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Mark M. Wilde
Louisiana State University

Haoyu Qi
Louisiana State University

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Energy-Constrained Private and Quantum Capacities of Quantum Channels

Mark M. Wilde¹, Senior Member, IEEE, and Haoyu Qi

Abstract—This paper establishes a general theory of energy-constrained quantum and private capacities of quantum channels. We begin by defining various energy-constrained communication tasks, including quantum communication with a uniform energy constraint, entanglement transmission with an average energy constraint, private communication with a uniform energy constraint, and secret key transmission with an average energy constraint. We develop several code conversions, which allow us to conclude non-trivial relations between the capacities corresponding to the above tasks. We then show how the regularized, energy-constrained coherent information is equal to the capacity for the first two tasks and is an achievable rate for the latter two tasks, whenever the energy observable satisfies the Gibbs condition of having a well-defined thermal state for all temperatures and the channel satisfies a finite output-entropy condition. For degradable channels satisfying these conditions, we find that the single-letter energy-constrained coherent information is equal to all of the capacities. We finally apply our results to degradable quantum Gaussian channels and recover several results already established in the literature (in some cases, we prove new results in this domain). Contrary to what may appear from some statements made in the literature recently, proofs of these results do not require the solution of any kind of minimum output entropy conjecture or entropy photon-number inequality.

Index Terms—Quantum capacity, private capacity, Gibbs observable, bosonic channels.

I. INTRODUCTION

THE capacity of a quantum channel to transmit quantum or private information is a fundamental characteristic of the channel that guides the design of practical communication protocols (see [1] for a review). The quantum capacity $Q(\mathcal{N})$ of a quantum channel \mathcal{N} is defined as the maximum rate at which qubits can be transmitted faithfully over many independent uses of \mathcal{N} , where the fidelity of transmission

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M. M. Wilde is with the Department of Physics and Astronomy, Hearne Institute for Theoretical Physics, Louisiana State University, Baton Rouge, LA 70803 USA, and also with the Center for Computation and Technology, Louisiana State University, Baton Rouge, LA 70803 USA (e-mail: mwilde@lsu.edu).

H. Qi is with the Department of Physics and Astronomy, Hearne Institute for Theoretical Physics, Louisiana State University, Baton Rouge, LA 70803 USA.

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tends to one in the limit as the number of channel uses tends to infinity [2]–[4]. Related, the private capacity $P(\mathcal{N})$ of \mathcal{N} is defined to be the maximum rate at which classical bits can be transmitted over many independent uses of \mathcal{N} such that 1) the receiver can decode the classical bits faithfully and 2) the environment of the channel cannot learn anything about the classical bits being transmitted [4], [5]. The quantum capacity is essential for understanding how fast we will be able to perform distributed quantum computations between remote locations, and the private capacity is connected to the ability to generate secret key between remote locations, as in quantum key distribution (see [6] for a review). Notions from classical information theory regarding wiretap channels are typically insightful for understanding private communication over quantum channels (see [7]–[14]). In general, there are connections between private capacity and quantum capacity of quantum channels [4] (see also [15]), but the results of [16]–[19] demonstrated that these concepts and the capacities can be very different. In fact, the most striking examples are channels for which their quantum capacity is equal to zero but their private capacity is strictly greater than zero [18], [19].

Bosonic Gaussian channels are some of the most important channels to consider, as they model practical communication links in which the mediators of information are photons (see [20], [21] for reviews). Recent years have seen advances in the quantum information theory of bosonic channels. For example, we now know the capacity for sending classical information over all single-mode phase-insensitive quantum Gaussian channels [22], [23] (and even the strong converse capacity [24]). The result of this theoretical development is that coherent states [25] of the light field suffice to achieve classical capacity of phase-insensitive bosonic Gaussian channels. Note that the classical capacity of these channels is non-trivial only when there is an energy constraint placed on the input signaling states [22], [23]—otherwise, it is equal infinity.

We have also seen advances related to quantum capacity of bosonic channels. Important statements, discussions, and critical steps concerning quantum capacity of single-mode quantum-limited attenuator and amplifier channels were reported in [26] and [27]. In particular, these papers stated a formula for the quantum capacity of these channels, whenever infinite energy is available at the transmitter. These formulas have been supported with a proof in [28, Th. 8], [29], and [30] (see Remark 4 of the present paper for further discussion of this point). However, in practice, no transmitter could ever use infinite energy to transmit quantum information, and so the results from [26] and [27] have limited

applicability to realistic scenarios. Given that the notion of quantum capacity itself is already somewhat removed from practice, as argued in [31], it seems that supplanting a sender and receiver with infinite energy in addition to perfect quantum computers and an infinite number of channel uses only serves to push this notion much farther away from practice. One of the main aims of the present paper is to continue the effort of bringing this notion closer to practice, by developing a general theory of energy-constrained quantum and private communication. Considering quantum and private capacity with a limited number of channel uses, as was done in [30] and [31], in addition to energy constraints, is left for future developments.

In light of the above discussion, we are thus motivated to understand both quantum and private communication over quantum channels with realistic energy constraints. References [32] and [33] were some of the earlier works to discuss quantum and private communication with energy constraints, in addition to other kinds of communication tasks. The more recent efforts in [28], [34], and [35] have considered energy-constrained communication in more general trade-off scenarios, but as special cases, they also furnished proofs for energy-constrained quantum and private capacities of quantum-limited attenuator and amplifier channels (see [28, Th. 8] and [35]). In more detail, let $Q(\mathcal{N}, N_S)$ and $P(\mathcal{N}, N_S)$ denote the respective quantum and private capacities of a quantum channel \mathcal{N} , such that the mean input photon number for each channel use cannot exceed $N_S \in [0, \infty)$. Reference [28, Th. 8] established that the quantum capacity of a pure-loss channel \mathcal{L}_η with transmissivity parameter $\eta \in [0, 1]$ is equal to

$$Q(\mathcal{L}_\eta, N_S) = \max\{g(\eta N_S) - g((1 - \eta)N_S), 0\}, \quad (1)$$

where $g(x)$ is the entropy of a thermal state with mean photon number x , defined as

$$g(x) \equiv (x + 1) \log_2(x + 1) - x \log_2 x. \quad (2)$$

The present paper (see (331)) establishes the private capacity formula for \mathcal{L}_η :

$$P(\mathcal{L}_\eta, N_S) = \max\{g(\eta N_S) - g((1 - \eta)N_S), 0\}. \quad (3)$$

A special case of the results of [35] established that the quantum and private capacities of a quantum-limited amplifier channel \mathcal{A}_κ with gain parameter $\kappa \in [1, \infty)$ are equal to

$$Q(\mathcal{A}_\kappa, N_S) = P(\mathcal{A}_\kappa, N_S) \quad (4)$$

$$= g(\kappa N_S + \kappa - 1) - g([\kappa - 1][N_S + 1]). \quad (5)$$

Taking the limit as $N_S \rightarrow \infty$, these formulas respectively converge to

$$\max\{\log_2(\eta/[1 - \eta]), 0\}, \quad (6)$$

$$\log_2(\kappa/[\kappa - 1]), \quad (7)$$

which were stated in [26] and [27] in the context of quantum capacity, with the latter proved in [29] and [30] for both quantum and private capacities. Figure 1 plots the ratios of the unconstrained to constrained quantum capacity formulas in (6) and (1), respectively. Figure 2 plots the ratios of the

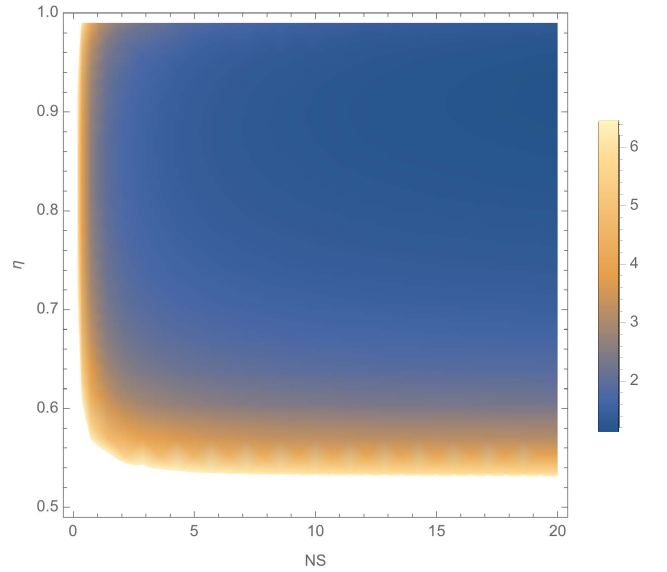


Fig. 1. Density plot of the ratio of the unconstrained and constrained quantum and private capacities of the pure-loss channel for $\eta \in [1/2, 1]$ and $N_S \in [0, 20]$. For lower photon numbers and higher loss $\eta \approx 0.5$, there is a large gap between these capacities.

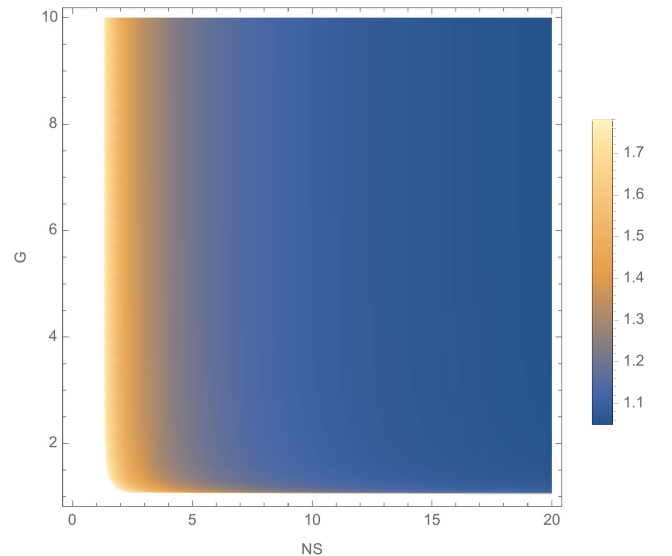


Fig. 2. Density plot of the ratio of the unconstrained and constrained quantum and private capacities of the pure-amplifier channel for $G \in [1, 10]$ and $N_S \in [0, 20]$. For lower photon numbers, there is a large gap between these capacities.

unconstrained to constrained quantum capacity formulas in (7) and (5), respectively.

The main purpose of the present paper is to go beyond bosonic channels and establish a general theory of energy-constrained quantum and private communication over quantum channels, in a spirit similar to that developed in [36]–[39] for other communication tasks. We first recall some preliminary background on quantum information in infinite-dimensional, separable Hilbert spaces in Section II. We now summarize the main contributions of our paper:

- In Section III, we define several energy-constrained communication tasks, including quantum communication with

a uniform energy constraint, entanglement transmission with an average energy constraint, private communication with a uniform energy constraint, and secret key transmission with an average energy constraint.

- In Section IV, we develop several code conversions between these various communication tasks, which allow us to conclude non-trivial relations between the capacities corresponding to them, as summarized in Section V and Theorem 1.
- Section VI proves that the regularized, energy-constrained coherent information is an achievable rate for all of the tasks, whenever the energy observable satisfies the Gibbs condition of having a well defined thermal state for all temperatures (Definition 3) and the channel satisfies a finite output-entropy condition (Condition 1). This result is stated as Theorem 2.
- For degradable channels satisfying the same conditions, we find in Section VII that the single-letter energy-constrained coherent information is equal to all of the capacities (stated as Theorem 3).
- Section VIII establishes a regularized converse for the energy-constrained private capacity (stated as Theorem 4), and it also establishes that the regularized, energy-constrained coherent information is equal to the capacity for quantum communication with a uniform energy constraint and entanglement transmission with an average energy constraint, under the same conditions on the energy observable and the channel. This latter result is stated as Theorem 5.
- We finally apply our results to quantum Gaussian channels in Section X and recover several results already established in the literature on Gaussian quantum information. In some cases, we establish new results, like the formula for private capacity in (3).
- In Section XI, we discuss how our general framework, along with recent developments in [40], allow for concluding estimates for the energy-constrained private and quantum capacities of particular non-Gaussian channels. Therein, we also consider alternative energy constraints for the pure-loss and quantum-limited amplifier channels, and we bound the capacities in these settings.

We conclude in Section XII with a summary and some open questions.

We would like to suggest that our contribution on this topic is timely. At the least, we think it should be a useful resource for the community of researchers working on related topics to have such a formalism and associated results written down explicitly, even though a skeptic might argue that they have been part of the folklore of quantum information theory for many years now. To support our viewpoint, we note that some statements made in several papers released in the past few years suggest that energy-constrained quantum and private capacities have not been sufficiently clarified in the existing literature. For example, in [41], one of the main results contributed was a non-tight upper bound on the private capacity of a pure-loss bosonic channel, in spite of the fact that (3) was already part of the folklore of quantum information theory. In [42], it is stated that the “entropy photon-number

inequality turns out to be crucial in the determining the classical capacity regions of the quantum bosonic broadcast and wiretap channels,” in spite of the fact that no such argument is needed to establish the quantum or private capacity of the pure-loss channel. Similarly, it is stated in [43] that the entropy photon-number inequality “conjecture is of particular significance in quantum information theory since if it were true then it would allow one to evaluate classical capacities of various bosonic channels, e.g. the bosonic broadcast channel and the wiretap channel.” Thus, it seems timely and legitimate to confirm that no such entropy photon-number inequality or minimum output-entropy conjecture is necessary in order to establish the results regarding quantum or private capacity of the pure-loss channel—the existing literature (specifically, [28, Th. 8] and now the previously folklore (331)) has established these capacities. The same is the case for the quantum-limited amplifier channel due to the results of [35]. The entropy photon-number inequality indeed implies formulas for quantum and private capacities of the quantum-limited attenuator and amplifier channels, but it appears to be much stronger than what is actually necessary to accomplish this goal. The different proof of these formulas that we give in the present paper (see Section X) is based on the monotonicity of quantum relative entropy, concavity of coherent information of degradable channels with respect to the input density operator, and covariance of Gaussian channels with respect to displacement operators.

II. QUANTUM INFORMATION PRELIMINARIES

A. Quantum States and Channels

Background on quantum information in infinite-dimensional systems is available in [39] (see also [37], [44]–[48]). We review some aspects here. We use \mathcal{H} throughout the paper to denote a separable Hilbert space, unless specified otherwise. Let $I_{\mathcal{H}}$ denote the identity operator acting on \mathcal{H} . Let $\mathcal{B}(\mathcal{H})$ denote the set of bounded linear operators acting on \mathcal{H} , and let $\mathcal{P}(\mathcal{H})$ denote the subset of $\mathcal{B}(\mathcal{H})$ that consists of positive semi-definite operators. Let $\mathcal{T}(\mathcal{H})$ denote the set of trace-class operators, those operators A for which the trace norm is finite: $\|A\|_1 \equiv \text{Tr}\{|A|\} < \infty$, where $|A| \equiv \sqrt{A^\dagger A}$. The Hilbert-Schmidt norm of A is defined as $\|A\|_2 \equiv \sqrt{\text{Tr}\{A^\dagger A\}}$. Let $\mathcal{D}(\mathcal{H})$ denote the set of density operators (states), which consists of the positive semi-definite, trace-class operators with trace equal to one. A state $\rho \in \mathcal{D}(\mathcal{H})$ is pure if there exists a unit vector $|\psi\rangle \in \mathcal{H}$ such that $\rho = |\psi\rangle\langle\psi|$. Every density operator $\rho \in \mathcal{D}(\mathcal{H})$ has a spectral decomposition in terms of some countable, orthonormal basis $\{|\phi_k\rangle\}_k$ as

$$\rho = \sum_k p(k) |\phi_k\rangle\langle\phi_k|, \quad (8)$$

where $p(k)$ is a probability distribution. The tensor product of two Hilbert spaces \mathcal{H}_A and \mathcal{H}_B is denoted by $\mathcal{H}_A \otimes \mathcal{H}_B$ or \mathcal{H}_{AB} . Given a multipartite density operator $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, we unambiguously write $\rho_A = \text{Tr}_{\mathcal{H}_B}\{\rho_{AB}\}$ for the reduced density operator on system A . Every density operator ρ has a purification $|\phi^\rho\rangle \in \mathcal{H}' \otimes \mathcal{H}$, for an auxiliary Hilbert space \mathcal{H}' , where $\|\phi^\rho\|_2 = 1$ and $\text{Tr}_{\mathcal{H}'}\{|\phi^\rho\rangle\langle\phi^\rho|\} = \rho$.

All purifications are related by an isometry acting on the purifying system. A state $\rho_{RA} \in \mathcal{D}(\mathcal{H}_R \otimes \mathcal{H}_A)$ extends $\rho_A \in \mathcal{D}(\mathcal{H}_A)$ if $\text{Tr}_{\mathcal{H}_R}\{\rho_{RA}\} = \rho_A$. We also say that ρ_{RA} is an extension of ρ_A . In what follows, we abbreviate notation like $\text{Tr}_{\mathcal{H}_R}$ as Tr_R .

For finite-dimensional Hilbert spaces \mathcal{H}_R and \mathcal{H}_S such that $\dim(\mathcal{H}_R) = \dim(\mathcal{H}_S) \equiv M$, we define the maximally entangled state $\Phi_{RS} \in \mathcal{D}(\mathcal{H}_R \otimes \mathcal{H}_S)$ of Schmidt rank M as

$$\Phi_{RS} \equiv \frac{1}{M} \sum_{m,m'} |m\rangle\langle m'|_R \otimes |m\rangle\langle m'|_S, \quad (9)$$

where $\{|m\rangle\}_m$ is an orthonormal basis for \mathcal{H}_R and \mathcal{H}_S . We define the maximally correlated state $\bar{\Phi}_{RS} \in \mathcal{D}(\mathcal{H}_R \otimes \mathcal{H}_S)$ as

$$\bar{\Phi}_{RS} \equiv \frac{1}{M} \sum_m |m\rangle\langle m|_R \otimes |m\rangle\langle m|_S, \quad (10)$$

which can be understood as arising by applying a completely dephasing channel $\sum_m |m\rangle\langle m|(\cdot)|m\rangle\langle m|$ to either system R or S of the maximally entangled state Φ_{RS} . We define the maximally mixed state of system S as $\pi_S \equiv I_S/M$.

A quantum channel $\mathcal{N} : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_B)$ is a completely positive, trace-preserving linear map. The Stinespring dilation theorem [49] implies that there exists another Hilbert space \mathcal{H}_E and a linear isometry $U : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$ such that for all $\tau \in \mathcal{T}(\mathcal{H}_A)$

$$\mathcal{N}(\tau) = \text{Tr}_E\{U\tau U^\dagger\}. \quad (11)$$

The Stinespring representation theorem also implies that every quantum channel has a Kraus representation with a countable set $\{K_l\}_l$ of bounded Kraus operators:

$$\mathcal{N}(\tau) = \sum_l K_l \tau K_l^\dagger, \quad (12)$$

where $\sum_l K_l^\dagger K_l = I_{\mathcal{H}_A}$. The Kraus operators are defined by the relation

$$\langle \varphi | K_l | \psi \rangle = \langle \varphi | \otimes \langle l | U | \psi \rangle, \quad (13)$$

for $|\varphi\rangle \in \mathcal{H}_B$, $|\psi\rangle \in \mathcal{H}_A$, and $\{|l\rangle\}_l$ some orthonormal basis for \mathcal{H}_E [50].

A complementary channel $\hat{\mathcal{N}} : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_E)$ of \mathcal{N} is defined for all $\tau \in \mathcal{T}(\mathcal{H}_A)$ as

$$\hat{\mathcal{N}}(\tau) = \text{Tr}_B\{U\tau U^\dagger\}. \quad (14)$$

Complementary channels are unique up to partial isometries acting on the Hilbert space \mathcal{H}_E .

A quantum channel $\mathcal{N} : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_B)$ is degradable [51] if there exists a quantum channel $\mathcal{D} : \mathcal{T}(\mathcal{H}_B) \rightarrow \mathcal{T}(\mathcal{H}_E)$, called a degrading channel, such that for some complementary channel $\hat{\mathcal{N}} : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_E)$ and all $\tau \in \mathcal{T}(\mathcal{H}_A)$:

$$\hat{\mathcal{N}}(\tau) = (\mathcal{D} \circ \mathcal{N})(\tau). \quad (15)$$

A positive operator-valued measure (POVM) is a set $\{\Lambda^x\}_x$ of positive semi-definite operators acting on a Hilbert space \mathcal{H} such that $\sum_x \Lambda^x = I_{\mathcal{H}}$.

B. Quantum Fidelity and Trace Distance

The fidelity of two quantum states $\rho, \sigma \in \mathcal{D}(\mathcal{H})$ is defined as [52]

$$F(\rho, \sigma) \equiv \|\sqrt{\rho}\sqrt{\sigma}\|_1^2. \quad (16)$$

Uhlmann's theorem is the statement that the fidelity has the following alternate expression as a probability overlap [52]:

$$F(\rho, \sigma) = \sup_U |\langle \phi^\rho | U \otimes I_{\mathcal{H}} | \phi^\sigma \rangle|^2, \quad (17)$$

where $|\phi^\rho\rangle \in \mathcal{H}' \otimes \mathcal{H}$ and $|\phi^\sigma\rangle \in \mathcal{H}'' \otimes \mathcal{H}$ are fixed purifications of ρ and σ , respectively, and the optimization is with respect to all partial isometries $U : \mathcal{H}'' \rightarrow \mathcal{H}'$. The fidelity is non-decreasing with respect to a quantum channel $\mathcal{N} : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_B)$, in the sense that for all $\rho, \sigma \in \mathcal{D}(\mathcal{H}_A)$:

$$F(\mathcal{N}(\rho), \mathcal{N}(\sigma)) \geq F(\rho, \sigma). \quad (18)$$

A simple modification of Uhlmann's theorem, found by combining (17) with the monotonicity property in (18), implies that for a given extension ρ_{AB} of ρ_A , there exists an extension σ_{AB} of σ_A such that

$$F(\rho_{AB}, \sigma_{AB}) = F(\rho_A, \sigma_A). \quad (19)$$

The trace distance between states ρ and σ is defined as $\|\rho - \sigma\|_1$. One can normalize the trace distance by multiplying it by 1/2 so that the resulting quantity lies in the interval $[0, 1]$. The trace distance obeys a direct-sum property: for an orthonormal basis $\{|x\rangle\}_x$ for an auxiliary Hilbert space \mathcal{H}_X , probability distributions $p(x)$ and $q(x)$, and sets $\{\rho^x\}_x$ and $\{\sigma^x\}_x$ of states in $\mathcal{D}(\mathcal{H}_B)$, which realize classical-quantum states

$$\rho_{XB} \equiv \sum_x p(x) |x\rangle\langle x|_X \otimes \rho_B^x, \quad (20)$$

$$\sigma_{XB} \equiv \sum_x q(x) |x\rangle\langle x|_X \otimes \sigma_B^x, \quad (21)$$

the following holds

$$\|\rho_{XB} - \sigma_{XB}\|_1 = \sum_x \|p(x)\rho_B^x - q(x)\sigma_B^x\|_1. \quad (22)$$

The trace distance is monotone non-increasing with respect to a quantum channel $\mathcal{N} : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_B)$, in the sense that for all $\rho, \sigma \in \mathcal{D}(\mathcal{H}_A)$:

$$\|\mathcal{N}(\rho) - \mathcal{N}(\sigma)\|_1 \leq \|\rho - \sigma\|_1. \quad (23)$$

The following equality holds for any two pure states $\phi, \psi \in \mathcal{D}(\mathcal{H})$:

$$\frac{1}{2} \|\phi - \psi\|_1 = \sqrt{1 - F(\phi, \psi)}. \quad (24)$$

For any two arbitrary states $\rho, \sigma \in \mathcal{D}(\mathcal{H})$, the following inequalities hold

$$1 - \sqrt{F(\rho, \sigma)} \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)}. \quad (25)$$

The inequality on the left is a consequence of the Powers-Stormer inequality [53, Lemma 4.1], which states that $\|P - Q\|_1 \geq \|P^{1/2} - Q^{1/2}\|_2^2$ for $P, Q \in \mathcal{P}(\mathcal{H})$. The inequality on the right follows from the monotonicity of

trace distance with respect to quantum channels, the identity in (24), and Uhlmann's theorem in (17). These inequalities are called Fuchs-van-de-Graaf inequalities, as they were established in [54] for finite-dimensional states.

C. Quantum Entropies and Information

The quantum entropy of a state $\rho \in \mathcal{D}(\mathcal{H})$ is defined as

$$H(\rho) \equiv \text{Tr}\{\eta(\rho)\}, \quad (26)$$

where $\eta(x) = -x \log_2 x$ if $x > 0$ and $\eta(0) = 0$. The trace in the above equation can be taken with respect to any countable orthonormal basis of \mathcal{H} [55, Definition 2]. The quantum entropy is a non-negative, concave, lower semicontinuous function on $\mathcal{D}(\mathcal{H})$ [56]. It is also not necessarily finite (see [57]). When ρ_A is assigned to a system A , we write $H(A)_\rho \equiv H(\rho_A)$.

The quantum relative entropy $D(\rho\|\sigma)$ of $\rho, \sigma \in \mathcal{D}(\mathcal{H})$ is defined as [58], [59]

$$D(\rho\|\sigma) \equiv [\ln 2]^{-1} \sum_{i,j} |\langle \phi_i | \psi_j \rangle|^2 \left[p(i) \ln \left(\frac{p(i)}{q(j)} \right) + q(j) - p(i) \right], \quad (27)$$

where $\rho = \sum_i p(i) |\phi_i\rangle\langle\phi_i|$ and $\sigma = \sum_j q(j) |\psi_j\rangle\langle\psi_j|$ are spectral decompositions of ρ and σ with $\{|\phi_i\rangle\}_i$ and $\{|\psi_j\rangle\}_j$ orthonormal bases. The prefactor $[\ln 2]^{-1}$ is there to ensure that the units of the quantum relative entropy are bits. We take the convention in (27) that $0 \ln 0 = 0 \ln \left(\frac{0}{0}\right) = 0$ but $\ln\left(\frac{c}{0}\right) = +\infty$ for $c > 0$. Each term in the sum in (27) is non-negative due to the inequality

$$x \ln(x/y) + y - x \geq 0 \quad (28)$$

holding for all $x, y \geq 0$ [58]. Thus, by Tonelli's theorem, the sums in (27) may be taken in either order as discussed in [58] and [59], and it follows that $D(\rho\|\sigma) \geq 0$ for all $\rho, \sigma \in \mathcal{D}(\mathcal{H})$, with equality holding if and only if $\rho = \sigma$ [58]. If the support of ρ is not contained in the support of σ , then $D(\rho\|\sigma) = +\infty$. The converse statement need not hold in general: there exist $\rho, \sigma \in \mathcal{D}(\mathcal{H})$ with the support of ρ contained in the support of σ such that $D(\rho\|\sigma) = +\infty$. For example, take ρ and σ diagonal in the same basis with the eigenvalues of ρ as in [57, eq. (7)] and those of σ as $\propto 1/n^2$ for $n \geq \lceil e \rceil$.

One of the most important properties of the quantum relative entropy $D(\rho\|\sigma)$ is that it is monotone with respect to a quantum channel $\mathcal{N} : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_B)$ [60]:

$$D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)). \quad (29)$$

The quantum mutual information $I(A; B)_\rho$ of a bipartite state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is defined as [59]

$$I(A; B)_\rho = D(\rho_{AB}\|\rho_A \otimes \rho_B), \quad (30)$$

and obeys the bound [59]

$$I(A; B)_\rho \leq 2 \min\{H(A)_\rho, H(B)_\rho\}. \quad (31)$$

The coherent information $I(A)B)_\rho$ of ρ_{AB} is defined as [45], [61]

$$I(A)B)_\rho \equiv I(A; B)_\rho - H(A)_\rho, \quad (32)$$

when $H(A)_\rho < \infty$. This expression reduces to

$$I(A)B)_\rho = H(B)_\rho - H(AB)_\rho \quad (33)$$

if $H(B)_\rho < \infty$ [45], [61].

The mutual information of a quantum channel $\mathcal{N} : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_B)$ with respect to a state $\rho \in \mathcal{D}(\mathcal{H}_A)$ is defined as [45]

$$I(\rho, \mathcal{N}) \equiv I(R; B)_\omega, \quad (34)$$

where $\omega_{RB} \equiv (\text{id}_R \otimes \mathcal{N}_{A \rightarrow B})(\psi_{RA}^\rho)$ and $\psi_{RA}^\rho \in \mathcal{D}(\mathcal{H}_R \otimes \mathcal{H}_A)$ is a purification of ρ , with $\mathcal{H}_R \simeq \mathcal{H}_A$. The coherent information of a quantum channel $\mathcal{N} : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_B)$ with respect to a state $\rho \in \mathcal{D}(\mathcal{H}_A)$ is defined as [45]

$$I_c(\rho, \mathcal{N}) \equiv I(R)B)_\omega, \quad (35)$$

with ω_{RB} defined as above. These quantities obey a data processing inequality, which is that for a quantum channel $\mathcal{M} : \mathcal{T}(\mathcal{H}_B) \rightarrow \mathcal{T}(\mathcal{H}_C)$ and ρ and \mathcal{N} as before, the following holds [45]

$$I(\rho, \mathcal{N}) \geq I(\rho, \mathcal{M} \circ \mathcal{N}), \quad (36)$$

$$I_c(\rho, \mathcal{N}) \geq I_c(\rho, \mathcal{M} \circ \mathcal{N}). \quad (37)$$

We require the following proposition for some of the developments in this paper:

Proposition 1: Let \mathcal{N} be a degradable quantum channel and $\hat{\mathcal{N}}$ a complementary channel for it. Let ρ_0 and ρ_1 be states and let $\rho_\lambda = \lambda\rho_0 + (1-\lambda)\rho_1$ for $\lambda \in [0, 1]$. Suppose that the entropies $H(\rho_\lambda)$ and $H(\mathcal{N}(\rho_\lambda))$ are finite for all $\lambda \in [0, 1]$. Then the coherent information of \mathcal{N} is concave with respect to these inputs, in the sense that

$$\lambda I_c(\rho_0, \mathcal{N}) + (1-\lambda) I_c(\rho_1, \mathcal{N}) \leq I_c(\rho_\lambda, \mathcal{N}). \quad (38)$$

Proof: This was established for the finite-dimensional case in [62]. We follow the proof given in [1, Th. 13.5.2]. First note that $H(\rho_\lambda)$ and $H(\mathcal{N}(\rho_\lambda))$ being finite for all $\lambda \in [0, 1]$ imply that $H(\hat{\mathcal{N}}(\rho_\lambda))$ is finite, by an application of the isometric invariance of the entropy, the Stinespring dilation theorem, and the entropy triangle inequality from [55, Th. 2], allowing us to conclude that

$$H(\hat{\mathcal{N}}(\rho_\lambda)) \leq H(\rho_\lambda) + H(\mathcal{N}(\rho_\lambda)). \quad (39)$$

Set $\bar{\lambda} \equiv 1 - \lambda$. Consider that

$$\begin{aligned} & I_c(\rho_\lambda, \mathcal{N}) - \lambda I_c(\rho_0, \mathcal{N}) - \bar{\lambda} I_c(\rho_1, \mathcal{N}) \\ &= H(\mathcal{N}(\rho_\lambda)) - H(\hat{\mathcal{N}}(\rho_\lambda)) - \lambda H(\mathcal{N}(\rho_0)) \\ & \quad + \lambda H(\hat{\mathcal{N}}(\rho_0)) - \bar{\lambda} H(\mathcal{N}(\rho_1)) + \bar{\lambda} H(\hat{\mathcal{N}}(\rho_1)). \end{aligned} \quad (40)$$

Defining the states

$$\rho_{UB} = \lambda |0\rangle\langle 0|_U \otimes \mathcal{N}(\rho_0) + \bar{\lambda} |1\rangle\langle 1|_U \otimes \mathcal{N}(\rho_1), \quad (41)$$

$$\sigma_{UE} = \lambda |0\rangle\langle 0|_U \otimes \hat{\mathcal{N}}(\rho_0) + \bar{\lambda} |1\rangle\langle 1|_U \otimes \hat{\mathcal{N}}(\rho_1), \quad (42)$$

we can then rewrite the last line above as

$$I(U; B)_\rho - I(U; E)_\sigma. \quad (43)$$

This quantity is non-negative from data processing of mutual information because we can apply the degrading channel $\mathcal{D}_{B \rightarrow E}$ to system B of ρ_{UB} and recover σ_{UE} :

$$\sigma_{UE} = \mathcal{D}_{B \rightarrow E}(\rho_{UB}). \quad (44)$$

This concludes the proof. \square

The conditional quantum mutual information (CQMI) of a finite-dimensional tripartite state ρ_{ABC} is defined as

$$I(A; B|C)_\rho \equiv H(AC)_\rho + H(BC)_\rho - H(ABC)_\rho - H(C)_\rho. \quad (45)$$

In the general case, it is defined as [47], [48]

$$I(A; B|C)_\rho \equiv \sup_{P_A} \left\{ I(A; BC)_{Q\rho Q} - I(A; C)_{Q\rho Q} : Q = P_A \otimes I_{BC} \right\}, \quad (46)$$

where the supremum is with respect to all finite-rank projections $P_A \in \mathcal{B}(\mathcal{H}_A)$ and we take the convention as in [47] and [48] that $I(A; BC)_{Q\rho Q} = \lambda I(A; BC)_{Q\rho Q/\lambda}$ where $\lambda = \text{Tr}\{Q\rho_{ABC}Q\}$. The above definition guarantees that many properties of CQMI in finite dimensions carry over to the general case [47], [48]. In particular, the following chain rule holds for a four-party state $\rho_{ABCD} \in \mathcal{D}(\mathcal{H}_{ABCD})$:

$$I(A; BC|D)_\rho = I(A; C|D)_\rho + I(A; B|CD)_\rho. \quad (47)$$

Fano's inequality [63] is the statement that for random variables X and Y with alphabets \mathcal{X} and \mathcal{Y} , respectively, the following inequality holds

$$H(X|Y) \leq \varepsilon \log_2(|\mathcal{X}| - 1) + h_2(\varepsilon), \quad (48)$$

where

$$\varepsilon \equiv \Pr\{X \neq Y\}, \quad (49)$$

$$h_2(\varepsilon) \equiv -\varepsilon \log_2 \varepsilon - (1 - \varepsilon) \log_2(1 - \varepsilon). \quad (50)$$

Observe that $\lim_{\varepsilon \rightarrow 0} h_2(\varepsilon) = 0$. Let $\rho_{AB}, \sigma_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ with $\dim(\mathcal{H}_A) < \infty$, $\varepsilon \in [0, 1]$, and suppose that $\|\rho_{AB} - \sigma_{AB}\|_1 / 2 \leq \varepsilon$. The Alicki–Fannes–Winter (AFW) inequality is as follows [64], [65]:

$$\left| H(A|B)_\rho - H(A|B)_\sigma \right| \leq 2\varepsilon \log_2 \dim(\mathcal{H}_A) + g(\varepsilon), \quad (51)$$

where

$$g(\varepsilon) \equiv (\varepsilon + 1) \log_2(\varepsilon + 1) - \varepsilon \log_2 \varepsilon. \quad (52)$$

Observe that $\lim_{\varepsilon \rightarrow 0} g(\varepsilon) = 0$. If the states are classical on the first system, as in (20)–(21), and $\dim(\mathcal{H}_X) < \infty$ and $\|\rho_{XB} - \sigma_{XB}\|_1 / 2 \leq \varepsilon$, then the inequality can be strengthened to [1, Th. 11.10.3]

$$\left| H(X|B)_\rho - H(X|B)_\sigma \right| \leq \varepsilon \log_2 \dim(\mathcal{H}_X) + g(\varepsilon). \quad (53)$$

III. ENERGY-CONSTRAINED QUANTUM AND PRIVATE CAPACITIES

In this section, we define various notions of energy-constrained quantum and private capacity of quantum channels. We start by defining an energy observable (see [39, Definition 11.3]):

Definition 1 (Energy Observable): Let G be a positive semi-definite operator, i.e., $G \in \mathcal{P}(\mathcal{H}_A)$. Throughout, we refer to G as an energy observable. In more detail, we define G as follows: let $\{|e_j\rangle\}_j$ be an orthonormal basis for a Hilbert space \mathcal{H} , and let $\{g_j\}_j$ be a sequence of non-negative real numbers bounded from below. Then the following formula

$$G|\psi\rangle = \sum_{j=1}^{\infty} g_j |e_j\rangle \langle e_j|\psi\rangle \quad (54)$$

defines a self-adjoint operator G on the dense domain $\{|\psi\rangle : \sum_{j=1}^{\infty} g_j^2 |\langle e_j|\psi\rangle|^2 < \infty\}$, for which $|e_j\rangle$ is an eigenvector with corresponding eigenvalue g_j .

For a state $\rho \in \mathcal{D}(\mathcal{H}_A)$, we follow the convention [38] that

$$\text{Tr}\{G\rho\} \equiv \sup_n \text{Tr}\{\Pi_n G \Pi_n \rho\}, \quad (55)$$

where Π_n denotes the spectral projection of G corresponding to the interval $[0, n]$.

Definition 2: The n th extension \overline{G}_n of an energy observable G is defined as

$$\overline{G}_n \equiv \frac{1}{n} [G \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes G], \quad (56)$$

where n is the number of factors in each tensor product above.

In the subsections that follow, let $\mathcal{N} : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_B)$ denote a quantum channel, and let G be an energy observable. Let $n \in \mathbb{N}$ denote the number of channel uses, $M \in \mathbb{N}$ the size of a code, $P \in [0, \infty)$ an energy parameter, and $\varepsilon \in [0, 1]$ an error parameter. In what follows, we discuss four different notions of capacity: quantum communication with a uniform energy constraint, entanglement transmission with an average energy constraint, private communication with a uniform energy constraint, and secret key transmission with an average energy constraint. Note that it is possible to consider other combinations, such as quantum communication with an average energy constraint, or secret key transmission with a uniform energy constraint, but we have decided to focus on the above four scenarios for simplicity.

A. Quantum Communication With a Uniform Energy Constraint

An $(n, M, G, P, \varepsilon)$ code for quantum communication with uniform energy constraint consists of an encoding channel $\mathcal{E}^n : \mathcal{T}(\mathcal{H}_S) \rightarrow \mathcal{T}(\mathcal{H}_A^{\otimes n})$ and a decoding channel $\mathcal{D}^n : \mathcal{T}(\mathcal{H}_B^{\otimes n}) \rightarrow \mathcal{T}(\mathcal{H}_S)$, where $M = \dim(\mathcal{H}_S)$. The energy constraint is uniform, in the sense that the following bound is required to hold for all states resulting from the output of the encoding channel \mathcal{E}^n :

$$\text{Tr}\{\overline{G}_n \mathcal{E}^n(\rho_S)\} \leq P, \quad (57)$$

where $\rho_S \in \mathcal{D}(\mathcal{H}_S)$. Note that

$$\text{Tr}\{\overline{G}_n \mathcal{E}^n(\rho_S)\} = \text{Tr}\{G \overline{\rho}_n\}, \quad (58)$$

where

$$\bar{\rho}_n \equiv \frac{1}{n} \sum_{i=1}^n \text{Tr}_{A^n \setminus A_i} \{ \mathcal{E}^n(\rho_S) \}. \quad (59)$$

due to the i.i.d. nature of the observable \bar{G}_n . Furthermore, the encoding and decoding channels are good for quantum communication, in the sense that for all pure states $\phi_{RS} \in \mathcal{D}(\mathcal{H}_R \otimes \mathcal{H}_S)$, where \mathcal{H}_R is isomorphic to \mathcal{H}_S , the following entanglement fidelity criterion holds

$$F(\phi_{RS}, (\text{id}_R \otimes [\mathcal{D}^n \circ \mathcal{N}^{\otimes n} \circ \mathcal{E}^n])(\phi_{RS})) \geq 1 - \varepsilon. \quad (60)$$

A rate R is achievable for quantum communication over \mathcal{N} subject to the uniform energy constraint P if for all $\varepsilon \in (0, 1)$, $\delta > 0$, and sufficiently large n , there exists an $(n, 2^{n[R-\delta]}, G, P, \varepsilon)$ quantum communication code with uniform energy constraint. The quantum capacity $Q(\mathcal{N}, G, P)$ of \mathcal{N} with uniform energy constraint is equal to the supremum of all achievable rates.

B. Entanglement Transmission With an Average Energy Constraint

An $(n, M, G, P, \varepsilon)$ code for entanglement transmission with average energy constraint is defined very similarly as above, except that the requirements are less stringent. The energy constraint holds on average, in the sense that it need only hold for the maximally mixed state π_S input to the encoding channel \mathcal{E}^n :

$$\text{Tr} \{ \bar{G}_n \mathcal{E}^n(\pi_S) \} \leq P. \quad (61)$$

Furthermore, we only demand that the particular maximally entangled state $\Phi_{RS} \in \mathcal{D}(\mathcal{H}_R \otimes \mathcal{H}_S)$, defined as

$$\Phi_{RS} \equiv \frac{1}{M} \sum_{m, m'=1}^M |m\rangle \langle m'|_R \otimes |m\rangle \langle m'|_S, \quad (62)$$

is preserved with good fidelity:

$$F(\Phi_{RS}, (\text{id}_R \otimes [\mathcal{D}^n \circ \mathcal{N}^{\otimes n} \circ \mathcal{E}^n])(\Phi_{RS})) \geq 1 - \varepsilon. \quad (63)$$

A rate R is achievable for entanglement transmission over \mathcal{N} subject to the average energy constraint P if for all $\varepsilon \in (0, 1)$, $\delta > 0$, and sufficiently large n , there exists an $(n, 2^{n[R-\delta]}, G, P, \varepsilon)$ entanglement transmission code with average energy constraint. The entanglement transmission capacity $E(\mathcal{N}, G, P)$ of \mathcal{N} with average energy constraint is equal to the supremum of all achievable rates.

From definitions, it immediately follows that quantum capacity with uniform energy constraint can never exceed entanglement transmission capacity with average energy constraint:

$$Q(\mathcal{N}, G, P) \leq E(\mathcal{N}, G, P). \quad (64)$$

In Section V, we establish the opposite inequality.

C. Private Communication With a Uniform Energy Constraint

An $(n, M, G, P, \varepsilon)$ code for private communication consists of a set $\{\rho_{A^n}^m\}_{m=1}^M$ of quantum states, each in $\mathcal{D}(\mathcal{H}_A^{\otimes n})$, and a POVM $\{\Lambda_{B^n}^m\}_{m=1}^M$ such that

$$\text{Tr} \{ \bar{G}_n \rho_{A^n}^m \} \leq P, \quad (65)$$

$$\text{Tr} \{ \Lambda_{B^n}^m \mathcal{N}^{\otimes n}(\rho_{A^n}^m) \} \geq 1 - \varepsilon, \quad (66)$$

$$\frac{1}{2} \left\| \hat{\mathcal{N}}^{\otimes n}(\rho_{A^n}^m) - \omega_{E^n} \right\|_1 \leq \varepsilon, \quad (67)$$

for all $m \in \{1, \dots, M\}$, with ω_{E^n} some fixed state in $\mathcal{D}(\mathcal{H}_E^{\otimes n})$. In the above, $\hat{\mathcal{N}}$ is a channel complementary to \mathcal{N} . Observe that

$$\text{Tr} \{ \bar{G}_n \rho_{A^n}^m \} = \text{Tr} \{ G \bar{\rho}_A^m \}, \quad (68)$$

where

$$\bar{\rho}_A^m \equiv \frac{1}{n} \sum_{i=1}^n \text{Tr}_{A^n \setminus A_i} \{ \rho_{A^n}^m \}. \quad (69)$$

A rate R is achievable for private communication over \mathcal{N} subject to uniform energy constraint P if for all $\varepsilon \in (0, 1)$, $\delta > 0$, and sufficiently large n , there exists an $(n, 2^{n[R-\delta]}, G, P, \varepsilon)$ private communication code. The private capacity $P(\mathcal{N}, G, P)$ of \mathcal{N} with uniform energy constraint is equal to the supremum of all achievable rates.

D. Secret Key Transmission With an Average Energy Constraint

An $(n, M, G, P, \varepsilon)$ code for secret key transmission with average energy constraint is defined very similarly as above, except that the requirements are less stringent. The energy constraint holds on average, in the sense that it need only hold for the average input state:

$$\frac{1}{M} \sum_{m=1}^M \text{Tr} \{ \bar{G}_n \rho_{A^n}^m \} \leq P. \quad (70)$$

Furthermore, we only demand that the conditions in (66)–(67) hold on average:

$$\frac{1}{M} \sum_{m=1}^M \text{Tr} \{ \Lambda_{B^n}^m \mathcal{N}^{\otimes n}(\rho_{A^n}^m) \} \geq 1 - \varepsilon, \quad (71)$$

$$\frac{1}{M} \sum_{m=1}^M \frac{1}{2} \left\| \hat{\mathcal{N}}^{\otimes n}(\rho_{A^n}^m) - \omega_{E^n} \right\|_1 \leq \varepsilon, \quad (72)$$

with ω_{E^n} some fixed state in $\mathcal{D}(\mathcal{H}_E^{\otimes n})$.

A rate R is achievable for secret key transmission over \mathcal{N} subject to the average energy constraint P if for all $\varepsilon \in (0, 1)$, $\delta > 0$, and sufficiently large n , there exists an $(n, 2^{n[R-\delta]}, G, P, \varepsilon)$ secret key transmission code with average energy constraint. The secret key transmission capacity $K(\mathcal{N}, G, P)$ of \mathcal{N} with average energy constraint is equal to the supremum of all achievable rates.

From definitions, it immediately follows that private capacity with uniform energy constraint can never exceed secret key transmission capacity with average energy constraint

$$P(\mathcal{N}, G, P) \leq K(\mathcal{N}, G, P). \quad (73)$$

In Section V, we establish the opposite inequality.

IV. CODE CONVERSIONS

In this section, we establish several code conversions, which allow for converting one type of code into another type of code along with some loss in the code parameters. In particular, in the forthcoming subsections, we show how to convert

- 1) an entanglement transmission code with an average energy constraint to a quantum communication code with a uniform energy constraint,
- 2) a quantum communication code with a uniform energy constraint to a private communication code with a uniform energy constraint,
- 3) and a secret key transmission code with an average energy constraint to a private communication code with a uniform energy constraint.

These code conversions then allow us to establish several non-trivial relations between the corresponding capacities, which we do in Section V.

A. Entanglement Transmission With an Average Energy Constraint to Quantum Communication With a Uniform Energy Constraint

In this subsection, we show how an entanglement transmission code with an average energy constraint implies the existence of a quantum communication code with a uniform energy constraint, such that there is a loss in performance in the resulting code with respect to several code parameters.

A result like this was first established in [66] and reviewed in [67]–[69], under the assumption that there is no energy constraint. Here we follow the proof approach available in [68] and [69], but we make several modifications in order to deal with going from an average energy constraint to a uniform energy constraint.

Proposition 2: For all $\delta \in (1/M, 1/2)$, the existence of an $(n, M, G, P, \varepsilon)$ entanglement transmission code with average energy constraint implies the existence of an $(n, \lfloor \delta M \rfloor, G, P/(1-2\delta), \min\{1, 2\sqrt{\varepsilon/(\delta-1/M)}\})$ quantum communication code with uniform energy constraint.

Proof: Suppose that an $(n, M, G, P, \varepsilon)$ entanglement transmission code with average energy constraint exists. This implies that the conditions in (61) and (63) hold. Let $\mathcal{C}^n : \mathcal{T}(\mathcal{H}_S) \rightarrow \mathcal{T}(\mathcal{H}_S)$ denote the finite-dimensional channel consisting of the encoding, communication channel, and decoding:

$$\mathcal{C}^n \equiv \mathcal{D}^n \circ \mathcal{N}^{\otimes n} \circ \mathcal{E}^n. \quad (74)$$

We proceed with the following algorithm:

- 1) Set $k = M$, $\mathcal{H}_M = \mathcal{H}_S$, and $\delta \in (1/M, 1/2)$. Suppose for now that δM is a positive integer.
- 2) Set $|\phi_k\rangle \in \mathcal{H}_k$ to be a state vector such that the input-output fidelity is minimized:

$$|\phi_k\rangle \equiv \arg \min_{|\phi\rangle \in \mathcal{H}_k} \langle \phi | \mathcal{C}^n(|\phi\rangle\langle\phi|) | \phi \rangle, \quad (75)$$

and set the fidelity F_k and energy E_k of $|\phi_k\rangle$ as follows:

$$F_k \equiv \min_{|\phi\rangle \in \mathcal{H}_k} \langle \phi | \mathcal{C}^n(|\phi\rangle\langle\phi|) | \phi \rangle \quad (76)$$

$$= \langle \phi_k | \mathcal{C}^n(|\phi_k\rangle\langle\phi_k|) | \phi_k \rangle, \quad (77)$$

$$E_k \equiv \text{Tr}\{\overline{G}_n \mathcal{E}^n(|\phi_k\rangle\langle\phi_k|)\}. \quad (78)$$

- 3) Set

$$\mathcal{H}_{k-1} \equiv \text{span}\{|\psi\rangle \in \mathcal{H}_k : |\langle\psi|\phi_k\rangle| = 0\}. \quad (79)$$

That is, \mathcal{H}_{k-1} is set to the orthogonal complement of $|\phi_k\rangle$ in \mathcal{H}_k , so that $\mathcal{H}_k = \mathcal{H}_{k-1} \oplus \text{span}\{|\phi_k\rangle\}$. Set $k := k - 1$.

- 4) Repeat steps 2-3 until $k = (1 - \delta)M$ after step 3.
- 5) Let $|\phi_k\rangle \in \mathcal{H}_k$ be a state vector such that the input energy is maximized:

$$|\phi_k\rangle \equiv \arg \max_{|\phi\rangle \in \mathcal{H}_k} \text{Tr}\{\overline{G}_n \mathcal{E}^n(|\phi\rangle\langle\phi|)\}, \quad (80)$$

and set the fidelity F_k and energy E_k of $|\phi_k\rangle$ as follows:

$$F_k \equiv \langle \phi_k | \mathcal{C}^n(|\phi_k\rangle\langle\phi_k|) | \phi_k \rangle \quad (81)$$

$$E_k \equiv \max_{|\phi\rangle \in \mathcal{H}_k} \text{Tr}\{\overline{G}_n \mathcal{E}^n(|\phi\rangle\langle\phi|)\} \quad (82)$$

$$= \text{Tr}\{\overline{G}_n \mathcal{E}^n(|\phi_k\rangle\langle\phi_k|)\}. \quad (83)$$

- 6) Set

$$\mathcal{H}_{k-1} \equiv \text{span}\{|\psi\rangle \in \mathcal{H}_k : |\langle\psi|\phi_k\rangle| = 0\}. \quad (84)$$

Set $k := k - 1$.

- 7) Repeat steps 5-6 until $k = 0$ after step 6.

The idea behind this algorithm is to successively remove minimum fidelity states from \mathcal{H}_S until $k = (1 - \delta)M$. By the structure of the algorithm and some analysis given below, we are then guaranteed for this k and lower that

$$1 - \min_{|\phi\rangle \in \mathcal{H}_k} \langle \phi | \mathcal{C}^n(|\phi\rangle\langle\phi|) | \phi \rangle \leq \varepsilon/\delta. \quad (85)$$

That is, the subspace \mathcal{H}_k is good for quantum communication with fidelity at least $1 - \varepsilon/\delta$. After this k , we then successively remove maximum energy states from \mathcal{H}_k until the algorithm terminates. Furthermore, the algorithm implies that

$$F_M \leq F_{M-1} \leq \dots \leq F_{(1-\delta)M+1}, \quad (86)$$

$$E_{(1-\delta)M} \geq E_{(1-\delta)M-1} \geq \dots \geq E_1, \quad (87)$$

$$\mathcal{H}_M \supseteq \mathcal{H}_{M-1} \supseteq \dots \supseteq \mathcal{H}_1. \quad (88)$$

Also, $\{|\phi_k\rangle\}_{k=1}^M$ is an orthonormal basis for \mathcal{H}_M , where $l \in \{1, \dots, M\}$.

We now analyze the result of this algorithm by employing Markov's inequality and some other tools. From the condition in (63) that the original code is good for entanglement transmission, we have that

$$F(\Phi_{RS}, (\text{id}_R \otimes \mathcal{C}^n)(\Phi_{RS})) \geq 1 - \varepsilon. \quad (89)$$

Since $\{|\phi_k\rangle\}_{k=1}^M$ is an orthonormal basis for \mathcal{H}_M , we can write

$$|\Phi\rangle_{RS} = \frac{1}{\sqrt{M}} \sum_{k=1}^M |\phi_k^*\rangle_R \otimes |\phi_k\rangle_S, \quad (90)$$

where $*$ denotes complex conjugate with respect to the basis in (62), and the reduced state can be written as $\Phi_S = \frac{1}{M} \sum_{k=1}^M |\phi_k\rangle\langle\phi_k|_S$. A consequence of [1, Exercise 9.5.1] is that

$$\begin{aligned} F(\Phi_{RS}, (\text{id}_R \otimes \mathcal{C}^n)(\Phi_{RS})) &\leq \frac{1}{M} \sum_k \langle \phi_k | \mathcal{C}^n(|\phi_k\rangle\langle\phi_k|) | \phi_k \rangle \\ &= \frac{1}{M} \sum_k F_k. \end{aligned} \quad (91)$$

So this means that

$$\frac{1}{M} \sum_k F_k \geq 1 - \varepsilon \Leftrightarrow \frac{1}{M} \sum_k (1 - F_k) \leq \varepsilon. \quad (92)$$

Now taking K as a uniform random variable with realizations $k \in \{1, \dots, M\}$ and applying Markov's inequality, we find that

$$\Pr\{1 - F_K \geq \varepsilon/\delta\} \leq \frac{\mathbb{E}_K\{1 - F_K\}}{\varepsilon/\delta} \leq \frac{\varepsilon}{\varepsilon/\delta} = \delta. \quad (93)$$

So this implies that $(1 - \delta)M$ of the F_k values are such that $F_k \geq 1 - \varepsilon/\delta$. Since they are ordered as given in (86), we can conclude that $\mathcal{H}_{(1-\delta)M}$ is a subspace good for quantum communication in the following sense:

$$\min_{|\phi\rangle \in \mathcal{H}_{(1-\delta)M}} \langle \phi | \mathcal{C}^n(|\phi\rangle\langle\phi|) |\phi\rangle \geq 1 - \varepsilon/\delta. \quad (94)$$

Now consider from the average energy constraint in (61) that

$$P \geq \text{Tr}\{\bar{G}_n \mathcal{E}^n(\pi_S)\} \quad (95)$$

$$= \frac{1}{M} \sum_{k=1}^M \text{Tr}\{\bar{G}_n \mathcal{E}^n(|\phi_k\rangle\langle\phi_k|_S)\} \quad (96)$$

$$= \frac{1}{M} \sum_{k=1}^M E_k \quad (97)$$

$$\geq \frac{1 - \delta}{(1 - \delta)M} \sum_{k=1}^{(1-\delta)M} E_k, \quad (98)$$

which we can rewrite as

$$\frac{1}{(1 - \delta)M} \sum_{k=1}^{(1-\delta)M} E_k \leq P/(1 - \delta). \quad (99)$$

Taking K' as a uniform random variable with realizations $k \in \{1, \dots, (1 - \delta)M\}$ and applying Markov's inequality, we find that

$$\Pr_{K'}\{E_{K'} \geq P/(1 - 2\delta)\} \leq \frac{P/(1 - \delta)}{P/(1 - 2\delta)} \quad (100)$$

$$= \frac{1 - 2\delta}{1 - \delta}. \quad (101)$$

Rewriting this, we find that

$$\Pr_{K'}\{E_{K'} \leq P/(1 - 2\delta)\} \geq 1 - \frac{1 - 2\delta}{1 - \delta} \quad (102)$$

$$= \frac{\delta}{1 - \delta}. \quad (103)$$

Thus, a fraction $\delta/(1 - \delta)$ of the remaining $(1 - \delta)M$ state vectors $|\phi_k\rangle$ are such that $E_k \leq P/(1 - 2\delta)$. Since they are ordered as in (87), this means that $\{|\phi_{\delta M}\rangle, \dots, |\phi_1\rangle\}$ have this property.

We can then conclude that the subspace $\mathcal{H}_{\delta M}$ is such that

$$\dim(\mathcal{H}_{\delta M}) = \delta M, \quad (104)$$

$$\min_{|\phi\rangle \in \mathcal{H}_{\delta M}} \langle \phi | \mathcal{C}^n(|\phi\rangle\langle\phi|) |\phi\rangle \geq 1 - \varepsilon/\delta, \quad (105)$$

$$\max_{|\phi\rangle \in \mathcal{H}_{\delta M}} \text{Tr}\{\bar{G}_n \mathcal{E}^n(|\phi\rangle\langle\phi|)\} \leq P/(1 - 2\delta). \quad (106)$$

Now applying Proposition 5 (in the appendix) to (105), we can conclude that the minimum entanglement fidelity obeys the following bound:

$$\min_{|\psi\rangle \in \mathcal{H}'_{\delta M} \otimes \mathcal{H}_{\delta M}} \langle \psi | (\text{id}_{\mathcal{H}'_{\delta M}} \otimes \mathcal{C}^n)(|\psi\rangle\langle\psi|) |\psi\rangle \geq 1 - 2\sqrt{\varepsilon/\delta}. \quad (107)$$

To finish off the proof, suppose that δM is not an integer. Then there exists a $\delta' < \delta$ such that $\delta' M = \lfloor \delta M \rfloor$ is a positive integer. By the above reasoning, there exists a code with parameters as given in (104)–(107), except with δ replaced by δ' . Then the code dimension is equal to $\lfloor \delta M \rfloor$. Using that $\delta' M = \lfloor \delta M \rfloor > \delta M - 1$, we find that $\delta' > \delta - 1/M$, which implies that $1 - 2\sqrt{\varepsilon/\delta'} > 1 - 2\sqrt{\varepsilon/[\delta - 1/M]}$. We also have that $P/(1 - 2\delta') < P/(1 - 2\delta)$. This concludes the proof. \square

B. Quantum Communication With a Uniform Energy Constraint Implies Private Communication With a Uniform Energy Constraint

This subsection establishes that a quantum communication code with uniform energy constraint can always be converted to one for private communication with uniform energy constraint, such that there is negligible loss with respect to code parameters.

Proposition 3: The existence of an $(n, M, G, P, \varepsilon)$ quantum communication code with uniform energy constraint implies the existence of an $(n, \lfloor M/2 \rfloor, G, P, \min\{1, 2\sqrt{\varepsilon}\})$ code for private communication with uniform energy constraint.

Proof: Starting from an $(n, M, G, P, \varepsilon)$ quantum communication code with uniform energy constraint, we can use it to transmit a maximally entangled state

$$\Phi_{RS} \equiv \frac{1}{M} \sum_{m, m'=1}^M |m\rangle\langle m'|_R \otimes |m\rangle\langle m'|_S \quad (108)$$

of Schmidt rank M faithfully, by applying (60):

$$F(\Phi_{RS}, (\text{id}_R \otimes \mathcal{D}^n \circ \mathcal{N}^{\otimes n} \circ \mathcal{E}^n)(\Phi_{RS})) \geq 1 - \varepsilon. \quad (109)$$

Consider that the state

$$\sigma_{RSE^n} \equiv (\text{id}_R \otimes \mathcal{D}^n \circ [\mathcal{U}^{\mathcal{N}}]^{\otimes n} \circ \mathcal{E}^n)(\Phi_{RS}) \quad (110)$$

extends the state output from the actual protocol. By Uhlmann's theorem (see (19)), there exists an extension of Φ_{RS} such that the fidelity between this extension and the state σ_{RSE^n} is equal to the fidelity in (109). However, the maximally entangled state Φ_{RS} is “unextendible” in the sense that the only possible extension is a tensor-product state $\Phi_{RS} \otimes \omega_{E^n}$ for some state ω_{E^n} . So, putting these statements together, we find that

$$F(\Phi_{RS} \otimes \omega_{E^n}, (\text{id}_R \otimes \mathcal{D}^n \circ [\mathcal{U}^{\mathcal{N}}]^{\otimes n} \circ \mathcal{E}^n)(\Phi_{RS})) \geq 1 - \varepsilon. \quad (111)$$

Furthermore, measuring the R and S systems locally in the Schmidt basis of Φ_{RS} only increases the fidelity, so that

$$F(\bar{\Phi}_{RS} \otimes \omega_{E^n}, (\text{id}_R \otimes \bar{\mathcal{D}}^n \circ [\mathcal{U}^{\mathcal{N}}]^{\otimes n} \circ \mathcal{E}^n)(\bar{\Phi}_{RS})) \geq 1 - \varepsilon, \quad (112)$$

where $\overline{\mathcal{D}}^n$ denotes the concatenation of the original decoder \mathcal{D}^n followed by the local measurement:

$$\overline{\mathcal{D}}^n(\cdot) \equiv \sum_m |m\rangle\langle m| \mathcal{D}^n(\cdot) |m\rangle\langle m| \quad (113)$$

$$= \sum_m \text{Tr}\{\mathcal{D}^{n\dagger}[|m\rangle\langle m|](\cdot)|m\rangle\langle m|\}. \quad (114)$$

Observe that $\{\mathcal{D}^{n\dagger}[|m\rangle\langle m|]\}_m$ is a valid POVM. Employing the inequalities in (25), we can conclude that

$$\frac{1}{2} \left\| \overline{\Phi}_{RS} \otimes \omega_{E^n} - (\text{id}_R \otimes \overline{\mathcal{D}}^n \circ [\mathcal{U}^{\mathcal{N}}]^{\otimes n} \circ \mathcal{E}^n)(\overline{\Phi}_{RS}) \right\|_1 \leq \sqrt{\varepsilon}. \quad (115)$$

Using the direct sum property of the trace distance from (22) and defining $\rho_{A^n}^m \equiv \mathcal{E}^n(|m\rangle\langle m|_S)$, we can then rewrite this as

$$\frac{1}{2M} \sum_{m=1}^M \left\| |m\rangle\langle m|_S \otimes \omega_{E^n} - (\overline{\mathcal{D}}^n \circ [\mathcal{U}^{\mathcal{N}}]^{\otimes n})(\rho_{A^n}^m) \right\|_1 \leq \sqrt{\varepsilon}. \quad (116)$$

Markov's inequality then guarantees that there exists a subset \mathcal{M}' of $[M]$ of size $\lfloor M/2 \rfloor$ such that the following condition holds for all $m \in \mathcal{M}'$:

$$\frac{1}{2} \left\| |m\rangle\langle m|_S \otimes \omega_{E^n} - (\overline{\mathcal{D}}^n \circ [\mathcal{U}^{\mathcal{N}}]^{\otimes n})(\rho_{A^n}^m) \right\|_1 \leq 2\sqrt{\varepsilon}. \quad (117)$$

We now define the private communication code to consist of codewords $\{\rho_{A^n}^m \equiv \mathcal{E}^n(|m\rangle\langle m|_S)\}_{m \in \mathcal{M}'}$ and the decoding POVM to be

$$\left\{ \Lambda_{B^n}^m \equiv \mathcal{D}^{n\dagger}(|m\rangle\langle m|) \right\}_{m \in \mathcal{M}'} \cup \left\{ \Lambda_{B^n}^0 \equiv \mathcal{D}^{n\dagger} \left(\sum_{m \notin \mathcal{M}'} |m\rangle\langle m| \right) \right\}. \quad (118)$$

Note that the energy constraint holds for all codewords

$$\text{Tr}\{\overline{G}_n \rho_{A^n}^m\} \leq P, \quad (119)$$

due to the assumption that we start from a quantum communication code with uniform energy constraint as given in (57). Applying monotonicity of partial trace to (117) with respect to system S , we find that the following condition holds for all $m \in \mathcal{M}'$:

$$\frac{1}{2} \left\| \omega_{E^n} - \hat{\mathcal{N}}^{\otimes n}(\rho_{A^n}^m) \right\|_1 \leq 2\sqrt{\varepsilon}, \quad (120)$$

which gives the desired security condition in (67). Applying monotonicity of partial trace to (117) with respect to system E^n gives that

$$\frac{1}{2} \left\| |m\rangle\langle m|_S - (\overline{\mathcal{D}}^n \circ \mathcal{N}^{\otimes n})(\rho_{A^n}^m) \right\|_1 \leq 2\sqrt{\varepsilon}, \quad (121)$$

for all $m \in \mathcal{M}'$. Abbreviating $\Gamma_{B^n}^{m'} \equiv \mathcal{D}^{n\dagger}(|m'\rangle\langle m'|)$, consider then that for all $m \in \mathcal{M}'$

$$\begin{aligned} & \frac{1}{2} \left\| |m\rangle\langle m|_S - (\overline{\mathcal{D}}^n \circ \mathcal{N}^{\otimes n})(\rho_{A^n}^m) \right\|_1 \\ &= \frac{1}{2} \left\| |m\rangle\langle m|_S - \sum_{m'=1}^M \text{Tr}\{\Gamma_{B^n}^{m'} \mathcal{N}^{\otimes n}(\rho_{A^n}^m)\} |m'\rangle\langle m'| \right\|_1 \\ &= \frac{1}{2} \left\| p_e |m\rangle\langle m|_S - \sum_{m' \neq m} \text{Tr}\{\Gamma_{B^n}^{m'} \mathcal{N}^{\otimes n}(\rho_{A^n}^m)\} |m'\rangle\langle m'| \right\|_1 \\ &= \frac{1}{2} \left(p_e + \sum_{m' \neq m} \text{Tr}\{\Gamma_{B^n}^{m'} \mathcal{N}^{\otimes n}(\rho_{A^n}^m)\} \right) \\ &= 1 - \text{Tr}\{\Lambda_{B^n}^m \mathcal{N}^{\otimes n}(\rho_{A^n}^m)\}, \end{aligned} \quad (122)$$

where $p_e \equiv 1 - \text{Tr}\{\Lambda_{B^n}^m \mathcal{N}^{\otimes n}(\rho_{A^n}^m)\}$. Combining this equality with (121) gives the desired reliable decoding condition in (66) for all $m \in \mathcal{M}'$

$$\text{Tr}\{\Lambda_{B^n}^m \mathcal{N}^{\otimes n}(\rho_{A^n}^m)\} \geq 1 - 2\sqrt{\varepsilon}. \quad (123)$$

Thus, we have shown that from an $(n, M, G, P, \varepsilon)$ quantum communication code with uniform energy constraint, one can realize an $(n, \lfloor M/2 \rfloor, G, P, 2\sqrt{\varepsilon})$ code for private communication with uniform energy constraint. \square

Remark 1: That a quantum communication code can be easily converted to a private communication code is part of the folklore of quantum information theory. Reference [4] proved that the unconstrained quantum capacity never exceeds the unconstrained private capacity, but we are not aware of an explicit code conversion statement of the form given in Proposition 3.

C. Secret Key Transmission With an Average Energy Constraint Implies Private Communication With a Uniform Energy Constraint

We finally establish that a secret key transmission code with average energy constraint can be converted to a private communication code with uniform energy constraint.

Proposition 4: For $\delta \in (1/M, 1/3)$, the existence of an $(n, M, G, P, \varepsilon)$ secret key transmission code with average energy constraint implies the existence of an $(n, \lfloor \delta M \rfloor, G, P/(1-3\delta), \min\{1, \varepsilon/[\delta-1/M]\})$ private communication code with uniform energy constraint.

Proof: To begin with, suppose that δM is an integer. The existence of an $(n, M, G, P, \varepsilon)$ secret key transmission code with average energy constraint implies that the following three conditions hold:

$$\frac{1}{M} \sum_{m=1}^M E_m \leq P, \quad \frac{1}{M} \sum_{m=1}^M T_m \geq 1 - \varepsilon, \quad (124)$$

$$\frac{1}{M} \sum_{m=1}^M D_m \leq \varepsilon, \quad (125)$$

where

$$E_m \equiv \text{Tr}\{\bar{G}_n \rho_{A^n}^m\}, \quad (126)$$

$$T_m \equiv \text{Tr}\{\Lambda_{B^n}^m \mathcal{N}^{\otimes n}(\rho_{A^n}^m)\}, \quad (127)$$

$$D_m \equiv \frac{1}{2} \left\| \hat{\mathcal{N}}^{\otimes n}(\rho_{A^n}^m) - \omega_{E^n} \right\|_1. \quad (128)$$

Now taking \hat{M} as a uniform random variable with realizations $m \in \{1, \dots, M\}$ and applying Markov's inequality, we have for $\delta \in (0, 1/3)$ that

$$\Pr_{\hat{M}}\{1 - T_{\hat{M}} \geq \varepsilon/\delta\} \leq \frac{\mathbb{E}_{\hat{M}}\{1 - T_{\hat{M}}\}}{\varepsilon/\delta} \leq \frac{\varepsilon}{\varepsilon/\delta}. \quad (129)$$

This implies that $(1 - \delta)M$ of the T_m values are such that $T_m \geq 1 - \varepsilon/\delta$. We then rearrange the order of T_m , D_m , and E_m using a label m' such that the first $(1 - \delta)M$ of the $T_{m'}$ variables satisfy the condition $T_{m'} \geq 1 - \varepsilon/\delta$. Now from (124), we have that

$$\varepsilon \geq \frac{1}{M} \sum_{m'=1}^M D_{m'} \geq \frac{1 - \delta}{(1 - \delta)M} \sum_{m'=1}^{(1-\delta)M} D_{m'}, \quad (130)$$

which can be rewritten as

$$\frac{1}{(1 - \delta)M} \sum_{m'=1}^{(1-\delta)M} D_{m'} \leq \frac{\varepsilon}{1 - \delta}. \quad (131)$$

Now taking \hat{M}' as a uniform random variable with realizations $m' \in \{1, \dots, (1 - \delta)M\}$ and applying Markov's inequality, we find that

$$\Pr_{\hat{M}'}\{D_{\hat{M}'} \geq \varepsilon/\delta\} \leq \frac{\mathbb{E}_{\hat{M}'}\{D_{\hat{M}'}\}}{\varepsilon/\delta} \quad (132)$$

$$\leq \frac{\varepsilon/(1 - \delta)}{\varepsilon/\delta} \quad (133)$$

$$= \frac{\delta}{1 - \delta}. \quad (134)$$

Thus a fraction $1 - [\delta/(1 - \delta)] = (1 - 2\delta)/(1 - \delta)$ of the first $(1 - \delta)M$ variables $D_{m'}$ satisfy $D_{\hat{M}'} \leq \varepsilon/\delta$. Now rearrange the order of $T_{m'}$, $D_{m'}$, and $E_{m'}$ with label m'' such that the first $(1 - 2\delta)M$ of them satisfy

$$T_{m''} \geq 1 - \varepsilon/\delta, \quad (135)$$

$$D_{m''} \leq \varepsilon/\delta. \quad (136)$$

From (124), we get that

$$P \geq \frac{1}{M} \sum_{m''=1}^M E_{m''} \geq \frac{1 - 2\delta}{(1 - 2\delta)M} \sum_{m''=1}^{(1-2\delta)M} E_{m''}, \quad (137)$$

which can be rewritten as

$$\frac{1}{(1 - 2\delta)M} \sum_{m''=1}^{(1-2\delta)M} E_{m''} \leq \frac{P}{1 - 2\delta}. \quad (138)$$

Taking \hat{M}'' as a uniform random variable with realizations $m'' \in \{1, \dots, (1 - 2\delta)M\}$ and applying Markov's inequality,

we find that

$$\Pr_{\hat{M}''}\{E_{\hat{M}''} \geq P/(1 - 3\delta)\} \leq \frac{\mathbb{E}_{\hat{M}''}\{E_{\hat{M}''}\}}{P/(1 - \delta)} \quad (139)$$

$$\leq \frac{P/(1 - 2\delta)}{P/(1 - 3\delta)} \quad (140)$$

$$= \frac{1 - 3\delta}{1 - 2\delta}. \quad (141)$$

Thus a fraction $1 - (1 - 3\delta)/(1 - 2\delta) = \delta/(1 - 2\delta)$ of the first $(1 - 2\delta)M$ variables $E_{m''}$ satisfy the condition $E_{\hat{M}''} \leq P/(1 - 3\delta)$. We can finally relabel $T_{m''}$, $D_{m''}$, and $E_{m''}$ with a label m''' such that the first δM of them satisfy

$$E_{m'''} \leq P/(1 - 3\delta), \quad (142)$$

$$T_{m'''} \geq 1 - \varepsilon/\delta, \quad (143)$$

$$D_{m'''} \leq \varepsilon/\delta. \quad (144)$$

The corresponding codewords then constitute an $(n, \delta M, G, P/(1 - 3\delta), \varepsilon/\delta)$ private communication code with uniform energy constraint.

To finish off the proof, suppose that δM is not an integer. Then there exists a $\delta' < \delta$ such that $\delta' M = \lfloor \delta M \rfloor$ is a positive integer. By the above reasoning, there exists a code with parameters as given in (142)–(144), except with δ replaced by δ' . Then the code size is equal to $\lfloor \delta M \rfloor$. Using that $\delta' M = \lfloor \delta M \rfloor > \delta M - 1$, we find that $\delta' > \delta - 1/M$, which implies that $1 - \varepsilon/\delta' > 1 - \varepsilon/[\delta - 1/M]$ and $\varepsilon/\delta' < \varepsilon/[\delta - 1/M]$. We also have that $P/(1 - 3\delta') < P/(1 - 3\delta)$. This concludes the proof. \square

V. IMPLICATIONS OF CODE CONVERSIONS FOR CAPACITIES

In this brief section, we show how the various code conversions from Section IV have implications for the capacities defined in Section III. The main result is the following theorem:

Theorem 1: Let $\mathcal{N} : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_B)$ be a quantum channel, $G \in \mathcal{P}(\mathcal{H}_A)$ an energy observable, and $P \in [0, \infty)$. Then the following relations hold for the capacities defined in Section III:

$$\begin{aligned} Q(\mathcal{N}, G, P) &= E(\mathcal{N}, G, P) \\ &\leq P(\mathcal{N}, G, P) = K(\mathcal{N}, G, P). \end{aligned} \quad (145)$$

Proof: As a consequence of the definitions of these capacities and as remarked in (64) and (73), we have that

$$Q(\mathcal{N}, G, P) \leq E(\mathcal{N}, G, P), \quad (146)$$

$$P(\mathcal{N}, G, P) \leq K(\mathcal{N}, G, P). \quad (147)$$

So it suffices to prove the following three inequalities:

$$Q(\mathcal{N}, G, P) \geq E(\mathcal{N}, G, P), \quad (148)$$

$$Q(\mathcal{N}, G, P) \leq P(\mathcal{N}, G, P), \quad (149)$$

$$P(\mathcal{N}, G, P) \geq K(\mathcal{N}, G, P). \quad (150)$$

These follow from Propositions 2, 3, and 4, respectively. Let us establish (148). Fix a constant $\delta \in (0, 1/2)$. Suppose that R is an achievable rate for entanglement transmission with an

average energy constraint $P(1-2\delta)$. This implies the existence of a sequence of $(n, M_n, G, P(1-2\delta), \varepsilon_n)$ codes such that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log M_n = R, \quad (151)$$

$$\lim_{n \rightarrow \infty} \varepsilon_n = 0. \quad (152)$$

Suppose that the sequence is such that M_n is non-decreasing with n (if it is not the case, then pick out a subsequence for which it is the case). Now pick n large enough such that $\delta \geq 1/M_n$. Invoking Proposition 2, there exists an $(n, \lfloor \delta M_n \rfloor, G, P, \min[1, 2\sqrt{\varepsilon_n/[\delta - 1/M_n]}])$ quantum communication code with uniform energy constraint. From the facts that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log (\lfloor \delta M_n \rfloor) = \liminf_{n \rightarrow \infty} \frac{1}{n} \log M_n \quad (153)$$

$$= R, \quad (154)$$

$$\limsup_{n \rightarrow \infty} 2\sqrt{\varepsilon_n/[\delta - 1/M_n]} = 0, \quad (155)$$

we can conclude that R is an achievable rate for quantum communication with uniform energy constraint P . So this implies that $Q(\mathcal{N}, G, P) \geq E(\mathcal{N}, G, P(1-2\delta))$. However, since we have shown this inequality to be true for all $\delta \in (0, 1/2)$, we can then take a supremum over $\delta \in (0, 1/2)$ to conclude that $Q(\mathcal{N}, G, P) \geq \sup_{\delta \in (0, 1/2)} E(\mathcal{N}, G, P(1-2\delta)) = E(\mathcal{N}, G, P)$. So we conclude (148). We can argue the other inequalities in (149) and (150) similarly, by applying Propositions 3 and 4, respectively. \square

VI. ACHIEVABILITY OF REGULARIZED, ENERGY-CONSTRAINED COHERENT INFORMATION FOR ENERGY-CONSTRAINED QUANTUM COMMUNICATION

The main result of this section is Theorem 2, which shows that the regularized energy-constrained coherent information is achievable for energy-constrained quantum communication. In order to do so, we need to restrict the energy observables and channels that we consider. We impose two arguably natural constraints: that the energy observable be a Gibbs observable as given in Definition 3 and that the channel have finite output entropy as given in Condition 1. Gibbs observables have been considered in several prior works [36], [37], [39], [65], [70], [71] as well as finite output-entropy channels [36], [37], [39].

When defining a Gibbs observable, we follow [39, Lemma 11.8] and [65, Sec. IV]:

Definition 3 (Gibbs Observable): Let G be an energy observable as given in Definition 1. Such an operator G is a Gibbs observable if for all $\beta > 0$, the following holds

$$\text{Tr}\{\exp(-\beta G)\} < \infty. \quad (156)$$

The above condition implies that a Gibbs observable G always has a finite value of the partition function $\text{Tr}\{\exp(-\beta G)\}$ for all $\beta > 0$ and thus a well defined thermal state for all $\beta > 0$, given by $e^{-\beta G} / \text{Tr}\{e^{-\beta G}\}$.

Condition 1 (Finite Output Entropy): Let G be a Gibbs observable and $P \in [0, \infty)$. A quantum channel \mathcal{N} satisfies

the finite-output entropy condition with respect to G and P if

$$\sup_{\rho: \text{Tr}\{G\rho\} \leq P} H(\mathcal{N}(\rho)) < \infty, \quad (157)$$

Lemma 1: Let \mathcal{N} denote a quantum channel satisfying Condition 1, G a Gibbs observable, and $P \in [0, \infty)$. Then any complementary channel $\hat{\mathcal{N}}$ of \mathcal{N} satisfies the finite-entropy condition

$$\sup_{\rho: \text{Tr}\{G\rho\} \leq P} H(\hat{\mathcal{N}}(\rho)) < \infty. \quad (158)$$

Proof: Let ρ be a density operator satisfying $\text{Tr}\{G\rho\} \leq P$, and let $\sum_i p_i |i\rangle\langle i|$ be a spectral decomposition of ρ . Let

$$\theta_\beta \equiv e^{-\beta G} / \text{Tr}\{e^{-\beta G}\} \quad (159)$$

denote a thermal state of G with inverse temperature $\beta > 0$. Consider that $H(\rho)$ is finite because a rewriting of $D(\rho \| \theta_\beta) \geq 0$ implies that

$$H(\rho) \leq \beta \text{Tr}\{G\rho\} + \log \text{Tr}\{e^{-\beta G}\} \quad (160)$$

$$\leq \beta P + \log \text{Tr}\{e^{-\beta G}\} < \infty, \quad (161)$$

where the last inequality follows from (156) and from the assumption that $P < \infty$. Consider that $|\psi^\rho\rangle = \sum_i \sqrt{p_i} |i\rangle \otimes |i\rangle$ is a purification of ρ and satisfies

$$H(\hat{\mathcal{N}}(\rho)) = H((\text{id} \otimes \mathcal{N})(|\psi^\rho\rangle\langle\psi^\rho|)) \quad (162)$$

$$\leq H(\rho) + H(\mathcal{N}(\rho)) < \infty. \quad (163)$$

The equality follows because the marginals of a pure bipartite state have the same entropy. The first inequality follows from subadditivity of entropy, and the last from (161) and the assumption that Condition 1 holds. We have shown that the entropy $H(\hat{\mathcal{N}}(\rho))$ is finite for all states satisfying $\text{Tr}\{G\rho\} \leq P$, and so (158) holds. \square

Theorem 2: Let $\mathcal{N} : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_B)$ denote a quantum channel satisfying Condition 1, G a Gibbs observable, and $P \in [0, \infty)$. Then the energy-constrained entanglement transmission capacity $E(\mathcal{N}, G, P)$ is bounded from below by the regularized energy-constrained coherent information of the channel \mathcal{N} :

$$E(\mathcal{N}, G, P) \geq \lim_{k \rightarrow \infty} \frac{1}{k} I_c(\mathcal{N}^{\otimes k}, \bar{G}_k, P),$$

where the energy-constrained coherent information of \mathcal{N} is defined as

$$I_c(\mathcal{N}, G, P) \equiv \sup_{\rho: \text{Tr}\{G\rho\} \leq P} H(\mathcal{N}(\rho)) - H(\hat{\mathcal{N}}(\rho)), \quad (164)$$

and $\hat{\mathcal{N}}$ denotes a complementary channel of \mathcal{N} .

Proof: The main challenge in proving this theorem is to have codes achieving the coherent information while meeting the average energy constraint. We prove the theorem by combining Klesse's technique for constructing entanglement transmission codes [68], [72] with an adaptation of Holevo's technique of approximation and constructing codes meeting an energy constraint [36], [37]. We follow their arguments very closely and show how to combine the techniques to achieve the desired result.

First, we recall what Klesse accomplished in [68] (see also the companion paper [72]). Let $\mathcal{M} : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_B)$

denote a quantum channel satisfying Condition 1 for some Gibbs observable and energy constraint, so that the receiver entropy is finite, as well as the environment entropy by Lemma 1. This implies that entropy-typical subspaces and sequences corresponding to these entropies are well defined and finite, a fact of which we make use. Let V denote a finite-dimensional linear subspace of \mathcal{H}_A . Set $L \equiv \dim(V)$, and let \mathcal{L} denote a channel defined to be the restriction of \mathcal{M} to states with support contained in V . Let $\{K_y\}_y$ be a set of Kraus operators for \mathcal{M} and define the probability $p_Y(y)$ by

$$p_Y(y) \equiv \frac{1}{L} \text{Tr}\{\Pi_V K_y^\dagger K_y \Pi_V\}, \quad (165)$$

where Π_V is a projection onto V . As discussed in [68], there is unitary freedom in the choice of the Kraus operators, and they can be chosen “diagonal,” so that $\text{Tr}\{\Pi_V K_y^\dagger K_x \Pi_V\} = 0$ for $x \neq y$. Let $T_Y^{n,\delta}$ denote the δ -entropy-typical set for p_Y , defined as

$$T_Y^{n,\delta} \equiv \{y^n : |-\log p_Y^n(y^n)/n - H(Y)| \leq \delta\}, \quad (166)$$

for integer $n \geq 1$ and real $\delta > 0$, where $p_Y^n(y^n) \equiv p_Y(y_1)p_Y(y_2)\cdots p_Y(y_n)$. Let $K_{y^n} \equiv K_{y_1} \otimes K_{y_2} \otimes \cdots \otimes K_{y_n}$. Now define the (trace-non-increasing) quantum operation $\mathcal{L}^{n,\delta}$ to be a map consisting of only the entropy-typical Kraus operators K_{y^n} such that $y^n \in T_Y^{n,\delta}$. The number of such Kraus operators is no larger than $2^{n[H(Y)+\delta]}$, and one can show that $H(Y) = H(\hat{\mathcal{M}}(\pi_V))$, where $\hat{\mathcal{M}}$ is a channel complementary to \mathcal{M} and $\pi_V \equiv \Pi_V/L$ denotes the maximally mixed state on V [68].

One can then further reduce the quantum operation $\mathcal{L}^{n,\delta}$ to another one $\tilde{\mathcal{L}}^{n,\delta}$ defined by projecting the output of $\mathcal{L}^{n,\delta}$ to the entropy-typical subspace of the density operator $\mathcal{L}(\pi_V) = \mathcal{M}(\pi_V)$. The entropy-typical subspace of a density operator σ with spectral decomposition $\sigma = \sum_z p_Z(z)|z\rangle\langle z|$ is defined as

$$T_\sigma^{n,\delta} \equiv \text{span}\{|z^n\rangle : |-\log p_Z^n(z^n)/n - H(\sigma)| \leq \delta\}, \quad (167)$$

for integer $n \geq 1$ and real $\delta > 0$. The resulting quantum operation $\tilde{\mathcal{L}}^{n,\delta}$ is thus finite-dimensional and has a finite number of Kraus operators. We then have the following bounds argued in [68]:

$$\tilde{L}^{n,\delta} \leq 2^{n[H(\hat{\mathcal{M}}(\pi_V))+\delta]}, \quad (168)$$

$$\text{Tr}\{\tilde{\mathcal{L}}^{n,\delta}(\pi_{V^{\otimes n}})\} \geq 1 - \varepsilon_1, \quad (169)$$

$$\|\tilde{\mathcal{L}}^{n,\delta}(\pi_{V^{\otimes n}})\|_2^2 \leq 2^{-n[H(\mathcal{M}(\pi_V))-3\delta]}, \quad (170)$$

$$F_e(C_n, \mathcal{L}^{\otimes n}) \geq F_e(C_n, \tilde{\mathcal{L}}^{n,\delta}), \quad (171)$$

where $\tilde{L}^{n,\delta}$ denotes the number of Kraus operators for $\tilde{\mathcal{L}}^{n,\delta}$ and the second inequality holds for all $\varepsilon_1 \in (0, 1)$ and sufficiently large n . Note that for this latter estimate, we require the law of large numbers to hold when we only know that the entropy is finite (this can be accomplished using the technique discussed in [73]). In the last line, we have written the entanglement fidelity of a code C_n (some subspace of $V^{\otimes n}$), which is defined as

$$F_e(C_n, \mathcal{L}^{\otimes n}) \equiv \sup_{\mathcal{R}^n} \langle \Phi_{C_n} | (\text{id} \otimes [\mathcal{R}^n \circ \mathcal{L}^{\otimes n}]) (\Phi_{C_n}) | \Phi_{C_n} \rangle, \quad (172)$$

where $|\Phi_{C_n}\rangle$ denotes a maximally entangled state built from an orthonormal basis of C_n and the optimization is with respect to recovery channels \mathcal{R}^n . Let $K_n \equiv \dim C_n$. From the developments in [68], the following bound holds

$$\begin{aligned} & \mathbb{E}_{U_{K_n}(V^{\otimes n})}\{F_e(U_{K_n}C_n, \tilde{\mathcal{L}}^{n,\delta})\} \\ & \geq \text{Tr}\{\tilde{\mathcal{L}}^{n,\delta}(\pi_{V^{\otimes n}})\} - \sqrt{K \tilde{L}^{n,\delta} \|\tilde{\mathcal{L}}^{n,\delta}(\pi_{V^{\otimes n}})\|_2^2}, \end{aligned} \quad (173)$$

where $\mathbb{E}_{U_{K_n}(V^{\otimes n})}$ denotes the expected entanglement fidelity when we apply a randomly selected unitary U_{K_n} to the codespace C_n , taking it to some different subspace of $V^{\otimes n}$. The unitary U_K is selected according to the unitarily invariant measure on the group $\mathbf{U}(V^{\otimes n})$ of unitaries acting on the subspace $V^{\otimes n}$. Combining with the inequalities in (168)–(171), we find that

$$\begin{aligned} & \mathbb{E}_{U_{K_n}(V^{\otimes n})}\{F_e(U_{K_n}C_n, \mathcal{L}^{\otimes n})\} \\ & \geq 1 - \varepsilon_1 - \left[2^{-n[H(\mathcal{M}(\pi_V)) - \hat{\mathcal{M}}(\pi_V) - R - 4\delta]}\right]^{\frac{1}{2}}, \end{aligned} \quad (174)$$

where the rate R of entanglement transmission is defined as $R \equiv [\log K_n]/n$. Thus, if we choose

$$R = H(\mathcal{M}(\pi_V)) - \hat{\mathcal{M}}(\pi_V) - 5\delta, \quad (175)$$

then we find that

$$\mathbb{E}_{U_{K_n}(V^{\otimes n})}\{F_e(U_{K_n}C_n, \tilde{\mathcal{L}}^{n,\delta})\} \geq 1 - \varepsilon_1 - 2^{-n\delta/2}, \quad (176)$$

and we see that the RHS can be made arbitrarily close to one by taking n large enough. We can then conclude that there exists a unitary U_{K_n} , such that the codespace defined by $U_{K_n}C_n$ achieves the same entanglement fidelity given above, implying that the rate $H(\mathcal{M}(\pi_V)) - \hat{\mathcal{M}}(\pi_V)$ is achievable for entanglement transmission over \mathcal{M} .

Now we apply the methods of Holevo [37] and further arguments of Klesse [68] to see how to achieve the rate given in the statement of the theorem for the channel \mathcal{N} while meeting the desired energy constraint. We follow the reasoning in [37] very closely. Consider that G is a non-constant operator. Thus, the image of the convex set of all density operators under the map $\rho \rightarrow \text{Tr}\{G\rho\}$ is an interval. Suppose first that P is not equal to the minimum eigenvalue of G . Then there exists a real number P' and a density operator ρ in $\mathcal{D}(\mathcal{H}_A)$ such that

$$\text{Tr}\{G\rho\} \leq P' < P. \quad (177)$$

Let $\rho = \sum_{j=1}^{\infty} \lambda_j |j\rangle\langle j|$ be a spectral decomposition of ρ , and define

$$\rho_d \equiv \sum_{j=1}^d \tilde{\lambda}_j |j\rangle\langle j|, \quad \text{where} \quad (178)$$

$$\tilde{\lambda}_j \equiv \lambda_j \left(\sum_{j=1}^d \lambda_j \right)^{-1}. \quad (179)$$

Then $\|\rho - \rho_d\|_1 \rightarrow 0$ as $d \rightarrow \infty$. Let $g(j) \equiv \langle j|G|j\rangle$, so that

$$\text{Tr}\{G\rho_d\} = \sum_{j=1}^d \tilde{\lambda}_j g(j) = P' + \varepsilon_d, \quad (180)$$

where $\varepsilon_d \rightarrow 0$ as $d \rightarrow \infty$. Consider the density operator $\rho_d^{\otimes m}$, and let $\Pi_d^{m,\delta}$ denote its strongly typical projector, defined as the projection onto the strongly typical subspace

$$\text{span}\{|j^m\rangle : |N(j|j^m)/m - \tilde{\lambda}_j| \leq \delta\}, \quad (181)$$

where $|j^m\rangle \equiv |j_1\rangle \otimes \cdots \otimes |j_m\rangle$ and $N(j|j^m)$ denotes the number of appearances of the symbol j in the sequence j^m . Let

$$\pi_d^{m,\delta} \equiv \Pi_d^{m,\delta} / \text{Tr}\{\Pi_d^{m,\delta}\} \quad (182)$$

denote the maximally mixed state on the strongly typical subspace. We then find that for positive integers m and n ,

$$\begin{aligned} & \text{Tr} \left\{ \overline{G}_{mn} \left(\left[\pi_d^{m,\delta} \right]^{\otimes n} - \rho_d^{\otimes mn} \right) \right\} \\ &= \text{Tr} \left\{ \overline{G}_m \left(\left[\pi_d^{m,\delta} \right]^{\otimes n} - \rho_d^{\otimes mn} \right) \right\} \end{aligned} \quad (183)$$

$$= \text{Tr} \left\{ \overline{G}_m \left(\pi_d^{m,\delta} - \rho_d^{\otimes m} \right) \right\} \leq \delta \max_{j \in [d]} g(j), \quad (184)$$

where $[d] \equiv \{1, \dots, d\}$ and the inequality follows from applying a bound from [74] (also called ‘‘typical average lemma’’ in [75]). Now we can apply the above inequality to find that

$$\text{Tr} \left\{ \overline{G}_{mn} \left[\pi_d^{m,\delta} \right]^{\otimes n} \right\} \leq \text{Tr}\{\overline{G}_m \rho_d^{\otimes m}\} + \delta \max_{j \in [d]} g(j) \quad (185)$$

$$= \text{Tr}\{G\rho_d\} + \delta \max_{j \in [d]} g(j) \quad (186)$$

$$= P' + \varepsilon_d + \delta \max_{j \in [d]} g(j). \quad (187)$$

For all d large enough, we can then find δ_0 such that the last line above is $\leq P/(1 + \delta_1)$ for $\delta, \delta_1 \in (0, \delta_0]$.

The quantum coding scheme we use is that of Klesse [68] discussed previously, now setting $\mathcal{M} = \mathcal{N}^{\otimes m}$ and the subspace V to be the frequency-typical subspace of $\rho_d^{\otimes m}$, so that $\Pi_V = \Pi_d^{m,\delta}$. Letting π_{C_n} denote the maximally mixed projector onto the codespace $C_n \subset V^{\otimes n}$, we find that [68, Sec. 5.3]

$$\mathbb{E}_{U_{K_n}(V^{\otimes n})}\{U_{K_n} \pi_{C_n} U_{K_n}^\dagger\} = \pi_{V^{\otimes n}} = \left[\pi_d^{m,\delta} \right]^{\otimes n}. \quad (188)$$

So this and the reasoning directly above imply that

$$\mathbb{E}_{U_{K_n}(V^{\otimes n})}\{\text{Tr}\{\overline{G}_{mn} U_{K_n} \pi_{C_n} U_{K_n}^\dagger\}\} \leq P/(1 + \delta_1), \quad (189)$$

for $\delta, \delta_1 \leq \delta_0$. Furthermore, from (176), for arbitrary $\varepsilon \in (0, 1)$ and sufficiently large n , we find that

$$\mathbb{E}_{U_{K_n}(V^{\otimes n})}\{1 - F_e(U_{K_n} C_n, \mathcal{N}^{\otimes mn})\} \leq \varepsilon, \quad (190)$$

as long as the rate

$$R = [H(\mathcal{N}^{\otimes m}(\pi_d^{m,\delta})) - H(\hat{\mathcal{N}}^{\otimes m}(\pi_d^{m,\delta}))]/m - \delta' \quad (191)$$

for $\delta' > 0$. At this point, we would like to argue the existence of a code that has arbitrarily small error and meets the energy constraint. Let E_0 denote the event $1 - F_e(U_{K_n} C_n, \mathcal{N}^{\otimes mn}) \leq \sqrt{\varepsilon}$ and let E_1 denote the event $\text{Tr}\{\overline{G}_{mn} U_{K_n} \pi_{C_n} U_{K_n}^\dagger\} \leq P$.

We can apply the union bound and Markov’s inequality to find that

$$\begin{aligned} & \Pr_{U_{K_n}(V^{\otimes n})} \{ \overline{E_0 \cap E_1} \} \\ &= \Pr_{U_{K_n}(V^{\otimes n})} \{ E_0^c \cup E_1^c \} \end{aligned} \quad (192)$$

$$\begin{aligned} & \leq \Pr_{U_{K_n}(V^{\otimes n})} \{ 1 - F_e(U_{K_n} C_n, \mathcal{N}^{\otimes mn}) \geq \sqrt{\varepsilon} \} \\ & \quad + \Pr_{U_{K_n}(V^{\otimes n})} \left\{ \text{Tr}\{\overline{G}_{mn} U_{K_n} \pi_{C_n} U_{K_n}^\dagger\} \geq P \right\} \end{aligned} \quad (193)$$

$$\begin{aligned} & \leq \frac{1}{\sqrt{\varepsilon}} \mathbb{E}_{U_{K_n}(V^{\otimes n})} \{ 1 - F_e(U_{K_n} C_n, \mathcal{N}^{\otimes mn}) \} \\ & \quad + \frac{1}{P} \mathbb{E}_{U_{K_n}(V^{\otimes n})} \{ \text{Tr}\{\overline{G}_{mn} U_{K_n} \pi_{C_n} U_{K_n}^\dagger\} \} \end{aligned} \quad (194)$$

$$\leq \sqrt{\varepsilon} + 1/(1 + \delta_1). \quad (195)$$

Since we can choose n large enough to have ε arbitrarily small, there exists such an n such that the last line is strictly less than one. This then implies the existence of a code C_n such that $F_e(C_n, \mathcal{N}^{\otimes mn}) \geq 1 - \sqrt{\varepsilon}$ and $\text{Tr}\{\overline{G}_{mn} \pi_{C_n}\} \leq P$ (i.e., it has arbitrarily good entanglement fidelity and meets the average energy constraint). Furthermore, the rate achievable using this code is equal to $[H(\mathcal{N}^{\otimes m}(\pi_d^{m,\delta})) - H(\hat{\mathcal{N}}^{\otimes m}(\pi_d^{m,\delta}))]/m$. We have shown that this rate is achievable for all $\delta > 0$ and all integer $m \geq 1$. By applying the limiting argument from [74] (see also [76]), we thus have that the following is an achievable rate as well:

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \lim_{m \rightarrow \infty} \frac{1}{m} [H(\mathcal{N}^{\otimes m}(\pi_d^{m,\delta})) - H(\hat{\mathcal{N}}^{\otimes m}(\pi_d^{m,\delta}))] \\ &= H(\mathcal{N}(\rho_d)) - H(\hat{\mathcal{N}}(\rho_d)), \end{aligned} \quad (196)$$

where $\text{Tr}\{G\rho_d\} \leq P' + \varepsilon_d \leq P$. Given that both $H(\mathcal{N}(\rho_d))$ and $H(\hat{\mathcal{N}}(\rho_d))$ are finite, we can apply (32)–(35) and rewrite

$$H(\mathcal{N}(\rho_d)) - H(\hat{\mathcal{N}}(\rho_d)) = I_c(\rho_d, \mathcal{N}). \quad (197)$$

Finally, we take the limit $d \rightarrow \infty$ and find that

$$\liminf_{d \rightarrow \infty} I_c(\rho_d, \mathcal{N}) \geq I_c(\rho, \mathcal{N}), \quad (198)$$

where we have used the representation

$$I_c(\rho_d, \mathcal{N}) = I(\rho_d, \mathcal{N}) - H(\rho_d), \quad (199)$$

applied that the mutual information is lower semicontinuous [45, Proposition 1], the entropy H is continuous for all states σ such that $\text{Tr}\{G\sigma\} < P$ (following from a variation of [39, Lemma 11.8]), and the fact that a purification $|\psi_d^\rho\rangle \equiv \sum_{j=1}^d \tilde{\lambda}_j^{1/2} |j\rangle \otimes |j\rangle$ has the convergence $\| |\psi_d^\rho\rangle \langle \psi_d^\rho| - |\psi^\rho\rangle \langle \psi^\rho| \|_1 \rightarrow 0$ as $d \rightarrow \infty$. Now since $H(\mathcal{N}(\rho))$ and $H(\hat{\mathcal{N}}(\rho))$ are each finite, we can rewrite

$$I_c(\rho, \mathcal{N}) = H(\mathcal{N}(\rho)) - H(\hat{\mathcal{N}}(\rho)). \quad (200)$$

We have thus proven that the rate $H(\mathcal{N}(\rho)) - H(\hat{\mathcal{N}}(\rho))$ is achievable for entanglement transmission with average energy constraint for all ρ satisfying $\text{Tr}\{G\rho\} < P$.

We can extend this argument to operators ρ such that $\text{Tr}\{G\rho\} = P$ by approximating them with operators $\rho_\xi = (1 - \xi)\rho + \xi|e\rangle\langle e|$, where $|e\rangle$ is chosen such that $\langle e|G|e\rangle < P$. Suppose now that P is the minimum eigenvalue of G . In this

case, the condition $\text{Tr}\{G\rho\} \leq P$ reduces to the support of ρ being contained in the spectral projection of G corresponding to this minimum eigenvalue. The condition in Definition 3 implies that the eigenvalues of G have finite multiplicity, and so the support of ρ is a fixed finite-dimensional subspace. Thus we can take $\rho_d = \rho$, and we can repeat the above argument with the equality $\text{Tr}\{G\rho\} = P$ holding at each step.

As a consequence, we can conclude that

$$\sup_{\text{Tr}\{G\rho\} \leq P} H(\mathcal{N}(\rho)) - H(\hat{\mathcal{N}}(\rho)) \quad (201)$$

is achievable as well. Finally, we can repeat the whole argument for all $\rho^{(k)} \in \mathcal{D}(\mathcal{H}_A^{\otimes k})$ satisfying $\text{Tr}\{\bar{G}_k \rho^{(k)}\} \leq P$, take the channel as $\mathcal{N}^{\otimes k}$, and conclude that the following rate is achievable:

$$\frac{1}{k} \sup_{\text{Tr}\{\bar{G}_k \rho^{(k)}\} \leq P} H(\mathcal{N}^{\otimes k}(\rho^{(k)})) - H(\hat{\mathcal{N}}^{\otimes k}(\rho^{(k)})). \quad (202)$$

Taking the limit as $k \rightarrow \infty$ gives the statement of the theorem. \square

VII. ENERGY-CONSTRAINED QUANTUM AND PRIVATE CAPACITY OF DEGRADABLE CHANNELS

It is unknown how to compute the quantum and private capacities of general channels, but if they are degradable, the task simplifies considerably. That is, it is known from [51] and [77], respectively, that both the unconstrained quantum and private capacities of a degradable channel \mathcal{N} are given by the following formula:

$$Q(\mathcal{N}) = P(\mathcal{N}) = \sup_{\rho} I_c(\rho, \mathcal{N}). \quad (203)$$

Here we prove the following theorem, which holds for the energy-constrained quantum and private capacities of a degradable channel \mathcal{N} :

Theorem 3: Let G be a Gibbs observable and $P \in [0, \infty)$. Let a quantum channel \mathcal{N} be degradable and satisfy Condition 1. Then the energy-constrained capacities $Q(\mathcal{N}, G, P)$, $E(\mathcal{N}, G, P)$, $P(\mathcal{N}, G, P)$, and $K(\mathcal{N}, G, P)$ are finite, equal, and given by the following formula:

$$\sup_{\rho: \text{Tr}\{G\rho\} \leq P} H(\mathcal{N}(\rho)) - H(\hat{\mathcal{N}}(\rho)), \quad (204)$$

where $\hat{\mathcal{N}}$ denotes a complementary channel of \mathcal{N} .

Proof: That the quantity in (204) is finite follows directly from the assumption in Condition 1 and Lemma 1. From Theorem 1, we have that

$$\begin{aligned} Q(\mathcal{N}, G, P) &= E(\mathcal{N}, G, P) \\ &\leq P(\mathcal{N}, G, P) = K(\mathcal{N}, G, P). \end{aligned} \quad (205)$$

Theorem 2 implies that the rate in (204) is achievable. So this gives that

$$\begin{aligned} \sup_{\rho: \text{Tr}\{G\rho\} \leq P} H(\mathcal{N}(\rho)) - H(\hat{\mathcal{N}}(\rho)) \\ \leq Q(\mathcal{N}, G, P) = E(\mathcal{N}, G, P). \end{aligned} \quad (206)$$

To establish the theorem, it thus suffices to prove the following converse inequality

$$K(\mathcal{N}, G, P) \leq \sup_{\rho: \text{Tr}\{G\rho\} \leq P} H(\mathcal{N}(\rho)) - H(\hat{\mathcal{N}}(\rho)). \quad (207)$$

To do so, we make use of several ideas from [4], [51], [62], and [77]. Consider an $(n, M, G, P, \varepsilon)$ code for secret key transmission with an average energy constraint, as described in Section III-D. Using such a code, we take a uniform distribution over the codewords, and the state resulting from an isometric extension of the channel is as follows:

$$\sigma_{\hat{M}B^n E^n} \equiv \frac{1}{M} \sum_{m=1}^M |m\rangle\langle m|_{\hat{M}} \otimes [\mathcal{U}^{\mathcal{N}}]^{\otimes n}(\rho_{A^n}^m). \quad (208)$$

Now consider that each codeword in such a code has a spectral decomposition as follows:

$$\rho_{A^n}^m \equiv \sum_{l=1}^{\infty} p_{L|\hat{M}}(l|m) |\psi^{l,m}\rangle\langle\psi^{l,m}|_{A^n}, \quad (209)$$

for a probability distribution $p_{L|\hat{M}}$ and some orthonormal basis $\{|\psi^{l,m}\rangle_{A^n}\}_l$ for \mathcal{H}_{A^n} . Then the state $\sigma_{\hat{M}B^n E^n}$ has the following extension:

$$\begin{aligned} \sigma_{L\hat{M}B^n E^n} &\equiv \frac{1}{M} \sum_{m=1}^M \sum_{l=1}^{\infty} p_{L|\hat{M}}(l|m) |l\rangle\langle l|_L \otimes |m\rangle\langle m|_{\hat{M}} \\ &\quad \otimes [\mathcal{U}^{\mathcal{N}}]^{\otimes n}(|\psi^{l,m}\rangle\langle\psi^{l,m}|_{A^n}). \end{aligned} \quad (210)$$

We can also define the state after the decoding measurement acts as

$$\begin{aligned} \sigma_{L\hat{M}M'E^n} &\equiv \frac{1}{M} \sum_{m,m'=1}^M \sum_{l=1}^{\infty} p_{L|\hat{M}}(l|m) |l\rangle\langle l|_L \otimes |m\rangle\langle m|_{\hat{M}} \\ &\quad \otimes \text{Tr}_{B^n} \{ \Lambda_{B^n}^{m'} [\mathcal{U}^{\mathcal{N}}]^{\otimes n} (|\psi^{l,m}\rangle\langle\psi^{l,m}|_{A^n}) \} \\ &\quad \otimes |m'\rangle\langle m'|_{M'}. \end{aligned} \quad (211)$$

Let $\bar{\rho}_A$ denote the average single-channel input state, defined as

$$\bar{\rho}_A \equiv \frac{1}{Mn} \sum_{m=1}^M \sum_{i=1}^n \text{Tr}_{A^n \setminus A_i} \{ \rho_{A^n}^m \}. \quad (212)$$

Applying the partial trace and the assumption in (70), it follows that

$$\text{Tr}\{G\bar{\rho}_A\} = \frac{1}{M} \sum_{m=1}^M \text{Tr}\{\bar{G}_n \rho_{A^n}^m\} \leq P. \quad (213)$$

Let $\bar{\sigma}_B$ denote the average single-channel output state:

$$\bar{\sigma}_B \equiv \mathcal{N}(\bar{\rho}_A) = \frac{1}{n} \sum_{i=1}^n \text{Tr}_{B^n \setminus B_i} \{ \sigma_{B^n} \}, \quad (214)$$

and let $\bar{\sigma}_E$ denote the average single-channel environment state:

$$\bar{\sigma}_E \equiv \hat{\mathcal{N}}(\bar{\rho}_A) = \frac{1}{n} \sum_{i=1}^n \text{Tr}_{E^n \setminus E_i} \{ \sigma_{E^n} \}. \quad (215)$$

It follows from non-negativity, subadditivity of entropy, concavity of entropy, (213), and the assumption that G is a Gibbs

observable that

$$\begin{aligned} 0 &\leq H\left(\frac{1}{M}\sum_{m=1}^M\rho_{A^n}^m\right) \\ &\leq\sum_{i=1}^n H\left(\frac{1}{M}\sum_{m=1}^M\text{Tr}_{A^n\setminus A_i}\{\rho_{A^n}^m\}\right) \\ &\leq nH(\bar{\rho}_A) < \infty. \end{aligned} \quad (216)$$

Similar reasoning but applying Condition 1 implies that

$$0 \leq H(B^n)_\sigma \leq \sum_{i=1}^n H(B_i)_\sigma \leq nH(B)_{\bar{\sigma}} < \infty. \quad (217)$$

Similar reasoning but applying Lemma 1 implies that

$$0 \leq H(E^n)_\sigma \leq \sum_{i=1}^n H(E_i)_\sigma \leq nH(E)_{\bar{\sigma}} < \infty. \quad (218)$$

Furthermore, the entropy $H(\hat{M})_\sigma = \log_2 M$ because the reduced state σ_M is maximally mixed with dimension equal to M .

Our analysis makes use of several other entropic quantities, each of which we need to argue is finitely bounded from above and below and thus can be added or subtracted at will in our analysis. The quantities involved are as follows, along with bounds for them [47], [59], [61]:

$$0 \leq I(\hat{M}; B^n)_\sigma \leq \min\{\log_2 M, nH(B)_{\bar{\sigma}}\}, \quad (219)$$

$$0 \leq I(\hat{M}; E^n)_\sigma \leq \min\{\log_2 M, nH(E)_{\bar{\sigma}}\}, \quad (220)$$

$$0 \leq H(\hat{M}|E^n)_\sigma \leq \log_2 M, \quad (221)$$

as well as

$$0 \leq I(\hat{M}L; B^n)_\sigma, \quad I(L; B^n|\hat{M})_\sigma, \quad H(B^n|L\hat{M})_\sigma \leq nH(B)_{\bar{\sigma}}, \quad (222)$$

and

$$0 \leq I(\hat{M}L; E^n)_\sigma, \quad I(L; E^n|\hat{M})_\sigma, \quad H(E^n|L\hat{M})_\sigma \leq nH(E)_{\bar{\sigma}}. \quad (223)$$

We now proceed with the converse proof:

$$\log_2 M = H(\hat{M})_\sigma \quad (224)$$

$$= I(\hat{M}; M')_\sigma + H(\hat{M}|M')_\sigma \quad (225)$$

$$\leq I(\hat{M}; M')_\sigma + h_2(\varepsilon) + \varepsilon \log_2(M - 1) \quad (226)$$

$$\leq I(\hat{M}; B^n)_\sigma + h_2(\varepsilon) + \varepsilon \log_2 M. \quad (227)$$

The first equality follows because the entropy of a uniform distribution is equal to the logarithm of its cardinality. The second equality is an identity. The first inequality follows from applying Fano's inequality in (48) to the condition in (71). The second inequality follows from applying the Holevo bound [78], [79]. The direct sum property of the trace distance and the security condition in (72) imply that

$$\begin{aligned} &\frac{1}{2} \|\sigma_{\hat{M}E^n} - \pi_{\hat{M}} \otimes \omega_{E^n}\|_1 \\ &= \frac{1}{M} \sum_{m=1}^M \frac{1}{2} \|\hat{\mathcal{N}}^{\otimes n}(\rho_{A^n}^m) - \omega_{E^n}\|_1 \leq \varepsilon, \end{aligned} \quad (228)$$

which, by the AFW inequality in (53) for classical–quantum states, means that

$$\left| H(\hat{M}|E^n)_{\pi \otimes \omega} - H(\hat{M}|E^n)_\sigma \right| \leq \varepsilon \log_2(M) + g(\varepsilon). \quad (229)$$

But

$$\begin{aligned} &H(\hat{M}|E^n)_{\pi \otimes \omega} - H(\hat{M}|E^n)_\sigma \\ &= H(\hat{M})_\pi - H(\hat{M}|E^n)_\sigma \end{aligned} \quad (230)$$

$$= H(\hat{M})_\sigma - H(\hat{M}|E^n)_\sigma \quad (231)$$

$$= I(\hat{M}; E^n)_\sigma, \quad (232)$$

so then

$$I(\hat{M}; E^n)_\sigma \leq \varepsilon \log_2(M) + g(\varepsilon). \quad (233)$$

Returning to (227) and inserting (233), we find that

$$\log_2 M \leq I(\hat{M}; B^n)_\sigma - I(\hat{M}; E^n)_\sigma + 2\varepsilon \log_2 M + h_2(\varepsilon) + g(\varepsilon). \quad (234)$$

We now focus on bounding the term $I(\hat{M}; B^n)_\sigma - I(\hat{M}; E^n)_\sigma$:

$$\begin{aligned} &I(\hat{M}; B^n)_\sigma - I(\hat{M}; E^n)_\sigma \\ &= I(\hat{M}L; B^n)_\sigma - I(L; B^n|\hat{M})_\sigma \\ &\quad - \left[I(\hat{M}L; E^n)_\sigma - I(L; E^n|\hat{M})_\sigma \right] \end{aligned} \quad (235)$$

$$\begin{aligned} &= I(\hat{M}L; B^n)_\sigma - I(\hat{M}L; E^n)_\sigma \\ &\quad - \left[I(L; B^n|\hat{M})_\sigma - I(L; E^n|\hat{M})_\sigma \right] \end{aligned} \quad (236)$$

$$\leq I(\hat{M}L; B^n)_\sigma - I(\hat{M}L; E^n)_\sigma \quad (237)$$

$$\begin{aligned} &= H(B^n)_\sigma - H(B^n|L\hat{M})_\sigma \\ &\quad - \left[H(E^n)_\sigma - H(E^n|L\hat{M})_\sigma \right] \end{aligned} \quad (238)$$

$$\begin{aligned} &= H(B^n)_\sigma - H(B^n|L\hat{M})_\sigma \\ &\quad - \left[H(E^n)_\sigma - H(B^n|L\hat{M})_\sigma \right] \end{aligned} \quad (239)$$

$$= H(B^n)_\sigma - H(E^n)_\sigma. \quad (240)$$

The first equality follows from the chain rule for mutual information. The second equality follows from a rearrangement. The first inequality follows from the assumption of degradability of the channel, which implies that Bob's mutual information is never smaller than Eve's: $I(L; B^n|\hat{M})_\sigma \geq I(L; E^n|\hat{M})_\sigma$. The third equality follows from definitions. The fourth equality follows because the marginal entropies of a pure state are equal, i.e.,

$$\begin{aligned} &H(B^n|L\hat{M})_\sigma \\ &= \frac{1}{M} \sum_{l,m} p_{L|\hat{M}}(l|m) H(\text{Tr}_{E^n} \{[\mathcal{U}^{\mathcal{N}}]^{\otimes n}(|\psi^{l,m}\rangle\langle\psi^{l,m}|_{A^n})\}) \\ &= \frac{1}{M} \sum_{l,m} p_{L|\hat{M}}(l|m) H(\text{Tr}_{B^n} \{[\mathcal{U}^{\mathcal{N}}]^{\otimes n}(|\psi^{l,m}\rangle\langle\psi^{l,m}|_{A^n})\}) \\ &= H(E^n|L\hat{M})_\sigma. \end{aligned} \quad (241)$$

Continuing, we have that

$$(240) = H(B_1)_\sigma - H(E_1)_\sigma + H(B_2 \cdots B_n)_\sigma \\ - H(E_1 \cdots E_n)_\sigma \\ - [I(B_1; B_2 \cdots B_n)_\sigma - I(E_1; E_2 \cdots E_n)_\sigma] \quad (242)$$

$$\leq H(B_1)_\sigma - H(E_1)_\sigma \\ + H(B_2 \cdots B_n)_\sigma - H(E_1 \cdots E_n)_\sigma \quad (243)$$

$$\leq \sum_{i=1}^n H(B_i)_\sigma - H(E_i)_\sigma \quad (244)$$

$$\leq n [H(B)_{\mathcal{U}(\bar{\rho})} - H(E)_{\mathcal{U}(\bar{\rho})}] \quad (245)$$

$$\leq n \left[\sup_{\rho: \text{Tr}\{G\rho\} \leq P} H(\mathcal{N}(\rho)) - H(\hat{\mathcal{N}}(\rho)) \right]. \quad (246)$$

The first equality follows by exploiting the definition of mutual information. The first inequality follows from the assumption of degradability, which implies that $I(B_1; B_2 \cdots B_n)_\sigma \geq I(E_1; E_2 \cdots E_n)_\sigma$. The second inequality follows by iterating the argument. The third inequality follows from the concavity of the coherent information for degradable channels (Proposition 1), with $\bar{\rho}_A$ defined as in (212) and satisfying (213). Thus, the final inequality follows because we can optimize the coherent information with respect all density operators satisfying the energy constraint.

Putting everything together and assuming that $\varepsilon < 1/2$, we find the following bound for all $(n, M, G, P, \varepsilon)$ private communication codes:

$$(1 - 2\varepsilon) \frac{1}{n} \log_2 M - \frac{1}{n} [h_2(\varepsilon) + g(\varepsilon)] \\ \leq \sup_{\rho: \text{Tr}\{G\rho\} \leq P} H(\mathcal{N}(\rho)) - H(\hat{\mathcal{N}}(\rho)). \quad (247)$$

Now taking the limit as $n \rightarrow \infty$ and then as $\varepsilon \rightarrow 0$, we can conclude the inequality in (207). This concludes the proof. \square

VIII. REGULARIZED CONVERSES FOR ENERGY-CONSTRAINED QUANTUM AND PRIVATE CAPACITY OF GENERAL CHANNELS

In this section, we establish regularized converses for the energy-constrained quantum and private capacities of general channels. We start with private capacity, but before doing so, we should give some further background (available in [39], [70], and [80]) and recall the definition of the energy-constrained private information of a channel [80]. A generalized (continuous) ensemble corresponds to a Borel probability measure on the set of quantum states. Let $\mathcal{M}(\mathcal{H})$ denote the set of all Borel probability measures on $\mathcal{D}(\mathcal{H})$ having the topology of weak convergence. The average state $\bar{\rho}(\mu)$ of a generalized ensemble $\mu \in \mathcal{M}(\mathcal{H})$ is the barycenter of the measure μ defined by the following Bochner integral:

$$\bar{\rho}(\mu) \equiv \int_{\mathcal{D}(\mathcal{H})} \mu(d\rho) \rho. \quad (248)$$

(The notation $\mu(d\rho)$ indicates that μ is a measure over all mixed states.) We let $\mathcal{N}(\mu)$ denote the generalized ensemble resulting from applying the channel to the states in the

generalized ensemble specified by μ . The Holevo quantity for a generalized ensemble is defined as

$$\chi(\mu) \equiv \int_{\mathcal{D}(\mathcal{H})} \mu(d\rho) D(\rho \| \bar{\rho}(\mu)). \quad (249)$$

The energy-constrained private information of a channel \mathcal{N} is then defined as [80]

$$C_p(\mathcal{N}, G, P) \equiv \sup_{\mu \in \mathcal{M}(\mathcal{H}): \text{Tr}\{G\bar{\rho}(\mu)\} \leq P} \chi(\mathcal{N}(\mu)) - \chi(\hat{\mathcal{N}}(\mu)), \quad (250)$$

where $\hat{\mathcal{N}}$ denotes a complementary channel of \mathcal{N} . We can now state our first result for general channels:

Theorem 4: Let G be a Gibbs observable and $P \in [0, \infty)$. Let a quantum channel \mathcal{N} satisfy Condition 1. Then the energy-constrained capacities $P(\mathcal{N}, G, P)$ and $K(\mathcal{N}, G, P)$ are finite, equal, and bounded from above by the regularized energy-constrained private information:

$$P(\mathcal{N}, G, P) = K(\mathcal{N}, G, P) \leq \lim_{k \rightarrow \infty} \frac{1}{k} C_p(\mathcal{N}^{\otimes k}, \bar{G}_k, P). \quad (251)$$

Proof: Theorem 1 implies that

$$P(\mathcal{N}, G, P) = K(\mathcal{N}, G, P). \quad (252)$$

To establish the theorem stated above, it thus suffices to prove the following converse inequality

$$K(\mathcal{N}, G, P) \leq \lim_{k \rightarrow \infty} \frac{1}{k} C_p(\mathcal{N}^{\otimes k}, \bar{G}_k, P). \quad (253)$$

To do so, we follow all of the steps of Theorem 3 until (234). Now let $\mu_0 \in \mathcal{M}(\mathcal{H}^{\otimes n})$ denote the discrete measure induced by the $(n, M, G, P, \varepsilon)$ secret-key transmission code. For this measure, the condition $\text{Tr}\{\bar{G}_n \bar{\rho}(\mu_0)\} \leq P$ holds by definition, being the same as (70). Thus, picking up from (234), we obtain the following:

$$I(\hat{M}; B^n)_\sigma - I(\hat{M}; E^n)_\sigma \\ = \chi(\mathcal{N}(\mu_0)) - \chi(\hat{\mathcal{N}}(\mu_0)) \quad (254)$$

$$\leq C_p(\mathcal{N}^{\otimes n}, \bar{G}_n, P), \quad (255)$$

with the inequality holding for the simple reason that we can never achieve a smaller value by optimizing over all generalized ensembles satisfying the energy constraint. We then conclude that

$$(1 - 2\varepsilon) \frac{1}{n} \log_2 M \leq \frac{1}{n} C_p(\mathcal{N}^{\otimes n}, \bar{G}_n, P) + \frac{1}{n} [h_2(\varepsilon) + g(\varepsilon)]. \quad (256)$$

Now taking the limit as $n \rightarrow \infty$ and then as $\varepsilon \rightarrow 0$, we can conclude the inequality in (253). \square

We now turn to the quantum capacity:

Theorem 5: Let G be a Gibbs observable and $P \in [0, \infty)$. Let a quantum channel \mathcal{N} satisfy Condition 1. Then the energy-constrained capacities $Q(\mathcal{N}, G, P)$ and $E(\mathcal{N}, G, P)$ are finite and equal to the regularized energy-constrained coherent information:

$$Q(\mathcal{N}, G, P) = E(\mathcal{N}, G, P) = \lim_{k \rightarrow \infty} \frac{1}{k} I_c(\mathcal{N}^{\otimes k}, \bar{G}_k, P). \quad (257)$$

Proof: Theorem 2 establishes the following lower bound:

$$Q(\mathcal{N}, G, P) \geq \lim_{k \rightarrow \infty} \frac{1}{k} I_c(\mathcal{N}^{\otimes k}, \overline{G}_k, P), \quad (258)$$

and Theorem 1 the following equality:

$$Q(\mathcal{N}, G, P) = E(\mathcal{N}, G, P). \quad (259)$$

We now establish the upper bound

$$E(\mathcal{N}, G, P) \leq \lim_{k \rightarrow \infty} \frac{1}{k} I_c(\mathcal{N}^{\otimes k}, \overline{G}_k, P). \quad (260)$$

Fix $\delta \in (0, 1)$. Consider an $(n, M, G, P(1 - \delta), \varepsilon)$ code for entanglement transmission with an average energy constraint, as described in Section III-B. Let

$$\omega_{RS} \equiv (\text{id}_R \otimes [\mathcal{D}^n \circ \mathcal{N}^{\otimes n} \circ \mathcal{E}^n])(\Phi_{RS}), \quad (261)$$

$$\kappa_{RB^n} \equiv (\text{id}_R \otimes [\mathcal{N}^{\otimes n} \circ \mathcal{E}^n])(\Phi_{RS}), \quad (262)$$

where the symbols on the right-hand side are described in Section III-B. Note that $M = \dim(\mathcal{H}_R)$, by definition. Let $\sum_l p(l) |\phi^l\rangle\langle\phi^l|_{RA^n}$ be a spectral decomposition of the state $(\text{id}_R \otimes \mathcal{E}^n)(\Phi_{RS})$, and define

$$\omega_{RS}^l \equiv (\text{id}_R \otimes [\mathcal{D}^n \circ \mathcal{N}^{\otimes n}])(|\phi^l\rangle\langle\phi^l|_{RA^n}), \quad (263)$$

$$\kappa_{RB^n}^l \equiv (\text{id}_R \otimes \mathcal{N}^{\otimes n})(|\phi^l\rangle\langle\phi^l|_{RA^n}), \quad (264)$$

so that

$$\omega_{RS} = \sum_l p(l) \omega_{RS}^l, \quad \kappa_{RB^n} = \sum_l p(l) \kappa_{RB^n}^l. \quad (265)$$

By the condition in (63), we have that

$$\varepsilon \geq 1 - \langle \Phi |_{RS} \omega_{RS} | \Phi \rangle_{RS} \quad (266)$$

$$= \sum_l p(l) \left[1 - \langle \Phi |_{RS} \omega_{RS}^l | \Phi \rangle_{RS} \right] \quad (267)$$

$$\equiv \sum_l p(l) \hat{F}_l. \quad (268)$$

Also, the energy constraint in (61) implies that

$$P(1 - \delta) \geq \text{Tr}\{\overline{G}_n \mathcal{E}^n(\pi_S)\} \quad (269)$$

$$= \sum_l p(l) \text{Tr}\{\overline{G}_n \phi_{A^n}^l\} \quad (270)$$

$$\equiv \sum_l p(l) E_l. \quad (271)$$

We would like to conclude that there exists at least one value of l for which the state $\phi_{RA^n}^l$ realizes a good entanglement generation code, in the sense of [4], while at the same time meeting the energy constraint. Let L be a random variable with probability distribution $p(l)$. By the union bound and Markov's inequality, for constant $\delta \in (0, 1)$, we have that

$$\begin{aligned} & \Pr_L \left\{ \left[E_L \leq P \cap \hat{F}_L \leq 2\varepsilon/\delta \right]^c \right\} \\ &= \Pr_L \left\{ E_L > P \cup \hat{F}_L > 2\varepsilon/\delta \right\} \end{aligned} \quad (272)$$

$$\leq \Pr_L \{E_L > P\} + \Pr_L \{\hat{F}_L > 2\varepsilon/\delta\} \quad (273)$$

$$\leq \frac{\mathbb{E}_L\{E_L\}}{P} + \frac{\mathbb{E}_L\{\hat{F}_L\}}{2\varepsilon/\delta} \quad (274)$$

$$\leq \frac{P(1 - \delta)}{P} + \frac{\varepsilon}{2\varepsilon/\delta} \quad (275)$$

$$= 1 - \delta + \delta/2 = 1 - \delta/2. \quad (276)$$

Thus, $\Pr_L\{E_L \leq P \cap \hat{F}_L \leq 2\varepsilon/\delta\} > \delta/2 > 0$, and we can conclude that there exists at least one realization l of L for which the conditions $E_l \leq P$ and $\hat{F}_l \leq 2\varepsilon/\delta$ hold. For this value, we have by (25) that

$$\frac{1}{2} \left\| \omega_{RS}^l - \Phi_{RS} \right\|_1 \leq \sqrt{2\varepsilon/\delta}. \quad (277)$$

Now consider that

$$\log \dim(\mathcal{H}_R) = I(R)S)_\Phi \quad (278)$$

$$\begin{aligned} & \leq I(R)S)_{\omega^l} + 2\sqrt{2\varepsilon/\delta} \log \dim(\mathcal{H}_R) \\ & \quad + g(\sqrt{2\varepsilon/\delta}). \end{aligned} \quad (279)$$

The equality follows from a direct calculation and the inequality from (277) and the continuity bound in (51). Continuing, we have that

$$I(R)S)_{\omega^l} \leq I(R)B^n)_{\kappa^l} \quad (280)$$

$$= H(\mathcal{N}^{\otimes n}(\phi_{A^n}^l)) - H(\hat{\mathcal{N}}^{\otimes n}(\phi_{A^n}^l)) \quad (281)$$

$$\leq I_c(\mathcal{N}^{\otimes n}, \overline{G}_n, P). \quad (282)$$

The first inequality follows from data processing of coherent information recalled in (37). The equality follows by rewriting the coherent information, given that the various entropies involved are finite. The final inequality follows because the definition of $I_c(\mathcal{N}^{\otimes n}, \overline{G}_n, P)$ involves an optimization with respect to all input states $\rho^{(n)}$ satisfying $\text{Tr}\{\overline{G}_n \rho^{(n)}\} \leq P$ and $\phi_{A^n}^l$ is one such state. Putting everything together, we find that

$$\begin{aligned} (1 - 2\sqrt{2\varepsilon/\delta}) \frac{1}{n} \log \dim(\mathcal{H}_R) & \leq \frac{1}{n} I_c(\mathcal{N}^{\otimes n}, \overline{G}_n, P) \\ & \quad + \frac{1}{n} g(\sqrt{2\varepsilon/\delta}). \end{aligned} \quad (283)$$

Now taking the limit as $n \rightarrow \infty$ and then as $\varepsilon \rightarrow 0$, we conclude that

$$E(\mathcal{N}, G, P(1 - \delta)) \leq \lim_{k \rightarrow \infty} \frac{1}{k} I_c(\mathcal{N}^{\otimes k}, \overline{G}_k, P). \quad (284)$$

However, we have proved that the above inequality holds for all $\delta \in (0, 1)$, and so we can take a supremum over $\delta \in (0, 1)$ and arrive at the conclusion that

$$\sup_{\delta \in (0, 1)} E(\mathcal{N}, G, P(1 - \delta)) = E(\mathcal{N}, G, P) \quad (285)$$

$$\leq \lim_{k \rightarrow \infty} \frac{1}{k} I_c(\mathcal{N}^{\otimes k}, \overline{G}_k, P), \quad (286)$$

which is the inequality in (260). This concludes the proof. \square

IX. THERMAL STATE AS THE OPTIMIZER

In this section, we prove that the function

$$\sup_{\text{Tr}\{G\rho\}=P} H(\mathcal{N}(\rho)) - H(\hat{\mathcal{N}}(\rho)) \quad (287)$$

is optimized by a thermal state input if the channel \mathcal{N} is degradable and satisfies certain other properties. In what follows, for a Gibbs observable G , we define the thermal state θ_β of inverse temperature $\beta > 0$ as

$$\theta_\beta \equiv \frac{e^{-\beta G}}{\text{Tr}\{e^{-\beta G}\}}. \quad (288)$$

Theorem 6: Let G be a Gibbs observable and $P \in [0, \infty)$. Let $\mathcal{N} : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_B)$ be a degradable quantum channel satisfying Condition 1. Let θ_β denote the thermal state of G , as in (288), satisfying $\text{Tr}\{G\theta_\beta\} = P$ for some $\beta > 0$. Suppose that \mathcal{N} and a complementary channel $\hat{\mathcal{N}} : \mathcal{T}(\mathcal{H}_A) \rightarrow \mathcal{T}(\mathcal{H}_E)$ are Gibbs preserving, in the sense that there exist $\beta_1, \beta_2 > 0$ such that

$$\mathcal{N}(\theta_\beta) = \theta_{\beta_1}, \quad \hat{\mathcal{N}}(\theta_\beta) = \theta_{\beta_2}. \quad (289)$$

Set

$$P_1 \equiv \text{Tr}\{G\mathcal{N}(\theta_\beta)\}, \quad P_2 \equiv \text{Tr}\{G\hat{\mathcal{N}}(\theta_\beta)\}. \quad (290)$$

Suppose further that \mathcal{N} and $\hat{\mathcal{N}}$ are such that, for all input states ρ such that $\text{Tr}\{G\rho\} = P$, the output energies satisfy

$$\text{Tr}\{G\mathcal{N}(\rho)\} \leq P_1, \quad \text{Tr}\{G\hat{\mathcal{N}}(\rho)\} \geq P_2. \quad (291)$$

Then the function

$$\sup_{\text{Tr}\{G\rho\}=P} H(\mathcal{N}(\rho)) - H(\hat{\mathcal{N}}(\rho)), \quad (292)$$

is optimized by the thermal state θ_β .

Proof: Let $\mathcal{D} : \mathcal{T}(\mathcal{H}_B) \rightarrow \mathcal{T}(\mathcal{H}_E)$ be a degrading channel such that $\mathcal{D} \circ \mathcal{N} = \hat{\mathcal{N}}$. Consider a state ρ such that $\text{Tr}\{G\rho\} = P$. The monotonicity of quantum relative entropy with respect to quantum channels (see (29)) implies that

$$D(\mathcal{N}(\rho) \parallel \mathcal{N}(\theta_\beta)) \geq D((\mathcal{D} \circ \mathcal{N})(\rho) \parallel (\mathcal{D} \circ \mathcal{N})(\theta_\beta)) \quad (293)$$

$$= D(\hat{\mathcal{N}}(\rho) \parallel \hat{\mathcal{N}}(\theta_\beta)). \quad (294)$$

By the assumption of the theorem, this means that

$$D(\mathcal{N}(\rho) \parallel \theta_{\beta_1}) \geq D(\hat{\mathcal{N}}(\rho) \parallel \theta_{\beta_2}), \quad (295)$$

where β_1 and β_2 are such that $\text{Tr}\{G\theta_{\beta_1}\} = P_1$ and $\text{Tr}\{G\theta_{\beta_2}\} = P_2$. After a rewriting using definitions and the fact that all terms below are finite, the inequality above becomes

$$\begin{aligned} \text{Tr}\{\hat{\mathcal{N}}(\rho) \log \theta_{\beta_2}\} - \text{Tr}\{\mathcal{N}(\rho) \log \theta_{\beta_1}\} \\ \geq H(\mathcal{N}(\rho)) - H(\hat{\mathcal{N}}(\rho)). \end{aligned} \quad (296)$$

Set $Z_1 \equiv \text{Tr}\{e^{-\beta_1 G}\}$ and $Z_2 \equiv \text{Tr}\{e^{-\beta_2 G}\}$. We can then rewrite the upper bound as

$$\begin{aligned} \text{Tr}\{\hat{\mathcal{N}}(\rho) \log \theta_{\beta_2}\} - \text{Tr}\{\mathcal{N}(\rho) \log \theta_{\beta_1}\} \\ = \text{Tr}\{\hat{\mathcal{N}}(\rho) \log [e^{-\beta_2 G} / Z_2]\} \\ - \text{Tr}\{\mathcal{N}(\rho) \log [e^{-\beta_1 G} / Z_1]\} \end{aligned} \quad (297)$$

$$= \log [Z_1 / Z_2] - \beta_2 \text{Tr}\{G\hat{\mathcal{N}}(\rho)\} + \beta_1 \text{Tr}\{G\mathcal{N}(\rho)\} \quad (298)$$

$$\leq \log [Z_1 / Z_2] - \beta_2 P_2 + \beta_1 P_1. \quad (299)$$

Thus, we have established a uniform upper bound on the coherent information of states subject to the constraints given in the theorem:

$$H(\mathcal{N}(\rho)) - H(\hat{\mathcal{N}}(\rho)) \leq \log [Z_1 / Z_2] - \beta_2 P_2 + \beta_1 P_1. \quad (300)$$

This bound is saturated when we choose the input $\rho = \theta_\beta$, where β is such that $\text{Tr}\{G\theta_\beta\} = P$, because

$$\log [Z_1 / Z_2] - \beta_2 P_2 + \beta_1 P_1 = H(\mathcal{N}(\theta_\beta)) - H(\hat{\mathcal{N}}(\theta_\beta)). \quad (301)$$

This concludes the proof. \square

Remark 2: Note that we can also conclude that $P_1 \geq P_2$ for channels satisfying the hypotheses of the above theorem because the channel is degradable, implying that $H(\theta_{\beta_1}) \geq H(\theta_{\beta_2})$, and the entropy of a thermal state is a strictly increasing function of the energy (and thus invertible) [65, Proposition 10].

Remark 3: The assumptions in Theorem 6 might seem somewhat artificial, but the next section demonstrates several natural examples of channels that satisfy the assumptions.

X. APPLICATION TO GAUSSIAN QUANTUM CHANNELS

We can now apply all of the results from previous sections to the particular case of quantum bosonic Gaussian channels [20], [21]. These channels model natural physical processes such as photon loss, photon amplification, thermalizing noise, or random kicks in phase space. They satisfy Condition 1 when the Gibbs observable for m modes is taken to be

$$\hat{E}_m \equiv \sum_{j=1}^m \omega_j \hat{a}_j^\dagger \hat{a}_j, \quad (302)$$

where $\omega_j > 0$ is the frequency of the j th mode and \hat{a}_j is the photon annihilation operator for the j th mode, so that $\hat{a}_j^\dagger \hat{a}_j$ is the photon number operator for the j th mode.

We start with a brief review of Gaussian states and channels (see [20], [21], and [81] for more comprehensive reviews, but note that here we mostly follow the conventions of [20]). Let

$$\hat{R} \equiv [\hat{q}_1, \dots, \hat{q}_m, \hat{p}_1, \dots, \hat{p}_m] \equiv [\hat{x}_1, \dots, \hat{x}_{2m}] \quad (303)$$

denote a row vector of position- and momentum-quadrature operators, satisfying the canonical commutation relations:

$$[\hat{R}_j, \hat{R}_k] = i\Omega_{j,k}, \quad \text{where } \Omega \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes I_m, \quad (304)$$

and I_m denotes the $m \times m$ identity matrix. We take the annihilation operator for the j th mode as $\hat{a}_j = (\hat{q}_j + i\hat{p}_j)/\sqrt{2}$. For z a column vector in \mathbb{R}^{2m} , we define the unitary displacement operator $D(z) = D^\dagger(-z) \equiv \exp(i\hat{R}z)$. Displacement operators satisfy the following relation:

$$D(z)D(z') = D(z + z') \exp\left(-\frac{i}{2}z^T \Omega z'\right). \quad (305)$$

Every state $\rho \in \mathcal{D}(\mathcal{H})$ has a corresponding Wigner characteristic function, defined as

$$\chi_\rho(z) \equiv \text{Tr}\{D(z)\rho\}, \quad (306)$$

and from which we can obtain the state ρ as

$$\rho = \int \frac{d^{2m}z}{(2\pi)^m} \chi_\rho(z) D^\dagger(z). \quad (307)$$

A quantum state ρ is Gaussian if its Wigner characteristic function has a Gaussian form as

$$\chi_\rho(\xi) = \exp\left(-\frac{1}{4}z^T V^\rho z + i[\mu^\rho]^T z\right), \quad (308)$$

where μ^ρ is the $2m \times 1$ mean vector of ρ , whose entries are defined by $\mu_j^\rho \equiv \langle \hat{R}_j \rangle_\rho$ and V^ρ is the $2m \times 2m$ covariance matrix of ρ , whose entries are defined as

$$V_{j,k}^\rho \equiv \langle \{\hat{R}_j - \mu_j^\rho, \hat{R}_k - \mu_k^\rho\} \rangle_\rho. \quad (309)$$

The following condition holds for a valid covariance matrix: $V \geq i\Omega$, which is a manifestation of the uncertainty principle.

A thermal Gaussian state θ_β of m modes with respect to \hat{E}_m from (302) and having inverse temperature $\beta > 0$ thus has the following form:

$$\theta_\beta = e^{-\beta \hat{E}_m} / \text{Tr}\{e^{-\beta \hat{E}_m}\}, \quad (310)$$

and has a mean vector equal to zero and a diagonal $2m \times 2m$ covariance matrix. One can calculate that the photon number in this state is equal to

$$\sum_j \frac{1}{e^{\beta \omega_j} - 1}. \quad (311)$$

It is also well known that thermal states can be written as a Gaussian mixture of displacement operators acting on the vacuum state:

$$\theta_\beta = \int d^{2m} \xi p(\xi) D(\xi) [|0\rangle\langle 0|]^{\otimes m} D^\dagger(\xi), \quad (312)$$

where $p(\xi)$ is a zero-mean, circularly symmetric Gaussian distribution. From this, it also follows that randomly displacing a thermal state in such a way leads to another thermal state of higher temperature:

$$\theta_\beta = \int d^{2m} \xi q(\xi) D(\xi) \theta_{\beta'} D^\dagger(\xi), \quad (313)$$

where $\beta' \geq \beta$ and $q(\xi)$ is a particular circularly symmetric Gaussian distribution.

A $2m \times 2m$ matrix S is symplectic if it preserves the symplectic form: $S\Omega S^T = \Omega$. According to Williamson's theorem [82], there is a diagonalization of the covariance matrix V^ρ of the form,

$$V^\rho = S^\rho (D^\rho \oplus D^\rho) (S^\rho)^T, \quad (314)$$

where S^ρ is a symplectic matrix and $D^\rho \equiv \text{diag}(v_1, \dots, v_m)$ is a diagonal matrix of symplectic eigenvalues such that $v_i \geq 1$ for all $i \in \{1, \dots, m\}$. Computing this decomposition is equivalent to diagonalizing the matrix $iV^\rho\Omega$ [83, Appendix A].

The entropy $H(\rho)$ of a quantum Gaussian state ρ is a direct function of the symplectic eigenvalues of its covariance matrix V^ρ [20]:

$$H(\rho) = \sum_{j=1}^m g((v_j - 1)/2) \equiv g(V^\rho), \quad (315)$$

where $g(\cdot)$ is defined in (52) and we have indicated a shorthand for this entropy as $g(V^\rho)$.

The Hilbert–Schmidt adjoint of a Gaussian quantum channel $\mathcal{N}_{X,Y}$ from m modes to m modes has the following effect on a displacement operator $D(z)$ [20]:

$$D(z) \mapsto D(Xz) \exp\left(-\frac{1}{4}z^T Yz + iz^T d\right), \quad (316)$$

where X is a real $2m \times 2m$ matrix, Y is a real $2m \times 2m$ positive semi-definite matrix, and $d \in \mathbb{R}^{2m}$, such that they satisfy

$$Y - i\Omega + iX^T\Omega X \geq 0. \quad (317)$$

The effect of the channel on the mean vector μ^ρ and the covariance matrix V^ρ is thus as follows:

$$\mu^\rho \mapsto X^T \mu^\rho + d, \quad (318)$$

$$V^\rho \mapsto X^T V^\rho X + Y. \quad (319)$$

All Gaussian channels are covariant with respect to displacement operators. That is, the following relation holds

$$\mathcal{N}_{X,Y}(D(z)\rho D^\dagger(z)) = D(X^T z) \mathcal{N}_{X,Y}(\rho) D^\dagger(X^T z). \quad (320)$$

Just as every quantum channel can be implemented as a unitary transformation on a larger space followed by a partial trace, so can Gaussian channels be implemented as a Gaussian unitary on a larger space with some extra modes prepared in the vacuum state, followed by a partial trace [20]. Given a Gaussian channel $\mathcal{N}_{X,Y}$ with Z such that $Y = ZZ^T$ we can find two other matrices X_E and Z_E such that there is a symplectic matrix

$$S = \begin{bmatrix} X^T & Z \\ X_E^T & Z_E \end{bmatrix}, \quad (321)$$

which corresponds to the Gaussian unitary transformation on a larger space. The complementary channel $\hat{\mathcal{N}}_{X_E, Y_E}$ from input to the environment then effects the following transformation on mean vectors and covariance matrices:

$$\mu^\rho \mapsto X_E^T \mu^\rho, \quad (322)$$

$$V^\rho \mapsto X_E^T V^\rho X_E + Y_E, \quad (323)$$

where $Y_E \equiv Z_E Z_E^T$.

A quantum Gaussian channel for which $X = X' \oplus X'$, $Y = Y' \oplus Y'$, and $d = d' \oplus d'$ is known as a phase-insensitive Gaussian channel, because it does not have a bias to either quadrature when applying noise to the input state.

The main result of this section is the following theorem, which gives an explicit expression for the energy-constrained capacities of all phase-insensitive degradable Gaussian channels that satisfy the conditions of Theorem 6 for all $\beta > 0$:

Theorem 7: Let $\mathcal{N}_{X,Y}$ be a phase-insensitive degradable Gaussian channel, having a dilation of the form in (321). Suppose that $\mathcal{N}_{X,Y}$ satisfies the conditions of Theorem 6 for all $\beta > 0$. Then its energy-constrained capacities $Q(\mathcal{N}_{X,Y}, \hat{E}_m, P)$, $E(\mathcal{N}_{X,Y}, \hat{E}_m, P)$, $P(\mathcal{N}_{X,Y}, \hat{E}_m, P)$, and $K(\mathcal{N}_{X,Y}, \hat{E}_m, P)$ are equal and given by the following formula:

$$g(X^T V^{\theta_\beta} X + Y) - g(X_E^T V^{\theta_\beta} X_E + Y_E), \quad (324)$$

where θ_β is a thermal state of mean photon number P .

Proof: Since the channel is degradable, satisfies Condition 1, and \hat{E}_m is a Gibbs observable, Theorem 3 applies and these capacities are given by the following formula:

$$\sup_{\rho: \text{Tr}\{\hat{E}_m \rho\} \leq P} H(\mathcal{N}_{X,Y}(\rho)) - H(\hat{\mathcal{N}}_{X_E, Y_E}(\rho)). \quad (325)$$

By assumption, the channel satisfies the conditions of Theorem 6 as well for all $\beta > 0$, so that the following function is optimized by a thermal state θ_β of mean photon number P :

$$\begin{aligned} \sup_{\rho: \text{Tr}\{\hat{E}_m \rho\}=P} H(\mathcal{N}_{X,Y}(\rho)) - H(\hat{\mathcal{N}}_{X_E,Y_E}(\rho)) \\ = H(\mathcal{N}_{X,Y}(\theta_\beta)) - H(\hat{\mathcal{N}}_{X_E,Y_E}(\theta_\beta)). \end{aligned} \quad (326)$$

It thus remains to prove that $H(\mathcal{N}_{X,Y}(\theta_\beta)) - H(\hat{\mathcal{N}}_{X_E,Y_E}(\theta_\beta))$ is increasing with decreasing β . This follows from the covariance property in (320), the concavity of coherent information in the input for degradable channels (Proposition 1), and the fact that thermal states can be realized by random Gaussian displacements of thermal states with lower temperature. Consider that

$$\begin{aligned} H(\mathcal{N}_{X,Y}(\theta_{\beta'})) - H(\hat{\mathcal{N}}_{X_E,Y_E}(\theta_{\beta'})) \\ = \int d^{2m} \xi q(\xi) \left[H(\mathcal{N}_{X,Y}(\theta_{\beta'})) - H(\hat{\mathcal{N}}_{X_E,Y_E}(\theta_{\beta'})) \right] \end{aligned} \quad (327)$$

$$\begin{aligned} = \int d^{2m} \xi q(\xi) \left[H(D(X\xi)\mathcal{N}_{X,Y}(\theta_{\beta'})D^\dagger(X\xi)) \right. \\ \left. - H(D(X_E\xi)\hat{\mathcal{N}}_{X_E,Y_E}(\theta_{\beta'})D^\dagger(X_E\xi)) \right] \end{aligned} \quad (328)$$

$$\begin{aligned} = \int d^{2m} \xi q(\xi) \left[H(\mathcal{N}_{X,Y}(D(\xi)\theta_{\beta'})D^\dagger(\xi)) \right. \\ \left. - H(\hat{\mathcal{N}}_{X_E,Y_E}(D(\xi)\theta_{\beta'})D^\dagger(\xi)) \right] \end{aligned} \quad (329)$$

$$\leq H(\mathcal{N}_{X,Y}(\theta_\beta)) - H(\hat{\mathcal{N}}_{X_E,Y_E}(\theta_\beta)). \quad (330)$$

The first equality follows by placing a probability distribution in front, and the second follows from the unitary invariance of quantum entropy. The third equality follows from the covariance property of quantum Gaussian channels, given in (320). The inequality follows because the coherent information of degradable channels is concave in the input state (Proposition 1) and from (313). \square

A. Special Cases: Single-Mode Pure-Loss and Quantum-Limited Amplifier Channels

We can now discuss some special cases of the above result, some of which have already been known in the literature. Suppose that the channel is a single-mode pure-loss channel \mathcal{L}_η , where $\eta \in [1/2, 1]$ characterizes the average fraction of photons that make it through the channel from sender to receiver.¹ In this case, the channel has $X = \sqrt{\eta}I_2$ and $Y = (1 - \eta)I_2$. We take the Gibbs observable to be the photon-number operator $\hat{a}^\dagger \hat{a}$ and the energy constraint to be $N_S \in [0, \infty)$. Such a channel is degradable [84] and was conjectured [33] to have energy-constrained quantum and private capacities equal to

$$g(\eta N_S) - g((1 - \eta)N_S). \quad (331)$$

This conjecture was proven for the quantum capacity in [28, Th. 8], and the present paper establishes the statement for private capacity. This was argued by exploiting particular properties of the g function (established in great detail

¹We do not consider transmissivities $\eta \in [0, 1/2]$ because the quantum capacity vanishes in this range since the channel becomes antidegradable.

in [85]) to show that the thermal state input is optimal for any fixed energy constraint. Here we can see this latter result as a consequence of the more general statements in Theorems 6 and 7, which are based on the monotonicity of relative entropy and other properties of this channel, such as covariance and degradability. Taking the limit $N_S \rightarrow \infty$, the formula in (331) converges to

$$\log_2(\eta/[1 - \eta]), \quad (332)$$

which is consistent with the formula stated in [27].

Suppose that the channel is a single-mode quantum-limited amplifier channel \mathcal{A}_κ of gain $\kappa \geq 1$. In this case, the channel has $X = \sqrt{\kappa}I_2$ and $Y = (\kappa - 1)I_2$. Again we take the energy operator and constraint as above. This channel is degradable [84] and was recently proven [35] to have energy-constrained quantum and private capacity equal to

$$g(\kappa N_S + \kappa - 1) - g([\kappa - 1][N_S + 1]). \quad (333)$$

The result was established by exploiting particular properties of the g function in addition to other arguments. However, we can again see this result as a consequence of the more general statements given in Theorems 6 and 7. Taking the limit $N_S \rightarrow \infty$, the formula converges to

$$\log_2(\kappa/[\kappa - 1]), \quad (334)$$

which is consistent with the formula stated in [27] and recently proven in [29] and [30].

Remark 4: Reference [27] has been widely accepted to have provided a complete proof of the unconstrained quantum capacity formulas given in (332) and (334). The important developments of [27] were to identify that it suffices to optimize coherent information of these channels with respect to a single channel use and Gaussian input states. The issue is that [27] relied on an “optimization procedure carried out in” [26] in order to establish the infinite-energy quantum capacity formula given there (see just before [27, eq. (12)]). However, a careful inspection of [26, Sec. V-B] reveals that no explicit optimization procedure is given there. The contentious point is that it is necessary to show that, among all Gaussian states, the thermal state is the input state optimizing the coherent information of the quantum-limited attenuator and amplifier channels. This point is not argued or in any way justified in [26, Sec. V-B] or in any subsequent work or review on the topic [39], [86]–[88]. As a consequence, we have been left to conclude that the proof from [27] features a gap which was subsequently closed in [28, Sec. III-G-1] and [35]. The result in [29] and [30] gives a completely different approach for establishing the unconstrained quantum and private capacities of the quantum-limited amplifier channel, which preceded the development in [35].

B. Special Cases: Multi-Mode Pure-Loss and Quantum-Limited Amplifier Channels

Our results from Theorems 6 and 7 allow for making more general statements, applicable to broadband scenarios considered in prior works for other capacities [32], [89], [90]. Let the Gibbs observable be \hat{E}_m , as given in (302), and

suppose that the energy constraint is $P \in [0, \infty)$. Suppose that the channel is an m -mode channel consisting of m parallel pure-loss channels \mathcal{L}_η , each with the same transmissivity $\eta \in [1/2, 1]$. Then for \hat{E}_m and such an m -mode channel, the conditions of Theorems 6 and 7 are satisfied, so that the energy-constrained quantum and private capacities are given by

$$\sum_{j=1}^m g(\eta N_j(\beta)) - g((1 - \eta)N_j(\beta)), \quad (335)$$

where

$$N_s(\beta) \equiv 1/(e^{\beta\omega_s} - 1), \quad (336)$$

and β is chosen such that $P = \sum_{j=1}^m N_j(\beta)$, so that the energy constraint is satisfied. A similar statement applies to m parallel quantum-limited amplifier channels each having the same gain $\kappa \geq 1$. In this case, the conditions of Theorems 6 and 7 are satisfied, so that the energy-constrained quantum and private capacities are given by

$$\sum_{j=1}^m g(\kappa N_j(\beta) + \kappa - 1) - g([\kappa - 1][N_j(\beta) + 1]), \quad (337)$$

where $N_j(\beta)$ is as defined above and β is chosen to satisfy $P = \sum_{j=1}^m N_j(\beta)$.

Theorems 6 and 7 can be applied indirectly to a more general scenario. Let $m = k + l$, where k and l are positive integers. Suppose that the channel consists of k pure-loss channels \mathcal{L}_{η_i} , each of transmissivity $\eta_i \in [1/2, 1]$, and l quantum-limited amplifier channels \mathcal{A}_{κ_j} , each of gain κ_j for $j \in \{1, \dots, l\}$. In this scenario, Theorems 6 and 7 apply to the individual channels, so that we know that a thermal state is the optimal input to each of them for a fixed input energy. The task is then to determine how to allocate the energy such that the resulting capacity is optimal. Let P denote the total energy budget, and suppose that a particular allocation $\{\{N_i\}_{i=1}^k, \{M_j\}_{j=1}^l\}$ is made such that

$$P = \sum_{i=1}^k \omega_i N_i + \sum_{j=1}^l \omega_j M_j. \quad (338)$$

Then Theorems 6 and 7 apply to the scenario when the allocation is fixed and imply that the resulting quantum and private capacities are equal and given by

$$\begin{aligned} & \sum_{i=1}^k g(\eta_i N_i) - g((1 - \eta_i)N_i) \\ & + \sum_{j=1}^l g(\kappa M_j + \kappa - 1) - g([\kappa - 1][M_j + 1]). \end{aligned} \quad (339)$$

However, we can then optimize this expression with respect to the energy allocation, leading to the following constrained

optimization problem:

$$\begin{aligned} & \max_{\{\{N_i\}_{i=1}^k, \{N_j\}_{j=1}^l\}} \sum_{i=1}^k g(\eta_i N_i) - g((1 - \eta_i)N_i) \\ & + \sum_{j=1}^l g(\kappa M_j + \kappa - 1) - g([\kappa - 1][M_j + 1]), \end{aligned} \quad (340)$$

such that

$$P = \sum_{i=1}^k \omega_i N_i + \sum_{j=1}^l \omega_j M_j. \quad (341)$$

This problem can be approached using Lagrange multiplier methods, and in some cases handled analytically, while others need to be handled numerically. Many different scenarios were considered already in [32], to which we point the interested reader. However, we should note that [32] was developed when the formulas above were only conjectured to be equal to the capacity and not proven to be so.

XI. DISCUSSION OF NON-GAUSSIAN CHANNELS AND OTHER ENERGY CONSTRAINTS

We stress again here that the framework for energy-constrained quantum and private capacity given in this paper applies to more general situations beyond bosonic Gaussian channels with photon number constraints, just as the frameworks from [36]–[38] and [70] do for other kinds of communication capacities. All we require for our theorems to apply is that the energy observable be a Gibbs observable (Definition 3) and the channel satisfy the finite output-entropy condition (Condition 1).

There are some interesting cases to consider. For example, it could be the case that the initial state of the environment in a thermal channel has not reached its equilibrium state and is in a non-Gaussian state different from a thermal state. This kind of channel is related to those presented and analyzed recently in [91]. If the initial environment state is an approximate thermal state (has trace distance close to a thermal state of a certain photon number), then the tools of the present paper, as well as those detailed in the recent work [40], could be used to estimate the quantum and private capacity of this non-equilibrium thermal channel.

Even in the bosonic setting, one could also consider other energy observables besides photon number observables. For example, one could consider the square or higher powers of the photon number observables, which might be relevant in situations in which the transmitter is highly sensitive to higher photon numbers. Using the square of the photon number would penalize higher photon numbers more severely than the typical photon number constraint.

For the case of the pure-loss and quantum-limited amplifier channels, we can give concrete bounds for energy-constrained quantum and private capacity, using \hat{n}^2 as the energy observable, by employing an idea put forward recently in [40, Remark 21], as well as other arguments. Suppose that the Gibbs observable is now \hat{n}^2 (the square of the photon number operator). We first discuss how to obtain an upper bound

on the capacities. Due to these channels being degradable, Theorem 3 applies, and it suffices to consider optimizing the single-copy energy-constrained coherent information in (204), subject to the constraint $\text{Tr}\{\hat{n}^2\rho\} \leq P$ on the input state ρ . By concavity of the square-root function, and due to the fact that $\text{Tr}\{\hat{n}^2\rho\} = \sum_{n=0}^{\infty} p(n)n^2$ for some probability distribution $p(n)$, it follows that every state satisfying $\text{Tr}\{\hat{n}^2\rho\} \leq P$ also satisfies $\text{Tr}\{\hat{n}\rho\} \leq \sqrt{P}$. Setting $N_S = \sqrt{P}$, we then find that the formulas in (331) and (333) with this value of N_S give an upper bound on the capacities.

To find a lower bound on the capacities, we can optimize the single-copy energy-constrained coherent information in (204) with respect to all Gaussian state inputs. The coherent information can also be rewritten in this case as a particular conditional entropy (see [27], [51]) that is a function of the input state ρ . Now we apply an argument from [40, Remark 21]. The pure-loss and quantum-limited amplifier channels and their complementary channels are phase-covariant, meaning that a unitary phase operator $e^{i\hat{n}\phi}$ acting on the input state commutes with the channels and acts as a unitary phase operator on the output. Since for any state ρ , the value of $\text{Tr}\{\hat{n}^2\rho\}$ is unchanged by applying a random phase to ρ , but the conditional entropy does not decrease under this operation and the phase-randomized state becomes number diagonal, it suffices to perform the optimization over all states that are both Gaussian (by assumption) and number diagonal. For a single-mode state, the only such possibility is a thermal state. Finally, since the functions in (331) and (333) are equal to the coherent informations of these channels when sending in a thermal state of mean photon number N_S , and these functions are monotone increasing with respect to N_S , it suffices to pick a thermal state $\theta(N_S)$ of mean photon number N_S that meets the energy constraint P with equality. Since $\text{Tr}\{\hat{n}^2\theta(N_S)\} = N_S(2N_S + 1)$, by solving the equation $N_S(2N_S + 1) = P$, we find that $N_S = \frac{1}{4}(\sqrt{1 + 8P} - 1)$ and then lower bounds on the energy-constrained quantum and private capacities of these channels are given by (331) and (333) with this value of N_S .

In summary, the energy-constrained quantum and private capacities of the pure-loss channel, with Gibbs observable set to \hat{n}^2 , are bounded from above by the function in (331) evaluated at $N_S = \sqrt{P}$ and from below by the formula in (331) evaluated at $N_S = \frac{1}{4}(\sqrt{1 + 8P} - 1)$. One obtains related bounds for the energy-constrained quantum and private capacities of the quantum-limited amplifier channel, with Gibbs observable set to \hat{n}^2 , by evaluating the formula in (333) at the same values of N_S .

We note that similar arguments can be employed for any power of the photon number operator \hat{n} , and one would find bounds for the energy-constrained capacities in a similar way.

Going beyond the bounds given above, it is an intriguing open question to identify the actual capacities with these modified Gibbs observables. In this scenario, it is not clear that the extremality of Gaussian states [92] applies, because the constraint is not on the covariance matrix, but rather on the expectation of a four-point correlator.

XII. CONCLUSION

This paper has provided a general theory of energy-constrained quantum and private communication over quantum channels. We defined several communication tasks (Section III), and then established ways of converting a code for one task to that of another task (Section IV). These code conversions have implications for capacities, establishing non-trivial relations between them (Section V). We showed that the regularized, energy-constrained coherent information is achievable for entanglement transmission with an average energy constraint, under the assumption that the energy observable is of the Gibbs form (Definition 3) and the channel satisfies the finite-output entropy condition (Condition 1). We then proved that the various quantum and private capacities of degradable channels are equal and characterized by the single-letter, energy-constrained coherent information (Section VII). We finally applied our results to Gaussian channels and recovered some results already known in the literature in addition to establishing new ones.

We have left open the question of proving that the regularized, energy-constrained private information, defined in (250), is an achievable rate for private communication. We think that this should certainly be possible. One particular method for doing so would be to extend the results of [11] such that they apply to coding with energy constraints and over infinite-dimensional channels. Other approaches, like that along the lines of [36] and [37] for public classical communication, in conjunction with the method from [4] and [5], could also be employed. The first approach mentioned above could potentially lead to a simpler proof of Theorem 2 (regarding quantum communication instead of private communication), but the details remain to be worked out.

Going forward from here, a great challenge is to establish a general theory of energy-constrained private and quantum communication with a limited number of channel uses. Recent progress in these scenarios without energy constraints [30], [31] suggests that this might be amenable to analysis. Another question is to identify and explore other physical systems, beyond bosonic channels, to which the general framework could apply. It could be interesting to explore generalizations of the results and settings from [93]–[97] regarding fermionic Gaussian channels. A more particular question we would like to see answered is whether concavity of coherent information of degradable channels could hold in settings beyond that considered in Proposition 1. We suspect that an approximation argument along the lines of that given in the proof of [45, Proposition 1] should make this possible. We also think it should be possible to establish an equality in Theorem 4, but we leave this for future endeavors.

APPENDIX

The following proposition states that a quantum code with good minimum fidelity implies that it has good minimum entanglement fidelity with negligible loss in parameters. This was first established in [66] and reviewed in [67]. Here we follow the proof available in [69], which therein established a

relation between trace distance and diamond distance between an arbitrary channel and the identity channel.

Proposition 5: Let $\mathcal{C} : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H})$ be a quantum channel with finite-dimensional input and output. Let \mathcal{H}' be a Hilbert space isomorphic to \mathcal{H} . If

$$\min_{|\phi\rangle \in \mathcal{H}} \langle \phi | \mathcal{C}(|\phi\rangle\langle\phi|) | \phi \rangle \geq 1 - \varepsilon, \quad (342)$$

then

$$\min_{|\psi\rangle \in \mathcal{H}' \otimes \mathcal{H}} \langle \psi | (\text{id}_{\mathcal{H}'} \otimes \mathcal{C})(|\psi\rangle\langle\psi|) | \psi \rangle \geq 1 - 2\sqrt{\varepsilon}, \quad (343)$$

where the optimizations are with respect to state vectors.

Proof: The inequality in (342) implies that the following inequality holds for all state vectors $|\phi\rangle \in \mathcal{H}$:

$$\langle \phi | [|\phi\rangle\langle\phi| - \mathcal{C}(|\phi\rangle\langle\phi|)] | \phi \rangle \leq \varepsilon. \quad (344)$$

By the inequalities in (25), this implies that

$$\| |\phi\rangle\langle\phi| - \mathcal{C}(|\phi\rangle\langle\phi|) \|_1 \leq 2\sqrt{\varepsilon}, \quad (345)$$

for all state vectors $|\phi\rangle \in \mathcal{H}$. We will show that

$$\left| \langle \phi | \left[|\phi\rangle\langle\phi^\perp| - \mathcal{C}(|\phi\rangle\langle\phi^\perp|) \right] | \phi^\perp \rangle \right| \leq 2\sqrt{\varepsilon}, \quad (346)$$

for every orthonormal pair $\{|\phi\rangle, |\phi^\perp\rangle\}$ of state vectors in \mathcal{H} . Set

$$|w_k\rangle \equiv \frac{|\phi\rangle + i^k |\phi^\perp\rangle}{\sqrt{2}} \quad (347)$$

for $k \in \{0, 1, 2, 3\}$. Then it follows that

$$|\langle \phi | \langle \phi^\perp | = \frac{1}{2} \sum_{k=0}^3 i^k |w_k\rangle\langle w_k|. \quad (348)$$

Consider now that

$$\begin{aligned} & \left| \langle \phi | \left[|\phi\rangle\langle\phi^\perp| - \mathcal{C}(|\phi\rangle\langle\phi^\perp|) \right] | \phi^\perp \rangle \right| \\ & \leq \left\| |\phi\rangle\langle\phi^\perp| - \mathcal{C}(|\phi\rangle\langle\phi^\perp|) \right\|_\infty \end{aligned} \quad (349)$$

$$\leq \frac{1}{2} \sum_{k=0}^3 \left\| |w_k\rangle\langle w_k| - \mathcal{C}(|w_k\rangle\langle w_k|) \right\|_\infty \quad (350)$$

$$\leq \frac{1}{4} \sum_{k=0}^3 \left\| |w_k\rangle\langle w_k| - \mathcal{C}(|w_k\rangle\langle w_k|) \right\|_1 \quad (351)$$

$$\leq 2\sqrt{\varepsilon}. \quad (352)$$

The first inequality follows from the characterization of the operator norm as $\|A\|_\infty = \sup_{|\phi\rangle, |\psi\rangle} |\langle \phi | A | \psi \rangle|$, where the optimization is with respect to state vectors $|\phi\rangle$ and $|\psi\rangle$. The second inequality follows from substituting (348) and applying the triangle inequality and homogeneity of the ∞ -norm. The third inequality follows because the ∞ -norm of a traceless Hermitian operator is bounded from above by half of its trace norm [98, Lemma 4]. The final inequality follows from applying (345).

Let $|\psi\rangle \in \mathcal{H}' \otimes \mathcal{H}$ be an arbitrary state vector. All such state vectors have a Schmidt decomposition of the following form:

$$|\psi\rangle = \sum_x \sqrt{p(x)} |\zeta_x\rangle \otimes |\varphi_x\rangle, \quad (353)$$

where $\{p(x)\}_x$ is a probability distribution and $\{|\zeta_x\rangle\}_x$ and $\{|\varphi_x\rangle\}_x$ are orthonormal sets, respectively. Then consider that

$$\begin{aligned} & 1 - \langle \psi | (\text{id}_{\mathcal{H}'} \otimes \mathcal{C})(|\psi\rangle\langle\psi|) | \psi \rangle \\ & = \langle \psi | (\text{id}_{\mathcal{H}'} \otimes \text{id}_{\mathcal{H}} - \text{id}_{\mathcal{H}'} \otimes \mathcal{C})(|\psi\rangle\langle\psi|) | \psi \rangle \\ & = \langle \psi | (\text{id}_{\mathcal{H}'} \otimes [\text{id}_{\mathcal{H}} - \mathcal{C}])(|\psi\rangle\langle\psi|) | \psi \rangle \\ & = \sum_{x,y} p(x)p(y) \langle \varphi_x | [|\varphi_x\rangle\langle\varphi_y| - \mathcal{C}(|\varphi_x\rangle\langle\varphi_y|)] | \varphi_y \rangle. \end{aligned} \quad (354)$$

Now applying the triangle inequality and (346), we find that

$$\begin{aligned} & 1 - \langle \psi | (\text{id}_{\mathcal{H}'} \otimes \mathcal{C})(|\psi\rangle\langle\psi|) | \psi \rangle \\ & = \left| \sum_{x,y} p(x)p(y) \langle \varphi_x | [|\varphi_x\rangle\langle\varphi_y| - \mathcal{C}(|\varphi_x\rangle\langle\varphi_y|)] | \varphi_y \rangle \right| \\ & \leq \sum_{x,y} p(x)p(y) |\langle \varphi_x | [|\varphi_x\rangle\langle\varphi_y| - \mathcal{C}(|\varphi_x\rangle\langle\varphi_y|)] | \varphi_y \rangle| \\ & \leq 2\sqrt{\varepsilon}. \end{aligned} \quad (355)$$

This concludes the proof. \square

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Mark M. Wilde (M'99–SM'13) is an Assistant Professor in the Department of Physics and Astronomy and the Center for Computation and Technology at Louisiana State University in Baton Rouge, Louisiana, USA. He is the recipient of a National Science Foundation Career Development Award and the APS-IUSSTF Professorship Award in Physics. His current research interests are in quantum Shannon theory, quantum optical communication, quantum computational complexity theory, and quantum error correction.

Haoyu Qi received his Ph.D. from the Department of Physics and Astronomy of Louisiana State University in 2017. He is currently a theoretical physicist at Xanadu, a Toronto-based company aiming to build the first optical quantum computer. His current research interests are in quantum algorithms, quantum annealing, Gaussian boson sampling, and quantum sensing.