}(\theta_{S_1A_1}\otimes\cdots\otimes\theta_{S_KA_K})$. Here we define mutual-information-like quantities

$$\tilde{I}(S(\mathcal{J});CS(\mathcal{J}^c))_\omega \equiv H_2(S(\mathcal{J}^c)C)_\omega - H(S(\mathcal{J}^c)C|S(\mathcal{J}))_\omega\ , \tag{4.70}$$

where $H_2(B)_\rho = -\log_2 \operatorname{Tr}\{\rho_B^2\}$ is the Rényi entropy of order two.

By combining Theorems 11 and 13, we obtain the following rate region that is achievable for unassisted classical communication over a quantum multiple-access channel when using a quantum simultaneous decoder:

Theorem 14 *The following rate region is achievable when using a quantum simultaneous decoder for unassisted classical communication over the K-sender, classical-input quantum-output multiple-access channel $x_1,\ldots,x_K\to\rho_{x_1,\ldots,x_K}$:*

$$\sum_{j\in\mathcal{J}} R_j \leq \tilde{I}(X(\mathcal{J});CX(\mathcal{J}^c))_\omega, \qquad \textit{for every } \mathcal{J}\subseteq[K]\ , \tag{4.71}$$

where

$$\omega_{X_1\cdots X_K C}\equiv\sum_{x_1,\ldots,x_K} p_{X_1}(x_1)\cdots p_{X_K}(x_K)|x_1\rangle\langle x_1|_{X_1}\otimes\cdots\otimes|x_K\rangle\langle x_K|_{X_K}\otimes\rho_{x_1,\ldots,x_K}. \tag{4.72}$$

The following alternative achievable rate region is generally different from the one in Theorem 13:

Theorem 15 *An achievable rate region for entanglement-assisted classical communication over the K-sender quantum multiple-access channel $\mathcal{N}_{A_1A_2,\ldots A_K\to C}$, by employing a quantum simultaneous decoder, is given by the following:*

$$\sum_{j\in\mathcal{J}} R_j \leq I'(S(\mathcal{J});CS(\mathcal{J}^c))_\omega, \qquad \textit{for every } \mathcal{J}\subseteq[K]\ , \tag{4.73}$$

where $\omega_{S_1\cdots S_K C}\equiv\mathcal{N}_{A_1\cdots A_K\to C}(\theta_{S_1A_1}\otimes\cdots\otimes\theta_{S_KA_K})$. Here we define mutual-information-like quantities

$$I'(S(\mathcal{J});CS(\mathcal{J}^c))_\omega \equiv H_2(C|S(\mathcal{J}^c))_\omega - H(C|S_1\cdots S_K)_\omega\ , \tag{4.74}$$

where $H_2(A|B)_\rho = -\log_2\operatorname{Tr}\{\rho_{AB}\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2}\}$ is the collision conditional entropy [DFW15].

Proof. We prove this theorem for the case of two senders, and then the extension to three or more senders is straightforward. The analysis proceeds similarly as in the proof of Theorem 12, and we pick up at (4.58). We find that

$$
\begin{aligned}
& L\ 2^{n[H(SC|R)_\omega+2\delta]} \operatorname{Tr}\{\Pi^{\omega,\delta}_{S^n}\omega^{\otimes n}_{SC}\Pi^{\omega,\delta}_{S^n}\omega^{\otimes n}_{SC}\} \\
& \leq L\ 2^{n[H(SC|R)_\omega+2\delta]}2^{-n[H(S)_\omega-\delta]} \operatorname{Tr}\{(\omega^{\otimes n}_S)^{-1/2}\omega^{\otimes n}_{SC}(\omega^{\otimes n}_S)^{-1/2}\omega^{\otimes n}_{SC}\} && (4.75)\\
& = 2^{nR_1}2^{n[H(C|RS)_\omega+3\delta]}\left[\operatorname{Tr}\{\omega_S^{-1/2}\omega_{SC}\omega_S^{-1/2}\omega_{SC}\}\right]^n && (4.76)\\
& = 2^{-n[H_2(C|S)_\omega-H(C|RS)_\omega-R_1-3\delta]}\ . && (4.77)
\end{aligned}
$$

The inequality follows because

$$
\begin{aligned}
& \operatorname{Tr}\{\Pi^{\omega,\delta}_{S^n}\omega^{\otimes n}_{SC}\Pi^{\omega,\delta}_{S^n}\omega^{\otimes n}_{SC}\} \\
& \leq 2^{-n[H(S)_\omega-\delta]/2} \operatorname{Tr}\{(\omega^{\otimes n}_S)^{-1/2}\omega^{\otimes n}_{SC}\Pi^{\omega,\delta}_{S^n}\omega^{\otimes n}_{SC}\} && (4.78)\\
& = 2^{-n[H(S)_\omega-\delta]/2} \operatorname{Tr}\{\omega^{\otimes n}_{SC}(\omega^{\otimes n}_S)^{-1/2}\omega^{\otimes n}_{SC}\Pi^{\omega,\delta}_{S^n}\} && (4.79)\\
& \leq 2^{-n[H(S)_\omega-\delta]/2}2^{-n[H(S)_\omega-\delta]/2} \operatorname{Tr}\{\omega^{\otimes n}_{SC}(\omega^{\otimes n}_S)^{-1/2}\omega^{\otimes n}_{SC}(\omega^{\otimes n}_S)^{-1/2}\}, && (4.80)
\end{aligned}
$$

where we apply (2.25) twice and the facts that $\omega^{\otimes n}_{SC}\Pi^{\omega,\delta}_{S^n}\omega^{\otimes n}_{SC} \geq 0$ and $\omega^{\otimes n}_{SC}(\omega^{\otimes n}_S)^{-1/2}\omega^{\otimes n}_{SC} \geq 0$. The equality in (4.76) follows because $H(S)_\omega = H(S|R)_\omega$ for the state ω and then because $H(SC|R)_\omega - H(S|R)_\omega = H(C|RS)_\omega$. The equality in (4.77) follows from the definition of the collision conditional entropy.

Following a similar analysis, we obtain the following upper bounds for the other two terms:

$$
M \operatorname{Tr}\{T_{R^nS^nC^n}[\mathcal{N}_{AB\to C}(\theta_{RA}\otimes\gamma_S\otimes\gamma_B)]^{\otimes n}\} \leq 2^{-n[H_2(C|R)_\omega-H(C|RS)_\omega-R_2-3\delta]}\ , \quad (4.81)
$$

$$
LM \operatorname{Tr}\{T_{R^nS^nC^n}[\mathcal{N}_{AB\to C}(\theta_R\otimes\theta_A\otimes\gamma_S\otimes\gamma_B)]^{\otimes n}\} \leq 2^{-n[H_2(C)_\omega-H(C|RS)_\omega-(R_1+R_2)-3\delta]}\ . \quad (4.82)
$$

The rest of the proof is then the same as in the proof of Theorem 12. ■

By combining Theorems 11 and 15, we obtain the following rate region that is achievable for unassisted classical communication over a quantum multiple-access channel when using a quantum simultaneous decoder:

Theorem 16 *The following rate region is achievable when using a quantum simultaneous decoder for unassisted classical communication over the K-sender, classical-input quantum-output multiple-access channel $x_1,\ldots,x_K \to \rho_{x_1,\ldots,x_K}$:*

$$
\sum_{j\in\mathcal{J}} R_j \leq I'(X(\mathcal{J});CX(\mathcal{J}^c))_\omega, \qquad \textit{for every } \mathcal{J}\subseteq[K]\ , \quad (4.83)
$$

where

$$
\omega_{X_1\cdots X_KC} \equiv \sum_{x_1,\ldots,x_K} p_{X_1}(x_1)\cdots p_{X_K}(x_K)|x_1\rangle\langle x_1|_{X_1}\otimes\cdots\otimes|x_K\rangle\langle x_K|_{X_K}\otimes\rho_{x_1,\ldots,x_K}. \quad (4.84)
$$

Observe that $H_2(A|Y)_\sigma = -\log\sum_y p(y)\operatorname{Tr}\{\sigma_y^2\}$ for a state $\sigma_{YA} = \sum_y p(y)|y\rangle\langle y|_Y\otimes\sigma_y$.

The rate region given in Theorem 16 is arguably an improvement over that from [FHS+12, Theorem 6]. By using the coding technique from [FHS+12, Theorem 6], for an m-sender multiple access channel, only m of the inequalities feature the conditional von Neumann entropy as the first term in the entropy differences, whereas the other $2^m - m - 1$ inequalities feature the conditional min-entropy. On the other hand, all $2^m - 1$ inequalities in Theorem 16 feature the collision conditional entropy (conditional Rényi entropy of order two), which is never smaller than the conditional min-entropy. Thus, as the number m of senders grows larger, the volume of the achievable rate region from Theorem 16 is generally significantly larger than the volume of the achievable rate region from [FHS+12, Theorem 6].

We can also compare the rate regions from Theorems 14 and 16. By using the identity

$$\begin{aligned} &H_2(X(\mathcal{J}^c)C)_\omega - H(X(\mathcal{J}^c)C|X(\mathcal{J}))_\omega \\ &= H_2(X(\mathcal{J}^c)C)_\omega - H(C|X_1\cdots X_K)_\omega - H(X(\mathcal{J}^c)|X(\mathcal{J}))_\omega \quad (4.85)\\ &= H_2(X(\mathcal{J}^c)C)_\omega - H(C|X_1\cdots X_K)_\omega - H(X(\mathcal{J}^c))_\omega, \quad (4.86)\end{aligned}$$

the difference between the information-theoretic terms in the inequalities in (4.73) and (4.69) is given by

$$H_2(C|X(\mathcal{J}^c))_\omega - H_2(X(\mathcal{J}^c)C)_\omega + H(X(\mathcal{J}^c))_\omega. \quad (4.87)$$

The above difference can sometimes be negative or positive, as one can find by some simple numerical tests with qubit states. Thus, the two rate regions from Theorems 14 and 16 are generally incomparable. However, we can easily make use of both results: a standard time-sharing argument establishes that the convex hull of the two rates regions is achievable.

Remark 17 *The quantum simultaneous decoding conjecture from [FHS+12, Wil11] is the statement that the Rényi entropies of order two in Theorems 14 and 16 can be replaced by quantum entropies (i.e., von Neumann entropies), while still employing a quantum simultaneous decoder. An explicit statement of the quantum simultaneous decoding conjecture is available in [FHS+12, Conjecture 4]. The conjecture has been solved in the case of two senders for unassisted classical communication in [FHS+12] and for entanglement-assisted classical communication in [XW13], but not for three or more senders in either case.*

5 Quantum simultaneous decoding for multiple-access channels and multiple quantum hypothesis testing

In this section, we establish explicit links between the quantum simultaneous decoding conjecture from [FHS+12, Wil11] and open questions from [AM14, BHOS15] in multiple quantum hypothesis testing. Recall that the general goal of quantum hypothesis testing is to minimize the error probability in identifying quantum states. In binary quantum hypothesis testing, one considers two hypotheses: the null hypothesis is that a quantum system is prepared in the state ρ, and the alternative hypothesis is that the quantum system is prepared in the state σ, where $\rho, \sigma \in \mathcal{D}(\mathcal{H})$. Operationally, the discriminator receives the state ρ with probability $p \in (0,1)$ and the state σ with probability $1-p$, and the task is to determine which state was prepared, by means of some quantum measurement $\{T, I-T\}$, where the test operator T satisfies $0 \leq T \leq I$. There are two kinds of errors: a Type I error occurs when the state is identified as σ when in fact ρ was prepared and a Type II error is the opposite kind of error. The error probabilities corresponding to the two types of errors are as follows:

$$\alpha(T,\rho) \equiv \mathrm{Tr}\{(I-T)\rho\}, \quad (5.1)$$

$$\beta(T,\sigma) \equiv \mathrm{Tr}\{T\sigma\}\ . \quad (5.2)$$

As in information theory, quantum hypothesis testing has been studied in the asymptotic i.i.d. setting. In the setting of *symmetric* hypothesis testing, we are interested in minimizing the overall error probability

$$P_e^*(p\rho,(1-p)\sigma) \equiv \inf_{T\,:\,0\leq T\leq I} \; p\,\alpha(T,\rho)+(1-p)\beta(T,\sigma) \tag{5.3}$$

$$= \frac{1}{2}\left(\operatorname{Tr}\{p\rho+(1-p)\sigma\}-\|p\rho-(1-p)\sigma\|_1\right). \tag{5.4}$$

In the i.i.d. setting, n quantum systems are prepared as either $\rho^{\otimes n}$ or $\sigma^{\otimes n}$, and the goal is to determine the optimal exponent of the error probability, defined as

$$\lim_{n\to\infty} -\frac{1}{n}\log_2 P_e^*(p\rho^{\otimes n},(1-p)\sigma^{\otimes n}). \tag{5.5}$$

One of the landmark results in quantum hypothesis testing is that the optimal exponent is equal to the quantum Chernoff distance [ACMnT+07, NS09]:

$$\lim_{n\to\infty} -\frac{1}{n}\log_2 P_e^*(p\rho^{\otimes n},(1-p)\sigma^{\otimes n}) = C(\rho,\sigma) \equiv \sup_{s\in[0,1]} -\log_2 \operatorname{Tr}\{\rho^s\sigma^{1-s}\}. \tag{5.6}$$

This development can be generalized to the setting in which ρ, σ, p, and $1-p$ can be replaced by positive semi-definite operators $\neq 0$ and positive constants. Indeed, for positive semi-definite A and B, we can define

$$P_e^*(A,B) \equiv \inf_{T\,:\,0\leq T\leq I} \; \operatorname{Tr}\{(I-T)A\}+\operatorname{Tr}\{TB\} \tag{5.7}$$

$$= \frac{1}{2}\left(\operatorname{Tr}\{A+B\}-\|A-B\|_1\right). \tag{5.8}$$

Then for positive constants $K_0, K_1 > 0$, we find that

$$\lim_{n\to\infty} -\frac{1}{n}\log_2 P_e^*(K_0A^{\otimes n},K_1B^{\otimes n}) = C(A,B) \equiv \sup_{s\in[0,1]} -\log_2 \operatorname{Tr}\{A^sB^{1-s}\}. \tag{5.9}$$

In the setting of *asymmetric* hypothesis testing, we are interested in the optimal exponent of the Type II error $\beta(T,\sigma)$, under a constraint on the Type I error, i.e., $\alpha(T,\sigma)\leq\varepsilon$, with $\varepsilon\in[0,1]$. That is, we are interested in the following quantity, now known as hypothesis testing relative entropy:

$$D_H^\varepsilon(\rho\|\sigma) \equiv -\log_2 \inf_T \{\beta(T,\sigma) : 0\leq T\leq I \wedge \alpha(T,\sigma)\leq\varepsilon\}. \tag{5.10}$$

The optimal exponential decay rate in the asymmetric setting is given by the quantum Stein's lemma [HP91, ON00], which establishes the following for all $\varepsilon\in(0,1)$:

$$\lim_{n\to\infty} \frac{1}{n} D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) = D(\rho\|\sigma), \tag{5.11}$$

giving the quantum relative entropy its fundamental operational interpretation.

As we can see from [AJW17b] and our developments in Section 3, position-based coding forges a direct connection between single-sender single-receiver communication and binary quantum hypothesis testing. Specifically, the Chernoff distance from symmetric hypothesis testing gives a lower bound on the entanglement-assisted error exponent; while the application of the results from asymmetric hypothesis testing leads to a lower bound on the one-shot entanglement-assisted capacity and in turn on the second-order coding rate for entanglement-assisted communication.

In what follows, we discuss the generalization of both asymmetric and symmetric hypothesis testing to multiple quantum states and their connections to quantum simultaneous decoding.

5.1 Symmetric multiple quantum hypothesis testing and quantum simultaneous decoding

We now tie one version of the quantum simultaneous decoding conjecture to [AM14, Conjecture 4.2], which has to do with distinguishing one state from a set of other possible states. To recall the setting of [AM14, Conjecture 4.2], suppose that a state ρ is prepared with probability $p \in (0,1)$ and with probability $1-p$ one state σ_i of r states is prepared with probability q_i, where $i \in \{1, \ldots, r\}$. The goal is to determine whether ρ was prepared or whether one of the other states was prepared, and the error probability in doing so is equal to

$$P_e^* \left(p\rho, (1-p) \sum_{i=1}^{r} q_i \sigma_i \right). \tag{5.12}$$

The measurement operator that achieves the minimum error probability is equal to

$$\left\{ p\rho - (1-p) \sum_{i=1}^{r} q_i \sigma_i \geq 0 \right\}. \tag{5.13}$$

As usual, we are interested in the i.i.d. case, in which ρ and σ_i are replaced by $\rho^{\otimes n}$ and $\sigma_i^{\otimes n}$ for large n, and [AM14, Conjecture 4.2] states that

$$\lim_{n\to\infty} -\frac{1}{n} \log_2 P_e^* \left(p\rho^{\otimes n}, (1-p) \sum_{i=1}^{r} q_i \sigma_i^{\otimes n} \right) = \min_i C(\rho, \sigma_i). \tag{5.14}$$

We now propose a slight generalization of [AM14, Conjecture 4.2] and (5.9), in which ρ and σ_i are replaced by positive semi-definite operators and p and $1-p$ are replaced by positive constants:

Conjecture 18 *Let $\{A, B_1, \ldots, B_r\}$ be a set of positive semi-definite operators with trace strictly greater than zero, and let K_0, ..., K_r be strictly positive constants. Then the following equality holds*

$$\lim_{n\to\infty} -\frac{1}{n} \log_2 P_e^* \left(K_0 A^{\otimes n}, \sum_{i=1}^{r} K_i B_i^{\otimes n} \right) = \min_i C(A, B_i)\ , \tag{5.15}$$

where P_e^* is defined in (5.7)–(5.8) and $C(A, B_i)$ in (5.9).

To see how Conjecture 18 is connected to quantum simultaneous decoding, recall from (4.14) our bound on the error probability when simultaneously decoding the message pair (l, m):

$$\begin{aligned} p_e(l,m) \leq\ & 4 \operatorname{Tr}\{(I - T_{RSC}) \mathcal{N}_{AB\to C}(\theta_{RA} \otimes \gamma_{SB})\} + 4\big[L \operatorname{Tr}\{T_{RSC} \mathcal{N}_{AB\to C}(\theta_R \otimes \theta_A \otimes \gamma_{SB})\} \\ & + M \operatorname{Tr}\{T_{RSC} \mathcal{N}_{AB\to C}(\theta_{RA} \otimes \gamma_S \otimes \gamma_B)\} + LM \operatorname{Tr}\{T_{RSC} \mathcal{N}_{AB\to C}(\theta_R \otimes \theta_A \otimes \gamma_S \otimes \gamma_B)\}\big]\ , \end{aligned} \tag{5.16}$$

where we have set $c = 1$ and used that $L - 1 < L$ and $M - 1 < M$. Now applying this bound to the i.i.d. case and setting $L = 2^{nR_1}$ and $M = 2^{nR_2}$, we find that the upper bound becomes

$$p_e(l,m) \leq 4 \left[\operatorname{Tr}\{(I - T)\rho^{\otimes n}\} + \operatorname{Tr}\{T \left(B_1^{\otimes n} + B_2^{\otimes n} + B_3^{\otimes n} \right)\} \right], \tag{5.17}$$

where

$$\rho \equiv \mathcal{N}_{AB\to C}(\theta_{RA} \otimes \gamma_{SB}), \tag{5.18}$$
$$B_1 \equiv 2^{R_1}\mathcal{N}_{AB\to C}(\theta_R \otimes \theta_A \otimes \gamma_{SB}), \tag{5.19}$$
$$B_2 \equiv 2^{R_2}\mathcal{N}_{AB\to C}(\theta_{RA} \otimes \gamma_S \otimes \gamma_B), \tag{5.20}$$
$$B_3 \equiv 2^{R_1+R_2}\mathcal{N}_{AB\to C}(\theta_R \otimes \theta_A \otimes \gamma_S \otimes \gamma_B). \tag{5.21}$$

To minimize the upper bound on the error probability, we should pick the test operator T as follows:

$$T \equiv \left\{\rho^{\otimes n} - \left(B_1^{\otimes n} + B_2^{\otimes n} + B_3^{\otimes n}\right) \geq 0\right\}, \tag{5.22}$$

and then the upper bound becomes

$$p_e(l,m) \leq 2\left(\mathrm{Tr}\{\rho^{\otimes n} + B_1^{\otimes n} + B_2^{\otimes n} + B_3^{\otimes n}\} - \left\|\rho^{\otimes n} - \left(B_1^{\otimes n} + B_2^{\otimes n} + B_3^{\otimes n}\right)\right\|_1\right) \tag{5.23}$$
$$= 4\ P_e^*\left(\rho^{\otimes n}, B_1^{\otimes n} + B_2^{\otimes n} + B_3^{\otimes n}\right). \tag{5.24}$$

The test operator T given in (5.22) was previously realized in [Wil10] and [HC17] to be relevant as a quantum simultaneous decoder in the context of unassisted classical communication over a quantum multiple access channel.

Now applying Conjecture 18 (provided it is true), we would find that the error probability $p_e(l,m)$ is bounded from above as $p_e(l,m) \lesssim e^{-nE(R_1,R_2)}$, with the error exponent $E(R_1,R_2)$ equal to

$$E(R_1,R_2) = \min\{C(\rho,B_1), C(\rho,B_2), C(\rho,B_3)\} \tag{5.25}$$
$$= \min\Big\{\sup_{s\in[0,1]} -\log_2 \mathrm{Tr}\{\rho^s B_1^{1-s}\},\ \sup_{s\in[0,1]} -\log_2 \mathrm{Tr}\{\rho^s B_2^{1-s}\},$$
$$\sup_{s\in[0,1]} -\log_2 \mathrm{Tr}\{\rho^s B_3^{1-s}\}\Big\} \tag{5.26}$$
$$= \min\Big\{\sup_{s\in[0,1]} (1-s)\Big[I_s(R;CS)_\omega - R_1\Big],\ \sup_{s\in[0,1]} (1-s)\Big[I_s(S;CR)_\omega - R_2\Big],$$
$$\sup_{s\in[0,1]} (1-s)\Big[I_s(RS;C)_\omega - (R_1+R_2)\Big]\ . \tag{5.27}$$

Thus the rate region (R_1,R_2) would be achievable as long as $E(R_1,R_2) > 0$. Now using the fact that, for a bipartite state ρ_{AB}

$$\lim_{s\to 1} I_s(A;B)_\rho = I(A;B)_\rho\ , \tag{5.28}$$

we would then find that the following rate region is achievable:

$$R_1 \leq I(R;CS)_\omega\ , \tag{5.29}$$
$$R_2 \leq I(S;CR)_\omega\ , \tag{5.30}$$
$$R_1 + R_2 \leq I(RS;C)_\omega\ , \tag{5.31}$$

where $\omega_{RSC} = \mathcal{N}_{AB\to C}(\theta_{RA} \otimes \gamma_{SB})$ and the above approach would solve the quantum simultaneous decoding conjecture. The method clearly extends to more than two senders. We remark that aspects of the above approach were discussed in the recent work [HC17] for the case of unassisted classical communication over a quantum multiple access channel, but there the connection to [AM14, Conjecture 4.2] or Conjecture 18 was not realized, nor was the entanglement-assisted case considered.

We end this section by noting that [AM14, Theorem 4.3] offers several suboptimal upper bounds on the error probability $P_e^*\big(\rho^{\otimes n}, B_1^{\otimes n} + B_2^{\otimes n} + B_3^{\otimes n}\big)$, which in turn could be used to establish suboptimal achievable rate regions for the quantum multiple-access channel. However, here we refrain from the details of what these regions would be, except to say that they would be in terms of the negative logarithm of the fidelity, replacing I_s in (5.27).

5.2 Asymmetric hypothesis testing with composite alternative hypothesis

We now tie the quantum simultaneous decoding problem to a different open question in asymmetric hypothesis testing. Recall our upper bound from (4.14) on the error probability for classical communication over a quantum multiple-access channel, as applied for the i.i.d. case:

$$p_e(l,m) \leq c_{\mathrm{I}} \operatorname{Tr}\{(I-T)\rho^{\otimes n}\} + c_{\mathrm{II}}\Big[\operatorname{Tr}\{T\left[B_1^{\otimes n} + B_2^{\otimes n} + B_3^{\otimes n}\right]\}\Big] , \tag{5.32}$$

where $L = 2^{nR_1}$, $M = 2^{nR_2}$, and the state ρ and the positive semi-definite operators B_1, B_2, and B_3 are given by

$$\rho = \mathcal{N}_{AB\to C}(\theta_{RA} \otimes \gamma_{SB}) , \tag{5.33}$$
$$B_1 = 2^{R_1}\mathcal{N}_{AB\to C}(\theta_R \otimes \theta_A \otimes \gamma_{SB}), \tag{5.34}$$
$$B_2 = 2^{R_2}\mathcal{N}_{AB\to C}(\theta_{RA} \otimes \gamma_S \otimes \gamma_B), \tag{5.35}$$
$$B_3 = 2^{R_1+R_2}\mathcal{N}_{AB\to C}(\theta_R \otimes \theta_A \otimes \gamma_S \otimes \gamma_B). \tag{5.36}$$

Rather than try to minimize the overall error probability as we did in the previous section, we could try to minimize all of the other error probabilities subject to a constraint on the error probability $\operatorname{Tr}\{(I-T)\rho^{\otimes n}\}$. Thus, we seek a test operator T which is capable of discriminating the state $\rho^{\otimes n}$ from the operator $B_1^{\otimes n} + B_2^{\otimes n} + B_3^{\otimes n}$. This kind of task is formally called asymmetric hypothesis testing with composite alternative hypothesis. The problem of a composite null hypothesis is that considered in the context of the quantum Sanov theorem, which was solved in [BDK+05] (see also [Hay02]), and finds application in communication over compound channels [BB09, Mos15] (see also [DD07, Hay09b]).

The following open question, strongly related to a question from [BHOS15], is relevant for asymmetric hypothesis testing with composite alternative hypothesis:

Question 19 *Consider a quantum state $\rho \in \mathcal{D}(\mathcal{H})$, a positive integer $r > 1$, and a finite set of positive semi-definite operators $\mathcal{B} \equiv \{B_i : 1 \leq i \leq r\}$, for which* $\operatorname{supp}(\rho) \subseteq \operatorname{supp}(B_i)$ $\forall B_i \in \mathcal{B}$ *and*

$$\min_i D(\rho\|B_i) > 0. \tag{5.37}$$

What is the most general form that ρ and $\mathcal{B}$ can take such that the following statement is true? For all $\varepsilon \in (0,1)$, $\delta > 0$, and sufficiently large n, there exists a binary test $\{T, I-T\}$ such that the Type I error is bounded from above by ε:

$$\operatorname{Tr}\{(I-T)\rho^{\otimes n}\} \leq \varepsilon , \tag{5.38}$$

and for all $B_i \in \mathcal{B}$, the exponential decay rate of the Type II error is bounded from below as follows:

$$-\frac{1}{n}\log_2 \operatorname{Tr}\{TB_i^{\otimes n}\} \geq \left[\min_i D(\rho\|B_i)\right] - \delta \ . \tag{5.39}$$

Below we prove the following special case:

Theorem 20 *The statement at the end of Question 19 is true when the set $\mathcal{B}$ forms a commuting set of operators (each of which need not commute with ρ).*

Proof. To this end, we employ the notion of a relative typical projector, which was used in [BSS03] to establish an alternate proof of the quantum Stein lemma (see also [BLW15] for a different use of relative typical projectors). Let $B_i = \sum_y f_Y^i(y)|\phi_y^i\rangle\langle\phi_y^i|$ denote a spectral decomposition of B_i. For a state ρ and positive semi-definite operator B_i, define the relative typical subspace $T_{\rho|B_i}^{\delta,n}$ for $\delta > 0$ and integer $n \geq 1$ as

$$T_{\rho|B_i}^{\delta,n} \equiv \operatorname{span}\left\{|\phi_{y^n}^i\rangle : \left|-\frac{1}{n}\log_2(f_{Y^n}^i(y^n)) + \operatorname{Tr}\{\rho\log_2 B_i\}\right| \leq \delta\right\}, \tag{5.40}$$

where

$$y^n \equiv y_1 \cdots y_n, \tag{5.41}$$

$$f_{Y^n}^i(y^n) \equiv \prod_{j=1}^n f_Y^i(y_j), \tag{5.42}$$

$$|\phi_{y^n}^i\rangle \equiv |\phi_{y_1}^i\rangle \otimes \cdots \otimes |\phi_{y_n}^i\rangle. \tag{5.43}$$

Let $\Pi_{\rho|B_i,\delta}^n$ denote the projection operator corresponding to the relative typical subspace $T_{\rho|B_i}^{\delta,n}$. The critical properties of the relative typical projector are as follows:

$$\operatorname{Tr}\{\Pi_{\rho|B_i,\delta}^n\rho^{\otimes n}\} \geq 1-\varepsilon, \tag{5.44}$$

$$2^{-n[-\operatorname{Tr}\{\rho\log_2 B_i\}+\delta]}\Pi_{\rho|B_i,\delta}^n \leq \Pi_{\rho|B_i,\delta}^n B_i^{\otimes n}\Pi_{\rho|B_i,\delta}^n \leq 2^{-n[-\operatorname{Tr}\{\rho\log_2 B_i\}-\delta]}\Pi_{\rho|B_i,\delta}^n, \tag{5.45}$$

with the first inequality holding for all $\varepsilon \in (0,1)$, $\delta > 0$, and sufficiently large n.

The main idea for the proof under the stated assumptions is to take the test operator T as

$$T = \Pi_{\rho|B_r,\delta}^n \cdots \Pi_{\rho|B_1,\delta}^n \Pi_{\rho,\delta}^n \Pi_{\rho|B_1,\delta}^n \cdots \Pi_{\rho|B_r,\delta}^n, \tag{5.46}$$

where $\Pi_{\rho,\delta}^n$ is the typical projector for ρ. Then we find that for all $\varepsilon \in (0,1)$, $\delta > 0$, and sufficiently large n

$$\operatorname{Tr}\{T\rho^{\otimes n}\} \geq \operatorname{Tr}\{\Pi_{\rho,\delta}^n\rho^{\otimes n}\} - \sum_{i=1}^r \left\|\Pi_{\rho|B_i,\delta}^n\rho^{\otimes n}\Pi_{\rho|B_i,\delta}^n - \rho^{\otimes n}\right\|_1 \tag{5.47}$$

$$\geq 1-\varepsilon-2r\sqrt{\varepsilon}, \tag{5.48}$$

which follows by applying Lemmas 1 and 2 and properties of typicality and relative typicality. To handle the other kind of error, consider that, from the assumption, all of the projectors $\Pi^n_{\rho|B_i,\delta}$ commute, so that

$$\begin{aligned}
\operatorname{Tr}\{TB_i^{\otimes n}\} &= \operatorname{Tr}\{\Pi^n_{\rho|B_r,\delta}\cdots\Pi^n_{\rho|B_1,\delta}\Pi^n_{\rho,\delta}\Pi^n_{\rho|B_1,\delta}\cdots\Pi^n_{\rho|B_r,\delta}B_i^{\otimes n}\} && (5.49)\\
&= \operatorname{Tr}\{\Pi^n_{\rho,\delta}\Pi^n_{\rho|B_1,\delta}\cdots\Pi^n_{\rho|B_r,\delta}B_i^{\otimes n}\Pi^n_{\rho|B_r,\delta}\cdots\Pi^n_{\rho|B_1,\delta}\} && (5.50)\\
&= \operatorname{Tr}\{\Pi^n_{\rho,\delta}\Pi^n_{\rho|B_1,\delta}\cdots\Pi^n_{\rho|B_r,\delta}\Pi^n_{\rho|B_i,\delta}B_i^{\otimes n}\Pi^n_{\rho|B_i,\delta}\Pi^n_{\rho|B_r,\delta}\cdots\Pi^n_{\rho|B_1,\delta}\} && (5.51)\\
&\le 2^{-n[-\operatorname{Tr}\{\rho\log_2 B_i\}-\delta]}\operatorname{Tr}\{\Pi^n_{\rho,\delta}\Pi^n_{\rho|B_1,\delta}\cdots\Pi^n_{\rho|B_r,\delta}\Pi^n_{\rho|B_i,\delta}\Pi^n_{\rho|B_r,\delta}\cdots\Pi^n_{\rho|B_1,\delta}\} && (5.52)\\
&\le 2^{-n[-\operatorname{Tr}\{\rho\log_2 B_i\}-\delta]}\operatorname{Tr}\{\Pi^n_{\rho,\delta}\} && (5.53)\\
&\le 2^{-n[-\operatorname{Tr}\{\rho\log_2 B_i\}-\delta]}2^{n[H(\rho)+\delta]} && (5.54)\\
&= 2^{-n[D(\rho\|B_i)-2\delta]}. && (5.55)
\end{aligned}$$

The statement of the theorem follows by setting $\varepsilon' \equiv \varepsilon + 2r\sqrt{\varepsilon}$ and $\delta' \equiv 2\delta$, considering that we have shown the existence of a test T for which

$$\operatorname{Tr}\{T\rho^{\otimes n}\} \ge 1-\varepsilon', \qquad \operatorname{Tr}\{TB_i^{\otimes n}\} \le 2^{-n[D(\rho\|B_i)-\delta']}, \tag{5.56}$$

and it is possible to satisfy this for any choice of $\varepsilon' \in (0,1)$ and $\delta' > 0$ by taking n sufficiently large. (Note that for the bound on the second kind of error probability to be decaying exponentially, we require $\delta' > 0$ to be small enough so that $\min_i D(\rho\|B_i) > \delta'$.) We remark that this conclusion is actually stronger than what is stated in Question 19 because here we conclude for all $B_i \in \mathcal{B}$, that

$$-\frac{1}{n}\log_2 \operatorname{Tr}\{TB_i^{\otimes n}\} \ge D(\rho\|B_i) - \delta' \ge \left[\min_i D(\rho\|B_i)\right] - \delta' \ . \tag{5.57}$$

This concludes the proof. ∎

To see how Question 19 is related to quantum simultaneous decoding of the multiple-access channel, consider that for ρ, B_1, …, B_3 as defined in (5.33)–(5.36), the inequality

$$\min_i D(\rho\|B_i) > 0 \tag{5.58}$$

is equivalent to the following set of inequalities:

$$\begin{aligned}
R_1 &< I(R;CS)_\omega \ , && (5.59)\\
R_2 &< I(S;CR)_\omega \ , && (5.60)\\
R_1 + R_2 &< I(RS;C)_\omega \ , && (5.61)
\end{aligned}$$

where $\omega_{RSC} = \mathcal{N}_{AB\to C}(\theta_{RA}\otimes\gamma_{SB})$. This equivalence holds because $\min_i D(\rho\|B_i) > 0$ is equivalent to the following three inequalities:

$$D(\rho\|B_1) > 0, \qquad D(\rho\|B_2) > 0, \qquad D(\rho\|B_3) > 0, \tag{5.62}$$

and

$$
\begin{aligned}
D(\rho\|B_1) &= D(\mathcal{N}_{AB\to C}(\theta_{RA}\otimes\gamma_{SB})\|2^{R_1}\mathcal{N}_{AB\to C}(\theta_R\otimes\theta_A\otimes\gamma_{SB})) && (5.63)\\
&= D(\mathcal{N}_{AB\to C}(\theta_{RA}\otimes\gamma_{SB})\|\mathcal{N}_{AB\to C}(\theta_R\otimes\theta_A\otimes\gamma_{SB})) - R_1 && (5.64)\\
&= I(R;CS)_\omega - R_1, && (5.65)\\
D(\rho\|B_2) &= D(\mathcal{N}_{AB\to C}(\theta_{RA}\otimes\gamma_{SB})\|2^{R_2}\mathcal{N}_{AB\to C}(\theta_{RA}\otimes\gamma_S\otimes\gamma_B)) && (5.66)\\
&= D(\mathcal{N}_{AB\to C}(\theta_{RA}\otimes\gamma_{SB})\|\mathcal{N}_{AB\to C}(\theta_{RA}\otimes\gamma_S\otimes\gamma_B)) - R_2, && (5.67)\\
&= I(S;CR)_\omega - R_2, && (5.68)\\
D(\rho\|B_3) &= D(\mathcal{N}_{AB\to C}(\theta_{RA}\otimes\gamma_{SB})\|2^{R_1+R_2}\mathcal{N}_{AB\to C}(\theta_R\otimes\theta_A\otimes\gamma_{SB})) && (5.69)\\
&= D(\mathcal{N}_{AB\to C}(\theta_{RA}\otimes\gamma_{SB})\|\mathcal{N}_{AB\to C}(\theta_R\otimes\theta_A\otimes\gamma_{SB})) - (R_1+R_2) && (5.70)\\
&= I(RS;C)_\omega - (R_1+R_2). && (5.71)
\end{aligned}
$$

Thus, if such a sequence of test operators existed as stated in Question 19, then the error probability $p_e(l,m)$ when decoding a multiple-access channel could be bounded from above as

$$
\begin{aligned}
p_e(l,m) &\leq c_{\mathrm{I}}\operatorname{Tr}\{(I-T)\rho^{\otimes n}\} + c_{\mathrm{II}}\left[\operatorname{Tr}\{T\left[B_1^{\otimes n}+B_2^{\otimes n}+B_3^{\otimes n}\right]\}\right] && (5.72)\\
&\leq c_{\mathrm{I}}\varepsilon + 3c_{\mathrm{II}}\left[2^{-n[\min_i D(\rho\|B_i)-\delta]}\right], && (5.73)
\end{aligned}
$$

where ρ, B_1, ..., B_3 are defined in (5.33)–(5.36). Then by choosing the rates R_1 and R_2 to satisfy

$$
\begin{aligned}
R_1 + 2\delta &\leq I(R;CS)_\omega, && (5.74)\\
R_2 + 2\delta &\leq I(S;CR)_\omega, && (5.75)\\
R_1 + R_2 + 2\delta &\leq I(RS;C)_\omega, && (5.76)
\end{aligned}
$$

which is equivalent to $\min_i D(\rho\|B_i) \geq 2\delta$, we would have

$$
p_e(l,m) \leq c_{\mathrm{I}}\varepsilon + 3c_{\mathrm{II}}2^{-n\delta}, \quad (5.77)
$$

and we could thus make the error probability as small as desired by taking n sufficiently large. Since $\delta > 0$ is arbitrary, we could then say that the rate region in (5.29)–(5.31) would be achievable. If the statement at the end of Question 19 holds for the states given above, then the method would clearly lead to a quantum simultaneous decoder for more than two senders, by a straightforward generalization of the above approach.

The authors of [BP10] considered a similar problem in asymmetric hypothesis testing for a specific family of states with certain permutation symmetry. We should point out that in [BP10], the lower bound for the exponential rate of the Type II error is given by a regularized version of $\min_i D(\rho\|\sigma_i)$. If a similar result held, along the lines stated in Question 19 and related to the conjecture in [BHOS15], without the need for regularization and for the operators in (5.33)–(5.36) (and more general ones relevant for more senders), then the developments in the present paper would immediately give bounds for the performance of quantum simultaneous decoding for the quantum multiple-access channel.

We end this section by remarking that our quantum simultaneous decoder in (4.50) gives a method for distinguishing $\rho^{\otimes n}$ from the operator $B_1^{\otimes n}+B_2^{\otimes n}+B_3^{\otimes n}$ with Type I error probability bounded, for sufficiently large n, by an arbitrary $\varepsilon \in (0,1)$ and the Type II error probability bounded, for a positive constant c, by

$$\approx c\ 2^{-n\min\left\{\tilde{I}(S;CR)_\omega - R_1,\ \tilde{I}(R;CS)_\omega - R_2,\ \tilde{I}(RC;S)_\omega - (R_1+R_2)\right\}}, \tag{5.78}$$

and

$$\approx c\ 2^{-n\min\{I'(S;CR)_\omega - R_1,\ I'(R;CS)_\omega - R_2,\ I'(RC;S)_\omega - (R_1+R_2)\}}, \tag{5.79}$$

where $\omega_{RSC} = \mathcal{N}_{AB\to C}(\theta_{RA} \otimes \gamma_{SB})$. To obtain the first statement, we apply the analysis in the proof of Theorem 12, and for the second, we apply the analysis in the proof of Theorem 15. The above statements have clear generalizations to more systems by invoking Theorems 13 and 15. Thus, our previous analysis in the context of communication applies for this interesting, special case of Question 19, albeit with suboptimal rates.

6 Conclusion

In this paper, we apply position-based coding to establish bounds of various quantities for classical communication. For entanglement-assisted classical communication over point-to-point quantum channels, we establish lower bounds on the one-shot error exponent, the one-shot capacity, and the second-order coding rate. We also find an alternative proof for an upper bound on one-shot entanglement-assisted classical capacity, which is arguably simpler than the approach from [MW14]. We give an achievable rate region for entanglement-assisted classical communication over multiple-access quantum channels. Furthermore, we explicitly show how to derandomize a randomness-assisted protocol (for multiple-access channel) to one without assistance from any extra resources. Our results indicate that position-based coding can be a powerful tool in achievability proofs of various communication protocols in quantum Shannon theory (for recent applications of position-based coding to private classical communication, see [Wil17a]). We finally tied some open questions in multiple quantum hypothesis testing to the quantum simultaneous decoding conjecture. Thus, we have shown that open problems in multiple quantum hypothesis testing are fundamental to the study of classical information transmission over quantum multiple-access channels.

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A Proof of Proposition 3 and quantum Stein's lemma

Here we provide a proof of Proposition 3. After doing that, we discuss briefly how Proposition 3 and [CMW16, Lemma 5] lead to a transparent proof of the quantum Stein's lemma [HP91, ON00].

Proof of Proposition 3. The statement of Proposition 3 is trivially true if $\rho\sigma = 0$, since both $D_H^\varepsilon(\rho\|\sigma) = \infty$ and $D_\alpha(\rho\|\sigma) = \infty$ in this case. So we consider the non-trivial case when this equality does not hold. We exploit Lemma 4 to establish the above bound. The proof also bears some similarities to a related proof in [AJW17a]. Recall from Lemma 4 that the following inequality holds for positive semi-definite operators A and B and for $\alpha \in (0,1)$:

$$\inf_{T:0\leq T\leq I} \operatorname{Tr}\{(I-T)A\} + \operatorname{Tr}\{TB\} = \frac{1}{2}\left(\operatorname{Tr}\{A+B\} - \|A-B\|_1\right) \tag{A.1}$$

$$\leq \operatorname{Tr}\{A^{\alpha}B^{1-\alpha}\}. \tag{A.2}$$

For $p \in (0,1)$, pick $A = p\rho$ and $B = (1-p)\,\sigma$. Plugging in to the above inequality, we find that there exists a measurement operator $T^* = T(p,\rho,\sigma)$ such that

$$p\operatorname{Tr}\{(I-T^*)\rho\} + (1-p)\operatorname{Tr}\{T^*\sigma\} \leq p^{\alpha}(1-p)^{1-\alpha}\operatorname{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\}. \tag{A.3}$$

This implies that

$$p\operatorname{Tr}\{(I-T^*)\rho\} \leq p^{\alpha}(1-p)^{1-\alpha}\operatorname{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\}, \tag{A.4}$$

and in turn that

$$\operatorname{Tr}\{(I-T^*)\rho\} \leq \left(\frac{1-p}{p}\right)^{1-\alpha}\operatorname{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\}. \tag{A.5}$$

For a given $\varepsilon \in (0,1)$ and $\alpha \in (0,1)$, we pick $p \in (0,1)$ such that

$$\left(\frac{1-p}{p}\right)^{1-\alpha}\operatorname{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\} = \varepsilon. \tag{A.6}$$

This is possible because we can rewrite the above equation as

$$\varepsilon = \left(\frac{1-p}{p}\right)^{1-\alpha}\operatorname{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\} = \left(\frac{1}{p}-1\right)^{1-\alpha}\operatorname{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\} \tag{A.7}$$

$$\Leftrightarrow \left(\frac{1}{p}-1\right)^{1-\alpha} = \frac{\varepsilon}{\operatorname{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\}} \tag{A.8}$$

$$\Leftrightarrow \frac{1}{p} = \left[\frac{\varepsilon}{\operatorname{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\}}\right]^{1/(1-\alpha)} + 1 \tag{A.9}$$

$$\Leftrightarrow p = \frac{1}{\left[\varepsilon/\operatorname{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\}\right]^{1/(1-\alpha)} + 1} \in (0,1)\,. \tag{A.10}$$

This means that $T(p,\rho,\sigma)$ with p selected as above is a measurement such that

$$\operatorname{Tr}\{(I-T^*)\rho\} \leq \varepsilon. \tag{A.11}$$

Now using the fact that the measurement T^{**} for the hypothesis testing relative entropy achieves the smallest type II error probability (by definition) and the fact that

$$(1-p)\operatorname{Tr}\{T^*\sigma\} \leq p^{\alpha}(1-p)^{1-\alpha}\operatorname{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\} \tag{A.12}$$

implies

$$\operatorname{Tr}\{T^*\sigma\} \leq \left(\frac{p}{1-p}\right)^{\alpha}\operatorname{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\}, \tag{A.13}$$

we find that

$$\mathrm{Tr}\{T^{**}\sigma\} \leq \left(\frac{p}{1-p}\right)^{\alpha} \mathrm{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\}. \tag{A.14}$$

Considering that

$$\varepsilon = \left(\frac{1-p}{p}\right)^{1-\alpha} \mathrm{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\} = \left(\frac{p}{1-p}\right)^{\alpha-1} \mathrm{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\} \tag{A.15}$$

implies that

$$\left[\frac{\varepsilon}{\mathrm{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\}}\right]^{1/(\alpha-1)} = \frac{p}{1-p}, \tag{A.16}$$

we get that

$$\begin{aligned}
\mathrm{Tr}\{T^{**}\sigma\} &\leq \left(\frac{p}{1-p}\right)^{\alpha} \mathrm{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\} && \text{(A.17)}\\
&= \left(\left[\frac{\varepsilon}{\mathrm{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\}}\right]^{1/(\alpha-1)}\right)^{\alpha} \mathrm{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\} && \text{(A.18)}\\
&= \varepsilon^{\alpha/(\alpha-1)} \left[\mathrm{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\}\right]^{\alpha/(1-\alpha)} \mathrm{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\} && \text{(A.19)}\\
&= \varepsilon^{\alpha/(\alpha-1)} \left[\mathrm{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\}\right]^{1/(1-\alpha)}. && \text{(A.20)}
\end{aligned}$$

Then, by taking a logarithm, we get that

$$\begin{aligned}
-\log_2 \mathrm{Tr}\{T^{**}\sigma\} &\geq -\log_2\left(\varepsilon^{\alpha/(\alpha-1)} \left[\mathrm{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\}\right]^{1/(1-\alpha)}\right) && \text{(A.21)}\\
&= -\frac{\alpha}{\alpha-1}\log_2(\varepsilon) + \frac{1}{\alpha-1}\log_2 \mathrm{Tr}\{\rho^{\alpha}\sigma^{1-\alpha}\} && \text{(A.22)}\\
&= -\frac{\alpha}{\alpha-1}\log_2(\varepsilon) + D_{\alpha}(\rho\|\sigma). && \text{(A.23)}
\end{aligned}$$

Putting everything together, we conclude the statement of Proposition 3. ■

We now briefly discuss how Proposition 3 and [CMW16, Lemma 5], once established, lead to a transparent proof of the quantum Stein's lemma [HP91, ON00] (see also [Hay07] in this context). Before doing so, let us recall that the quantum Stein's lemma (with strong converse) can be summarized as the following equality holding for all $\varepsilon \in (0,1)$, states ρ, and positive semi-definite operators σ:

$$\lim_{n\to\infty} \frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n}\|\sigma^{\otimes n}) = D(\rho\|\sigma), \tag{A.24}$$

thus giving the quantum relative entropy its most fundamental operational meaning as the optimal Type II error exponent in asymmetric quantum hypothesis testing. Before giving the transparent proof, let us recall the sandwiched Rényi relative entropy [MLDS+13, WWY14], defined for $\alpha \in (1,\infty)$ as

$$\widetilde{D}_{\alpha}(\rho\|\sigma) \equiv \frac{1}{\alpha-1}\log_2 \mathrm{Tr}\{(\sigma^{(1-\alpha)/2\alpha}\rho\sigma^{(1-\alpha)/2\alpha})^{\alpha}\}. \tag{A.25}$$

whenever $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$ and set to $+\infty$ otherwise. For $\alpha \in (0,1)$, it is defined as above. The sandwiched Rényi relative entropy obeys the following limit [MLDS+13, WWY14]: $\lim_{\alpha\to 1} \widetilde{D}_{\alpha}(\rho\|\sigma) =$

$D(\rho\|\sigma)$. [CMW16, Lemma 5] is the statement that the following inequality holds for all $\alpha > 1$ and $\varepsilon \in (0,1)$:

$$D_H^\varepsilon(\rho\|\sigma) \leq \widetilde{D}_\alpha(\rho\|\sigma) + \frac{\alpha}{\alpha-1}\log_2\left(\frac{1}{1-\varepsilon}\right). \tag{A.26}$$

Employing Proposition 3 and [CMW16, Lemma 5] leads to a direct proof of the quantum Stein's lemma [HP91, ON00]. Applying Proposition 3, we find that the following inequality holds for all $\alpha \in (0,1)$ and $\varepsilon \in (0,1)$:

$$\frac{1}{n}D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) \geq \frac{\alpha}{n(\alpha-1)}\log_2\left(\frac{1}{\varepsilon}\right) + \frac{1}{n}D_\alpha(\rho^{\otimes n}\|\sigma^{\otimes n}) \tag{A.27}$$

$$= \frac{\alpha}{n(\alpha-1)}\log_2\left(\frac{1}{\varepsilon}\right) + D_\alpha(\rho\|\sigma). \tag{A.28}$$

Taking the limit as $n \to \infty$ gives the following inequality holding for all $\alpha \in (0,1)$:

$$\lim_{n\to\infty}\frac{1}{n}D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) \geq D_\alpha(\rho\|\sigma). \tag{A.29}$$

We can then take the limit as $\alpha \to 1$ to get that

$$\lim_{n\to\infty}\frac{1}{n}D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) \geq D(\rho\|\sigma). \tag{A.30}$$

Applying [CMW16, Lemma 5], we find that the following holds for all $\alpha > 1$ and $\varepsilon \in (0,1)$:

$$\frac{1}{n}D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) \leq \frac{\alpha}{n(\alpha-1)}\log_2\left(\frac{1}{1-\varepsilon}\right) + \frac{1}{n}\widetilde{D}_\alpha(\rho^{\otimes n}\|\sigma^{\otimes n}) \tag{A.31}$$

$$= \frac{\alpha}{n(\alpha-1)}\log_2\left(\frac{1}{1-\varepsilon}\right) + \widetilde{D}_\alpha(\rho\|\sigma). \tag{A.32}$$

Taking the limit $n \to \infty$, we find that the following holds for all $\alpha > 1$

$$\lim_{n\to\infty}\frac{1}{n}D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) \leq \widetilde{D}_\alpha(\rho\|\sigma). \tag{A.33}$$

Then taking the limit as $\alpha \to 1$, we get that

$$\lim_{n\to\infty}\frac{1}{n}D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) \leq D(\rho\|\sigma). \tag{A.34}$$

We note here that a slightly different approach would be to set $\alpha = 1 + 1/\sqrt{n}$ in both (A.28) and (A.32) and then take the limit $n \to \infty$.

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