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Algorithms Related to Triangle Groups

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Abstract

Given a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, one can talk about the different properties of this subgroup. These properties have been studied extensively in an attempt to classify these subgroups. Tim Hsu created an algorithm to determine whether a subgroup is a congruence subgroup by using permutations [9]. Lang, Lim, and Tan also created an algorithm to determine if a subgroup is a congruence subgroup by using Farey Symbols [15]. Sebbar classified torsion-free congruence subgroups of genus 0 [25]. Pauli and Cummins computed and tabulated all congruence subgroups of genus less than 24 [3]. However, there are still some problems left to be solved.

In the first part of this thesis, we will use the concept of Farey Symbols and bipartite cuboid graphs to determine when two subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ are in the same conjugacy class in $\mathrm{PSL}_2(\mathbb{Z})$. We implemented this algorithm, and other related algorithms, with SageMath [23]. In the second part of the thesis, we will extend these ideas to general triangle groups. Specifically, we will classify some small index conjugacy classes of subgroups of the triangle group $\overline{\triangle}(2,4,6)$.

Chapter 1. Preliminaries

1.1. Hyperbolic Geometry

In this section, we will recall some basic facts about hyperbolic geometry following the notations of Katok [11]. Let $\mathcal{H} = \{z = x + iy \in \mathbb{C} | \mathrm{Im}(z) > 0\}$ be the upper half complex plane with the differential form

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}.$$

The length of a curve C parameterized by $x(t)$ and $y(t)$ for $0 \leq t \leq 1$ is defined by

$$h(C) = \int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)} dt.$$

The hyperbolic distance between $z_1, z_2 \in \mathcal{H}$ is given by $\rho(z_1, z_2) = \inf h(C)$ where C is all differentiable curves connecting z_1 and z_2.

Definition 1. For $z_1, z_2 \in \mathcal{H}$, the shortest curve connecting z_1 and z_2 with respect to this metric is called the **geodesics** between z_1 and z_2.

It is well known that geodesics on $\mathcal{H}$ are either vertical lines or semicircles with endpoints on $\mathbb{R} \cup \{\infty\}$. Then it is not hard to see that for any two distinct points $z_1, z_2 \in \mathcal{H}$, there exists a geodesic connecting z_1 and z_2 and that the geodesic is unique. The hyperbolic area for $A \subset \mathcal{H}$ is defined to be

$$\mu(A) = \int_A \frac{dxdy}{y^2}.$$

A hyperbolic triangle is a region in $\mathcal{H}$ that is bounded by three geodesics.

Theorem 1 (Gauss-Bonnet [11])**.** *Let $\triangle$ be a hyperbolic triangle with angles $\frac{\pi}{a}, \frac{\pi}{b}, \frac{\pi}{c}$ where $a, b, c > 0 \in \mathbb{Z}$. Then $\mu(\triangle) = \pi - \frac{\pi}{a} - \frac{\pi}{b} - \frac{\pi}{c}$.*

Note that as $\mu(\triangle) > 0$, the above theorem implies that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$. When $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ (resp. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$), $\triangle$ is euclidean (resp. spherical) and we can classify (a, b, c) into a finite number of cases [17]. When $\triangle$ is hyperbolic, there are infinitely many cases. We will provide an algorithm for classifying specific cases when $\triangle$ is hyperbolic in the latter half of this thesis.

1.2. Triangle Groups

Let $a, b, c \geq 2 \in \mathbb{Z}$ and $\triangle$ to be a hyperbolic triangle with angles $\frac{\pi}{a}, \frac{\pi}{b}, \frac{\pi}{c}$. One can also suppose that $a \leq b \leq c$. Let G be a group generated by reflections across each sides of the triangle $\triangle$. Note that by reflections, one can tile the upper half plane using the triangle $\triangle$. A triangle group $\triangle(a, b, c)$ is a subgroup of G containing only the orientation preserving elements. Note that depending on context, we will use $\triangle(a, b, c)$ to either denote the triangle group or the hyperbolic triangle with angles $\frac{\pi}{a}, \frac{\pi}{b}, \frac{\pi}{c}$.

Recall that $\mathrm{SL}_2(\mathbb{R})$ is the group of 2 by 2 matrices with entries in $\mathbb{R}$ and determinant 1. There is an action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}$ given by the Möbius transformation,

$$\gamma z := \frac{az + b}{cz + d} \text{ for } z \in \mathcal{H} \text{ and } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Note that for $-I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $-Iz = Iz = z$. Sometimes it will be more convenient to consider $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{I, -I\}$. Since orientation preserving isometries on $\mathcal{H}$ are Möbius transformations which are represented by some $\pm M \in \mathrm{SL}_2(\mathbb{R})$, $\triangle(a, b, c)$ is then a subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Following the notations presented in [2], $\triangle(a, b, c)$ have presentation

$$\triangle(a, b, c) = \langle \sigma_a, \sigma_b, \sigma_c | \sigma_a^a = \sigma_b^b = \sigma_c^c = \sigma_a \sigma_b \sigma_c = -1 \rangle. \tag{1.1}$$

We will define $\overline{\triangle}(a, b, c) = \triangle(a, b, c)/\{\pm 1\}$. We also note that we only need 2 generators to describe $\triangle(a, b, c)$ since there is a relation among the 3 generators.

Let $\mathbb{P}^1(\mathbb{R})$ denote the set $\mathbb{R} \cup \{\infty\}$. $\mathrm{SL}_2(\mathbb{R})$ also acts on $\mathbb{P}^1(\mathbb{R})$. For a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$, the action of Γ will split $\mathbb{P}^1(\mathbb{R})$ into equivalence classes. We will denote $\mathcal{H} \cup \mathbb{P}^1(\mathbb{R})$ as $\overline{\mathcal{H}}$. For $z_1, z_2 \in \overline{\mathcal{H}}$, z_1 is equivalent to z_2 if there exists $\gamma \in \Gamma$ such that $\gamma z_1 = z_2$.

Recall that the modular group is defined as

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}.$$

Let $\mathrm{PSL}_2(\mathbb{Z}) := \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$, then $\mathrm{SL}_2(\mathbb{Z}) \cong \triangle(2,3,\infty)$ and $\mathrm{PSL}_2(\mathbb{Z}) \cong \overline{\triangle}(2,3,\infty)$. We will discuss more about why we would want to use $\mathrm{PSL}_2(\mathbb{Z})$ in a later section.

Example 1.2.1. Using the Gauss-Bonnet theorem, we have that $\mu(\triangle(2,3,\infty)) = \pi - \frac{\pi}{2} - \frac{\pi}{3} - \frac{\pi}{\infty} = \frac{\pi}{6}$, where we take $\frac{\pi}{\infty}$ to be 0.

Let be $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $p_M(x)$ be the charateristic polynomial of M. Then $p_M(x) = x^2 - \mathrm{tr}(M)x + 1$. We apply the quadratic formula to find that the discriminant is $\mathrm{tr}(M)^2 - 4$. If $|\mathrm{tr}(M)| < 2$, the eigenvalues of the matrix M are not real. The other cases, $|\mathrm{tr}(M)| = 2$ and $|\mathrm{tr}(M)| > 2$, give us either 1 real eigenvalue or 2 real eigenvalues. Now let us look at the fixed points of M. A point $z \in \overline{\mathcal{H}}$ is fixed by M if

$$\frac{az + b}{cz + d} = z.$$

Then we have

$$cz^2 + (d - a)z - b = 0.$$

The discriminant of this equation is

$$(d - a)^2 + 4bc = (a + d)^2 - 4(ad - bc) = \mathrm{tr}(M) - 4\det(M) = \mathrm{tr}(M) - 4.$$

These facts give us a classification for the matrices in $\mathrm{SL}_2(\mathbb{R})$.

Definition 2. A non-identity element $\gamma \in \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ is called **elliptic, parabolic, or hyperbolic**, if

$$|\mathrm{tr}(\gamma)| < 2, \ |\mathrm{tr}(\gamma)| = 2, \ \text{or } |\mathrm{tr}(\gamma)| > 2,$$

respectively. Equivalent, $\gamma \in \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ is called elliptic, parabolic, or hyperbolic if γ has one fixed point in $\mathcal{H}$, γ has one fixed point on $\mathbb{P}^1(\mathbb{R})$, or γ has two distinct fixed points on $\mathbb{P}^1(\mathbb{R})$ respectively.

Definition 3. Let $\tau \in \mathcal{H} \cup \mathbb{R} \cup \{\infty\}$. This element τ is called an **elliptic point** if it is fixed by an elliptic element. It is called a **cusp** if it is fixed by a parabolic element.

Definition 4. Given a discrete subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$, the **modular curve** Y_Γ is defined as the quotient space

$$Y_\Gamma := \Gamma \backslash \mathcal{H}.$$

This curve can be viewed as a noncompact Riemann surface. In order to compactify Y_Γ, we have to add back in the cusps of Γ. This gives us a compact Riemann surface for the curve $X_\Gamma := \Gamma \backslash \overline{\mathcal{H}}$. A more in-depth discussion can be found in chapters 2 and 3 of [6].

Definition 5. The **genus** of a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ is defined as the genus of X_Γ as a compact Riemann surface.

For a given triangulation of X, the Euler number is defined as

$$V - E + F,$$

where V is number of vertices, E is the number of edges, and F is the number of faces. The Euler characteristic can be similarly defined for connected plane graphs. The genus

g_X of X is defined to be

$$2 - 2g_X = V - E + F.$$

We note that the genus of $\triangle(a, b, c)$ is 0.

To compute the genera of finite index subgroups of a given triangle group, one can use a powerful tool called the Riemann-Hurwitz formula. The following are from chapters 2 and 3 of [6]. Let $f : X \to Y$ be a nonconstant holomorphic map between compact orientable Riemann surfaces and d be the degree of f. For $x \in X$, let e_x be the ramification degree of f at x. Let g_X and g_Y be the genera of X and Y. Then the Riemann-Hurwitz formula state

$$2g_X - 2 = d(2g_Y - 2) + \sum_{x \in X}(e_x - 1). \tag{1.2}$$

In fact, the derivation of the Riemann-Hurwitz formula does involve the Euler characteristic of the surfaces X and Y. A short discussion can be found in chapter 3 of [6].

1.3. Generators of Triangle Groups

Given $\triangle(a, b, c)$ with the presentation given in equation (1.1), there exists an embedding

$$\triangle(a, b, c) \hookrightarrow \mathrm{SL}_2(\mathbb{R})$$

that is unique up to conjugation in $\mathrm{SL}_2(\mathbb{R})$. The embedding is given by Petersson [22] (see Clark and Voight [2] or Petersson [22]). For $s \geq 2 \in \mathbb{Z}$, let $\zeta_s = e^{\frac{2\pi i}{s}}$. Then

$$\lambda_s = \zeta_s + \frac{1}{\zeta_s} = 2\cos(\frac{2\pi}{s}) \text{ and } \mu_s = 2\sin(\frac{2\pi}{s}) = -i\left(\zeta_s - \frac{1}{\zeta_s}\right)$$

where $\zeta_\infty = 1, \lambda_\infty = 2,$ and $\mu_\infty = 0$. Finally, for $\triangle(a,b,c) \hookrightarrow \mathrm{SL}_2(\mathbb{R})$ with $\triangle(a,b,c)$ having the presentation given in 1.1,

$$\sigma_a \mapsto g_a := \frac{1}{2}\begin{bmatrix} \lambda_{2a} & \mu_{2a} \\ -\mu_{2a} & \lambda_{2a} \end{bmatrix} \tag{1.3}$$

$$\sigma_b \mapsto g_b := \frac{1}{2}\begin{bmatrix} \lambda_{2b} & t\mu_{2b} \\ -\mu_{2b}/t & \lambda_{2b} \end{bmatrix} \tag{1.4}$$

where

$$t + 1/t = 2\frac{\lambda_{2a}\lambda_{2b} + 2\lambda_{2c}}{\mu_{2a}\mu_{2b}}. \tag{1.5}$$

Let $c = \frac{\lambda_{2a}\lambda_{2b}+2\lambda_{2c}}{\mu_{2a}\mu_{2b}}$. Then $t = c \pm \sqrt{c^2-1}$. Notice that there are two choices for t. The choice of t correspond to conjugacy in the embedding. Let $t_+ = c + \sqrt{c^2-1}$ and $t_- = c - \sqrt{c^2-1}$ and $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Some computation will shows that $t_+t_- = 1$. Then

$$S^{-1}\begin{bmatrix} \lambda_{2b} & t_+\mu_{2b} \\ -\mu_{2b}/t_+ & \lambda_{2b} \end{bmatrix} S = \begin{bmatrix} \lambda_{2b} & \mu_{2b}/t_+ \\ -t_+\mu_{2b} & \lambda_{2b} \end{bmatrix} = \begin{bmatrix} \lambda_{2b} & t_-\mu_{2b} \\ -\mu_{2b}/t_- & \lambda_{2b} \end{bmatrix}.$$

So up to conjugation, we'll choose t_+ for the map. We can also apply S to the mapping of σ_a to get

$$S^{-1}\begin{bmatrix} \lambda_{2a} & \mu_{2a} \\ -\mu_{2a} & \lambda_{2a} \end{bmatrix} S = \begin{bmatrix} \lambda_{2a} & -\mu_{2a} \\ \mu_{2a} & \lambda_{2a} \end{bmatrix} = \begin{bmatrix} \lambda_{2a} & \mu_{2a} \\ -\mu_{2a} & \lambda_{2a} \end{bmatrix}^{-1}.$$

This mapping gives us a set of generators g_a, g_b that generate a subgroup of $\mathrm{SL}_2(\mathbb{R})$ isomorphic to $\triangle(a,b,c) = \langle \sigma_a, \sigma_b, \sigma_c | \sigma_a^a = \sigma_b^b = (\sigma_a\sigma_b)^c = -1\rangle$. Therefore, g_a and g_b also have the property that $g_a^a = g_b^b = (g_bg_a)^c = -I$.

Example 1.3.1. Recall that $\mathrm{SL}_2(\mathbb{Z}) \cong \triangle(2,3,\infty)$. Using the embedding above gives us the matrices

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1/2 & 3/2 \\ -1/2 & 1/2 \end{bmatrix}.$$

The conjugation in $\mathrm{SL}_2(\mathbb{R})$ is $M = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

$$M \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} M^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$M \begin{bmatrix} 1/2 & 3/2 \\ -1/2 & 1/2 \end{bmatrix} M^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

1.4. Permutation Representation

Let $\overline{\triangle}(a,b,c)$ be a group with generators g_a and g_b satisfying the relations $g_a^a = g_b^b = (g_a g_b)^c = \pm 1$. Let Γ be a discrete index n subgroup of $\overline{\triangle}(a,b,c)$ and let $\{\gamma_i\}$, with $\gamma_1 = I$, be a set of left coset representatives in $\overline{\triangle}(a,b,c)$. Then $\overline{\triangle}(a,b,c) = \bigcup_{i=1}^{n} \gamma_i \Gamma$. For $g \in \overline{\triangle}(a,b,c)$, g acts on the left cosets of Γ by multiplication on the left. This action is the same as group homomorphisms from Γ to a subgroup of the symmetric group S_n. So there exists a homomorphism

$$\theta : \overline{\triangle}(a,b,c) \to S_n$$

where if $\theta(g) = \sigma_g$, then $g\gamma_i\Gamma = \gamma_{\theta_g(i)}\Gamma$. Then by defining $\theta(g_a) = \sigma_a$ and $\theta(g_b) = \sigma_b$, we can represent any finite index n subgroups as a pair of permutations σ_a and σ_b satisfying $\sigma_a^a = \sigma_b^b = (\sigma_a\sigma_b)^c = 1$. The transitivity of the group generated by σ_a and σ_b comes from g_a and g_b generating $\overline{\triangle}(a,b,c)$.

Conversely, suppose that σ_a and σ_b, with relations $\sigma_a^a = \sigma_b^b = (\sigma_a\sigma_b)^c = 1$, generate a transitive subgroup of S_n with a homomorphism θ as defined above. Let

$$\Gamma = \{g \in \overline{\triangle}(a,b,c) \mid \theta(g)(1) = 1\}.$$

Note that $g \in \overline{\triangle}(a,b,c)$ is in Γ if and only if $g\Gamma = \Gamma$, i.e. $\sigma_g(1) = 1$. Therefore, if $\theta(g)(1) \neq 1$, then $g \notin \Gamma$ and $g\Gamma$ is a coset. Then by transitivity of σ_a and σ_b, Γ is an index

n subgroup of $\overline{\triangle}(a, b, c)$. A more general discussion can be found in Hall's *The Theory of Groups* in chapter 5 [18].

1.5. Fundamental Domain

Definition 6. For a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$, a **fundamental domain** for Γ is a connected hyperbolic polygon F with boundary ∂F on $\overline{\mathcal{H}}$ such that

- $\forall z \in \overline{\mathcal{H}}$, there exists $\gamma \in \Gamma$ such that $\gamma z \in F$;
- If $z, z' \in F$ and $z' = \gamma z$ for some nonscalar $\gamma \in \Gamma$, then z and z' belong to ∂F.

Proposition 1. Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, F a fundamental domain of Γ, and $\Gamma' \subset \Gamma$ be a finite index subgroup. Let $\{\gamma_i\}$ be the set of left cosets for Γ' such that $\bigcup_i \gamma_i^{-1} F$ is connected and write Γ as a disjoint union of cosets

$$\Gamma = \bigcup_i \gamma_i \Gamma'.$$

Then

$$F' = \bigcup_i \gamma_i^{-1} F$$

is a fundamental domain for Γ'.

Proof. Let $z \in \overline{\mathcal{H}}$. Then $z' = \gamma z$ where $z' \in F$ and $\gamma \in \Gamma$. We also have that $\gamma = \gamma_i \gamma'$ where $\gamma' \in \Gamma$. Then $\gamma' z \in \gamma_i^{-1} F$.

Now suppose that to the contrary, for $z_1, z_2 \in F'$ and $z_2 = \gamma z_1$ for some nonscalar $\gamma \in \Gamma'$, but z_1 and z_2 is not in $\partial F'$. Then $z_1 = \gamma_i^{-1} y_1$ and $z_2 = \gamma_j^{-1} y_2$ for $y_i \in F$. Then $y_2 = \gamma_j \gamma \gamma_i^{-1} y_1$ is an equivalence relation in Γ which contradicts the fact that F is a fundamental domain. $\square$

Note that we can choose $\{\gamma_i\}$ so that F' is connected by propagating F using the generators of Γ.

For a triangle group $\triangle(a, b, c)$ with a fundamental domain F, the elliptic points of $\triangle(a, b, c)$ lie on the boundary of F. By labeling each fixed point of order a (resp. order b) with a white (resp. black) dot, we can obtain a bipartite graph from the fundamental domain. We can also find a bipartite graph corresponding to subgroups of $\triangle(a, b, c)$ by propagating the fundamental domain of $\triangle(a, b, c)$. A more in-depth discussion of the bipartite graphs arising from fundamental domains will appear in Chapter 3.

1.6. Modular Group and Its Finite Index Subgroups

There are many different classes of triangle groups that have been studied over the years. One important class of triangle groups is arithmetic triangle groups. These are groups which parameterize certain abelian surfaces admitting quarternionic multiplication [26]. The classification for these arithmetic triangle groups has been done by Takeuchi [28]. A related discussion in terms of generalized Legendre curves is given in this paper [4]. Another class of triangle groups is of the form $\triangle(2, b, \infty)$ where $b \geq 2 \in \mathbb{Z}$. These groups are called Hecke groups. Hecke groups are special because it contains cusps, which does not exists if $a, b, c < \infty$. Recall that cusps are fixed by parabolic elements, and any parabolic element is conjugate by some matrix $A \in \mathrm{SL}_2(\mathbb{R})$ to $\pm \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ for some $b \in \mathbb{R}$ [11][Section 1.4]. Therefore, cusps are fixed points of elements with infinite order, which does not exists in the case of $a, b, c < \infty$ where all the generators have finite order. In [14], C.L. Lang and M.L. Lang studied Hecke groups and produced an algorithm to find the Hecke Farey Symbol, an extension of the classical Farey Symbol. We will discuss Farey Symbol in a later section. Within the class of Hecke groups, one specific group is the group $\mathrm{SL}_2(\mathbb{Z}) \cong \triangle(2, 3, \infty)$ which is called the modular group. This group parame-

terizes the isomorphism classes of elliptic curves [6][Theorem 2.5.1] and it will be the discussion point for the majority of this thesis.

First, we recall some basic facts regarding $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{PSL}_2(\mathbb{Z})$. The proof of these statements can be found in [6].

Theorem 2. *Let* $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ *and* $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. *The group* $SL_2(\mathbb{Z})$ *is generated by the matrices* S *and* T.

Proposition 2. The cusps of $\mathrm{PSL}_2(\mathbb{Z})$ are $\mathbb{P}^1(\mathbb{Q})$, all of which are equivalent in $\mathrm{PSL}_2(\mathbb{Z})$.

Proposition 3. Every elliptic element of $\mathrm{PSL}_2(\mathbb{Z})$ has order 2 or 3. Moreover, the is one inequivalent class of order 2 and one inequivalent class of order 3.

Definition 7. Let Γ be a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ and z be a cusp of Γ. Then there exists an element $M \in \mathrm{PSL}_2(\mathbb{Z})$ such that MTM^{-1} fixes z. The smallest positive integer n such that $MT^nM^{-1} \in \Gamma$ is called the **width** of the cusp z.

For a positive integer N, here are some classical subgroups of $\mathrm{SL}_2(\mathbb{Z})$:

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod N \right\}$$

$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \bmod N \right\}$$

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \bmod N \right\}$$

Definition 8. A subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is called a **congruence subgroup** if it contains $\Gamma(N)$ for some N. The smallest N such that the containment holds is called the **level** of the subgroup.

The groups $\Gamma(N), \Gamma_1(N)$, and $\Gamma_0(N)$ are all congruence subgroups. Dennin [5] showed that for a given genus, the number of congruence subgroups is finite. On the other

hand, Jones [10] showed that there are infinitely many noncongruence subgroups for a given genus.

A standard fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})$ is

$$F = \left\{ z \in \mathcal{H} : \frac{-1}{2} \leq \Re(z) \leq \frac{1}{2} \text{ and } |z| \geq 1 \right\}.$$

A visual representation of F is given in Figure 1.1.

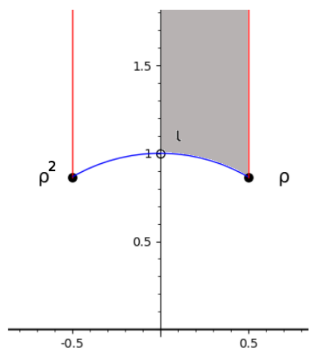

Figure 1.1: A standard fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})$ with $\rho = \frac{1+\sqrt{-3}}{2}$

From the given fundamental domain, ρ is an elliptic point of order 3 fixed by $ST^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ and is equivalent to ρ^2 by $S\rho = \rho^2$. The point i is an elliptic point of order 2 fixed by S. The cusp is ∞ fixed by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. This fundamental domain also contains 2 copies of the hyperbolic triangle for $\triangle(2,3,\infty)$ (one copy is the shaded part in Figure 1.1) with internal angles $\pi/2, \pi/3$, and 0. Note that we need 2 copies to satisfy the orientation preserving condition. Recall from before that $\triangle(2,3,\infty)$ have area $\frac{\pi}{6}$, so the area of this fundamental domain is $\frac{\pi}{3}$. As a corollary of Proposition 3, we see that the elliptic points of $\mathrm{PSL}_2(\mathbb{Z})$ are equivalent to either i or ρ.

Using the fundamental domain F for $\mathrm{PSL}_2(\mathbb{Z})$, we can cover the upper half complex plane using F and the generators of $\mathrm{PSL}_2(\mathbb{Z})$ as given in Figure 1.2.

Example 1.6.1. Figure 1.3 is a fundamental domain for $\Gamma_0(4)$. The transformations on

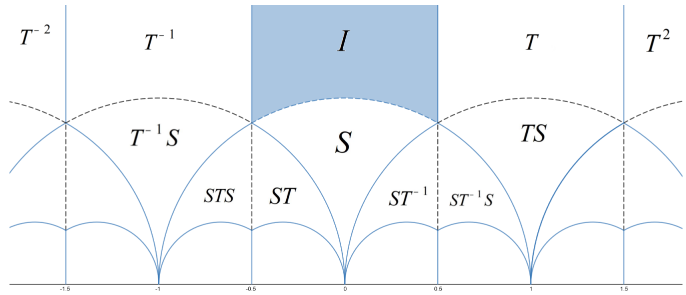

Figure 1.2: A partial tessellation of $\overline{\mathcal{H}}$

the boundary of the domain produce the following set of matrices

$$\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ 4 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ -4 & 1\end{bmatrix}, \begin{bmatrix}-1 & 1\\ -4 & 3\end{bmatrix},$$

pairing a to b, c to h, d to g, and e to f respectively in an orientation reversing manner. Note that up to signs, we have that

$$\begin{bmatrix}1 & 0\\ -4 & 1\end{bmatrix} = -\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}-1 & 1\\ -4 & 3\end{bmatrix}^{-1}, \begin{bmatrix}1 & 0\\ 4 & 1\end{bmatrix} = -\begin{bmatrix}-1 & 1\\ -4 & 3\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}.$$

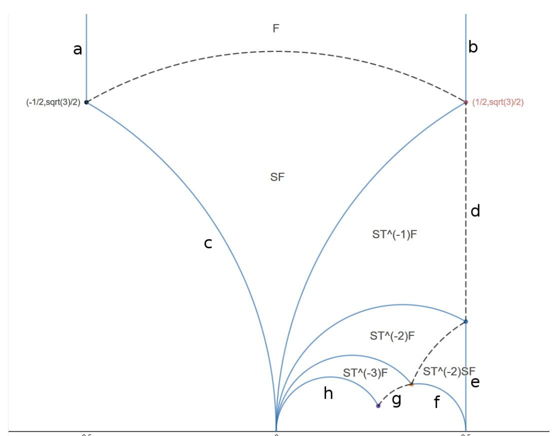

Figure 1.3: A fundamental domain of $\Gamma_0(4)$ using cosets

We can obtain another representation for a fundamental domain of $\mathrm{PSL}_2(\mathbb{Z})$ as follows by applying S to the left side of Figure 1.1 downward to get Figure 1.4. The new polygon is a special polygon (see Definition 9).

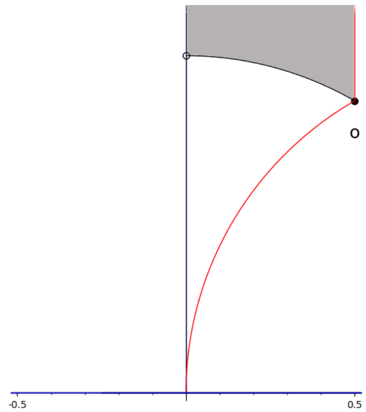

Figure 1.4: A special polygon for $\mathrm{PSL}_2(\mathbb{Z})$ with $o = \frac{1+\sqrt{-3}}{2}$

In Figure 1.4, we note that i and o are still elliptic points of order 2 and 3 respectively. However, there are now two vertices that correspond to the cusps, ∞ and 0. In this case, those two cusps are equivalent by the matrix S. The angle between the two red lines meeting at o is $\frac{2\pi}{3}$.

By propagating the special polygon in Figure 1.4 using some Möbius transformations from $\mathrm{PSL}_2(\mathbb{Z})$, we attain Figure 1.5 as a partial tessellation of $\overline{\mathcal{H}}$.

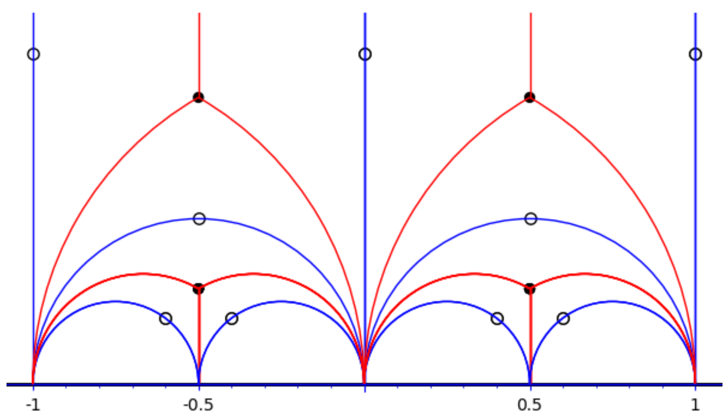

Figure 1.5: A partial tessellation of $\overline{\mathcal{H}}$ using the special polygon in Figure 1.4

In this thesis, the fundamental domains for triangle groups are hyperbolic triangles. Consequently, the tessellation obtained from a fundamental domain is made up of hyperbolic geodesics. More than that, the geodesics that connect the cusps are semicircles, or

vertical lines, and their endpoints are on $\mathbb{Q} \cup \{\infty\}$. Any two cusps $\frac{a}{b}$ and $\frac{c}{d}$ in the lowest form that are joined by a blue line satisfy the condition that $|ad - bc| = 1$. Note the points in the middle of the blue line (marked by $\circ$) are images of i and the points formed by the intersections of the red arcs (marked by $\bullet$) which are images of ρ.

Example 1.6.2. Figure 1.6 is another fundamental domain for $\Gamma_0(4)$. The side pairings labeled by 1's and 2's give rise to the matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$$

respectively, which generate $\Gamma_0(4)$ in $\mathrm{PSL}_2(\mathbb{Z})$.

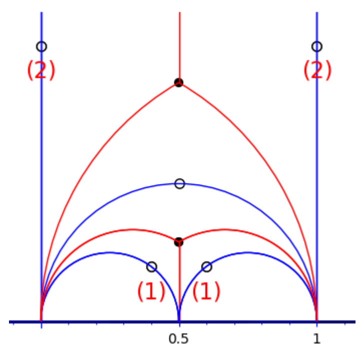

Figure 1.6: $\Gamma_0(4)$ with images of elliptic points labeled

Compared to Example 1.6.1, the set of generators given the side pairings of a special polygon is minimal. This is the motivation for using special polygons as our fundamental domains. For a more general discussion, refer to Maskit's [19] paper on Poincaré's polygons.

We denote the hyperbolic geodesic from 0 to i (resp. from 0 to $\rho = \frac{1+\sqrt{-3}}{2}$) by A_e (resp. A_o). For any $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$, γA_e (resp. γA_0) is called an **even (resp. odd) edge**. In Figure 1.6, the blue line consists of 2 even edges connected at i and it is called an **even line**. The red line contains 3 odd edges connected at ρ. Two even lines are considered

paired if they share the same integer labeling as in Figure 1.6. The paired lines (the lines with the same labeling) are called **free sides**. The fundamental domain given above for $\Gamma_0(4)$ is a special polygon.

Definition 9. A **special polygon** P is a convex hyperbolic polygon with boundary ∂P which is a union of even and odd edges satisfying the following:

P_1) The even edges in ∂P come in connected pairs and form a semi-circle or vertical lines.

P_2) The odd edges in ∂P come in connected pairs. An odd edge a is paired to an odd edge b which makes an internal angle of $\frac{2\pi}{3}$ with a.

P_3) Let e, f be two even edges in ∂P forming an even line. Then either e is paired to f, or e and f form a *free side* of P and is paired to another free side of P.

P_4) 0 and ∞ are two vertices of P.

Definition 10. Let P be a special polygon. The **inner tessellation** of P consists of even and odd edges contained inside P but not including the boundary.

The vertices of a special polygon can either be a cusp on $\mathbb{P}^1(\mathbb{Q})$ or an elliptic point located in $\mathcal{H}$ located on the boundary of the special polygon. The odd and even pairings come from the fact that the elliptic points can be order 2 or 3.

Note that in the picture for $\Gamma_0(4)$ (i.e. Figure 1.6), $\frac{1}{2}$ can be computed as the mediant between 0 and 1.

Definition 11. For $\frac{a}{b}, \frac{c}{d} \in \mathbb{P}^1(\mathbb{Q})$ in simplest form, where we denote ∞ (resp. $-\infty$) as $\frac{1}{0}$ (resp. $\frac{-1}{0}$), the **mediant** of $\frac{a}{b}$ and $\frac{c}{d}$ is denoted by mediant $\left(\frac{a}{b}, \frac{c}{d}\right)$ and computed by $\frac{a+c}{b+d}$.

For a geodesic with endpoints $\frac{a}{b}, \frac{c}{d} \in \mathbb{P}^1(\mathbb{Q})$ in lowest form and $|ad - bc| = 1$, using the endpoints and the mediant as vertices, we obtain a hyperbolic triangle. By Gauss-Bonnet, the area of this hyperbolic triangle is π.

Theorem 3 (Kulkarni [12] Theorem 3.2)**.** *A special polygon is a fundamental domain for the subgroup of $PSL_2(\mathbb{Z})$ generated by the side-pairing transformations and these transformations form an independent set of generators for the subgroup. Conversely every subgroup of finite index in $PSL_2(\mathbb{Z})$ admits a special polygon as a fundamental domain.*

Note that special polygons for a given finite index group Γ of $\mathrm{PSL}_2(\mathbb{Z})$ are not unique.

Example 1.6.3. Figure 1.7 is another special polygon corresponding to $\Gamma_0(4)$. Note that while the special polygons for Figure 1.6 and Figure 1.7 share the same number of hyperbolic triangles, their vertices are different.

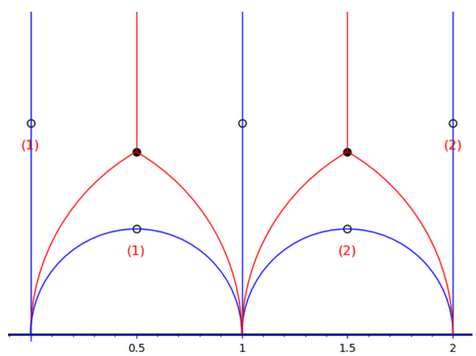

Figure 1.7: Another special polygon for $\Gamma_0(4)$ with different vertices

1.7. Overview

There are two major ideas that will be used throughout this thesis. One is that a fundamental domain for a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ can be obtained by propagating a fundamental domain of $\mathrm{PSL}_2(\mathbb{Z})$. The other is that the set of generators obtained from a special polygon is minimal. Using these two ideas, we will discuss and develop computational methods for working with finite index subgroups of $\mathrm{PSL}_2(\mathbb{Z})$. We will look at these methods from three different angles. One will be from the perspective of Farey

Symbols. Another will be using permutation representations. The last will be using bipartite cuboid graphs.

For the second part, we will try to generalize these ideas to general triangle groups $\overline{\triangle}(a, b, c)$. The extension of permutation representations and graphs can be extended in an obvious manner. However, we may not have Farey Symbols, which rely heavily on cusps. Using these theories, we will develop algorithms involving triangle groups and finite index subgroups of triangle groups. The algorithms include finding a set of generators and determining whether a given matrix is contained in a given finite index subgroup of $\overline{\triangle}(a, b, c)$. We will also classify nonequivalent isomorphism classes of subgroups of $\overline{\triangle}(2, 4, 6)$ for index up to 11.

Chapter 2. Farey Symbols

For a special polygon, one can describe the polygon using its vertices. This description is called a **Farey Symbol**. The following discussion is based on the work of Kulkarni [12].

2.1. Introduction

We recall the following classical definition of a Farey Sequence.

Definition 12. A **Farey Sequence** is a sequence of completely reduced rational numbers between 0 and 1 in ascending order where each adjacent pair of numbers have cross-ratio 1. The **cross-ratio** of $\frac{x_1}{y_1}, \frac{x_2}{y_2} \in \mathbb{P}^1(\mathbb{Q})$ is denoted as $|x_1y_2 - x_2y_1|$.

Lemma 1. Let $\frac{x_1}{y_1}, \frac{x_2}{y_2} \in \mathbb{P}^1(\mathbb{Q})$ be in simplest form with cross-ratio 1 and z be the mediant$\left(\frac{x_1}{y_1}, \frac{x_2}{y_2}\right)$. Then the cross-ratio of $\left(\frac{x_1}{y_1}, z\right)$ and the cross-ratio of $\left(z, \frac{x_2}{y_2}\right)$ are also 1.

Proof. Suppose $a, b \in \mathbb{P}^1(\mathbb{Q})$ with $a = \frac{x_1}{y_1}$ and $b = \frac{x_2}{y_2}$ and $|x_1y_2 - x_2y_1| = 1$. Let $c = \frac{x_1+x_2}{y_1+y_2}$. Then the cross-ratio of a and c is $|x_1(y_1 + y_2) - y_1(x_1 + x_2)| = |x_1y_2 - x_2y_1| = 1$. Similarly, the cross-ratio of b and c is also 1. □

Lemma 2. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\frac{x_1}{y_1}, \frac{x_2}{y_2} \in \mathbb{P}^1(\mathbb{Q})$ be in simplest form. Then mediant$\left(M\left(\frac{x_1}{y_1}\right), M\left(\frac{x_2}{y_2}\right)\right) = M\left(\text{mediant}\left(\frac{x_1}{y_1}, \frac{x_2}{y_2}\right)\right)$.

Proof.

$$\begin{aligned}
\text{mediant}\left(M\left(\frac{x_1}{y_1}\right), M\left(\frac{x_2}{y_2}\right)\right) &= \frac{ax_1 + by_1 + ax_2 + by_2}{cx_1 + dy_1 + cx_2 + dy_2} \\
&= \frac{a(x_1 + x_2) + b(y_1 + y_2)}{c(x_1 + x_2) + d(y_1 + y_2)} \\
&= M\left(\frac{x_1 + x_2}{y_1 + y_2}\right) \\
&= M\left(\text{mediant}\left(\frac{x_1}{y_1}, \frac{x_2}{y_2}\right)\right)
\end{aligned}$$

$\square$

Definition 13. A **generalized Farey Sequence** is an expression of the form of

$$\{-\infty = \frac{``-1"}{0}, x_1, x_2, \ldots, x_n, \frac{``1"}{0} = \infty\}$$

where

i) x_1 and x_n are integers and for exactly one x_i, $x_i = 0$.

ii) $x_i = \frac{a_i}{b_i}$ are rational numbers in their reduced form and in an increasing order, such that

$$|a_{i+1}b_i - a_i b_{i+1}| = 1, \quad i = 1, 2, \ldots, n-1.$$

Definition 14. A **Farey Symbol** is a generalized Farey Sequence with pairing information between any two adjacent entries. The pairing is labeled as follow:

- If x_i and x_{i+1} are endpoints of two paired even edges, then the pairing is labeled $\circ$.
- If x_i and x_{i+1} are endpoints of two paired odd edges, then the pairing is labeled $\bullet$.
- If x_i and x_{i+1} are endpoints of a free side and paired with x_i' and x_{i+1}', then the pairing is labeled by an integer. Different pairings are labeled by different integers.

Example 2.1.1. A Farey Symbol for $\Gamma_0(4)$ corresponding to Figure 1.6 is

$$-\infty \underbrace{\quad}_{2} \tfrac{0}{1} \underbrace{\quad}_{1} \tfrac{1}{2} \underbrace{\quad}_{1} \tfrac{1}{1} \underbrace{\quad}_{2} \infty$$

We can compute the matrices for the pairings explicitly by finding the Möbius transformations for the edges. For 1, we need to find a, b, c, d such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{2} = \frac{1}{2}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} 0 = 1$, and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. For 2, we need a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ that fixes ∞ and sends 0 to 1. We find that 1 corresponds to the matrix $\begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$ and 2 corresponds to the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Example 2.1.2. Consider the following Farey Symbol

$$\begin{array}{ccccccccc} -\infty & & \frac{0}{1} & & \frac{1}{2} & & \frac{1}{1} & & \infty \\ & 1 & & \bullet & & \bullet & & 1 & \end{array}$$

This Farey Symbol shares the same entries as Example 2.1.1. However, by changing the pairings, this Farey Symbol now represents a different subgroup. The subgroup corresponding to this Farey Symbol is now of index 8. Similar to Example 2.1.1, we can compute the matrices for the pairings explicitly. The matrix corresponding to 1 is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. For the odd pairing between 0 and $\frac{1}{2}$, we first note that the odd edges forming this pairing lie inside a hyperbolic triangle with vertices $0, \frac{1}{3}$, and $\frac{1}{2}$. Therefore, we need to find $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ that permutes these vertices. The matrix is $\pm \begin{bmatrix} 2 & -1 \\ 7 & -3 \end{bmatrix}$. For the second odd pairing, we need a matrix that permutes $\frac{1}{2}, \frac{2}{3}$, and $\frac{1}{1}$. The matrix is $\pm \begin{bmatrix} 4 & -3 \\ 7 & -5 \end{bmatrix}$. An explicit algorithm is given in Theorem 4.

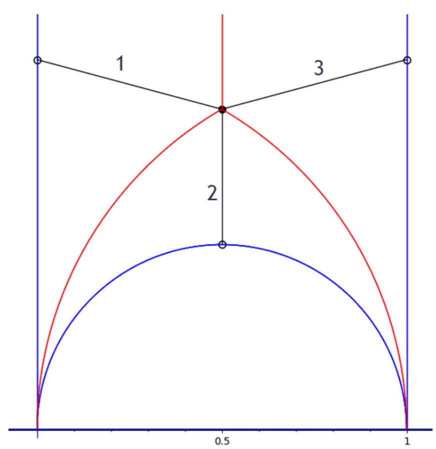

Figure 2.1: A partial special polygon with vertices $0, 1$, and ∞

2.2. Invariants From Farey Symbols

Given a finite index subgroup Γ of $\mathrm{PSL}_2(\mathbb{Z})$, one can use the Farey Symbol to represent the subgroup in a concise way. From the Farey Symbol, we can obtain invariants for

the given subgroup. The number of even (resp. odd) pairing is the number of inequivalent elliptic points of order 2 (resp. order 3).

Let F be a Farey Symbol and $\{-\infty, x_1, \ldots, x_n, \infty\}$ be the corresponding generalized Farey Sequence with $x_i = \frac{a_i}{b_i}$. The generalized Farey Sequence will also have a cyclic order. This means that $x_{n+1} = \infty = -\infty = x_0$. The width of each vertex x_i can be calculated from the Farey Symbol as follows:

$$\mathrm{width}(x_i) = |a_{i-1}b_{i+1} - a_{i+1}b_{i-1}| + c$$

where c is $0, \frac{1}{2}$, or 1 if x_i is adjacent to $0, 1$, or 2 odd edges respectively. The width of the vertex ∞ is

$$\mathrm{width}(\infty) = |a_n b_1 - a_1 b_n| + c.$$

The vertices can be partitioned into equivalent classes $\{x_i\}$, which are cusps. If x_i and x_{i+1} are paired by an even or odd pairing, then x_i and x_{i+1} are equivalent cusps. If x_i and x_{i+1} are paired with x'_i and x'_{i+1} by a free pairing, then x_i and x'_{i+1} are equivalent and x_{i+1} and x'_i are equivalent.

Then the width of a cusp $\{x_i\}$ is the sum of the width of $x_j \in \{x_i\}$. Let n be the index of a finite index subgroup with a set of inequivalent cusps $\{x_i\}$. Then

$$n = \sum_i \mathrm{width}(x_i).$$

We can also obtain the genus of Γ by applying the Riemann-Hurwitz formula given in equation (1.2) [12]. Let Γ be an index n subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ and e_2, e_3, c be the numbers of inequivalent order 2 elliptic points, order 3 elliptic points, and cusps respectively.

Then the genus g of Γ is given by

$$g = 1 + \frac{n}{12} - \frac{e_2}{4} - \frac{e_3}{3} - \frac{c}{2}.$$

The generators of Γ are given by the pairing information of the Farey Symbol.

Theorem 4 (Kulkarni [12] Theorem 6.1)**.** *Suppose $\frac{a_i}{b_i}$ and $\frac{a_{i+1}}{b_{i+1}}$ are two adjacent entries of a Farey Symbol F. Then if the pairing between them, p_{i+1}, is an even pairing, let:*

$$G_{i+1} = \begin{bmatrix} a_{i+1}b_{i+1} + a_ib_i & -a_i^2 - a_{i+1}^2 \\ b_i^2 + b_{i+1}^2 & -a_{i+1}b_{i+1} - a_ib_i \end{bmatrix}.$$

If p_{i+1} is an odd pairing, let:

$$G_{i+1} = \begin{bmatrix} a_{i+1}b_{i+1} + a_ib_{i+1} + a_ib_i & -a_i^2 - a_ia_{i+1} - a_{i+1}^2 \\ b_i^2 + b_ib_{i+1} + b_{i+1}^2 & -a_{i+1}b_{i+1} - a_{i+1}b_i - a_ib_i \end{bmatrix}.$$

If p_{i+1} is a free pairing that is paired with the side between $\frac{a_k}{b_k}$ and $\frac{a_{k+1}}{b_{k+1}}$, let:

$$G_{i+1} = \begin{bmatrix} a_{k+1}b_{i+1} + a_kb_i & -a_ka_i - a_{k+1}a_{i+1} \\ b_kb_i + b_{k+1}b_{i+1} & -a_{i+1}b_{k+1} - a_ib_k \end{bmatrix}.$$

Then G_{i+1} is the side transformation corresponding to the pairing between p_{i+1}.

Example 2.2.1. The Farey Symbol for $\Gamma_0(4)$ is

$$\begin{array}{ccccccccc} -\infty & & \frac{0}{1} & & \frac{1}{2} & & \frac{1}{1} & & \infty \\ & 2 & & 1 & & 1 & & 2 & \end{array}$$

The generators are $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$. There are no elliptic points. The cusps 0 and 1 are equivalent. So there are 3 inequivalent cusps: 0, $\frac{1}{2}$, and ∞. The cusp 0 has width 4, $\frac{1}{2}$ has width 1, and ∞ has width 1. The index is 6. The genus is 0.

2.3. Algorithm for Constructing Farey Symbol

The following algorithm was given by Kurth [13] based on the work of Kulkarni. Currently in SageMath, there exists the KFarey package [24] for producing a Farey Symbol along with the invariants for a given finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. We expand

upon that algorithm to produce a bipartite cuboid graph with labelings. We will discuss more about bipartite cuboid graphs in the next chapter.

There are two facts to be noted prior to discussing the algorithm. The first fact is Lemma 2 . Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $0, 1, \infty$ be the vertices of a hyperbolic triangle. Then the image of the hyperbolic triangle under the action of M will have vertices $\frac{a}{c}, \frac{a+b}{c+d}, \frac{b}{d}$. Therefore, to find the image of the hyperbolic triangle with vertices $0, 1, \infty$ under M, we only need to find the image of 0 and ∞. The last vertex comes from the mediant of the image of 0 and ∞. The second fact is the following. Let $\frac{b}{d} < \frac{a}{c} \in \mathbb{P}^1(\mathbb{Q})$ in simplest form with $|ad - bc| = 1$. Then $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ takes 0 to $\frac{b}{d}$ and ∞ to $\frac{a}{c}$.

Using these two facts, we can propagate our special polygon (Figure 2.1) along any of its sides by finding the mediant between the corresponding two vertices.

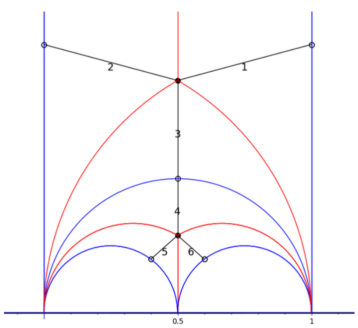

Figure 2.2: Figure 2.1 reflected down

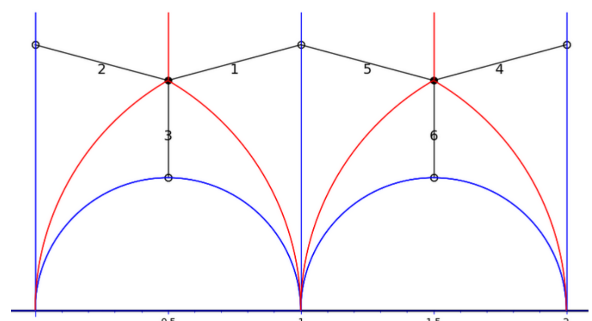

Figure 2.3: Figure 2.1 reflected right

Example 2.3.1. Recall Figure 2.1. If we want to reflect the triangle downward, we only need to find the mediant of 0 and 1. The mediant$(0, 1)$ is $\frac{1}{2}$. The vertices $0, 1$, and $\frac{1}{2}$ produce the polygon in Figure 2.2. If we want to reflect the triangle to the right, we find the mediant between 1 and ∞. The mediant$(1, \infty)$ is 2 and the polygon is the one in Figure 2.3.

Algorithm 1 Produce a Farey Symbol given a finite index subgroup Γ of $\mathrm{PSL}_2(\mathbb{Z})$ and a fundamental domain

Check if $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is in Γ

if Yes **then**

  Return

$$-\infty \underbrace{\qquad}_{\circ} 0 \underbrace{\qquad}_{\bullet} \infty$$

  Return Figure 1.4

  Terminate

else if $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$ are in Γ **then**

  $\Gamma = \Gamma_2$, where Γ_2 is the unique subgroup of index 2

  Return

$$-\infty \underbrace{\qquad}_{\bullet} \tfrac{0}{1} \underbrace{\qquad}_{\bullet} \infty$$

  Terminate

end if

Make a partial Farey Symbol:

$$-\infty \underbrace{\qquad} \tfrac{0}{1} \underbrace{\qquad} \tfrac{1}{1} \underbrace{\qquad} \infty.$$

Let F be the hyperbolic triangle with vertices $\{0, 1, \infty\}$ with labeling given in Figure 2.1

$n = 3$

while There are unpaired entries **do**

  Check whether you can pair the entries. If yes, assign the appropriate pairing

  if Pairing between $\frac{p_i}{q_i}$ and $\frac{p_{i+1}}{q_{i+1}}$ is odd **then**

Reflect one portion of F over the geodesic with endpoints $\{\frac{p_i}{q_i}, \frac{p_{i+1}}{q_{i+1}}\}$. Specifically, let

$$G_{i+1} = \begin{bmatrix} p_{i+1}q_{i+1} + p_iq_{i+1} + p_iq_i & -p_i^2 - p_ip_{i+1} - p_{i+1}^2 \\ q_i^2 + q_iq_{i+1} + q_{i+1}^2 & -p_{i+1}q_{i+1} - p_{i+1}q_i - p_iq_i \end{bmatrix}$$

and let pt be the fixed point of G_{i+1}. The new hyperbolic triangle will have vertices $\{\frac{p_i}{q_i}, pt, \frac{p_{i+1}}{q_{i+1}}\}$

Label the new region $n + 1$

Let $n = n + 1$

end if

if all entries have been paired **then**

terminate

else if There are any unpaired entries, say $\frac{p_i}{q_i}$ and $\frac{p_{i+1}}{q_{i+1}}$ **then**

We add a new entry, $\frac{p_i+p_{i+1}}{q_i+q_{i+1}}$

Reflect F along the geodesic with endpoints $\frac{p_i}{q_i}$ and $\frac{p_{i+1}}{q_{i+1}}$ forming a new hyperbolic triangle with vertices $\{\frac{p_i}{q_i}, \frac{p_i+p_{i+1}}{q_i+q_{i+1}}, \frac{p_{i+1}}{q_{i+1}}\}$

Label the region with $n + 1, n + 2, n + 3$ in a counterclockwise manner

$n = n + 3$

end if

end while

Example 2.3.2. We apply the algorithm to $\Gamma_0(15)$ and it will produce Figure 2.4. We can also call upon functions from the KFarey package to find the invariants. The dash lines indicate the points that will be paired in the corresponding bipartite graph. We will discuss bipartite cuboid graphs in the next chapter.

```
import sage_farey_to_graph_5_4
from sage.all import *
from importlib import reload
reload(sage_farey_to_graph_5_4)

farey = sage_farey_to_graph_5_4.findfarey(Gamma0(15))
print("A Farey Symbol: ")
print(farey)
print("Order 2 elliptic points: ",FareySymbol(Gamma0(15)).nu2())
print("Order 3 elliptic points: ",FareySymbol(Gamma0(15)).nu3())
print("Cusps: ",FareySymbol(Gamma0(15)).cusps())
print("Cusps width: ", FareySymbol(Gamma0(15)).cusp_widths())
print("Genus: ",FareySymbol(Gamma0(15)).genus())
sage_farey_to_graph_5_4.fareygraph(farey,boundary = True, interior = True, bcgraph=True,figuresize = 20)
```

```
A Farey Symbol:
[[-Infinity, 0, '1'], [0, 1/6, '3'], [1/6, 1/5, '5'], [1/5, 1/4, '4'], [1/4, 1/3, '2'], [1/3, 1/2, '3'], [1/2, 3
/5, '4'], [3/5, 2/3, '5'], [2/3, 1, '2'], [1, +Infinity, '1']]
Order 2 elliptic points:  0
Order 3 elliptic points:  0
Cusps:  [0, 1/6, 1/5, Infinity]
Cusps width:  [15, 5, 3, 1]
Genus:  1
```

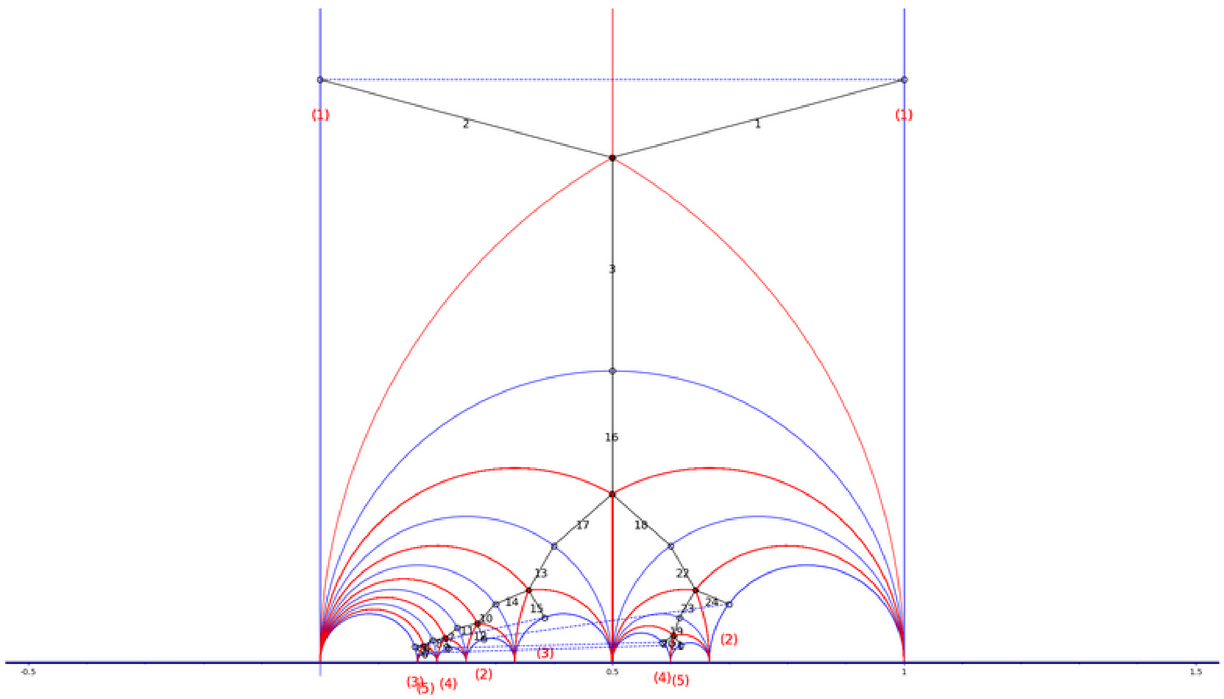

Figure 2.4: The Farey Symbol and special polygon for $\Gamma_0(15)$ using Algorithm 1

Chapter 3. Bipartite Cuboid Graphs

3.1. Interpreting Fundamental Domains of Subgroups of $\mathbf{PSL_2}(\mathbb{Z})$ as Graphs

Definition 15. A **bipartite cuboid graph** (BCG) is a finite connected bipartite graph, which can be embedded on an orientable surface, such that

- every white (resp. black) vertex is of degree 1 or 2 (resp. 1 or 3),
- there is a cyclic order on the edges incident at each vertex.

We will choose counterclockwise as our direction for the cyclic order on the edges for the rest of this paper.

By using Algorithm 1, one can produce a fundamental domain for a finite index subgroup Γ of $\mathrm{PSL}_2(\mathbb{Z})$. Given a fundamental domain F with the inner tessellation (definition 10), let the images of the elliptic points of order 3 (resp. order 2) of $\mathrm{PSL}_2(\mathbb{Z})$ under the actions arising from the left cosets be represented as the black (resp. white) vertices. One can obtain a graph from a fundamental domain by connecting the black vertices to their neighboring white vertices. We will demonstrate this process by the following example. Note that the number of edges added to connect black and white vertices is the number of $\overline{\triangle}(2,3,\infty)$ used to tessellate the fundamental domain of Γ, which is also the index of Γ in $\mathrm{PSL}_2(\mathbb{Z})$.

Example 3.1.1. Consider the fundamental domain with its inner tessellation for $\Gamma_0(4)$ in Figure 3.1. Then we can construct the associated BCG by adding edges connecting each black vertex to its neighboring white vertices (Figure 3.2). We also give a label to each newly added edge. If there exists a dashed line between two vertices indicating there is a pairing between them, we pair those two vertices. The resulting BCG is Figure 3.3.

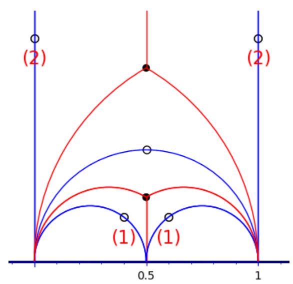

Figure 3.1: $\Gamma_0(4)$ with images of the elliptic points

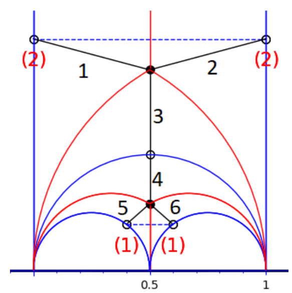

Figure 3.2: $\Gamma_0(4)$ with labels on the black edges

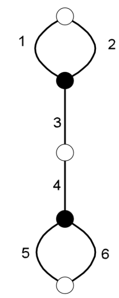

Figure 3.3: $\Gamma_0(4)$ represented as a BCG

The figures in Example 3.1.1 can also be obtained by applying Algorithm 1.

Lemma 3. Given an index n subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, the corresponding BCG has n edges.

Proof. Recall Proposition 1. For an index n subgroup Γ of $\mathrm{PSL}_2(\mathbb{Z})$, the fundamental domain of Γ contains n copies of the fundamental domain of $\mathrm{PSL}_2(\mathbb{Z})$. From the construction of the BCG for Γ, each edge of the BCG corresponds to a copy of the fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})$. □

From a BCG, we can obtain the same invariants that we found from using Farey Symbols. An even (resp. odd) leaf is a white (resp. black) vertex of degree 1. The invariants can be obtained as follows:

$$e_2 = \text{number of even leaves},$$

$$e_3 = \text{number of odd leaves},$$

$$c = \text{number of faces}.$$

Recall that for a planar graph, a face is a region enclosed by a cycle. There is also the infinite face which is the region "outside" of the graph.

Example 3.1.2. In Figure 3.3, the subgraph given by edges 1 and 2 is called a face. This face corresponds to a cusp of width 1. Another face is the subgraph with edges 5 and 6.

This face corresponds to another cusp of width 1. There is also the infinite face corresponding to a cusp of width 4.

Note that not all BCG's are planar. Since BCG's come from finite index subgroups of $\mathrm{PSL}_2(\mathbb{Z})$, there are BCG's with positive genus. A positive genus for a BCG means that the BCG contains edges that cross each other. Then the faces for that BCG become harder to identify visually. We will present a way to extend the definition of faces and identify them by using permutations in a later discussion.

Example 3.1.3. Figure 3.4 is the BCG corresponding to the subgroup $\Gamma_0(15)$. While we know the genus and the cusps of this subgroup from Figure 2.4, it is not easy to determine those invariants from the BCG.

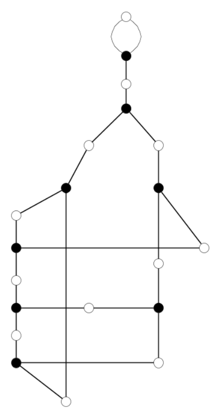

Figure 3.4: The corresponding BCG for $\Gamma_0(15)$ in Figure 2.4

From Figure 3.4, one might be tempted to move the edges and vertices to make the graph into a plane graph. However, this might not work. Figure 3.4 is not a standard graph, but rather a BCG (recall Definition 15), so there is a cyclic ordering on the edges. By removing the crossings, the new BCG might not be isomorphic to the original BCG. We define isomorphisms between BCG's below.

Definition 16. Two BCG's are **isomorphic** if there exists a bijective map between the BCG's that sends edges to edges while keeping edge adjacency and the cyclic ordering

around the vertices.

Theorem 5 (Kulkarni [12] Theorem 4.2)**.** *There is a bijective correspondence between isomorphism classes of bipartite cuboid graphs and conjugacy classes of finite index subgroups of $PSL_2(\mathbb{Z})$.*

Example 3.1.4. Figure 3.5 shows two BCG's that are isomorphic to each other. Note that the cyclic ordering around each vertex is preserved along with edge adjacency. Figure 3.6 shows two BCG's that are not isomorphic to each other. As standard graphs in the plane, they are isomorphic. However, as BCG's, there is no way to label the edges so that both adjacency and cyclic ordering will be simultaneously preserved.

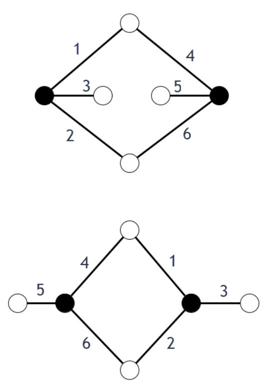

Figure 3.5: Isomorphic

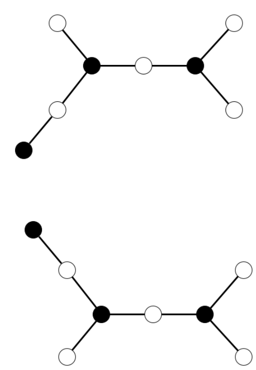

Figure 3.6: Non-isomorphic

Example 3.1.5. Figure 3.7 and Figure 3.8 are not isomorphic as BCG's. The crossing in Figure 3.8 indicates the genus of that graph is 1.

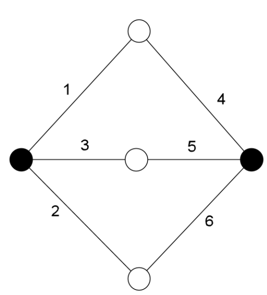

Figure 3.7: BCG with Genus 0

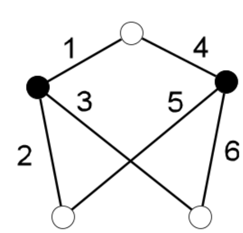

Figure 3.8: BCG with Genus 1

Examples 3.1.4 and 3.1.5 show that the process of determining the isomorphism between two BCG's is not particularly straightforward. This process only grows in difficulty as the BCG's increase in size or they become more homogeneous. Consider Figure 3.9, 3.10, and 3.11. We would like to determine if there are any isomorphisms among those three BCG's. We will use permutation representations to help us find the isomorphisms if there are any. We will find that only Figures 3.9 and 3.10 are isomorphic to each other.

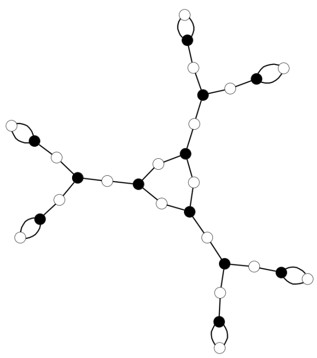

Figure 3.9: A BCG with 36 edges

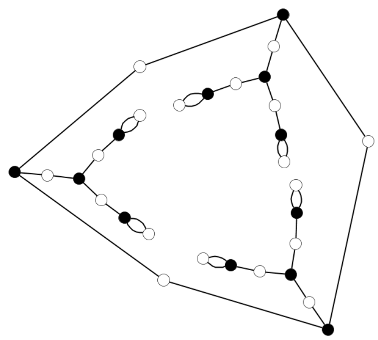

Figure 3.10: Another BCG with 36 edges

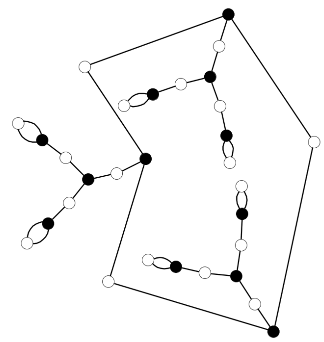

Figure 3.11: A third BCG with 36 edges

3.2. Interpreting Graphs as Permutations

So far, we have seen how to consider finite index subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ as Farey Symbols and BCG's. Now we consider these subgroups as permutations. Recall from Section 1.4 that we can associate a permutation representation to a finite index subgroup of $\overline{\triangle}(a, b, c)$. In the case for finite index subgroups of $\mathrm{PSL}_2(\mathbb{Z})$, we state the following theorem from Millington [20].

Let X be a finite set of n letters and x_1 a fixed member of X. A pairing $(\sigma_2, \sigma_3)_{x_1}$ is defined as an equivalence class of pairs of permutations σ_2, σ_3 acting on X under the equivalence relation $\sim$, where

- $\sigma_2^2 = \sigma_3^3 = I$,

- the group generated by σ_2 and σ_3 is transitive on X,
- $(\sigma_2, \sigma_3) \sim (\sigma_2', \sigma_3')$ if there exists a permutation $\tau \in S_n$ such that

$$\tau\sigma_2\tau^{-1} = \sigma_2', \quad \tau\sigma_3\tau^{-1} = \sigma_3', \quad \tau x_1 = x_1.$$

Theorem 6 (Millington [20] Theorem 1). *Suppose that $\{\sigma_2, \sigma_3\} \in S_n$ generate a transitive subgroup on X, a finite set of n letters. Also suppose that $\sigma_2^2 = \sigma_3^3 = 1$. Then there is a one-to-one correspondence between finite index subgroups Γ of $PSL_2(\mathbb{Z})$ and pairings $(\sigma_2, \sigma_3)_{x_1}$ acting on a set X of n letters. Moreover, Γ has invariants n, g, e_2, e_3, c satisfying the Riemann-Hurwitz formula and $n = \sum_{\alpha_i} width(\alpha_i)$, where α_i are the inequivalent cusps, if and only if*

1. *σ_2 fixes e_2 letters of X,*
2. *σ_3 fixes e_3 letters of X,*
3. *$\sigma_2\sigma_3$ consists of c disjoint cycles of length $width(\alpha_i)$ for inequivalent cusps α_i.*

From a permutation representation, one can also obtain the standard invariants (e_2, e_3, c, etc.) of the corresponding finite index subgroup Γ of $\mathrm{PSL}_2(\mathbb{Z})$ by the results of Theorem 6. One can obtain a set of permutations for a given subgroup Γ by following the discussion in Section 1.4.

Another way to find σ_2 and σ_3 is by looking at BCG's. Recall that we can represent a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ as a BCG. For the BCG, we can give it a labeling on the edges. We can find σ_2 and σ_3 from a BCG with labels by reading the labels around each vertex in a counterclockwise orientation. The labels around each white (resp. black) vertex form a cycle in an order 2 (resp. 3) permutation. We present an example of this process below.

Example 3.2.1. For $\Gamma_0(4)$ as given in Figure 3.3, the permutations are $\sigma_2 = (1,2)(3,4)(5,6)$,

$\sigma_3 = (1,3,2)(4,5,6)$, and $\sigma_2\sigma_3 = (1)(6)(2,3,5,4)$. From Millington's Theorem 6, we can find the invariants of this subgroup. This subgroup is torsion-free. There are 3 inequivalent cusps of width 1, 1, and 4. The genus is 0.

Example 3.2.2. For $\Gamma_0(13)$, the permutations are

$$\sigma_2 = (1,3)(2,4)(5,7)(6,10)(8,13)(12,14)(9)(11)$$

$$\sigma_3 = (1,2,3)(4,5,6)(7,8,9)(10,11,12)(13)(14).$$

$$\sigma_2\sigma_3 = (1)(2,5,8,13,9,7,6,11,12,14,10,4,3)$$

This subgroup has 2 order 2 elliptic points, 2 order 3 elliptic points, and 2 cusps. One cusp is of width 1 and the other is width 13. The genus is 0.

By using permutations, we can give a new definition for faces of a BCG.

Definition 17. For a BCG with a set of permutations $\{\sigma_2, \sigma_3\}$, the **faces** of the BCG bijectively correspond to the disjoint cycles of the permutation $\sigma_2\sigma_3$.

3.3. Isomorphisms Between BCG's

In the previous section, we established a way to go from a BCG with labelings to a set of permutations. In this section, we will work with permutations. However, associated to the permutations is a BCG. In other words, determining whether two BCG's are isomorphic is equivalent to finding the conjugation between two sets of permutations if it exists. We will establish an algorithm to go from permutations back to BCG's in a later section.

Definition 18. For permutations $\{\sigma_2, \sigma_3\}$ and $\{\sigma_2', \sigma_3'\}$ such that $\langle\sigma_2, \sigma_3\rangle$ generate a transitive subgroup of S_n for some $n \in \mathbb{N}$, we say that these two sets are **conjugates** of each other if there exists a $\tau \in S_n$ such that $\tau\{\sigma_2, \sigma_3\}\tau^{-1} = \{\sigma_2', \sigma_3'\}$.

Example 3.3.1. Let

$$\sigma_2 = (1,2)(3,4)(5)(6),\ \ \sigma_3 = (1,2,3)(4,5,6),$$

$$\sigma_2' = (1,2)(5,6)(3)(4),\ \ \sigma_3' = (1,5,6)(2,3,4)$$

in S_6. Then $\tau = (6,4,2)(3,1,5)$ will satisfy $\tau\{\sigma_2, \sigma_3\}\tau^{-1} = \{\sigma_2', \sigma_3'\}$.

Example 3.3.2. Let

$$\sigma_2 = (1,4)(2,6)(3,5),\ \ \sigma_3 = (1,2,3)(4,5,6),$$

$$\sigma_2' = (1,4)(3,6)(2,5),\ \ \sigma_3' = (1,2,3)(4,5,6)$$

in S_6. Since conjugation does not change cycle type, if there exists a $\tau \in S_6$ that conjugates the permutations, then $\sigma_2\sigma_3$ must have the same cycle type as $\sigma_2'\sigma_3'$. However, $\sigma_2\sigma_3 = (1,5)(2,4)(3,6)$ and $\sigma_2'\sigma_3' = (1,5,3,4,2,6)$. Therefore, there is no $\tau \in S_6$ such that $\tau\{\sigma_2, \sigma_3\}\tau^{-1} = \{\sigma_2', \sigma_3'\}$.

As n increases, trying to find a conjugation $\tau \in S_n$ becomes exponentially harder if one were to naively find τ by checking all elements in S_n. We will introduce the concept of paths to produce a more sophisticated method of finding τ.

Definition 19. Let X be a set of n letters and $\sigma_2, \sigma_3 \in S_n$, a **path** starting from $k \leq n$ is a sequence $\{x_i\}$, $x_i \in X$ with no repeated entries where

$$x_1 = k, x_2 = \sigma_3(x_1), x_3 = \sigma_2(x_2), x_4 = \sigma_3(x_3), x_5 = \sigma_2(x_4), ..., x_i = \sigma_j(x_{i-1})$$

where $j = 3$ if i is even and $j = 2$ if i is odd. A path is called **maximal** if $\{x_1, x_2, \ldots, x_i\}$ is a path but $x_{i+1} = x_j$ for some $j \leq i$.

In the definition of paths, σ_2 and σ_3 only need to be elements of S_n. However, for σ_2 and σ_3 arising from a BCG, it is necessary that $\langle\sigma_2, \sigma_3\rangle$ is a transitive subgroup of S_n.

From now, the set X will be the set of labels of a BCG. Also note that in the definition of paths, the starting direction is fixed.

Definition 20. Given a maximal path $p = \{x_1, \ldots, x_i\}$, if $\sigma_j(x_i) = x_k$ where $k \leq i$, then we say the **path type** of this path is $(k, i - k + 1)$.

Example 3.3.3. Recall the graph for $\Gamma_0(4)$ (Figure 3.3) and the permutation representation $\sigma_2 = (1,2)(3,4)(5,6)$ and $\sigma_3 = (1,3,2)(4,5,6)$. A path staring at 1 would be $\{1,3,4,5,6\}$ with path type $(3,3)$. A path starting at 3 would be $\{3,2,1\}$ with path type $(1,3)$. To be more precise, we can track the numbers as follows:

$$3 \xmapsto{\sigma_3} 2 \xmapsto{\sigma_2} 1 \xmapsto{\sigma_3} 3$$

$$1 \xmapsto{\sigma_3} 3 \xmapsto{\sigma_2} 4 \xmapsto{\sigma_3} 5 \xmapsto{\sigma_2} 6 \xmapsto{\sigma_3} 4.$$

Let P be a set containing maximal paths starting at i for $1 \leq i \leq n$.

Lemma 4. If $\{\sigma_2, \sigma_3\}$ is conjugate to $\{\sigma_2', \sigma_3'\}$ in S_n, then for all $p \in P$, there exists $p' \in P'$ such that $|p| = |p'|$ and p and p' have the same path type.

Proof. Suppose that $\{\sigma_2, \sigma_3\}$ is conjugate to $\{\sigma_2', \sigma_3'\}$. Then there exists a $\tau \in S_n$ such that $\tau\sigma_2\tau^{-1} = \sigma_2'$ and $\tau\sigma_3\tau^{-1} = \sigma_3'$. Let $p = \{x_1, x_2, \ldots, x_k\}$ be a path in P. Let $x_1' = \tau x_1$. Since $1 \leq x_1' \leq n$, there is a path in P' that start at x_1'. For the path starting at x_1', $x_2' = \sigma_3'(x_1') = \tau\sigma_3\tau^{-1}(\tau x_1) = \tau\sigma_3(x_1) = \tau(x_2)$. Therefore, for a path $p \in P$, there exists a path $p' = \tau p$ in P' with the same length with the same path type. $\square$

Note that if x_i is a label corresponding to a leaf of a BCG, then x_i is either the starting point of a path or it is the ending point of a path.

Lemma 5. Let X be a set of n labels and $\{\sigma_2, \sigma_3\} \in S_n$. Let P be defined as above for X. Then $|\{\text{paths of length } 1\}| \leq \frac{n}{3} + 2$.

Proof. By definition of a path, paths of length 1 correspond to elliptic points of order 3.

Then consider the Riemann-Hurwitz formula for $\mathrm{PSL}_2(\mathbb{Z})$

$$g = 1 + \frac{n}{12} - \frac{e_2}{4} - \frac{e_3}{3} - \frac{c}{2}$$

where n is the index, g is the genus, e_2, e_3 are the number of elliptic points of the corresponding order, and c is the number of cusps. Some rearrangement yields

$$e_2 + \frac{4e_3}{3} = 4 + \frac{n}{3} - 2c - 4g.$$

So we have

$$\begin{aligned} e_2 + e_3 &\leq 4 + \frac{n}{3} - 2c - 4g \\ &\leq 4 + \frac{n}{3} - 2 - 4g \\ &\leq \frac{n}{3} + 2. \end{aligned}$$

□

Definition 21. A **covering** of a set X with n labels is a minimal set $\{p_i\} \in P$ such that $\cup_i p_i \supseteq \{1, 2 \ldots, n\}$.

By using the definitions above and Lemma 4, we can now describe how to determine whether or not two sets of permutations are conjugate of one another, and correspondingly, whether or not two BCG's are isomorphic to each other. First we want to note that for $\{\sigma_2, \sigma_3\}$ and $\{\sigma_2', \sigma_3'\}$ in S_n, the sets P and P' are finite. Specifically, $|P| = |P'| = n$. The set P also contains a longest path, say p_1. By Lemma 4, if there exists a conjugation τ, then p_1 is mapped to some p_1' in P' with the same length and path type. We then attempt to construct τ by looking at the mappings of the paths. In the worst-case scenario, we have to look at all mappings of p_1 into P'. Then we use a depth-first search starting from p_1 to get a covering C. In terms of graphs, this is the same as finding a spanning tree. We find C' using the same depth-first search. If at any time where p_i

does not match up with p'_i, then we have to choose a new path p'_1 and do the depth-first search again. Therefore, by the end of the algorithm, we either found a possible candidate τ or we exhausted all possible mappings for p_1. Then we verify whether τ is indeed the conjugation.

Theorem 7. *Algorithm 2 terminates and produces a conjugate τ between $\{\sigma_2, \sigma_3\}$ and $\{\sigma'_2, \sigma'_3\}$ if it exists. This algorithm have time complexity $\mathcal{O}(n^3)$ where n is size of the set X.*

Proof. Suppose that the time it takes for a program to compare two elements or to do one computation is 1 time unit. For a set X with n labels, finding P takes n^2 time units. Finding P' would also take n^2 time units. Therefore, finding P and P' would take $2n^2$ time units. In the worst-case scenario, we have to map $p_1 \in P$ to all paths in P' before finding the correct map or conclude that such a map does not exist. This will take at most n time units. Finding a covering will take n^2 time units. At worse, we need to find a covering C' for each mapping of p_1. Altogether, it would take time $\mathcal{O}(n^3)$ to determine whether or not a conjugation element τ exists. □

Note that if there are invariants between $\{\sigma_2, \sigma_3\}$ and $\{\sigma'_2, \sigma'_3\}$ that do not correlate, then $\{\sigma_2, \sigma_3\}$ and $\{\sigma'_2, \sigma'_3\}$ are not conjugates. We only go through the full algorithm when the corresponding BCG's are extremely similar (for example Figure 3.6).

Algorithm 2 Determining Isomorphism Between 2 BCG's

1: Given 2 BCG's, find the corresponding permutation representation $\sigma = \{\sigma_2, \sigma_3\}$ and $\sigma' = \{\sigma'_2, \sigma'_3\}$

2: Check possible invariants between σ and σ'

if Some invariants do not match up **then**

  Output ”Not conjugate” and Terminate

end if

$i = 1$, $C' = \{\}$

Pick one of the longest paths $p_1 \in P$, and let $C = \{p_1\}$

while There is no isomorphism **do**

  Find $p_i' \in P'$ that matches $p_i \in C$ by depth-first search

  if p_i' exists and i = 1 **then**

    Add p_i' to C'

    Remove p_i' from P'

  else if p_i' does not exist and $i = 1$ **then**

    Break and print ”No conjugate”

  end if

  if no such p_i' exists and $i \neq 1$ **then**

    $i = 1$

    $C' = \{\}$

    Go back to start of While

  else

    if C is not a covering and $|C| = i$ **then**

      Find $p_{i+1} \in P$ by depth first search

      Add p_{i+1} to C

    else if C is a covering and $|C| = i$ **then**

      Find and check τ using C and C'.

If τ is a conjugate, output τ and Terminate.
If not, $i = 1$, $C' = \{\}$ and go back to start of while
end if
$i = i + 1$
end if
end while

The implementation of Algorithm 2 can be found on the author's website [23] in the file of the same name. We can then use this algorithm to classify subgroups of triangle groups. Note that algorithms for classifying subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ already exist (for example [3] and [30]). However, those algorithms are restricted to $\mathrm{PSL}_2(\mathbb{Z})$. On the other hand, Algorithm 2 works for subgroups of any triangle group with permutation representation $\{\sigma_a, \sigma_b\}$. We will demonstrate this in Section 4.4 for classifying subgroups of $\overline{\triangle}(2,4,6)$.

Example 3.3.4. Let

$$\sigma_2 = (1,3)(2,4)(5,13)(6,7)(8,21)(9,10)(11,12)(14,16)(15,19)(17,18)(20,22)(23,24)$$

$$\sigma_3 = (1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)(19,20,21)(22,23,24)$$

$$\sigma_2\sigma_3 = (2,5,14,17,16,15,20,23,22,21,9,11,10,7,4,3)(6,8,19,13)$$

$$\sigma_2' = (1,3)(2,4)(5,13)(6,7)(8,21)(9,10)(11,12)(14,16)(15,19)(17,18)(20,22)(23,24)$$

$$\sigma_3' = (1,2,3)(4,6,5)(7,9,8)(10,11,12)(13,15,14)(16,17,18)(19,21,20)(22,23,24)$$

$$\sigma_2'\sigma_3' = (2,6,9,11,10,8,20,23,22,19,14,17,16,13,4,3)(5,15,21,7)$$

These permutations represent two conjugacy classes of index 24 subgroups of $\mathrm{PSL}_2(\mathbb{Z})$. Since there is no differences in the obvious invariants such as the number of

```
In [6]: import sage_bcg_isom_general
        from sage.all import *
        from importlib import reload
        reload(sage_bcg_isom_general)

        n=24
        G = SymmetricGroup(n)
        siga = G('(1,3)(2,4)(6,7)(9,10)(11,12)(5,13)(8,21)(14,16)(17,18)(15,19)(20,22)(23,24)')
        sigb = G('(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)(19,20,21)(22,23,24)')
        sigaprime = G('(1,3)(2,4)(6,7)(9,10)(11,12)(5,13)(8,21)(14,16)(17,18)(15,19)(20,22)(23,24)')
        sigbprime = G('(1,2,3)(5,4,6)(7,9,8)(10,11,12)(13,15,14)(16,17,18)(19,21,20)(22,23,24)')

        sage_bcg_isom_general.isom(siga,sigb,sigaprime,sigbprime,n)

        siga (1,3)(2,4)(5,13)(6,7)(8,21)(9,10)(11,12)(14,16)(15,19)(17,18)(20,22)(23,24)
        sigb (1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)(19,20,21)(22,23,24)
        sigaprime (1,3)(2,4)(5,13)(6,7)(8,21)(9,10)(11,12)(14,16)(15,19)(17,18)(20,22)(23,24)
        sigbprime (1,2,3)(4,6,5)(7,9,8)(10,11,12)(13,15,14)(16,17,18)(19,21,20)(22,23,24)
        Isomorphic with tau =  (5,6)(7,13)(8,15)(9,14)(10,16)(11,17)(12,18)(19,21)
```

Figure 3.12: Example 1 of Algorithm 2

elliptic points or cusps, we cannot yet determine if these two conjugacy classes are the same. We apply the algorithm to determine that they are conjugates (Figure 3.12).

Example 3.3.5. Let

$$\sigma_2 = (1,2)(3,4)(5,10)(6,7)(8,9)(11,16)(12,13)(14,15)(17,19)(18,24)(20,23)(21,22)$$

$$\sigma_3 = (1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)(19,20,21)(22,23,24)$$

$$\sigma_2\sigma_3 = (1,3,5,11,17,20,24,16,12,14,13,10,6,8,7,4)(18,22,19)(21,23)$$

$$\sigma_2' = (1,2)(3,4)(5,10)(6,7)(8,9)(11,16)(12,13)(14,15)(17,19)(18,24)(20,23)(21,22)$$

$$\sigma_3' = (1,2,3)(4,5,6)(7,8,9)(10,12,11)(13,14,15)(16,17,18)(19,20,21)(22,23,24)$$

$$\sigma_2'\sigma_3' = (1,3,5,12,14,13,11,17,20,24,16,10,6,8,7,4)(18,22,19)(21,23)$$

Once more, the two sets of permutations produce the same set of invariants. However, the algorithm determines that these two sets of permutations are not conjugates (Figure 3.13). By looking at the paths, the longest path for $\{\sigma_2, \sigma_3\}$ starts at 2. The path

```
In [9]: import sage_bcg_isom_general
        from sage.all import *
        from importlib import reload
        reload(sage_bcg_isom_general)

        n=24
        G = SymmetricGroup(n)
        siga = G('(1,2)(3,4)(6,7)(8,9)(5,10)(12,13)(14,15)(11,16)(17,19)(18,24)(21,22)(20,23)')
        sigb = G('(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)(19,20,21)(22,23,24)')
        sigaprime = G('(1,2)(3,4)(6,7)(8,9)(5,10)(12,13)(14,15)(11,16)(17,19)(18,24)(21,22)(20,23)')
        sigbprime = G('(1,2,3)(4,5,6)(7,8,9)(10,12,11)(13,14,15)(16,17,18)(19,20,21)(22,23,24)')

        sage_bcg_isom_general.isom(siga,sigb,sigaprime,sigbprime,n)

        siga (1,2)(3,4)(5,10)(6,7)(8,9)(11,16)(12,13)(14,15)(17,19)(18,24)(20,23)(21,22)
        sigb (1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)(19,20,21)(22,23,24)
        sigaprime (1,2)(3,4)(5,10)(6,7)(8,9)(11,16)(12,13)(14,15)(17,19)(18,24)(20,23)(21,22)
        sigbprime (1,2,3)(4,5,6)(7,8,9)(10,12,11)(13,14,15)(16,17,18)(19,20,21)(22,23,24)
        Nonisomorphic
```

Figure 3.13: Second example of Algorithm 2

has length 13 with path type $(7, 7)$. The longest path for $\{\sigma_2', \sigma_3'\}$ starts at 19 and also have length 13. However, the path type is $(11, 3)$.

3.4. Permutations Back to BCG's

From a BCG with labelings, we saw how one can obtain the corresponding set of permutations that represent the given BCG. Now, we can apply the idea of paths and covering to obtain a BCG from a set of permutations. The main idea of the algorithm is to reverse the process of finding a covering. For the algorithm, any newly added edges will be added in a clockwise manner.

Algorithm 3 Produce a BCG from a set of permutations

Given $\{\sigma_2, \sigma_3\} \in S_n$, find the set P

Find a covering C as in Algorithm 2

For $c_1 \in C$, draw a path for c_1 including the path type and label the edges

for c_i in C with $i > 1$ **do**

    Draw c_i including the path type. Anytime a new edge is found, it is added in a clockwise manner

end for

Output the BCG corresponding to the set of permutations

For a given covering C, the algorithm runs through C without having to look at any element more than once. This will give us linear time for the for-loop in the algorithm. The majority of the time cost will be in finding P and C. As before, finding P and C will take $\mathcal{O}(n^2)$ time.

3.5. BCG's Back to Generalized Farey Symbols

Given a BCG, one can also obtain a Farey Symbol. The following algorithm is from Caranica's thesis [1].

Algorithm 4 Produce a Farey Symbol given a graph

Break up the graph at even vertices until the graph is a tree

Pick a leaf as the starting vertex

if starting vertex is odd **then**

    Assign the vertex and the adjacent vertex the set $a = \{\frac{0}{1}, \frac{1}{0}\}$

else

    Assign the vertex the set $a = \{\frac{0}{1}, \frac{1}{0}\}$

end if

while there are leaves without a set **do**

    Pick an odd vertex that is adjacent to an even vertex that has been assigned a set

    Set a to be the set of that even vertex

    In a counterclockwise manner starting from the chosen odd vertex, assign the adjacent even vertices that does not have a set the set $b = \{\text{mediant}(a), \min(a)\}$ and $c = \{\text{mediant}(a), \max(a)\}$ respectively

end while

For any remaining odd leaves, assign to it the same set as the one for their adjacent even vertex.

Note that in the algorithm above, the choice of the starting leaf was not defined. By choosing different leaves as the starting point, one might obtain different Farey Symbols which represent different subgroups. However, those subgroups will be in the same conjugacy class.

Definition 22. Any BCG with a fixed starting vertex is called a **marked bipartite cuboid graph**.

Theorem 8 (Vidal [30])**.** *There is a one-to-one correspondence between the marked bipartite cuboid graphs and finite index subgroups of $PSL_2(\mathbb{Z})$.*

Example 3.5.1. Consider the subgroup $\Gamma_0(3)$. The BCG corresponding to $\Gamma_0(3)$ is Figure 3.14.

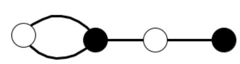

Figure 3.14: The BCG for $\Gamma_0(3)$

We apply Caranica's Algorithm 4 to the BCG to find a Farey Symbol. We begin by breaking up the cycles.

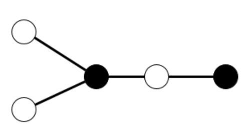

Figure 3.15: The tree for $\Gamma_0(3)$

Now we have choices for a starting leaf. We choose the two even leaves and find the corresponding Farey Symbols using the algorithm.

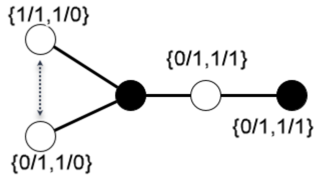

Figure 3.16: Choosing the bottom leaf as the starting leaf

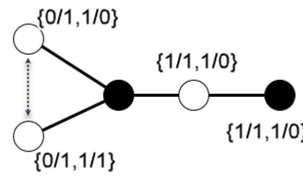

Figure 3.17: Choosing the top leaf as the starting leaf

The dashed lines in Figures 3.16 and 3.17 are there to remind us that we need to pair up the corresponding entries in the Farey Symbols. The Farey Symbols for both graphs are below.

$$-\infty \underbrace{\quad}_{1} 0 \underbrace{\quad}_{\bullet} 1 \underbrace{\quad}_{1} \infty$$

Figure 3.18: Farey Symbol for Figure 3.16

$$-\infty \underbrace{\quad}_{1} 0 \underbrace{\quad}_{1} 1 \underbrace{\quad}_{\bullet} \infty$$

Figure 3.19: Farey Symbol for Figure 3.17

The generators for the left Farey Symbol are $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}$. The generators for the right Farey Symbol are $\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$. These generators have the following relations

$$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}.$$

These two Farey Symbols represent different subgroups in the same conjugacy class in $\mathrm{PSL}_2(\mathbb{Z})$.

As we discussed in previous sections, there is a bijection between labeled BCG's and sets of permutations. This allows us to translate Algorithm 4 into an algorithm that takes in a set of permutations and produces a Farey Symbol.

Algorithm 5 Produce a Farey Symbol given a set of permutations

Given $\{\sigma_2, \sigma_3\} \in S_n$. Create σ_2' by splitting all possible length-2 cycles of σ_2 into length-1 cycles such that $\langle \sigma_2', \sigma_3 \rangle$ remains a transitive subgroup of S_n

Pick a 1 cycle from σ_2' or σ_3 as the starting point and denoted it as x_1

unchecked = []

if x_1 is in σ_3 **then**

    Assign x_1 and $x_2 = \sigma_2(x_1)$ the set $a_1 = \{\frac{0}{1}, \frac{1}{0}\}$

    Add x_2 to unchecked

else

    Assign x_1 the set $a_1 = \{\frac{0}{1}, \frac{1}{0}\}$

    Add x_1 to unchecked

end if

checked = []

while there are still x_i in unchecked[] **do**

    Find $x_{i+1} = \sigma_3(x_i)$ and $x_{i+2} = \sigma_3(x_{i+1})$

    if $x_{i+1} \mathrel{!}= x_{i+2}$ **then**

        Assign x_{i+1} (resp. x_{i+2}) the set $a_{i+1} = \{\text{mediant}(a_i), \min(a_i)\}$ (resp. $a_{i+2} = \{\text{mediant}(a_i), \max(a_i)\}$)

        Assign $\sigma_2'(x_{i+1})$ (resp.$\sigma_2'(x_{i+2})$) the set $a_{i+1} = \{\text{mediant}(a_i), \min(a_i)\}$ (resp. $a_{i+2} = \{\text{mediant}(a_i), \max(a_i)\}$)

        If $\sigma_2'(x_{i+1}) \mathrel{!}= x_{i+1}$, add $\sigma_2'(x_{i+1})$ to unchecked[]. Else, add x_{i+1} to checked[]

        If $\sigma_2'(x_{i+2}) \mathrel{!}= x_{i+2}$, add $\sigma_2'(x_{i+2})$ to unchecked[]. Else, add x_{i+2} to checked[]

        Remove x_i from unchecked[]

```
        Add x_i to checked[]
    else
        Remove x_i from unchecked[]
        Add x_i to checked[]
    end if
end while
```

For x_i corresponding to length-1 cycles in σ_2' or σ_3, the associated set a_i gives the entries in the Farey symbols.

If $\sigma_3(x_i) = x_i$, then it is an odd pairing. If $\sigma_2(x_i) = x_i$, then it is an even pairing. If x_i, x_j are in a 2-cycle in σ_2, then it is a free pairing.

Example 3.5.2. Let

$$\sigma_2 = (1,7)(2,11)(3,4)(5,10)(6,8)(9,12)$$

$$\sigma_3 = (1,2,3)(4,5,6)(7,8,9)(10,11,12).$$

Then we have

$$\sigma_2' = (1)(7)(2)(11)(3,4)(5)(10)(6,8)(9,12).$$

Let $x_1 = 1$ and attach $a_1 = \{\frac{0}{1}, \frac{1}{0}\}$.

The algorithm will produce the list in Table 3.1.

$(1,\{\frac{0}{1},\frac{1}{0}\})$	$(4,\{1,\frac{1}{0}\})$	$(8,\{2,\frac{1}{0}\})$	$(12,\{3,2\})$
$(2,\{\frac{0}{1},1\})$	$(5,\{2,1\})$	$(9,\{3,2\})$	$(10,\{\frac{5}{2},2\})$
$(3,\{1,\frac{1}{0}\})$	$(6,\{2,\frac{1}{0}\})$	$(7,\{3,\frac{1}{0}\})$	$(11,\{\frac{5}{2},3\})$

Table 3.1: Edges and labelings from Algorithm 5

Since $1, 2, 5, 7, 10, 11$ are 1-cycle in σ_2', the generalized Farey Sequence is

$$-\infty \quad 0 \quad 1 \quad 2 \quad \tfrac{5}{2} \quad 3 \quad \infty$$

Then we use σ_2 and σ_3 to get the appropriate pairings. A Farey Symbol for a subgroup in this conjugacy class is

$$\begin{array}{ccccccccccccc} -\infty & & 0 & & 1 & & 2 & & \frac{5}{2} & & 3 & & \infty \\ & 1 & & 2 & & 3 & & 3 & & 2 & & 1 & \end{array}$$

This Farey Symbol has generators

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 8 & -3 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} 7 & -12 \\ 3 & -5 \end{bmatrix}.$$

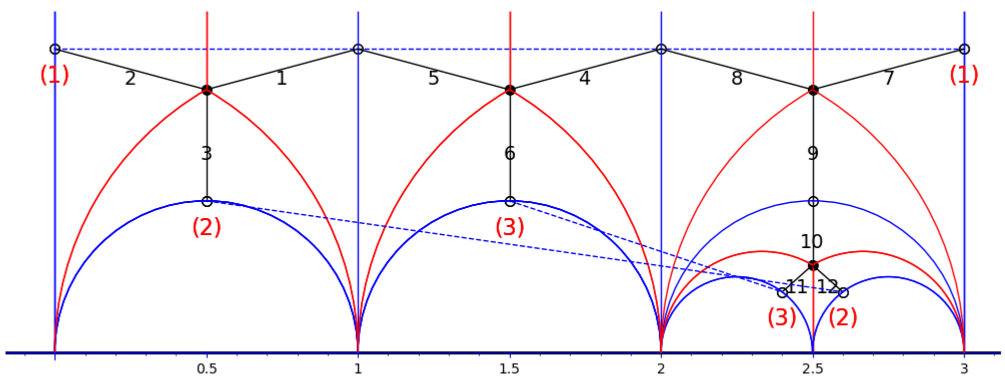

Figure 3.20: A special polygon corresponding to the Farey Symbol in Example 3.5.2

Note that the labelings on the graph in Figure 3.20 does not match the given σ_2. However, they are conjugates by $\tau = (4, 12, 8, 6, 11, 7, 5, 10, 9)$.

Once again, the implementation of these algorithms can be found on the author's website [23].

Chapter 4. Extension to General Triangle Groups

4.1. Introduction

In this section, we will discuss general triangle groups. As these groups arise naturally from hyperbolic geometry, they are used in many other contexts. As mentioned previously, these triangle groups are not only groups but can also be interpreted as Riemann surfaces and modular curves. Shimura's works on these subgroups led to a class of algebraic curves known as Shimura curves [26]. Takeuchi [28] considered a special class of triangle groups called arithmetic triangle groups and produced a classification for all arithmetic triangle groups. Clark and Voight [2] worked with these curves and their corresponding Belyĭ maps to extend the idea of congruence subgroups to finite index subgroups of hyperbolic triangle groups. Triangle groups also arise in Tu and Yang's works [29] [33] on transformations of hypergeometric functions. Note that this list is not exhaustive as there are many more people who are working on this topic such as Elkies [7], Sijsling [27], Milne [21], Fuselier, Long, Ramakrishna, Swisher and Tu [8], and Voight [31, 32, etc].

4.2. Drawing Fundamental Domains for Finite Index Subgroups of $\overline{\triangle}(a, b, c)$

Let Γ be a finite index subgroup of $\overline{\triangle}(a, b, c)$ represented by σ_a and σ_b. Ling Long, Fang-Ting Tu, and I collaborated to produce the following algorithm to draw a fundamental domain for Γ. The implementation of this algorithm and the algorithms above can be obtained on the author's website [23]. This algorithm will also output a set of generators and the associated pairings of the boundaries. The method used in this algorithm relies on the discussions in Sections 1.3, 1.4, and 1.5.

Note that John Voight also created a variant of this algorithm using quaternion

algebras [16, 31, 32].

Algorithm 6 Draw a fundamental domain for a given finite index subgroup of $\overline{\triangle}(a, b, c)$

Given a finite index subgroup of $\overline{\triangle}(a, b, c)$ represented as a pair of permutations $\{\sigma_a, \sigma_b\}$, use the embeddings given in (1.3) and (1.4) to obtain generators $\langle g_a, g_b \rangle$ for $\overline{\triangle}(a, b, c)$ with $g_a^a == g_b^b == (g_a g_b)^c == \pm I$

Using fixed points of g_a, g_b, and $g_a g_b$, draw a hyperbolic triangle $\triangle(a, b, c)$. Then we reflect the triangle across the geodesic connecting the fixed points of g_a and g_b. The union of the two copies is a fundamental domain for $\overline{\triangle}(a, b, c)$.

For a given subgroup of $\overline{\triangle}(a, b, c)$ of index n given by permutations σ_a and σ_b where $\sigma_a^a == \sigma_b^b == (\sigma_a \sigma_b)^c == \pm 1$, check if σ_a and σ_b form a transitive subgroup of S_n. If yes, continue. If not, Terminate.

Pick $i = 1$ to represent the fundamental domain of $\overline{\triangle}(a, b, c)$

checkedlist = [1]

matrixlist = [I]

k = a

while len(checkedlist)$< n$ **do**

    for i in checkedlist **do**

        if $\sigma_k(i)$ not in checkedlist **then**

            Add $\sigma_k(i)$ to checkedlist

            Add matrixlist[checkedlist.index(i)]$*g_k$ to matrixlist

        end if

    end for

    if k==a **then**

        k=b

    else

        k=a

    end if

    Draw the translations of $\overline{\triangle}(a, b, c)$ using matrices in matrixlist[]

end while

Find the pairing for the boundaries and the matrices corresponding to the fixed points

Output a fundamental domain and a set of generators

Examples for Algorithm 6 can be found in the Appendix.

4.3. Group Membership

Lemma 6. For $\{g_a, g_b\}$ defined in equations (1.3) and (1.4), the fixed points of g_a and g_b have real parts equal to 0.

Proof. Let $g_a = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ and $g_b = \begin{bmatrix} a & tb \\ -b/t & a \end{bmatrix}$ where t came from the embeddings in Section 1.3. Finding the fixed point(s) z for g_a gives us $z^2 = -1$. Finding the fixed point(s) z for g_b gives us $z^2 = -t^2$ where t is defined in equation (1.5). $\square$

From the proof of Lemma 6, we can see that i will be a fixed point of the fundamental domain of $\overline{\triangle}(a, b, c)$ given by g_a and g_b. This lemma allows us to use the idea of Lim, Lang, and Tan [15] to create an algorithm to check whether a matrix $M \in \mathrm{SL}_2(\mathbb{R})$ is in $\overline{\triangle}(a, b, c)$ under the embeddings given in Section 1.3.

We will briefly discuss Lim, Lang, and Tan's algorithm. For a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, there exists a geodesic with endpoints $\frac{b}{d}$ and $\frac{a}{c}$. This geodesic is the

image of the geodesic $\{0, \infty\}$ under the action of M . For a finite index subgroup Γ of $\mathrm{PSL}_2(\mathbb{Z})$, let F be a special polygon corresponding to Γ and $\{g_i\}$ be the corresponding generating set. Then using powers of $\{g_i\}$, the algorithm will move the geodesic $\{\frac{b}{d}, \frac{a}{c}\}$ back into the special polygon. After a finite number of steps, we obtain a matrix M_k such that $M_k(\{\frac{b}{d}, \frac{a}{c}\})$ intersects F. If $M_k(\{\frac{b}{d}, \frac{a}{c}\})$ lies on the boundary of F, then M is in Γ. Otherwise, M is not in Γ.

We extend this algorithm to general $\overline{\triangle}(a, b, c)$ with generating set $\{g_a, g_b\}$ by first drawing a fundamental domain F and picking a starting line. Let l be the partial geodesic containing i on the boundary of F with endpoints x and y fixed by $g_a g_b$ and $g_b g_a$ respectively. For $M \in \mathrm{SL}_2(\mathbb{R})$, let $l' = M(l)$. Then we apply powers of the generators $\{g_a, g_b\}$ to l' to bring it back into F. At each step, we choose the power that reduces the hyperbolic distance between i and the current location of the image of i. After a finite number of steps, we obtain a matrix M_k such that $M_k(l')$ intersects F. If $M_k(l')$ is a boundary of F, then M is in $\overline{\triangle}(a, b, c)$. If $M_k(l')$ intersects F at either x or y, apply powers of $g_a g_b$ or $g_b g_a$ and check if $(g_a g_b)^j M_k(l')$ (or $(g_b g_a)^j M_k(l')$) for $1 \leq j \leq \mathrm{order}(g_a g_b)$ is a boundary of F. If $M_k(l')$ intersects F at any other points, M is not in $\overline{\triangle}(a, b, c)$. This process will also give M as a word using the generators of $\overline{\triangle}(a, b, c)$.

Algorithm 7 Determine whether a matrix $M \in \mathrm{SL}_2(\mathbb{R})$ is in $\overline{\triangle}(a, b, c)$

Use Algorithm 6 to get a fundamental domain F and a set of generators $\{g_a, g_b\}$ for $\overline{\triangle}(a, b, c)$ with $g_a^a == g_b^b == (g_a g_b)^c == \pm I$

if M is a generator or $\pm I$ **then**

    Terminate and output "In group"

end if

Pick the boundary on the fundamental domain with endpoints $\{x, y\}$ where $x = \text{Fixedpoint}(g_a g_b)$ and $y = \text{Fixedpoint}(g_b g_a)$

$\{x1, y1\} = M\{x, y\}$

$mid = \text{Midpoint}(x, y)$

$mid1 = \text{Midpoint}(x1, y1)$

$d1 = \text{HyperbolicDistance}(mid, mid1)$

word = []

while $d1 > 0$ **do**

  for k from 1 to $b - 1$ **do**

    Find the distance d between $g_b^k mid1$ and mid

    if $d < d1$ **then**

      $d1 = d$; $mid1 = g_b^k mid1$; $\{x1, y1\} = g_b^k \{x, y\}$; gtemp $= g_b^k$

    end if

  end for

  Add gtemp to word[]

  if $\{x1, y1\}$ intersects F but $x1 \neq x$ and $y1 \neq y$ **then**

    Terminate and output "Not in group"

  end if

  if $a \neq 2$ **then**

    for k from 1 to $a - 1$ **do**

      Find the distance d between $g_a^k mid1$ and mid

      if $d < d1$ **then**

        $d1 = d$; $mid1 = g_a^k mid1$; $\{x1, y1\} = g_a^k \{x, y\}$; gtemp $= g_a^k$

end if

end for

Add gtemp to word[]

if $\{x1, y1\}$ intersects F but $x1 \neq x$ and $y1 \neq y$ **then**

Terminate and output "Not in group"

end if

else

Find the distance d between $g_a mid1$ and mid

$d1 = d$

$mid1 = g_1 mid1$

$\{x1, y1\} = g_1\{x, y\}$

Add g_a to word[]

end if

if $\{x1, y1\}$ intersects F but $x1 \neq x$ and $y1 \neq y$ **then**

Terminate and output "Not in group"

end if

end while

if $\{x1, y1\}$ intersects F but $x1 \neq x$ and $y1 \neq y$ **then**

Terminate and output "Not in group"

else

Terminate, output "In group", output word[]

end if

Example 4.3.1. Let

$$M = \begin{bmatrix} \frac{-16-10\sqrt{3}}{\left(1+\sqrt{3}\right)^3} & \frac{\sqrt{2}\left(16+9\sqrt{3}\right)}{\left(1+\sqrt{3}\right)^2} \\ \frac{\left(-22-12\sqrt{3}\right)\sqrt{2}}{\left(1+\sqrt{3}\right)^4} & \frac{34+20\sqrt{3}}{\left(1+\sqrt{3}\right)^3} \end{bmatrix},$$

$$g_a = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; g_b = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{1}{2}+\frac{\sqrt{3}}{2} \\ -\frac{1}{1+\sqrt{3}} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

Figures 4.1 to 4.5 show the steps that the algorithm takes to determine whether a matrix $M \in \mathrm{SL}_2(\mathbb{R})$ is in $\overline{\triangle}(2,4,6)$ with generating set $\{g_a, g_b\}$.

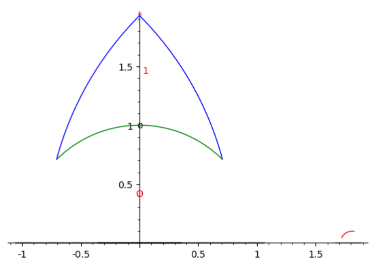

Figure 4.1: Step 1

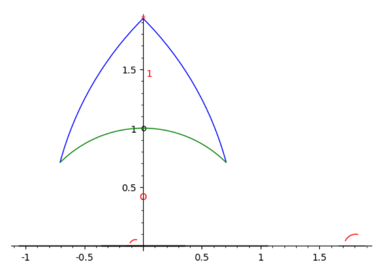

Figure 4.2: Step 2

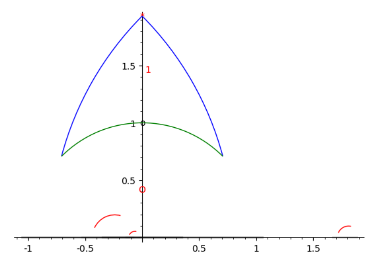

Figure 4.3: Step 3

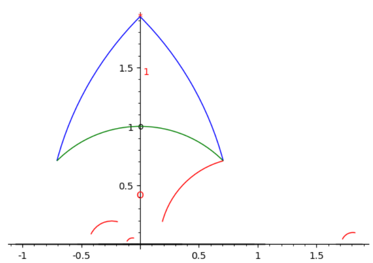

Figure 4.4: Step 4

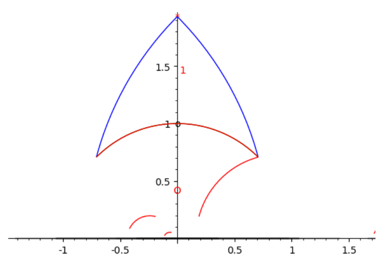

Figure 4.5: Step 5

```
g1 =  [0.000000000000000  1.00000000000000]
[-1.00000000000000 0.000000000000000]
g2 =  [ 0.707106781186547   1.36602540378444]
[-0.366025403784439  0.707106781186547]
word = [-1.63397459621556  5.98502364871592]
[-1.08604416314956  3.36602540378444]
word in string
g2g1g2^2g1g2g1g2^3  =  [-1.63397459621556  5.98502364871591]
[-1.08604416314956  3.36602540378444] /{+/-I}
```

Figure 4.6: Results

Figure 4.6 shows M written as a word using $g_1 = g_a$ and $g_2 = g_b$

$$M = g_2 g_1 g_2^2 g_1 g_2 g_1 g_2^3.$$

4.4. Classifications of Subgroups of $\overline{\triangle}(2,4,6)$ up to Index 11

In this section, we will list the conjugacy classes of finite index subgroups of $\overline{\triangle}(2,4,6)$ up to index 11. We present the conjugacy classes in terms of their permutation

representations.

For a given index n, we fix an order 2 permutation σ_2. Then we iterate through S_n to find all possible order 4 permutations σ_4 satisfying the relations in equation (1.1) and produce a transitive subgroup in S_n. This process gives us a list of pairs of permutations. Then we use Algorithm 2 to check for isomorphisms between the pairs of permutations which then give a complete list of conjugacy classes of index n subgroups.

The invariants of a given subgroup can be obtained from the corresponding permutations σ_2 and σ_4. The elliptic points of $\overline{\triangle}(2,4,6)$ have order dividing $2, 3, 4$, or 6. If we write the permutations σ_2, σ_4, and $\sigma_2\sigma_4$ as products of disjoint cycles, then the numbers of elliptic points of order 2, 3, 4, and 6 are:

$$\begin{aligned}
e_2 =& \text{number of length-1 cycles of } \sigma_2 \\
&+ \text{number of length-2 cycles of } \sigma_4, \\
&+ \text{number of length-3 cycles of } \sigma_2\sigma_4, \\
e_3 =& \text{number of length-2 cycles of } \sigma_2\sigma_4, \\
e_4 =& \text{number of length-1 cycles of } \sigma_4 \\
e_6 =& \text{number of length-1 cycles of } \sigma_2\sigma_4.
\end{aligned}$$

Using the Riemann-Hurwitz formula 1.2, we find that the genus formula for subgroups of $\overline{\triangle}(2,4,6)$ is

$$g = \frac{1}{24}n - \frac{3}{8}e_4 - \frac{1}{2}e_3 - \frac{1}{4}e_2 - \frac{5}{12}e_6 + 1.$$

Table 4.1: A Table of Conjugacy Classes of Subgroups of $\overline{\triangle}(2,4,6)$, Index 1-7

Index	Order 2	Order 4	Order 6	Genus
2	(1,2)	(1)(2)	(1,2)	0
	(1,2)	(1,2)	(1)(2)	0
	(1)(2)	(1,2)	(1,2)	0
3	(1,2)	(2,3)	(1,3,2)	0
4	(1,2)(3)	(1,3,2,4)	(1,4)(2,3)	0
	(1,2)(3)	(1,3,4,2)	(2,3,4)	0
	(1,2)(3,4)	(1,3)(2,4)	(1,4)(2,3)	0
	(1,2)(3,4)	(1,3,4,2)	(2,3)	0
5	(1,2)(3,4)	(2,4,5,3)	(1,4,2)(3,5)	0
6	(1,2)	(1,5)(2,3,4,6)	(1,3,4,6,2,5)	0
	(1,2)(3,4)	(1,4)(2,5)(3,6)	(1,5,2,4,6,3)	0
	(1,2)(3,4)	(1,4,5,6)	(1,2,4,3,5,6)	0
	(1,2)(3,4)	(1,4)(2,6,3,5)	(1,6,3)(2,4,5)	0
	(1,2)(3,4)	(1,5,4,6)	(1,2,5,4,3,6)	0
	(1,2)(3,4)(5,6)	(1,4)(2,5)(3,6)	(1,5,3)(2,4,6)	0
	(1,2)(3,4)(5,6)	(1,5,2,4)(3,6)	(1,4,6,2,5,3)	1
	(1,2)(3,4)(5,6)	(1,5)(2,3,4,6)	(1,3,6)(2,5)	0
	(1,2)(3,4)(5,6)	(2,5)(4,6)	(1,5,4,3,6,2)	0
	(1,2)(3,4)(5,6)	(1,6,2,3)	(1,3,4)(2,6,5)	0
7	(1,2)(3,4)(5,6)	(2,4,5,6)(3,7)	(1,4,7,3,5,2)	0
	(1,2)(3,4)(5,6)	(2,5,6,4)(3,7)	(1,5,4,7,3,2)	0
	(1,2)(3,4)(5,6)	(1,2,7,3)(4,5)	(1,7,3,5,6,4)	0
	(1,2)(3,4)(5,6)	(1,2,3,7)(4,5)	(1,3,5,6,4,7)	0

Table 4.2: A Table of Conjugacy Classes of Subgroups of $\overline{\triangle}(2,4,6)$, Index 8-9

Index	Order 2	Order 4	Order 6	Genus
8	(1,2)(3,4)	(1,6,8,3)(2,4,5,7)	(1,4)(2,6,8,3,5,7)	0
	(1,2)(3,4)	(1,3,6,4)(2,8,7,5)	(1,8,7,5,2,3)(4,6)	0
	(1,2)(3,4)(5,6)	(1,4,8,5)(2,6)(3,7)	(1,6)(2,4,7,3,8,5)	0
	(1,2)(3,4)(5,6)	(1,4,7,3)(2,6)(5,8)	(1,6,8,5,2,4)(3,7)	0
	(1,2)(3,4)(5,6)	(1,2,7,5)(3,4,8,6)	(1,7,5,3,8,6)	0
	(1,2)(3,4)(5,6)	(1,6,8,3)(2,4,5,7)	(1,4)(2,6,7)(3,5,8)	0
	(1,2)(3,4)(5,6)	(1,8,4,6)(2,5,7,3)	(1,5)(2,8,4)(3,6,7)	0
	(1,2)(3,4)(5,6)	(1,7,3,5)(2,6)(4,8)	(1,6)(2,7,3,8,4,5)	0
	(1,2)(3,4)(5,6)	(1,2,5,7)(3,4,6,8)	(1,5,8,3,6,7)	0
	(1,2)(3,4)(5,6)	(1,2,4,7)(3,5,6,8)	(1,4,5,8,3,7)	0
	(1,2)(3,4)(5,6)(7,8)	(1,8,3,6)(2,4,5,7)	(1,4,6,7,3,5)(2,8)	1
	(1,2)(3,4)(5,6)(7,8)	(1,7)(2,3,6,8)	(1,3,4,6,5,8)(2,7)	0
	(1,2)(3,4)(5,6)(7,8)	(1,2,7,5)(3,4,8,6)	(1,7,6)(3,8,5)	0
	(1,2)(3,4)(5,6)(7,8)	(1,4,8,5)(2,6,7,3)	(1,6)(2,4)(3,8)(5,7)	0
9	(1,2)(3,4)(5,6)	(1,8,6,4)(2,9,7,5)	(1,9,7,5,4,3)(2,8,6)	0
	(1,2)(3,4)(5,6)	(1,9,3,7)(2,4,6,8)	(1,4,7)(2,9,3,6,5,8)	0
	(1,2)(3,4)(5,6)	(1,9,5,7)(2,6,8,4)	(1,6,7)(2,9,5,8,4,3)	0
	(1,2)(3,4)(5,6)	(1,7,8,9)(2,3,5,4)	(1,3,2,7,8,9)(4,5,6)	0
	(1,2)(3,4)(5,6)(7,8)	(1,8)(2,3,7,9)(4,5)	(1,3,5,6,4,7)(2,8,9)	0
	(1,2)(3,4)(5,6)(7,8)	(1,6)(2,7)(4,9,5,8)	(1,7,4,3,9,5)(2,6,8)	0
	(1,2)(3,4)(5,6)(7,8)	(2,5,6,7)(3,9,4,8)	(1,5,7,3,8,2)(4,9)	0
	(1,2)(3,4)(5,6)(7,8)	(1,3,2,7)(5,8)(6,9)	(1,7,5,9,6,8)(2,3,4)	0
	(1,2)(3,4)(5,6)(7,8)	(1,7,9,4)(2,5,6,8)	(1,5,8,9,4,3)(2,7)	0
	(1,2)(3,4)(5,6)(7,8)	(1,6,5,4)(2,3,8,9)	(1,3)(2,6,4,8,7,9)	0
	(1,2)(3,4)(5,6)(7,8)	(2,7,4,3)(5,9,6,8)	(1,7,5,8,4,2)(6,9)	0
	(1,2)(3,4)(5,6)(7,8)	(1,5,8,9)(3,6)(4,7)	(1,2,5,3,7,9)(4,6,8)	0

Table 4.3: A Table of Conjugacy Classes of Subgroups of $\overline{\triangle}(2,4,6)$, Index 10-11

Index	Order 2	Order 4	Order 6	Genus
10	(1,2)(3,4)(5,6)(7,8)	(1,5,10,6)(2,7)(3,9,4,8)	(1,7,3,8,2,5)(4,9)(6,10)	0
	(1,2)(3,4)(5,6)(7,8)	(1,2,7,3)(4,9,5,8)(6,10)	(1,7,4)(3,9,5,10,6,8)	0
	(1,2)(3,4)(5,6)(7,8)	(1,2,8,4)(3,6,10,7)(5,9)	(1,8,3)(4,6,9,5,10,7)	0
	(1,2)(3,4)(5,6)(7,8)	(1,8,5,2)(3,10,9,6)(4,7)	(2,8,4,10,9,6)(3,7,5)	0
	(1,2)(3,4)(5,6)(7,8)	(1,8,3,6)(2,9,4,7)(5,10)	(1,9,4,6,10,5)(2,8)(3,7)	0
	(1,2)(3,4)(5,6)(7,8)	(1,5,10,6)(2,9,4,7)(3,8)	(1,9,4,8,2,5)(3,7)(6,10)	0
	(1,2)(3,4)(5,6)(7,8)	(1,2,9,6)(3,10,8,5)(4,7)	(1,9,6,3,7,5)(4,10,8)	0
	(1,2)(3,4)(5,6)(7,8)	(1,6,7,8)(2,4,5,9)(3,10)	(1,4,10,3,5,7)(2,6,9)	0
	(1,2)(3,4)(5,6)(7,8)	(1,8,4,6)(2,10,9,7)(3,5)	(1,10,9,7,4,5)(2,8)(3,6)	0
	(1,2)(3,4)(5,6)(7,8)	(1,8)(2,6,7,9)(3,5,10,4)	(1,6,10,4,5,7)(2,8,9)	0
	(1,2)(3,4)(5,6)(7,8)	(1,4,5,6)(2,10,9,7)(3,8)	(1,10,9,7,3,5)(2,4,8)	0
	(1,2)(3,4)(5,6)(7,8)	(1,9,4,7)(2,8)(3,6,10,5)	(1,8)(2,9,4,6,3,7)(5,10)	0
	(1,2)(3,4)(5,6)(7,8)(9,10)	(1,6,9,4)(2,7)(3,10,5,8)	(1,7,3)(2,6,8)(4,10)(5,9)	0
	(1,2)(3,4)(5,6)(7,8)(9,10)	(1,7,5,2)(3,9,8,6)	(2,7,6)(3,4,9,10,8,5)	0
	(1,2)(3,4)(5,6)(7,8)(9,10)	(1,10,2,3)(4,7,6,5)	(1,3,7,8,6,4)(2,10,9)	0
	(1,2)(3,4)(5,6)(7,8)(9,10)	(1,10,2,3)(4,8,7,6)	(1,3,8,6,5,4)(2,10,9)	0
	(1,2)(3,4)(5,6)(7,8)(9,10)	(1,10,4,8)(2,5,3,9)	(1,5,6,3,8,7)(2,10)(4,9)	0
11	(1,2)(3,4)(5,6)(7,8)(9,10)	(1,9)(2,4,7,10)(3,6,8,11)	(1,4,6,5,8,10)(2,9)(3,7,11)	0
	(1,2)(3,4)(5,6)(7,8)(9,10)	(1,6,11,7)(2,8)(4,10,5,9)	(1,8)(2,6,9,5,11,7)(3,10,4)	0
	(1,2)(3,4)(5,6)(7,8)(9,10)	(2,9,4,7)(3,6,11,8)(5,10)	(1,9,5,11,8,2)(3,7)(4,6,10)	0
	(1,2)(3,4)(5,6)(7,8)(9,10)	(2,8,3,7)(4,9)(5,11,6,10)	(1,8,2)(3,9,5,10,4,7)(6,11)	0
	(1,2)(3,4)(5,6)(7,8)(9,10)	(1,11,6,8)(2,7,9,4)(5,10)	(1,7)(2,11,6,10,4,3)(5,8,9)	0
	(1,2)(3,4)(5,6)(7,8)(9,10)	(1,11,6,8)(2,7,10,5)(4,9)	(1,7)(2,11,6)(3,9,5,8,10,4)	0
	(1,2)(3,4)(5,6)(7,8)(9,10)	(1,8,11,5)(2,6,7,9)(4,10)	(1,6)(2,8,9,4,3,10)(5,7,11)	0
	(1,2)(3,4)(5,6)(7,8)(9,10)	(1,7,11,6)(2,8)(4,10,5,9)	(1,8,11,6,9,5)(2,7)(3,10,4)	0

Appendix. Examples for $\overline{\triangle}(2, 4, 6)$

Example 0.0.1. Using Algorithm 6, we can get a fundamental domain for the triangle group $\overline{\triangle}(2, 4, 6)$ as given in Figure 1.

```
The permutations are ta= () tb= () tc= () with a,b,c =  2 4 6
ta elliptic points: [1]
tb elliptic points: [1]
ta pairs: []
tb pairs: []
A list of generators:

[
[0.000000000000000  1.00000000000000]
[-1.00000000000000 0.000000000000000],

[ 0.707106781186547   1.36602540378444]
[-0.366025403784439  0.707106781186547]
]

The boundaries of the fundamental domain: [1, 1, 1, 1]
```

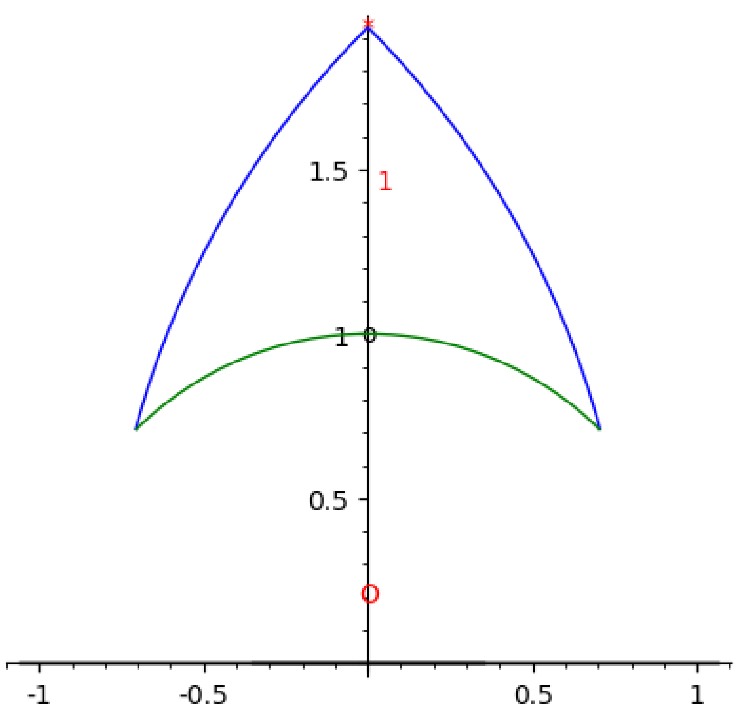

Figure 1: A fundamental domain for $\overline{\triangle}(2, 4, 6)$

Example 0.0.2. Let

$$\sigma_2 = (1,4)(2,5)(3,6)$$

$$\sigma_4 = (1,2,3,4)(5,6)$$

$$\sigma_6 = (1)(2,6,4)(3,5).$$

Then Algorithm 6 produced a fundamental domain and generators given in Figure 2.

```
The permutations are ta= (1,4)(2,5)(3,6) tb= (1,2,3,4)(5,6) tc= (2,6,4)(3,5) with a,b,c =  2 4 6
ta elliptic points: []
tb elliptic points: [5]
ta pairs: [[1, 4], [3, 6]]
tb pairs: [[5, 6]]
A list of generators:

[
[-0.517638090205041   2.73205080756888]
[-0.732050807568877   1.93185165257814],

[-0.707106781186547   1.36602540378444]
[-0.366025403784439 -0.707106781186547],

[-0.707106781186547   2.36602540378444]
[-0.633974596215561  0.707106781186547]
]

The boundaries of the fundamental domain: [1, 1, 0, 0][2, 0, 0, 0][3, 1, 0, 0][4, 1, 0, 0]
[5, 0, 0, 1][6, 1, 1, 0]
```

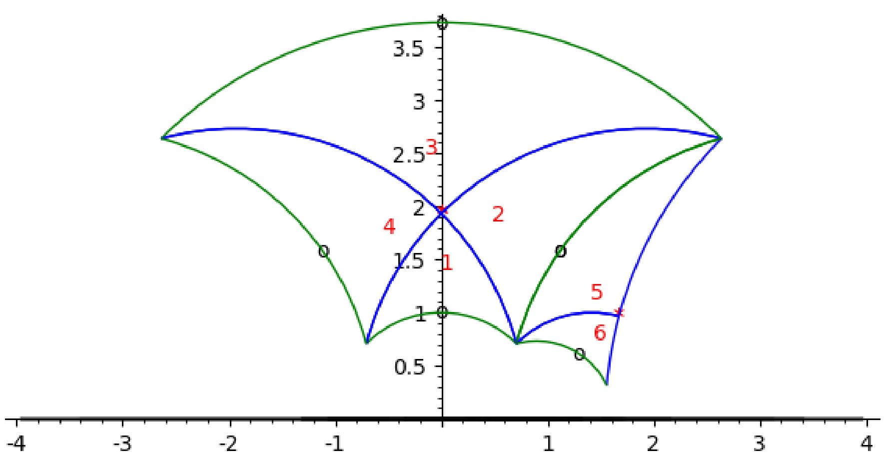

Figure 2: A fundamental domain for a subgroup of $\overline{\triangle}(2,4,6)$

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Vita

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