

7-1-2019

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Recommended Citation

Ding, D., Quek, Y., Shor, P., & Wilde, M. (2019). Entropy Bound for the Classical Capacity of a Quantum Channel Assisted by Classical Feedback. *IEEE International Symposium on Information Theory - Proceedings, 2019-July*, 250-254. <https://doi.org/10.1109/ISIT.2019.8849604>

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Citation	Ding, Dawei et al. "Entropy Bound for the Classical Capacity of a Quantum Channel Assisted by Classical Feedback." Paper presented at the 2019 IEEE International Symposium on Information Theory (ISIT), Paris, France, 7-12 July 2019, IEEE © 2019 The Author(s)
As Published	10.1109/ISIT.2019.8849604
Publisher	IEEE
Version	Author's final manuscript
Citable link	https://hdl.handle.net/1721.1/126673
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Detailed Terms	http://creativecommons.org/licenses/by-nc-sa/4.0/

Entropy Bound for the Classical Capacity of a Quantum Channel Assisted by Classical Feedback

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Abstract—We prove that the classical capacity of an arbitrary quantum channel assisted by a free classical feedback channel is bounded from above by the maximum average output entropy of the quantum channel. As a consequence of this bound, we conclude that a classical feedback channel does not improve the classical capacity of a quantum erasure channel, and by taking into account energy constraints, we conclude the same for a pure-loss bosonic channel. The method for establishing the aforementioned entropy bound involves identifying an information measure having two key properties: 1) it does not increase under a one-way local operations and classical communication channel from the receiver to the sender and 2) a quantum channel from sender to receiver cannot increase the information measure by more than the maximum output entropy of the channel. This information measure can be understood as the sum of two terms, with one corresponding to classical correlation and the other to entanglement.

I. INTRODUCTION

A famous result of Shannon is that a free feedback channel does not increase the capacity of a classical channel for communication [1]. That is, the feedback-assisted capacity is equal to the channel's mutual information. Shannon's result indicates that the mutual information formula for capacity is particularly robust, in the sense that, *a priori*, one might consider a feedback channel to be a strong resource for assisting communication.

With the rise of quantum information theory, several researchers have found variations and generalizations of Shannon's aforementioned result, in the context of communication over quantum channels. For example, Bowen proved that the capacity of a quantum channel for sending classical messages, when assisted by a free quantum feedback channel, is equal to the channel's entanglement-assisted capacity [2], which is in turn equal to the mutual information of a quantum channel [3]–[5]. This result indicates that the mutual information of a quantum channel is robust, in a sense similar to that mentioned above. The result also indicates that the best strategy, in the limit of many channel uses, is to use the quantum feedback channel once in order to establish sufficient shared entanglement between the sender and receiver, and to subsequently employ an entanglement-assisted communication protocol [3]–[5]. Bowen's result was strengthened to a

strong converse statement in [6], [7]. Bowen *et al.* proved that the capacity of an entanglement-breaking channel for sending classical messages is not increased by a free classical feedback channel [8], and this result was strengthened to a strong converse statement in [9]. Ref. [10] discussed several inequalities relating the classical capacity assisted by classical feedback to other capacities. At the same time, it is known that in general there can be an arbitrarily large gap between the unassisted classical capacity and the classical capacity assisted by classical feedback [11].

Our aim here is to go beyond [8] to establish an upper bound on the classical capacity of an *arbitrary*, not just entanglement-breaking, quantum channel assisted by a classical feedback channel. Due to the fact that a quantum feedback channel is a stronger resource than a classical feedback channel, an immediate consequence of Bowen's result [2] is that the entanglement-assisted capacity is an upper bound on the classical capacity assisted by classical feedback. However, since a quantum channel can, in general, establish quantum entanglement [12]–[14] and entanglement can increase capacity [3]–[5], in such cases it may appear difficult to establish an upper bound on this capacity other than the entanglement-assisted capacity. Our main result is that the latter is actually possible: we prove here that the maximum output entropy of a quantum channel is an upper bound on its classical capacity assisted by classical feedback. As a generalization of this result, we find that the maximum average output entropy is an upper bound on the same capacity for a channel that is a probabilistic mixture of other channels.

The approach that we take for establishing the aforementioned bounds is similar in spirit to approaches used to bound other assisted capacities or protocols [15]–[18]. We identify an information measure that has two key properties: 1) it does not increase under a free operation, which in this case is a one-way local operations and classical communication (1W-LOCC) channel from the receiver to the sender, and 2) a quantum channel from sender to receiver cannot increase the information measure by more than the maximum output entropy of the channel. This information measure can be understood as the sum of two terms, with one corresponding

to classical correlation and the other to entanglement.

We organize the rest of the paper as follows. Section II provides a formal definition of a protocol for classical communication over a quantum channel assisted by classical feedback. Section III discusses explicitly how to purify such a protocol, which is an important conceptual step for our analysis. Section IV introduces our key information measure and several important supplementary lemmas regarding it. Section V then employs this information measure and the supplementary lemmas to establish the maximum output entropy bound for classical capacity assisted by classical feedback. We apply this bound to the erasure channel and pure-loss bosonic channel in Section VI. We conclude in Section VII.

II. PROTOCOL FOR CLASSICAL COMMUNICATION OVER A QUANTUM CHANNEL ASSISTED BY CLASSICAL FEEDBACK

To begin with, let $n, M \in \mathbb{N}$, let $\varepsilon \in [0, 1]$, and let $E \geq 0$. Let $\mathcal{N}_{A \rightarrow B}$ be a quantum channel, and let H be a Hamiltonian acting on the input system A of $\mathcal{N}_{A \rightarrow B}$. An $(n, M, H, E, \varepsilon)$ protocol for classical communication over a quantum channel $\mathcal{N}_{A \rightarrow B}$ consists of n uses of the quantum channel $\mathcal{N}_{A \rightarrow B}$, along with the assistance of a classical feedback channel from the receiver Bob to the sender Alice, in order for Alice to send one of M messages to Bob such that the error probability is no larger than ε . Furthermore, the average state at the input of each channel use should have energy no larger than E , when taken with respect to the Hamiltonian H .

In more detail, the protocol consists of an initial classical-quantum state $\sigma_{F_0 B'_1}$, with F_0 classical and B'_1 quantum, of the form

$$\sigma_{F_0 B'_1} = \sum_{f_0} p(f_0) |f_0\rangle \langle f_0|_{F_0} \otimes \sigma_{B'_1}^{f_0}. \quad (1)$$

It also involves n encoding channels, with each one denoted by $\mathcal{E}_{A'_{i-1} F_{i-1} \rightarrow A'_i A_i}$ for $i \in \{1, \dots, n\}$, as well as n decoding channels, with each of them denoted by $\mathcal{D}_{B_i B'_i \rightarrow F_i B'_{i+1}}$ for $i \in \{1, \dots, n-1\}$. Note that all F systems are classical because the feedback channel is constrained to be a classical channel. So this means that each decoding channel is a quantum instrument. The final decoding is denoted by $\mathcal{D}_{B_n B'_n \rightarrow \hat{W}}$.

We now detail the form of such a protocol. It begins with Alice preparing the following classical-quantum state:

$$\rho_{W A'_0} \equiv \frac{1}{M} \sum_{m=1}^M |m\rangle \langle m|_W \otimes \rho_{A'_0}^m, \quad (2)$$

for some set $\{\rho_{A'_0}^m\}_m$ of quantum states. The global initial state is then $\rho_{W A'_0} \otimes \sigma_{F_0 B'_1}$. Alice then performs the encoding channel $\mathcal{E}_{A'_0 F_0 \rightarrow A'_1 A_1}$ and the state becomes as follows:

$$\omega_{W A'_1 A_1 B'_1}^{(1)} \equiv \mathcal{E}_{A'_0 F_0 \rightarrow A'_1 A_1}(\rho_{W A'_0} \otimes \sigma_{F_0 B'_1}). \quad (3)$$

Alice transmits the A_1 system through the first use of the channel $\mathcal{N}_{A_1 \rightarrow B_1}$, resulting in the following state:

$$\rho_{W A'_1 B_1 B'_1}^{(1)} \equiv \mathcal{N}_{A_1 \rightarrow B_1}(\omega_{W A'_1 A_1 B'_1}^{(1)}). \quad (4)$$

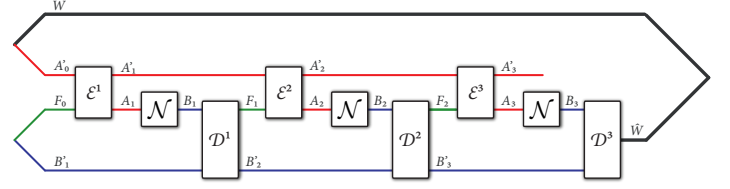


Fig. 1. A protocol for classical communication over three uses of a quantum channel $\mathcal{N}_{A \rightarrow B}$, when assisted by a classical feedback channel.

Bob processes his systems $B_1 B'_1$ with the decoding channel $\mathcal{D}_{B_1 B'_1 \rightarrow F_1 B'_2}^1$ and Alice acts with the encoding channel $\mathcal{E}_{A'_1 F_1 \rightarrow A'_2 A_2}^2$, resulting in the state

$$\omega_{W A'_2 A_2 B'_2}^{(2)} \equiv (\mathcal{E}_{A'_1 F_1 \rightarrow A'_2 A_2}^2 \circ \mathcal{D}_{B_1 B'_1 \rightarrow F_1 B'_2}^1)(\rho_{W A'_1 B_1 B'_1}^{(1)}). \quad (5)$$

This process iterates $n - 2$ more times, resulting in the following states:

$$\rho_{W A'_i B_i B'_i}^{(i)} \equiv \mathcal{N}_{A_i \rightarrow B_i}(\omega_{W A'_i A_i B'_i}^{(i)}), \quad (6)$$

$$\omega_{W A'_{i+1} A_{i+1} B'_{i+1}}^{(i+1)} \equiv (\mathcal{E}_{A'_i F_i \rightarrow A'_{i+1} A_{i+1}}^{i+1} \circ \mathcal{D}_{B_i B'_i \rightarrow F_i B'_{i+1}}^i)(\rho_{W A'_i B_i B'_i}^{(i)}), \quad (7)$$

for $i \in \{2, \dots, n-1\}$. The final decoding (measurement) channel $\mathcal{D}_{B_n B'_n \rightarrow \hat{W}}^n$ results in the following state:

$$\rho_{W \hat{W}} \equiv (\text{Tr}_{A'_n} \circ \mathcal{D}_{B_n B'_n \rightarrow \hat{W}}^n)(\rho_{W A'_n B_n B'_n}^{(n)}). \quad (8)$$

Figure 1 depicts the above protocol for $n = 3$.

For an $(n, M, H, E, \varepsilon)$ protocol, the following is satisfied

$$\frac{1}{2} \|\bar{\Phi}_{W \hat{W}} - \rho_{W \hat{W}}\|_1 \leq \varepsilon, \quad (9)$$

where $\bar{\Phi}_{W \hat{W}} \equiv \frac{1}{M} \sum_{m=1}^M |m\rangle \langle m|_W \otimes |m\rangle \langle m|_{\hat{W}}$ is the maximally classically correlated state. Note that $\frac{1}{2} \|\bar{\Phi}_{W \hat{W}} - \rho_{W \hat{W}}\|_1 = \Pr\{\hat{W} \neq W\}$, where W here denotes the uniform random variable corresponding to the message choice and \hat{W} denotes the random variable corresponding to the classical value in the register \hat{W} of the state $\rho_{W \hat{W}}$. Furthermore, the following energy constraint applies as well:

$$\text{Tr}\{H \bar{\omega}_A\} \leq E, \quad \bar{\omega}_A \equiv \frac{1}{n} \sum_{i=1}^n \omega_{A'_i}^{(i)}, \quad (10)$$

which limits the energy of the average input state.

III. PURIFIED PROTOCOL

Our goal is to bound the rate of such a protocol. With this in mind, we can devise a protocol that simulates the above one. It consists of purifying each step of the above protocol and Bob keeping a copy of the classical feedback, such that at each time step, conditioned on the value of the message in W and the feedback in the existing systems labeled by F , the state is pure. To be clear, we go through the steps of the purified protocol. In order to simplify notation, we let \hat{A} be a joint system throughout, referring to both the original A'

system as well as a purifying system, and we take the same convention for \hat{B} . The initial state of Alice is as follows:

$$\rho_{W\hat{A}_0} \equiv \frac{1}{M} \sum_{m=1}^M |m\rangle\langle m|_W \otimes \psi_{\hat{A}_0}^m, \quad (11)$$

where $\psi_{\hat{A}_0}^m$ is a purification of $\rho_{\hat{A}_0}^m$, such that tracing over a subsystem of $\psi_{\hat{A}_0}^m$ gives $\rho_{\hat{A}_0}^m$. The initial state of Bob is as follows:

$$\sigma_{F_0 F'_0 \hat{B}_1} \equiv \sum_{f_0} p(f_0) |f_0\rangle\langle f_0|_{F_0} \otimes |f_0\rangle\langle f_0|_{F'_0} \otimes \varphi_{\hat{B}_1}^{f_0}, \quad (12)$$

where $\varphi_{\hat{B}_1}^{f_0}$ is a purification of $\sigma_{F'_0}^{f_0}$, such that tracing over a subsystem of $\varphi_{\hat{B}_1}^{f_0}$ gives $\sigma_{F'_0}^{f_0}$, and he keeps an extra copy F'_0 of the classical data. Let $\mathcal{U}_{\hat{A}_{i-1}F_{i-1} \rightarrow \hat{A}_i A_i}$ denote an isometric channel extending the encoding channel $\mathcal{E}_{\hat{A}_{i-1}F_{i-1} \rightarrow A'_i A_i}$, for $i \in \{1, \dots, n\}$. Since the system F_i is classical, for $i \in \{1, \dots, n-1\}$, the decoding channel $\mathcal{D}_{B_i B'_i \rightarrow F_i B'_{i+1}}$ can be written explicitly as

$$\mathcal{D}_{B_i B'_i \rightarrow F_i B'_{i+1}} = \sum_{f_i} \mathcal{D}_{B_i B'_i \rightarrow B'_{i+1}}^{i, f_i} \otimes |f_i\rangle\langle f_i|_{F_i}, \quad (13)$$

such that $\{\mathcal{D}_{B_i B'_i \rightarrow B'_{i+1}}^{i, f_i}\}_{f_i}$ is a collection of completely positive maps such that the sum map $\sum_{f_i} \mathcal{D}_{B_i B'_i \rightarrow B'_{i+1}}^{i, f_i}$ is trace preserving. Let $V_{B_i \hat{B}_i \rightarrow \hat{B}_{i+1}}^{i, f_i}$ be a linear map such that tracing over a subsystem of $V_{B_i \hat{B}_i \rightarrow \hat{B}_{i+1}}^{i, f_i}(\cdot)[V_{B_i \hat{B}_i \rightarrow \hat{B}_{i+1}}^{i, f_i}]^\dagger$ gives the original map $\mathcal{D}_{B_i B'_i \rightarrow B'_{i+1}}^{i, f_i}$, and define the map

$$\mathcal{V}_{B_i \hat{B}_i \rightarrow \hat{B}_{i+1}}^{i, f_i}(\tau_{B_i \hat{B}_i}) \equiv V_{B_i \hat{B}_i \rightarrow \hat{B}_{i+1}}^{i, f_i} \tau_{B_i \hat{B}_i} [V_{B_i \hat{B}_i \rightarrow \hat{B}_{i+1}}^{i, f_i}]^\dagger.$$

Then we define the enlarged decoding channel $\mathcal{V}_{B_i \hat{B}_i \rightarrow F_i \hat{B}_{i+1} F'_i}^i$ as

$$\mathcal{V}_{B_i \hat{B}_i \rightarrow F_i \hat{B}_{i+1} F'_i}^i \equiv \sum_{f_i} \mathcal{V}_{B_i \hat{B}_i \rightarrow \hat{B}_{i+1}}^{i, f_i} \otimes |f_i\rangle\langle f_i|_{F_i} \otimes |f_i\rangle\langle f_i|_{F'_i}.$$

Note that this enlarged decoding channel keeps an extra copy of the classical feedback value for Bob in the register F'_i . The final decoding channel in the original protocol is equivalent to a measurement channel, and thus can be written as

$$\mathcal{D}_{B_n B'_n \rightarrow \hat{W}}^n(\tau_{B_n B'_n}) = \sum_w \text{Tr}\{\Lambda_{B_n B'_n}^w \tau_{B_n B'_n}\} |w\rangle\langle w|_{\hat{W}},$$

where $\{\Lambda_{B_n B'_n}^w\}_w$ is a POVM. We enlarge it as follows in the simulation protocol:

$$\mathcal{V}_{B_n \hat{B}_n \rightarrow \hat{B}_{n+1} \hat{W}}^n(\tau_{B_n \hat{B}_n}) = \sum_w \sqrt{\Lambda_{B_n B'_n}^w} \tau_{B_n \hat{B}_n} \sqrt{\Lambda_{B_n B'_n}^w} \otimes |w\rangle\langle w|_{\hat{W}}, \quad (14)$$

where the meaning of the notation is that the map $\sqrt{\Lambda_{B_n B'_n}^w}(\cdot)\sqrt{\Lambda_{B_n B'_n}^w}$ acts nontrivially on the subsystems $B_n B'_n$ in the original protocol and trivially on all other B subsystems, while mapping all B systems to a system \hat{B}_{n+1} large enough to accommodate all of them. In the

simulation protocol, we also consider an isometric channel $\mathcal{U}_{A \rightarrow BE}^{\mathcal{N}}$ that simulates the original channel $\mathcal{N}_{A \rightarrow B}$ as follows: $\mathcal{N}_{A \rightarrow B} = \text{Tr}_E \circ \mathcal{U}_{A \rightarrow BE}^{\mathcal{N}}$.

Thus, the various states involved in the purified protocol are as follows. The global initial state is $\rho_{W\hat{A}_0} \otimes \sigma_{F_0 F'_0 \hat{B}_1}$. Alice performs the enlarged encoding channel $\mathcal{U}_{\hat{A}_0 F_0 \rightarrow \hat{A}_1 A_1}^1$ and the state becomes as follows:

$$\omega_{W\hat{A}_1 A_1 \hat{B}_1 F'_0}^{(1)} \equiv \mathcal{U}_{\hat{A}_0 F_0 \rightarrow \hat{A}_1 A_1}^1 (\rho_{W\hat{A}_0} \otimes \sigma_{F_0 F'_0 \hat{B}_1}). \quad (15)$$

Alice transmits the A_1 system through the first use of the extended channel $\mathcal{U}_{A_1 \rightarrow B_1 E_1}^{\mathcal{N}}$, resulting in the following state:

$$\rho_{W\hat{A}_1 B_1 \hat{B}_1 E_1 F'_0}^{(1)} = \mathcal{U}_{A_1 \rightarrow B_1 E_1}^{\mathcal{N}} (\omega_{W\hat{A}_1 A_1 \hat{B}_1 F'_0}^{(1)}). \quad (16)$$

Bob processes his systems $B_1 \hat{B}_1$ with the enlarged decoding channel $\mathcal{V}_{B_1 \hat{B}_1 \rightarrow F_1 F'_1 \hat{B}_2}^1$ and Alice acts with the enlarged encoding channel $\mathcal{U}_{\hat{A}_1 F_1 \rightarrow \hat{A}_2 A_2}^2$, resulting in the state $\omega_{W\hat{A}_2 A_2 \hat{B}_2 E_1 F'_0 F'_1}^{(2)} \equiv (\mathcal{U}_{\hat{A}_1 F_1 \rightarrow \hat{A}_2 A_2}^2 \circ \mathcal{V}_{B_1 \hat{B}_1 \rightarrow F_1 F'_1 \hat{B}_2}^1)(\rho_{W\hat{A}_1 B_1 \hat{B}_1 E_1 F'_0}^{(1)})$. This process iterates $n-2$ more times, resulting in the following states:

$$\begin{aligned} \rho_{W\hat{A}_i B_i \hat{B}_i E_i^i [F_0^{i-1}]'} &\equiv \mathcal{U}_{A_i \rightarrow B_i E_i}^{\mathcal{N}} (\omega_{W\hat{A}_i A_i \hat{B}_i E_i^{i-1} [F_0^{i-1}]'}^{(i)}), \\ \omega_{W\hat{A}_{i+1} A_{i+1} \hat{B}_{i+1} E_i^i [F_0^i]'} &\equiv \\ (\mathcal{U}_{\hat{A}_i F_i \rightarrow \hat{A}_{i+1} A_{i+1}}^{i+1} \circ \mathcal{V}_{B_i \hat{B}_i \rightarrow F_i F'_i \hat{B}_{i+1}}^i) &(\rho_{W\hat{A}_i B_i \hat{B}_i E_i^i [F_0^{i-1}]'}^{(i)}), \end{aligned}$$

for $i \in \{2, \dots, n-1\}$. The final enlarged decoding channel $\mathcal{V}_{B_n \hat{B}_n \rightarrow \hat{B}_{n+1} \hat{W}}^n$ results in the following state: $\rho_{W\hat{A}_n \hat{B}_n \hat{B}_{n+1} \hat{W} E_1^n [F_0^{n-1}]'} \equiv \mathcal{V}_{B_n \hat{B}_n \rightarrow \hat{B}_{n+1} \hat{W}}^n (\rho_{W\hat{A}_n B_n \hat{B}_n E_1^n [F_0^{n-1}]'}^{(n)})$. Note that we recover each state of the original protocol from Section II by performing particular partial traces.

IV. INFORMATION MEASURE FOR ANALYSIS OF PROTOCOL

The key information measure that we use to analyze this protocol is as follows:

$$I(W; CF)_\tau + S(C|WF)_\tau, \quad (17)$$

where τ_{WFC} is a classical-quantum state of the form

$$\tau_{WFC} = \sum_{w,f} p(w, f) |w\rangle\langle w|_W \otimes |f\rangle\langle f|_F \otimes \tau_C^{w,f}. \quad (18)$$

The first term in (17) represents the classical correlation between system W and systems CF , while the second term represents the average entanglement between the system C of the state $\tau_C^{w,f}$ and a purifying reference system.

We now establish some properties of the information measure in (17). Let us first recall the following lemma from [19]:

Lemma 1: Let ϕ_{AB} be a pure bipartite state, and let $\{p(x), \varphi_{A'B'}^x\}$ be an ensemble of pure bipartite states obtained from ϕ_{AB} by means of a 1W-LOCC channel of the form

$$\sum_x \mathcal{U}_{A \rightarrow A'}^x \otimes \mathcal{V}_{B \rightarrow B'}^x \otimes |x\rangle\langle x|_X, \quad (19)$$

where $\{\mathcal{V}_{B \rightarrow B'}^x\}_x$ is a collection of completely positive trace non-increasing maps with $\mathcal{V}_{B \rightarrow B'}^x(\cdot) = V_{B \rightarrow B'}^x(\cdot)[V_{B \rightarrow B'}^x]^\dagger$ and $\{\mathcal{U}_{A \rightarrow A'}^x\}_x$ is a collection of isometric channels, so that

$$\varphi_{A'B'}^x \equiv \frac{1}{p(x)} (\mathcal{U}_{A \rightarrow A'}^x \otimes \mathcal{V}_{B \rightarrow B'}^x)(\phi_{AB}), \quad (20)$$

$$p(x) \equiv \text{Tr}\{(\mathcal{U}_{A \rightarrow A'}^x \otimes \mathcal{V}_{B \rightarrow B'}^x)(\phi_{AB})\}. \quad (21)$$

Then the following inequality holds $S(B)_\phi \geq S(B'|X)_\tau$, for $\tau_{XA'B'} \equiv \sum_x p(x)|x\rangle\langle x|_X \otimes \varphi_{A'B'}^x$.

The above lemma leads to the following one, which is the statement that the quantity in (17) is monotone with respect to 1W-LOCC channels:

Lemma 2: Let τ_{WFAB} be a classical–quantum state, with classical systems WF and quantum systems AB pure when conditioned on WF , and let $\mathcal{M}_{AB \rightarrow A'B'X}$ be a 1W-LOCC channel of the form in (19). Then the following holds

$$I(W; BF)_\tau + S(B|WF)_\tau \geq I(W; B'FX)_\theta + S(B'|WF)_\theta,$$

where $\theta_{WFA'B'X} \equiv \mathcal{M}_{AB \rightarrow A'B'X}(\tau_{WFAB})$.

Proof. The inequality $I(W; BF)_\tau \geq I(W; B'FX)_\theta$ holds from data processing. In more detail, consider that $\theta_{WFB'X}$ is equal to

$$\begin{aligned} &= \text{Tr}_{A'} \left\{ \sum_x (\mathcal{U}_{A \rightarrow A'}^x \otimes \mathcal{V}_{B \rightarrow B'}^x)(\tau_{WFAB}) \otimes |x\rangle\langle x|_X \right\} \\ &= \sum_x \mathcal{V}_{B \rightarrow B'}^x(\tau_{WFB}) \otimes |x\rangle\langle x|_X, \end{aligned} \quad (22)$$

where the last equality follows because each map $\mathcal{U}_{A \rightarrow A'}^x$ is trace preserving. So the state $\theta_{WFB'X}$ can be understood as arising from the action of the quantum instrument $\sum_x \mathcal{V}_{B \rightarrow B'}^x \otimes |x\rangle\langle x|_X$ on the state τ_{WFB} , and since this is a channel from B to $B'X$, the data processing inequality applies so that $I(W; BF)_\tau \geq I(W; B'FX)_\theta$. The inequality $S(B|WF)_\tau \geq S(B'|WF)_\theta$ follows from an application of Lemma 1, by conditioning on the classical systems WF . ■

The following lemma places an entropic upper bound on the amount by which the quantity in (17) can increase by the action of a channel $\mathcal{N}_{A \rightarrow B}$:

Lemma 3: Let $\tau_{WFAB'}$ be a classical–quantum state of the following form:

$$\tau_{WFAB'} = \sum_{w,f} p(w,f) |w\rangle\langle w|_W \otimes |f\rangle\langle f|_F \otimes \tau_{AB'}^{w,f}. \quad (23)$$

Then

$$\begin{aligned} &I(W; BB'F)_\omega + S(BB'|WF)_\omega \\ &\quad - [I(W; B'F)_\tau + S(B'|WF)_\tau] \leq S(B)_\omega, \end{aligned} \quad (24)$$

where $\omega_{WFB'B'} \equiv \mathcal{N}_{A \rightarrow B}(\tau_{WFAB'})$.

Proof. Consider that

$$\begin{aligned} &I(W; BB'F)_\omega + S(BB'|WF)_\omega \\ &\quad - [I(W; B'F)_\tau + S(B'|WF)_\tau] \\ &= I(W; BB'F)_\omega + S(BB'|WF)_\omega \\ &\quad - [I(W; B'F)_\omega + S(B'|WF)_\omega] \end{aligned} \quad (25)$$

$$= I(W; B|B'F)_\omega + S(B|B'WF)_\omega \quad (26)$$

$$= S(B|B'F)_\omega - S(B|B'WF)_\omega + S(B|B'WF)_\omega \quad (27)$$

$$= S(B|B'F)_\omega \leq S(B)_\omega. \quad (28)$$

All inequalities follow from definitions and applying chain rules for mutual information and entropy. The final inequality follows because conditioning does not increase entropy. ■

V. MAXIMUM OUTPUT ENTROPY BOUND

Now that we have identified a quantity that does not increase under 1W-LOCC from Bob to Alice and cannot increase by more than the output entropy of a channel under its action, we can use these properties to establish the following upper bound on the rate of a feedback-assisted communication protocol:

Theorem 4: For an $(n, M, H, E, \varepsilon)$ protocol for classical communication over a quantum channel $\mathcal{N}_{A \rightarrow B}$ assisted by classical feedback, of the form described in Section II, the following bound applies

$$(1 - \varepsilon) \log_2 M \leq n \cdot \sup_{\rho: \text{Tr}\{H\rho\} \leq E} S(\mathcal{N}(\rho)) + h_2(\varepsilon). \quad (29)$$

Proof. Let us consider the purified simulation of a given $(n, M, H, E, \varepsilon)$ protocol, as given in Section III. We start with

$$\log_2 M = I(W; \hat{W})_{\overline{\Phi}} \quad (30)$$

$$\leq I(W; \hat{W})_\rho + \varepsilon \log_2 M + h_2(\varepsilon), \quad (31)$$

where we have applied the condition in (9) and standard entropy inequalities. Continuing, we find that

$$\begin{aligned} &I(W; \hat{W})_\rho \\ &\leq I(W; B_n \hat{B}_n [F_0^{n-1}]')_{\rho^{(n)}} + S(B_n \hat{B}_n [[F_0^{n-1}]'] W)_{\rho^{(n)}} \end{aligned} \quad (32)$$

$$\begin{aligned} &= I(W; B_n \hat{B}_n [F_0^{n-1}]')_{\rho^{(n)}} + S(B_n \hat{B}_n [[F_0^{n-1}]'] W)_{\rho^{(n)}} \\ &\quad - [I(W; \hat{B}_1 F'_0)_{\omega^{(1)}} + S(\hat{B}_1 | F'_0 W)_{\omega^{(1)}}] \end{aligned} \quad (33)$$

$$\begin{aligned} &= I(W; B_n \hat{B}_n [F_0^{n-1}]')_{\rho^{(n)}} + S(B_n \hat{B}_n [[F_0^{n-1}]'] W)_{\rho^{(n)}} \\ &\quad - [I(W; \hat{B}_1 F'_0)_{\omega^{(1)}} + S(\hat{B}_1 | F'_0 W)_{\omega^{(1)}}] \end{aligned}$$

$$+ \sum_{i=2}^n I(W; \hat{B}_i [F_0^{i-1}]')_{\omega^{(i)}} + S(\hat{B}_i [[F_0^{i-1}]'] W)_{\omega^{(i)}}$$

$$- [I(W; \hat{B}_i [F_0^{i-1}]')_{\omega^{(i)}} + S(\hat{B}_i [[F_0^{i-1}]'] W)_{\omega^{(i)}}].$$

The first inequality follows from data processing and non-negativity of entropy. The first equality follows because $I(W; \hat{B}_1 F'_0)_{\omega^{(1)}} + S(\hat{B}_1 | F'_0 W)_{\omega^{(1)}} = 0$ for the initial state $\omega_{W \hat{A}_1 A_1 \hat{B}_1 F'_0}^{(1)}$ (there is no classical correlation between W and $\hat{B}_1 F'_0$, and the state on system \hat{B}_1 is pure when conditioned on $F'_0 W$). The last equality follows by adding and subtracting the same term. Continuing, we find that the quantity in the last line above is bounded as

$$\begin{aligned} &\leq I(W; B_n \hat{B}_n [F_0^{n-1}]')_{\rho^{(n)}} + S(B_n \hat{B}_n [[F_0^{n-1}]'] W)_{\rho^{(n)}} \\ &\quad - [I(W; \hat{B}_1 F'_0)_{\omega^{(1)}} + S(\hat{B}_1 | F'_0 W)_{\omega^{(1)}}] \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=2}^n I(W; B_{i-1} \hat{B}_{i-1} [F_0^{i-2}]')_{\rho^{(i-1)}} \\
& \quad + S(B_{i-1} \hat{B}_{i-1} | [F_0^{i-2}]' W)_{\rho^{(i-1)}} \\
& \quad - \left[I(W; \hat{B}_i [F_0^{i-1}]')_{\omega^{(i)}} + S(\hat{B}_i | [F_0^{i-1}]' W)_{\omega^{(i)}} \right] \\
& = \sum_{i=1}^n I(W; B_i \hat{B}_i [F_0^{i-1}]')_{\rho^{(i)}} + S(B_i \hat{B}_i | [F_0^{i-1}]' W)_{\rho^{(i)}} \\
& \quad - \left[I(W; \hat{B}_i [F_0^{i-1}]')_{\omega^{(i)}} + S(\hat{B}_i | [F_0^{i-1}]' W)_{\omega^{(i)}} \right] \quad (34) \\
& \leq \sum_{i=1}^n S(B_i)_{\rho^{(i)}} \leq n S(B)_{\mathcal{N}(\bar{\omega})} \leq n \sup_{\rho: \text{Tr}\{H\rho\} \leq E} S(\mathcal{N}(\rho)). \quad (35)
\end{aligned}$$

The first inequality follows from Lemma 2. The first equality follows by collecting terms. The second inequality follows from Lemma 3. The third inequality follows from concavity of entropy and the definition of $\bar{\omega}_A$ in (10). The final inequality follows from the energy constraint in (10), and by optimizing over all input states that satisfy this energy constraint. By combining (30)–(31) and (32)–(35), we arrive at (29). ■

In Appendix A, we show how to extend this result to the maximum average output entropy:

Theorem 5: Let $\mathcal{N}_{A \rightarrow B} = \sum_x p_X(x) \mathcal{N}_{A \rightarrow B}^x$, where p_X is a probability distribution and $\{\mathcal{N}_{A \rightarrow B}^x\}_x$ is a set of channels. For an $(n, M, H, E, \varepsilon)$ protocol for classical communication over the channel $\mathcal{N}_{A \rightarrow B}$ assisted by classical feedback, of the form described in Section II, the following bound applies

$$(1 - \varepsilon) \log_2 M \leq n \cdot \sup_{\rho: \text{Tr}\{H\rho\} \leq E} \sum_x p_X(x) S(\mathcal{N}^x(\rho)) + h_2(\varepsilon).$$

VI. EXAMPLES

From the upper bound in Theorem 4, we conclude that the feedback-assisted capacity of a noiseless qudit channel of dimension d is $\log_2 d$.

Furthermore, consider a pure-loss bosonic channel [20] with transmissivity $\eta \in [0, 1]$. Taking the Hamiltonian as the photon number operator and energy constraint $N_S \geq 0$, it is known from [20] that this channel's energy constrained classical capacity and maximum output entropy are equal to $g(\eta N_S)$, where $g(x) \equiv (x + 1) \log_2(x + 1) - x \log_2 x$. Applying these results and Theorem 4, we conclude that classical feedback does not increase the energy-constrained classical capacity of the pure-loss bosonic channel.

A quantum erasure channel is defined as $\rho \rightarrow (1 - p)\rho + p|e\rangle\langle e|$ for $p \in [0, 1]$, where ρ is the state of a d -dimensional input system and $|e\rangle\langle e|$ is an erasure state orthogonal to all inputs. Applying Theorem 5, we conclude that the classical capacity of the erasure channel assisted by classical feedback is equal to $(1 - p) \log_2 d$, so that classical feedback does not increase the classical capacity of the erasure channel.

VII. CONCLUSION

Our main result is that the maximum average output entropy of a quantum channel is an upper bound on its classical capacity assisted by classical feedback. Note that the bound

is a weak converse bound. Going forward from here, it would be good to find strong converse and tighter bounds on the classical capacity assisted by classical feedback.

Acknowledgements. We acknowledge discussions with Xin Wang, Patrick Hayden, and Tsachy Weissman. DD is supported by a National Defense Science and Engineering Graduate Fellowship. YQ is supported by a Stanford Graduate Fellowship and a National University of Singapore Overseas Graduate Scholarship. PWS is supported by the NSF under Grant No. CCF-1525130 and through the NSF Science and Technology Center for Science of Information under Grant No. CCF0-939370. MMW acknowledges NSF grant no. 1350397. DD and MMW thank God for all His provisions.

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APPENDIX A
MAXIMUM AVERAGE OUTPUT ENTROPY BOUND FOR
PROBABILISTIC MIXTURE OF CHANNELS

In this appendix, we provide a simple proof of Theorem 5. The main idea behind the proof is to observe that any feedback-assisted protocol of the form discussed in Section II, which is for communication over a probabilistic mixture channel $\mathcal{N}_{A \rightarrow B} = \sum_z p_Z(z) \mathcal{N}_{A \rightarrow B}^z$, has a simulation of the following form:

- 1) Before the i th use of the channel $\mathcal{N}_{A \rightarrow B}$ in the feedback-assisted protocol, Bob selects a random variable Z_i independently according to the distribution p_Z . He transmits Z_i over the classical feedback channel to Alice.
- 2) Each channel use $\mathcal{N}_{A \rightarrow B}$ from the original protocol is replaced by a simulation in terms of another channel $\mathcal{M}_{AZ' \rightarrow B}$, which accepts a quantum input on system A and a classical input on system Z' . Conditioned on the value z in system Z' , the channel $\mathcal{M}_{AZ' \rightarrow B}$ applies $\mathcal{N}_{A \rightarrow B}^z$ to the quantum system A . Thus, if the random variable $Z \sim p_Z$ is fed into the input system Z' of $\mathcal{M}_{AZ' \rightarrow B}$, then the channel $\mathcal{M}_{AZ' \rightarrow B}$ is indistinguishable from the original channel $\mathcal{N}_{A \rightarrow B}$.
- 3) Alice feeds a copy of the classical random variable Z_i into the i th use of the channel $\mathcal{M}_{AZ' \rightarrow B}$.
- 4) All other aspects of the protocol are executed in the same way as before. Namely, even though it would be an advantage to Alice to modify her encodings and Bob to modify later decodings based on the realizations of Z_i , they do not do so, and they instead blindly operate all other aspects of the simulation protocol as they are in the original protocol.

Our goal now is to establish the inequality in Theorem 5, relating the n, M, E, ε parameters of the original $(n, M, H, E, \varepsilon)$ protocol by using the above simulation.

The main observation to make from here is that the same proof from Lemma 3 gives the following bound:

$$I(W; BB'FZ)_\omega + S(BB'|WFZ)_\omega - [I(W; B'FZ)_\tau + S(B'|WFZ)_\tau] \leq S(B|Z)_\omega, \quad (36)$$

where $\omega_{WFZBB'}$ is the following state:

$$\omega_{WFZBB'} \equiv \mathcal{M}_{AZ' \rightarrow B}(\tau_{WFZZ'AB'}) \quad (37)$$

$$\tau_{WFZZ'AB'} \equiv \sum_{w,f,z} p(w, f, z) |w, f, z, z\rangle \langle w, f, z, z|_{W,F,Z,Z'} \otimes \tau_{AB'}^{w,f,z}. \quad (38)$$

This follows by grouping Z with F , but then discarding only F and B' at the end of the proof. We then apply this bound, and the same reasoning in the proof of Theorem 4, except that the variables Z_0, \dots, Z_i are grouped together with the feedback variables $[F_0^{i-1}]'$ and then the same reasoning in (32)–(34) applies. At this point, we invoke (36) and find that

$$(1 - \varepsilon) \log_2 M \leq \sum_{i=1}^n S(B_i|Z_i)_{\rho^{(i)}} + h_2(\varepsilon). \quad (39)$$

We can then bound the sum over entropies as follows:

$$\sum_{i=1}^n S(B_i|Z_i)_{\rho^{(i)}} \leq nS(B|Z)_{\bar{\rho}} \quad (40)$$

$$= n \sum_z p_Z(z) S(\mathcal{N}^z(\bar{\omega})) \quad (41)$$

$$\leq n \sup_{\rho: \text{Tr}\{H\rho\} \leq E} \sum_z p_Z(z) S(\mathcal{N}^z(\rho)). \quad (42)$$

The first inequality is by concavity of conditional entropy, and the conditional entropy is defined on the averaged channel output state over uses $\bar{\rho}_{BZ} \equiv \sum_z p_Z(z) |z\rangle \langle z| \otimes \mathcal{N}^z(\bar{\omega})$, $\bar{\omega}_A = \frac{1}{n} \sum_{i=1}^n \omega_{A_i}^{(i)}$. The second equality is by definition of conditional entropy. The third inequality follows from optimizing over states that satisfy the energy constraint in (10). This concludes the proof of Theorem 5.