

9-1-2020

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### Recommended Citation

Wang, X., & Wilde, M. (2020).  $\alpha$  -logarithmic negativity. *Physical Review A*, 102 (3) <https://doi.org/10.1103/PhysRevA.102.032416>

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# $\alpha$ -Logarithmic negativity

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(Dated: September 25, 2020)*

The logarithmic negativity of a bipartite quantum state is a widely employed entanglement measure in quantum information theory, due to the fact that it is easy to compute and serves as an upper bound on distillable entanglement. More recently, the  $\kappa$ -entanglement of a bipartite state was shown to be the first entanglement measure that is both easily computable and has a precise information-theoretic meaning, being equal to the exact entanglement cost of a bipartite quantum state when the free operations are those that completely preserve the positivity of the partial transpose [Wang and Wilde, Phys. Rev. Lett. **125**(4):040502, July 2020]. In this paper, we provide a non-trivial link between these two entanglement measures, by showing that they are the extremes of an ordered family of  $\alpha$ -logarithmic negativity entanglement measures, each of which is identified by a parameter  $\alpha \in [1, \infty]$ . In this family, the original logarithmic negativity is recovered as the smallest with  $\alpha = 1$ , and the  $\kappa$ -entanglement is recovered as the largest with  $\alpha = \infty$ . We prove that the  $\alpha$ -logarithmic negativity satisfies the following properties: entanglement monotone, normalization, faithfulness, and subadditivity. We also prove that it is neither convex nor monogamous. Finally, we define the  $\alpha$ -logarithmic negativity of a quantum channel as a generalization of the notion for quantum states, and we show how to generalize many of the concepts to arbitrary resource theories.

## I. INTRODUCTION

Establishing a quantitative theory of entanglement has long been one of the central concerns of quantum information theory [1–4]. Starting with the original developments in [5], there now exists a large body of knowledge on this topic [6, 7].

The traditional approaches to quantifying entanglement are the axiomatic approach and the operational (information-theoretic or resource-theoretic) approach. In the axiomatic approach, one identifies a list of desirable properties that a measure of entanglement should possess. Most prominent among these is that a measure of entanglement should not increase under the action of a quantum channel realized by local operations and classical communication (LOCC); if a measure of entanglement satisfies this property, it is called an entanglement monotone. In the operational approach, one identifies a meaningful information-processing task involving entanglement as a resource, as well as some set of physical operations that are allowed for free. For example, one could fix the free operations to be LOCC [5], separable operations [8–10], or operations that completely preserve the positivity of the partial transpose (C-PPT-P operations) [11]. Examples of information-processing tasks include entanglement distillation [5], for which the goal is to use many copies of a quantum state and free op-

erations to produce as many high quality Bell states as possible. In the opposite task, known as entanglement dilution [5, 12], the goal is to use as few Bell states as possible, along with LOCC, to produce as many high-fidelity copies of a quantum state as possible.

One of the most well known measures of entanglement is the logarithmic negativity [13, 14], defined for a bipartite state  $\rho_{AB}$  as

$$E_N(\rho_{AB}) \equiv \log_2 \|T_B(\rho_{AB})\|_1, \quad (1)$$

where  $T_B$  is the partial transpose map [15], defined for an orthonormal basis  $\{|i\rangle_B\}_i$  as

$$T_B(Y_{AB}) \equiv \sum_{i,j} (I_A \otimes |i\rangle\langle j|_B) Y_{AB} (I_A \otimes |j\rangle\langle i|_B), \quad (2)$$

and  $\|X\|_1 \equiv \text{Tr}[\sqrt{X^\dagger X}]$  denotes the trace norm of an operator  $X$ . The logarithmic negativity satisfies a number of properties, the most important of which is that it is an entanglement monotone [13, 14]. The widespread use of the logarithmic negativity is due to the ease with which it can be computed, and the fact that it provides an upper bound on the distillable entanglement of a bipartite state [13, 16]. In this context, it should be mentioned that an entanglement measure alternative to the logarithmic negativity was recently proposed in [17] (also known as the max-Rains relative entropy [18]): it is easy to compute via semi-definite programming, it satisfies all of the properties that the logarithmic negativity does, and it provides a generally tighter upper bound on distillable entanglement.

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Another entanglement measure proposed in recent work is the  $\kappa$ -entanglement of a quantum state [19, 20], defined as follows:

$$E_\kappa(\rho_{AB}) \equiv \log_2 \inf\{\text{Tr}[S_{AB}] : -T_B(S_{AB}) \leq T_B(\rho_{AB}) \leq T_B(S_{AB}), S_{AB} \geq 0\}. \quad (3)$$

The  $\kappa$ -entanglement can be computed easily by semi-definite programming [19, 20]. In the context of the resource theory of entanglement, the  $\kappa$ -entanglement measure can be regarded as a breakthrough: not only is it easily computable and satisfies a number of desirable properties, but it also has a precise information-theoretic interpretation as the exact entanglement cost of a bipartite state  $\rho_{AB}$  when C-PPT-P operations are allowed for free (see [19, 20] for details and see [21, 22] for earlier work on this information-processing task). No other entanglement measure is known to have these properties, which makes  $\kappa$ -entanglement desirable from both the axiomatic and information-theoretic perspectives.

It is known that the  $\kappa$ -entanglement and logarithmic negativity coincide for two-qubit states and for bosonic Gaussian states [19, 20]. This reduction raises the question of whether there might be a deeper connection between the two measures. *In this paper, we show that this is indeed the case, by defining a whole family of ordered entanglement measures that interpolate between the logarithmic negativity, the smallest in the family, and the  $\kappa$ -entanglement, the largest in the family.* We call each member of the family the  $\alpha$ -logarithmic negativity, where  $\alpha$  is a parameter in the interval  $[1, \infty]$ . The  $\alpha$ -logarithmic negativities are ordered, and the usual logarithmic negativity is recovered when  $\alpha = 1$ , whereas the  $\kappa$ -entanglement is recovered when  $\alpha = \infty$ . In this sense, and following earlier traditions in quantum information theory [23–26], we can refer to the  $\kappa$ -entanglement alternatively as the *max-logarithmic negativity*.

Our results in this paper are related to the precedent of Rényi [27], who established an interpolating family of entropic measures based on a parameter  $\alpha$ . However, it should be noted that we cannot interpret  $\alpha$ -logarithmic negativity in terms of the traditional definitions of quantum Rényi relative entropies [28–30], for two reasons:

1. The normalization prefactor for  $\alpha$ -logarithmic negativity is different from that employed for quantum Rényi relative entropies.
2. More importantly, the quantum Rényi relative entropies are functions of quantum states, or more generally positive semi-definite operators, whereas the  $\alpha$ -logarithmic negativity is based on an information measure that is a function of a (not necessarily positive semi-definite) Hermitian operator.

The second reason given above for the difference between  $\alpha$ -logarithmic negativity and Rényi relative entropy is already apparent in the definition of the logarithmic negativity in (1), since the argument is  $T_B(\rho_{AB})$ , which is

a (not necessarily positive semi-definite) Hermitian operator. Nevertheless, we discuss connections between the  $\alpha$ -logarithmic negativity and sandwiched Rényi relative entropy [29, 30] in the case that one allows for the first argument of the sandwiched Rényi relative entropy to be a Hermitian operator.

We remark here that the technical development in our paper heavily relies on results from [31] and [32]. In particular, our proofs that the  $\alpha$ -logarithmic negativities are ordered and are entanglement monotones strongly rely on the methods of [31], which therein were only applied to quantum states or positive semi-definite operators. Here, we simply observe that the methods of [31] apply when one of the operators is Hermitian, not unlike how the authors of [33] observed that the methods of [31] apply more generally to positive trace-preserving maps (rather than just the strict subset of completely positive trace-preserving maps). Also, in order to establish that the  $\alpha$ -logarithmic negativity can be computed by convex optimization, we rely on a general theorem established in [32]. Thus, given the benefits of the methods of [31] and [32] for establishing entanglement measures in quantum information, it seems fruitful to continue the mathematical physics directions considered in [31] and [32] for future work, as has already been done in several papers [34–37].

We note here that other papers [38–40] have pursued various generalizations of logarithmic negativity, but their status as entanglement monotones remains unclear.

In the rest of the paper, we provide a detailed exposition of the  $\alpha$ -logarithmic negativity. In particular, we define it in Section II. In Section III, we detail several properties of the quantities underlying  $\alpha$ -logarithmic negativity, and in Section IV, we prove that the  $\alpha$ -logarithmic negativities are ordered (i.e., monotone increasing with respect to  $\alpha \in [1, \infty]$ ). Then in Section V, we prove that the logarithmic negativity and the  $\kappa$ -entanglement are special cases of the  $\alpha$ -logarithmic negativity when  $\alpha = 1$  and  $\alpha = \infty$ , respectively. In Section VI, we establish several properties of the  $\alpha$ -logarithmic negativity, including normalization, faithfulness, subadditivity, and that it is an entanglement monotone. We also prove that it is neither convex nor monogamous. In Section VII, we define the  $\alpha$ -logarithmic negativity of a quantum channel as a generalization of the measure for states. Therein, we also discuss other generalizations of the  $\alpha$ -logarithmic negativity. We finally conclude in Section VIII with a brief summary and some open questions.

## II. $\alpha$ -LOGARITHMIC NEGATIVITY

In this section, we define the  $\alpha$ -logarithmic negativity of a bipartite state  $\rho_{AB}$ .

First, let us define the following quantities, which are functions of  $\alpha \geq 1$ , a Hermitian operator  $X \neq 0$ , and a

positive semi-definite operator  $\sigma \neq 0$ :

$$\mu_\alpha(X\|\sigma) \equiv \begin{cases} \left\| \sigma^{\frac{1-\alpha}{2\alpha}} X \sigma^{\frac{1-\alpha}{2\alpha}} \right\|_\alpha & \text{if } \text{supp}(X) \subseteq \text{supp}(\sigma) \\ +\infty & \text{else} \end{cases}, \quad (4)$$

$$\nu_\alpha(X\|\sigma) \equiv \log_2 \mu_\alpha(X\|\sigma), \quad (5)$$

where the  $\alpha$ -norm of an operator  $Y$  is defined for  $\alpha \geq 1$  as

$$\|Y\|_\alpha \equiv (\text{Tr}[|Y|^\alpha])^{1/\alpha}, \quad (6)$$

$$|Y| \equiv \sqrt{Y^\dagger Y}, \quad (7)$$

and the inverse  $\sigma^{(1-\alpha)/2\alpha}$  is understood in the generalized sense (i.e., taken on the support of  $\sigma$ ). The definition in (4) is consistent with the following limit:

$$\mu_\alpha(X\|\sigma) = \lim_{\varepsilon \searrow 0} \mu_\alpha(X\|(1-\varepsilon)\sigma + \varepsilon\theta), \quad (8)$$

where  $\theta$  is a positive definite state. A proof for the equality in (8) follows the same steps given in the proof of [29, Lemma 13].

We define

$$\mu_\infty(X\|\sigma) \equiv \lim_{\alpha \rightarrow \infty} \mu_\alpha(X\|\sigma) \quad (9)$$

$$= \left\| \sigma^{-1/2} X \sigma^{-1/2} \right\|_\infty \quad (10)$$

$$= \inf \{ \lambda : -\lambda\sigma \leq X \leq \lambda\sigma \}, \quad (11)$$

and

$$\nu_\infty(X\|\sigma) \equiv \log_2 \mu_\infty(X\|\sigma). \quad (12)$$

Both of the above formulas are defined as above in the case that  $\text{supp}(X) \subseteq \text{supp}(\sigma)$ , and  $\mu_\infty(X\|\sigma)$  and  $\nu_\infty(X\|\sigma)$  are set to  $+\infty$  otherwise.

The function  $\nu_\alpha(X\|\sigma)$  is related to the sandwiched Rényi relative entropy  $\tilde{D}_\alpha(X\|\sigma)$  [29, 30] of a Hermitian operator  $X \neq 0$  and a positive semi-definite operator  $\sigma \neq 0$  as follows:

$$\tilde{D}_\alpha(X\|\sigma) = \frac{\alpha}{\alpha-1} \nu_\alpha(X\|\sigma). \quad (13)$$

Also, we have that

$$\nu_\infty(X\|\sigma) = D_{\max}(X\|\sigma) \quad (14)$$

$$\equiv \log_2 \left\| \sigma^{-1/2} X \sigma^{-1/2} \right\|_\infty \quad (15)$$

$$= \log \inf \{ \lambda : -\lambda\sigma \leq X \leq \lambda\sigma \}, \quad (16)$$

where  $D_{\max}(X\|\sigma)$  is the max-relative entropy [24] of a Hermitian operator  $X \neq 0$  and a positive semi-definite operator  $\sigma \neq 0$ . Note that  $D_{\max}(X\|\sigma) = +\infty$  if  $\text{supp}(X) \not\subseteq \text{supp}(\sigma)$ . We also note here that both the sandwiched Rényi relative entropy and max-relative entropy have only been considered in prior work when  $X$  is positive semi-definite, and so (13) and (14) represent strict generalizations of the previously considered definitions.

We are now ready to define the  $\alpha$ -logarithmic negativity of a bipartite state as follows:

**Definition 1 ( $\alpha$ -logarithmic negativity)** Let  $\rho_{AB}$  be a bipartite state. We define its  $\alpha$ -logarithmic negativity as

$$E_N^\alpha(\rho_{AB}) \equiv \inf_{\sigma_{AB} \in \text{PPT}(A:B)} \nu_\alpha(T_B(\rho_{AB})\|\sigma_{AB}), \quad (17)$$

where  $T_B$  is the partial transpose map, defined in (2), and the set  $\text{PPT}(A:B)$  is the set of positive partial transpose states, defined as [41–44]

$$\text{PPT}(A:B) \equiv \{ \sigma_{AB} : \sigma_{AB} \geq 0, T_B(\sigma_{AB}) \geq 0, \text{Tr}[\sigma_{AB}] = 1 \}. \quad (18)$$

It can be useful as a proof tool to take the infimum in Definition 1 over only the faithful states in  $\text{PPT}(A:B)$ , and the following lemma establishes that this is possible:

**Lemma 1** The following identity holds

$$E_N^\alpha(\rho_{AB}) = \inf_{\sigma_{AB} \in \text{PPT}_{\text{inv}}(A:B)} \nu_\alpha(T_B(\rho_{AB})\|\sigma_{AB}), \quad (19)$$

where  $\text{PPT}_{\text{inv}}(A:B)$  consists of the faithful states in  $\text{PPT}(A:B)$ :

$$\text{PPT}_{\text{inv}}(A:B) \equiv \{ \sigma_{AB} : \sigma_{AB} > 0, T_B(\sigma_{AB}) > 0, \text{Tr}[\sigma_{AB}] = 1 \}. \quad (20)$$

**Proof.** This follows because the set  $\text{PPT}_{\text{inv}}(A:B)$  is dense in  $\text{PPT}(A:B)$  and the  $\alpha$ -norm is continuous, so that the  $\alpha$ -norm of any state in  $\text{PPT}(A:B)$  can be approximated arbitrarily well by that of a state in  $\text{PPT}_{\text{inv}}(A:B)$ . ■

### III. PROPERTIES OF THE FUNCTIONS

#### $\mu_\alpha(X\|\sigma)$ AND $\nu_\alpha(X\|\sigma)$

In this section, we establish several properties of the functions  $\mu_\alpha(X\|\sigma)$  and  $\nu_\alpha(X\|\sigma)$  that are used in the rest of the paper. We also employ the following abbreviations: CP stands for completely positive, TP stands for trace-preserving, and CPTP stands for completely positive and trace preserving.

Throughout the section,  $X \neq 0$  is a Hermitian operator and  $\sigma$  is a positive definite operator. Following [31, 45], for  $p \geq 1$ , we define the following norm

$$\|X\|_{p,\sigma} \equiv \left\| \sigma^{1/2p} X \sigma^{1/2p} \right\|_p. \quad (21)$$

Note that

$$\|X\|_{\infty,\sigma} \equiv \lim_{p \rightarrow \infty} \|X\|_{p,\sigma} = \|X\|_\infty. \quad (22)$$

We also define the following CP map [31]:

$$\Gamma_\sigma(X) \equiv \sigma^{1/2} X \sigma^{1/2}, \quad (23)$$

so that

$$\Gamma_\sigma^{-1}(X) = \sigma^{-1/2} X \sigma^{-1/2}. \quad (24)$$

**Lemma 2** Let  $X \neq 0$  be a Hermitian operator, and let  $\sigma$  be a positive definite operator. Let  $\mathcal{P}$  be a positive and trace-non-increasing map. Then the following inequality holds for all  $\alpha \geq 1$ :

$$\nu_\alpha(X\|\sigma) \geq \nu_\alpha(\mathcal{P}(X)\|\mathcal{P}(\sigma)). \quad (25)$$

**Proof.** This follows as a direct consequence of the reasoning given in [31, Theorem 6] (see also [33, Theorem 2]). We repeat the argument here for completeness.

Since  $\alpha \geq 1$  and the logarithm function is monotone increasing, (25) is equivalent to

$$\|\Gamma_\sigma^{-1}(X)\|_{\alpha,\sigma} \geq \|\Gamma_{\mathcal{P}(\sigma)}^{-1}(\mathcal{P}(X))\|_{\alpha,\mathcal{P}(\sigma)}. \quad (26)$$

Observe that

$$\Gamma_{\mathcal{P}(\sigma)}^{-1}(\mathcal{P}(X)) = (\Gamma_{\mathcal{P}(\sigma)}^{-1} \circ \mathcal{P} \circ \Gamma_\sigma)(\Gamma_\sigma^{-1}(X)). \quad (27)$$

As a result,

$$\begin{aligned} & \left\| \Gamma_{\mathcal{P}(\sigma)}^{-1}(\mathcal{P}(X)) \right\|_{\alpha,\mathcal{P}(\sigma)} \\ & \leq \left\| \Gamma_{\mathcal{P}(\sigma)}^{-1} \circ \mathcal{P} \circ \Gamma_\sigma \right\|_{(\alpha,\sigma) \rightarrow (\alpha,\mathcal{P}(\sigma))} \cdot \|\Gamma_\sigma^{-1}(X)\|_{\alpha,\sigma}, \end{aligned} \quad (28)$$

where, for a map  $\mathcal{M}$ ,

$$\|\mathcal{M}\|_{(\alpha,\sigma) \rightarrow (\alpha',\sigma')} \equiv \sup_{Y \neq 0} \frac{\|\mathcal{M}(Y)\|_{\alpha',\sigma'}}{\|Y\|_{\alpha,\sigma}}. \quad (29)$$

We also define

$$\|\mathcal{M}\|_{\alpha \rightarrow \alpha'} \equiv \|\mathcal{M}\|_{(\alpha,I) \rightarrow (\alpha',I)}. \quad (30)$$

So it suffices to establish that

$$\left\| \Gamma_{\mathcal{P}(\sigma)}^{-1} \circ \mathcal{P} \circ \Gamma_\sigma \right\|_{(\alpha,\sigma) \rightarrow (\alpha,\mathcal{P}(\sigma))} \leq 1. \quad (31)$$

By employing [31, Theorem 4], it is only necessary to establish (31) for  $\alpha = 1$  and  $\alpha = \infty$ .

For  $\alpha = 1$ , it follows that

$$\left\| \Gamma_{\mathcal{P}(\sigma)}^{-1} \circ \mathcal{P} \circ \Gamma_\sigma \right\|_{(1,\sigma) \rightarrow (1,\mathcal{P}(\sigma))} = \|\mathcal{P}\|_{1 \rightarrow 1} \leq 1 \quad (32)$$

because  $\mathcal{P}$  is positive and trace non-increasing. For  $\alpha = \infty$ , it follows that

$$\begin{aligned} & \left\| \Gamma_{\mathcal{P}(\sigma)}^{-1} \circ \mathcal{P} \circ \Gamma_\sigma \right\|_{(\infty,\sigma) \rightarrow (\infty,\mathcal{P}(\sigma))} \\ & = \left\| \Gamma_{\mathcal{P}(\sigma)}^{-1} \circ \mathcal{P} \circ \Gamma_\sigma \right\|_{\infty \rightarrow \infty}. \end{aligned} \quad (33)$$

Since  $\Gamma_{\mathcal{P}(\sigma)}^{-1} \circ \mathcal{P} \circ \Gamma_\sigma$  is a positive map, by the Russo–Dye theorem (Corollary 2.3.8 of [46]), we have that

$$\begin{aligned} & \left\| \Gamma_{\mathcal{P}(\sigma)}^{-1} \circ \mathcal{P} \circ \Gamma_\sigma \right\|_{\infty \rightarrow \infty} \\ & = \left\| (\Gamma_{\mathcal{P}(\sigma)}^{-1} \circ \mathcal{P} \circ \Gamma_\sigma)(I) \right\|_{\infty} \end{aligned} \quad (34)$$

$$= \left\| (\Gamma_{\mathcal{P}(\sigma)}^{-1} \circ \mathcal{P})(\sigma) \right\|_{\infty} \quad (35)$$

$$= \left\| (\mathcal{P}(\sigma))^{-1/2} \mathcal{P}(\sigma) (\mathcal{P}(\sigma))^{-1/2} \right\|_{\infty} \quad (36)$$

$$= 1, \quad (37)$$

concluding the proof. ■

**Lemma 3** Let

$$Y_{XB} \equiv \sum_x p(x) |x\rangle\langle x|_X \otimes Y_B^x, \quad (38)$$

$$\sigma_{XB} \equiv \sum_x q(x) |x\rangle\langle x|_X \otimes \sigma_B^x, \quad (39)$$

where  $\{Y_B^x\}_x$  is a set of Hermitian operators such that  $Y_B^x \neq 0$  for all  $x$ ,  $\{p(x)\}_x$  is a probability distribution,  $\{\sigma_B^x\}$  is a set of positive definite operators, and  $\{q(x)\}_x$  is a set of strictly positive reals. Then for  $\alpha \geq 1$ , we have that

$$\begin{aligned} \nu_\alpha(Y_{XB}\|\sigma_{XB}) & \geq \sum_x p(x) \nu_\alpha(Y_B^x\|\sigma_B^x) \\ & \quad + \left( \frac{\alpha - 1}{\alpha} \right) D(p\|q), \end{aligned} \quad (40)$$

where  $D(p\|q) := \sum_x p(x) \log_2(p(x)/q(x))$  is the classical relative entropy.

**Proof.** The case  $\alpha = 1$  follows directly because

$$\nu_{\alpha=1}(Y_{XB}\|\sigma_{XB}) = \log_2 \|Y_{XB}\|_1 \quad (41)$$

$$= \log_2 \left[ \sum_x p(x) \|Y_B^x\|_1 \right] \quad (42)$$

$$\geq \sum_x p(x) \log_2 \|Y_B^x\|_1 \quad (43)$$

$$= \sum_x p(x) \nu_{\alpha=1}(Y_B^x\|\sigma_B^x), \quad (44)$$

where the inequality follows from concavity of the logarithm.

For the case  $\alpha > 1$ , consider that

$$\begin{aligned} \nu_\alpha(Y_{XB}\|\sigma_{XB}) & = \log_2 \mu_\alpha(Y_{XB}\|\sigma_{XB}) \\ & = \frac{1}{\alpha} \log_2 [\mu_\alpha(Y_{XB}\|\sigma_{XB})]^\alpha \end{aligned} \quad (45)$$

$$= \frac{1}{\alpha} \log_2 \sum_x [\mu_\alpha(p(x)Y_B^x\|q(x)\sigma_B^x)]^\alpha \quad (46)$$

$$= \frac{1}{\alpha} \log_2 \sum_x p(x)^\alpha q(x)^{1-\alpha} [\mu_\alpha(Y_B^x\|\sigma_B^x)]^\alpha \quad (47)$$

$$= \frac{1}{\alpha} \log_2 \sum_x p(x)^\alpha q(x)^{1-\alpha} [\mu_\alpha(Y_B^x\|\sigma_B^x)]^\alpha \quad (48)$$

$$= \frac{1}{\alpha} \log_2 \sum_x p(x) \left( \frac{p(x)}{q(x)} \right)^{\alpha-1} [\mu_\alpha(Y_B^x\|\sigma_B^x)]^\alpha \quad (49)$$

$$\geq \frac{1}{\alpha} \sum_x p(x) \log_2 \left[ \left( \frac{p(x)}{q(x)} \right)^{\alpha-1} [\mu_\alpha(Y_B^x\|\sigma_B^x)]^\alpha \right] \quad (50)$$

$$\begin{aligned}
&= \left(\frac{\alpha-1}{\alpha}\right) \sum_x p(x) \log_2 \left(\frac{p(x)}{q(x)}\right) \\
&\quad + \frac{1}{\alpha} \sum_x p(x) \log_2 [\mu_\alpha(Y_B^x \|\sigma_B^x)]^\alpha \quad (51)
\end{aligned}$$

$$= \left(\frac{\alpha-1}{\alpha}\right) D(p\|q) + \sum_x p(x) \nu_\alpha(Y_B^x \|\sigma_B^x). \quad (52)$$

The third equality follows from definitions and the fact that  $\left\|\sum_y |y\rangle\langle y| \otimes B^y\right\|_\alpha^\alpha = \sum_y \|B^y\|_\alpha^\alpha$  for a set  $\{B^y\}_y$  of operators. The sole inequality is a consequence of concavity of the logarithm. ■

**Lemma 4** *Let  $X \neq 0$  be a Hermitian operator, and let  $\sigma$  be a positive definite operator. Then the following inequality holds for all  $\alpha \geq 1$ :*

$$\mu_1(X\|\sigma) = \|X\|_1 \leq \mu_\alpha(X\|\sigma) \cdot (\text{Tr}[\sigma])^{(\alpha-1)/\alpha}. \quad (53)$$

Equivalently,

$$\log_2 \|X\|_1 \leq \nu_\alpha(X\|\sigma) + \left(\frac{\alpha-1}{\alpha}\right) \log_2 \text{Tr}[\sigma]. \quad (54)$$

**Proof.** The proof of this lemma closely follows the proof of [31, Theorem 5]. Set  $\alpha' = \alpha/(\alpha-1)$ . Consider that

$$\begin{aligned}
&\mu_\alpha(X\|\sigma) \cdot (\text{Tr}[\sigma])^{1/\alpha'} \\
&= \left\| \sigma^{(1-\alpha)/2\alpha} X \sigma^{(1-\alpha)/2\alpha} \right\|_\alpha \cdot (\text{Tr}[\sigma])^{1/\alpha'} \quad (55)
\end{aligned}$$

$$= \left\| \sigma^{-1/2\alpha'} X \sigma^{-1/2\alpha'} \right\|_\alpha \cdot (\text{Tr}[\sigma])^{1/\alpha'} \quad (56)$$

$$= \left\| \sigma^{1/2\alpha'} \right\|_{2\alpha'} \left\| \sigma^{-1/2\alpha'} X \sigma^{-1/2\alpha'} \right\|_\alpha \left\| \sigma^{1/2\alpha'} \right\|_{2\alpha'} \quad (57)$$

$$\geq \left\| \sigma^{1/2\alpha'} \sigma^{-1/2\alpha'} X \sigma^{-1/2\alpha'} \sigma^{1/2\alpha'} \right\|_1 \quad (58)$$

$$= \|X\|_1, \quad (59)$$

where we used that  $(\text{Tr}[\sigma])^{1/\alpha'} = \left\| \sigma^{1/2\alpha'} \right\|_{2\alpha'}^2$  and the inequality is a consequence of Hölder's inequality. ■

**Lemma 5** *Let  $X \neq 0$  be a Hermitian operator, and let  $\sigma$  be a positive definite operator. Then the following inequality holds for all  $\beta > \alpha > 1$ :*

$$\left[ \frac{\mu_\alpha(X\|\sigma)}{\|X\|_1} \right]^{\frac{\alpha}{\alpha-1}} \leq \left[ \frac{\mu_\beta(X\|\sigma)}{\|X\|_1} \right]^{\frac{\beta}{\beta-1}}. \quad (60)$$

Equivalently,

$$\begin{aligned}
&\frac{\alpha}{\alpha-1} [\nu_\alpha(X\|\sigma) - \log_2 \|X\|_1] \\
&\leq \frac{\beta}{\beta-1} [\nu_\beta(X\|\sigma) - \log_2 \|X\|_1] \quad (61)
\end{aligned}$$

**Proof.** The proof of this lemma follows the proof of [31, Theorem 7] closely. Since  $1 < \alpha < \beta$ , it follows that  $\frac{1}{\beta} < \frac{1}{\alpha} < 1$ , so that there exists  $\theta \in (0, 1)$  such that

$$\frac{1}{\alpha} = 1 - \theta + \frac{\theta}{\beta}. \quad (62)$$

We then find that by simple manipulations that

$$\theta \left( \frac{\alpha}{\alpha-1} \right) = \left( \frac{\beta}{\beta-1} \right). \quad (63)$$

Consider from [31, Corollary 3] that the following inequality holds

$$\|\Gamma_\sigma^{-1}(X)\|_{\alpha,\sigma} \leq \|\Gamma_\sigma^{-1}(X)\|_{1,\sigma}^{1-\theta} \|\Gamma_\sigma^{-1}(X)\|_{\beta,\sigma}^\theta. \quad (64)$$

Consider that

$$\|\Gamma_\sigma^{-1}(X)\|_{1,\sigma} = \|X\|_1, \quad (65)$$

$$\|\Gamma_\sigma^{-1}(X)\|_{\alpha,\sigma} = \mu_\alpha(X\|\sigma), \quad (66)$$

$$\|\Gamma_\sigma^{-1}(X)\|_{\beta,\sigma} = \mu_\beta(X\|\sigma), \quad (67)$$

from which we conclude that

$$\mu_\alpha(X\|\sigma) \leq \|X\|_1^{1-\theta} \cdot \mu_\beta(X\|\sigma)^\theta, \quad (68)$$

so that

$$\mu_\alpha(X\|\sigma)^{\frac{\alpha}{\alpha-1}} \leq \|X\|_1^{\frac{(1-\theta)\alpha}{\alpha-1}} \cdot \mu_\beta(X\|\sigma)^{\frac{\theta\alpha}{\alpha-1}} \quad (69)$$

$$= \|X\|_1^{\frac{\alpha}{\alpha-1} - \frac{\beta}{\beta-1}} \cdot \mu_\beta(X\|\sigma)^{\frac{\beta}{\beta-1}}. \quad (70)$$

Rewriting this, we find that

$$\left[ \frac{\mu_\alpha(X\|\sigma)}{\|X\|_1} \right]^{\frac{\alpha}{\alpha-1}} \leq \left[ \frac{\mu_\beta(X\|\sigma)}{\|X\|_1} \right]^{\frac{\beta}{\beta-1}}. \quad (71)$$

Taking a logarithm, we get that

$$\begin{aligned}
&\frac{\alpha}{\alpha-1} [\nu_\alpha(X\|\sigma) - \log_2 \|X\|_1] \\
&\leq \frac{\beta}{\beta-1} [\nu_\beta(X\|\sigma) - \log_2 \|X\|_1], \quad (72)
\end{aligned}$$

concluding the proof. ■

**Corollary 1** *Let  $X \neq 0$  be a Hermitian operator, and let  $\sigma$  be a positive definite operator. Then the following inequality holds for all  $\beta > \alpha > 1$ :*

$$\mu_\alpha(X\|\sigma) \leq \mu_\beta(X\|\sigma). \quad (73)$$

Equivalently,

$$\nu_\alpha(X\|\sigma) \leq \nu_\beta(X\|\sigma). \quad (74)$$

**Proof.** This follows easily from the fact that

$$\frac{\beta}{\beta-1} \leq \frac{\alpha}{\alpha-1} \quad (75)$$

for  $\beta > \alpha > 1$  and by applying Lemma 5. ■

**Lemma 6** *Let  $X \neq 0$  be a Hermitian operator, and let  $\sigma$  be a positive definite operator. Then for all  $\alpha \geq 1$ , the following function is convex:*

$$\sigma \mapsto [\mu_\alpha(X\|\sigma)]^\alpha. \quad (76)$$

**Proof.** Consider that

$$[\mu_\alpha(X\|\sigma)]^\alpha = \left\| \sigma^{\frac{1-\alpha}{2\alpha}} X \sigma^{\frac{1-\alpha}{2\alpha}} \right\|_\alpha^\alpha \quad (77)$$

$$= \text{Tr} \left[ \left( \sigma^{\frac{1-\alpha}{2\alpha}} X \sigma^{\frac{1-\alpha}{2\alpha}} X \sigma^{\frac{1-\alpha}{2\alpha}} \right)^{\frac{\alpha}{2}} \right]. \quad (78)$$

A general theorem [32, Theorem 5.2] states that the following function of positive definite operators  $A$  and  $B$  is jointly convex in  $A$  and  $B$ :

$$\text{Tr} \left[ \left( [\mathcal{P}_1(A^p)]^{1/2} \mathcal{P}_2(B^q) [\mathcal{P}_1(A^p)]^{1/2} \right)^s \right], \quad (79)$$

for positive maps  $\mathcal{P}_1$  and  $\mathcal{P}_2$ ,  $p, q \in [-1, 0]$ , and  $s \geq 0$ . (For this statement, please consult Theorem 5.2 of [32] and the brief remarks stated before Theorem 5.3 therein). Then we see that the convexity of  $[\mu_\alpha(X\|\sigma)]^\alpha$  in  $\sigma$  for  $\alpha \geq 1$  follows as a special case of [32, Theorem 5.2] by taking

$$\mathcal{P}_1 = \text{id}, \quad (80)$$

$$\mathcal{P}_2(\cdot) = X(\cdot)X, \quad (81)$$

$$p = q = (1 - \alpha) / \alpha, \quad (82)$$

$$s = \alpha / 2, \quad (83)$$

$$A = B = \sigma, \quad (84)$$

concluding the proof. ■

#### IV. ORDERING OF THE $\alpha$ -LOGARITHMIC NEGATIVITY

Recall that the logarithmic negativity of a bipartite state  $\rho_{AB}$  is defined as [13, 14]

$$E_N(\rho_{AB}) \equiv \log_2 \|T_B(\rho_{AB})\|_1. \quad (85)$$

**Proposition 1** *Let  $\rho_{AB}$  be a bipartite quantum state, and let  $1 \leq \alpha \leq \beta$ . Then*

$$E_N(\rho_{AB}) \leq E_N^\alpha(\rho_{AB}) \leq E_N^\beta(\rho_{AB}). \quad (86)$$

**Proof.** Let  $\sigma_{AB}$  be an arbitrary state in  $\text{PPT}_{\text{inv}}(A : B)$ . Applying Lemma 4, we find that

$$E_N(\rho_{AB}) = \log_2 \|T_B(\rho_{AB})\|_1 \quad (87)$$

$$\leq \nu_\alpha(T_B(\rho_{AB})\|\sigma_{AB}). \quad (88)$$

Since the inequality holds for all  $\sigma_{AB} \in \text{PPT}_{\text{inv}}(A : B)$ , by applying Lemma 1 and an infimum, we conclude the first inequality in (86).

To establish the second inequality in (86), let  $\sigma_{AB}$  be an arbitrary state in  $\text{PPT}_{\text{inv}}(A : B)$ . Then applying Definition 1 and Corollary 1, we find that

$$E_N^\alpha(\rho_{AB}) \leq \nu_\alpha(T_B(\rho_{AB})\|\sigma_{AB}) \quad (89)$$

$$\leq \nu_\beta(T_B(\rho_{AB})\|\sigma_{AB}). \quad (90)$$

Since the inequality holds for all  $\sigma_{AB} \in \text{PPT}_{\text{inv}}(A : B)$ , we conclude that  $E_N^\alpha(\rho_{AB}) \leq E_N^\beta(\rho_{AB})$ . ■

#### V. LIMITS OF THE $\alpha$ -LOGARITHMIC NEGATIVITY

In this section, we consider two limits of the  $\alpha$ -logarithmic negativity, when  $\alpha \rightarrow 1$  and  $\alpha \rightarrow \infty$ . The former limit converges to the well known logarithmic negativity [13, 14] and the latter converges to the  $\kappa$ -entanglement of a bipartite quantum state [19, 20]. Recall that the  $\kappa$ -entanglement of a quantum state is defined as [19, 20]

$$E_\kappa(\rho_{AB}) \equiv \log_2 \inf \{ \text{Tr}[S_{AB}] :$$

$$-T_B(S_{AB}) \leq T_B(\rho_{AB}) \leq T_B(S_{AB}), S_{AB} \geq 0 \}. \quad (91)$$

**Proposition 2** *Let  $\rho_{AB}$  be a bipartite quantum state. Then*

$$\lim_{\alpha \rightarrow 1} E_N^\alpha(\rho_{AB}) = E_N(\rho_{AB}). \quad (92)$$

**Proof.** Let  $\sigma_{AB}$  be a state in  $\text{PPT}_{\text{inv}}(A : B)$ . Then

$$\begin{aligned} & \lim_{\alpha \rightarrow 1} E_N^\alpha(\rho_{AB}) \\ &= \lim_{\alpha \rightarrow 1} \inf_{\sigma_{AB} \in \text{PPT}_{\text{inv}}(A:B)} \nu_\alpha(T_B(\rho_{AB})\|\sigma_{AB}) \end{aligned} \quad (93)$$

$$\leq \lim_{\alpha \rightarrow 1} \nu_\alpha(T_B(\rho_{AB})\|\sigma_{AB}) \quad (94)$$

$$= \lim_{\alpha \rightarrow 1} \log_2 \left\| \sigma_{AB}^{(1-\alpha)/2\alpha} T_B(\rho_{AB}) \sigma_{AB}^{(1-\alpha)/2\alpha} \right\|_\alpha \quad (95)$$

$$= \log_2 \|T_B(\rho_{AB})\|_1 \quad (96)$$

$$= E_N(\rho_{AB}). \quad (97)$$

Combining with the inequality  $E_N^\alpha(\rho_{AB}) \geq E_N(\rho_{AB})$  from Proposition 1, we conclude the proof. ■

**Definition 2 (Max-logarithmic negativity)** *For  $\rho_{AB}$  a bipartite quantum state, the max-logarithmic negativity  $E_N^{\text{max}}(\rho_{AB})$  is defined as*

$$E_N^{\text{max}}(\rho_{AB}) \equiv \inf_{\sigma_{AB} \in \text{PPT}(A:B)} \nu_\infty(T_B(\rho_{AB})\|\sigma_{AB}), \quad (98)$$

and  $\nu_\infty$  is defined in (12).

**Proposition 3** *Let  $\rho_{AB}$  be a bipartite quantum state. Then*

$$E_\kappa(\rho_{AB}) = E_N^{\text{max}}(\rho_{AB}) = \lim_{\alpha \rightarrow \infty} E_N^\alpha(\rho_{AB}). \quad (99)$$

**Proof.** We first prove the first equality in (99). Consider that by the substitution  $S_{AB} \rightarrow T_B(S_{AB})$  in (91), we find that

$$\begin{aligned} & E_\kappa(\rho_{AB}) = \log_2 \inf \{ \text{Tr}[T_B(S_{AB})] : \\ & -S_{AB} \leq T_B(\rho_{AB}) \leq S_{AB}, T_B(S_{AB}) \geq 0 \}. \end{aligned} \quad (100)$$

Since  $\text{Tr}[T_B(S_{AB})] = \text{Tr}[S_{AB}]$ , it follows that

$$\begin{aligned} & E_\kappa(\rho_{AB}) = \log_2 \inf \{ \text{Tr}[S_{AB}] : \\ & -S_{AB} \leq T_B(\rho_{AB}) \leq S_{AB}, T_B(S_{AB}) \geq 0 \}. \end{aligned} \quad (101)$$

From the condition  $-S_{AB} \leq T_B(\rho_{AB}) \leq S_{AB}$ , it follows that  $-S_{AB} \leq S_{AB}$  and thus that  $S_{AB} \geq 0$ . By approximation (the fact that positive definite operators are dense in the set of positive semi-definite ones), it follows that we can take the infimum over  $S_{AB} > 0$ . Now make the substitution  $S_{AB} \rightarrow \mu\sigma_{AB}$  in (101), where  $\mu > 0$  and  $\sigma_{AB} \in \text{PPT}_{\text{inv}}(A : B)$ . Then we find that

$$E_\kappa(\rho_{AB}) = \log_2 \inf \left\{ \mu : \begin{aligned} -\mu\sigma_{AB} \leq T_B(\rho_{AB}) \leq \mu\sigma_{AB}, \\ \sigma_{AB} \in \text{PPT}_{\text{inv}}(A : B) \end{aligned} \right\}. \quad (102)$$

The condition  $-\mu\sigma_{AB} \leq T_B(\rho_{AB}) \leq \mu\sigma_{AB}$  is equivalent to  $-\mu I_{AB} \leq \sigma_{AB}^{-1/2} T_B(\rho_{AB}) \sigma_{AB}^{-1/2} \leq \mu I_{AB}$ , which is in turn equivalent to

$$\left\| \sigma_{AB}^{-1/2} T_B(\rho_{AB}) \sigma_{AB}^{-1/2} \right\|_\infty \leq \mu. \quad (103)$$

So then

$$E_\kappa(\rho_{AB}) = \log_2 \inf \left\{ \mu : \left\| \sigma_{AB}^{-1/2} T_B(\rho_{AB}) \sigma_{AB}^{-1/2} \right\|_\infty \leq \mu, \right. \\ \left. \sigma_{AB} \in \text{PPT}_{\text{inv}}(A : B) \right\}. \quad (104)$$

Then it follows that

$$E_\kappa(\rho_{AB}) \\ = \inf_{\sigma_{AB} \in \text{PPT}_{\text{inv}}(A : B)} \log_2 \left\| \sigma_{AB}^{-1/2} T_B(\rho_{AB}) \sigma_{AB}^{-1/2} \right\|_\infty \quad (105)$$

$$= \inf_{\sigma_{AB} \in \text{PPT}_{\text{inv}}(A : B)} \nu_\infty(T_B(\rho_{AB}) \| \sigma_{AB}), \quad (106)$$

thus establishing (99).

Now we establish the second equality in (99). Consider that

$$\lim_{\alpha \rightarrow \infty} E_N^\alpha(\rho_{AB}) \\ = \sup_{\alpha \in [1, \infty)} \inf_{\sigma_{AB} \in \text{PPT}(A : B)} \nu_\alpha(T_B(\rho_{AB}) \| \sigma_{AB}) \quad (107)$$

$$= \inf_{\sigma_{AB} \in \text{PPT}(A : B)} \sup_{\alpha \in [1, \infty)} \nu_\alpha(T_B(\rho_{AB}) \| \sigma_{AB}) \quad (108)$$

$$= \inf_{\sigma_{AB} \in \text{PPT}(A : B)} \nu_\infty(T_B(\rho_{AB}) \| \sigma_{AB}) \quad (109)$$

$$= E_N^{\text{max}}(\rho_{AB}). \quad (110)$$

The first equality follows from the ordering inequality from Proposition 1, using which we can replace  $\lim_{\alpha \rightarrow \infty}$  with  $\sup_{\alpha \in [1, \infty)}$ . The second (critical) equality above is a consequence of the ordering inequality from Proposition 1, the quasi-convexity of  $\nu_\alpha(T_B(\rho_{AB}) \| \sigma_{AB})$  with respect to  $\sigma_{AB}$  (Proposition 5), and the convexity and compactness of the set  $\text{PPT}(A : B)$ . All of these properties allow for applying the minimax theorem from [47, Corollary A.2], concluding the proof of the second equality in (99). ■

Putting together previous results, we conclude the following:

**Proposition 4** *If  $\rho_{AB}$  satisfies the condition  $T_B(|T_B(\rho_{AB})|) \geq 0$ , then all  $\alpha$ -logarithmic negativities are equal; i.e., the following equality holds for all  $\alpha \geq 1$ :*

$$E_N^\alpha(\rho_{AB}) = E_N(\rho_{AB}). \quad (111)$$

**Proof.** It is known from [19, Proposition 3] that  $E_N(\rho_{AB}) = E_\kappa(\rho_{AB})$  if  $\rho_{AB}$  satisfies the condition  $T_B(|T_B(\rho_{AB})|) \geq 0$ . Then the equality in (111) follows as a consequence of the ordering inequality from Proposition 1, as well as Propositions 2 and 3. ■

**Remark 1** *Since all pure states [48], two-qubit states [49], Werner states [21], and bosonic Gaussian states [21] satisfy the condition  $T_B(|T_B(\rho_{AB})|) \geq 0$ , we conclude that the equality in (111) holds for such states.*

## VI. PROPERTIES OF THE $\alpha$ -LOGARITHMIC NEGATIVITY

Here we prove that the  $\alpha$ -logarithmic negativity obeys several fundamental properties, making it an interesting entanglement measure to consider in quantum information theory. These properties include the following:

1. Entanglement monotone under selective C-PPT-P channels (which includes selective LOCC channels as a special case).
2. It can be calculated by convex optimization.
3. Normalization on maximally entangled states.
4. Faithfulness.
5. Subadditivity.

We also prove by counterexample that the  $\alpha$ -logarithmic negativities are neither convex nor monogamous.

### A. Entanglement monotonicity under selective completely positive partial transpose preserving channels

Let us first recall the class of completely positive partial transpose preserving (C-PPT-P) channels [11, 50], defined as bipartite channels  $\mathcal{N}_{AB \rightarrow A'B'}$  such that  $\mathcal{N}_{AB \rightarrow A'B'}$  is CPTP and the map  $T_{B'} \circ \mathcal{N}_{AB \rightarrow A'B'} \circ T_B$  is CP. Related to this, a C-PPT-P quantum instrument consists of the collection  $\{\mathcal{N}_{AB \rightarrow A'B'}^x\}_x$ , where each  $\mathcal{N}_{AB \rightarrow A'B'}^x$  is CP, the map  $T_{B'} \circ \mathcal{N}_{AB \rightarrow A'B'}^x \circ T_B$  is CP, and the sum map  $\sum_x \mathcal{N}_{AB \rightarrow A'B'}^x$  is TP. It is well known that the set of C-PPT-P channels contains the set of LOCC channels, as well as the set of separable channels [11, 50].

The following fundamental theorem establishes that the  $\alpha$ -logarithmic negativities are entanglement monotones for all  $\alpha \geq 1$ :

**Theorem 1 (Entanglement monotone)** *Let  $\{\mathcal{N}_{AB \rightarrow A'B'}^x\}_x$  be a C-PPT-P quantum instrument, and let  $\rho_{AB}$  be a bipartite state. Then the  $\alpha$ -logarithmic negativity is an entanglement monotone; i.e., the following inequality holds for all  $\alpha \geq 1$ :*

$$E_N^\alpha(\rho_{AB}) \geq \sum_{x:p(x)>0} p(x) E_N^\alpha(\rho_{A'B'}^x), \quad (112)$$

where

$$p(x) \equiv \text{Tr}[\mathcal{N}_{AB \rightarrow A'B'}^x(\rho_{AB})], \quad (113)$$

$$\rho_{A'B'}^x \equiv \frac{1}{p(x)} \mathcal{N}_{AB \rightarrow A'B'}^x(\rho_{AB}). \quad (114)$$

**Proof.** Let  $\sigma_{AB}$  be an arbitrary state in  $\text{PPT}_{\text{inv}}(A : B)$ . Let

$$\begin{aligned} \mathcal{N}_{AB \rightarrow A'B'X}^T(Y_{AB}) &\equiv \sum_x \mathcal{N}_{AB \rightarrow A'B'}^{T,x}(Y_{AB}) \otimes |x\rangle\langle x|_X, \\ \mathcal{N}_{AB \rightarrow A'B'}^{T,x}(Y_{AB}) &\equiv (T_{B'} \circ \mathcal{N}_{AB \rightarrow A'B'}^x \circ T_B)(Y_{AB}). \end{aligned} \quad (115)$$

Note that the map  $\mathcal{N}_{AB \rightarrow A'B'X}^T$  is completely positive and trace preserving, which is a consequence of each map  $\mathcal{N}_{AB \rightarrow A'B'}^{T,x}$  being CP and the sum map  $\sum_x \mathcal{N}_{AB \rightarrow A'B'}^{T,x}$  being TP. Let

$$\sigma_{A'B'}^x \equiv \frac{1}{q(x)} \mathcal{N}_{AB \rightarrow A'B'}^{T,x}(\sigma_{AB}), \quad (116)$$

where the probability distribution  $\{q(x)\}_x$  is defined as

$$q(x) \equiv \text{Tr}[\mathcal{N}_{AB \rightarrow A'B'}^{T,x}(\sigma_{AB})]. \quad (117)$$

Note that  $\sigma_{A'B'}^x \in \text{PPT}(A' : B')$  because

$$\sigma_{A'B'}^x = \frac{1}{q(x)} \mathcal{N}_{AB \rightarrow A'B'}^{T,x}(\sigma_{AB}) \geq 0, \quad (118)$$

since  $\sigma_{AB} \geq 0$  and  $\mathcal{N}_{AB \rightarrow A'B'}^{T,x}$  is CP and

$$T_{B'}(\sigma_{A'B'}^x) = \frac{1}{q(x)} \mathcal{N}_{AB \rightarrow A'B'}^x(T_B(\sigma_{AB})) \geq 0, \quad (119)$$

since  $T_B(\sigma_{AB}) \geq 0$  and  $\mathcal{N}_{AB \rightarrow A'B'}^x$  is CP. Also,  $\text{Tr}[\sigma_{A'B'}^x] = 1$  by definition. Then consider that

$$\begin{aligned} &\nu_\alpha(T_B(\rho_{AB}) \| \sigma_{AB}) \\ &\geq \nu_\alpha(\mathcal{N}_{AB \rightarrow A'B'X}^T(T_B(\rho_{AB})) \| \mathcal{N}_{AB \rightarrow A'B'X}^T(\sigma_{AB})) \end{aligned} \quad (120)$$

$$\geq \left( \frac{\alpha - 1}{\alpha} \right) D(p \| q) + \sum_x p(x) \nu_\alpha(T_{B'}(\rho_{A'B'}^x) \| \sigma_{A'B'}^x) \quad (121)$$

$$\geq \sum_x p(x) \nu_\alpha(T_{B'}(\rho_{A'B'}^x) \| \sigma_{A'B'}^x) \quad (122)$$

$$\geq \sum_x p(x) E_N^\alpha(\rho_{A'B'}^x). \quad (123)$$

The first inequality follows from Lemma 2 (data processing). The second inequality follows from the facts that

$$\begin{aligned} \mathcal{N}_{AB \rightarrow A'B'X}^T(T_B(\rho_{AB})) &= \sum_x p(x) |x\rangle\langle x|_X \otimes T_{B'}(\rho_{A'B'}^x), \\ \mathcal{N}_{AB \rightarrow A'B'X}^T(\sigma_{AB}) &= \sum_x q(x) |x\rangle\langle x|_X \otimes \sigma_{A'B'}^x, \end{aligned} \quad (124)$$

and by applying Lemma 3. The third inequality follows because the classical relative entropy  $D(p \| q)$  is non-negative. The final inequality follows from Definition 1.

Since the chain of inequalities holds for an arbitrary state  $\sigma_{AB} \in \text{PPT}_{\text{inv}}(A : B)$ , we conclude (112). ■

Note that Theorem 1 applies to the case in which the C-PPT-P instrument consists of a single element, i.e., when the C-PPT-P instrument is really just a C-PPT-P channel. We remark here that if the goal is to establish monotonicity under the action of a C-PPT-P channel (and not the more general case of an instrument), then the proof is slightly simpler than above. For clarity and due to its brevity, we show the few steps explicitly now. Let  $\mathcal{N}_{AB \rightarrow A'B'}$  be a C-PPT-P channel, and let  $\sigma_{AB} \in \text{PPT}_{\text{inv}}(A : B)$ . By definition,  $\mathcal{N}_{AB \rightarrow A'B'}^T \equiv T_{B'} \circ \mathcal{N}_{AB \rightarrow A'B'} \circ T_B$  is a quantum channel, and  $\mathcal{N}_{AB \rightarrow A'B'}^T(\sigma_{AB}) \in \text{PPT}(A' : B')$  by the same reasoning given in (118) and (119). Then consider that

$$\begin{aligned} &\nu_\alpha(T_B(\rho_{AB}) \| \sigma_{AB}) \\ &\geq \nu_\alpha(\mathcal{N}_{AB \rightarrow A'B'}^T(T_B(\rho_{AB})) \| \mathcal{N}_{AB \rightarrow A'B'}^T(\sigma_{AB})) \end{aligned} \quad (125)$$

$$= \nu_\alpha(T_{B'}(\mathcal{N}_{AB \rightarrow A'B'}(\rho_{AB})) \| \mathcal{N}_{AB \rightarrow A'B'}^T(\sigma_{AB})) \quad (126)$$

$$\geq E_N^\alpha(\mathcal{N}_{AB \rightarrow A'B'}(\rho_{AB})), \quad (127)$$

where the first inequality follows from Lemma 2 and the second from the fact that  $\mathcal{N}_{AB \rightarrow A'B'}^T(\sigma_{AB}) \in \text{PPT}(A' : B')$  and Definition 1. Since the inequalities hold for all  $\sigma_{AB} \in \text{PPT}_{\text{inv}}(A : B)$ , we conclude that

$$E_N^\alpha(\rho_{AB}) \geq E_N^\alpha(\mathcal{N}_{AB \rightarrow A'B'}(\rho_{AB})). \quad (128)$$

## B. Convex optimization

**Proposition 5** *Let  $\rho_{AB}$  be a bipartite quantum state, and let  $\alpha \geq 1$ . Then the  $\alpha$ -logarithmic negativity  $E_N^\alpha(\rho_{AB})$  can be calculated by convex optimization.*

**Proof.** We can rewrite  $E_N^\alpha(\rho_{AB})$  as follows:

$$\begin{aligned} &E_N^\alpha(\rho_{AB}) \\ &= \inf_{\sigma_{AB} \in \text{PPT}(A:B)} \log_2 \mu_\alpha(T_B(\rho_{AB}) \| \sigma_{AB}) \end{aligned} \quad (129)$$

$$= \inf_{\sigma_{AB} \in \text{PPT}(A:B)} \frac{1}{\alpha} \log_2 [\mu_\alpha(T_B(\rho_{AB}) \| \sigma_{AB})]^\alpha \quad (130)$$

$$= \frac{1}{\alpha} \log_2 \inf_{\sigma_{AB} \in \text{PPT}(A:B)} [\mu_\alpha(T_B(\rho_{AB}) \| \sigma_{AB})]^\alpha. \quad (131)$$

The statement of the proposition then follows as a consequence of Lemma 6. ■

### C. Normalization

**Proposition 6 (Normalization)** For a maximally entangled state  $\Phi_{AB}^d$  of Schmidt rank  $d \geq 2$ :

$$\Phi_{AB}^d \equiv \frac{1}{d} \sum_{i,j} |i\rangle\langle j|_A \otimes |i\rangle\langle j|_B, \quad (132)$$

where  $\{|i\rangle_A\}_i$  and  $\{|i\rangle_B\}_i$  are orthonormal bases, the following equality holds for all  $\alpha \geq 1$

$$E_N^\alpha(\Phi_{AB}^d) = \log_2 d. \quad (133)$$

**Proof.** This is a direct consequence of  $E_N(\Phi_{AB}^d) = E_\kappa(\Phi_{AB}^d) = \log_2 d$  [19], the ordering inequality in Proposition 1, and Proposition 3. ■

### D. Faithfulness

It is known that the logarithmic negativity is faithful, meaning that  $E_N(\rho_{AB}) \geq 0$  and  $E_N(\rho_{AB}) = 0$  if and only if  $\rho_{AB} \in \text{PPT}(A : B)$ . To see that  $E_N(\rho_{AB}) \geq 0$ , consider that

$$E_N(\rho_{AB}) = \log_2 \|T_B(\rho_{AB})\|_1 \quad (134)$$

$$\geq \log_2 \text{Tr}[T_B(\rho_{AB})] \quad (135)$$

$$= \log_2 \text{Tr}[\rho_{AB}] = 0. \quad (136)$$

The implication  $\rho_{AB} \in \text{PPT}(A : B) \Rightarrow E_N(\rho_{AB}) = 0$  follows easily from the fact that  $\|T_B(\rho_{AB})\|_1 = \text{Tr}[T_B(\rho_{AB})]$  for such states, and the opposite implication was shown, e.g., in [19, Proposition 5]. Using this, we can conclude faithfulness for the  $\alpha$ -logarithmic negativity:

**Proposition 7 (Faithfulness)** Let  $\rho_{AB}$  be a bipartite quantum state, and let  $\alpha \geq 1$ . Then  $E_N^\alpha(\rho_{AB}) \geq 0$  and  $E_N^\alpha(\rho_{AB}) = 0$  if and only if  $\rho_{AB} \in \text{PPT}(A : B)$ .

**Proof.** The inequality  $E_N^\alpha(\rho_{AB}) \geq 0$  follows from Proposition 1 and the fact that  $E_N(\rho_{AB}) \geq 0$ .

To see the other statement, let  $\rho_{AB} \in \text{PPT}(A : B)$ . Then we can pick  $\sigma_{AB} = T_B(\rho_{AB}) \in \text{PPT}(A : B)$ , and we find that

$$E_N^\alpha(\rho_{AB}) \leq \nu_\alpha(T_B(\rho_{AB})\|\sigma_{AB}) \quad (137)$$

$$= \nu_\alpha(T_B(\rho_{AB})\|T_B(\rho_{AB})) \quad (138)$$

$$= \log_2 \left\| \left( (T_B(\rho_{AB}))^{\frac{1-\alpha}{2\alpha}} T_B(\rho_{AB}) (T_B(\rho_{AB}))^{\frac{1-\alpha}{2\alpha}} \right) \right\|_\alpha \quad (139)$$

$$= \log_2 \left\| \left( (T_B(\rho_{AB}))^{\frac{1}{\alpha}} \right) \right\|_\alpha \quad (140)$$

$$= 0. \quad (141)$$

We then conclude that  $E_N^\alpha(\rho_{AB}) = 0$  if  $\rho_{AB} \in \text{PPT}(A : B)$ .

Now suppose that  $E_N^\alpha(\rho_{AB}) = 0$ . Then this means that  $E_N(\rho_{AB}) = 0$  by Proposition 1, and we conclude that  $\rho_{AB} \in \text{PPT}(A : B)$  as a consequence of the faithfulness of logarithmic negativity. ■

### E. Subadditivity

In this section, we establish subadditivity of the  $\alpha$ -logarithmic negativity  $E_N^\alpha$ :

**Proposition 8 (Subadditivity)** Let  $\rho_{A_1 B_1}$  and  $\omega_{A_2 B_2}$  be bipartite states. Then the following subadditivity inequality holds for all  $\alpha \geq 1$ :

$$E_N^\alpha(\rho_{A_1 B_1} \otimes \omega_{A_2 B_2}) \leq E_N^\alpha(\rho_{A_1 B_1}) + E_N^\alpha(\omega_{A_2 B_2}), \quad (142)$$

where the bipartition for  $E_N^\alpha(\rho_{A_1 B_1} \otimes \omega_{A_2 B_2})$  is understood to be  $A_1 A_2 | B_1 B_2$ .

**Proof.** Let  $\sigma_{A_1 B_1}^{(1)}$  and  $\sigma_{A_2 B_2}^{(2)}$  be arbitrary PPT states in  $\text{PPT}(A_1 : B_1)$  and  $\text{PPT}(A_2 : B_2)$ , respectively. Then it follows that  $\sigma_{A_1 B_1}^{(1)} \otimes \sigma_{A_2 B_2}^{(2)} \in \text{PPT}(A_1 A_2 : B_1 B_2)$ , so that

$$\begin{aligned} E_N^\alpha(\rho_{A_1 B_1} \otimes \omega_{A_2 B_2}) &\leq \nu_\alpha(T_{B_1}(\rho_{A_1 B_1}) \otimes T_{B_2}(\omega_{A_2 B_2})\|\sigma_{A_1 B_1}^{(1)} \otimes \sigma_{A_2 B_2}^{(2)}) \\ &= \nu_\alpha(T_{B_1}(\rho_{A_1 B_1})\|\sigma_{A_1 B_1}^{(1)}) + \nu_\alpha(T_{B_2}(\omega_{A_2 B_2})\|\sigma_{A_2 B_2}^{(2)}). \end{aligned} \quad (143)$$

where we have exploited the additivity of  $\nu_\alpha$  to establish the equality. Since the inequality holds for arbitrary  $\sigma_{A_1 B_1}^{(1)}$  and  $\sigma_{A_2 B_2}^{(2)}$ , the inequality in (142) follows. ■

It is not clear to us whether the opposite inequality (superadditivity) holds in general. It is well known that the logarithmic negativity is additive [13], and it has been shown recently in [19] that the max-logarithmic negativity ( $\kappa$ -entanglement) is additive also. So by Remark 1, it follows that the  $\alpha$ -logarithmic negativities are additive for the states mentioned there. However, establishing additivity in general (or a counterexample) is a problem that we leave open for future work.

### F. No convexity

As a consequence of the counterexample given in [19, Proposition 6] in addition to Remark 1, it follows that the  $\alpha$ -logarithmic negativity is not generally convex for any choice of  $\alpha \in [1, \infty]$ . Indeed, by picking

$$\rho_{AB}^1 \equiv \Phi_{AB}^2, \quad (145)$$

$$\rho_{AB}^2 \equiv \frac{1}{2} (|00\rangle\langle 00|_{AB} + |11\rangle\langle 11|_{AB}), \quad (146)$$

$$\bar{\rho}_{AB} \equiv \frac{1}{2} (\rho_{AB}^1 + \rho_{AB}^2), \quad (147)$$

we find for all  $\alpha \in [1, \infty]$  that

$$E_N^\alpha(\rho_{AB}^1) = 1, \quad (148)$$

$$E_N^\alpha(\rho_{AB}^2) = 0, \quad (149)$$

$$E_N^\alpha(\bar{\rho}_{AB}) = \log_2 \frac{3}{2}, \quad (150)$$

which implies that

$$E_N^\alpha(\bar{\rho}_{AB}) > \frac{1}{2} (E_N^\alpha(\rho_{AB}^1) + E_N^\alpha(\rho_{AB}^2)). \quad (151)$$

### G. No monogamy

An entanglement measure  $E$  is said to be monogamous [51–53] if the following inequality holds for all tripartite states  $\rho_{ABC}$ :

$$E(\rho_{A:B}) + E(\rho_{A:C}) \leq E(\rho_{A:BC}), \quad (152)$$

where the bipartition is indicated by a colon.

As a consequence of the counterexample given in [19, Proposition 7] in addition to Propositions 1 and 3 and Remark 1, it follows that the  $\alpha$ -logarithmic negativity is not generally monogamous for any choice of  $\alpha \in [1, \infty]$ . Indeed, consider the following state of three qubits:

$$|\psi\rangle_{ABC} \equiv \frac{1}{2} (|000\rangle_{ABC} + |011\rangle_{ABC} + \sqrt{2}|110\rangle_{ABC}). \quad (153)$$

It was shown in [19, Proposition 7] that the following inequality holds

$$E_\kappa(\psi_{A:B}) + E_\kappa(\psi_{A:C}) > E_\kappa(\psi_{A:BC}). \quad (154)$$

Since the reduced states  $\psi_{AB}$  and  $\psi_{AC}$  are two-qubit states, it follows from Remark 1 and (154) for all  $\alpha \geq 1$  that

$$E_N^\alpha(\psi_{A:B}) + E_N^\alpha(\psi_{A:C}) > E_N^\alpha(\psi_{A:BC}) \quad (155)$$

$$\geq E_N^\alpha(\psi_{A:BC}), \quad (156)$$

where the last inequality is a consequence of Propositions 1 and 3. So monogamy does not hold for any of the  $\alpha$ -logarithmic negativities.

## VII. GENERALIZATIONS

### A. $\alpha$ -Logarithmic negativity of a quantum channel

We can also generalize the notion of  $\alpha$ -logarithmic negativity from bipartite quantum states to point-to-point quantum channels. Before doing so, let us recall that the logarithmic negativity of a quantum channel  $\mathcal{N}_{A \rightarrow B}$  is defined as [54]

$$E_N(\mathcal{N}) \equiv \log_2 \|T_B \circ \mathcal{N}_{A \rightarrow B}\|_\diamond. \quad (157)$$

Recall that the diamond norm of a Hermitian preserving map  $\mathcal{M}_{A \rightarrow B}$  is defined as [55]

$$\|\mathcal{M}_{A \rightarrow B}\|_\diamond \equiv \sup_{\psi_{RA}} \|\mathcal{M}_{A \rightarrow B}(\psi_{RA})\|_1, \quad (158)$$

with the optimization over pure bipartite states  $\psi_{RA}$  such that the reference system  $R$  is isomorphic to the channel input system  $A$ . By applying definitions, we see that we can write the logarithmic negativity of a quantum channel as an optimized version of the logarithmic negativity of quantum states:

$$E_N(\mathcal{N}) = \sup_{\psi_{RA}} E_N(\omega_{RB}), \quad (159)$$

where  $\omega_{RB} \equiv \mathcal{N}_{A \rightarrow B}(\psi_{RA})$ . Note that this kind of channel generalization of state measures is quite common in quantum information theory [18, 56–58].

Continuing in this spirit, we define the following:

**Definition 3 ( $\alpha$ -log. negativity of a channel)** *The  $\alpha$ -logarithmic negativity of a quantum channel is defined for  $\alpha \geq 1$  as*

$$E_N^\alpha(\mathcal{N}) = \sup_{\psi_{RA}} E_N^\alpha(\omega_{RB}), \quad (160)$$

with  $\omega_{RB} \equiv \mathcal{N}_{A \rightarrow B}(\psi_{RA})$ .

We could define it more generally with an optimization over mixed input states  $\psi_{RA}$  with unbounded reference system  $R$ . However, the maximal value is always achieved by a pure bipartite input state with reference system  $R$  isomorphic to the channel input system  $A$ , as a consequence of the monotonicity inequality in Theorem 1, the Schmidt decomposition theorem, and the invariance of  $E_N^\alpha$  with respect to isometric channels acting on the reference system  $R$  (this latter statement itself being a consequence of Theorem 1).

By the above observation and Remark 1, it follows that

$$E_N^\alpha(\mathcal{N}) = E_N(\mathcal{N}) \quad (161)$$

for all  $\alpha \in [1, \infty]$  if  $\mathcal{N}$  is qubit channel (with qubit input and qubit output), if  $\mathcal{N}$  is a unitary channel, or if  $\mathcal{N}$  is a Werner–Holevo channel [59–61]. Recall that a Werner–Holevo channel  $\mathcal{W}_{A \rightarrow B}^{(p,d)}$  with parameters  $p \in [0, 1]$  and integer  $d \geq 2$  is defined as [59–61]

$$\mathcal{W}_{A \rightarrow B}^{(p,d)} \equiv (1-p) \mathcal{W}_{A \rightarrow B}^{(0,d)} + p \mathcal{W}_{A \rightarrow B}^{(1,d)}, \quad (162)$$

where the channels  $\mathcal{W}_{A \rightarrow B}^{(0,d)}$  and  $\mathcal{W}_{A \rightarrow B}^{(1,d)}$  are defined as

$$\mathcal{W}_{A \rightarrow B}^{(0,d)}(X_A) \equiv \frac{\text{Tr}[X_A] I_B + \text{id}_{A \rightarrow B}(T_A(X_A))}{d+1}, \quad (163)$$

$$\mathcal{W}_{A \rightarrow B}^{(1,d)}(X_A) \equiv \frac{\text{Tr}[X_A] I_B - \text{id}_{A \rightarrow B}(T_A(X_A))}{d-1}, \quad (164)$$

and  $T_A$  denotes the partial transpose map on system  $A$ . Note that the Choi state of the Werner–Holevo channel  $\mathcal{W}_{A \rightarrow B}^{p,d}$  is a Werner state [62]:

$$\mathcal{W}_{A \rightarrow B}^{(p,d)}(\Phi_{RA}) \equiv (1-p) \frac{2}{d(d+1)} \Pi_{RB}^S + p \frac{2}{d(d-1)} \Pi_{RB}^A, \quad (165)$$

where

$$\Pi_{RB}^S \equiv (I_{RB} + F_{RB})/2, \quad (166)$$

$$\Pi_{RB}^A \equiv (I_{RB} - F_{RB})/2, \quad (167)$$

and  $F_{RB}$  is the unitary swap operator.

The claim stated after (161) follows easily from Remark 1 for qubit-qubit channels, for unitary channels because the output state is pure (and then applying Remark 1), and for Werner–Holevo channels by employing their covariance symmetry [61] and Theorem 1 to conclude that the optimal state  $\psi_{RA}$  in (160) is the maximally entangled state, so that the output of the channel is a Werner state, from which we conclude the claim by applying Remark 1. For the Werner–Holevo channel, we can in fact give the following equality for all  $\alpha \geq 1$  [19, 21]:

$$E_N^\alpha \left( \mathcal{W}_{A \rightarrow B}^{(p,d)} \right) = \begin{cases} \log_2 \left( \frac{2}{d} [2p - 1] + 1 \right) & \text{if } p \geq \frac{1}{2} \\ 0 & \text{if } p \leq \frac{1}{2} \end{cases}. \quad (168)$$

Also, as a consequence of the ordering inequality in Proposition 1 and [19, Theorem 27], the following equalities hold for all  $\alpha \geq 1$ :

$$E_N^\alpha (\mathcal{L}_{\eta, N_B}) = \log_2 \left( \frac{1 + \eta}{(1 - \eta)(2N_B + 1)} \right), \quad (169)$$

$$E_N^\alpha (\mathcal{A}_{G, N_B}) = \log_2 \left( \frac{G + 1}{(G - 1)(2N_B + 1)} \right), \quad (170)$$

$$E_N^\alpha (\mathcal{T}_\xi) = \log_2 (1/\xi), \quad (171)$$

where  $\mathcal{L}_{\eta, N_B}$  is a thermal channel with transmissivity  $\eta \in (0, 1)$  and thermal photon number  $N_B \in (0, \eta/[1 - \eta])$ ,  $\mathcal{A}_{G, N_B}$  is an amplifier channel with gain  $G > 1$  and thermal photon number  $N_B \in (0, 1/[G - 1])$ , and  $\mathcal{T}_\xi$  is an additive-noise channel with noise variance  $\xi \in (0, 1)$ . (See [19, Section VII] for more details of these channels.)

## B. Other positive but not completely positive maps

We note briefly here that the main definition in this paper can be generalized to other maps, besides the partial transpose map, that are positive but not completely positive. Let  $\mathcal{P}_B$  be a such a map. Then we can define the set of free states as follows:

$$P(A : B) \equiv \{ \sigma_{AB} : \sigma_{AB}, \mathcal{P}_B(\sigma_{AB}) \geq 0, \text{Tr}[\sigma_{AB}] = 1 \}, \quad (172)$$

and we can define a generalized logarithmic negativity of a bipartite state  $\rho_{AB}$  as follows for  $\alpha \geq 1$ :

$$E_P^\alpha (\rho_{AB}) \equiv \inf_{\sigma_{AB} \in P(A:B)} \nu_\alpha (\mathcal{P}_B(\rho_{AB}) \| \sigma_{AB}). \quad (173)$$

The same notion can be generalized to channels as in Definition 3. Such a concept could be interesting to explore further.

## C. Generalizations to other resource theories

Recently, there has been a concerted effort to generalize concepts developed in particular quantum resource theories to more general ones (see [63] for a recent review and [64–71] for other papers in this spirit). With this in mind, it seems fruitful to generalize the concepts developed in this paper to arbitrary resource theories, beyond the resource theory of entanglement.

To this end, consider a resource theory with a set  $\mathcal{S}_F$  of free states and a set  $\mathcal{O}_F$  of free operations (a quantum operation being a completely positive, trace non-increasing map). Furthermore, suppose that  $\mathcal{P}$  is a Hermiticity-preserving and trace-preserving map satisfying

$$\mathcal{F} \circ \mathcal{P} = \mathcal{P} \circ \mathcal{F}, \quad (174)$$

for all free operations  $\mathcal{F} \in \mathcal{O}_F$ , as well as

$$\mathcal{P}(\sigma) \in \mathcal{S}_F \quad \text{if } \sigma \in \mathcal{S}_F. \quad (175)$$

(It should be clear that the map  $\mathcal{P}$  mentioned above should generalize the partial transpose operation considered earlier.) We could demand further that

$$\sigma \in \mathcal{S}_F \quad \text{if } \mathcal{P}(\sigma) \in \mathcal{S}_F. \quad (176)$$

Then for  $\alpha \geq 1$ , we define the  $\alpha$ -logarithmic resourcefulness of a state  $\rho$  with respect to  $\mathcal{P}$  as

$$R_\alpha^\mathcal{P}(\rho) \equiv \inf_{\sigma \in \mathcal{S}_F} \nu_\alpha (\mathcal{P}(\rho) \| \sigma). \quad (177)$$

This measure satisfies many of the properties that the  $\alpha$ -logarithmic negativity does. By the same proof given for Proposition 1, we have that the ordering property is satisfied

$$R_\alpha^\mathcal{P}(\rho) \leq R_\beta^\mathcal{P}(\rho), \quad (178)$$

for all states  $\rho$  and for  $1 \leq \alpha \leq \beta$ .

We also have the following regarding faithfulness:

**Proposition 9** *Fix  $\alpha \geq 1$ . The  $\alpha$ -logarithmic resourcefulness is faithful, meaning that  $R_\alpha^\mathcal{P}(\rho) \geq 0$  and  $R_\alpha^\mathcal{P}(\rho) = 0$  if  $\rho \in \mathcal{S}_F$ . If (176) holds and  $\alpha > 1$ , then  $\rho \in \mathcal{S}_F$  if  $R_\alpha^\mathcal{P}(\rho) = 0$ .*

**Proof.** The proof follows along the same lines as that given for Proposition 7. To see that  $R_\alpha^\mathcal{P}(\rho) \geq 0$  for an arbitrary state  $\rho$ , consider that

$$R_\alpha^\mathcal{P}(\rho) \geq R_1^\mathcal{P}(\rho) \quad (179)$$

$$= \log_2 \|\mathcal{P}(\rho)\|_1 \quad (180)$$

$$\geq \log_2 \text{Tr}[\mathcal{P}(\rho)] \quad (181)$$

$$= \log_2 \text{Tr}[\rho] \quad (182)$$

$$= 0. \quad (183)$$

For the first inequality, we used the ordering inequality in (178). The second inequality follows because

$\|X\|_1 \geq \text{Tr}[X]$  for an arbitrary square operator  $X$ . The second equality exploits the assumption that  $\mathcal{P}$  is trace preserving.

If  $\rho \in \mathcal{S}_F$ , then pick  $\sigma = \mathcal{P}(\rho) \in \mathcal{S}_F$  (following from (175)), and we find that

$$R_\alpha^{\mathcal{P}}(\rho) \leq \nu_\alpha(\mathcal{P}(\rho)\|\sigma) \quad (184)$$

$$= \nu_\alpha(\mathcal{P}(\rho)\|\mathcal{P}(\rho)) \quad (185)$$

$$= \log_2 \left\| [\mathcal{P}(\rho)]^{1/\alpha} \right\|_\alpha \quad (186)$$

$$= 0. \quad (187)$$

If (176) holds and  $R_\alpha^{\mathcal{P}}(\rho) = 0$ , then the ordering inequality in (178) implies that  $R_1^{\mathcal{P}}(\rho) = \log_2 \|\mathcal{P}(\rho)\|_1 = 0$ , which means that  $\|\mathcal{P}(\rho)\|_1 = 1$ . Let a Jordan–Hahn decomposition of  $\mathcal{P}(\rho)$  be  $\mathcal{P}(\rho) = P - N$  (i.e.,  $P, N \geq 0$  and  $PQ = 0$ ). Consider that  $\text{Tr}[\mathcal{P}(\rho)] = \text{Tr}[P] - \text{Tr}[N] = 1$ . Also,  $1 = \|\mathcal{P}(\rho)\|_1 = \text{Tr}[P] + \text{Tr}[N]$ . Subtracting these equations gives that  $\text{Tr}[N] = 0 \Rightarrow N = 0$ . Then  $\mathcal{P}(\rho) \geq 0$  and  $\text{Tr}[\mathcal{P}(\rho)] = 1$ . Since we know by assumption that  $R_\alpha^{\mathcal{P}}(\rho) = 0$  for some  $\alpha > 1$  and  $\mathcal{P}(\rho)$  is a quantum state, from the relation in (13), the definition in (177), and the equality conditions for sandwiched Rényi relative entropy [29–31], it follows that  $\mathcal{P}(\rho) \in \mathcal{S}_F$ , and so (176) implies that  $\rho \in \mathcal{S}_F$ . ■

Let  $\{\mathcal{F}^x\}_x$  denote a free quantum instrument, i.e., such that  $\mathcal{F}^x \in \mathcal{O}_F$  and  $\sum_x \mathcal{F}^x$  is a quantum channel. The most prominent property of  $\alpha$ -logarithmic resourcefulness is that it is monotone under the action of a free quantum instrument, in the following sense:

$$R_\alpha^{\mathcal{P}}(\rho) \geq \sum_{x:p(x)>0} p(x) R_\alpha^{\mathcal{P}}(\rho^x), \quad (188)$$

where

$$p(x) \equiv \text{Tr}[\mathcal{F}^x(\rho)], \quad (189)$$

$$\rho^x \equiv \frac{1}{p(x)} \mathcal{F}^x(\rho). \quad (190)$$

The proof of this inequality is nearly identical to that given for Theorem 1, with the main change being that we use the defining property in (174) in the proof of Theorem 1. The proof is simpler in the case that one is interested in establishing monotonicity under a free quantum channel rather than a free quantum instrument, and so we detail it briefly here for clarity. Let  $\sigma \in \mathcal{S}_F$  and  $\mathcal{F} \in \mathcal{C}_F$ , where  $\mathcal{C}_F$  denotes the set of free channels. Then

$$\nu_\alpha(\mathcal{P}(\rho)\|\sigma) \geq \nu_\alpha((\mathcal{F} \circ \mathcal{P})(\rho)\|\mathcal{F}(\sigma)) \quad (191)$$

$$= \nu_\alpha((\mathcal{P} \circ \mathcal{F})(\rho)\|\mathcal{F}(\sigma)) \quad (192)$$

$$\geq R_\alpha^{\mathcal{P}}(\mathcal{F}(\rho)), \quad (193)$$

where the first inequality employs Lemma 2 and the sole equality employs (174). Since the inequality holds for all  $\sigma \in \mathcal{S}_F$ , we conclude that

$$R_\alpha^{\mathcal{P}}(\rho) \geq R_\alpha^{\mathcal{P}}(\mathcal{F}(\rho)). \quad (194)$$

We can also define the  $\alpha$ -logarithmic resourcefulness of a quantum channel  $\mathcal{N}$  for all  $\alpha \geq 1$  and with respect to  $\mathcal{P}$  as

$$R_\alpha^{\mathcal{P}}(\mathcal{N}) \equiv \sup_{\rho} R_\alpha^{\mathcal{P}}(\mathcal{N}(\rho)). \quad (195)$$

We think it is interesting to explore particular instantiations of this resourcefulness measure for particular resource theories, but we leave this for future work.

## VIII. CONCLUSION

In summary, we have defined an ordered family of  $\alpha$ -logarithmic negativity entanglement measures that interpolate between the logarithmic negativity [13, 14] and the  $\kappa$ -entanglement [19, 20]. We proved that this family of entanglement measures satisfies LOCC monotonicity, normalization, faithfulness, subadditivity, and can be computed by convex optimization. The proofs of these properties built upon prior results from [31] and [32]. We also proved that it is generally not convex nor is it monogamous. Finally, we defined the  $\alpha$ -logarithmic negativity of a quantum channel as a generalization of the measure for bipartite states, and we showed how to generalize many of the concepts to arbitrary resource theories.

Going forward from here, we think that it is worthwhile to establish the definition and properties of the  $\alpha$ -logarithmic negativity of quantum states in the von Neumann algebra setting. The tools developed in [37] should be useful for this task. Note here that we do think that it is necessary to use the approach of [37] over that given in [36], because our definition of  $\alpha$ -logarithmic negativity requires working with a Hermitian operator and a positive semi-definite one. Since [37] builds upon the approach given in [31] for the finite-dimensional case, we suspect that the methods of [37] should lead to a cogent notion of  $\alpha$ -logarithmic negativity of quantum states in the von Neumann algebra setting. One could also define and explore the  $\alpha$ -logarithmic negativity of a quantum channel in the von Neumann algebra setting, generalizing Definition 3.

As the logarithmic negativity is frequently used to study entanglement in quantum many-body physics [38, 39, 72] and holographic field theories [73, 74], the  $\alpha$ -logarithmic negativity may also shed light on these areas.

It is an interesting open question to determine whether  $\alpha$ -logarithmic negativity can be experimentally measured via witnessing. This is true for the logarithmic negativity [75] (see also [76]) and the  $\kappa$ -entanglement [20], and so it remains open to determine the answer for  $\alpha \in (1, \infty)$ .

We also think that it is interesting to explore generalizations of quantum information measures that accept Hermitian operators as input, rather than just positive semi-definite ones, the latter being the traditional approach in quantum information theory. The former approach has been employed fruitfully not only here, but also in recent work that gave exact characterizations of

the one-shot distillable entanglement of a bipartite quantum state [77] and the one-shot distillable coherence of a quantum state [78].

*Note:* After a preprint of our paper appeared online as [79], we noticed another work [80] that proposed different generalizations of the logarithmic negativity.

## ACKNOWLEDGMENTS

We are grateful to Nilanjana Datta and Gilad Gour for feedback during the Workshop *Mathematical Aspects in Current Quantum Information Theory 2019* (MAQIT 2019) held at Seoul National University, Korea. Part of this work was done when XW was at the University of Maryland. MMW acknowledges support from the National Science Foundation under Award nos. 1350397 and 1907615.

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