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## Geometric distinguishability measures limit quantum channel estimation and discrimination

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# Geometric distinguishability measures limit quantum channel estimation and discrimination

Vishal Katariya\*

Mark M. Wilde<sup>\*,†</sup>

March 2, 2021

## Abstract

Quantum channel estimation and discrimination are fundamentally related information processing tasks of interest in quantum information science. In this paper, we analyze these tasks by employing the right logarithmic derivative Fisher information and the geometric Rényi relative entropy, respectively, and we also identify connections between these distinguishability measures. A key result of our paper is that a chain-rule property holds for the right logarithmic derivative Fisher information and the geometric Rényi relative entropy for the interval  $\alpha \in (0, 1)$  of the Rényi parameter  $\alpha$ . In channel estimation, these results imply a condition for the unattainability of Heisenberg scaling, while in channel discrimination, they lead to improved bounds on error rates in the Chernoff and Hoeffding error exponent settings. More generally, we introduce the amortized quantum Fisher information as a conceptual framework for analyzing general sequential protocols that estimate a parameter encoded in a quantum channel. We then use this framework, beyond the aforementioned application, to show that Heisenberg scaling is not possible when a parameter is encoded in a classical–quantum channel. We then identify a number of other conceptual and technical connections between the tasks of estimation and discrimination and the distinguishability measures involved in analyzing each. As part of this work, we present a detailed overview of the geometric Rényi relative entropy of quantum states and channels, as well as its properties, which may be of independent interest.

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## 1 Introduction

Quantum channel discrimination and estimation are fundamental tasks in quantum information science. Channel discrimination refers to the task of distinguishing two (or more) quantum channels, while quantum channel estimation is a generalization of this scenario. Instead of determining an unknown channel selected from a finite set, the goal of channel estimation is to estimate a particular member chosen from a continuously parameterized set of quantum channels. The simplest channel discrimination task consists of discriminating two channels selected from a set  $\{\mathcal{N}_\theta\}_{\theta \in \{1,2\}}$ , whereas the simplest estimation task consists of identifying a particular member of a continuously parameterized set of channels  $\{\mathcal{N}_\theta\}_{\theta \in \Theta}$ , where  $\Theta \subseteq \mathbb{R}$ . Theoretical studies in both the discrimination and estimation of quantum channels have been applied in a variety of settings, including quantum illumination [1], phase estimation using optical interferometry [2–4], and gravitational wave detection [5–8].

In classical parameter estimation, the unknown parameter  $\theta$  is encoded in a probability distribution  $p_\theta(x)$  with associated random variable  $X$ . One tries to guess its value from a realization  $x$  of  $X$  by calculating an estimator  $\hat{\theta}(x)$  of the true value  $\theta$ . The most common

measure of performance employed in estimation theory is the mean-squared error, defined as  $\mathbb{E}[(\hat{\theta}(X) - \theta)^2]$ . For an unbiased estimator satisfying  $\mathbb{E}[\hat{\theta}(X)] = \theta$ , the mean-squared error is equal to  $\text{Var}(\hat{\theta}(X))$ , and one of the fundamental results of classical estimation theory is the Cramer-Rao lower bound (CRB) on the mean-squared error of an unbiased estimator:

$$\text{Var}(\hat{\theta}(X)) \geq \frac{1}{I_F(\theta; \{p_\theta\}_\theta)}. \quad (1.1)$$

The lower bound features the Fisher information, defined as the following function of the probability distribution family  $\{p_\theta\}_\theta$ :

$$I_F(\theta; \{p_\theta\}_\theta) = \mathbb{E}[(\partial_\theta \ln p_\theta(X))^2] = \int dx p_\theta(x) (\partial_\theta \ln p_\theta(x))^2, \quad (1.2)$$

where we employ the shorthand  $\partial_\theta(\cdot) \equiv \frac{\partial}{\partial \theta}(\cdot)$ . Recalling the interpretation of  $-\ln p_\theta(x)$  as the surprisal of the realization  $x$ , it follows that  $\partial_\theta[-\ln p_\theta(x)]$  is the rate of change of the surprisal with the parameter  $\theta$  (surprisal rate). After noticing that the expected surprisal rate vanishes, by applying the conservation of probability, it follows that the Fisher information is equal to the variance of the surprisal rate, thus characterizing its fluctuations [9, 10]. If one generates  $n$  independent samples  $x^n \equiv x_1, \dots, x_n$  of  $p_\theta(x)$ , described by the random sequence  $X^n \equiv X_1, \dots, X_n$ , and forms an unbiased estimator  $\hat{\theta}(x^n)$ , then the Fisher information increases linearly with  $n$  and the CRB becomes as follows:

$$\text{Var}(\hat{\theta}(X^n)) \geq \frac{1}{n I_F(\theta; \{p_\theta\}_\theta)}, \quad (1.3)$$

which is how it is commonly employed in applications.

In quantum estimation, the parameter  $\theta$  is encoded in a quantum state  $\rho_\theta$  or a quantum channel  $\mathcal{N}_\theta$ , and generally, it is possible to attain better-than-classical scaling in error by using quantum resources such as entanglement and collective measurements. When formulating a quantum generalization of the Cramer-Rao bound and Fisher information, it is necessary to find a quantum generalization of the logarithmic derivative  $\partial_\theta \ln p_\theta(x)$  in (1.2). However, the noncommutative nature of quantum mechanics yields an infinite number of logarithmic derivatives of  $\rho_\theta$ . To demonstrate this point, consider that we can define a family of parameterized logarithmic derivative operators  $\{D_\theta^{(p)}\}_p$  with  $p \in [0, 1/2]$  as follows:  $\partial_\theta \rho_\theta := p D_\theta^{(p)} \rho_\theta + (1 - p) \rho_\theta D_\theta^{(p)}$ . Each  $D_\theta^{(p)}$  collapses to the scalar logarithmic derivative in the classical case. The two most studied logarithmic derivatives are specific instances of  $D_\theta^{(p)}$ : the symmetric logarithmic derivative (SLD) corresponding to  $p = 1/2$  [11] and the right logarithmic derivative (RLD) corresponding to  $p = 0$  [12]. At least two quantum Fisher informations can be defined based on these specific possibilities. By far, the SLD Fisher information has been the most studied, on account of it providing the tightest quantum Cramer-Rao bound (QCRB) in single parameter estimation of quantum states, while also being achievable when many copies of the state are available. The recent review [13] provides an in-depth study of these and other notions in quantum estimation.

In this paper, we focus on the task of estimating a single unknown parameter  $\theta$  encoded in a quantum channel  $\mathcal{N}_\theta$ . This task has been studied extensively in prior work [14–29], and the most general setting for this problem is known as the sequential setting [23, 30–32], in which one can interact with the channel  $n$  independent times in the most general way allowed by quantum mechanics. Heisenberg scaling refers to the quantum Fisher information scaling as  $n^2$ , where  $n$  is the number of channel uses, or as  $t^2$ , where  $t$  is the total probing time. One fundamental question for channel estimation is whether Heisenberg scaling can be achieved when estimating a particular quantum channel.

Our approach to the channel estimation problem involves defining the amortized Fisher information of a family of channels, which is in the same spirit as the amortized channel divergence introduced in [33]. The amortized Fisher information provides a compact mathematical framework for studying the difference between sequential and parallel estimation strategies, just as the amortized channel divergence does for channel discrimination [33]. Specifically, we prove that the amortized Fisher information is a generic bound for all channel estimation protocols (called the “meta-converse” for channel estimation).

One key result of our paper is a chain rule for the RLD Fisher information, with a consequence being that amortization does not increase the RLD Fisher information of quantum channels. Importantly, when combining this result with the aforementioned meta-converse, it follows that Heisenberg scaling is unattainable for a channel family if its RLD Fisher information is finite. This latter result generalizes a finding of [20] beyond parallel strategies for channel estimation to the more general sequential strategies. Let us also note that evaluating the finiteness condition for the RLD Fisher information is a simpler task than evaluating the RLD (or SLD) Fisher information itself.

Turning to the related task of channel discrimination, a key tool that we employ for this purpose is the geometric Rényi relative entropy. This distinguishability measure has its roots in [34], and it was further developed in [35, 36] (see also [37, 38]). It was given the name “geometric Rényi relative entropy” in [39] because it is a function of the matrix geometric mean of its arguments. It was also used to great effect in [39] to bound quantum channel capacities and error rates of channel discrimination in the asymmetric setting. We continue to use it in this vein, in particular, by improving upper bounds on error rates of channel discrimination in the symmetric setting (specifically, the Chernoff and Hoeffding error exponents). Due to the chain rule of the geometric Rényi relative entropy (and hence amortization collapse of the related channel function), the bounds that we report here are both single-letter and efficiently computable via semi-definite programs. Our bounds also improve upon those found recently in [33, 40].

As mentioned earlier, channel estimation is a generalization of channel discrimination to the case in which the unknown parameter is continuous. We devote the last section of our paper to bringing out connections between the two tasks. We observe that the RLD Fisher information arises from taking the limit of the geometric Rényi relative entropy of two infinitesimally close elements of a family of channels. Therefore, in this sense, we see that the QCRB arising from the RLD Fisher information has the geometric Rényi relative entropy underlying it. Further, we connect properties of the SLD and RLD Fisher

informations to the corresponding properties of their underlying distance measures (fidelity and geometric Rényi relative entropy, respectively).

Our paper is structured as follows. First, we present a more detailed, yet brief overview of our results in Section 2. In Section 3, we review some notation and mathematical identities used throughout our paper. In Section 4, we present the information-processing tasks of channel estimation and discrimination. Section 5 contains all of our results regarding bounds on channel estimation. Section 6 introduces the geometric Rényi relative entropy and contains our bounds on channel discrimination. Section 7 brings out connections between estimation and discrimination, building on our results from the previous two sections. In Section 8, we conclude by summarizing our results and outlining future work. The appendices of our paper contain many detailed mathematical proofs, as well as a detailed overview of the geometric Rényi relative entropy of quantum states and channels (Appendices H and I).

## 2 Summary of Results

Here we summarize the main contributions and results of our paper:

1. In Section 5.1, we provide definitions for the SLD and RLD Fisher informations of quantum state families. These definitions are accompanied by specific conditions that govern the finiteness of the quantities. We also prove that the SLD and RLD Fisher informations are physically consistent, i.e., that the definitions provided are consistent with a limiting procedure in which some additive noise vanishes.
2. In Section 5.3, we define the generalized Fisher information of quantum state and channel families, with the aim of establishing a number of properties that arise solely from data processing. We also provide finiteness conditions for the SLD and RLD Fisher informations of quantum channels, which are helpful for determining whether Heisenberg scaling can occur in channel estimation. In this same section, we also introduce the idea of and define the amortized Fisher information of quantum channel families, as a generalization of the amortized channel divergence introduced in [33]. We then establish a meta-converse for all channel estimation protocols, which demonstrates that amortized Fisher information is a generic bound for all such protocols.
3. In Section 5.4, we cast the SLD and RLD Fisher informations as optimization problems. Specifically, we cast the SLD Fisher information of quantum states as a semi-definite program, the SLD Fisher information of quantum channels as a bilinear program, and the RLD Fisher information of both quantum states and channels as a semi-definite program. We also provide a quadratically constrained program for the root SLD Fisher information of quantum states, whose formulation is used to establish the chain rule property of the root SLD Fisher information. We provide duals to our semi-definite programs in all cases.

4. In Section 5.5.1, we show that sequential estimation strategies provide no advantage over parallel estimation strategies for classical-quantum channel families.
5. In Sections 5.5 and 5.6, we utilize the SLD and RLD Fisher information of quantum channels to place lower bounds on the error of sequential parameter estimation protocols. We prove chain rule properties for the RLD Fisher information and the root SLD Fisher information, which imply an amortization collapse for these quantities.
6. An important corollary of the amortization collapse of the RLD Fisher information is a condition for the unattainability of Heisenberg scaling. Specifically, we prove that if the RLD Fisher information of a channel family is finite, then Heisenberg scaling is unattainable for it. Thus, we provide an operational consequence of the finiteness condition for the RLD Fisher information of quantum channels.
7. When estimating a single parameter, the RLD Fisher information is never smaller than the SLD Fisher information. We study an example in Section 5.7 regarding the effectiveness of the RLD Fisher information as a performance bound when estimating various parameters encoded in a generalized amplitude damping channel.
8. In Sections 6.1 and 6.2, we provide a limit-based formula for the geometric Rényi relative entropy, and then we establish consistency of this formula with more explicit formulas for the whole range  $\alpha \in (0, 1) \cup (1, \infty)$ . We review existing and also establish new properties of the geometric Rényi relative entropy of quantum states and channels.
9. In the rest of Section 6, we use the geometric Rényi relative entropy to improve currently known upper bounds on error rates in quantum channel discrimination. We (a) use the geometric fidelity to place an upper bound on the error exponent in the symmetric Chernoff setting and (b) introduce the Belavkin–Staszewski divergence sphere as an upper bound on the Hoeffding error exponent. We also study a task called “sequential channel discrimination with repetition” and establish an upper bound on its Chernoff and Hoeffding error exponents.
10. Finally, in Section 7, we bring out a number of conceptual and technical connections between the tasks of channel estimation and discrimination.

### 3 Quantum information preliminaries

We begin by recalling some basic facts and identities that appear often in this paper and more generally in quantum information. For further background, we refer to the textbooks [41–44].

A quantum state is described by a density operator, which is a positive semi-definite operator with trace equal to one and often denoted by  $\rho$ ,  $\sigma$ ,  $\tau$ , etc. A quantum channel



$\mathcal{N}_{A \rightarrow B}$  taking an input quantum system  $A$  to an output quantum system  $B$  is described by a completely positive, trace-preserving map. In this paper, we deal exclusively with finite-dimensional systems, but it is clear that many of the concepts and results should generalize to quantum states and channels acting on separable Hilbert spaces.

Let  $|\Gamma\rangle_{RA}$  denote the unnormalized maximally entangled vector:

$$|\Gamma\rangle_{RA} := \sum_i |i\rangle_R |i\rangle_A, \quad (3.1)$$

where  $\{|i\rangle_R\}_i$  and  $\{|i\rangle_A\}_i$  are orthonormal bases for the isomorphic Hilbert spaces  $\mathcal{H}_R$  and  $\mathcal{H}_A$ . We repeatedly use the fact that a pure bipartite state  $|\psi\rangle_{RA}$  can be written as  $(X_R \otimes I_A)|\Gamma\rangle_{RA}$  where  $X_R$  is an operator satisfying  $\text{Tr}[X_R^\dagger X_R] = 1$ . For a linear operator  $M$ , the following transpose trick identity holds

$$(I_R \otimes M_A) |\Gamma\rangle_{RA} = (M_R^T \otimes I_A) |\Gamma\rangle_{RA}, \quad (3.2)$$

where  $M^T$  denotes the transpose of  $M$  with respect to the orthonormal basis  $\{|i\rangle_R\}_i$ . For a linear operator  $K_R$ , the following identity holds

$$\langle \Gamma|_{RA} (K_R \otimes I_A) |\Gamma\rangle_{RA} = \text{Tr}[K_R]. \quad (3.3)$$

The Choi operator  $\Gamma_{RB}^{\mathcal{N}}$  of a quantum channel  $\mathcal{N}_{A \rightarrow B}$  is defined as

$$\Gamma_{RB}^{\mathcal{N}} := \mathcal{N}_{A \rightarrow B}(\Gamma_{RA}), \quad (3.4)$$

where

$$\Gamma_{RA} := |\Gamma\rangle\langle\Gamma|_{RA}. \quad (3.5)$$

The Choi operator is positive semi-definite and satisfies the following property as a consequence of  $\mathcal{N}_{A \rightarrow B}$  being trace preserving:

$$\text{Tr}_B[\Gamma_{RB}^{\mathcal{N}}] = I_R. \quad (3.6)$$

The following post-selected teleportation identity [45] allows for writing the output of a quantum channel  $\mathcal{N}_{A \rightarrow B}$  on an input quantum state  $\rho_{RA}$  in the following way:

$$\mathcal{N}_{A \rightarrow B}(\rho_{RA}) = \langle \Gamma|_{AS} \rho_{RA} \otimes \Gamma_{SB}^{\mathcal{N}} |\Gamma\rangle_{AS}, \quad (3.7)$$

where  $S$  is a system isomorphic to the channel input system  $A$ .

## 4 Setting of quantum channel parameter estimation and discrimination

We now recall the two related tasks of channel parameter estimation and discrimination. In the first task, one is interested in estimating an unknown channel selected from a continuously parameterized family of channels, while in the latter, the goal is the same but the unknown channel is selected from a finite set. The metrics used to quantify performance are different and are explained below. Also, in this paper, we focus exclusively on channel discrimination of just two quantum channels.

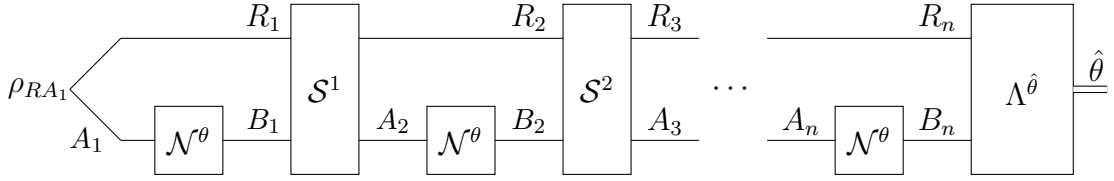


Figure 1: Processing  $n$  uses of channel  $\mathcal{N}^\theta$  in a sequential or adaptive manner is the most general approach to channel parameter estimation or discrimination. The  $n$  uses of the channel are interleaved with  $n$  quantum channels  $\mathcal{S}^1$  through  $\mathcal{S}^{n-1}$ , which can also share memory systems with each other. The final measurement's outcome is then used to obtain an estimate of the unknown parameter  $\theta$ . If  $\theta \in \Theta \subseteq \mathbb{R}$ , then this is the task of parameter estimation. If, on the other hand,  $\theta \in \{1, 2\}$ , then this task corresponds to channel discrimination.

## 4.1 Quantum channel parameter estimation

Let us now discuss channel parameter estimation in more detail. Let  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  denote a family of quantum channels with input system  $A$  and output system  $B$ , such that each channel in the family is parameterized by a single real parameter  $\theta \in \Theta \subseteq \mathbb{R}$ , where  $\Theta$  is the parameter set. The problem we consider is this: given a particular unknown channel  $\mathcal{N}_{A \rightarrow B}^\theta$ , how well can we estimate  $\theta$  when allowed to probe the channel  $n$  times? There are various ways that one can probe the quantum channel  $n$  times, but each such procedure results in a probability distribution  $p_\theta(x)$  for a final measurement outcome  $x$ , with corresponding random variable  $X$ . This distribution  $p_\theta(x)$  depends on the unknown parameter  $\theta$ . Using the measurement outcome  $x$ , one formulates an estimate  $\hat{\theta}(x)$  of the unknown parameter. An unbiased estimator satisfies  $\mathbb{E}_{p_\theta}[\hat{\theta}(X)] = \theta$ . For an unbiased estimator (on which we focus exclusively here), the mean squared error (MSE) is a commonly considered measure of performance:

$$\text{Var}(\hat{\theta}(X)) := \mathbb{E}[(\hat{\theta}(X) - \theta)^2] = \int dx p_\theta(x) (\hat{\theta}(x) - \theta)^2. \quad (4.1)$$

One major question of interest is to ascertain the optimal scaling of the MSE with the number  $n$  of channel uses. We note that much work has been done on this topic, with an inexhaustive reference list given by [14–24, 26–28]. We also clarify that our approach adopts the frequentist approach to parameter estimation. In general, the MSE and Cramer–Rao bounds may depend on the value of the unknown parameter, in contrast with the more general paradigm of Bayesian parameter estimation [46]. This is alleviated by enforcing the unbiasedness condition.

The most general channel estimation procedure is depicted in Figure 1. A sequential or adaptive strategy that makes  $n$  calls to the channel is specified in terms of an input quantum state  $\rho_{R_1 A_1}$ , a set of interleaved channels  $\{\mathcal{S}_{R_i B_i \rightarrow R_{i+1} A_{i+1}}^i\}_{i=1}^{n-1}$ , and a final quantum measurement  $\{\Lambda_{R_n B_n}^{\hat{\theta}}\}_{\hat{\theta}}$  that outputs an estimate  $\hat{\theta}$  of the unknown parameter (here we incorporate any classical post-processing of a preliminary measurement outcome  $x$  to

generate the estimate  $\hat{\theta}$  as part of the final measurement). Note that any particular strategy  $\{\rho_{R_1 A_1}, \{\mathcal{S}_{R_i B_i \rightarrow R_{i+1} A_{i+1}}^i\}_{i=1}^{n-1}, \{\Lambda_{R_n B_n}^{\hat{\theta}}\}_{\hat{\theta}}\}$  employed does not depend on the actual value of the unknown parameter  $\theta$ . We make the following abbreviation for a fixed strategy in what follows:

$$\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}}\} \equiv \{\rho_{R_1 A_1}, \{\mathcal{S}_{R_i B_i \rightarrow R_{i+1} A_{i+1}}^i\}_{i=1}^{n-1}, \{\Lambda_{R_n B_n}^{\hat{\theta}}\}_{\hat{\theta}}\}. \quad (4.2)$$

The strategy begins with the estimator preparing the input quantum state  $\rho_{R_1 A_1}$  and sending the  $A_1$  system into the channel  $\mathcal{N}_{A_1 \rightarrow B_1}^{\theta}$ . The first channel  $\mathcal{N}_{A_1 \rightarrow B_1}^{\theta}$  outputs the system  $B_1$ , which is then available to the estimator. The resulting state is

$$\rho_{R_1 B_1}^{\theta} := \mathcal{N}_{A_1 \rightarrow B_1}^{\theta}(\rho_{R_1 A_1}). \quad (4.3)$$

The estimator adjoins the system  $B_1$  to system  $R_1$  and applies the channel  $\mathcal{S}_{R_1 B_1 \rightarrow R_2 A_2}^1$ , leading to the state

$$\rho_{R_2 A_2}^{\theta} := \mathcal{S}_{R_1 B_1 \rightarrow R_2 A_2}^1(\rho_{R_1 B_1}^{\theta}). \quad (4.4)$$

The channel  $\mathcal{S}_{R_1 B_1 \rightarrow R_2 A_2}^1$  can take an action conditioned on information in the system  $B_1$ , which itself might contain some partial information about the unknown parameter  $\theta$ . The estimator then inputs the system  $A_2$  into the second use of the channel  $\mathcal{N}_{A_2 \rightarrow B_2}^{\theta}$ , which outputs a system  $B_2$  and gives the state

$$\rho_{R_2 B_2}^{\theta} := \mathcal{N}_{A_2 \rightarrow B_2}^{\theta}(\rho_{R_2 A_2}^{\theta}). \quad (4.5)$$

This process repeats  $n - 2$  more times, for which we have the intermediate states

$$\rho_{R_i B_i}^{\theta} := \mathcal{N}_{A_i \rightarrow B_i}^{\theta}(\rho_{R_i A_i}^{\theta}), \quad (4.6)$$

$$\rho_{R_i A_i}^{\theta} := \mathcal{S}_{R_{i-1} B_{i-1} \rightarrow R_i A_i}^{i-1}(\rho_{R_{i-1} B_{i-1}}^{\theta}), \quad (4.7)$$

for  $i \in \{3, \dots, n\}$ , and at the end, the estimator has systems  $R_n$  and  $B_n$ . We define  $\omega_{R_n B_n}^{\theta}$  to be the final state of the estimation protocol before the final measurement  $\{\Lambda_{R_n B_n}^{\hat{\theta}}\}_{\hat{\theta}}$ :

$$\omega_{R_n B_n}^{\theta} := (\mathcal{N}_{A_n \rightarrow B_n}^{\theta} \circ \mathcal{S}_{R_{n-1} B_{n-1} \rightarrow R_n A_n}^{n-1} \circ \dots \circ \mathcal{S}_{R_1 B_1 \rightarrow R_2 A_2}^1 \circ \mathcal{N}_{A_1 \rightarrow B_1}^{\theta})(\rho_{R_1 A_1}). \quad (4.8)$$

The estimator finally performs a measurement  $\{\Lambda_{R_n B_n}^{\hat{\theta}}\}_{\hat{\theta}}$  that outputs an estimate  $\hat{\theta}$  of the unknown parameter  $\theta$ . The conditional probability for the estimate  $\hat{\theta}$  given the unknown parameter  $\theta$  is given by the Born rule:

$$p_{\theta}(\hat{\theta}) = \text{Tr}[\Lambda_{R_n B_n}^{\hat{\theta}} \omega_{R_n B_n}^{\theta}]. \quad (4.9)$$

As we stated above, any particular strategy does not depend on the value of the unknown parameter  $\theta$ , but the states at each step of the protocol do depend on  $\theta$  through the successive probings of the underlying channel  $\mathcal{N}_{A \rightarrow B}^{\theta}$ .

Note that such a sequential strategy contains a parallel or non-adaptive strategy as a special case: the system  $R_1$  can be arbitrarily large and divided into subsystems, with the only role of the interleaved channels  $\mathcal{S}_{R_i B_i \rightarrow R_{i+1} A_{i+1}}^i$  being that they redirect these subsystems to

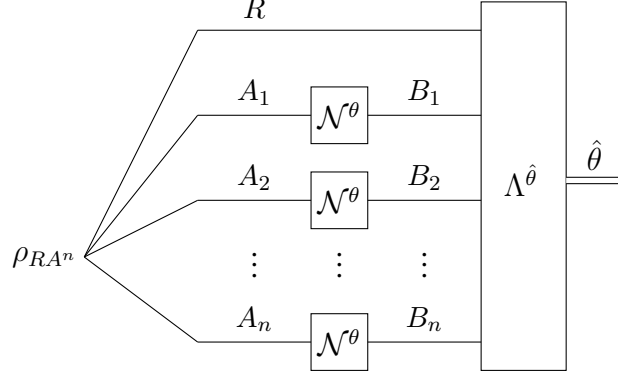


Figure 2: Processing  $n$  uses of channel  $\mathcal{N}^\theta$  in a parallel manner. The  $n$  channels are called in parallel, allowing for entanglement to be shared among input systems  $A_1$  through  $A_n$ , along with a quantum memory system  $R$ . A collective measurement is made, with its outcome being an estimate  $\hat{\theta}$  for the unknown parameter  $\theta$ . Parallel strategies form a special case of sequential ones, and therefore parallel strategies are no more powerful than sequential ones.

be the inputs of future calls to the channel (as would be the case in any non-adaptive strategy for estimation or discrimination). Figure 2 depicts a parallel or non-adaptive channel estimation strategy.

One main goal of the present paper is to place a lower bound on the MSE of a general sequential strategy for channel parameter estimation, such that the lower bound is a function solely of the channel family  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  and the number  $n$  of channel uses. Such a bound indicates a fundamental limitation for channel estimation that cannot be improved upon by any possible estimation strategy.

## 4.2 Quantum channel discrimination

The operational setting for quantum channel discrimination is exactly as described above, and the only difference is that  $\theta \in \Theta = \{1, \dots, d\}$  for some integer  $d$ . In this work, we focus exclusively on the case  $d = 2$  for channel discrimination.

### 4.2.1 Symmetric setting

In this subsection, we recall the setting of symmetric or Bayesian channel discrimination in which there is a prior probability distribution for  $\theta$ :  $\Pr[\theta = 1] = p \in (0, 1)$  and  $\Pr[\theta = 2] = 1 - p$ . The relevant measure of performance of a given channel discrimination strategy  $\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}}\}$  is the expected error probability:

$$\begin{aligned} p_e^{(n)}(\{\mathcal{N}^\theta\}_\theta, \{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}}\}) \\ = \Pr[\hat{\theta} \neq \theta] \end{aligned} \tag{4.10}$$

$$= \Pr[\theta = 1] \Pr[\hat{\theta} = 2|\theta = 1] + \Pr[\theta = 2] \Pr[\hat{\theta} = 1|\theta = 2] \quad (4.11)$$

$$= p \operatorname{Tr}[\Lambda_{R_n B_n}^{\hat{\theta}=2} \omega_{R_n B_n}^{\theta=1}] + (1-p) \operatorname{Tr}[\Lambda_{R_n B_n}^{\hat{\theta}=1} \omega_{R_n B_n}^{\theta=2}] \quad (4.12)$$

$$= p \operatorname{Tr}[(I_{R_n B_n} - \Lambda_{R_n B_n}) \omega_{R_n B_n}^{\theta=1}] + (1-p) \operatorname{Tr}[\Lambda_{R_n B_n} \omega_{R_n B_n}^{\theta=2}], \quad (4.13)$$

where  $\omega_{R_n B_n}^{\theta}$  is the state at the end of the protocol, as defined in (4.8), and we made the abbreviation  $\Lambda_{R_n B_n} \equiv \Lambda_{R_n B_n}^{\hat{\theta}=1}$ . We can also write the error probability in conventional notation as follows:

$$p_e^{(n)}(\{\mathcal{N}^{\theta}\}_{\theta}, \{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}}\}) := p \alpha_n(\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}}\}) + (1-p) \beta_n(\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}}\}), \quad (4.14)$$

where  $\alpha_n$  is called the Type I error probability and  $\beta_n$  the Type II error probability:

$$\alpha_n(\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}}\}) := \operatorname{Tr}[(I_{R_n B_n} - \Lambda_{R_n B_n}) \omega_{R_n B_n}^{\theta=1}], \quad (4.15)$$

$$\beta_n(\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}}\}) := \operatorname{Tr}[\Lambda_{R_n B_n} \omega_{R_n B_n}^{\theta=2}]. \quad (4.16)$$

By optimizing the final measurement, we arrive at the following optimized error probability:

$$p_e^{(n)}(\{\mathcal{N}^{\theta}\}_{\theta}, \mathcal{S}^{(n)}) := \inf_{\{\Lambda^{\hat{\theta}}\}_{\hat{\theta}}} p_e^{(n)}(\{\mathcal{N}^{\theta}\}_{\theta}, \mathcal{S}^{(n)}, \Lambda^{\hat{\theta}}) \quad (4.17)$$

$$= \frac{1}{2} (1 - \|p \omega_{R_n B_n}^{\theta=1} - (1-p) \omega_{R_n B_n}^{\theta=2}\|_1), \quad (4.18)$$

where the last equality follows from a standard result in quantum state discrimination theory [47–49]. We can perform a further optimization over all discrimination strategies to arrive at the optimal expected error probability:

$$p_e^{(n)}(\{\mathcal{N}^{\theta}\}_{\theta}) := \inf_{\mathcal{S}^{(n)}} p_e^{(n)}(\{\mathcal{N}^{\theta}\}_{\theta}, \mathcal{S}^{(n)}) \quad (4.19)$$

$$= \frac{1}{2} (1 - \|p(\mathcal{N}^{\theta=1})^{(n)} - (1-p)(\mathcal{N}^{\theta=2})^{(n)}\|_{\diamond n}), \quad (4.20)$$

where the quantum strategy distance [50–52] (see also [53, 54]) is defined as

$$\|p(\mathcal{N}^{\theta=1})^{(n)} - (1-p)(\mathcal{N}^{\theta=2})^{(n)}\|_{\diamond n} := \sup_{\mathcal{S}^{(n)}} \|p \omega_{R_n B_n}^{\theta=1} - (1-p) \omega_{R_n B_n}^{\theta=2}\|_1. \quad (4.21)$$

Although the strategy distance can be computed by means of a semi-definite program [52], this fact is only useful for small  $n$  and small-dimensional channels because the difficulty in calculating grows quickly as  $n$  becomes larger (see [55] for explicit examples of the calculation of the strategy distance).

As such, we are interested in the exponential rate at which the expected error probability converges to zero in the limit as  $n$  becomes larger:

$$\xi_n(p, \{\mathcal{N}^{\theta}\}_{\theta}) := -\frac{1}{n} \ln p_e^{(n)}(\{\mathcal{N}^{\theta}\}_{\theta}). \quad (4.22)$$

This quantity is called the non-asymptotic Chernoff exponent of quantum channels [33], and its asymptotic counterparts are defined as

$$\underline{\xi}(\{\mathcal{N}^\theta\}_\theta) := \liminf_{n \rightarrow \infty} \xi_n(p, \{\mathcal{N}^\theta\}_\theta), \quad \bar{\xi}(\{\mathcal{N}^\theta\}_\theta) := \limsup_{n \rightarrow \infty} \xi_n(p, \{\mathcal{N}^\theta\}_\theta). \quad (4.23)$$

The asymptotic quantities  $\underline{\xi}(\{\mathcal{N}^\theta\}_\theta)$  and  $\bar{\xi}(\{\mathcal{N}^\theta\}_\theta)$  are independent of the particular value of  $p \in (0, 1)$ .

Another goal of the present paper is to establish an improved upper bound on  $\xi_n(p, \{\mathcal{N}^\theta\}_\theta)$  and thus on  $\bar{\xi}(\{\mathcal{N}^\theta\}_\theta)$ .

#### 4.2.2 Asymmetric setting – Hoeffding error exponent

Another setting of interest for channel discrimination is called the Hoeffding error exponent setting (see, e.g., [33, 56]). In this case, there is no assumed prior probability on the parameter  $\theta$ . In this setting, the Type II error probability  $\beta_n$  in (4.16) is constrained to decrease exponentially at a fixed rate  $r > 0$ , and the objective is to determine the optimal exponential rate of decay for the Type I error probability  $\alpha_n$  in (4.15), subject to this constraint. Formally, the non-asymptotic Hoeffding error exponent is defined as follows [33]:

$$B_n(r, \{\mathcal{N}^\theta\}_\theta) := \sup_{\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}}\}} \left\{ -\frac{1}{n} \ln \alpha_n(\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}}\}) \mid -\frac{1}{n} \ln \beta_n(\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}}\}) \geq r \right\}, \quad (4.24)$$

and its asymptotic variants as follows:

$$\underline{B}(r, \{\mathcal{N}^\theta\}_\theta) := \liminf_{n \rightarrow \infty} B_n(r, \{\mathcal{N}^\theta\}_\theta), \quad \bar{B}(r, \{\mathcal{N}^\theta\}_\theta) := \limsup_{n \rightarrow \infty} B_n(r, \{\mathcal{N}^\theta\}_\theta). \quad (4.25)$$

#### 4.2.3 Sequential channel discrimination with repetition

As a variation of the general channel discrimination setting discussed in Section 4.2, we can consider a more specialized setting that we call sequential channel discrimination with repetition. In this setting, the general,  $n$ -round channel discrimination protocol discussed in Section 4.2 is repeated  $m$  times, such that the final state of the protocol is  $(\omega_{R_n B_n}^\theta)^{\otimes m}$ , where  $\omega_{R_n B_n}^\theta$  is defined in (4.8). One can then perform a collective measurement  $\{\Lambda_{(R_n B_n)^m}^{\hat{\theta}}\}$  on this final state, where the notation  $(R_n B_n)^m$  is a shorthand for all of the remaining systems at the end of the  $nm$  calls to the channel. We abbreviate such a protocol with the notation  $\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}, (m)}\}$ , which indicates that the protocol  $\mathcal{S}^{(n)}$  is fixed, but the final measurement is performed on  $m$  systems. The two kinds of errors in such a protocol are then defined as follows:

$$\alpha_{n,m}(\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}, (m)}\}) := \text{Tr}[(I_{R_n B_n} - \Lambda_{(R_n B_n)^m}^{\hat{\theta}=1})(\omega_{R_n B_n}^{\theta=1})^{\otimes m}], \quad (4.26)$$

$$\beta_{n,m}(\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}, (m)}\}) := \text{Tr}[\Lambda_{(R_n B_n)^m}^{\hat{\theta}=1}(\omega_{R_n B_n}^{\theta=2})^{\otimes m}]. \quad (4.27)$$

This somewhat specialized setting has been considered in the context of quantum channel estimation [32]. We refer to such a protocol as an  $(n, m)$  protocol for sequential channel discrimination with repetition.

Of course, sequential channel discrimination with repetition is special kind of channel discrimination protocol of the form discussed in Section 4.2, in which the channel is called  $nm$  times. Thus, the optimal error probabilities involved in  $(n, m)$  sequential channel discrimination with repetition cannot be smaller than the optimal error probabilities in a general channel discrimination protocol that calls the channel  $nm$  times. At the same time, a general channel discrimination protocol that calls the channel  $n$  times is trivially an  $(n, 1)$  sequential channel discrimination protocol with repetition (however, the phrase “with repetition” is not particular apt in this specialized instance).

We can define the non-asymptotic Chernoff and Hoeffding error exponents in a similar way to how they were defined in the previous section. The non-asymptotic Chernoff exponent is defined as

$$\xi_{n,m}(p, \{\mathcal{N}^\theta\}_\theta) := -\frac{1}{nm} \ln p_e^{(n,m)}(\{\mathcal{N}^\theta\}_\theta), \quad (4.28)$$

where

$$p_e^{(n,m)}(\{\mathcal{N}^\theta\}_\theta) := \inf_{\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta},(m)}\}} p \alpha_{n,m}(\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta},(m)}\}) + (1-p) \beta_{n,m}(\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta},(m)}\}), \quad (4.29)$$

and the non-asymptotic Hoeffding exponent as

$$B_{n,m}(r, \{\mathcal{N}^\theta\}_\theta) := \sup_{\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta},(m)}\}} \left\{ -\frac{1}{nm} \ln \alpha_{n,m}(\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta},(m)}\}) \left| -\frac{1}{nm} \ln \beta_{n,m}(\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta},(m)}\}) \geq r \right. \right\}. \quad (4.30)$$

From these non-asymptotic quantities, one can then define asymptotic quantities similar to (4.23) and (4.25). However, note that they might possibly depend on the order of limits (whether one takes  $\lim_{m \rightarrow \infty}$  or  $\lim_{n \rightarrow \infty}$  first). Another contribution of our paper is to establish upper bounds on the asymptotic versions of  $\xi_{n,m}(p, \{\mathcal{N}^\theta\}_\theta)$  and  $B_{n,m}(r, \{\mathcal{N}^\theta\}_\theta)$  that hold in the case that we take the limit  $\lim_{m \rightarrow \infty}$  first, followed by the limit  $\lim_{n \rightarrow \infty}$ .

## 5 Limits on quantum channel parameter estimation

### 5.1 Classical and quantum Fisher information

#### 5.1.1 Classical Fisher information and its operational relevance

Let us first recall some fundamental results well known in classical estimation theory [57, 58] (see also [59]). Here, we suppose that there is a family  $\{p_\theta\}_\theta$  of probability distributions that are a function of the unknown parameter  $\theta \in \Theta \subseteq \mathbb{R}$ , and the goal is to produce an

estimate  $\hat{\theta}$  of  $\theta$  from  $n$  independent samples of the distribution  $p_\theta(x)$ . It is clear that the estimate can improve as the number  $n$  of samples becomes large, but we are interested in how the MSE scales with  $n$ , as well as particular scaling factors.

Let  $\{p_\theta(x)\}_\theta$  denote a family of probability density functions. Suppose that the family  $\{p_\theta(x)\}_\theta$  is differentiable with respect to the parameter  $\theta$ , so that  $\partial_\theta p_\theta(x)$  exists for all values of  $\theta$  and  $x$ , where  $\partial_\theta \equiv \frac{\partial}{\partial \theta}$ . The classical Fisher information  $I_F(\theta; \{p_\theta\}_\theta)$  of the family  $\{p_\theta(x)\}_\theta$  is defined as follows:

$$I_F(\theta; \{p_\theta\}_\theta) := \begin{cases} \int_\Omega dx \frac{1}{p_\theta(x)} (\partial_\theta p_\theta(x))^2 & \text{if } \text{supp}(\partial_\theta p_\theta) \subseteq \text{supp}(p_\theta) \\ +\infty & \text{otherwise} \end{cases}, \quad (5.1)$$

where  $\Omega$  is the sample space for the probability density function  $p_\theta(x)$ . When the support condition

$$\text{supp}(\partial_\theta p_\theta) \subseteq \text{supp}(p_\theta) \quad (5.2)$$

is satisfied (understood as “essential support”), the classical Fisher information has the following alternative expression:

$$I_F(\theta; \{p_\theta\}_\theta) = \int_\Omega dx p_\theta(x) (\partial_\theta \ln p_\theta(x))^2 = \mathbb{E}_{p_\theta}[(\partial_\theta \ln p_\theta(X))^2], \quad (5.3)$$

interpreted as the variance of the surprisal rate  $\partial_\theta[-\ln p_\theta(x)]$ .

One of the fundamental results of classical estimation theory [57–59] is the Cramer–Rao lower bound on the MSE of an unbiased estimator of  $\theta$ :

$$\text{Var}(\hat{\theta}) \geq \frac{1}{n I_F(\theta; \{p_\theta\}_\theta)}. \quad (5.4)$$

The Cramer–Rao bound can be saturated, in the sense that there exists an estimator, the maximum likelihood estimator, having an MSE that achieves the lower bound in the large  $n$  limit of many independent trials [60].

### 5.1.2 SLD Fisher information and its operational relevance

The classical Cramer–Rao bound (CRB) can be generalized to a quantum scenario [11, 47, 49] (see also [61]). Let  $\{\rho_\theta\}_\theta$  denote a family of quantum states into which the parameter  $\theta \in \Theta \subseteq \mathbb{R}$  is encoded. One can then subject  $n$  copies of this state  $\rho_\theta$  to a quantum measurement  $\{\Lambda_x\}_x$  to yield a classical probability distribution according to the Born rule:

$$p_\theta(x) = \text{Tr}[\Lambda_x \rho_\theta^{\otimes n}], \quad (5.5)$$

from which one then forms an estimate  $\hat{\theta}$ . Suppose that the family  $\{\rho_\theta\}_\theta$  of quantum states is differentiable with respect to  $\theta$ , so that  $\partial_\theta \rho_\theta$  exists for all values of  $\theta$ . We can then apply the classical CRB as given in (5.4), but it is desirable in the quantum case to perform the best possible measurement in order to know the scaling of any possible quantum estimation



strategy. The optimal measurement leads to the *most informative* CRB, which is called the quantum CRB (QCRB) and is given as the following bound on the variance of an unbiased estimator of  $\theta$ :

$$\text{Var}(\hat{\theta}) \geq \frac{1}{n I_F(\theta; \{\rho_\theta\}_\theta)}, \quad (5.6)$$

where  $I_F(\theta; \{\rho_\theta\}_\theta)$  is the symmetric logarithmic derivative (SLD) quantum Fisher information, given in Definition 1 below, and we have applied the additivity relation  $I_F(\theta; \{\rho_\theta^{\otimes n}\}_\theta) = n I_F(\theta; \{\rho_\theta\}_\theta)$ . The lower bound in (5.6) is achievable in the large  $n$  limit of many copies of the state  $\rho_\theta$  [62, 63].

**Definition 1 (SLD Fisher information)** *Let  $\{\rho_\theta\}_\theta$  be a differentiable family of quantum states. Then the SLD Fisher information is defined as follows:*

$$I_F(\theta; \{\rho_\theta\}_\theta) = \begin{cases} 2 \left\| (\rho_\theta \otimes I + I \otimes \rho_\theta^T)^{-\frac{1}{2}} ((\partial_\theta \rho_\theta) \otimes I) |\Gamma\rangle \right\|_2^2 & \text{if } \Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp = 0 \\ +\infty & \text{otherwise} \end{cases}, \quad (5.7)$$

where  $\Pi_{\rho_\theta}^\perp$  denotes the projection onto the kernel of  $\rho_\theta$ ,  $|\Gamma\rangle = \sum_i |i\rangle|i\rangle$  is the unnormalized maximally entangled vector,  $\{|i\rangle\}_i$  is any orthonormal basis, the transpose in (5.7) is with respect to this basis, and the inverse is taken on the support of  $\rho_\theta \otimes I + I \otimes \rho_\theta^T$ .

Let the spectral decomposition of  $\rho_\theta$  be given as

$$\rho_\theta = \sum_j \lambda_\theta^j |\psi_\theta^j\rangle\langle\psi_\theta^j|, \quad (5.8)$$

which includes the indices for which  $\lambda_\theta^j = 0$ . Then the projection  $\Pi_{\rho_\theta}^\perp$  onto the kernel of  $\rho_\theta$  is given by

$$\Pi_{\rho_\theta}^\perp := \sum_{j: \lambda_\theta^j = 0} |\psi_\theta^j\rangle\langle\psi_\theta^j|. \quad (5.9)$$

With this notation, the SLD quantum Fisher information can also be written as follows, as discussed in Appendix B:

$$I_F(\theta; \{\rho_\theta\}_\theta) = \begin{cases} 2 \sum_{j,k: \lambda_\theta^j + \lambda_\theta^k > 0} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{\lambda_\theta^j + \lambda_\theta^k} & \text{if } \Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp = 0 \\ +\infty & \text{otherwise} \end{cases}. \quad (5.10)$$

The formula in (5.7) has the advantage that it is basis independent, with no need to perform a spectral decomposition in order to calculate the SLD Fisher information. It also leads to a semi-definite program for calculating the SLD Fisher information, as we show in Section 5.4.1.

As we discuss in more detail in Appendix C, the finiteness condition

$$\Pi_{\rho_\theta}^\perp \partial_\theta \rho_\theta \Pi_{\rho_\theta}^\perp = 0 \quad (5.11)$$

in (5.7) is equivalent to the following condition:

$$\forall j, k : \langle \psi_j^\theta | (\partial_\theta \rho_\theta) | \psi_k^\theta \rangle = 0 \text{ if } \lambda_j^\theta + \lambda_k^\theta = 0, \quad (5.12)$$

which is helpful for understanding the formula in (5.10).

Note that the condition  $\Pi_{\rho_\theta}^\perp \partial_\theta \rho_\theta \Pi_{\rho_\theta}^\perp = 0$  is not equivalent to  $\text{supp}(\partial_\theta \rho_\theta) \subseteq \text{supp}(\rho_\theta)$ . The latter condition  $\text{supp}(\partial_\theta \rho_\theta) \subseteq \text{supp}(\rho_\theta)$  is equivalent to  $\Pi_{\rho_\theta}^\perp \partial_\theta \rho_\theta = \partial_\theta \rho_\theta \Pi_{\rho_\theta}^\perp = 0$  and implies  $\Pi_{\rho_\theta}^\perp \partial_\theta \rho_\theta \Pi_{\rho_\theta}^\perp = 0$ , but the converse is not necessarily true. To elaborate on this point, consider that we can write the operator  $\partial_\theta \rho_\theta$  with respect to the Hilbert space decomposition  $\text{supp}(\rho_\theta) \oplus \ker(\rho_\theta)$  in the following matrix form:

$$\partial_\theta \rho_\theta = \begin{bmatrix} (\partial_\theta \rho_\theta)_{0,0} & (\partial_\theta \rho_\theta)_{0,1} \\ (\partial_\theta \rho_\theta)_{0,1}^\dagger & (\partial_\theta \rho_\theta)_{1,1} \end{bmatrix}, \quad (5.13)$$

where

$$(\partial_\theta \rho_\theta)_{0,0} := \Pi_{\rho_\theta} \partial_\theta \rho_\theta \Pi_{\rho_\theta}, \quad (\partial_\theta \rho_\theta)_{0,1} := \Pi_{\rho_\theta} \partial_\theta \rho_\theta \Pi_{\rho_\theta}^\perp, \quad (\partial_\theta \rho_\theta)_{1,1} := \Pi_{\rho_\theta}^\perp \partial_\theta \rho_\theta \Pi_{\rho_\theta}^\perp. \quad (5.14)$$

The constraint  $\text{supp}(\partial_\theta \rho_\theta) \subseteq \text{supp}(\rho_\theta)$  implies that both  $(\partial_\theta \rho_\theta)_{0,1}$  and  $(\partial_\theta \rho_\theta)_{1,1}$  are zero, whereas the constraint  $\Pi_{\rho_\theta}^\perp \partial_\theta \rho_\theta \Pi_{\rho_\theta}^\perp = 0$  implies that  $(\partial_\theta \rho_\theta)_{1,1}$  is zero.

As we also show in Appendix C, when the finiteness condition in (5.11) holds, a formula alternative but equal to (5.10) is as follows:

$$I_F(\theta; \{\rho_\theta\}_\theta) = 2 \sum_{j,k: \lambda_j^\theta, \lambda_k^\theta > 0} \frac{|\langle \psi_j^\theta | (\partial_\theta \rho_\theta) | \psi_k^\theta \rangle|^2}{\lambda_j^\theta + \lambda_k^\theta} + 4 \sum_{j: \lambda_j^\theta > 0} \frac{\langle \psi_j^\theta | (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) | \psi_j^\theta \rangle}{\lambda_j^\theta}. \quad (5.15)$$

For a differentiable family  $\{|\varphi_\theta\rangle\langle\varphi_\theta|\}_\theta$  of pure states, we discuss in Appendix C.1 how the formula in (5.10) reduces to the well known expression [63, 64]:

$$I_F(\theta; \{|\varphi_\theta\rangle\langle\varphi_\theta|\}_\theta) = 4 [\langle \partial_\theta \phi_\theta | \partial_\theta \phi_\theta \rangle - |\langle \partial_\theta \phi_\theta | \phi_\theta \rangle|^2]. \quad (5.16)$$

That is, for all pure-state differentiable families, the finiteness condition in (5.11) always holds and one can employ the formula in (5.10) to arrive at the expression in (5.16).

The following proposition demonstrates that the definition in (5.10) is physically consistent, in the sense that it is the result of a limiting procedure in which some constant additive noise vanishes:

**Proposition 2** *Let  $\{\rho_\theta\}_\theta$  be a differentiable family of quantum states. Then the SLD Fisher information in (5.10) is given by the following limit:*

$$I_F(\theta; \{\rho_\theta\}_\theta) = \lim_{\varepsilon \rightarrow 0} I_F(\theta; \{\rho_\theta^\varepsilon\}_\theta), \quad (5.17)$$

where

$$\rho_\theta^\varepsilon := (1 - \varepsilon) \rho_\theta + \varepsilon \pi_d, \quad (5.18)$$

and  $\pi_d := I/d$  is the maximally mixed state, with  $d$  large enough so that  $\text{supp}(\rho_\theta) \subseteq \text{supp}(\pi)$  for all  $\theta$ .

**Proof.** See Appendix C. ■

In the case that the condition  $\Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp = 0$  holds, we can also write the SLD Fisher information as follows:

$$I_F(\theta; \{\rho_\theta\}_\theta) = \text{Tr}[L_\theta^2 \rho_\theta] = \text{Tr}[L_\theta(\partial_\theta \rho_\theta)], \quad (5.19)$$

where the operator  $L_\theta$  is the symmetric logarithmic derivative (SLD) [11], defined through the following differential equation:

$$\partial_\theta \rho_\theta := \frac{1}{2} (\rho_\theta L_\theta + L_\theta \rho_\theta). \quad (5.20)$$

In Appendix B, we revisit the derivation of [65] and show how (5.19) is a consequence of (5.7) when the finiteness condition in (5.11) holds. By sandwiching (5.20) on the left and right by  $\langle \psi_k^\theta |$  and  $|\psi_j^\theta\rangle$ , with  $|\psi_j^\theta\rangle, |\psi_k^\theta\rangle \in \text{supp}(\rho_\theta)$ , one can check that the SLD has the following unique and explicit form on the subspace  $\text{span}\{|\psi\rangle\langle\varphi| : |\psi\rangle, |\varphi\rangle \in \text{supp}(\rho_\theta)\}$ :

$$L_\theta = 2 \sum_{j,k: \lambda_j^\theta + \lambda_k^\theta > 0} \frac{\langle \psi_j^\theta | (\partial_\theta \rho_\theta) | \psi_k^\theta \rangle}{\lambda_j^\theta + \lambda_k^\theta} |\psi_j^\theta\rangle\langle\psi_k^\theta|. \quad (5.21)$$

Then, in the case that the finiteness condition in (5.11) holds, after evaluating (5.19), we arrive at the explicit formula for the SLD Fisher information in (5.10).

As indicated above, in the case that (5.11) holds, the following equality holds between the basis-independent formula in (5.7) and the basis-dependent formula in (5.10):

$$I_F(\theta; \{\rho_\theta\}_\theta) = 2 \sum_{j,k: \lambda_j^\theta + \lambda_k^\theta > 0} \frac{|\langle \psi_j^\theta | (\partial_\theta \rho_\theta) | \psi_k^\theta \rangle|^2}{\lambda_j^\theta + \lambda_k^\theta} \quad (5.22)$$

$$= 2 \langle \Gamma | ((\partial_\theta \rho_\theta) \otimes I) (\rho_\theta \otimes I + I \otimes \rho_\theta^T)^{-1} ((\partial_\theta \rho_\theta) \otimes I) | \Gamma \rangle \quad (5.23)$$

$$= 2 \left\| (\rho_\theta \otimes I + I \otimes \rho_\theta^T)^{-\frac{1}{2}} ((\partial_\theta \rho_\theta) \otimes I) | \Gamma \rangle \right\|_2^2. \quad (5.24)$$

This basis-independent formula was explicitly given in [65]. Arguably, it is implicitly given in [66, 67], being a consequence of (a) the general theory presented in [66] in terms of monotone metrics and the relative modular operator formalism [68] and (b) the well known isomorphism connecting the Hilbert–Schmidt inner product to an extended vector-space inner product [69], which is called Ando’s identity in [70] (see also [71–74]). The formula in (5.23) was presented in [67, Remark 4] in the relative modular operator formalism and in [65] in the extended Hilbert space formalism (as given above). As indicated above, we discuss this equality in more detail in Appendix B.

The explicit formula in (5.10) can be difficult to evaluate in practice because it requires performing a spectral decomposition of  $\rho_\theta$ . The same is true for the formula in (5.7) due to the presence of a matrix inverse. To get around these problems, we show in Section 5.4.1 how the SLD Fisher information can be evaluated by means of a semi-definite program that takes  $\rho_\theta$  and  $\partial_\theta \rho_\theta$  as input (that is, with this approach, there is no need to perform a diagonalization of  $\rho_\theta$  or a matrix inverse). See [43, 75] for general background on semi-definite programming.

### 5.1.3 RLD Fisher information

The quantum Cramer–Rao bound (QCRB) provides a technique to bound the MSE in estimating a parameter by using the SLD Fisher information. As mentioned previously, there is in fact an infinite number of QCRBs, with each of them arising from a particular noncommutative generalization of the classical Fisher information in (5.1). Another noncommutative generalization of the classical Fisher information is the *right* logarithmic derivative (RLD) Fisher information:

**Definition 3 (RLD Fisher information)** *Let  $\{\rho_\theta\}_\theta$  be a differentiable family of quantum states. Then the RLD Fisher information is defined as follows:*

$$\widehat{I}_F(\theta; \{\rho_\theta\}_\theta) = \begin{cases} \text{Tr}[(\partial_\theta \rho_\theta)^2 \rho_\theta^{-1}] & \text{if } \text{supp}(\partial_\theta \rho_\theta) \subseteq \text{supp}(\rho_\theta) \\ +\infty & \text{otherwise} \end{cases}, \quad (5.25)$$

where the inverse  $\rho_\theta^{-1}$  is taken on the support of  $\rho_\theta$ .

Note that the support condition  $\text{supp}(\partial_\theta \rho_\theta) \subseteq \text{supp}(\rho_\theta)$  is equivalent to  $\Pi_{\rho_\theta}^\perp \partial_\theta \rho_\theta = \partial_\theta \rho_\theta \Pi_{\rho_\theta}^\perp = 0$ , which implies that  $\Pi_{\rho_\theta}^\perp \partial_\theta \rho_\theta \Pi_{\rho_\theta}^\perp = 0$ .

For a differentiable family  $\{|\varphi_\theta\rangle\langle\varphi_\theta|\}_\theta$  of pure states, the RLD Fisher information has trivial behavior due to the finiteness condition in (5.25). If the family is constant, such that  $|\varphi_\theta\rangle = |\varphi\rangle$  for all  $\theta$ , then the RLD Fisher information is finite and equal to zero. Otherwise, the RLD Fisher information is infinite. We show this in more detail in Appendix C.1. Thus, the RLD Fisher information is a degenerate and uninteresting information measure for pure-state families.

Similar to Proposition 2, the following proposition demonstrates that the definition in (5.25) is physically consistent, in the sense that it is the result of a limiting procedure in which some constant additive noise vanishes:

**Proposition 4** *Let  $\{\rho_\theta\}_\theta$  be a differentiable family of quantum states. Then the RLD Fisher information in (5.25) is given by the following limit:*

$$\widehat{I}_F(\theta; \{\rho_\theta\}_\theta) = \lim_{\varepsilon \rightarrow 0} \widehat{I}_F(\theta; \{\rho_\theta^\varepsilon\}_\theta), \quad (5.26)$$

where

$$\rho_\theta^\varepsilon := (1 - \varepsilon) \rho_\theta + \varepsilon \pi_d, \quad (5.27)$$

and  $\pi_d := I/d$  is the maximally mixed state, with  $d$  large enough so that  $\text{supp}(\rho_\theta) \subseteq \text{supp}(\pi)$  for all  $\theta$ .

**Proof.** See Appendix C. ■

In the case that the following support condition holds

$$\text{supp}(\partial_\theta \rho_\theta) \subseteq \text{supp}(\rho_\theta), \quad (5.28)$$

then the RLD Fisher information can also be defined in the following way:

$$\widehat{I}_F(\theta; \{\rho_\theta\}_\theta) := \text{Tr}[R_\theta R_\theta^\dagger \rho_\theta] = \text{Tr}[(\partial_\theta \rho_\theta) R_\theta^\dagger], \quad (5.29)$$

where the RLD operator [12] is defined through the following differential equation:

$$\partial_\theta \rho_\theta = \rho_\theta R_\theta. \quad (5.30)$$

By observing from (5.30) that  $\Pi_{\rho_\theta} R_\theta = \rho_\theta^{-1} \partial_\theta \rho_\theta$ , where  $\Pi_{\rho_\theta}$  is the projection onto the support of  $\rho_\theta$ , the RLD Fisher information can be written explicitly as  $\widehat{I}_F(\theta; \{\rho_\theta\}_\theta) := \text{Tr}[(\partial_\theta \rho_\theta)^2 \rho_\theta^{-1}]$ , consistent with Definition 3. This formula is thus a more direct quantum generalization of the classical formula in (5.3).

The SLD Fisher information never exceeds the RLD Fisher information:

$$I_F(\theta; \{\rho_\theta\}_\theta) \leq \widehat{I}_F(\theta; \{\rho_\theta\}_\theta), \quad (5.31)$$

which can be seen from the operator convexity of the function  $x^{-1}$  for  $x > 0$ . That is, for full-rank  $\rho_\theta$ , we have that

$$2(\rho_\theta \otimes I + I \otimes \rho_\theta^T)^{-1} = \left( \frac{1}{2} \rho_\theta \otimes I + \frac{1}{2} I \otimes \rho_\theta^T \right)^{-1} \quad (5.32)$$

$$\leq \frac{1}{2} (\rho_\theta \otimes I)^{-1} + \frac{1}{2} (I \otimes \rho_\theta^T)^{-1} \quad (5.33)$$

$$= \frac{1}{2} (\rho_\theta^{-1} \otimes I) + \frac{1}{2} (I \otimes \rho_\theta^{-T}), \quad (5.34)$$

and then (5.7), (3.2), (3.3), and the limit formulas in Propositions 2 and 4 lead to (5.31). Thus, as a consequence of (5.6) and (5.31), the RLD Fisher information leads to another lower bound on the MSE of an unbiased estimator:

$$\text{Var}(\hat{\theta}) \geq \frac{1}{n \widehat{I}_F(\theta; \{\rho_\theta\}_\theta)}. \quad (5.35)$$

Although the inequality above is not generally achievable, the RLD Fisher information possesses an operational meaning in terms of a task called reverse estimation [76].

The formula in (5.25) may be difficult to evaluate in practice due to the presence of a matrix inverse. In Section 5.4.1, we show how this quantity can be evaluated by means of a semi-definite program that takes  $\rho_\theta$  and  $\partial_\theta \rho_\theta$  as input, thus obviating the need to perform the inverse.

## 5.2 Basic properties of SLD and RLD Fisher information of quantum states

Here we collect some basic properties of SLD and RLD Fisher information of quantum states, which include faithfulness, data processing, additivity, and decomposition on classical–quantum states.

### 5.2.1 Faithfulness

**Proposition 5 (Faithfulness)** *For a differentiable family  $\{\rho_A^\theta\}_\theta$  of quantum states, the SLD and RLD Fisher informations are equal to zero:*

$$I_F(\theta; \{\rho_A\}_\theta) = \widehat{I}_F(\theta; \{\rho_A\}_\theta) = 0 \quad \forall \theta \in \Theta, \quad (5.36)$$

*if and only if  $\rho_A^\theta$  has no dependence on the parameter  $\theta$  (i.e.,  $\rho_A^\theta = \rho_A$  for all  $\theta$ ).*

**Proof.** The if-part follows directly from plugging into the definitions after observing that  $\partial_\theta \rho_\theta = 0$  for a constant family. So we now prove the only-if part. If  $I_F(\theta; \{\rho_A\}_\theta) = 0$ , then it is necessary for the finiteness condition in (5.11) to hold (otherwise we would have a contradiction). Then this means that

$$\Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp = 0, \quad (5.37)$$

$$2 \sum_{j,k: \lambda_j^\theta + \lambda_k^\theta > 0} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{\lambda_j^\theta + \lambda_k^\theta} = 0 \quad \forall \theta. \quad (5.38)$$

By sandwiching the first equation by  $\langle \psi_\theta^j |$  and  $| \psi_\theta^k \rangle$  for which  $\lambda_j^\theta, \lambda_k^\theta = 0$ , we find that these matrix elements  $\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle$  of  $\partial_\theta \rho_\theta$  are equal to zero. Since  $\lambda_j^\theta + \lambda_k^\theta > 0$  in the latter expression, the latter equality implies the following

$$|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2 = 0 \quad (5.39)$$

for all  $\lambda_j^\theta$  and  $\lambda_k^\theta$  satisfying  $\lambda_j^\theta + \lambda_k^\theta > 0$ . This implies that these matrix elements  $\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle$  of  $\partial_\theta \rho_\theta$  are equal to zero. These are all possible matrix elements, and so we conclude that  $\partial_\theta \rho_\theta = 0$ . This in turn implies that  $\rho_\theta$  is a constant family (i.e.,  $\rho_A^\theta = \rho_A$  for all  $\theta$ ). If  $\widehat{I}_F(\theta; \{\rho_A\}_\theta) = 0$ , then by the inequality in (5.31),  $I_F(\theta; \{\rho_A\}_\theta) = 0$ . Then by what we have just shown,  $\rho_\theta$  is a constant family in this case also. ■

### 5.2.2 Data processing

The SLD and RLD Fisher informations obey the following data-processing inequalities:

$$I_F(\theta; \{\rho_A^\theta\}_\theta) \geq I_F(\theta; \{\mathcal{N}_{A \rightarrow B}(\rho_A^\theta)\}_\theta), \quad (5.40)$$

$$\widehat{I}_F(\theta; \{\rho_A^\theta\}_\theta) \geq \widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}(\rho_A^\theta)\}_\theta), \quad (5.41)$$

where  $\mathcal{N}_{A \rightarrow B}$  is a quantum channel independent of the parameter  $\theta$  (more generally, these hold if  $\mathcal{N}_{A \rightarrow B}$  is a two-positive, trace-preserving map). The data-processing inequalities for  $I_F$  and  $\widehat{I}_F$  were established in [66]. In fact, the inequality in (5.41) is an immediate consequence of [77, Proposition 4.1].

### 5.2.3 Additivity

**Proposition 6** *Let  $\{\rho_A^\theta\}_\theta$  and  $\{\sigma_B^\theta\}_\theta$  be differentiable families of quantum states. Then the SLD and RLD Fisher informations are additive in the following sense:*

$$I_F(\theta; \{\rho_A^\theta \otimes \sigma_B^\theta\}_\theta) = I_F(\theta; \{\rho_A^\theta\}_\theta) + I_F(\theta; \{\sigma_B^\theta\}_\theta), \quad (5.42)$$

$$\hat{I}_F(\theta; \{\rho_A^\theta \otimes \sigma_B^\theta\}_\theta) = \hat{I}_F(\theta; \{\rho_A^\theta\}_\theta) + \hat{I}_F(\theta; \{\sigma_B^\theta\}_\theta). \quad (5.43)$$

**Proof.** See Appendix D. ■

### 5.2.4 Decomposition for classical–quantum families

**Proposition 7** *Let  $\{\rho_{XB}^\theta\}_\theta$  be a differentiable family of classical–quantum states, where*

$$\rho_{XB}^\theta := \sum_x p_\theta(x) |x\rangle\langle x|_X \otimes \rho_\theta^x. \quad (5.44)$$

*Then the following decompositions hold for the SLD and RLD Fisher informations:*

$$I_F(\theta; \{\rho_{XB}^\theta\}_\theta) = I_F(\theta; \{p_\theta\}_\theta) + \sum_x p_\theta(x) I_F(\theta; \{\rho_\theta^x\}_\theta), \quad (5.45)$$

$$\hat{I}_F(\theta; \{\rho_{XB}^\theta\}_\theta) = I_F(\theta; \{p_\theta\}_\theta) + \sum_x p_\theta(x) \hat{I}_F(\theta; \{\rho_\theta^x\}_\theta). \quad (5.46)$$

**Proof.** See Appendix E. ■

We note here that the extended convexity inequality reported in [78, Eq. (4)] is a consequence of (5.45). That is, one recovers the extended convexity inequality of [78] by performing a partial trace over the classical register  $X$  on the left-hand side of (5.45) and applying the data-processing inequality in (5.40).

## 5.3 Generalized Fisher information and a meta-converse for channel parameter estimation

Since data processing is such a fundamental and powerful tool, it can be fruitful to define and develop a generalized distinguishability measure based on this property alone (this is also called generalized divergence [79, 80]). This approach has been employed for some time now in quantum communication [39, 80–89] and distinguishability [33, 90–92] theory. Here we extend the approach to quantum estimation theory.

### 5.3.1 Generalized Fisher information of states

Let  $\mathcal{D}$  denote the set of density operators and  $\Theta$  the parameter set. We define the generalized Fisher information of quantum states as follows:

**Definition 8 (Generalized Fisher information of quantum states)** *The generalized Fisher information  $\mathbf{I}_F(\theta; \{\rho_A^\theta\}_\theta)$  of a family  $\{\rho_A^\theta\}_\theta$  of quantum states is a function  $\mathbf{I}_F : \Theta \times \mathcal{D} \rightarrow \mathbb{R}$  that does not increase under the action of a parameter-independent quantum channel  $\mathcal{N}_{A \rightarrow B}$ :*

$$\mathbf{I}_F(\theta; \{\rho_A^\theta\}_\theta) \geq \mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}(\rho_A^\theta)\}_\theta). \quad (5.47)$$

It follows from (5.40) and (5.41) that the SLD and RLD Fisher informations in (5.7) and (5.25) are particular examples because they possess this basic property. Furthermore, the generalized divergence of [79, 80] is a special case of generalized Fisher information when the parameter  $\theta$  takes on only two values.

An immediate consequence of Definition 8 is that the generalized Fisher information is equal to a constant, minimal value for a state family that has no dependence on the parameter  $\theta$ :

$$\mathbf{I}_F(\theta; \{\rho_A\}_\theta) = c. \quad (5.48)$$

This follows because one can get from one fixed family  $\{\rho_A\}_\theta$  to another  $\{\sigma_A\}_\theta$  by means of a trace and replace channel  $(\cdot) \rightarrow \text{Tr}[(\cdot)]\sigma_A$ , and then we apply the data-processing inequality. If this constant  $c$  is equal to zero, then we say that the generalized Fisher information is *weakly faithful*.

A generalized Fisher information obeys the *direct-sum property* if the following equality holds

$$\mathbf{I}_F\left(\theta; \left\{ \sum_x p(x) |x\rangle\langle x| \otimes \rho_\theta^x \right\}_\theta\right) = \sum_x p(x) \mathbf{I}_F(\theta; \{\rho_\theta^x\}_\theta), \quad (5.49)$$

where, for each  $x$ , the family  $\{\rho_\theta^x\}_\theta$  of quantum states is differentiable. Observe that the probability distribution  $p(x)$  has no dependence on the parameter  $\theta$ . If a generalized Fisher information obeys the direct-sum property, then it is also convex in the following sense:

$$\sum_x p(x) \mathbf{I}_F(\theta; \{\rho_\theta^x\}_\theta) \geq \mathbf{I}_F(\theta; \{\bar{\rho}_\theta\}_\theta), \quad (5.50)$$

where  $\bar{\rho}_\theta := \sum_x p(x) \rho_\theta^x$ . This follows by applying (5.49) and the data-processing inequality with a partial trace over the classical register. Thus, due to (5.40), (5.41), and Proposition 7, the SLD and RLD Fisher informations are convex.

### 5.3.2 Generalized Fisher information of channels

From the generalized Fisher information of states, we can define the generalized Fisher information of channels:

**Definition 9 (Generalized Fisher information of quantum channels)** *The generalized Fisher information of a family  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  of quantum channels is defined in terms of the following optimization:*

$$\mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) := \sup_{\rho_{RA}} \mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA})\}_\theta). \quad (5.51)$$



In the above definition, we take the supremum over arbitrary states  $\rho_{RA}$  with unbounded reference system  $R$ .

The SLD Fisher information of quantum channels was defined in [93] and the RLD Fisher information of quantum channels in [20]; these are special cases of (5.51). The generalized channel divergence of [56, 91] is a special case of generalized Fisher information of channels when the parameter  $\theta$  takes on only two values.

**Remark 10** *As is the case for all information measures that obey the data-processing inequality, we can employ the data-processing inequality in (5.47) with respect to partial trace and the Schmidt decomposition theorem to conclude that it suffices to perform the optimization in (5.51) with respect to pure bipartite states  $\psi_{RA}$  with system  $R$  isomorphic to system  $A$ , so that*

$$\mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) = \sup_{\psi_{RA}} \mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\psi_{RA})\}_\theta). \quad (5.52)$$

Some basic properties of the generalized Fisher information of quantum channels are as follows:

**Proposition 11** *Let  $\{\mathcal{N}_{A \rightarrow B}\}_\theta$  be a family of quantum channels that has no dependence on the parameter  $\theta$ , and suppose that the underlying generalized Fisher information is weakly faithful. Then*

$$\mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}\}_\theta) = 0. \quad (5.53)$$

**Proof.** This follows as an immediate consequence of the definition, (5.48), and the weak faithfulness assumption. ■

**Proposition 12 (Reduction to states)** *Let  $\{\rho_B^\theta\}_\theta$  be a family of quantum states, and define the family  $\{\mathcal{R}_{A \rightarrow B}^\theta\}_\theta$  of replacer channels as*

$$\mathcal{R}_{A \rightarrow B}^\theta(\omega_A) = \text{Tr}[\omega_A] \rho_B^\theta. \quad (5.54)$$

*Then*

$$\mathbf{I}_F(\theta; \{\mathcal{R}_{A \rightarrow B}^\theta\}_\theta) = \mathbf{I}_F(\theta; \{\rho_B^\theta\}_\theta). \quad (5.55)$$

**Proof.** This follows from the definition and the data-processing inequality. Consider that

$$\mathbf{I}_F(\theta; \{\mathcal{R}_{A \rightarrow B}^\theta\}_\theta) = \sup_{\psi_{RA}} \mathbf{I}_F(\theta; \{\mathcal{R}_{A \rightarrow B}^\theta(\psi_{RA})\}_\theta) \quad (5.56)$$

$$= \sup_{\psi_{RA}} \mathbf{I}_F(\theta; \{\psi_R \otimes \rho_B^\theta\}_\theta) \quad (5.57)$$

$$= \mathbf{I}_F(\theta; \{\rho_B^\theta\}_\theta). \quad (5.58)$$

The last equality follows because

$$\mathbf{I}_F(\theta; \{\rho_B^\theta\}_\theta) \geq \mathbf{I}_F(\theta; \{\psi_R \otimes \rho_B^\theta\}_\theta), \quad (5.59)$$

$$\mathbf{I}_F(\theta; \{\rho_B^\theta\}_\theta) \leq \mathbf{I}_F(\theta; \{\psi_R \otimes \rho_B^\theta\}_\theta), \quad (5.60)$$

with the first inequality following from the fact that there is a parameter-independent preparation channel such that  $\rho_B^\theta \rightarrow \psi_R \otimes \rho_B^\theta$ , while the second inequality follows from data-processing under partial trace over the reference system  $R$ . ■

**Proposition 13** *Let  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  be a family of quantum channels, and suppose that the underlying generalized Fisher information is weakly faithful and obeys the direct-sum property. Then the following inequalities hold*

$$\mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\Phi_{RA})\}_\theta) \leq \mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) \leq d \cdot \mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\Phi_{RA})\}_\theta), \quad (5.61)$$

where  $\Phi_{RA}$  is the maximally entangled state and  $d$  is the dimension of the channel input system  $A$ .

**Proof.** The first inequality is trivial, following from the definition in (5.51). So we prove the second one and note that it follows from a quantum steering or remote state preparation argument. Let  $\psi_{RA}$  be an arbitrary pure bipartite input state. To each such state, there exists an operator  $Z_R$  satisfying

$$\psi_{RA} = d \cdot Z_R \Phi_{RA} Z_R^\dagger, \quad (5.62)$$

$$\text{Tr}[Z_R^\dagger Z_R] = 1. \quad (5.63)$$

Let  $\mathcal{P}_{R \rightarrow XR}$  denote the following steering quantum channel:

$$\mathcal{P}_{R \rightarrow XR}(\omega_R) := |0\rangle\langle 0|_X \otimes Z_R \omega_R Z_R^\dagger + |1\rangle\langle 1|_X \otimes \sqrt{I_R - Z_R^\dagger Z_R} \omega_R \sqrt{I_R - Z_R^\dagger Z_R}, \quad (5.64)$$

and consider that

$$\mathcal{P}_{R \rightarrow XR}(\Phi_{RA}) = \frac{1}{d} |0\rangle\langle 0|_X \otimes \psi_{RA} + \left(1 - \frac{1}{d}\right) |1\rangle\langle 1|_X \otimes \sigma_{RA}, \quad (5.65)$$

where

$$\sigma_{RA} := \left(1 - \frac{1}{d}\right)^{-1} \sqrt{I_R - Z_R^\dagger Z_R} \Phi_{RA} \sqrt{I_R - Z_R^\dagger Z_R}. \quad (5.66)$$

This implies that

$$\begin{aligned} & \mathcal{P}_{R \rightarrow XR}(\mathcal{N}_{A \rightarrow B}^\theta(\Phi_{RA})) \\ &= \mathcal{N}_{A \rightarrow B}^\theta(\mathcal{P}_{R \rightarrow XR}(\Phi_{RA})) \end{aligned} \quad (5.67)$$

$$= \frac{1}{d} |0\rangle\langle 0|_X \otimes \mathcal{N}_{A \rightarrow B}^\theta(\psi_{RA}) + \left(1 - \frac{1}{d}\right) |1\rangle\langle 1|_X \otimes \mathcal{N}_{A \rightarrow B}^\theta(\sigma_{RA}). \quad (5.68)$$

Then we find that

$$\mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\Phi_{RA})\}_\theta)$$

$$\geq \mathbf{I}_F(\theta; \{\mathcal{P}_{R \rightarrow XR}(\mathcal{N}_{A \rightarrow B}^\theta(\Phi_{RA}))\}_\theta) \quad (5.69)$$

$$= \frac{1}{d} \mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\psi_{RA})\}_\theta) + \left(1 - \frac{1}{d}\right) \mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\sigma_{RA})\}_\theta) \quad (5.70)$$

$$\geq \frac{1}{d} \mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\psi_{RA})\}_\theta). \quad (5.71)$$

The first inequality follows from data processing. The equality follows from (5.68) and the direct-sum property in (5.49). The last inequality follows from the assumption that  $\mathbf{I}_F$  is weakly faithful, so that  $\mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\sigma_{RA})\}_\theta) \geq 0$ . Since the inequality holds for all pure bipartite states  $\psi_{RA}$ , we conclude the second inequality in (5.61). ■

**Remark 14** Note that a special case of (5.61) occurs when the parameter  $\theta$  takes on only two values. So the argument above applies to all generalized channel divergences [91] that are weakly faithful and obey the direct-sum property, which includes diamond distance, relative entropy, negative root fidelity, and Petz-, sandwiched, and geometric Rényi relative quasi-entropies.

**Remark 15** Supposing that a generalized Fisher information is weakly faithful and obeys the direct-sum property, a consequence of Proposition 13 is that, in order to determine whether the corresponding generalized Fisher information of channels is finite, it is only necessary to check the value of the quantity on the maximally entangled input state.

Particular generalized Fisher informations of channels of interest include the SLD and RLD ones. Due to (5.40)–(5.41), Propositions 5, 7, and 13, and Remark 15, we can write them respectively as follows:

$$I_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) = \begin{cases} \sup_{\psi_{RA}} I_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\psi_{RA})\}_\theta) & \text{if } \Pi_{\Gamma_{RB}^{\mathcal{N}^\theta}}^\perp (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \Pi_{\Gamma_{RB}^{\mathcal{N}^\theta}}^\perp = 0 \\ +\infty & \text{otherwise.} \end{cases}, \quad (5.72)$$

$$\widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) = \begin{cases} \left\| \text{Tr}_B[(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1}(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})] \right\|_\infty & \text{if } \text{supp}(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \subseteq \text{supp}(\Gamma_{RB}^{\mathcal{N}^\theta}) \\ +\infty & \text{otherwise.} \end{cases}, \quad (5.73)$$

where  $\Gamma_{RB}^{\mathcal{N}^\theta}$  is the Choi operator of the channel  $\mathcal{N}_{A \rightarrow B}^\theta$ . The explicit expression above for  $\widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta)$  was given in [20] and is recalled in Proposition 29 below. It is unclear to us at the moment how to obtain a more explicit form for  $I_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta)$  in terms of its Choi operator.

The finiteness conditions in (5.72) and (5.73) have interesting implications for a differentiable family  $\{\mathcal{U}_\theta\}_\theta$  of isometric or unitary channels. When such a family acts on one share of a maximally entangled state, it induces a differentiable family of pure states. Now applying what was stated previously in Sections 5.1.2 and 5.1.3 for such families, it follows that the SLD Fisher information of  $\{\mathcal{U}_\theta\}_\theta$  is always finite, whereas the RLD Fisher

information of  $\{\mathcal{U}_\theta\}_\theta$  is finite if and only if it is equal to zero (i.e., when the family  $\{\mathcal{U}_\theta\}_\theta$  is a constant family  $\{\mathcal{U}\}_\theta$  independent of the parameter  $\theta$ ). So in this sense, the RLD Fisher information of isometric or unitary channels is a degenerate and uninteresting information measure.

### 5.3.3 Amortized Fisher information

The generalized Fisher information of quantum channels is motivated by channel parameter estimation, and in particular, by the parallel setting of channel estimation. Now motivated by the more general sequential setting of channel parameter estimation, we define the following amortized Fisher information of quantum channels:

**Definition 16 (Amortized Fisher information of quantum channels)** *The amortized Fisher information of a family  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  of quantum channels is defined as follows:*

$$\mathbf{I}_F^A(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) := \sup_{\{\rho_{RA}^\theta\}_\theta} [\mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) - \mathbf{I}_F(\theta; \{\rho_{RA}^\theta\}_\theta)], \quad (5.74)$$

where the supremum is with respect to arbitrary state families  $\{\rho_{RA}^\theta\}_\theta$  with unbounded reference system  $R$ .

The idea behind this quantity is the same as that of the amortized channel divergence of [33]. We allow for a resource at the channel input in order to help with the estimation task, but then we subtract off the value of this resource in order to account for the amount of resource that is strictly present in the channel family. In this case, the resource is estimability, as proposed in [76]. This kind of idea has been useful in the analysis of feedback-assisted or sequential protocols in other areas of quantum information science [39, 86, 92, 94–99], and here we see how it is useful in the context of channel parameter estimation. Also, we should indicate here that the amortized channel divergence of [33] is a special case of the amortized Fisher information in which the parameter  $\theta$  takes on only two values.

**Proposition 17** *Let  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  be a family of quantum channels, and suppose that the underlying generalized Fisher information is weakly faithful. Then the generalized Fisher information does not exceed the amortized one:*

$$\mathbf{I}_F^A(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) \geq \mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta). \quad (5.75)$$

**Proof.** This follows because we can always pick the input family  $\{\rho_{RA}^\theta\}_\theta$  in (5.74) to have no dependence on the parameter  $\theta$ . Then we find that

$$\mathbf{I}_F^A(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) \geq \mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA})\}_\theta) - \mathbf{I}_F(\theta; \{\rho_{RA}\}_\theta) \quad (5.76)$$

$$= \mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA})\}_\theta), \quad (5.77)$$

where we applied the weak faithfulness assumption to arrive at the equality. Since the inequality holds for all input states  $\rho_{RA}$ , we conclude (5.75). ■

We now connect the amortized Fisher information to sequential channel estimation through the following meta-converse, which generalizes the related meta-converse of [33]:

**Theorem 18** *Consider a general sequential channel estimation protocol of the form discussed in Section 4.1. Suppose that the generalized Fisher information  $\mathbf{I}_F$  is weakly faithful. Then the following inequality holds*

$$\mathbf{I}_F(\theta; \{\omega_{R_n B_n}^\theta\}_\theta) \leq n \cdot \mathbf{I}_F^A(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta), \quad (5.78)$$

where  $\omega_{R_n B_n}^\theta$  is the final state of the estimation protocol, as given in (4.8).

**Proof.** Consider that

$$\begin{aligned} & \mathbf{I}_F(\theta; \{\omega_{R_n B_n}^\theta\}_\theta) \\ &= \mathbf{I}_F(\theta; \{\omega_{R_n B_n}^\theta\}_\theta) - \mathbf{I}_F(\theta; \{\rho_{R_1 A_1}\}_\theta) \end{aligned} \quad (5.79)$$

$$\begin{aligned} &= \mathbf{I}_F(\theta; \{\omega_{R_n B_n}^\theta\}_\theta) - \mathbf{I}_F(\theta; \{\rho_{R_1 A_1}\}_\theta) + \sum_{i=2}^n (\mathbf{I}_F(\theta; \{\rho_{R_i A_i}^\theta\}_\theta) - \mathbf{I}_F(\theta; \{\rho_{R_{i-1} A_{i-1}}^\theta\}_\theta)) \end{aligned} \quad (5.80)$$

$$\begin{aligned} &= \mathbf{I}_F(\theta; \{\omega_{R_n B_n}^\theta\}_\theta) - \mathbf{I}_F(\theta; \{\rho_{R_1 A_1}\}_\theta) \\ &\quad + \sum_{i=2}^n \left( \mathbf{I}_F(\theta; \{\mathcal{S}_{R_{i-1} B_{i-1} \rightarrow R_i A_i}^{i-1}(\rho_{R_{i-1} B_{i-1}}^\theta)\}_\theta) - \mathbf{I}_F(\theta; \{\rho_{R_i A_i}^\theta\}_\theta) \right) \end{aligned} \quad (5.81)$$

$$\begin{aligned} &\leq \mathbf{I}_F(\theta; \{\omega_{R_n B_n}^\theta\}_\theta) - \mathbf{I}_F(\theta; \{\rho_{R_1 A_1}\}_\theta) \\ &\quad + \sum_{i=2}^n \left( \mathbf{I}_F(\theta; \{\rho_{R_{i-1} B_{i-1}}^\theta\}_\theta) - \mathbf{I}_F(\theta; \{\rho_{R_i A_i}^\theta\}_\theta) \right) \end{aligned} \quad (5.82)$$

$$= \sum_{i=1}^n (\mathbf{I}_F(\theta; \{\rho_{R_i B_i}^\theta\}_\theta) - \mathbf{I}_F(\theta; \{\rho_{R_i A_i}^\theta\}_\theta)) \quad (5.83)$$

$$= \sum_{i=1}^n (\mathbf{I}_F(\theta; \{\mathcal{N}_{A_i \rightarrow B_i}^\theta(\rho_{R_i A_i}^\theta)\}_\theta) - \mathbf{I}_F(\theta; \{\rho_{R_i A_i}^\theta\}_\theta)) \quad (5.84)$$

$$\leq n \cdot \sup_{\{\rho_{RA}^\theta\}_\theta} [\mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) - \mathbf{I}_F(\theta; \{\rho_{RA}^\theta\}_\theta)] \quad (5.85)$$

$$= n \cdot \mathbf{I}_F^A(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta). \quad (5.86)$$

The first equality follows from the weak faithfulness assumption and because the initial state of the protocol has no dependence on the parameter  $\theta$ . The inequality follows from the data-processing inequality. The other steps are straightforward manipulations. ■

For some particular choices of the generalized Fisher information, the inequality in (5.75) can be reversed, which is called an “amortization collapse.” Theorem 18 makes

such a collapse useful for establishing limits on the performance of sequential estimation protocols if the underlying Fisher information has a relation to the MSE through a CRB. We show later that the following equalities hold for the root SLD and RLD Fisher informations for all differentiable families  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  of quantum channels:

$$\sqrt{I_F^A}(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) = \sqrt{I_F}(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta), \quad (5.87)$$

$$\widehat{I}_F^A(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) = \widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta). \quad (5.88)$$

Also, for differentiable families  $\{\mathcal{N}_{X \rightarrow B}^\theta\}_\theta$  of classical–quantum channels, the following equality holds for the SLD Fisher information:

$$I_F^A(\theta; \{\mathcal{N}_{X \rightarrow B}^\theta\}_\theta) = I_F(\theta; \{\mathcal{N}_{X \rightarrow B}^\theta\}_\theta). \quad (5.89)$$

### 5.3.4 Environment-parameterized and environment-seizable channel families

In this section, we recall the notion of environment-parameterized and environment-seizable channel families, as discussed in [33, 90, 100], and we show that the amortized Fisher information collapses for environment-seizable channel families. Environment-parameterized channel families are also known as programmable channel families [101].

**Definition 19 (Environment-parameterized family)** *A family  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  is called environment-parameterized if there exists a family  $\{\rho_E^\theta\}_\theta$  of states and a parameter-independent quantum channel  $\mathcal{M}_{AE \rightarrow B}$  such that the action of  $\mathcal{N}_{A \rightarrow B}^\theta$  on any channel input  $\omega_A$  can be written as follows:*

$$\mathcal{N}_{A \rightarrow B}^\theta(\omega_A) = \mathcal{M}_{AE \rightarrow B}(\omega_A \otimes \rho_E^\theta). \quad (5.90)$$

It is important to highlight that *every* channel family is environment parameterized in a trivial way, as discussed in [100] for a finite set. Indeed, set  $\rho_E^\theta = |\theta\rangle\langle\theta|_E$ , where the vectors  $\{|\theta\rangle_E\}_\theta$  are an orthonormal family, and set

$$\mathcal{M}_{AE \rightarrow B}(\tau_{AE}) = \int d\theta \mathcal{N}_{A \rightarrow B}^\theta(\langle\theta|_E \tau_{AE} |\theta\rangle_E). \quad (5.91)$$

This simulation can be thought of as preparing a classical register  $E$  with the parameter value  $\theta$ , and then the parameter-independent channel  $\mathcal{M}_{AE \rightarrow B}$  observes the value  $\theta$  in the classical register and performs the channel  $\mathcal{N}_{A \rightarrow B}^\theta$  on the input system  $A$ . However, this construction is not useful for obtaining upper bounds on the performance of channel families for quantum estimation, because the classical Fisher information of the classical background family  $\{|\theta\rangle\langle\theta|_E\}_\theta$  is equal to infinity.

The notion of environment-parameterized channels only becomes interesting or useful for obtaining bounds on the performance of channel estimation in the case that the background environment states  $\rho_E^\theta$  are not perfectly distinguishable, as considered in [17, 23, 90]. That is, this concept is only useful for obtaining bounds if the Fisher information of the state family  $\{\rho_E^\theta\}_\theta$  is finite. In a general sense, performance bounds in the general sequential setting can be understood as being a consequence of the following proposition:

**Proposition 20** *Let  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  be an environment-parameterized channel family with associated environment state family  $\{\rho_E^\theta\}_\theta$ . Suppose that the underlying generalized Fisher information is subadditive on product-state families. Then the amortized Fisher information obeys the following bound:*

$$\mathbf{I}_F^A(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) \leq \mathbf{I}_F(\theta; \{\rho_E^\theta\}_\theta). \quad (5.92)$$

**Proof.** Let  $\{\omega_{RA}^\theta\}_\theta$  be an arbitrary input state family. Then the following chain of inequalities holds

$$\mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\omega_{RA}^\theta)\}_\theta) = \mathbf{I}_F(\theta; \{\mathcal{M}_{AE \rightarrow B}(\omega_{RA}^\theta \otimes \rho_E^\theta)\}_\theta) \quad (5.93)$$

$$\leq \mathbf{I}_F(\theta; \{\omega_{RA}^\theta \otimes \rho_E^\theta\}_\theta) \quad (5.94)$$

$$\leq \mathbf{I}_F(\theta; \{\omega_{RA}^\theta\}_\theta) + \mathbf{I}_F(\theta; \{\rho_E^\theta\}_\theta). \quad (5.95)$$

The equality follows by applying (5.90). The first inequality follows from data processing, and the second inequality follows from the assumption of subadditivity of  $\mathbf{I}_F$  on product-state families. Since the inequality holds for an arbitrary state family  $\{\omega_{RA}^\theta\}_\theta$ , we conclude (5.92). ■

Perhaps the most interesting case of environment-parameterized channel families is when the environment states are seizable by a pre- and post-processing of the channel [33, 100]:

**Definition 21 (Environment-seizable family)** *An environment-parameterized channel family  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  with associated environment state family  $\{\rho_E^\theta\}_\theta$  is called environment seizable if there exists a parameter-independent input state  $\zeta_{RA}$  and post-processing channel  $\mathcal{D}_{RB \rightarrow E}$  that can be used to seize the background state  $\rho_E^\theta$  in the following sense:*

$$\mathcal{D}_{RB \rightarrow E}(\mathcal{N}_{A \rightarrow B}^\theta(\zeta_{RA})) = \rho_E^\theta. \quad (5.96)$$

Simple examples of these channel families, along with simple environment-seizing procedures, were discussed in [33]. These examples include erasure and dephasing channels, with the underlying parameter being the noise parameter of the channel.

As indicated by Definition 21, environment-seizable channel families are fully identified with their background environment states. That is, for such channel families, the most powerful procedure for estimating them is to seize the background states first and then perform processing on these background environment states. One way to formalize this is with the following proposition:

**Proposition 22** *Let  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  be an environment-seizable channel family with associated environment state family  $\{\rho_E^\theta\}_\theta$ . Suppose that the underlying generalized Fisher information is subadditive on product-state families and weakly faithful. Then the amortized Fisher information is equal to the generalized Fisher information of the environment state family:*

$$\mathbf{I}_F^A(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) = \mathbf{I}_F(\theta; \{\rho_E^\theta\}_\theta). \quad (5.97)$$

**Proof.** The inequality  $\leq$  was established by Proposition 20. To see the opposite inequality, pick  $\{\rho_{RA}^\theta\}_\theta$  in the definition of  $\mathbf{I}_F^A(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta)$  to be the parameter-independent family  $\{\zeta_{RA}\}_\theta$ . Then it follows that

$$\mathbf{I}_F^A(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) \geq \mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\zeta_{RA})\}_\theta) - \mathbf{I}_F(\theta; \{\zeta_{RA}\}_\theta) \quad (5.98)$$

$$= \mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\zeta_{RA})\}_\theta) \quad (5.99)$$

$$\geq \mathbf{I}_F(\theta; \{\mathcal{D}_{RB \rightarrow E}(\mathcal{N}_{A \rightarrow B}^\theta(\zeta_{RA}))\}_\theta) \quad (5.100)$$

$$= \mathbf{I}_F(\theta; \{\rho_E^\theta\}_\theta). \quad (5.101)$$

The first inequality follows from Definition 16. The first equality follows from the weak faithfulness assumption. The second inequality follows from data processing. The final equality follows from Definition 21. ■

For these channel families, we can then employ the SLD Fisher information to arrive at the following conclusion, the first part of which was already given in [23]:

**Conclusion 23** *Let  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  be an environment-parameterized channel family with associated environment state family  $\{\rho_E^\theta\}_\theta$ . As a direct consequence of the QCRB in (5.6), the meta-converse from Theorem 18, and the bound in Proposition 20, we conclude the following bound on the MSE of an unbiased estimator  $\hat{\theta}$  of  $\theta$  that results from an  $n$ -round sequential estimation protocol:*

$$\text{Var}(\hat{\theta}) \geq \frac{1}{n \mathbf{I}_F(\theta; \{\rho_E^\theta\}_\theta)}. \quad (5.102)$$

*If the channel family is environment seizable as well, then this bound is achievable in the large  $n$  limit.*

## 5.4 Optimizing the SLD and RLD Fisher information of quantum states and channels

Particular generalized Fisher informations of interest in applications, due to the bounds in (5.6), (5.31), and (5.35), are the SLD and RLD ones. In this section, we show how these quantities, along with their dynamic channel versions, can be cast as optimization problems. In some cases, we find semi-definite programs, which implies that these quantities can be efficiently computed [102–105] (we should clarify that, by “efficient,” we mean the computational run time is polynomial in the dimension of the states or channels under consideration). Thus, in these cases, there is no need to compute spectral decompositions or matrix inverses in order to evaluate the Fisher information quantities.

### 5.4.1 Semi-definite program for SLD Fisher information of quantum states

We begin with the SLD Fisher information, establishing that it can be evaluated by means of a semi-definite program.



**Proposition 24** *The SLD Fisher information of a differentiable family  $\{\rho_\theta\}_\theta$  of states satisfying the finiteness condition in (5.11) can be evaluated by means of the following semi-definite program:*

$$I_F(\theta; \{\rho_\theta\}_\theta) = 2 \cdot \inf \left\{ \mu \in \mathbb{R} : \begin{bmatrix} \mu & \langle \Gamma | (\partial_\theta \rho_\theta \otimes I) \\ (\partial_\theta \rho_\theta \otimes I) | \Gamma \rangle & \rho_\theta \otimes I + I \otimes \rho_\theta^T \end{bmatrix} \geq 0 \right\}. \quad (5.103)$$

*The dual semi-definite program is as follows:*

$$2 \cdot \sup_{\lambda, |\varphi\rangle, Z} 2 \operatorname{Re}[\langle \varphi | (\partial_\theta \rho_\theta \otimes I) | \Gamma \rangle] - \operatorname{Tr}[(\rho_\theta \otimes I + I \otimes \rho_\theta^T) Z], \quad (5.104)$$

*subject to  $\lambda \in \mathbb{R}$ ,  $|\varphi\rangle$  an arbitrary complex vector,  $Z$  Hermitian, and*

$$\lambda \leq 1, \quad \begin{bmatrix} \lambda & \langle \varphi | \\ | \varphi \rangle & Z \end{bmatrix} \geq 0. \quad (5.105)$$

**Proof.** The primal semi-definite program is a direct consequence of the formula in (5.23) and Lemma 57. The dual program is a consequence of Lemma 58. ■

#### 5.4.2 Root SLD Fisher information of quantum states as a quadratically constrained optimization

In this section, we find that the root SLD Fisher information of quantum states can be computed by means of a quadratically constrained optimization. These optimization problems are difficult to solve in general, but heuristic methods are available [106]. In any case, the particular optimization formula in Proposition 25 is helpful for establishing the chain rule property of the root SLD Fisher information, which we discuss in Section 5.5.2.

**Proposition 25** *Let  $\{\rho_\theta\}_\theta$  be a differentiable family of quantum states. Then the root SLD Fisher information can be written as the following optimization:*

$$\sqrt{I_F}(\theta; \{\rho_\theta\}_\theta) = \sqrt{2} \sup_X \{ |\operatorname{Tr}[X(\partial_\theta \rho_\theta)]| : \operatorname{Tr}[(XX^\dagger + X^\dagger X)\rho_\theta] \leq 1 \}. \quad (5.106)$$

*If the finiteness condition in (5.11) is not satisfied, then the optimization formula evaluates to  $+\infty$ .*

**Proof.** Let us begin by supposing that the finiteness condition in (5.11) is satisfied (i.e.,  $\Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp = 0$ ). Recall from (5.23) the following formula for SLD Fisher information:

$$I_F(\theta; \{\rho_\theta\}_\theta) = 2 \langle \Gamma | (\partial_\theta \rho_\theta \otimes I) (\rho_\theta \otimes I + I \otimes \rho_\theta^T)^{-1} (\partial_\theta \rho_\theta \otimes I) | \Gamma \rangle, \quad (5.107)$$

so that

$$\frac{1}{\sqrt{2}} \sqrt{I_F}(\theta; \{\rho_\theta\}_\theta)$$

$$= \sqrt{\langle \Gamma | (\partial_\theta \rho_\theta \otimes I) (\rho_\theta \otimes I + I \otimes \rho_\theta^T)^{-1} (\partial_\theta \rho_\theta \otimes I) | \Gamma \rangle} \quad (5.108)$$

$$= \left\| (\rho_\theta \otimes I + I \otimes \rho_\theta^T)^{-\frac{1}{2}} (\partial_\theta \rho_\theta \otimes I) | \Gamma \rangle \right\|_2 \quad (5.109)$$

$$= \sup_{|\psi\rangle: \|\psi\rangle\|_2=1} \left| \langle \psi | (\rho_\theta \otimes I + I \otimes \rho_\theta^T)^{-\frac{1}{2}} (\partial_\theta \rho_\theta \otimes I) | \Gamma \rangle \right|. \quad (5.110)$$

Observe that the projection onto the support of  $\rho_\theta \otimes I + I \otimes \rho_\theta^T$  is

$$\Pi_{\rho_\theta} \otimes \Pi_{\rho_\theta^T} + \Pi_{\rho_\theta}^\perp \otimes \Pi_{\rho_\theta^T}^\perp + \Pi_{\rho_\theta} \otimes \Pi_{\rho_\theta^T}^\perp = I \otimes I - \Pi_{\rho_\theta}^\perp \otimes \Pi_{\rho_\theta^T}^\perp. \quad (5.111)$$

Thus, it suffices to optimize over  $|\psi\rangle$  satisfying

$$|\psi\rangle = (I \otimes I - \Pi_{\rho_\theta}^\perp \otimes \Pi_{\rho_\theta^T}^\perp) |\psi\rangle \quad (5.112)$$

because

$$\begin{aligned} & (\rho_\theta \otimes I + I \otimes \rho_\theta^T)^{-\frac{1}{2}} (\partial_\theta \rho_\theta \otimes I) | \Gamma \rangle \\ &= (I \otimes I - \Pi_{\rho_\theta}^\perp \otimes \Pi_{\rho_\theta^T}^\perp) (\rho_\theta \otimes I + I \otimes \rho_\theta^T)^{-\frac{1}{2}} (\partial_\theta \rho_\theta \otimes I) | \Gamma \rangle. \end{aligned} \quad (5.113)$$

Now define

$$|\psi'\rangle := (\rho_\theta \otimes I + I \otimes \rho_\theta^T)^{-\frac{1}{2}} |\psi\rangle, \quad (5.114)$$

which implies that

$$|\psi\rangle = (I \otimes I - \Pi_{\rho_\theta}^\perp \otimes \Pi_{\rho_\theta^T}^\perp) |\psi\rangle = (\rho_\theta \otimes I + I \otimes \rho_\theta^T)^{\frac{1}{2}} |\psi'\rangle, \quad (5.115)$$

because  $I \otimes I - \Pi_{\rho_\theta}^\perp \otimes \Pi_{\rho_\theta^T}^\perp$  is the projection onto the support of  $\rho_\theta \otimes I + I \otimes \rho_\theta^T$ . Thus, the following equivalence holds

$$\|\psi\rangle\|_2 = 1 \iff \left\| (\rho_\theta \otimes I + I \otimes \rho_\theta^T)^{\frac{1}{2}} |\psi'\rangle \right\|_2 = 1 \quad (5.116)$$

$$\iff \langle \psi' | (\rho_\theta \otimes I + I \otimes \rho_\theta^T) | \psi' \rangle = 1. \quad (5.117)$$

Now fix the operator  $X$  such that

$$|\psi'\rangle = (X \otimes I) | \Gamma \rangle. \quad (5.118)$$

Then the last condition above is the same as the following:

$$1 = \langle \Gamma | (X^\dagger \otimes I) (\rho_\theta \otimes I + I \otimes \rho_\theta^T) (X \otimes I) | \Gamma \rangle \quad (5.119)$$

$$= \langle \Gamma | (X^\dagger \rho_\theta X \otimes I + X^\dagger X \otimes \rho_\theta^T) | \Gamma \rangle \quad (5.120)$$

$$= \langle \Gamma | (X^\dagger \rho_\theta X \otimes I + X^\dagger X \rho_\theta \otimes I) | \Gamma \rangle \quad (5.121)$$

$$= \text{Tr}[X^\dagger \rho_\theta X] + \text{Tr}[X^\dagger X \rho_\theta] \quad (5.122)$$

$$= \text{Tr}[(X X^\dagger + X^\dagger X) \rho_\theta], \quad (5.123)$$

where we used (3.2) and (3.3). So then the optimization problem in (5.110) is equal to the following:

$$\begin{aligned} & \sup_{X: \text{Tr}[(X X^\dagger + X^\dagger X) \rho_\theta] = 1} |\langle \Gamma | (X \otimes I) (\partial_\theta \rho_\theta \otimes I) | \Gamma \rangle| \\ &= \sup_{X: \text{Tr}[(X X^\dagger + X^\dagger X) \rho_\theta] = 1} |\langle \Gamma | (X (\partial_\theta \rho_\theta) \otimes I) | \Gamma \rangle| \end{aligned} \quad (5.124)$$

$$= \sup_X \{ |\text{Tr}[X (\partial_\theta \rho_\theta)]| : \text{Tr}[(X X^\dagger + X^\dagger X) \rho_\theta] = 1 \}, \quad (5.125)$$

where again we used (3.3). Now suppose that  $\text{Tr}[(X X^\dagger + X^\dagger X) \rho_\theta] = c$ , with  $c \in (0, 1)$ . Then we can multiply  $X$  by  $\sqrt{1/c}$ , and the new operator satisfies the equality constraint while the value of the objective function increases. So we can write

$$\sqrt{I_F}(\theta; \{\rho_\theta\}_\theta) = \sqrt{2} \sup_X \{ |\text{Tr}[X (\partial_\theta \rho_\theta)]| : \text{Tr}[(X X^\dagger + X^\dagger X) \rho_\theta] \leq 1 \}. \quad (5.126)$$

Finally, in this form, note that we can trivially include  $X = 0$  as part of the optimization because it leads to a generally suboptimal value of zero for the objective function.

Suppose that  $\Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp \neq 0$ . Then we can pick  $X = c \Pi_{\rho_\theta}^\perp + d I$  where  $c, d > 0$  and  $2d^2 = 1$ . We find that

$$\text{Tr}[(X X^\dagger + X^\dagger X) \rho_\theta] = 2 \text{Tr}[(c \Pi_{\rho_\theta}^\perp + d I)^2 \rho_\theta] \quad (5.127)$$

$$= 2 \text{Tr}[(c^2 + 2cd) \Pi_{\rho_\theta}^\perp + d^2 I] \rho_\theta \quad (5.128)$$

$$= 2d^2 = 1. \quad (5.129)$$

for this case, so that the constraint in (5.106) is satisfied. The objective function then evaluates to

$$|\text{Tr}[X (\partial_\theta \rho_\theta)]| = |\text{Tr}[(c \Pi_{\rho_\theta}^\perp + d I) (\partial_\theta \rho_\theta)]| \quad (5.130)$$

$$= |c \text{Tr}[\Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta)] + d \text{Tr}[\partial_\theta \rho_\theta]| \quad (5.131)$$

$$= c |\text{Tr}[\Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta)]|. \quad (5.132)$$

Then we can pick  $c > 0$  arbitrarily large to get that (5.106) evaluates to  $+\infty$  in the case that  $\Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp \neq 0$ . ■

We can use the optimization formula in Proposition 25 to conclude that the data-processing inequality holds for all two-positive, trace-preserving maps, which includes quantum channels as a special case. This was already observed in [66], but here we give a different proof based on the optimization formula in Proposition 25.

**Proposition 26** *Let  $\{\rho_\theta\}_\theta$  be a differentiable family of quantum states, and let  $\mathcal{P}$  be a two-positive, trace-preserving map. Then the following data-processing inequality holds*

$$I_F(\theta; \{\rho_\theta\}_\theta) \geq I_F(\theta; \{\mathcal{P}(\rho_\theta)\}_\theta). \quad (5.133)$$

**Proof.** Let  $X$  be an operator satisfying

$$\text{Tr}[(XX^\dagger + X^\dagger X)\mathcal{P}(\rho_\theta)] \leq 1. \quad (5.134)$$

Then it follows that

$$1 \geq \text{Tr}[(XX^\dagger + X^\dagger X)\mathcal{P}(\rho_\theta)] \quad (5.135)$$

$$= \text{Tr}[\mathcal{P}^\dagger(XX^\dagger + X^\dagger X)\rho_\theta] \quad (5.136)$$

$$\geq \text{Tr}[(\mathcal{P}^\dagger(X)\mathcal{P}^\dagger(X^\dagger) + \mathcal{P}^\dagger(X^\dagger)\mathcal{P}^\dagger(X))\rho_\theta], \quad (5.137)$$

where the last inequality follows because  $\rho_\theta \geq 0$  and

$$\mathcal{P}^\dagger(XX^\dagger) \geq \mathcal{P}^\dagger(X)\mathcal{P}^\dagger(X^\dagger), \quad \mathcal{P}^\dagger(X^\dagger X) \geq \mathcal{P}^\dagger(X^\dagger)\mathcal{P}^\dagger(X). \quad (5.138)$$

The latter inequalities are a consequence of the Schwarz inequality, which holds for two-positive, unital maps [107, Eq. (3.14)]. (Note that two-positive, unital maps are the Hilbert–Schmidt adjoints of two-positive, trace-preserving maps). Furthermore,

$$|\text{Tr}[X(\partial_\theta \mathcal{P}(\rho_\theta))]| = |\text{Tr}[X\mathcal{P}(\partial_\theta \rho_\theta)]| \quad (5.139)$$

$$= |\text{Tr}[\mathcal{P}^\dagger(X)(\partial_\theta \rho_\theta)]| \quad (5.140)$$

$$\leq \sup_Z \{|\text{Tr}[Z(\partial_\theta \rho_\theta)]| : \text{Tr}[(ZZ^\dagger + Z^\dagger Z)\rho_\theta] \leq 1\} \quad (5.141)$$

$$= \frac{1}{\sqrt{2}} \sqrt{I_F(\theta; \{\rho_\theta\}_\theta)}. \quad (5.142)$$

Since the inequality holds for all  $X$  satisfying (5.134), we conclude that

$$\sqrt{I_F(\theta; \{\rho_\theta\}_\theta)} \geq \sqrt{I_F(\theta; \{\mathcal{P}(\rho_\theta)\}_\theta)}. \quad (5.143)$$

This concludes the proof. ■

### 5.4.3 Bilinear program for SLD Fisher information of quantum channels

We can exploit Proposition 24 and a number of manipulations to arrive at a bilinear program for the SLD Fisher information of channels:

**Proposition 27** *The SLD Fisher information of a differentiable family  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  of channels satisfying the finiteness condition in (5.72) can be evaluated by means of the following bilinear program:*

$$I_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}) = 2 \sup_{\substack{\lambda, |\varphi\rangle_{RR'B'}, \\ W_{RR'B'}, Y_R, \sigma_R}} \left( 2 \text{Re}[\langle \varphi |_{RR'B'} (\partial_\theta \Gamma_{RB}^{\mathcal{N}_R^\theta}) | \Gamma \rangle_{RR'BB'}] - \text{Tr}[Y_R \Phi(W_{RR'B'})] \right) \quad (5.144)$$

subject to

$$\sigma_R \geq 0, \quad \text{Tr}[\sigma_R] = 1, \quad \lambda \leq 1, \quad \begin{bmatrix} \lambda & \langle \varphi |_{RBR'B'} \\ | \varphi \rangle_{RBR'B'} & W_{RBR'B'} \end{bmatrix} \geq 0, \quad \begin{bmatrix} \sigma_R & I_R \\ I_R & Y_R \end{bmatrix} \geq 0. \quad (5.145)$$

where

$$|\Gamma\rangle_{RR'BB'} := |\Gamma\rangle_{RR'} \otimes |\Gamma\rangle_{BB'}, \quad (5.146)$$

$$\begin{aligned} \Phi(W_{RBR'B'}) &:= (\text{Tr}_{BR'B'}[\Gamma_{R'B'}^{\mathcal{N}^\theta} (F_{RR'} \otimes F_{BB'}) W_{RBR'B'} (F_{RR'} \otimes F_{BB'})^\dagger])^T \\ &\quad + \text{Tr}_{BR'B'}[(\Gamma_{R'B'}^{\mathcal{N}^\theta})^T W_{RBR'B'}], \end{aligned} \quad (5.147)$$

and  $F_{RR'}$  is the flip or swap operator that swaps systems  $R$  and  $R'$ , with a similar definition for  $F_{BB'}$  but for  $B$  and  $B'$ .

**Proof.** See Appendix F. ■

The optimization above is a jointly constrained semi-definite bilinear program [108] because the variables  $Y_R$  and  $W_{RBR'B'}$  are operators involved in the optimization and they multiply each other in the last expression in (5.144). This kind of optimization can be approached with a heuristic “seesaw” method, but more advanced methods are available in [108].

#### 5.4.4 Semi-definite programs for RLD Fisher information of quantum states and channels

We now give semi-definite programs for the RLD Fisher information of quantum states:

**Proposition 28** *The RLD Fisher information of a differentiable family  $\{\rho_\theta\}_\theta$  of states satisfying the support condition in (5.28) can be evaluated by means of the following semi-definite program:*

$$\widehat{I}_F(\theta; \{\rho_A^\theta\}) = \inf \left\{ \text{Tr}[M] : M \geq 0, \begin{bmatrix} M & \partial_\theta \rho_\theta \\ \partial_\theta \rho_\theta & \rho_\theta \end{bmatrix} \geq 0 \right\}. \quad (5.148)$$

The dual semi-definite program is as follows:

$$\sup_{X, Y, Z} 2 \text{Re}[\text{Tr}[Y(\partial_\theta \rho_\theta)]] - \text{Tr}[Z \rho_\theta], \quad (5.149)$$

subject to  $X$  and  $Y$  being Hermitian and

$$X \leq I, \quad \begin{bmatrix} X & Y^\dagger \\ Y & Z \end{bmatrix} \geq 0. \quad (5.150)$$

**Proof.** The primal semi-definite program is a direct consequence of the RLD formula in (5.25) and Lemma 57. The dual program is found by applying Lemma 58. ■

The following formula for the RLD Fisher information of quantum channels is known from [20]. It comes about by manipulating the RLD formula in (5.25) by means of Lemma 59. We review its proof in Appendix G.

**Proposition 29** *Let  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  be a differentiable family of quantum channels such that the support condition in (5.73) holds. Then the RLD Fisher information of quantum channels has the following explicit form:*

$$\hat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) = \left\| \text{Tr}_B[(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1}(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})] \right\|_\infty, \quad (5.151)$$

where  $\Gamma_{RB}^{\mathcal{N}^\theta}$  is the Choi operator of the channel  $\mathcal{N}_{A \rightarrow B}^\theta$ .

We then find the following semi-definite program for the RLD Fisher information of quantum channels:

**Proposition 30** *Let  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  be a differentiable family of quantum channels such that the support condition in (5.73) holds. Then the RLD Fisher information of quantum channels can be calculated by means of the following semi-definite program:*

$$\hat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) = \inf \lambda \in \mathbb{R}^+, \quad (5.152)$$

subject to

$$\lambda I_R \geq \text{Tr}_B[M_{RB}], \quad \begin{bmatrix} M_{RB} & \partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta} \\ \partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta} & \Gamma_{RB}^{\mathcal{N}^\theta} \end{bmatrix} \geq 0. \quad (5.153)$$

The dual program is given by

$$\sup_{\rho_R \geq 0, P_{RB}, Z_{RB}, Q_{RB}} 2 \text{Re}[\text{Tr}[Z_{RB}(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})]] - \text{Tr}[Q_{RB} \Gamma_{RB}^{\mathcal{N}^\theta}], \quad (5.154)$$

subject to

$$\text{Tr}[\rho_R] \leq 1, \quad \begin{bmatrix} P_{RB} & Z_{RB}^\dagger \\ Z_{RB} & Q_{RB} \end{bmatrix} \geq 0, \quad P_{RB} \leq \rho_R \otimes I_B. \quad (5.155)$$

**Proof.** The form of the primal program follows directly from (5.151), Lemma 57, and from the following characterization of the infinity norm of a positive semi-definite operator  $W$ :

$$\|W\|_\infty = \inf \{ \lambda \geq 0 : W \leq \lambda I \}. \quad (5.156)$$

To arrive at the dual program, we use the standard forms of primal and dual semi-definite programs for Hermitian operators  $A$  and  $B$  and a Hermiticity-preserving map  $\Phi$  [43]:

$$\sup_{X \geq 0} \{ \text{Tr}[AX] : \Phi(X) \leq B \}, \quad \inf_{Y \geq 0} \{ \text{Tr}[BY] : \Phi^\dagger(Y) \geq A \}. \quad (5.157)$$

From (5.152)–(5.153), we identify

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} \lambda & 0 \\ 0 & M_{RB} \end{bmatrix}, \quad \Phi^\dagger(Y) = \begin{bmatrix} \lambda I_R - \text{Tr}_B[M_{RB}] & 0 & 0 \\ 0 & M_{RB} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (5.158)$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta} \\ 0 & -\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta} & -\Gamma_{RB}^{\mathcal{N}^\theta} \end{bmatrix}. \quad (5.159)$$

Setting

$$X = \begin{bmatrix} \rho_R & 0 & 0 \\ 0 & P_{RB} & Z_{RB}^\dagger \\ 0 & Z_{RB} & Q_{RB} \end{bmatrix}, \quad (5.160)$$

we find that

$$\text{Tr}[X\Phi^\dagger(Y)] = \text{Tr} \left[ \begin{bmatrix} \rho_R & 0 & 0 \\ 0 & P_{RB} & Z_{RB}^\dagger \\ 0 & Z_{RB} & Q_{RB} \end{bmatrix} \begin{bmatrix} \lambda I_R - \text{Tr}_B[M_{RB}] & 0 & 0 \\ 0 & M_{RB} & 0 \\ 0 & 0 & 0 \end{bmatrix} \right] \quad (5.161)$$

$$= \text{Tr}[\rho_R(\lambda I_R - \text{Tr}_B[M_{RB}])] + \text{Tr}[P_{RB}M_{RB}] \quad (5.162)$$

$$= \lambda \text{Tr}[\rho_R] + \text{Tr}[(P_{RB} - \rho_R \otimes I_B)M_{RB}] \quad (5.163)$$

$$= \text{Tr} \left[ \begin{bmatrix} \lambda & 0 \\ 0 & M_{RB} \end{bmatrix} \begin{bmatrix} \text{Tr}[\rho_R] & 0 \\ 0 & P_{RB} - \rho_R \otimes I_B \end{bmatrix} \right], \quad (5.164)$$

which implies that

$$\Phi(X) = \begin{bmatrix} \text{Tr}[\rho_R] & 0 \\ 0 & P_{RB} - \rho_R \otimes I_B \end{bmatrix}. \quad (5.165)$$

Then plugging into the left-hand side of (5.157), we find that the dual is given by

$$\sup_{\rho_R, P_{RB}, Z_{RB}, Q_{RB}} \text{Tr} \left[ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta} \\ 0 & -\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta} & -\Gamma_{RB}^{\mathcal{N}^\theta} \end{bmatrix} \begin{bmatrix} W_R & 0 & 0 \\ 0 & P_{RB} & Z_{RB}^\dagger \\ 0 & Z_{RB} & Q_{RB} \end{bmatrix} \right], \quad (5.166)$$

subject to

$$\begin{bmatrix} \rho_R & 0 & 0 \\ 0 & P_{RB} & Z_{RB}^\dagger \\ 0 & Z_{RB} & Q_{RB} \end{bmatrix} \geq 0, \quad \begin{bmatrix} \text{Tr}[\rho_R] & 0 \\ 0 & P_{RB} - \rho_R \otimes I_B \end{bmatrix} \leq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (5.167)$$

Upon making the swap  $Z_{RB} \rightarrow -Z_{RB}$ , which does not change the optimal value, and simplifying, we find the following form:

$$\sup_{\rho_R \geq 0, P_{RB}, Z_{RB}, Q_{RB}} 2 \text{Re}[\text{Tr}[Z_{RB}(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})]] - \text{Tr}[Q_{RB} \Gamma_{RB}^{\mathcal{N}^\theta}], \quad (5.168)$$

subject to

$$\text{Tr}[\rho_R] \leq 1, \quad \begin{bmatrix} P_{RB} & -Z_{RB}^\dagger \\ -Z_{RB} & Q_{RB} \end{bmatrix} \geq 0, \quad P_{RB} \leq \rho_R \otimes I_B. \quad (5.169)$$

Then we note that

$$\begin{bmatrix} P_{RB} & -Z_{RB}^\dagger \\ -Z_{RB} & Q_{RB} \end{bmatrix} \geq 0 \iff \begin{bmatrix} P_{RB} & Z_{RB}^\dagger \\ Z_{RB} & Q_{RB} \end{bmatrix} \geq 0 \quad (5.170)$$

This concludes the proof. ■

## 5.5 SLD Fisher information limits on quantum channel parameter estimation

### 5.5.1 SLD Fisher information limit on parameter estimation of classical–quantum channels

We first consider the special case of a family  $\{\mathcal{N}_{X \rightarrow B}^\theta\}_\theta$  of classical–quantum channels of the following form:

$$\mathcal{N}_{X \rightarrow B}^\theta(\sigma_X) := \sum_x \langle x|_X \sigma_X |x\rangle_X \omega_B^{x,\theta}, \quad (5.171)$$

where  $\{|x\rangle\}_x$  is an orthonormal basis and  $\{\omega_B^{x,\theta}\}_x$  is a collection of states prepared at the channel output conditioned on the value of the unknown parameter  $\theta$  and on the result of the measurement of the channel input. The key aspect of these channels is that the measurement at the input is the same regardless of the value of the parameter  $\theta$ . We find the following amortization collapse for these channels:

**Theorem 31** *Let  $\{\mathcal{N}_{X \rightarrow B}^\theta\}_\theta$  be a family of differentiable classical–quantum channels. Then the following amortization collapse occurs*

$$I_F(\theta; \{\mathcal{N}_{X \rightarrow B}^\theta\}_\theta) = I_F^A(\theta; \{\mathcal{N}_{X \rightarrow B}^\theta\}_\theta) = \sup_x I_F(\theta; \{\omega_B^{x,\theta}\}_\theta). \quad (5.172)$$

**Proof.** If the finiteness condition in (5.72) does not hold, then all quantities are trivially equal to  $+\infty$ . So let us suppose that the finiteness condition in (5.72) holds. Note that the finiteness condition is equivalent to

$$\Pi_{\omega_B^{x,\theta}}^\perp (\partial_\theta \omega_B^{x,\theta}) \Pi_{\omega_B^{x,\theta}}^\perp = 0 \quad \forall x. \quad (5.173)$$

First, consider that the following inequality holds

$$I_F(\theta; \{\mathcal{N}_{X \rightarrow B}^\theta\}_\theta) \geq \sup_x I_F(\theta; \{\omega_B^{x,\theta}\}_\theta) \quad (5.174)$$

because we can input the state  $|x\rangle\langle x|_X$  to the channel  $\mathcal{N}_{X \rightarrow B}^\theta$  and obtain the output state  $\mathcal{N}_{X \rightarrow B}^\theta(|x\rangle\langle x|_X) = \omega_B^{x,\theta}$ . Then we can optimize over  $x \in \mathcal{X}$  and obtain the bound above.



We now prove the less trivial inequality

$$I_F^A(\theta; \{\mathcal{N}_{X \rightarrow B}^\theta\}_\theta) \leq \sup_x I_F(\theta; \{\omega_B^{x,\theta}\}_\theta). \quad (5.175)$$

Let  $\{\rho_{RA}^\theta\}_\theta$  be a differentiable family of quantum states. If the classical–quantum channel  $\mathcal{N}_{X \rightarrow B}^\theta$  acts on  $\rho_{RA}^\theta$  (identifying  $X = A$ ), the output state is as follows:

$$\mathcal{N}_{X \rightarrow B}^\theta(\rho_{RA}^\theta) = \sum_x p_\theta(x) \rho_R^{x,\theta} \otimes \omega_B^{x,\theta}, \quad (5.176)$$

where

$$\rho_R^{x,\theta} := \frac{1}{p_\theta(x)} \langle x|_X \rho_{RA}^\theta |x\rangle_X, \quad p_\theta(x) := \text{Tr}[\langle x|_X \rho_{RA}^\theta |x\rangle_X]. \quad (5.177)$$

Then consider that

$$\begin{aligned} & I_F(\theta; \{\mathcal{N}_{X \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) \\ &= I_F\left(\theta; \left\{ \sum_x p_\theta(x) \rho_R^{x,\theta} \otimes \omega_B^{x,\theta} \right\}_\theta\right) \end{aligned} \quad (5.178)$$

$$\leq I_F\left(\theta; \left\{ \sum_x p_\theta(x) |x\rangle\langle x|_X \otimes \rho_R^{x,\theta} \otimes \omega_B^{x,\theta} \right\}_\theta\right) \quad (5.179)$$

$$= I_F(\theta; \{p_\theta\}_\theta) + \sum_x p_\theta(x) I_F(\theta; \{\rho_R^{x,\theta} \otimes \omega_B^{x,\theta}\}_\theta) \quad (5.180)$$

$$= I_F(\theta; \{p_\theta\}_\theta) + \sum_x p_\theta(x) I_F(\theta; \{\rho_R^{x,\theta}\}_\theta) + \sum_x p_\theta(x) I_F(\theta; \{\omega_B^{x,\theta}\}_\theta) \quad (5.181)$$

$$\leq I_F(\theta; \{p_\theta\}_\theta) + \sum_x p_\theta(x) I_F(\theta; \{\rho_R^{x,\theta}\}_\theta) + \sup_x I_F(\theta; \{\omega_B^{x,\theta}\}_\theta) \quad (5.182)$$

$$= I_F\left(\theta; \left\{ \sum_x p_\theta(x) |x\rangle\langle x|_X \otimes \rho_R^{x,\theta} \right\}_\theta\right) + \sup_x I_F(\theta; \{\omega_B^{x,\theta}\}_\theta) \quad (5.183)$$

$$\leq I_F(\theta; \{\rho_{RA}^\theta\}_\theta) + \sup_x I_F(\theta; \{\omega_B^{x,\theta}\}_\theta). \quad (5.184)$$

The first inequality follows from the data-processing inequality for Fisher information with respect to partial trace over the  $X$  system. The second equality follows from Proposition 7. The third equality follows from the additivity of SLD Fisher information for product states (Proposition 6). The second inequality follows from the fact that the average cannot exceed the maximum. The last equality follows again from Proposition 7. The final inequality follows from the data-processing inequality under the action of the measurement channel  $(\cdot) \rightarrow \sum_x |x\rangle\langle x|_X (\cdot) |x\rangle\langle x|_X$  on the state  $\rho_{RA}$ . Thus, the following inequality holds for an arbitrary family  $\{\rho_{RA}^\theta\}_\theta$  of states:

$$I_F(\theta; \{\mathcal{N}_{X \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) - I_F(\theta; \{\rho_{RA}^\theta\}_\theta) \leq \sup_x I_F(\theta; \{\omega_B^{x,\theta}\}_\theta). \quad (5.185)$$

Since the inequality in (5.185) holds for an arbitrary family  $\{\rho_{RA}^\theta\}_\theta$  of states, we conclude (5.175). Combining (5.174) and (5.175), along with the general inequality in (5.75), we conclude (5.172). ■

**Conclusion 32** *As a direct consequence of the QCRB in (5.6), the meta-converse from Theorem 18, and the amortization collapse from Theorem 31, we conclude the following bound on the MSE of an unbiased estimator  $\hat{\theta}$  for classical–quantum channel families defined in (5.171) and for which the finiteness condition in (5.173) holds:*

$$\text{Var}(\hat{\theta}) \geq \frac{1}{n \sup_x I_F(\theta; \{\omega_B^{x,\theta}\}_\theta)}. \quad (5.186)$$

*Thus, there is no advantage that sequential estimation strategies bring over parallel estimation strategies for this class of channels. In fact, an optimal parallel estimation strategy consists of picking the same optimal input letter  $x$  to each channel use in order to estimate  $\theta$ .*

### 5.5.2 Root SLD Fisher information limit for quantum channel parameter estimation

We begin by showing that the root SLD Fisher information obeys the following chain rule:

**Proposition 33 (Chain rule)** *Let  $\{\rho_\theta\}_\theta$  be a differentiable family of quantum states, and let  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  be a differentiable family of quantum channels. Then the following chain rule holds for the root SLD Fisher information:*

$$\sqrt{I_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta)} \leq \sqrt{I_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta)} + \sqrt{I_F(\theta; \{\rho_{RA}^\theta\}_\theta)}. \quad (5.187)$$

**Proof.** If the finiteness conditions in (5.11) and (5.72) do not hold, then the inequality is trivially satisfied. So let us suppose that the finiteness conditions (5.11) and (5.72) hold.

By invoking Proposition 25 and Remark 10, first consider that the root SLD Fisher information of channels has the following representation as an optimization:

$$\begin{aligned} & \frac{1}{\sqrt{2}} \sqrt{I_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta)} \\ &= \frac{1}{\sqrt{2}} \sup_{\rho_{RA}} \sqrt{I_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA})\}_\theta)} \end{aligned} \quad (5.188)$$

$$= \sup_{\rho_{RA}} \sup_{X_{RB}} \left\{ \begin{array}{l} |\text{Tr}[X_{RB}(\partial_\theta \mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}))]| : \\ \text{Tr}[(X_{RB} X_{RB}^\dagger + X_{RB}^\dagger X_{RB}) \mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA})] \leq 1 \end{array} \right\} \quad (5.189)$$

$$= \sup_{\rho_{RA}, X_{RB}} \left\{ \begin{array}{l} |\text{Tr}[X_{RB}(\partial_\theta \mathcal{N}_{A \rightarrow B}^\theta)(\rho_{RA})]| : \\ \text{Tr}[(X_{RB} X_{RB}^\dagger + X_{RB}^\dagger X_{RB}) \mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA})] \leq 1 \end{array} \right\}, \quad (5.190)$$

where the distinction between the third and last line is that  $\partial_\theta \mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}) = (\partial_\theta \mathcal{N}_{A \rightarrow B}^\theta)(\rho_{RA})$  (i.e., for fixed  $\rho_{RA}$ , the state  $\rho_{RA}$  is constant with respect to the partial derivative).

Now recall the post-selected teleportation identity from (3.7):

$$\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta) = \langle \Gamma|_{AS} \rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta} |\Gamma\rangle_{AS}. \quad (5.191)$$

This implies that

$$\begin{aligned} \partial_\theta(\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)) &= \partial_\theta(\langle \Gamma|_{AS} \rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta} |\Gamma\rangle_{AS}) \end{aligned} \quad (5.192)$$

$$= \langle \Gamma|_{AS} \partial_\theta(\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta}) |\Gamma\rangle_{AS} \quad (5.193)$$

$$= \langle \Gamma|_{AS} [(\partial_\theta \rho_{RA}^\theta) \otimes \Gamma_{SB}^{\mathcal{N}^\theta} + \rho_{RA}^\theta \otimes (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})] |\Gamma\rangle_{AS} \quad (5.194)$$

$$= \langle \Gamma|_{AS} [(\partial_\theta \rho_{RA}^\theta) \otimes \Gamma_{SB}^{\mathcal{N}^\theta} |\Gamma\rangle_{AS} + \langle \Gamma|_{AS} \rho_{RA}^\theta \otimes (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta}) |\Gamma\rangle_{AS}] \quad (5.195)$$

$$= \mathcal{N}_{A \rightarrow B}^\theta(\partial_\theta \rho_{RA}^\theta) + (\partial_\theta \mathcal{N}_{A \rightarrow B}^\theta)(\rho_{RA}^\theta). \quad (5.196)$$

Let  $X_{RB}$  be an arbitrary operator satisfying

$$\text{Tr}[(X_{RB} X_{RB}^\dagger + X_{RB}^\dagger X_{RB}) \mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)] \leq 1. \quad (5.197)$$

Working with the left-hand side of the inequality, we find that

$$\begin{aligned} \text{Tr}[(X_{RB} X_{RB}^\dagger + X_{RB}^\dagger X_{RB}) \mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)] &= \text{Tr}[(\mathcal{N}_{A \rightarrow B}^\theta)^\dagger (X_{RB} X_{RB}^\dagger + X_{RB}^\dagger X_{RB}) (\rho_{RA}^\theta)] \end{aligned} \quad (5.198)$$

$$\geq \text{Tr}[(Z_{RA} Z_{RA}^\dagger + Z_{RA}^\dagger Z_{RA}) (\rho_{RA}^\theta)], \quad (5.199)$$

where we set

$$Z_{RA} := (\mathcal{N}_{A \rightarrow B}^\theta)^\dagger (X_{RB}). \quad (5.200)$$

The equality follows because  $(\mathcal{N}_{A \rightarrow B}^\theta)^\dagger$  is the Hilbert–Schmidt adjoint of  $\mathcal{N}_{A \rightarrow B}^\theta$ , and the inequality follows because  $\rho_{RA}^\theta \geq 0$  and

$$(\mathcal{N}_{A \rightarrow B}^\theta)^\dagger (X_{RB}^\dagger) (\mathcal{N}_{A \rightarrow B}^\theta)^\dagger (X_{RB}) \leq (\mathcal{N}_{A \rightarrow B}^\theta)^\dagger (X_{RB}^\dagger X_{RB}), \quad (5.201)$$

$$(\mathcal{N}_{A \rightarrow B}^\theta)^\dagger (X_{RB}) (\mathcal{N}_{A \rightarrow B}^\theta)^\dagger (X_{RB}^\dagger) \leq (\mathcal{N}_{A \rightarrow B}^\theta)^\dagger (X_{RB} X_{RB}^\dagger), \quad (5.202)$$

which themselves follow from the Schwarz inequality for completely positive unital maps [107, Eq. (3.14)]. So we conclude that

$$\text{Tr}[(Z_{RA} Z_{RA}^\dagger + Z_{RA}^\dagger Z_{RA}) (\rho_{RA}^\theta)] \leq 1. \quad (5.203)$$

Then consider that

$$\begin{aligned} &|\text{Tr}[X_{RB}(\partial_\theta(\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)))]| \\ &= |\text{Tr}[X_{RB}((\partial_\theta \mathcal{N}_{A \rightarrow B}^\theta)(\rho_{RA}^\theta))] + \text{Tr}[X_{RB} \mathcal{N}_{A \rightarrow B}^\theta(\partial_\theta \rho_{RA}^\theta)]| \end{aligned} \quad (5.204)$$

$$= |\text{Tr}[X_{RB}((\partial_\theta \mathcal{N}_{A \rightarrow B}^\theta)(\rho_{RA}^\theta))] + \text{Tr}[(\mathcal{N}_{A \rightarrow B}^\theta)^\dagger (X_{RB}) (\partial_\theta \rho_{RA}^\theta)]| \quad (5.205)$$

$$\leq |\text{Tr}[X_{RB}((\partial_\theta \mathcal{N}_{A \rightarrow B}^\theta)(\rho_{RA}^\theta))]| + |\text{Tr}[(\mathcal{N}_{A \rightarrow B}^\theta)^\dagger(X_{RB})(\partial_\theta \rho_{RA}^\theta)]|. \quad (5.206)$$

By applying (5.190), we find that

$$\sqrt{2} |\text{Tr}[X_{RB}((\partial_\theta \mathcal{N}_{A \rightarrow B}^\theta)(\rho_{RA}^\theta))]| \leq \sqrt{I_F}(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta). \quad (5.207)$$

Since the operator  $(\mathcal{N}_{A \rightarrow B}^\theta)^\dagger(X_{RB}) = Z_{RA}$  satisfies (5.203), by applying the optimization in (5.106), we find that

$$\sqrt{2} |\text{Tr}[(\mathcal{N}_{A \rightarrow B}^\theta)^\dagger(X_{RB})(\partial_\theta \rho_{RA}^\theta)]| \leq \sqrt{I_F}(\theta; \{\rho_{RA}^\theta\}_\theta). \quad (5.208)$$

So we conclude that

$$\sqrt{2} |\text{Tr}[X_{RB}(\partial_\theta (\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)))]| \leq \sqrt{I_F}(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) + \sqrt{I_F}(\theta; \{\rho_{RA}^\theta\}_\theta). \quad (5.209)$$

Since  $X_{RB}$  is an arbitrary operator satisfying (5.197), we can optimize over all such operators to conclude the chain rule inequality in (5.187). ■

**Corollary 34** *Let  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  be a family of differentiable quantum channels. Then the following amortization collapse occurs for the root SLD Fisher information of quantum channels:*

$$\sqrt{I_F}^{\mathcal{A}}(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) = \sqrt{I_F}(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta), \quad (5.210)$$

where

$$\sqrt{I_F}^{\mathcal{A}}(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) := \sup_{\{\rho_{RA}^\theta\}_\theta} \left[ \sqrt{I_F}(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) - \sqrt{I_F}(\theta; \{\rho_{RA}^\theta\}_\theta) \right]. \quad (5.211)$$

**Proof.** If the finiteness condition in (5.72) does not hold, then the equality trivially holds. So let us suppose that the finiteness condition in (5.72) holds. The inequality  $\geq$  follows from Proposition 17 and the fact that the root SLD Fisher information is faithful (see (5.36)). The opposite inequality  $\leq$  is a consequence of the chain rule from Proposition 33. Let  $\{\rho_{RA}^\theta\}_\theta$  be a family of quantum states on a systems  $RA$ . Then it follows from Proposition 33 that

$$\sqrt{I_F}(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) - \sqrt{I_F}(\theta; \{\rho_{RA}^\theta\}_\theta) \leq \sqrt{I_F}(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta). \quad (5.212)$$

Since the family  $\{\rho_{RA}^\theta\}_\theta$  is arbitrary, we can take a supremum of the left-hand side over all such families, and conclude that

$$\sqrt{I_F}^{\mathcal{A}}(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) \leq \sqrt{I_F}(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta). \quad (5.213)$$

This concludes the proof. ■

**Corollary 35** *Let  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  and  $\{\mathcal{M}_{B \rightarrow C}^\theta\}_\theta$  be differentiable families of quantum channels. Then the root SLD Fisher information of quantum channels is subadditive with respect to serial composition, in the following sense:*

$$\sqrt{I_F}(\theta; \{\mathcal{M}_{B \rightarrow C}^\theta \circ \mathcal{N}_{A \rightarrow B}^\theta\}_\theta) \leq \sqrt{I_F}(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) + \sqrt{I_F}(\theta; \{\mathcal{M}_{B \rightarrow C}^\theta\}_\theta). \quad (5.214)$$

**Proof.** If the finiteness condition in (5.72) does not hold for either channel, then the inequality trivially holds. So let us suppose that the finiteness condition in (5.72) holds for both channels. Pick an arbitrary input state  $\omega_{RA}$ . Now apply Proposition 33 to find that

$$\begin{aligned} & \sqrt{I_F}(\theta; \{\mathcal{M}_{B \rightarrow C}^\theta(\mathcal{N}_{A \rightarrow B}^\theta(\omega_{RA}))\}_\theta) \\ & \leq \sqrt{I_F}(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\omega_{RA})\}_\theta) + \sqrt{I_F}(\theta; \{\mathcal{M}_{B \rightarrow C}^\theta\}_\theta) \end{aligned} \quad (5.215)$$

$$\leq \sup_{\omega_{RA}} \sqrt{I_F}(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\omega_{RA})\}_\theta) + \sqrt{I_F}(\theta; \{\mathcal{M}_{B \rightarrow C}^\theta\}_\theta) \quad (5.216)$$

$$= \sqrt{I_F}(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) + \sqrt{I_F}(\theta; \{\mathcal{M}_{B \rightarrow C}^\theta\}_\theta). \quad (5.217)$$

Since the inequality holds for all input states, we conclude that

$$\sup_{\omega_{RA}} \sqrt{I_F}(\theta; \{\mathcal{M}_{B \rightarrow C}^\theta(\mathcal{N}_{A \rightarrow B}^\theta(\omega_{RA}))\}_\theta) \leq \sqrt{I_F}(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) + \sqrt{I_F}(\theta; \{\mathcal{M}_{B \rightarrow C}^\theta\}_\theta), \quad (5.218)$$

which implies (5.214). ■

The following bound in (5.219) was reported recently in [32]. Here, we see how it is a consequence of the QCRB in (5.6), the meta-converse from Theorem 18, and the amortization collapse from Corollary 34. At the same time, our approach offers a technical improvement over the result of [32], in that the families of quantum channels to which the bound applies need only be differentiable rather than second-order differentiable, the latter being required by the approach of [32].

**Conclusion 36** *As a direct consequence of the QCRB in (5.6), the meta-converse from Theorem 18, and the amortization collapse from Corollary 34, we conclude the following bound on the MSE of an unbiased estimator  $\hat{\theta}$  for all differentiable quantum channel families:*

$$\text{Var}(\hat{\theta}) \geq \frac{1}{n^2 I_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta)}. \quad (5.219)$$

*This bound thus poses a “Heisenberg” limitation on sequential estimation protocols for all differentiable quantum channel families satisfying the finiteness condition in (5.72).*

## 5.6 RLD Fisher information limit on quantum channel parameter estimation

### 5.6.1 RLD Fisher information of quantum channels and its properties

We now recall and establish some properties of the RLD Fisher information of quantum channels. Following [20] and the general prescription in Definition 9, it is defined as follows:

$$\hat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) := \sup_{\rho_{RA}} \hat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA})\}_\theta), \quad (5.220)$$

but note that the optimization can be restricted to pure bipartite states, due to Remark 10. Recall that the RLD Fisher information of quantum channels has an explicit formula, as given in (5.151).

The following additivity relation was established in [20], and we review its proof in Appendix G.

**Proposition 37** *Let  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  and  $\{\mathcal{M}_{C \rightarrow D}^\theta\}_\theta$  be differentiable families of quantum channels. Then the RLD Fisher information of quantum channels is additive in the following sense:*

$$\widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta \otimes \mathcal{M}_{C \rightarrow D}^\theta\}_\theta) = \widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) + \widehat{I}_F(\theta; \{\mathcal{M}_{C \rightarrow D}^\theta\}_\theta). \quad (5.221)$$

The RLD Fisher information of quantum states and channels obeys the following chain rule:

**Proposition 38 (Chain rule)** *Let  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  be a differentiable family of quantum channels, and let  $\{\rho_{RA}^\theta\}_\theta$  be a differentiable family of quantum states on systems  $RA$ , with the system  $R$  of arbitrary size. Then the following chain rule holds*

$$\widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) \leq \widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) + \widehat{I}_F(\theta; \{\rho_{RA}^\theta\}_\theta). \quad (5.222)$$

**Proof.** If the finiteness conditions in (5.28) and (5.73) do not hold, then the inequality is trivially satisfied. So let us suppose that the finiteness conditions (5.28) and (5.73) hold. Recall the following post-selected teleportation identity from (3.7):

$$\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta) = \langle \Gamma |_{AS} \rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta} | \Gamma \rangle_{AS}. \quad (5.223)$$

Then we can write

$$\begin{aligned} & \widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) \\ &= \text{Tr}[(\partial_\theta \mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta))^2 (\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta))^{-1}] \end{aligned} \quad (5.224)$$

$$= \text{Tr}[(\partial_\theta (\langle \Gamma |_{AS} \rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta} | \Gamma \rangle_{AS}))^2 (\langle \Gamma |_{AS} \rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta} | \Gamma \rangle_{AS})^{-1}] \quad (5.225)$$

$$= \text{Tr}[(\langle \Gamma |_{AS} \partial_\theta (\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta}) | \Gamma \rangle_{AS})^2 (\langle \Gamma |_{AS} \rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta} | \Gamma \rangle_{AS})^{-1}] \quad (5.226)$$

$$\leq \text{Tr}[\langle \Gamma |_{AS} (\partial_\theta (\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta})) (\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta})^{-1} (\partial_\theta (\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta})) | \Gamma \rangle_{AS}] \quad (5.227)$$

$$= \text{Tr}_{RB}[\langle \Gamma |_{AS} (\partial_\theta (\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta})) (\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta})^{-1} (\partial_\theta (\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta})) | \Gamma \rangle_{AS}] \quad (5.228)$$

$$= \langle \Gamma |_{AS} \text{Tr}_{RB}[(\partial_\theta (\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta})) (\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta})^{-1} (\partial_\theta (\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta}))] | \Gamma \rangle_{AS}. \quad (5.229)$$

The second equality follows from applying (5.223), and the inequality is a consequence of the transformer inequality in Lemma 59, with

$$L = \langle \Gamma |_{AS} \otimes I_{RB}, \quad (5.230)$$

$$X = \partial_\theta (\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta}), \quad (5.231)$$

$$Y = \rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta}. \quad (5.232)$$

Now consider that

$$\partial_\theta(\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta}) = (\partial_\theta \rho_{RA}^\theta) \otimes \Gamma_{SB}^{\mathcal{N}^\theta} + \rho_{RA}^\theta \otimes (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta}).$$

Right multiplying this by  $(\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta})^{-1}$  gives

$$\begin{aligned} & (\partial_\theta(\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta}))(\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta})^{-1} \\ &= (\partial_\theta \rho_{RA}^\theta)(\rho_{RA}^\theta)^{-1} \otimes \Gamma_{SB}^{\mathcal{N}^\theta}(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1} + \rho_{RA}^\theta(\rho_{RA}^\theta)^{-1} \otimes (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1} \end{aligned} \quad (5.233)$$

$$= (\partial_\theta \rho_{RA}^\theta)(\rho_{RA}^\theta)^{-1} \otimes \Pi_{\Gamma^{\mathcal{N}^\theta}} + \Pi_{\rho_{RA}^\theta} \otimes (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1}. \quad (5.234)$$

Right multiplying the last line by  $(\partial_\theta(\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta}))$  gives

$$\begin{aligned} & \left[ (\partial_\theta \rho_{RA}^\theta)(\rho_{RA}^\theta)^{-1} \otimes \Pi_{\Gamma^{\mathcal{N}^\theta}} + \Pi_{\rho_{RA}^\theta} \otimes (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1} \right] (\partial_\theta(\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta})) \\ &= \left[ (\partial_\theta \rho_{RA}^\theta)(\rho_{RA}^\theta)^{-1} \otimes \Pi_{\Gamma^{\mathcal{N}^\theta}} + \Pi_{\rho_{RA}^\theta} \otimes (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1} \right] \\ & \quad \times \left[ (\partial_\theta \rho_{RA}^\theta) \otimes \Gamma_{SB}^{\mathcal{N}^\theta} + \rho_{RA}^\theta \otimes (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta}) \right] \end{aligned} \quad (5.235)$$

$$\begin{aligned} &= (\partial_\theta \rho_{RA}^\theta)(\rho_{RA}^\theta)^{-1} (\partial_\theta \rho_{RA}^\theta) \otimes \Gamma_{SB}^{\mathcal{N}^\theta} + (\partial_\theta \rho_{RA}^\theta)(\rho_{RA}^\theta)^{-1} \rho_{RA}^\theta \otimes \Pi_{\Gamma^{\mathcal{N}^\theta}} (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta}) \\ & \quad + \Pi_{\rho_{RA}^\theta} (\partial_\theta \rho_{RA}^\theta) \otimes (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1} \Gamma_{SB}^{\mathcal{N}^\theta} + \rho_{RA}^\theta \otimes (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1} (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta}) \\ &= (\partial_\theta \rho_{RA}^\theta)(\rho_{RA}^\theta)^{-1} (\partial_\theta \rho_{RA}^\theta) \otimes \Gamma_{SB}^{\mathcal{N}^\theta} + (\partial_\theta \rho_{RA}^\theta) \Pi_{\rho_{RA}^\theta} \otimes \Pi_{\Gamma^{\mathcal{N}^\theta}} (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta}) \\ & \quad + \Pi_{\rho_{RA}^\theta} (\partial_\theta \rho_{RA}^\theta) \otimes (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta}) \Pi_{\Gamma^{\mathcal{N}^\theta}} + \rho_{RA}^\theta \otimes (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1} (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta}). \end{aligned} \quad (5.236)$$

Since the finiteness conditions  $\Pi_{\rho_{RA}^\theta}^\perp (\partial_\theta \rho_{RA}^\theta) = (\partial_\theta \rho_{RA}^\theta) \Pi_{\rho_{RA}^\theta}^\perp = 0$  and  $\Pi_{\Gamma^{\mathcal{N}^\theta}}^\perp (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta}) = (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta}) \Pi_{\Gamma^{\mathcal{N}^\theta}}^\perp = 0$  hold, we can “add in” extra zero terms to the two middle terms above to conclude that

$$\begin{aligned} & (\partial_\theta(\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta}))(\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta})^{-1} (\partial_\theta(\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta})) = (\partial_\theta \rho_{RA}^\theta)(\rho_{RA}^\theta)^{-1} (\partial_\theta \rho_{RA}^\theta) \otimes \Gamma_{SB}^{\mathcal{N}^\theta} \\ & \quad + 2(\partial_\theta \rho_{RA}^\theta) \otimes (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta}) + \rho_{RA}^\theta \otimes (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1} (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta}). \end{aligned} \quad (5.237)$$

Now taking the partial trace over  $RB$ , we find the following for each term:

$$\text{Tr}_{RB}[(\partial_\theta \rho_{RA}^\theta)(\rho_{RA}^\theta)^{-1} (\partial_\theta \rho_{RA}^\theta) \otimes \Gamma_{SB}^{\mathcal{N}^\theta}] = \text{Tr}_R[(\partial_\theta \rho_{RA}^\theta)(\rho_{RA}^\theta)^{-1} (\partial_\theta \rho_{RA}^\theta)] \otimes I_S, \quad (5.238)$$

$$\text{Tr}_{RB}[2(\partial_\theta \rho_{RA}^\theta) \otimes (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})] = 2 \text{Tr}_R[(\partial_\theta \rho_{RA}^\theta)] \otimes \text{Tr}_B[(\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})] \quad (5.239)$$

$$= 2 \text{Tr}_R[(\partial_\theta \rho_{RA}^\theta)] \otimes (\partial_\theta \text{Tr}_B[\Gamma_{SB}^{\mathcal{N}^\theta}]) \quad (5.240)$$

$$= 2 \text{Tr}_R[(\partial_\theta \rho_{RA}^\theta)] \otimes (\partial_\theta(I_S)) \quad (5.241)$$

$$= 0, \quad (5.242)$$

$$\text{Tr}_{RB}[\rho_{RA}^\theta \otimes (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1} (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})] = \rho_A^\theta \otimes \text{Tr}_B[(\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1} (\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})]. \quad (5.243)$$

Now applying the sandwich  $\langle \Gamma|_{AS}(\cdot)|\Gamma\rangle_{AS}$ , the first and last terms become as follows:

$$\begin{aligned} & \langle \Gamma|_{AS} \text{Tr}_R[(\partial_\theta \rho_{RA}^\theta)(\rho_{RA}^\theta)^{-1}(\partial_\theta \rho_{RA}^\theta)] \otimes I_S |\Gamma\rangle_{AS} \\ &= \text{Tr}[(\partial_\theta \rho_{RA}^\theta)(\rho_{RA}^\theta)^{-1}(\partial_\theta \rho_{RA}^\theta)] \end{aligned} \quad (5.244)$$

$$= \text{Tr}[(\partial_\theta \rho_{RA}^\theta)^2(\rho_{RA}^\theta)^{-1}], \quad (5.245)$$

and

$$\begin{aligned} & \langle \Gamma|_{AS} \rho_A^\theta \otimes \text{Tr}_B[(\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1}(\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})] |\Gamma\rangle_{AS} \\ &= \text{Tr}[(\rho_S^\theta)^T \text{Tr}_B[(\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1}(\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})]]. \end{aligned} \quad (5.246)$$

Plugging back into (5.229), we find that

$$\begin{aligned} & \langle \Gamma|_{AS} \text{Tr}_{RB}[(\partial_\theta(\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta}))(\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta})^{-1}(\partial_\theta(\rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta}))] |\Gamma\rangle_{AS} \\ &= \text{Tr}[(\partial_\theta \rho_{RA}^\theta)^2(\rho_{RA}^\theta)^{-1}] + \text{Tr}[(\rho_S^\theta)^T \text{Tr}_B[(\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1}(\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})]] \end{aligned} \quad (5.247)$$

$$\leq \text{Tr}[(\partial_\theta \rho_{RA}^\theta)^2(\rho_{RA}^\theta)^{-1}] + \left\| \text{Tr}_B[(\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1}(\partial_\theta \Gamma_{SB}^{\mathcal{N}^\theta})] \right\|_\infty \quad (5.248)$$

$$= \widehat{I}_F(\theta; \{\rho_{RA}^\theta\}_\theta) + \widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta). \quad (5.249)$$

This concludes the proof. ■

**Corollary 39** *Let  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  be a differentiable family of quantum channels. Then amortization does not increase the RLD Fisher information of quantum channels, in the following sense:*

$$\widehat{I}_F^A(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) = \widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta). \quad (5.250)$$

**Proof.** If the finiteness condition in (5.73) does not hold, then the equality trivially holds. So let us suppose that the finiteness condition in (5.73) holds. The inequality  $\geq$  follows from Proposition 17 and the fact that the RLD Fisher information is faithful (see (5.36)). The opposite inequality  $\leq$  is a consequence of the chain rule from Proposition 38. Let  $\{\rho_{RA}^\theta\}_\theta$  be a family of quantum states on systems  $RA$ . Then it follows from Proposition 38 that

$$\widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) - \widehat{I}_F(\theta; \{\rho_{RA}^\theta\}_\theta) \leq \widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta). \quad (5.251)$$

Since the family  $\{\rho_{RA}^\theta\}_\theta$  is arbitrary, we can take a supremum over the left-hand side over all such families, and conclude that

$$\widehat{I}_F^A(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) \leq \widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta). \quad (5.252)$$

This concludes the proof. ■

**Corollary 40** *Let  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  and  $\{\mathcal{M}_{B \rightarrow C}^\theta\}_\theta$  be differentiable families of quantum channels. Then the RLD Fisher information of quantum channels is subadditive with respect to serial composition, in the following sense:*

$$\widehat{I}_F(\theta; \{\mathcal{M}_{B \rightarrow C}^\theta \circ \mathcal{N}_{A \rightarrow B}^\theta\}_\theta) \leq \widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) + \widehat{I}_F(\theta; \{\mathcal{M}_{B \rightarrow C}^\theta\}_\theta). \quad (5.253)$$



**Proof.** If the finiteness condition in (5.73) does not hold for both channels, then the inequality is trivially satisfied. So let us suppose that the finiteness condition in (5.73) holds for both channels. Pick an arbitrary input state  $\omega_{RA}$ . Now apply Proposition 38 to find that

$$\begin{aligned} & \widehat{I}_F(\theta; \{\mathcal{M}_{B \rightarrow C}^\theta(\mathcal{N}_{A \rightarrow B}^\theta(\omega_{RA}))\}_\theta) \\ & \leq \widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\omega_{RA})\}_\theta) + \widehat{I}_F(\theta; \{\mathcal{M}_{B \rightarrow C}^\theta\}_\theta) \end{aligned} \quad (5.254)$$

$$\leq \sup_{\omega_{RA}} \widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\omega_{RA})\}_\theta) + \widehat{I}_F(\theta; \{\mathcal{M}_{B \rightarrow C}^\theta\}_\theta) \quad (5.255)$$

$$= \widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) + \widehat{I}_F(\theta; \{\mathcal{M}_{B \rightarrow C}^\theta\}_\theta). \quad (5.256)$$

Since the inequality holds for all input states, we conclude that

$$\sup_{\omega_{RA}} \widehat{I}_F(\theta; \{\mathcal{M}_{B \rightarrow C}^\theta(\mathcal{N}_{A \rightarrow B}^\theta(\omega_{RA}))\}_\theta) \leq \widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) + \widehat{I}_F(\theta; \{\mathcal{M}_{B \rightarrow C}^\theta\}_\theta), \quad (5.257)$$

which implies (5.253). ■

### 5.6.2 RLD Fisher information bound for general channel parameter estimation

**Conclusion 41** *As a direct consequence of the QCRB in (5.35), the meta-converse from Theorem 18, and the amortization collapse from Corollary 39, we conclude the following bound on the MSE of an unbiased estimator  $\hat{\theta}$  for all quantum channel families  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$ :*

$$\text{Var}(\hat{\theta}) \geq \frac{1}{n \widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta)}. \quad (5.258)$$

*This bound thus poses a strong limitation on sequential estimation protocols for all differentiable quantum channel families satisfying the finiteness condition in (5.73).*

Conclusion 41 strengthens one of the results of [20]. There, it was proved that the RLD Fisher information of quantum channels is a limitation for parallel estimation protocols, but Conclusion 41 establishes it as a limitation for the more general sequential estimation protocols.

Conclusion 41 establishes (5.73) as a sufficient condition for the unattainability of Heisenberg scaling. In a recent paper [109, Theorem 1] concurrent to ours, a necessary and sufficient condition for the unattainability of Heisenberg scaling with general sequential estimation protocols has been established.

## 5.7 Example: Estimating the parameters of the generalized amplitude damping channel

We now apply the bound in (5.258) to a particular example, the generalized amplitude damping channel [110]. This channel has been studied previously in the context of quantum estimation theory [15, 16], where the SLD Fisher information of quantum channels was

studied. Our goal now is to compute the RLD Fisher information of this channel with respect to its parameters.

Recall that a generalized amplitude damping channel is defined in terms of its loss  $\gamma \in (0, 1)$  and noise  $N \in (0, 1)$  as

$$\mathcal{A}_{\gamma,N}(\rho) := K_1 \rho K_1^\dagger + K_2 \rho K_2^\dagger + K_3 \rho K_3^\dagger + K_4 \rho K_4^\dagger, \quad (5.259)$$

where

$$K_1 := \sqrt{1-N} \left( |0\rangle\langle 0| + \sqrt{1-\gamma} |1\rangle\langle 1| \right), \quad (5.260)$$

$$K_2 := \sqrt{\gamma(1-N)} |0\rangle\langle 1|, \quad (5.261)$$

$$K_3 := \sqrt{N} \left( \sqrt{1-\gamma} |0\rangle\langle 0| + |1\rangle\langle 1| \right), \quad (5.262)$$

$$K_4 := \sqrt{\gamma N} |1\rangle\langle 0|. \quad (5.263)$$

The Choi operator of the channel is then given by

$$\Gamma_{RB}^{\mathcal{A}_{\gamma,N}} := (\text{id}_R \otimes \mathcal{A}_{\gamma,N})(\Gamma_{RA}) \quad (5.264)$$

$$= (1-\gamma N) |00\rangle\langle 00| + \sqrt{1-\gamma} (|00\rangle\langle 11| + |11\rangle\langle 00|) + \gamma N |01\rangle\langle 01| \\ + \gamma(1-N) |10\rangle\langle 10| + (1-\gamma(1-N)) |11\rangle\langle 11| \quad (5.265)$$

$$= \begin{bmatrix} 1-\gamma N & 0 & 0 & \sqrt{1-\gamma} \\ 0 & \gamma N & 0 & 0 \\ 0 & 0 & \gamma(1-N) & 0 \\ \sqrt{1-\gamma} & 0 & 0 & 1-\gamma(1-N) \end{bmatrix}. \quad (5.266)$$

### 5.7.1 Estimating loss

Let us apply this approach to the generalized amplitude damping channel, and in particular, with the goal of finding limits on estimating the loss parameter  $\gamma \in (0, 1)$ . By direct evaluation, we find that

$$\partial_\gamma \Gamma_{RB}^{\mathcal{A}_{\gamma,N}} = \begin{bmatrix} -N & 0 & 0 & -\frac{1}{2\sqrt{1-\gamma}} \\ 0 & N & 0 & 0 \\ 0 & 0 & 1-N & 0 \\ -\frac{1}{2\sqrt{1-\gamma}} & 0 & 0 & -(1-N) \end{bmatrix}. \quad (5.267)$$

Then we evaluate the expression in (5.151), which for our case is as follows:

$$\hat{I}_F(\gamma; \{\mathcal{A}_{\gamma,N}\}_\gamma) = \left\| \text{Tr}_B \left[ \left( \partial_\gamma \Gamma_{RB}^{\mathcal{A}_{\gamma,N}} \right) \left( \Gamma_{RB}^{\mathcal{A}_{\gamma,N}} \right)^{-1} \left( \partial_\gamma \Gamma_{RB}^{\mathcal{A}_{\gamma,N}} \right) \right] \right\|_\infty. \quad (5.268)$$

Using the fact that

$$\left( \Gamma_{RB}^{\mathcal{A}_{\gamma,N}} \right)^{-1} = \begin{bmatrix} \frac{1-\gamma(1-N)}{(1-N)N\gamma^2} & 0 & 0 & \frac{-\sqrt{1-\gamma}}{(1-N)N\gamma^2} \\ 0 & \frac{1}{\gamma N} & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma(1-N)} & 0 \\ \frac{-\sqrt{1-\gamma}}{(1-N)N\gamma^2} & 0 & 0 & \frac{1-\gamma N}{(1-N)N\gamma^2} \end{bmatrix}, \quad (5.269)$$

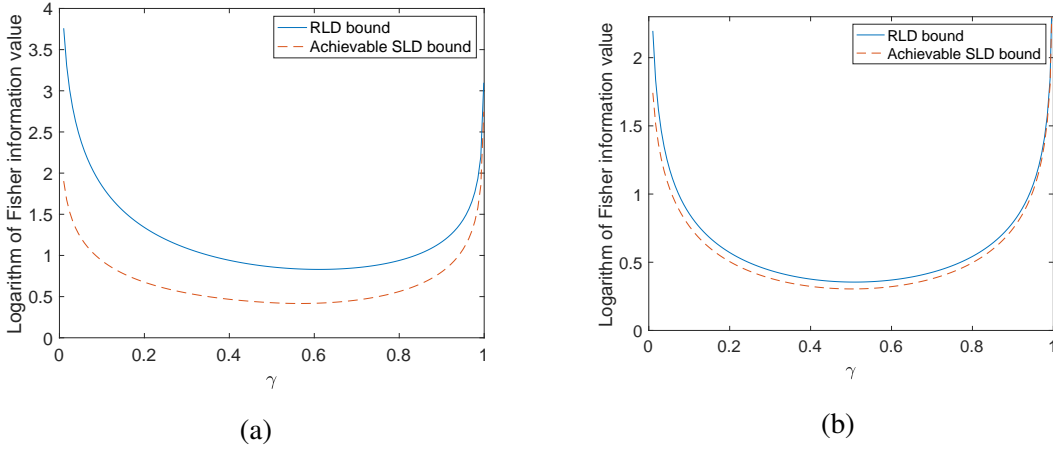


Figure 3: (a) Logarithm of RLD bound and achievable SLD bound versus loss  $\gamma$  for noise  $N = 0.2$ , when estimating the loss  $\gamma$ . (b) Logarithm of the RLD bound and achievable SLD bound versus loss  $\gamma$  for noise  $N = 0.45$ , when estimating the loss  $\gamma$ .

we find that

$$\text{Tr}_B \left[ \left( \partial_\gamma \Gamma_{RB}^{\mathcal{A}_{\gamma,N}} \right) \left( \Gamma_{RB}^{\mathcal{A}_{\gamma,N}} \right)^{-1} \left( \partial_\gamma \Gamma_{RB}^{\mathcal{A}_{\gamma,N}} \right) \right] = \begin{bmatrix} f_1(\gamma, N) & 0 \\ 0 & f_2(\gamma, N) \end{bmatrix}, \quad (5.270)$$

where

$$f_1(\gamma, N) := \frac{\frac{1}{N-\gamma N} + \frac{1}{1-N} - 4}{4\gamma^2}, \quad (5.271)$$

$$f_2(\gamma, N) := \frac{\frac{1}{(1-\gamma)(1-N)} + \frac{1}{N} - 4}{4\gamma^2}. \quad (5.272)$$

Note that if  $N \leq 1/2$ , then  $f_1(\gamma, N) \geq f_2(\gamma, N)$ , while if  $N > 1/2$ , then  $f_1(\gamma, N) < f_2(\gamma, N)$ . It then follows that

$$\hat{I}_F(\gamma; \{\mathcal{A}_{\gamma,N}\}_\gamma) = \begin{cases} f_1(\gamma, N) & N \leq 1/2 \\ f_2(\gamma, N) & N > 1/2 \end{cases}. \quad (5.273)$$

Thus, it follows from (5.258) that the formula in (5.273) provides a fundamental limitation on any protocol that attempts to estimate the loss parameter  $\gamma$ . For the noise parameter  $N$  equal to 0.2 and 0.45, Figure 3 depicts the logarithm of this bound, as well as the logarithm of the achievable bound from the SLD Fisher information of channels, corresponding to a parallel strategy that estimates  $\gamma$ . The RLD bound becomes better as  $N$  approaches  $1/2$ , and we find numerically that the RLD and SLD bounds coincide at  $N = 1/2$ .

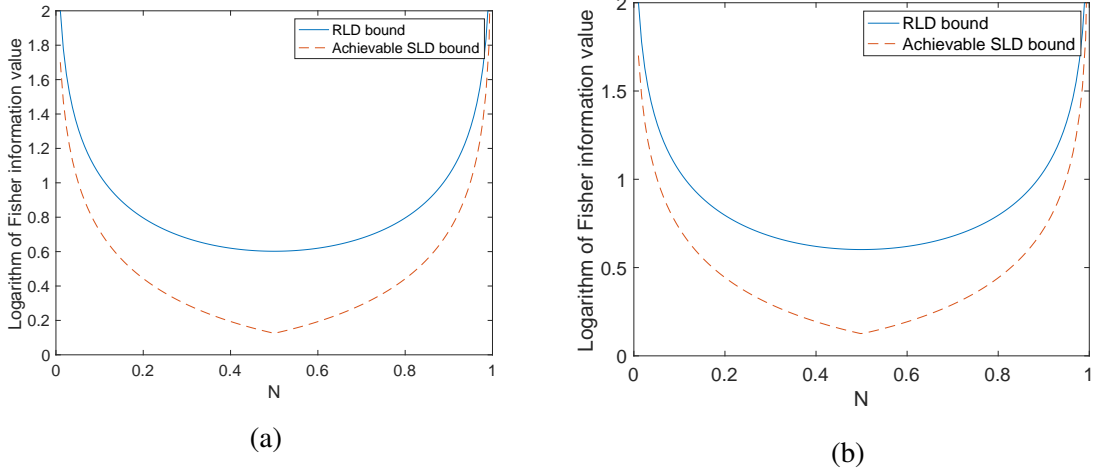


Figure 4: (a) Logarithm of RLD bound and achievable SLD bound versus noise  $N$  for loss  $\gamma = 0.5$ , when estimating the noise  $N$ . (b) Logarithm of the RLD bound and achievable SLD bound versus noise  $N$  for loss  $\gamma = 0.8$ , when estimating the noise  $N$ .

### 5.7.2 Estimating noise

Now suppose that we are interested in estimating the noise parameter  $N$  of a generalized amplitude damping channel. We find that

$$\partial_N \Gamma_{RB}^{\mathcal{A}_{\gamma,N}} = -\gamma (I_2 \otimes \sigma_Z). \quad (5.274)$$

Then by exploiting (5.269), we find that

$$\text{Tr}_B \left[ \left( \partial_N \Gamma_{RB}^{\mathcal{A}_{\gamma,N}} \right) \left( \Gamma_{RB}^{\mathcal{A}_{\gamma,N}} \right)^{-1} \left( \partial_N \Gamma_{RB}^{\mathcal{A}_{\gamma,N}} \right) \right] = \begin{bmatrix} \frac{1}{N(1-N)} & 0 \\ 0 & \frac{1}{N(1-N)} \end{bmatrix}. \quad (5.275)$$

Thus we have

$$\hat{I}_F(N; \{\mathcal{A}_{\gamma,N}\}_N) = \frac{1}{N(1-N)}. \quad (5.276)$$

For the loss parameter  $\gamma$  equal to 0.5 and 0.8, Figure 4 depicts the logarithm of the RLD bound, as well as the logarithm of the achievable bound from the SLD Fisher information of channels, corresponding to a parallel strategy that estimates  $N$ . The RLD bound becomes better as  $\gamma$  approaches 1.

### 5.7.3 Estimating a phase in loss and noise

Now let us suppose that we have a combination of a coherent process and the generalized amplitude damping channel. In particular, let us suppose that a phase  $\phi$  is encoded in a unitary  $e^{-i\phi\sigma_Z}$ , and this is followed by the generalized amplitude damping channel. Then this process is

$$\mathcal{A}_{\phi,\gamma,N}(\rho) := \mathcal{A}_{\gamma,N}(e^{-i\phi\sigma_Z} \rho e^{i\phi\sigma_Z}). \quad (5.277)$$

The goal is to estimate the phase  $\phi$ .

The Choi operator is given by

$$\Gamma_{RB}^{\mathcal{A}_{\phi,\gamma,N}} := \begin{bmatrix} 1 - \gamma N & 0 & 0 & e^{-i2\phi} \sqrt{1 - \gamma} \\ 0 & \gamma N & 0 & 0 \\ 0 & 0 & \gamma(1 - N) & 0 \\ e^{i2\phi} \sqrt{1 - \gamma} & 0 & 0 & 1 - \gamma(1 - N) \end{bmatrix}, \quad (5.278)$$

and we find that

$$\partial_{\phi} \Gamma_{RB}^{\mathcal{A}_{\phi,\gamma,N}} = \begin{bmatrix} 0 & 0 & 0 & -2ie^{-i2\phi} \sqrt{1 - \gamma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2ie^{i2\phi} \sqrt{1 - \gamma} & 0 & 0 & 0 \end{bmatrix} \quad (5.279)$$

Using the fact that

$$\left(\Gamma_{RB}^{\mathcal{A}_{\phi}}\right)^{-1} = \begin{bmatrix} \frac{1 - \gamma(1 - N)}{(1 - N)N\gamma^2} & 0 & 0 & \frac{-e^{-2i\phi} \sqrt{1 - \gamma}}{(1 - N)N\gamma^2} \\ 0 & \frac{1}{\gamma N} & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma(1 - N)} & 0 \\ \frac{-e^{2i\phi} \sqrt{1 - \gamma}}{(1 - N)N\gamma^2} & 0 & 0 & \frac{1 - \gamma N}{(1 - N)N\gamma^2} \end{bmatrix}, \quad (5.280)$$

we find that

$$\text{Tr}_B \left[ \left( \partial_{\phi} \Gamma_{RB}^{\mathcal{A}_{\phi,\gamma,N}} \right) \left( \Gamma_{RB}^{\mathcal{A}_{\phi,\gamma,N}} \right)^{-1} \left( \partial_{\phi} \Gamma_{RB}^{\mathcal{A}_{\phi,\gamma,N}} \right) \right] = \begin{bmatrix} \frac{4(1 - \gamma)(1 - \gamma N)}{(1 - N)N\gamma^2} & 0 \\ 0 & \frac{4(1 - \gamma)(1 - \gamma(1 - N))}{(1 - N)N\gamma^2} \end{bmatrix}. \quad (5.281)$$

Then if  $N > 1/2$ , we have that

$$\left\| \text{Tr}_B \left[ \left( \partial_{\phi} \Gamma_{RB}^{\mathcal{A}_{\phi,\gamma,N}} \right) \left( \Gamma_{RB}^{\mathcal{A}_{\phi,\gamma,N}} \right)^{-1} \left( \partial_{\phi} \Gamma_{RB}^{\mathcal{A}_{\phi,\gamma,N}} \right) \right] \right\|_{\infty} = \frac{4(1 - \gamma)(1 - \gamma(1 - N))}{(1 - N)N\gamma^2}, \quad (5.282)$$

while if  $N \leq 1/2$ , then

$$\left\| \text{Tr}_B \left[ \left( \partial_{\phi} \Gamma_{RB}^{\mathcal{A}_{\phi,\gamma,N}} \right) \left( \Gamma_{RB}^{\mathcal{A}_{\phi,\gamma,N}} \right)^{-1} \left( \partial_{\phi} \Gamma_{RB}^{\mathcal{A}_{\phi,\gamma,N}} \right) \right] \right\|_{\infty} = \frac{4(1 - \gamma)(1 - \gamma N)}{(1 - N)N\gamma^2}. \quad (5.283)$$

So we conclude that

$$\hat{I}_F(\phi; \{\mathcal{A}_{\phi,\gamma,N}\}_{\phi}) = \frac{4(1 - \gamma)(1 - \gamma(N + (1 - 2N)u(2N - 1)))}{(1 - N)N\gamma^2}, \quad (5.284)$$

where

$$u(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}. \quad (5.285)$$

For the noise parameter  $N$  equal to 0.2 and 0.45, Figure 5 depicts the logarithm of the RLD bound, as well as the logarithm of the achievable bound from the SLD Fisher information of channels, corresponding to a parallel strategy that estimates the phase  $\phi$  at  $\phi = 0.1$ . The RLD bound becomes better as  $\gamma$  approaches 1.

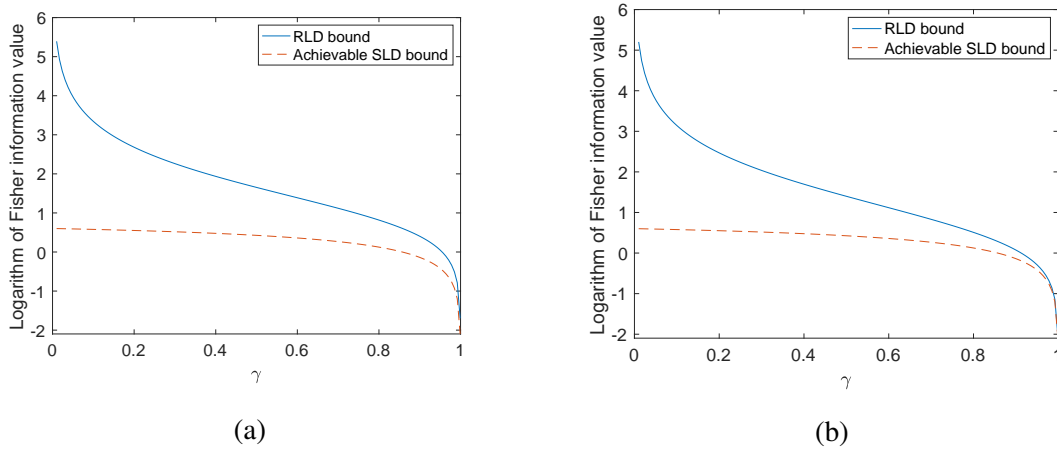


Figure 5: (a) Logarithm of RLD bound and achievable SLD bound versus loss  $\gamma$  for noise  $N = 0.2$ , when estimating the phase  $\phi = 0.1$ . (b) Logarithm of the RLD bound and achievable SLD bound versus loss  $\gamma$  for noise  $N = 0.45$ , when estimating the phase  $\phi = 0.1$ .

## 6 Limits on quantum channel discrimination

In this section, we shift to quantum channel discrimination, which has some close ties to the theory of quantum channel estimation, as discussed in Section 4. The main tool that we use for the analysis here is the geometric Rényi relative entropy, which we review in what follows and in more detail in Appendix H.

### 6.1 Geometric Rényi relative entropy

The geometric Rényi relative entropy is a key distinguishability measure that we employ in the context of quantum channel discrimination, and it is even connected to the RLD Fisher information, as we discuss in the forthcoming Section 7. The geometric Rényi relative entropy has its roots in the early work [34], and the specific form given below was introduced by [35, 36]. It has been reviewed briefly in [37] and in more detail in [38] (in particular, see [38, Example 4.5]). See also [111] for a more recent review. It has been used effectively in recent work to obtain upper bounds on quantum channel capacities [39] and rates of channel discrimination in the asymmetric setting [39, Appendix D]. This latter paper has thus established the geometric Rényi relative entropy as a useful tool in bounding rates of operational tasks.

We define the geometric Rényi relative entropy as follows:

**Definition 42 (Geometric Rényi relative entropy)** *Let  $\rho$  be a state,  $\sigma$  a positive semi-definite operator, and  $\alpha \in (0, 1) \cup (1, \infty)$ . The geometric Rényi relative quasi-entropy*

is defined as

$$\widehat{Q}_\alpha(\rho\|\sigma) := \lim_{\varepsilon \rightarrow 0^+} \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right], \quad (6.1)$$

where  $\sigma_\varepsilon := \sigma + \varepsilon I$ . The geometric Rényi relative entropy is then defined as

$$\widehat{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \ln \widehat{Q}_\alpha(\rho\|\sigma). \quad (6.2)$$

It is called the geometric Rényi relative entropy [39] because it can be written in terms of the weighted operator geometric mean as

$$\widehat{Q}_\alpha(\rho\|\sigma) = \text{Tr}[G_\alpha(\sigma, \rho)], \quad (6.3)$$

where the weighted operator geometric mean is defined as

$$G_\alpha(\sigma, \rho) := \lim_{\varepsilon \rightarrow 0^+} \sigma_\varepsilon^{\frac{1}{2}} \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \sigma_\varepsilon^{\frac{1}{2}}. \quad (6.4)$$

See, e.g., [112] for a review of operator geometric means.

When the condition  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$  holds, the geometric Rényi relative entropy can be written for  $\alpha \in (0, 1) \cup (1, \infty)$  as

$$\widehat{Q}_\alpha(\rho\|\sigma) := \text{Tr} \left[ \sigma \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right)^\alpha \right]. \quad (6.5)$$

For  $\alpha \in (0, 1)$ , if the condition  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$  does not hold, then the explicit formula for it is more complicated, given by [113, 114]

$$\widehat{Q}_\alpha(\rho\|\sigma) := \text{Tr} \left[ \sigma \left( \sigma^{-\frac{1}{2}} \tilde{\rho} \sigma^{-\frac{1}{2}} \right)^\alpha \right], \quad (6.6)$$

where

$$\tilde{\rho} := \rho_{0,0} - \rho_{0,1} \rho_{1,1}^{-1} \rho_{0,1}^\dagger, \quad (6.7)$$

$$\rho_{0,0} := \Pi_\sigma \rho \Pi_\sigma, \quad \rho_{0,1} := \Pi_\sigma \rho \Pi_\sigma^\perp, \quad \rho_{1,1} := \Pi_\sigma^\perp \rho \Pi_\sigma^\perp, \quad (6.8)$$

$\Pi_\sigma$  is the projection onto the support of  $\sigma$ ,  $\Pi_\sigma^\perp$  the projection onto the kernel of  $\sigma$ , and all inverses are evaluated on the supports of the operators. We detail how this explicit formula follows from (6.1) in Appendix H. For  $\alpha \in (1, \infty)$ , if the condition  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$  does not hold, then it is equal to  $+\infty$ .

A special case of the geometric Rényi relative entropy of interest to us here, for  $\alpha = 1/2$ , involves the geometric fidelity [113, 115]:

$$\widehat{D}_{1/2}(\rho\|\sigma) := -2 \ln \widehat{Q}_{1/2}(\rho\|\sigma) = -\ln \widehat{F}(\rho, \sigma), \quad (6.9)$$

where the geometric fidelity of  $\rho$  and  $\sigma$  is defined as

$$\widehat{F}(\rho, \sigma) := \left( \lim_{\varepsilon \rightarrow 0^+} \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} \right)^{\frac{1}{2}} \right] \right)^2. \quad (6.10)$$

A recent paper has explored the geometric fidelity (therein called Matsumoto fidelity) and its relation to semi-definite programming [116].

The geometric Rényi relative entropy has a number of fundamental properties that make it a worthwhile quantity to study. Although it is not known to have an information-theoretic interpretation on its own, it is an upper bound on other information quantities that are connected to operational tasks. The important properties of geometric Rényi relative entropy are as follows:

- Convergence to the Belavkin–Staszewski relative entropy [117] in the limit  $\alpha \rightarrow 1$ :

$$\lim_{\alpha \rightarrow 1} \hat{D}_\alpha(\rho\|\sigma) = \hat{D}(\rho\|\sigma), \quad (6.11)$$

where the Belavkin–Staszewski relative entropy  $\hat{D}(\rho\|\sigma)$  is defined as

$$\hat{D}(\rho\|\sigma) := \begin{cases} \text{Tr}[\rho \ln(\rho^{\frac{1}{2}} \sigma^{-1} \rho^{\frac{1}{2}})] & \text{if } \text{supp}(\rho) \subseteq \text{supp}(\sigma) \\ +\infty & \text{otherwise} \end{cases}. \quad (6.12)$$

- Convergence to the max-relative entropy [118] in the limit  $\alpha \rightarrow \infty$ :

$$\lim_{\alpha \rightarrow \infty} \hat{D}_\alpha(\rho\|\sigma) = D_{\max}(\rho\|\sigma), \quad (6.13)$$

as proven in Appendix H, where

$$D_{\max}(\rho\|\sigma) := \begin{cases} \ln \inf\{\lambda \geq 0 : \rho \leq \lambda \sigma\} & \text{if } \text{supp}(\rho) \subseteq \text{supp}(\sigma) \\ +\infty & \text{otherwise} \end{cases}. \quad (6.14)$$

- For all  $\alpha \in (0, 1) \cup (1, 2]$ , data-processing inequality [35, 36]:

$$\hat{D}_\alpha(\rho\|\sigma) \geq \hat{D}_\alpha(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)), \quad (6.15)$$

where  $\rho$  is quantum state,  $\sigma$  is a positive semi-definite operator, and  $\mathcal{N}$  is a quantum channel.

- Monotonicity in  $\alpha$  for  $\alpha \in (0, 1) \cup (1, \infty)$ . That is,

$$\hat{D}_\alpha(\rho\|\sigma) \leq \hat{D}_\beta(\rho\|\sigma), \quad (6.16)$$

for  $0 < \alpha \leq \beta$ , as proven in Appendix H.

- Not smaller than the sandwiched Rényi relative entropy for all  $\alpha \in (0, 1) \cup (1, \infty)$  [37, 89]:

$$\tilde{D}_\alpha(\rho\|\sigma) \leq \hat{D}_\alpha(\rho\|\sigma), \quad (6.17)$$

where the sandwiched Rényi relative entropy is defined as [81, 119]

$$\tilde{D}_\alpha(\rho\|\sigma) := \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\alpha - 1} \ln \text{Tr}[(\sigma_\varepsilon^{\frac{1-\alpha}{2\alpha}} \rho \sigma_\varepsilon^{\frac{1-\alpha}{2\alpha}})^\alpha]. \quad (6.18)$$

Special case for  $\alpha = 1/2$ : geometric fidelity is not larger than the fidelity [120]:

$$\hat{F}(\rho, \sigma) \leq F(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1^2. \quad (6.19)$$



- The geometric Rényi relative entropy of two quantum states  $\rho$  and  $\sigma$  is computable via a semi-definite program [39].

As indicated above, we provide a detailed review of the geometric Rényi relative entropy and its properties in Appendix H.

## 6.2 Properties of geometric Rényi relative entropy of quantum channels

In this section, we discuss some properties of the geometric Rényi relative entropy of quantum channels. These properties were established in [39] for the interval  $\alpha \in (1, 2]$  and implicitly under suitable support conditions on the Choi operators of the channels, but the interval  $\alpha \in (0, 1)$  was not discussed in [39], nor the case when the support conditions do not hold. Our main observation here is that the same properties hold for the full interval  $\alpha \in (0, 1) \cup (1, 2]$  and without support conditions, by following essentially the same proofs from [39]. For completeness, we provide proofs in Appendix I.

As observed in [56, 91], any state distinguishability measure can be generalized to quantum channels by optimizing over all input states to the channel. Thus, the geometric Rényi relative entropy of quantum channels is defined as follows:

**Definition 43** *For a quantum channel  $\mathcal{N}_{A \rightarrow B}$  and a completely positive map  $\mathcal{M}_{A \rightarrow B}$ , their geometric Rényi relative entropy is defined for  $\alpha \in (0, 1) \cup (1, \infty)$  as*

$$\widehat{D}_\alpha(\mathcal{N} \parallel \mathcal{M}) := \sup_{\rho_{RA}} \widehat{D}_\alpha(\mathcal{N}_{A \rightarrow B}(\rho_{RA}) \parallel \mathcal{M}_{A \rightarrow B}(\rho_{RA})). \quad (6.20)$$

By applying Remark 10, the formula simplifies as follows for the data-processing interval  $\alpha \in (0, 1) \cup (1, 2]$ :

$$\widehat{D}_\alpha(\mathcal{N} \parallel \mathcal{M}) = \sup_{\psi_{RA}} \widehat{D}_\alpha(\mathcal{N}_{A \rightarrow B}(\psi_{RA}) \parallel \mathcal{M}_{A \rightarrow B}(\psi_{RA})), \quad (6.21)$$

where the supremum is with respect to all pure bipartite states  $\psi_{RA}$  with system  $R$  isomorphic to system  $A$ .

In fact, the formula simplifies further:

**Proposition 44** *Let  $\mathcal{N}_{A \rightarrow B}$  and  $\mathcal{M}_{A \rightarrow B}$  be quantum channels, and let  $\Gamma_{RB}^{\mathcal{N}}$  and  $\Gamma_{RB}^{\mathcal{M}}$  be their respective Choi operators. For  $\alpha \in (0, 1) \cup (1, 2]$ , the geometric Rényi relative entropy of quantum channels  $\mathcal{N}_{A \rightarrow B}$  and  $\mathcal{M}_{A \rightarrow B}$  has the following explicit form:*

$$\widehat{D}_\alpha(\mathcal{N} \parallel \mathcal{M}) = \frac{1}{\alpha - 1} \ln \widehat{Q}_\alpha(\mathcal{N} \parallel \mathcal{M}), \quad (6.22)$$

where

$$\widehat{Q}_\alpha(\mathcal{N}||\mathcal{M}) := \begin{cases} \lambda_{\min}(\text{Tr}_B[G_\alpha(\Gamma_{RB}^{\mathcal{M}}, \Gamma_{RB}^{\mathcal{N}})]) & \text{if } \alpha \in (0, 1) \\ & \text{and } \text{supp}(\Gamma_{RB}^{\mathcal{N}}) \subseteq \text{supp}(\Gamma_{RB}^{\mathcal{M}}) \\ \|\text{Tr}_B[G_\alpha(\Gamma_{RB}^{\mathcal{M}}, \Gamma_{RB}^{\mathcal{N}})]\|_\infty & \text{if } \alpha \in (1, 2] \\ & \text{and } \text{supp}(\Gamma_{RB}^{\mathcal{N}}) \subseteq \text{supp}(\Gamma_{RB}^{\mathcal{M}}) \\ \lambda_{\min}(\text{Tr}_B[G_\alpha(\Gamma_{RB}^{\mathcal{M}}, \widetilde{\Gamma_{RB}^{\mathcal{N}}})]) & \text{if } \alpha \in (0, 1) \\ & \text{and } \text{supp}(\Gamma_{RB}^{\mathcal{N}}) \not\subseteq \text{supp}(\Gamma_{RB}^{\mathcal{M}}) \\ +\infty & \text{if } \alpha \in (1, 2] \\ & \text{and } \text{supp}(\Gamma_{RB}^{\mathcal{N}}) \not\subseteq \text{supp}(\Gamma_{RB}^{\mathcal{M}}) \end{cases}, \quad (6.23)$$

$\lambda_{\min}$  denotes the minimum eigenvalue of its argument,

$$G_\alpha(X, Y) := X^{1/2}(X^{-1/2}YX^{-1/2})^\alpha X^{1/2}, \quad (6.24)$$

$$\widetilde{\Gamma_{RB}^{\mathcal{N}}} := (\Gamma_{RB}^{\mathcal{N}})_{0,0} - (\Gamma_{RB}^{\mathcal{N}})_{0,1}(\Gamma_{RB}^{\mathcal{N}})_{1,1}^{-1}[(\Gamma_{RB}^{\mathcal{N}})_{0,1}]^\dagger, \quad (6.25)$$

$$(\Gamma_{RB}^{\mathcal{N}})_{0,0} := \Pi_{\Gamma^{\mathcal{M}}} \Gamma_{RB}^{\mathcal{N}} \Pi_{\Gamma^{\mathcal{M}}}, \quad (6.26)$$

$$(\Gamma_{RB}^{\mathcal{N}})_{0,1} := \Pi_{\Gamma^{\mathcal{M}}} \Gamma_{RB}^{\mathcal{N}} \Pi_{\Gamma^{\mathcal{M}}}^\perp, \quad (6.27)$$

$$(\Gamma_{RB}^{\mathcal{N}})_{1,1} := \Pi_{\Gamma^{\mathcal{M}}}^\perp \Gamma_{RB}^{\mathcal{N}} \Pi_{\Gamma^{\mathcal{M}}}^\perp, \quad (6.28)$$

$\Pi_{\Gamma^{\mathcal{M}}}$  is the projection onto the support of  $\Gamma_{RB}^{\mathcal{M}}$ ,  $\Pi_{\Gamma^{\mathcal{M}}}^\perp$  is the projection onto its kernel, and all inverses are taken on the support. For  $\alpha \in (0, 1)$ , we have the following alternative form:

$$\widehat{Q}_\alpha(\mathcal{N}||\mathcal{M}) = \lim_{\varepsilon \rightarrow 0^+} \lambda_{\min}(\text{Tr}_B[G_\alpha(\Gamma_{RB}^{\mathcal{M}_\varepsilon}, \Gamma_{RB}^{\mathcal{N}})]), \quad (6.29)$$

where  $\Gamma_{RB}^{\mathcal{M}_\varepsilon} := \Gamma_{RB}^{\mathcal{M}} + \varepsilon I_{RB}$ .

**Proof.** See Appendix I. ■

It is known from [39] that the geometric Rényi relative entropy of quantum channels  $\mathcal{N}_{A \rightarrow B}$  and  $\mathcal{M}_{A \rightarrow B}$  converges to the Belavkin–Staszewski relative entropy of channels in the limit as  $\alpha \rightarrow 1$ :

$$\lim_{\alpha \rightarrow 1} \widehat{D}_\alpha(\mathcal{N}_{A \rightarrow B}||\mathcal{M}_{A \rightarrow B}) = \widehat{D}(\mathcal{N}_{A \rightarrow B}||\mathcal{M}_{A \rightarrow B}), \quad (6.30)$$

and the Belavkin–Staszewski relative entropy of channels has the following explicit expression:

$$\widehat{D}(\mathcal{N}_{A \rightarrow B}||\mathcal{M}_{A \rightarrow B}) := \|\text{Tr}_B[(\Gamma_{RB}^{\mathcal{N}})^{1/2} \log_2((\Gamma_{RB}^{\mathcal{N}})^{1/2}(\Gamma_{RB}^{\mathcal{M}})^{-1}(\Gamma_{RB}^{\mathcal{N}})^{1/2})(\Gamma_{RB}^{\mathcal{N}})^{1/2}]\|_\infty \quad (6.31)$$

if  $\text{supp}(\Gamma_{RB}^{\mathcal{N}}) \subseteq \text{supp}(\Gamma_{RB}^{\mathcal{M}})$  and  $\widehat{D}(\mathcal{N}_{A \rightarrow B}||\mathcal{M}_{A \rightarrow B}) := +\infty$  otherwise.

**Proposition 45 (Chain rule)** For  $\rho_{RA}$  a quantum state,  $\sigma_{RA}$  a positive semi-definite operator,  $\mathcal{N}_{A \rightarrow B}$  a quantum channel, and  $\mathcal{M}_{A \rightarrow B}$  a completely positive map, the following chain rule holds for  $\alpha \in (0, 1) \cup (1, 2]$ :

$$\widehat{D}_\alpha(\mathcal{N}_{A \rightarrow B}(\rho_{RA}) \| \mathcal{M}_{A \rightarrow B}(\sigma_{RA})) \leq \widehat{D}_\alpha(\mathcal{N} \| \mathcal{M}) + \widehat{D}_\alpha(\rho_{RA} \| \sigma_{RA}). \quad (6.32)$$

**Proof.** See Appendix I. ■

**Corollary 46** The geometric Rényi relative entropy does not increase under amortization for all  $\alpha \in (0, 1) \cup (1, 2]$ :

$$\widehat{D}_\alpha(\mathcal{N} \| \mathcal{M}) = \widehat{D}_\alpha^A(\mathcal{N} \| \mathcal{M}), \quad (6.33)$$

where the amortized geometric Rényi relative entropy is defined from the general approach given in [33]:

$$\widehat{D}_\alpha^A(\mathcal{N} \| \mathcal{M}) := \sup_{\rho_{RA}, \sigma_{RA}} \left[ \widehat{D}_\alpha(\mathcal{N}_{A \rightarrow B}(\rho_{RA}) \| \mathcal{M}_{A \rightarrow B}(\sigma_{RA})) - \widehat{D}_\alpha(\rho_{RA} \| \sigma_{RA}) \right]. \quad (6.34)$$

**Proof.** The proof is the same as that given for Corollaries 34 and 39. ■

**Proposition 47** The geometric Rényi relative entropy is subadditive under serial concatenation of quantum channels for  $\alpha \in (0, 1) \cup (1, 2]$ , in the following sense:

$$\widehat{D}_\alpha(\mathcal{N}_2 \circ \mathcal{N}_1 \| \mathcal{M}_2 \circ \mathcal{M}_1) \leq \widehat{D}_\alpha(\mathcal{N}_2 \| \mathcal{M}_2) + \widehat{D}_\alpha(\mathcal{N}_1 \| \mathcal{M}_1), \quad (6.35)$$

where  $\mathcal{N}_1$  and  $\mathcal{N}_2$  are quantum channels and  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are completely positive maps.

**Proof.** The proof is the same as that given for Corollaries 35 and 40. ■

Just as the geometric fidelity of quantum states is a special case of geometric Rényi relative entropy, so is the geometric fidelity of quantum channels:

$$\widehat{F}(\mathcal{N}, \mathcal{M}) := \inf_{\psi_{RA}} \widehat{F}(\mathcal{N}_{A \rightarrow B}(\psi_{RA}), \mathcal{M}_{A \rightarrow B}(\psi_{RA})), \quad (6.36)$$

where  $\mathcal{N}_{A \rightarrow B}$  is a quantum channel and  $\mathcal{M}_{A \rightarrow B}$  is a completely positive map. By employing Proposition 44, we find the following formula for the geometric fidelity of channels:

$$\widehat{F}(\mathcal{N}, \mathcal{M}) = \left[ \lim_{\varepsilon \rightarrow 0^+} \lambda_{\min} \left( \text{Tr}_B [(\Gamma_{RB}^{\mathcal{M}_\varepsilon})^{1/2} ((\Gamma_{RB}^{\mathcal{M}_\varepsilon})^{-1/2} \Gamma_{RB}^{\mathcal{N}} (\Gamma_{RB}^{\mathcal{M}_\varepsilon})^{-1/2})^{1/2} (\Gamma_{RB}^{\mathcal{M}_\varepsilon})^{1/2}] \right) \right]^2. \quad (6.37)$$

By exploiting this formula, we arrive at the following semi-definite program for the geometric fidelity of quantum channels:

**Proposition 48** *The geometric channel fidelity of a quantum channel  $\mathcal{N}$  and a full-rank completely positive map  $\mathcal{M}$  can be calculated by means of the following semi-definite program:*

$$\sqrt{\widehat{F}}(\mathcal{N}, \mathcal{M}) = \sup_{\mu \geq 0, X_{RB} \geq 0} \mu, \quad (6.38)$$

subject to

$$\begin{bmatrix} \Gamma_{RB}^{\mathcal{N}} & X_{RB} \\ X_{RB} & \Gamma_{RB}^{\mathcal{M}} \end{bmatrix} \geq 0, \quad \mu I_R \leq \text{Tr}_B[X_{RB}]. \quad (6.39)$$

The dual program is given by

$$\inf_{\rho_R \geq 0, Y_{RB}, W_{RB}, Z_{RB}} \text{Tr}[\Gamma_{RB}^{\mathcal{N}} Y_{RB}] + \text{Tr}[\Gamma_{RB}^{\mathcal{M}} Z_{RB}] \quad (6.40)$$

subject to

$$\begin{bmatrix} Y_{RB} & W_{RB}^\dagger \\ W_{RB} & Z_{RB} \end{bmatrix} \geq 0, \quad W_{RB} + W_{RB}^\dagger \geq \rho_R \otimes I_B, \quad \text{Tr}[\rho_R] = 1. \quad (6.41)$$

**Proof.** As argued above, the geometric fidelity of quantum channels is given by the expression in (6.37), which involves the standard operator geometric mean of  $\Gamma_{RB}^{\mathcal{M}}$  and  $\Gamma_{RB}^{\mathcal{N}}$ :

$$G_{\frac{1}{2}}(\Gamma_{RB}^{\mathcal{M}}, \Gamma_{RB}^{\mathcal{N}}) := (\Gamma_{RB}^{\mathcal{M}})^{1/2} ((\Gamma_{RB}^{\mathcal{M}})^{-1/2} \Gamma_{RB}^{\mathcal{N}} (\Gamma_{RB}^{\mathcal{M}})^{-1/2})^{1/2} (\Gamma_{RB}^{\mathcal{M}})^{1/2} \quad (6.42)$$

and the minimum eigenvalue of its partial trace over system  $B$ . The following characterization of  $G_{\frac{1}{2}}(\Gamma_{RB}^{\mathcal{M}}, \Gamma_{RB}^{\mathcal{N}})$  is well known [107]

$$G_{\frac{1}{2}}(\Gamma_{RB}^{\mathcal{M}}, \Gamma_{RB}^{\mathcal{N}}) = \sup \left\{ X_{RB} \geq 0 : \begin{bmatrix} \Gamma_{RB}^{\mathcal{N}} & X_{RB} \\ X_{RB} & \Gamma_{RB}^{\mathcal{M}} \end{bmatrix} \geq 0 \right\}, \quad (6.43)$$

where the ordering is with respect to the operator order (Löwner order). Additionally, the minimum eigenvalue of a positive semi-definite operator  $L$  is given by

$$\lambda_{\min}(L) = \sup \{ \mu \geq 0 : \mu I \leq L \}. \quad (6.44)$$

Putting together (6.43) and (6.44), we conclude (6.38)–(6.39).

To find the dual program, consider that the dual characterization of the minimum eigenvalue  $\lambda_{\min}(L)$  of an operator  $L$  is as follows:

$$\lambda_{\min}(L) = \inf_{\rho \geq 0, \text{Tr}[\rho]=1} \text{Tr}[L\rho], \quad (6.45)$$

so that

$$\sqrt{\widehat{F}}(\mathcal{N}, \mathcal{M}) = \inf_{\rho_R \geq 0, \text{Tr}[\rho_R]=1} \sup_{X_{RB} \geq 0} \text{Tr}[\rho_R \text{Tr}_B[X_{RB}]] \quad (6.46)$$

subject to

$$\begin{bmatrix} \Gamma_{RB}^{\mathcal{N}} & X_{RB} \\ X_{RB} & \Gamma_{RB}^{\mathcal{M}} \end{bmatrix} \geq 0. \quad (6.47)$$

For fixed  $\rho_R$ , we can then consider finding the dual of the following program:

$$\sup_{X_{RB} \geq 0} \text{Tr}[\rho_R \text{Tr}_B[X_{RB}]] \quad (6.48)$$

subject to

$$\begin{bmatrix} \Gamma_{RB}^{\mathcal{N}} & X_{RB} \\ X_{RB} & \Gamma_{RB}^{\mathcal{M}} \end{bmatrix} \geq 0. \quad (6.49)$$

Considering that

$$\begin{bmatrix} \Gamma_{RB}^{\mathcal{N}} & X_{RB} \\ X_{RB} & \Gamma_{RB}^{\mathcal{M}} \end{bmatrix} \geq 0 \quad \Longleftrightarrow \quad \begin{bmatrix} \Gamma_{RB}^{\mathcal{N}} & -X_{RB} \\ -X_{RB} & \Gamma_{RB}^{\mathcal{M}} \end{bmatrix} \geq 0 \quad (6.50)$$

$$\Longleftrightarrow \quad \begin{bmatrix} \Gamma_{RB}^{\mathcal{N}} & 0 \\ 0 & \Gamma_{RB}^{\mathcal{M}} \end{bmatrix} \geq \begin{bmatrix} 0 & X_{RB} \\ X_{RB} & 0 \end{bmatrix}, \quad (6.51)$$

the standard form of the SDP is

$$\sup_{X \geq 0} \{ \text{Tr}[AX] : \Phi(X) \leq B \}, \quad (6.52)$$

with

$$A = \rho_R \otimes I_B, \quad \Phi(X_{RB}) = \begin{bmatrix} 0 & X_{RB} \\ X_{RB} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \Gamma_{RB}^{\mathcal{N}} & 0 \\ 0 & \Gamma_{RB}^{\mathcal{M}} \end{bmatrix}. \quad (6.53)$$

Then the dual map  $\Phi^\dagger$  is given by

$$\text{Tr}[Y\Phi(X)] = \text{Tr}[\Phi^\dagger(Y)X], \quad (6.54)$$

so that

$$\text{Tr} \left[ \begin{bmatrix} Y_{RB} & W_{RB}^\dagger \\ W_{RB} & Z_{RB} \end{bmatrix} \Phi(X_{RB}) \right] = \text{Tr} \left[ \begin{bmatrix} Y_{RB} & W_{RB}^\dagger \\ W_{RB} & Z_{RB} \end{bmatrix} \begin{bmatrix} 0 & X_{RB} \\ X_{RB} & 0 \end{bmatrix} \right] \quad (6.55)$$

$$= \text{Tr} \left[ \left( W_{RB} + W_{RB}^\dagger \right) X_{RB} \right], \quad (6.56)$$

and we thus identify

$$\Phi^\dagger(Y) = W_{RB} + W_{RB}^\dagger. \quad (6.57)$$

Then plugging in to the standard form of the dual program

$$\inf_{Y \geq 0} \{ \text{Tr}[BY] : \Phi^\dagger(Y) \geq A \}, \quad (6.58)$$

we find that it is given by

$$\inf \text{Tr}[\Gamma_{RB}^{\mathcal{N}} Y_{RB}] + \text{Tr}[\Gamma_{RB}^{\mathcal{M}} Z_{RB}] \quad (6.59)$$

subject to

$$\begin{bmatrix} Y_{RB} & W_{RB}^\dagger \\ W_{RB} & Z_{RB} \end{bmatrix} \geq 0, \quad W_{RB} + W_{RB}^\dagger \geq \rho_R \otimes I_B. \quad (6.60)$$

So applying strong duality to assert equality of (6.48) and (6.59) and combining this with (6.46), the geometric fidelity of quantum channels  $\mathcal{N}$  and  $\mathcal{M}$  can be computed as

$$\inf_{\rho_R \geq 0} \text{Tr}[\Gamma_{RB}^{\mathcal{N}} Y_{RB}] + \text{Tr}[\Gamma_{RB}^{\mathcal{M}} Z_{RB}] \quad (6.61)$$

subject to

$$\begin{bmatrix} Y_{RB} & W_{RB}^\dagger \\ W_{RB} & Z_{RB} \end{bmatrix} \geq 0, \quad W_{RB} + W_{RB}^\dagger \geq \rho_R \otimes I_B, \quad \text{Tr}[\rho_R] = 1. \quad (6.62)$$

Strong duality holds because we can choose  $\rho_R = \pi_R$  (maximally mixed state)  $W_{RB} = I_{RB}$  and  $Y_{RB} = Z_{RB} = 2I_{RB}$  so that all constraints in the dual program are strict. This concludes the proof. ■

### 6.3 Geometric fidelity of quantum channels as a limit on symmetric channel discrimination

One main use of the geometric fidelity of quantum channels is as a limit on the error exponent of symmetric channel discrimination:

**Conclusion 49** *As a direct consequence of Eq. (159) of [33], the inequality in (6.17), the meta-converse from [33, Lemma 14], and the amortization collapse in Corollary 46, the following bound holds for the non-asymptotic Chernoff error exponent  $\xi_n(p, \mathcal{N}, \mathcal{M})$  of symmetric channel discrimination of quantum channels  $\mathcal{N}_{A \rightarrow B}$  and  $\mathcal{M}_{A \rightarrow B}$ :*

$$\xi_n(p, \mathcal{N}, \mathcal{M}) \leq \widehat{D}_{1/2}(\mathcal{N} \parallel \mathcal{M}) - \frac{1}{n} \ln[p(1-p)], \quad (6.63)$$

where  $\xi_n(p, \mathcal{N}, \mathcal{M})$  is defined in (4.22). Thus, we conclude the following bound on the asymptotic exponent:

$$\bar{\xi}(\mathcal{N}, \mathcal{M}) \leq \widehat{D}_{1/2}(\mathcal{N} \parallel \mathcal{M}). \quad (6.64)$$

This result is a significant improvement over the bound from [33, Proposition 21] because  $\widehat{D}_{1/2}(\mathcal{N} \parallel \mathcal{M}) \leq \min\{D_{\max}(\mathcal{N} \parallel \mathcal{M}), D_{\max}(\mathcal{M} \parallel \mathcal{N})\}$ , due to (6.13) and (6.16). It is also efficiently computable, so that it improves as well upon the amortized fidelity bound from [33, Proposition 21] and [40].

An achievable rate for symmetric channel discrimination is given by the Chernoff information of quantum channels [33], defined as

$$C(\mathcal{N} \parallel \mathcal{M}) := \sup_{\psi_{RA}, \alpha \in (0,1)} (1 - \alpha) D_\alpha(\mathcal{N}_{A \rightarrow B}(\psi_{RA}) \parallel \mathcal{M}_{A \rightarrow B}(\psi_{RA})), \quad (6.65)$$

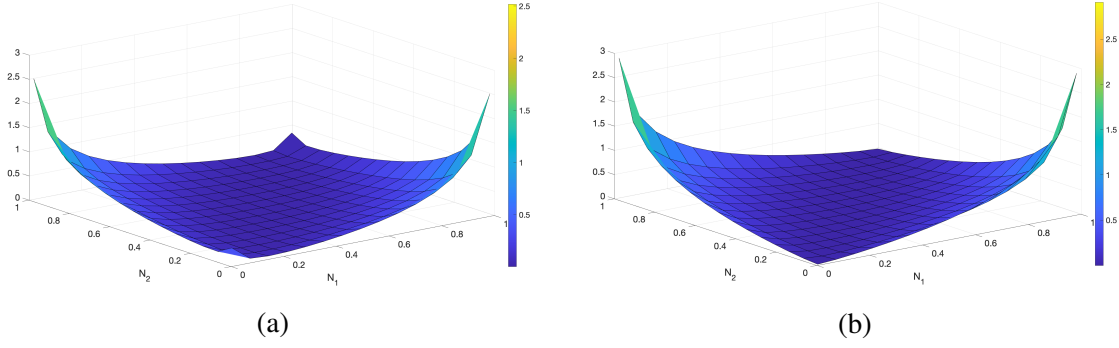


Figure 6: (a) Difference of the geometric fidelity upper bound and Chernoff information lower bound for generalized amplitude damping channels with fixed loss  $\gamma_1 = 0.8$  and  $\gamma_2 = 0.7$ . (b) Same plot but fixed loss  $\gamma_1 = 0.5$  and  $\gamma_2 = 0.5$ .

where the Petz–Rényi relative entropy  $D_\alpha(\rho\|\sigma)$  of a quantum state and a positive semi-definite operator  $\sigma$  is defined for  $\alpha \in (0, 1) \cup (1, \infty)$  as [71, 121]

$$D_\alpha(\rho\|\sigma) = \begin{cases} \frac{1}{\alpha-1} \ln \text{Tr}[\rho^\alpha \sigma^{1-\alpha}] & \text{if } \alpha \in (0, 1) \text{ or} \\ & \alpha \in (1, \infty) \text{ and } \text{supp}(\rho) \subseteq \text{supp}(\sigma) \\ +\infty & \text{otherwise} \end{cases} \quad (6.66)$$

This corresponds to a parallel discrimination strategy in which we feed in one share of a state  $\psi_{RA}$  to each use of the channel and then perform a collective measurement on all of the output systems (this is even a special case of what is depicted in Figure 2). That is, we have that

$$C(\mathcal{N}\|\mathcal{M}) \leq \underline{\xi}(\mathcal{N}, \mathcal{M}) \leq \bar{\xi}(\mathcal{N}, \mathcal{M}) \leq \hat{D}_{1/2}(\mathcal{N}\|\mathcal{M}). \quad (6.67)$$

In Figures 6 and 7, we compare the achievable lower bound given by  $C(\mathcal{N}\|\mathcal{M})$  with the general upper bound set by  $\hat{D}_{1/2}(\mathcal{N}\|\mathcal{M})$  for the case of the generalized amplitude damping channel defined in (5.259), for various values of the loss and noise parameters.

## 6.4 Belavkin–Staszewski divergence sphere as a limit on error exponent of quantum channel discrimination

In this section, we establish a limit on the asymptotic Hoeffding error exponent for quantum channel discrimination. We find a generic upper bound for arbitrary quantum channels in terms of what we call the Belavkin–Staszewski divergence sphere formula.

**Proposition 50** *For quantum channels  $\mathcal{N}$  and  $\mathcal{M}$ , the Belavkin–Staszewski divergence sphere is an upper bound on their asymptotic Hoeffding error exponent for quantum channel discrimination:*

$$\overline{B}(r, \mathcal{N}, \mathcal{M}) \leq \inf_{\mathcal{T}: \hat{D}(\mathcal{T}\|\mathcal{M}) \leq r} \hat{D}(\mathcal{T}\|\mathcal{N}). \quad (6.68)$$

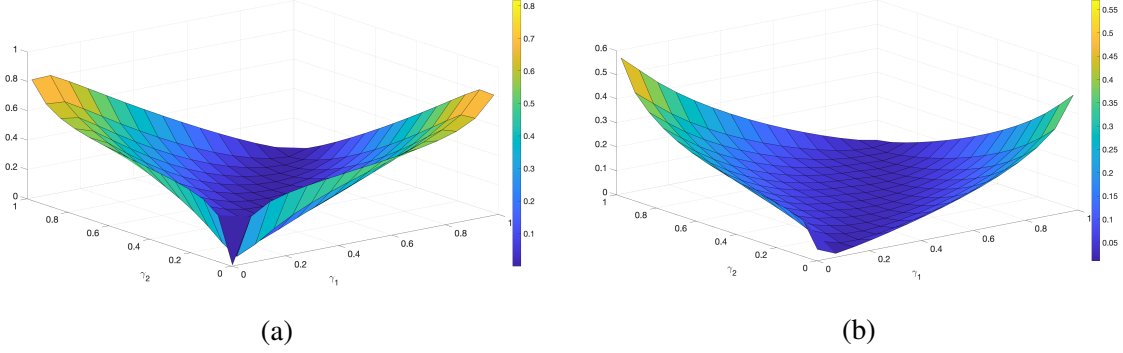


Figure 7: a) Difference of the geometric fidelity upper bound and Chernoff information lower bound for generalized amplitude damping channels with fixed noise  $N_1 = 0.2$  and  $N_2 = 0.2$ . (b) Same plot but fixed noise  $N_1 = 0.3$  and  $N_2 = 0.5$ .

**Proof.** The argument is the same as that given in [41, Exercise 3.15], [122, Eq. (16)], and [33, Proposition 30], but here we use the fact that the Belavkin–Staszewski relative entropy of quantum channels is a strong converse upper bound for asymmetric quantum channel discrimination [39, Theorem 49]. Fix  $\delta > 0$ . Let  $\mathcal{T}$  be a quantum channel such that  $\hat{D}(\mathcal{T} \parallel \mathcal{M}) \leq r - \delta$ . By construction, it follows that  $r > \hat{D}(\mathcal{T} \parallel \mathcal{M})$ . Let  $(\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}}\})_n$  denote a sequence of channel discrimination strategies for  $\mathcal{T}$  and  $\mathcal{M}$ , and let us denote the associated Type I and II error probabilities by

$$\alpha_n^{\mathcal{T} \parallel \mathcal{M}}(\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}}\}), \quad \beta_n^{\mathcal{T} \parallel \mathcal{M}}(\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}}\}), \quad (6.69)$$

respectively. By applying [39, Theorem 49], that the Belavkin–Staszewski relative entropy is a strong converse upper bound for asymmetric channel discrimination of  $\mathcal{T}$  and  $\mathcal{M}$ , if  $(\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}}\})_n$  is a sequence of channel discrimination strategies for these channels such that

$$\limsup_{n \rightarrow \infty} -\frac{1}{n} \ln \beta_n^{\mathcal{T} \parallel \mathcal{M}}(\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}}\}) \geq r, \quad (6.70)$$

then necessarily, we have that

$$\limsup_{n \rightarrow \infty} \alpha_n^{\mathcal{T} \parallel \mathcal{M}}(\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}}\}) = 1. \quad (6.71)$$

However, this implies that  $\{\mathcal{S}^{(n)}, I - \Lambda^{\hat{\theta}}\}$  can be used as a channel discrimination strategy for the channels  $\mathcal{T}$  and  $\mathcal{M}$ , and let us denote the associated Type I and II error probabilities by

$$\alpha_n^{\mathcal{T} \parallel \mathcal{N}}(\{\mathcal{S}^{(n)}, I - \Lambda^{\hat{\theta}}\}), \quad \beta_n^{\mathcal{T} \parallel \mathcal{N}}(\{\mathcal{S}^{(n)}, I - \Lambda^{\hat{\theta}}\}). \quad (6.72)$$

By applying (6.71), we conclude that

$$\limsup_{n \rightarrow \infty} \alpha_n^{\mathcal{T} \parallel \mathcal{N}}(\{\mathcal{S}^{(n)}, I - \Lambda^{\hat{\theta}}\}) = 0, \quad (6.73)$$



and by again invoking the strong converse from [39, Theorem 49], it is necessary that

$$\limsup_{n \rightarrow \infty} -\frac{1}{n} \ln \beta_n^{\mathcal{T} \parallel \mathcal{N}}(\{\mathcal{S}^{(n)}, \Lambda^{\hat{\theta}}\}) \leq \widehat{D}(\mathcal{T} \parallel \mathcal{N}). \quad (6.74)$$

Thus, we find the following bound holding for an arbitrary quantum channel  $\mathcal{T}$  for which  $r > \widehat{D}(\mathcal{T} \parallel \mathcal{M})$ :

$$\overline{B}(r, \mathcal{N}, \mathcal{M}) \leq \inf_{\mathcal{T}: \widehat{D}(\mathcal{T} \parallel \mathcal{M}) \leq r - \delta} \widehat{D}(\mathcal{T} \parallel \mathcal{N}). \quad (6.75)$$

Since  $\delta > 0$  is arbitrary in the above argument, we can employ the facts that the Belavkin–Staszewski relative entropy is continuous in its first argument to arrive at the bound stated in (6.68). ■

## 6.5 Bounds for sequential channel discrimination with repetition

In this section, we establish upper bounds on the asymptotic error exponents for sequential channel discrimination with repetition, as defined in Section 4.2.3. The main idea is to exploit the amortization collapse for the geometric Rényi channel divergence from Corollary 46, the meta-converse from [33, Lemma 14], and the finite-sample bounds from [123].

**Proposition 51** *For quantum channels  $\mathcal{N}$  and  $\mathcal{M}$ , the following asymptotic Chernoff exponent for sequential channel discrimination with repetition is bounded for all  $p \in (0, 1)$  as follows:*

$$\limsup_{n \rightarrow \infty} \limsup_{m \rightarrow \infty} \xi_{n,m}(p, \mathcal{N}, \mathcal{M}) \leq \widehat{C}(\mathcal{N} \parallel \mathcal{M}), \quad (6.76)$$

where the upper bound in (6.76) holds only for the particular order of limits of  $n$  and  $m$  given and the geometric Chernoff information of quantum channels is defined as follows:

$$\widehat{C}(\mathcal{N} \parallel \mathcal{M}) := \sup_{\alpha \in (0,1)} (1 - \alpha) \widehat{D}_\alpha(\mathcal{N} \parallel \mathcal{M}). \quad (6.77)$$

The following asymptotic Hoeffding exponent for sequential channel discrimination with repetition is bounded as follows:

$$\limsup_{n \rightarrow \infty} \limsup_{m \rightarrow \infty} B_{n,m}(r, \mathcal{N}, \mathcal{M}) \leq \sup_{\alpha \in (0,1)} \frac{\alpha - 1}{\alpha} \left( r - \widehat{D}_\alpha(\mathcal{N} \parallel \mathcal{M}) \right). \quad (6.78)$$

**Proof.** The method for establishing both bounds is the same. By applying [123, Theorem 4.7], the following upper bound holds for the error exponent in the Chernoff setting:

$$\begin{aligned} -\frac{1}{nm} \ln p_e^{(n,m)}(\mathcal{S}^{(n)}, \Lambda^{\hat{\theta},m}) &\leq \frac{1}{m} C(\omega_{R_n B_n}^{\theta=1} \parallel \omega_{R_n B_n}^{\theta=2}) \\ &\quad + \frac{3(d^2 - 1)}{2} \frac{\ln n}{nm} + \frac{c}{nm} + \frac{1}{nm(12n + 1)}, \end{aligned} \quad (6.79)$$

where  $C(\omega_{R_n B_n}^{\theta=1} \parallel \omega_{R_n B_n}^{\theta=2})$  is the Chernoff information of the final states of the discrimination protocol,  $d$  is the dimension of this output state, and  $c$  is a constant that depends on the final output states. Now applying the meta-converse from [33, Lemma 14], we find that

$$C(\omega_{R_n B_n}^{\theta=1} \parallel \omega_{R_n B_n}^{\theta=2}) \leq \widehat{C}(\omega_{R_n B_n}^{\theta=1} \parallel \omega_{R_n B_n}^{\theta=2}) \quad (6.80)$$

$$= \sup_{\alpha \in (0,1)} (1 - \alpha) \widehat{D}_\alpha(\omega_{R_n B_n}^{\theta=1} \parallel \omega_{R_n B_n}^{\theta=2}) \quad (6.81)$$

$$\leq m \sup_{\alpha \in (0,1)} (1 - \alpha) \widehat{D}_\alpha^A(\mathcal{N} \parallel \mathcal{M}) \quad (6.82)$$

$$= m \sup_{\alpha \in (0,1)} (1 - \alpha) \widehat{D}_\alpha(\mathcal{N} \parallel \mathcal{M}) \quad (6.83)$$

$$= m \widehat{C}(\mathcal{N} \parallel \mathcal{M}). \quad (6.84)$$

The second-to-last equality follows from Corollary 46. Combining with the above, we find the following bound

$$-\frac{1}{nm} \ln p_e^{(n,m)}(\mathcal{S}^{(n)}, \Lambda^{\hat{\theta},m}) \leq \widehat{C}(\mathcal{N} \parallel \mathcal{M}) + \frac{3(d^2 - 1) \ln n}{2nm} + \frac{c}{nm} + \frac{1}{nm(12n + 1)}. \quad (6.85)$$

By taking the limit as  $m \rightarrow \infty$ , we get the following uniform bound:

$$\limsup_{m \rightarrow \infty} \left[ -\frac{1}{nm} \ln p_e^{(n,m)}(\mathcal{S}^{(n)}, \Lambda^{\hat{\theta},m}) \right] \leq \widehat{C}(\mathcal{N} \parallel \mathcal{M}). \quad (6.86)$$

Then taking the limit as  $n \rightarrow \infty$ , we arrive at (6.76). The proof of (6.77) is essentially the same, except that we start from the other bound in [123, Theorem 4.7] (having to do with the Hoeffding exponent). ■

## 7 Connections between estimation and discrimination of quantum channels

In this section, we outline connections between channel estimation and discrimination, which indicate how one could derive many of the results in Sections 5.5 and 5.6 based on properties of the quantum fidelity and geometric Rényi relative entropy. To do so, one however needs the stronger assumption that the family of states or channels is second-order differentiable with respect to the parameter  $\theta$ . This is the main reason that we have avoided this approach in our earlier developments, because we have shown that it is possible to develop them under the assumption of first-order differentiability only. Nevertheless, the connections are interesting and so we go through them here.

### 7.1 Limit formulas for SLD and RLD Fisher informations

The starting point is the following limit formula for the SLD Fisher information:

**Proposition 52** *Let  $\{\rho_\theta\}_\theta$  be a second-order differentiable family of quantum states. Then the following holds*

$$I_F(\theta; \{\rho_\theta\}_\theta) = \lim_{\varepsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \frac{8}{\delta^2} \left( 1 - \sqrt{F}(\rho_\theta^\varepsilon, \rho_{\theta+\delta}^\varepsilon) \right), \quad (7.1)$$

$$= \lim_{\varepsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \frac{4}{\delta^2} \left( -\ln F(\rho_\theta^\varepsilon, \rho_{\theta+\delta}^\varepsilon) \right), \quad (7.2)$$

where

$$\rho_\theta^\varepsilon := (1 - \varepsilon) \rho_\theta + \varepsilon \pi_d, \quad (7.3)$$

with  $\pi_d$  the maximally mixed state.

The first expression without the  $\varepsilon \rightarrow 0$  limit was given in [124], where it was assumed that the family  $\{\rho_\theta\}_\theta$  is full rank. A different proof was then given in [125], in which the full rank assumption is made as well. We can then apply these former results and Proposition 2 to arrive at the limiting expression in (7.1). The limit in (7.2) is also well known (see, e.g., [126, Section 6] and [90]), and we recall a proof of this due to [127] in Appendix J.

The exchange of limits in (7.1) has implicitly been the subject of more recent investigations [28, 128–130], starting with [128] and concluding with [28]. The main claim of [28] is that the limit exchange is possible for any second-order differentiable family if one modifies (7.1) from a forward shift to a central shift:

$$I_F(\theta; \{\rho_\theta\}_\theta) = \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{8}{\delta^2} \left( 1 - \sqrt{F}(\rho_{\theta-\delta/2}^\varepsilon, \rho_{\theta+\delta/2}^\varepsilon) \right) \quad (7.4)$$

$$= \lim_{\delta \rightarrow 0} \frac{8}{\delta^2} \left( 1 - \sqrt{F}(\rho_{\theta-\delta/2}, \rho_{\theta+\delta/2}) \right). \quad (7.5)$$

Implicitly the finiteness condition in (5.11) has been assumed in the derivation of [28].

The RLD Fisher information has been connected to the geometric Rényi relative entropy via a limit formula of the form in (7.1) (see [115, Section 11] and [35, 36, Section 6.4]). In this case, we have the following:

**Proposition 53** *Let  $\{\rho_\theta\}_\theta$  be a second-order differentiable family of quantum states. Then the following equalities hold for all  $\alpha \in (0, 1) \cup (1, \infty)$ :*

$$\widehat{I}_F(\theta; \{\rho_\theta\}_\theta) = \lim_{\varepsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \frac{2}{\alpha(\alpha - 1)\delta^2} \left( \widehat{Q}_\alpha(\rho_{\theta+\delta}^\varepsilon \| \rho_\theta^\varepsilon) - 1 \right), \quad (7.6)$$

$$= \lim_{\varepsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \frac{2}{\delta^2 \alpha} \widehat{D}_\alpha(\rho_{\theta+\delta}^\varepsilon \| \rho_\theta^\varepsilon), \quad (7.7)$$

where

$$\rho_\theta^\varepsilon := (1 - \varepsilon) \rho_\theta + \varepsilon \pi_d, \quad (7.8)$$

with  $\pi_d$  the maximally mixed state. Additionally, we have that

$$\widehat{I}_F(\theta; \{\rho_\theta\}_\theta) = \lim_{\varepsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \frac{2}{\delta^2} \widehat{D}(\rho_{\theta+\delta}^\varepsilon \| \rho_\theta^\varepsilon). \quad (7.9)$$

**Proof.** Due to the particular order of limits given above, we can assume that  $\rho_\theta$  is full rank. Let us define

$$d\rho_\theta := \rho_{\theta+\delta} - \rho_\theta, \quad (7.10)$$

and observe that

$$\text{Tr}[d\rho_\theta] = 0. \quad (7.11)$$

Then by plugging into (6.5), we find that

$$\widehat{Q}_\alpha(\rho_{\theta+\delta}||\rho_\theta) = \text{Tr} \left[ \rho_\theta \left( \rho_\theta^{-1/2} \rho_{\theta+\delta} \rho_\theta^{-1/2} \right)^\alpha \right] \quad (7.12)$$

$$= \text{Tr} \left[ \rho_\theta \left( \rho_\theta^{-1/2} (\rho_\theta + d\rho_\theta) \rho_\theta^{-1/2} \right)^\alpha \right] \quad (7.13)$$

$$= \text{Tr} \left[ \rho_\theta \left( I + \rho_\theta^{-1/2} d\rho_\theta \rho_\theta^{-1/2} \right)^\alpha \right]. \quad (7.14)$$

Now, by using the expansion

$$(1+x)^\alpha = 1 + \alpha x + \frac{1}{2}(\alpha-1)\alpha x^2 + O(x^3), \quad (7.15)$$

we evaluate the innermost expression of (7.14):

$$\begin{aligned} \left( I + \rho_\theta^{-1/2} d\rho_\theta \rho_\theta^{-1/2} \right)^\alpha &= I + \alpha \rho_\theta^{-1/2} d\rho_\theta \rho_\theta^{-1/2} \\ &\quad + \frac{1}{2}(\alpha-1)\alpha \left( \rho_\theta^{-1/2} d\rho_\theta \rho_\theta^{-1/2} \right)^2 + O((d\rho_\theta)^3). \end{aligned} \quad (7.16)$$

Now left-multiplying by  $\rho_\theta$  and taking the trace gives

$$\begin{aligned} \widehat{Q}_\alpha(\rho_{\theta+\delta}||\rho_\theta) &= \text{Tr} \left[ \rho_\theta \left( I + \rho_\theta^{-1/2} d\rho_\theta \rho_\theta^{-1/2} \right)^\alpha \right] \end{aligned} \quad (7.17)$$

$$= \text{Tr} \left[ \rho_\theta \left( I + \alpha \rho_\theta^{-1/2} d\rho_\theta \rho_\theta^{-1/2} + \frac{1}{2}(\alpha-1)\alpha \left( \rho_\theta^{-1/2} d\rho_\theta \rho_\theta^{-1/2} \right)^2 + O((d\rho_\theta)^3) \right) \right] \quad (7.18)$$

$$= \text{Tr}[\rho_\theta] + \alpha \text{Tr}[d\rho_\theta] + \frac{1}{2}(\alpha-1)\alpha \text{Tr} \left[ \rho_\theta \left( \rho_\theta^{-1/2} d\rho_\theta \rho_\theta^{-1/2} \right)^2 \right] + O((d\rho_\theta)^3) \quad (7.19)$$

$$= 1 + \frac{1}{2}(\alpha-1)\alpha \text{Tr} [d\rho_\theta \rho_\theta^{-1} d\rho_\theta] + O((d\rho_\theta)^3). \quad (7.20)$$

So then

$$\frac{2}{\alpha(\alpha-1)\delta^2} \left( \widehat{Q}_\alpha(\rho_{\theta+\delta}||\rho_\theta) - 1 \right) = \text{Tr} \left[ \frac{d\rho_\theta}{\delta} \rho_\theta^{-1} \frac{d\rho_\theta}{\delta} \right] + \frac{2}{\alpha(\alpha-1)\delta^2} O((d\rho_\theta)^3).$$

For a second-order differentiable family, the following limit holds

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta^2} O(\|d\rho_\theta\|_\infty^3) = \lim_{\delta \rightarrow 0} \delta O(\| [d\rho_\theta/\delta]_\infty \|^3) = 0. \quad (7.21)$$

Then we find that

$$\lim_{\delta \rightarrow 0} \frac{2}{\alpha(\alpha-1)\delta^2} \left( \widehat{Q}_\alpha(\rho_{\theta+\delta} \| \rho_\theta) - 1 \right) = \lim_{\delta \rightarrow 0} \text{Tr} \left[ \frac{d\rho_\theta}{\delta} \rho_\theta^{-1} \frac{d\rho_\theta}{\delta} \right] \quad (7.22)$$

$$= \text{Tr} [(\partial_\theta \rho_\theta)^2 \rho_\theta^{-1}] \quad (7.23)$$

$$= \widehat{I}_F(\theta; \{\rho_\theta\}_\theta), \quad (7.24)$$

as claimed.

The equality between (7.6) and (7.7) is similar to the equality between (7.1) and (7.2) and is shown in Appendix J. Defining  $\eta(x) = x \ln x$ , the last equality in (7.9) follows because

$$\widehat{D}_\alpha(\rho_{\theta+\delta} \| \rho_\theta) = \text{Tr}[\rho_{\theta+\delta} \ln \rho_{\theta+\delta}^{1/2} \rho_\theta^{-1} \rho_{\theta+\delta}^{1/2}] \quad (7.25)$$

$$= \text{Tr}[\rho_\theta \eta(\rho_\theta^{-1/2} \rho_{\theta+\delta} \rho_\theta^{-1/2})] \quad (7.26)$$

$$= \text{Tr}[\rho_\theta \eta(\rho_\theta^{-1/2} (\rho_\theta + d\rho_\theta) \rho_\theta^{-1/2})] \quad (7.27)$$

$$= \text{Tr}[\rho_\theta \eta(I + \rho_\theta^{-1/2} d\rho_\theta \rho_\theta^{-1/2})] \quad (7.28)$$

$$= \text{Tr}[\rho_\theta (\rho_\theta^{-1/2} d\rho_\theta \rho_\theta^{-1/2} + [\rho_\theta^{-1/2} d\rho_\theta \rho_\theta^{-1/2}]^2/2)] + O((d\rho_\theta)^3) \quad (7.29)$$

$$= \text{Tr}[d\rho_\theta] + \text{Tr}[d\rho_\theta \rho_\theta^{-1} d\rho_\theta]/2 + O((d\rho_\theta)^3) \quad (7.30)$$

$$= \text{Tr}[d\rho_\theta \rho_\theta^{-1} d\rho_\theta]/2 + O((d\rho_\theta)^3), \quad (7.31)$$

where we used that  $\eta(1+x) = x + x^2/2 + O(x^3)$ . The reasoning to arrive at (7.9) is similar to what was given previously. ■

## 7.2 Linking properties of Fisher informations and Rényi relative entropies

The limit formulas in Propositions 52 and 53 allow us to connect properties of the SLD and RLD Fisher informations to the fidelity and geometric Rényi relative entropy, respectively. This only occurs when the family of states or channels is second-order differentiable, because the limit formulas in Propositions 52 and 53 only apply under such a circumstance.

We list the connections now:

- Data processing for the SLD and RLD Fisher informations in (5.40)–(5.41) follows from data processing for the fidelity and the geometric Rényi relative entropy in (6.15), respectively.
- Additivity of the SLD and RLD Fisher informations in (5.42) and (5.43) follows from the limit formulas in (7.2) and (7.7), respectively, and additivity of these quantities.
- The decomposition of SLD and RLD Fisher informations for classical–quantum states in Proposition 7 follows from the limit formulas in (7.1) and (7.6), respectively, and also because the underlying quantities have the same decomposition for classical–quantum states.

- The amortization collapse in Theorem 31 for the SLD Fisher information of classical–quantum channels is a consequence of the amortization collapse for the sandwiched Rényi relative entropy given in [33, Lemma 26].
- The chain rule for the root SLD Fisher information in Proposition 33 is a consequence of the limit formula in (7.1), the triangle inequality for the Bures distance, and the related chain-rule inequality given in [33, Lemma 44].
- The additivity of the RLD Fisher information in Proposition 37 is a consequence of the limit formula in (7.7) and the additivity of the geometric Rényi relative entropy of quantum channels (the latter can either be shown directly or as a consequence of Proposition 47).
- The simple formula for the RLD Fisher information in Proposition 29 can be seen as a consequence of the limit formula in (7.7) and the simple formula for the geometric Rényi relative entropy of quantum channels. This is shown explicitly in Appendix K.
- The chain rule for the RLD Fisher information in Proposition 38 is a consequence of the limit formula in (7.7) and the chain rule for the geometric Rényi relative entropy from Proposition 45.

### 7.3 Semi-definite programs for channel fidelity and SLD Fisher information of quantum channels

In this section, we show how the fidelity of quantum channels can be computed by means of a semi-definite program. This was already shown in [32], but here we arrive at semi-definite programs that are functions of the Choi operators of the channels involved. Once the semi-definite program for fidelity of channels is established, one can then use it and generalizations of the limit formulas from Proposition 52 to approximate the SLD Fisher information of quantum channels.

Our starting point is the following semi-definite program and its dual for the root fidelity of quantum states [131]:

**Proposition 54** *Let  $\rho$  and  $\sigma$  be quantum states. Then their root fidelity  $\sqrt{F}(\rho, \sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1$  can be calculated by means of the following semi-definite program*

$$\sup_Q \left\{ \operatorname{Re}[\operatorname{Tr}[Q]] : \begin{bmatrix} \rho & Q^\dagger \\ Q & \sigma \end{bmatrix} \geq 0 \right\}, \quad (7.32)$$

and its dual is given by

$$\frac{1}{2} \inf_{W, Z} \left\{ \operatorname{Tr}[\rho W] + \operatorname{Tr}[\sigma Z] : \begin{bmatrix} W & I \\ I & Z \end{bmatrix} \geq 0 \right\}. \quad (7.33)$$

Using this semi-definite program, we then find the following for the root fidelity of quantum channels:

**Proposition 55** *Let  $\mathcal{N}_{A \rightarrow B}$  and  $\mathcal{M}_{A \rightarrow B}$  be quantum channels with respective Choi operators  $\Gamma_{RB}^{\mathcal{N}}$  and  $\Gamma_{RB}^{\mathcal{M}}$ . Then their root channel fidelity*

$$\sqrt{F}(\mathcal{N}_{A \rightarrow B}, \mathcal{M}_{A \rightarrow B}) := \inf_{\psi_{RA}} \sqrt{F}(\mathcal{N}_{A \rightarrow B}(\psi_{RA}), \mathcal{M}_{A \rightarrow B}(\psi_{RA})) \quad (7.34)$$

*can be calculated by means of the following semi-definite program:*

$$\sup_{\lambda \geq 0, Q_{RB}} \left\{ \lambda : \lambda I_R \leq \text{Re}[\text{Tr}_B[Q_{RB}]], \quad \begin{bmatrix} \Gamma_{RB}^{\mathcal{N}} & Q_{RB}^\dagger \\ Q_{RB} & \Gamma_{RB}^{\mathcal{M}} \end{bmatrix} \geq 0 \right\}. \quad (7.35)$$

*and its dual is given by*

$$\frac{1}{2} \inf_{\rho_R, W_{RB}, Z_{RB}} \text{Tr}[\Gamma_{RB}^{\mathcal{N}} W_{RB}] + \text{Tr}[\Gamma_{RB}^{\mathcal{M}} Z_{RB}], \quad (7.36)$$

*subject to*

$$\rho_R \geq 0, \quad \text{Tr}[\rho_R] = 1, \quad \begin{bmatrix} W_{RB} & \rho_R \otimes I_B \\ \rho_R \otimes I_B & Z_{RB} \end{bmatrix} \geq 0. \quad (7.37)$$

*The expression in (7.35) is equal to*

$$\sup_{Q_{RB}} \left\{ \lambda_{\min}(\text{Re}[\text{Tr}_B[Q_{RB}]]): \begin{bmatrix} \Gamma_{RB}^{\mathcal{N}} & Q_{RB}^\dagger \\ Q_{RB} & \Gamma_{RB}^{\mathcal{M}} \end{bmatrix} \geq 0 \right\}, \quad (7.38)$$

*where  $\lambda_{\min}$  denotes the minimum eigenvalue of its argument.*

**Proof.** See Appendix L. ■

**Remark 56** *Now combining Proposition 55 with the following limit formula for SLD Fisher information of quantum channels*

$$I_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) = \lim_{\delta \rightarrow 0} \frac{8}{\delta^2} (1 - \sqrt{F}(\mathcal{N}_{A \rightarrow B}^\theta, \mathcal{N}_{A \rightarrow B}^{\theta+\delta})), \quad (7.39)$$

*we can approximate  $I_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta)$  numerically by picking  $\delta \approx 10^{-3}$  or  $\delta \approx 10^{-4}$  and calculating  $\sqrt{F}(\mathcal{N}_{A \rightarrow B}^\theta, \mathcal{N}_{A \rightarrow B}^{\theta+\delta})$  by means of the semi-definite program in Proposition 55.*

## 8 Conclusion

In this paper, we have used geometric distinguishability measures to place limits on the related tasks of quantum channel estimation and discrimination. By proving chain rules

for the RLD Fisher information, as well as the root SLD Fisher information, we have established single-letter quantum Cramer–Rao bounds on the performance of estimating a parameter encoded in a quantum channel. In particular, the chain rule for the RLD Fisher information implies a simple condition to determine if a particular family of channels can admit Heisenberg scaling in error, complementing other conditions that have been presented previously in various settings [18–20, 25, 26].

We have also used the geometric Rényi relative entropy to improve the bounds of [33, 40] in the realm of quantum channel discrimination, particularly in both the Chernoff and Hoeffding settings. Finally, we have detailed some conceptual and technical connections between estimation and discrimination. The conceptual connections are due to the fact that one task can be seen as a generalization of the other. The technical connections are due to the divergence measures that underlie each Fisher information quantity, whenever the family under question is second-order differentiable.

Extending our results to multiparameter estimation has been accomplished in [132]. In future work, we will include energy constraints in our formalism and study the behavior of QFI quantities in the presence of energy constraints on the probe state. That is, the operational quantity to be developed further in future work is the energy-constrained generalized Fisher information of a quantum channel family, defined as follows:

$$\mathbf{I}_{F,E}(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) = \sup_{\rho_{RA}: \text{Tr}[H_A \rho_A] \leq E} \mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA})\}_\theta) \quad (8.1)$$

where  $H_A$  is a Hamiltonian acting on the input system of the channel  $\mathcal{N}_{A \rightarrow B}^\theta$ . This definition generalizes the energy-constrained channel divergence introduced in [133]. Furthermore, a relevant information quantity for sequential channel estimation with energy constraints is the following energy-constrained amortized Fisher information:

$$\mathbf{I}_{F,E}^A(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) = \sup_{\{\rho_{RA}^\theta\}_\theta: \text{Tr}[H_A \rho_A^\theta] \leq E} \mathbf{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) - \mathbf{I}_F(\theta; \{\rho_{RA}^\theta\}_\theta) \quad (8.2)$$

We will study properties of these energy-constrained Fisher informations analogous to their corresponding unconstrained versions.

It is an interesting open question to determine whether sequential channel discrimination strategies offer any benefit over parallel discrimination strategies in the limit of a large number of channel uses and in the Chernoff and Hoeffding error exponent settings. It is known that, in asymmetric quantum channel discrimination, sequential strategies offer no advantage over parallel ones in the limit of a large number channel uses [33, 92, 122, 134]. In a recent paper [109] concurrent to ours, it was established that sequential estimation strategies offer no advantage over parallel ones in the limit of a large number of channel uses whenever Heisenberg scaling is unattainable. What remains open is to determine whether sequential strategies can outperform parallel strategies in the case when Heisenberg scaling is attainable.

We also leave open the question of determining an operational interpretation of the RLD Fisher information of channels as the optimal classical Fisher information needed to



simulate the channel family in a local way (inspired by the question addressed in [76] for quantum state families). This task connects to coherence distillation of quantum channels from a resource-theoretic perspective [135].

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## A Technical lemmas

Here we collect some technical lemmas used throughout the paper.

**Lemma 57** *Let  $X$  be a linear operator and let  $Y$  be a positive definite operator. Then*

$$X^\dagger Y^{-1} X = \min \left\{ M : \begin{bmatrix} M & X^\dagger \\ X & Y \end{bmatrix} \geq 0 \right\}, \quad (\text{A.1})$$

where the ordering for the minimization is understood in the operator interval sense (Löwner order).

**Proof.** This is a direct consequence of the Schur complement lemma, which states that

$$\begin{bmatrix} M & X^\dagger \\ X & Y \end{bmatrix} \geq 0 \iff Y \geq 0, \quad M \geq X^\dagger Y^{-1} X. \quad (\text{A.2})$$

This concludes the proof. ■

**Lemma 58** *Let  $K$  and  $Z$  be Hermitian operators, and let  $W$  be a linear operator. Then the dual of the following semi-definite program*

$$\inf_M \left\{ \text{Tr}[KM] : \begin{bmatrix} M & W^\dagger \\ W & Z \end{bmatrix} \geq 0 \right\}, \quad (\text{A.3})$$

with  $M$  Hermitian, is given by

$$\sup_{P, Q, R} \left\{ 2 \text{Re}(\text{Tr}[W^\dagger Q]) - \text{Tr}[ZR] : P \leq K, \begin{bmatrix} P & Q^\dagger \\ Q & R \end{bmatrix} \geq 0 \right\}, \quad (\text{A.4})$$

where  $Q$  is a linear operator and  $P$  and  $R$  are Hermitian.

**Proof.** The standard forms of a primal and dual semi-definite program, for  $A$  and  $B$  Hermitian and  $\Phi$  a Hermiticity-preserving map, are respectively as follows [43]:

$$\inf_{Y \geq 0} \left\{ \text{Tr}[BY] : \Phi^\dagger(Y) \geq A \right\}, \quad (\text{A.5})$$

$$\sup_{X \geq 0} \left\{ \text{Tr}[AX] : \Phi(X) \leq B \right\}, \quad (\text{A.6})$$

where  $\Phi^\dagger$  is the Hilbert–Schmidt adjoint of  $\Phi$ . Noting that

$$\begin{bmatrix} M & W^\dagger \\ W & Z \end{bmatrix} \geq 0 \iff \begin{bmatrix} M & -W^\dagger \\ -W & Z \end{bmatrix} \geq 0 \iff \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \geq \begin{bmatrix} 0 & W^\dagger \\ W & -Z \end{bmatrix}, \quad (\text{A.7})$$

we conclude the statement of the lemma after making the following identifications:

$$B = K, \quad Y = M, \quad \Phi^\dagger(M) = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}, \quad (\text{A.8})$$

$$A = \begin{bmatrix} 0 & W^\dagger \\ W & -Z \end{bmatrix}, \quad X = \begin{bmatrix} P & Q^\dagger \\ Q & R \end{bmatrix}, \quad \Phi(X) = P. \quad (\text{A.9})$$

This concludes the proof. ■

**Lemma 59** *Let  $X$  be a linear square operator, let  $Y$  be a positive definite operator, and let  $L$  be a linear operator. Then*

$$LX^\dagger L^\dagger (LYL^\dagger)^{-1} LXL^\dagger \leq LX^\dagger Y^{-1} XL^\dagger, \quad (\text{A.10})$$

where the inverse on the left hand side is taken on the image of  $L$ . If  $L$  is invertible, then the following equality holds

$$LX^\dagger L^\dagger (LYL^\dagger)^{-1} LXL^\dagger = LX^\dagger Y^{-1} XL^\dagger. \quad (\text{A.11})$$

**Proof.** Fix an operator  $M \geq 0$  satisfying

$$\begin{bmatrix} M & X^\dagger \\ X & Y \end{bmatrix} \geq 0. \quad (\text{A.12})$$

Since the maps  $(\cdot) \rightarrow L(\cdot)L^\dagger$  and  $(\cdot) \rightarrow (I_2 \otimes L)(\cdot)(I_2 \otimes L)^\dagger$  are positive, the condition  $M \geq 0$  and that in (A.12) imply the following conditions:

$$LML^\dagger \geq 0, \quad (\text{A.13})$$

$$\begin{bmatrix} LML^\dagger & LX^\dagger L^\dagger \\ LXL^\dagger & LY L^\dagger \end{bmatrix} = (I_2 \otimes L) \begin{bmatrix} M & X^\dagger \\ X & Y \end{bmatrix} (I_2 \otimes L)^\dagger \geq 0. \quad (\text{A.14})$$

Applying (A.1), we conclude that

$$LML^\dagger \geq \min \left\{ W \geq 0 : \begin{bmatrix} W & LX^\dagger L^\dagger \\ LXL^\dagger & LY L^\dagger \end{bmatrix} \geq 0 \right\} \quad (\text{A.15})$$

$$= LX^\dagger L^\dagger (LY L^\dagger)^{-1} LXL^\dagger. \quad (\text{A.16})$$

Since  $M$  is an arbitrary operator that satisfies  $M \geq 0$  and (A.12), we can pick it to be the smallest and set it to  $X^\dagger Y^{-1} X$ . Thus we conclude (A.10).

If  $L$  is invertible, then consider that

$$LX^\dagger L^\dagger (LY L^\dagger)^{-1} LXL^\dagger = LX^\dagger L^\dagger L^{-\dagger} Y^{-1} L^{-1} LXL^\dagger \quad (\text{A.17})$$

$$= LX^\dagger Y^{-1} XL^\dagger, \quad (\text{A.18})$$

so that (A.11) follows. ■

**Lemma 60** *For positive semi-definite operators  $X$  and  $Y$ ,*

$$\|X \otimes I + I \otimes Y\|_\infty = \|X\|_\infty + \|Y\|_\infty. \quad (\text{A.19})$$

**Proof.** This follows because

$$\|X \otimes I + I \otimes Y\|_\infty$$

$$= \sup_{|\psi\rangle: \|\psi\|_2=1} \langle \psi | (X \otimes I + I \otimes Y) | \psi \rangle \quad (\text{A.20})$$

$$\geq \sup_{|\phi\rangle, |\varphi\rangle: \|\phi\|_2=\|\varphi\|_2=1} ((\langle \phi | \otimes \langle \varphi |) (X \otimes I + I \otimes Y) (|\phi\rangle \otimes |\varphi\rangle)) \quad (\text{A.21})$$

$$= \sup_{|\phi\rangle, |\varphi\rangle: \|\phi\|_2=\|\varphi\|_2=1} \langle \phi | X | \phi \rangle + \langle \varphi | Y | \varphi \rangle \quad (\text{A.22})$$

$$= \sup_{|\phi\rangle: \|\phi\|_2=1} \langle \phi | X | \phi \rangle + \sup_{|\varphi\rangle: \|\varphi\|_2=1} \langle \varphi | Y | \varphi \rangle \quad (\text{A.23})$$

$$= \|X\|_\infty + \|Y\|_\infty. \quad (\text{A.24})$$

On the other hand, from the triangle inequality for the infinity norm, we have that

$$\|X \otimes I + I \otimes Y\|_\infty \leq \|X \otimes I\|_\infty + \|I \otimes Y\|_\infty \quad (\text{A.25})$$

$$= \|X\|_\infty + \|Y\|_\infty, \quad (\text{A.26})$$

thus establishing (A.19). ■

**Lemma 61** *Let  $L$  be a square operator and  $f$  a function such that the squares of the singular values of  $L$  are in the domain of  $f$ . Then*

$$Lf(L^\dagger L) = f(LL^\dagger)L. \quad (\text{A.27})$$

**Proof.** This is a direct consequence of the singular value decomposition theorem. Let  $L = UDV$  be a singular value decomposition of  $L$ , where  $U$  and  $V$  are unitary operators and  $D$  is a diagonal, positive semi-definite operator. Then

$$Lf(L^\dagger L) = UDVf((UDV)^\dagger UDV) \quad (\text{A.28})$$

$$= UDVf(V^\dagger DU^\dagger UDV) \quad (\text{A.29})$$

$$= UDVV^\dagger f(D^2)V \quad (\text{A.30})$$

$$= UDf(D^2)V \quad (\text{A.31})$$

$$= Uf(D^2)DV \quad (\text{A.32})$$

$$= Uf(DVV^\dagger D)U^\dagger UDV \quad (\text{A.33})$$

$$= f(UDVV^\dagger DU^\dagger)UDV \quad (\text{A.34})$$

$$= f(LL^\dagger)L. \quad (\text{A.35})$$

This concludes the proof. ■

The following lemma builds upon [28, Lemma 3], wherein the essential proof ideas are given.

**Lemma 62** *Let  $A$  be an invertible Hermitian operator,  $B$  a linear operator,  $C$  a Hermitian operator, and let  $\varepsilon > 0$ . Then with*

$$M(\varepsilon) := \begin{bmatrix} A & \varepsilon B \\ \varepsilon B^\dagger & \varepsilon^2 C \end{bmatrix}, \quad (\text{A.36})$$

$$D(\varepsilon) := \begin{bmatrix} A + \varepsilon^2 \operatorname{Re}[A^{-1}BB^\dagger] & 0 \\ 0 & \varepsilon^2 (C - B^\dagger A^{-1}B) \end{bmatrix}, \quad (\text{A.37})$$

$$G := \begin{bmatrix} 0 & -iA^{-1}B \\ iB^\dagger A^{-1} & 0 \end{bmatrix}, \quad (\text{A.38})$$

the following inequality holds

$$\|M(\varepsilon) - e^{-i\varepsilon G} D(\varepsilon) e^{i\varepsilon G}\|_\infty \leq o(\varepsilon^2). \quad (\text{A.39})$$

**Proof.** Observe that  $G$  is Hermitian and consider that

$$e^{i\varepsilon G} M(\varepsilon) e^{-i\varepsilon G} = \left( I + i\varepsilon G - \frac{\varepsilon^2}{2} G^2 \right) M(\varepsilon) \left( I - i\varepsilon G - \frac{\varepsilon^2}{2} G^2 \right) + o(\varepsilon^2). \quad (\text{A.40})$$

Then we find that

$$\begin{aligned} \left( I + i\varepsilon G - \frac{\varepsilon^2}{2} G^2 \right) M(\varepsilon) \left( I - i\varepsilon G - \frac{\varepsilon^2}{2} G^2 \right) &= M(\varepsilon) + i\varepsilon [GM(\varepsilon) - M(\varepsilon)G] \\ &\quad + \varepsilon^2 \left[ GM(\varepsilon)G - \frac{1}{2} G^2 M(\varepsilon) - \frac{1}{2} M(\varepsilon) G^2 \right] + o(\varepsilon^2). \end{aligned} \quad (\text{A.41})$$

Now observe that

$$GM(\varepsilon) = \begin{bmatrix} 0 & -iA^{-1}B \\ iB^\dagger A^{-1} & 0 \end{bmatrix} \begin{bmatrix} A & \varepsilon B \\ \varepsilon B^\dagger & \varepsilon^2 C \end{bmatrix} \quad (\text{A.42})$$

$$= \begin{bmatrix} -i\varepsilon A^{-1}BB^\dagger & -i\varepsilon^2 A^{-1}BC \\ iB^\dagger & i\varepsilon B^\dagger A^{-1}B \end{bmatrix} \quad (\text{A.43})$$

$$= \begin{bmatrix} -i\varepsilon A^{-1}BB^\dagger & o(\varepsilon) \\ iB^\dagger & i\varepsilon B^\dagger A^{-1}B \end{bmatrix}, \quad (\text{A.44})$$

$$M(\varepsilon)G = [GM(\varepsilon)]^\dagger \quad (\text{A.45})$$

$$= \begin{bmatrix} i\varepsilon BB^\dagger A^{-1} & -iB \\ o(\varepsilon) & -i\varepsilon B^\dagger A^{-1}B \end{bmatrix}, \quad (\text{A.46})$$

which implies that

$$\begin{aligned} i\varepsilon [GM(\varepsilon) - M(\varepsilon)G] &= i\varepsilon \left( \begin{bmatrix} -i\varepsilon A^{-1}BB^\dagger & o(\varepsilon) \\ iB^\dagger & i\varepsilon B^\dagger A^{-1}B \end{bmatrix} - \begin{bmatrix} i\varepsilon BB^\dagger A^{-1} & -iB \\ o(\varepsilon) & -i\varepsilon B^\dagger A^{-1}B \end{bmatrix} \right) \end{aligned} \quad (\text{A.47})$$

$$= \begin{bmatrix} 2\varepsilon^2 \operatorname{Re}[A^{-1}BB^\dagger] & -\varepsilon B + o(\varepsilon^2) \\ -\varepsilon B^\dagger + o(\varepsilon^2) & -2\varepsilon^2 B^\dagger A^{-1}B \end{bmatrix}. \quad (\text{A.48})$$

Also, observe that

$$GM(\varepsilon)G = \begin{bmatrix} o(1) & o(\varepsilon) \\ iB^\dagger & o(1) \end{bmatrix} \begin{bmatrix} 0 & -iA^{-1}B \\ iB^\dagger A^{-1} & 0 \end{bmatrix} \quad (\text{A.49})$$



$$= \begin{bmatrix} o(\varepsilon) & o(1) \\ o(1) & B^\dagger A^{-1} B \end{bmatrix}, \quad (\text{A.50})$$

$$G^2 M(\varepsilon) = G[GM(\varepsilon)] \quad (\text{A.51})$$

$$= \begin{bmatrix} 0 & -iA^{-1}B \\ iB^\dagger A^{-1} & 0 \end{bmatrix} \begin{bmatrix} o(1) & o(\varepsilon) \\ iB^\dagger & o(1) \end{bmatrix} \quad (\text{A.52})$$

$$= \begin{bmatrix} A^{-1}BB^\dagger & o(1) \\ o(1) & o(\varepsilon) \end{bmatrix}, \quad (\text{A.53})$$

$$M(\varepsilon)G^2 = [G^2 M(\varepsilon)]^\dagger \quad (\text{A.54})$$

$$= \begin{bmatrix} BB^\dagger A^{-1} & o(1) \\ o(1) & o(\varepsilon) \end{bmatrix}. \quad (\text{A.55})$$

So then we find that

$$\begin{aligned} & \varepsilon^2 \left[ GM(\varepsilon)G - \frac{1}{2}G^2 M(\varepsilon) - \frac{1}{2}M(\varepsilon)G^2 \right] \\ &= \varepsilon^2 \left( \begin{bmatrix} o(\varepsilon) & o(1) \\ o(1) & B^\dagger A^{-1} B \end{bmatrix} - \frac{1}{2} \begin{bmatrix} A^{-1}BB^\dagger & o(1) \\ o(1) & o(\varepsilon) \end{bmatrix} - \frac{1}{2} \begin{bmatrix} BB^\dagger A^{-1} & o(1) \\ o(1) & o(\varepsilon) \end{bmatrix} \right) \end{aligned} \quad (\text{A.56})$$

$$= \begin{bmatrix} -\varepsilon^2 \operatorname{Re}[A^{-1}BB^\dagger] + o(\varepsilon^3) & o(\varepsilon^2) \\ o(\varepsilon^2) & \varepsilon^2 B^\dagger A^{-1} B + o(\varepsilon^3) \end{bmatrix}. \quad (\text{A.57})$$

So then

$$\begin{aligned} & \left( I + i\varepsilon G - \frac{\varepsilon^2}{2}G^2 \right) M(\varepsilon) \left( I - i\varepsilon G - \frac{\varepsilon^2}{2}G^2 \right) \\ &= M(\varepsilon) + i\varepsilon [GM(\varepsilon) - M(\varepsilon)G] \\ & \quad + \varepsilon^2 \left[ GM(\varepsilon)G - \frac{1}{2}G^2 M(\varepsilon) - \frac{1}{2}M(\varepsilon)G^2 \right] + o(\varepsilon^2) \end{aligned} \quad (\text{A.58})$$

$$\begin{aligned} &= \begin{bmatrix} A & \varepsilon B \\ \varepsilon B^\dagger & \varepsilon^2 C \end{bmatrix} + \begin{bmatrix} 2\varepsilon^2 \operatorname{Re}[A^{-1}BB^\dagger] & -\varepsilon B + o(\varepsilon^2) \\ -\varepsilon B^\dagger + o(\varepsilon^2) & -2\varepsilon^2 B^\dagger A^{-1} B \end{bmatrix} \\ & \quad + \begin{bmatrix} -\varepsilon^2 \operatorname{Re}[A^{-1}BB^\dagger] + o(\varepsilon^3) & o(\varepsilon^2) \\ o(\varepsilon^2) & \varepsilon^2 B^\dagger A^{-1} B + o(\varepsilon^3) \end{bmatrix} + o(\varepsilon^2) \end{aligned} \quad (\text{A.59})$$

$$= \begin{bmatrix} A + \varepsilon^2 \operatorname{Re}[A^{-1}BB^\dagger] & 0 \\ 0 & \varepsilon^2 (C - B^\dagger A^{-1} B) \end{bmatrix} + o(\varepsilon^2) \quad (\text{A.60})$$

$$= D(\varepsilon) + o(\varepsilon^2). \quad (\text{A.61})$$

So we conclude that

$$e^{i\varepsilon G} M(\varepsilon) e^{-i\varepsilon G} = D(\varepsilon) + o(\varepsilon^2), \quad (\text{A.62})$$

which in turn implies that

$$M(\varepsilon) = e^{-i\varepsilon G} D(\varepsilon) e^{i\varepsilon G} + o(\varepsilon^2), \quad (\text{A.63})$$

from which we conclude the claim in (A.39). ■

## B Basis-dependent and basis-independent formulas for SLD Fisher information

Here we review the proof of the following equality, mentioned in (5.22)–(5.23), which was reported in [65] and holds when  $\Pi_{\rho_\theta}^\perp(\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp = 0$ :

$$\frac{1}{2} I_F(\theta; \{\rho_\theta\}_\theta) = \sum_{j,k: \lambda_\theta^j + \lambda_\theta^k > 0} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{\lambda_\theta^j + \lambda_\theta^k} \quad (\text{B.1})$$

$$= \langle \Gamma | ((\partial_\theta \rho_\theta) \otimes I) (\rho_\theta \otimes I + I \otimes \rho_\theta^T)^{-1} ((\partial_\theta \rho_\theta) \otimes I) | \Gamma \rangle. \quad (\text{B.2})$$

Consider that

$$\begin{aligned} & \rho_\theta \otimes I + I \otimes \rho_\theta^T \\ &= \sum_{j: \lambda_j > 0} \lambda_\theta^j |\psi_\theta^j\rangle \langle \psi_\theta^j| \otimes I + I \otimes \left( \sum_{k: \lambda_k > 0} \lambda_\theta^k |\psi_\theta^k\rangle \langle \psi_\theta^k| \right)^T \end{aligned} \quad (\text{B.3})$$

$$= \sum_{j: \lambda_j > 0} \lambda_\theta^j |\psi_\theta^j\rangle \langle \psi_\theta^j| \otimes I + I \otimes \sum_{k: \lambda_k > 0} \lambda_\theta^k |\overline{\psi_\theta^k}\rangle \langle \overline{\psi_\theta^k}| \quad (\text{B.4})$$

$$= \sum_{j: \lambda_j > 0, k} \lambda_\theta^j |\psi_\theta^j\rangle \langle \psi_\theta^j| \otimes |\overline{\psi_\theta^k}\rangle \langle \overline{\psi_\theta^k}| + \sum_{j, k: \lambda_k > 0} \lambda_\theta^k |\psi_\theta^j\rangle \langle \psi_\theta^j| \otimes |\overline{\psi_\theta^k}\rangle \langle \overline{\psi_\theta^k}| \quad (\text{B.5})$$

$$= \sum_{j, k: \lambda_j + \lambda_k > 0} (\lambda_\theta^j + \lambda_\theta^k) |\psi_\theta^j\rangle \langle \psi_\theta^j| \otimes |\overline{\psi_\theta^k}\rangle \langle \overline{\psi_\theta^k}|, \quad (\text{B.6})$$

where  $|\overline{\psi_\theta^k}\rangle$  denotes the complex conjugate of  $|\psi_\theta^k\rangle$  with respect to the orthonormal basis  $\{|i\rangle\}_i$  for the unnormalized maximally entangled vector  $|\Gamma\rangle$ . Then it follows that

$$(\rho_\theta \otimes I + I \otimes \rho_\theta^T)^{-1} = \sum_{j, k: \lambda_j + \lambda_k > 0} \frac{1}{\lambda_\theta^j + \lambda_\theta^k} |\psi_\theta^j\rangle \langle \psi_\theta^j| \otimes |\overline{\psi_\theta^k}\rangle \langle \overline{\psi_\theta^k}|, \quad (\text{B.7})$$

and we find that

$$\begin{aligned} & \langle \Gamma | ((\partial_\theta \rho_\theta) \otimes I) (\rho_\theta \otimes I + I \otimes \rho_\theta^T)^{-1} ((\partial_\theta \rho_\theta) \otimes I) | \Gamma \rangle \\ &= \langle \Gamma | ((\partial_\theta \rho_\theta) \otimes I) \left( \sum_{j, k: \lambda_j + \lambda_k > 0} \frac{1}{\lambda_\theta^j + \lambda_\theta^k} |\psi_\theta^j\rangle \langle \psi_\theta^j| \otimes |\overline{\psi_\theta^k}\rangle \langle \overline{\psi_\theta^k}| \right) ((\partial_\theta \rho_\theta) \otimes I) | \Gamma \rangle \end{aligned} \quad (\text{B.8})$$

$$= \sum_{j, k: \lambda_j + \lambda_k > 0} \frac{1}{\lambda_\theta^j + \lambda_\theta^k} \langle \Gamma | ((\partial_\theta \rho_\theta) \otimes I) \left( |\psi_\theta^j\rangle \langle \psi_\theta^j| \otimes |\overline{\psi_\theta^k}\rangle \langle \overline{\psi_\theta^k}| \right) ((\partial_\theta \rho_\theta) \otimes I) | \Gamma \rangle \quad (\text{B.9})$$

$$= \sum_{j, k: \lambda_j + \lambda_k > 0} \frac{1}{\lambda_\theta^j + \lambda_\theta^k} \langle \Gamma | ((\partial_\theta \rho_\theta) |\psi_\theta^j\rangle \langle \psi_\theta^j| (\partial_\theta \rho_\theta) \otimes |\overline{\psi_\theta^k}\rangle \langle \overline{\psi_\theta^k}|) | \Gamma \rangle \quad (\text{B.10})$$

$$= \sum_{j,k:\lambda_j+\lambda_k>0} \frac{1}{\lambda_\theta^j + \lambda_\theta^k} \langle \Gamma | ((\partial_\theta \rho_\theta) |\psi_\theta^j\rangle \langle \psi_\theta^j| (\partial_\theta \rho_\theta) |\psi_\theta^k\rangle \langle \psi_\theta^k| \otimes I) | \Gamma \rangle \quad (\text{B.11})$$

$$= \sum_{j,k:\lambda_j+\lambda_k>0} \frac{1}{\lambda_\theta^j + \lambda_\theta^k} \text{Tr}[(\partial_\theta \rho_\theta) |\psi_\theta^j\rangle \langle \psi_\theta^j| (\partial_\theta \rho_\theta) |\psi_\theta^k\rangle \langle \psi_\theta^k|] \quad (\text{B.12})$$

$$= \sum_{j,k:\lambda_j+\lambda_k>0} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{\lambda_\theta^j + \lambda_\theta^k}, \quad (\text{B.13})$$

where we used (3.2) and (3.3).

Following the approach given in [65], we can also see how the formula in (5.19) arises from the differential equation in (5.20) and the formula in (B.2). Again, this development is only relevant when the finiteness condition  $\Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp = 0$  holds. Consider that the SLD operator  $L_\theta$  is defined from the following differential equation:

$$\partial_\theta \rho_\theta = \frac{1}{2} (\rho_\theta L_\theta + L_\theta \rho_\theta). \quad (\text{B.14})$$

Then this is equivalent to the following vectorized form:

$$(\partial_\theta \rho_\theta \otimes I) |\Gamma\rangle = \left( \frac{1}{2} (\rho_\theta L_\theta + L_\theta \rho_\theta) \otimes I \right) |\Gamma\rangle \quad (\text{B.15})$$

$$= \frac{1}{2} (\rho_\theta L_\theta \otimes I + L_\theta \rho_\theta \otimes I) |\Gamma\rangle \quad (\text{B.16})$$

$$= \frac{1}{2} (\rho_\theta L_\theta \otimes I + L_\theta \otimes \rho_\theta^T) |\Gamma\rangle \quad (\text{B.17})$$

$$= \frac{1}{2} (\rho_\theta \otimes I + I \otimes \rho_\theta^T) (L_\theta \otimes I) |\Gamma\rangle. \quad (\text{B.18})$$

Consider that

$$(\Pi_{\rho_\theta}^\perp \otimes \Pi_{\rho_\theta^T}^\perp) (\partial_\theta \rho_\theta \otimes I) |\Gamma\rangle = 0 \quad (\text{B.19})$$

because

$$(\Pi_{\rho_\theta}^\perp \otimes \Pi_{\rho_\theta^T}^\perp) (\partial_\theta \rho_\theta \otimes I) |\Gamma\rangle = \left( \Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) \otimes \Pi_{\rho_\theta^T}^\perp \right) |\Gamma\rangle \quad (\text{B.20})$$

$$= (\Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) \otimes (\Pi_{\rho_\theta}^\perp)^T) |\Gamma\rangle \quad (\text{B.21})$$

$$= (\Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp \otimes I) |\Gamma\rangle \quad (\text{B.22})$$

$$= 0. \quad (\text{B.23})$$

Thus,  $(\partial_\theta \rho_\theta \otimes I) |\Gamma\rangle$  is only non-zero on the space onto which  $I \otimes I - \Pi_{\rho_\theta}^\perp \otimes \Pi_{\rho_\theta^T}^\perp$  projects, i.e.,

$$\left( I \otimes I - \Pi_{\rho_\theta}^\perp \otimes \Pi_{\rho_\theta^T}^\perp \right) (\partial_\theta \rho_\theta \otimes I) |\Gamma\rangle = (\partial_\theta \rho_\theta \otimes I) |\Gamma\rangle. \quad (\text{B.24})$$

Furthermore, note that the support of the operator  $\rho_\theta \otimes I + I \otimes \rho_\theta^T$  is given by

$$\Pi_{\rho_\theta} \otimes \Pi_{\rho_\theta^T} + \Pi_{\rho_\theta}^\perp \otimes \Pi_{\rho_\theta^T} + \Pi_{\rho_\theta} \otimes \Pi_{\rho_\theta^T}^\perp = I \otimes I - \Pi_{\rho_\theta}^\perp \otimes \Pi_{\rho_\theta^T}^\perp. \quad (\text{B.25})$$

Thus, by applying the inverse of the operator  $\frac{1}{2}(\rho_\theta \otimes I + I \otimes \rho_\theta^T)$  on its support on both sides, we find that

$$2(\rho_\theta \otimes I + I \otimes \rho_\theta^T)^{-1}(\partial_\theta \rho_\theta \otimes I)|\Gamma\rangle = (I \otimes I - \Pi_{\rho_\theta}^\perp \otimes \Pi_{\rho_\theta^T}^\perp)(L_\theta \otimes I)|\Gamma\rangle. \quad (\text{B.26})$$

Next, we use the fact that

$$\text{Tr}[X^\dagger Y] = \langle \Gamma | (X \otimes I)^\dagger (Y \otimes I) | \Gamma \rangle, \quad (\text{B.27})$$

and we find that

$$\text{Tr}[L_\theta(\partial_\theta \rho_\theta)] = \langle \Gamma | (\partial_\theta \rho_\theta \otimes I)(L_\theta \otimes I) | \Gamma \rangle \quad (\text{B.28})$$

$$= \langle \Gamma | (\partial_\theta \rho_\theta \otimes I) \left( I \otimes I - \Pi_{\rho_\theta}^\perp \otimes \Pi_{\rho_\theta^T}^\perp \right) (L_\theta \otimes I) | \Gamma \rangle \quad (\text{B.29})$$

$$= 2\langle \Gamma | (\partial_\theta \rho_\theta \otimes I) (\rho_\theta \otimes I + I \otimes \rho_\theta^T)^{-1} (\partial_\theta \rho_\theta \otimes I) | \Gamma \rangle, \quad (\text{B.30})$$

where we used (B.24) and (B.26). This concludes the proof that

$$I_F(\theta; \{\rho_A^\theta\}) = \text{Tr}[L_\theta(\partial_\theta \rho_\theta)]. \quad (\text{B.31})$$

## C Physical consistency of SLD and RLD Fisher informations of quantum states

We begin by establishing the equivalence of the conditions in (5.11) and (5.12). Suppose that  $\Pi_{\rho_\theta}^\perp(\partial_\theta \rho_\theta)\Pi_{\rho_\theta}^\perp = 0$  holds. Then consider that

$$\Pi_{\rho_\theta}^\perp = \sum_{j:\lambda_\theta^j=0} |\psi_\theta^j\rangle\langle\psi_\theta^j|, \quad (\text{C.1})$$

so that

$$0 = \Pi_{\rho_\theta}^\perp(\partial_\theta \rho_\theta)\Pi_{\rho_\theta}^\perp \quad (\text{C.2})$$

$$= \left( \sum_{j:\lambda_\theta^j=0} |\psi_\theta^j\rangle\langle\psi_\theta^j| \right) (\partial_\theta \rho_\theta) \left( \sum_{k:\lambda_\theta^k=0} |\psi_\theta^k\rangle\langle\psi_\theta^k| \right) \quad (\text{C.3})$$

$$= \sum_{j:\lambda_\theta^j=0} \sum_{k:\lambda_\theta^k=0} |\psi_\theta^j\rangle\langle\psi_\theta^j| (\partial_\theta \rho_\theta) |\psi_\theta^k\rangle\langle\psi_\theta^k| \quad (\text{C.4})$$

$$= \sum_{j,k:\lambda_\theta^j+\lambda_\theta^k=0} \langle\psi_\theta^j|(\partial_\theta \rho_\theta)|\psi_\theta^k\rangle |\psi_\theta^j\rangle\langle\psi_\theta^k|. \quad (\text{C.5})$$

The last equality follows because  $\lambda_\theta^j \geq 0$  for all  $j$ , so that  $\lambda_\theta^j + \lambda_\theta^k = 0$  is equivalent to  $\lambda_\theta^j = 0 \wedge \lambda_\theta^k = 0$ . Then it follows that  $\langle\psi_\theta^j|(\partial_\theta \rho_\theta)|\psi_\theta^k\rangle = 0$  if  $\lambda_\theta^j + \lambda_\theta^k = 0$ . This

establishes (5.11)  $\Rightarrow$  (5.12). The opposite implication follows from running the proof above backwards.

The equality in (5.15) is established in (C.34)–(C.36) of the proof given below.

**Proof of Proposition 2.** First, it is helpful to write the spectral decomposition of  $\rho_\theta$  as follows:

$$\rho_\theta = \sum_{j \in \mathcal{S}} \lambda_\theta^j |\psi_\theta^j\rangle\langle\psi_\theta^j| + \sum_{j \in \mathcal{K}} \lambda_\theta^j |\psi_\theta^j\rangle\langle\psi_\theta^j|, \quad (\text{C.6})$$

where  $\mathcal{S}$  is the set of indices for which  $\lambda_\theta^j > 0$  and  $\mathcal{K}$  is the set of indices for which  $\lambda_\theta^j = 0$  ( $\mathcal{S}$  and  $\mathcal{K}$  are meant to refer to support and kernel, respectively). Let us define

$$\Pi_{\rho_\theta} := \sum_{j \in \mathcal{S}} |\psi_\theta^j\rangle\langle\psi_\theta^j|, \quad \Pi_{\rho_\theta}^\perp := I - \Pi_{\rho_\theta} = \sum_{j \in \mathcal{K}} |\psi_\theta^j\rangle\langle\psi_\theta^j|. \quad (\text{C.7})$$

Then

$$\rho_\theta^\varepsilon = (1 - \varepsilon) \rho_\theta + \varepsilon \pi \quad (\text{C.8})$$

$$= \sum_{j \in \mathcal{S}} [(1 - \varepsilon) \lambda_\theta^j] |\psi_\theta^j\rangle\langle\psi_\theta^j| + \frac{\varepsilon}{d} \sum_j |\psi_\theta^j\rangle\langle\psi_\theta^j| \quad (\text{C.9})$$

$$= \sum_{j \in \mathcal{S}} \left[ (1 - \varepsilon) \lambda_\theta^j + \frac{\varepsilon}{d} \right] |\psi_\theta^j\rangle\langle\psi_\theta^j| + \frac{\varepsilon}{d} \sum_{j \in \mathcal{K}} |\psi_\theta^j\rangle\langle\psi_\theta^j| \quad (\text{C.10})$$

$$= \sum_{j \in \mathcal{S}} \left[ (1 - \varepsilon) \lambda_\theta^j + \frac{\varepsilon}{d} \right] |\psi_\theta^j\rangle\langle\psi_\theta^j| + \frac{\varepsilon}{d} \Pi_{\rho_\theta}^\perp. \quad (\text{C.11})$$

Let  $\{\lambda_{\theta,\varepsilon}^j\}_j$  denote the eigenvalues of  $\rho_\theta^\varepsilon$ , so that  $\lambda_{\theta,\varepsilon}^j = (1 - \varepsilon) \lambda_\theta^j + \frac{\varepsilon}{d}$  for  $j \in \mathcal{S}$  and  $\lambda_{\theta,\varepsilon}^j = \frac{\varepsilon}{d}$  for  $j \in \mathcal{K}$ . Observe that the state  $\rho_\theta^\varepsilon$  has full support. Also, observe that

$$\partial_\theta \rho_\theta^\varepsilon = (1 - \varepsilon) \partial_\theta \rho_\theta. \quad (\text{C.12})$$

Plugging into the formula in (5.10), we find that

$$\begin{aligned} & \frac{1}{2(1 - \varepsilon)^2} I_F(\theta; \{\rho_\theta^\varepsilon\}_\theta) \\ &= \frac{1}{(1 - \varepsilon)^2} \sum_{j,k} \frac{|\langle\psi_\theta^j|(\partial_\theta \rho_\theta^\varepsilon)|\psi_\theta^k\rangle|^2}{\lambda_{\theta,\varepsilon}^j + \lambda_{\theta,\varepsilon}^k} \end{aligned} \quad (\text{C.13})$$

$$= \sum_{j,k} \frac{|\langle\psi_\theta^j|(\partial_\theta \rho_\theta)|\psi_\theta^k\rangle|^2}{\lambda_{\theta,\varepsilon}^j + \lambda_{\theta,\varepsilon}^k} \quad (\text{C.14})$$

$$\begin{aligned} &= \sum_{j \in \mathcal{S}, k \in \mathcal{S}} \frac{|\langle\psi_\theta^j|(\partial_\theta \rho_\theta)|\psi_\theta^k\rangle|^2}{\lambda_{\theta,\varepsilon}^j + \lambda_{\theta,\varepsilon}^k} + \sum_{j \in \mathcal{S}, k \in \mathcal{K}} \frac{|\langle\psi_\theta^j|(\partial_\theta \rho_\theta)|\psi_\theta^k\rangle|^2}{\lambda_{\theta,\varepsilon}^j + \lambda_{\theta,\varepsilon}^k} \\ &\quad + \sum_{j \in \mathcal{K}, k \in \mathcal{S}} \frac{|\langle\psi_\theta^j|(\partial_\theta \rho_\theta)|\psi_\theta^k\rangle|^2}{\lambda_{\theta,\varepsilon}^j + \lambda_{\theta,\varepsilon}^k} + \sum_{j \in \mathcal{K}, k \in \mathcal{K}} \frac{|\langle\psi_\theta^j|(\partial_\theta \rho_\theta)|\psi_\theta^k\rangle|^2}{\lambda_{\theta,\varepsilon}^j + \lambda_{\theta,\varepsilon}^k} \end{aligned} \quad (\text{C.15})$$

Let us consider the terms one at a time, starting with the first one:

$$\sum_{j \in \mathcal{S}, k \in \mathcal{S}} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{\lambda_{\theta, \varepsilon}^j + \lambda_{\theta, \varepsilon}^k} = \sum_{j \in \mathcal{S}, k \in \mathcal{S}} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{(1 - \varepsilon) [\lambda_\theta^j + \lambda_\theta^k] + \frac{2\varepsilon}{d}} \quad (\text{C.16})$$

The second term simplifies as follows:

$$\begin{aligned} & \sum_{j \in \mathcal{S}, k \in \mathcal{K}} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{\lambda_{\theta, \varepsilon}^j + \lambda_{\theta, \varepsilon}^k} \\ &= \sum_{j \in \mathcal{S}, k \in \mathcal{K}} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{(1 - \varepsilon) \lambda_\theta^j + \frac{2\varepsilon}{d}} \end{aligned} \quad (\text{C.17})$$

$$= \sum_{j \in \mathcal{S}, k \in \mathcal{K}} \frac{\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle \langle \psi_\theta^k | (\partial_\theta \rho_\theta) | \psi_\theta^j \rangle}{(1 - \varepsilon) \lambda_\theta^j + \frac{2\varepsilon}{d}} \quad (\text{C.18})$$

$$= \sum_{j \in \mathcal{S}} \frac{1}{(1 - \varepsilon) \lambda_\theta^j + \frac{2\varepsilon}{d}} \langle \psi_\theta^j | (\partial_\theta \rho_\theta) \left( \sum_{k \in \mathcal{K}} |\psi_\theta^k \rangle \langle \psi_\theta^k| \right) (\partial_\theta \rho_\theta) | \psi_\theta^j \rangle \quad (\text{C.19})$$

$$= \sum_{j \in \mathcal{S}} \frac{\langle \psi_\theta^j | (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) | \psi_\theta^j \rangle}{(1 - \varepsilon) \lambda_\theta^j + \frac{2\varepsilon}{d}}. \quad (\text{C.20})$$

Similarly, due to symmetry, we find the following for the third term:

$$\sum_{j \in \mathcal{K}, k \in \mathcal{S}} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{\lambda_{\theta, \varepsilon}^j + \lambda_{\theta, \varepsilon}^k} = \sum_{j \in \mathcal{S}} \frac{\langle \psi_\theta^j | (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) | \psi_\theta^j \rangle}{(1 - \varepsilon) \lambda_\theta^j + \frac{2\varepsilon}{d}}. \quad (\text{C.21})$$

For the last term, we find that

$$\begin{aligned} & \sum_{j \in \mathcal{K}, k \in \mathcal{K}} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{\lambda_{\theta, \varepsilon}^j + \lambda_{\theta, \varepsilon}^k} \\ &= \sum_{j \in \mathcal{K}, k \in \mathcal{K}} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{\frac{2\varepsilon}{d}} \end{aligned} \quad (\text{C.22})$$

$$= \frac{d}{2\varepsilon} \sum_{j \in \mathcal{K}, k \in \mathcal{K}} |\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2 \quad (\text{C.23})$$

$$= \frac{d}{2\varepsilon} \sum_{j \in \mathcal{K}, k \in \mathcal{K}} \langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle \langle \psi_\theta^k | (\partial_\theta \rho_\theta) | \psi_\theta^j \rangle \quad (\text{C.24})$$

$$= \frac{d}{2\varepsilon} \sum_{j \in \mathcal{K}, k \in \mathcal{K}} \text{Tr} [|\psi_\theta^j \rangle \langle \psi_\theta^j| (\partial_\theta \rho_\theta) |\psi_\theta^k \rangle \langle \psi_\theta^k| (\partial_\theta \rho_\theta)] \quad (\text{C.25})$$

$$= \frac{d}{2\varepsilon} \text{Tr} \left[ \left( \sum_{j \in \mathcal{K}} |\psi_\theta^j \rangle \langle \psi_\theta^j| \right) (\partial_\theta \rho_\theta) \left( \sum_{k \in \mathcal{K}} |\psi_\theta^k \rangle \langle \psi_\theta^k| \right) (\partial_\theta \rho_\theta) \right] \quad (\text{C.26})$$

$$= \frac{d}{2\varepsilon} \text{Tr} [\Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta)] \quad (\text{C.27})$$

$$= \frac{d}{2\varepsilon} \text{Tr} [(\Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp)^2] \quad (\text{C.28})$$

$$= \frac{d}{2\varepsilon} \|\Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp\|_2^2, \quad (\text{C.29})$$

where  $\|A\|_2 := \sqrt{\text{Tr}[A^\dagger A]}$  is the Hilbert–Schmidt norm of an operator  $A$ . Putting everything together, we find that

$$\begin{aligned} I_F(\theta; \{\rho_\theta^\varepsilon\}_\theta) &= 2(1-\varepsilon)^2 \sum_{j \in \mathcal{S}, k \in \mathcal{S}} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{(1-\varepsilon) [\lambda_\theta^j + \lambda_\theta^k] + \frac{2\varepsilon}{d}} \\ &\quad + 4(1-\varepsilon)^2 \sum_{j \in \mathcal{S}} \frac{\langle \psi_\theta^j | (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) | \psi_\theta^j \rangle}{(1-\varepsilon) \lambda_\theta^j + \frac{2\varepsilon}{d}} + \frac{d(1-\varepsilon)^2}{\varepsilon} \|\Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp\|_2^2. \end{aligned} \quad (\text{C.30})$$

Now consider that

$$\|\Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp\|_2^2 = 0 \quad \Longleftrightarrow \quad \Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp = 0. \quad (\text{C.31})$$

If this condition holds, then the last term vanishes and we find that

$$\lim_{\varepsilon \rightarrow 0} I_F(\theta; \{\rho_\theta^\varepsilon\}_\theta) = 2 \sum_{j \in \mathcal{S}, k \in \mathcal{S}} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{\lambda_\theta^j + \lambda_\theta^k} + 4 \sum_{j \in \mathcal{S}} \frac{\langle \psi_\theta^j | (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) | \psi_\theta^j \rangle}{\lambda_\theta^j}. \quad (\text{C.32})$$

However, if this condition does not hold, then  $\|\Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp\|_2 > 0$  and the following limit holds

$$\lim_{\varepsilon \rightarrow 0} I_F(\theta; \{\rho_\theta^\varepsilon\}_\theta) = +\infty. \quad (\text{C.33})$$

Now consider that

$$\begin{aligned} &2 \sum_{j, k: \lambda_j + \lambda_k > 0} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{\lambda_\theta^j + \lambda_\theta^k} \\ &= 2 \sum_{j, k: (j \notin \mathcal{K} \wedge k \notin \mathcal{K})} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{\lambda_\theta^j + \lambda_\theta^k} \end{aligned} \quad (\text{C.34})$$

$$= 2 \left[ \sum_{j \in \mathcal{S}, k \in \mathcal{S}} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{\lambda_\theta^j + \lambda_\theta^k} + \sum_{j \in \mathcal{S}, k \in \mathcal{K}} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{\lambda_\theta^j + \lambda_\theta^k} + \sum_{j \in \mathcal{K}, k \in \mathcal{S}} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{\lambda_\theta^j + \lambda_\theta^k} \right] \quad (\text{C.35})$$

$$= 2 \left[ \sum_{j \in \mathcal{S}, k \in \mathcal{S}} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{\lambda_\theta^j + \lambda_\theta^k} + 2 \sum_{j \in \mathcal{S}} \frac{\langle \psi_\theta^j | (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) | \psi_\theta^j \rangle}{\lambda_\theta^j} \right], \quad (\text{C.36})$$

where we arrived at the last line by applying the previous reasoning. Thus, we find that if  $\Pi_{\rho_\theta}^\perp (\partial_\theta \rho_\theta) \Pi_{\rho_\theta}^\perp = 0$ , then

$$\lim_{\varepsilon \rightarrow 0} I_F(\theta; \{\rho_\theta^\varepsilon\}_\theta) = 2 \sum_{j,k: \lambda_j + \lambda_k > 0} \frac{|\langle \psi_\theta^j | (\partial_\theta \rho_\theta) | \psi_\theta^k \rangle|^2}{\lambda_\theta^j + \lambda_\theta^k}. \quad (\text{C.37})$$

This concludes the proof. ■

**Proof of Proposition 4.** Following the notation from the previous proof, it follows that

$$(\partial_\theta \rho_\theta^\varepsilon)^2 = (1 - \varepsilon)^2 (\partial_\theta \rho_\theta)^2 \quad (\text{C.38})$$

$$(\rho_\theta^\varepsilon)^{-1} = \sum_{j \in \mathcal{S}} \frac{1}{(1 - \varepsilon) \lambda_\theta^j + \frac{\varepsilon}{d}} |\psi_\theta^j\rangle\langle\psi_\theta^j| + \frac{d}{\varepsilon} \Pi_{\rho_\theta}^\perp, \quad (\text{C.39})$$

so that

$$\widehat{I}_F(\theta; \{\rho_\theta^\varepsilon\}_\theta) = \text{Tr}[(\partial_\theta \rho_\theta^\varepsilon)^2 (\rho_\theta^\varepsilon)^{-1}] \quad (\text{C.40})$$

$$\begin{aligned} &= (1 - \varepsilon)^2 \text{Tr} \left[ (\partial_\theta \rho_\theta)^2 \left( \sum_{j \in \mathcal{S}} \frac{1}{(1 - \varepsilon) \lambda_\theta^j + \frac{\varepsilon}{d}} |\psi_\theta^j\rangle\langle\psi_\theta^j| \right) \right] \\ &\quad + \frac{d(1 - \varepsilon)^2}{\varepsilon} \text{Tr}[(\partial_\theta \rho_\theta)^2 \Pi_{\rho_\theta}^\perp]. \end{aligned} \quad (\text{C.41})$$

The condition  $\text{Tr}[(\partial_\theta \rho_\theta)^2 \Pi_{\rho_\theta}^\perp] = 0$  is equivalent to the condition  $(\partial_\theta \rho_\theta)^2 \Pi_{\rho_\theta}^\perp = 0$  because both  $(\partial_\theta \rho_\theta)^2$  and  $\Pi_{\rho_\theta}^\perp$  are positive semi-definite. The condition  $(\partial_\theta \rho_\theta)^2 \Pi_{\rho_\theta}^\perp = 0$  is equivalent to the condition  $\text{supp}((\partial_\theta \rho_\theta)^2) \subseteq \text{supp}(\rho_\theta)$ . Since  $\text{supp}((\partial_\theta \rho_\theta)^2) = \text{supp}(\partial_\theta \rho_\theta)$ , this condition is in turn equivalent to  $\text{supp}(\partial_\theta \rho_\theta) \subseteq \text{supp}(\rho_\theta)$ . Thus,

$$\text{supp}(\partial_\theta \rho_\theta) \subseteq \text{supp}(\rho_\theta) \iff \text{Tr}[(\partial_\theta \rho_\theta)^2 \Pi_{\rho_\theta}^\perp] = 0, \quad (\text{C.42})$$

and we find that if  $\text{supp}(\partial_\theta \rho_\theta) \subseteq \text{supp}(\rho_\theta)$ , then

$$\lim_{\varepsilon \rightarrow 0} \widehat{I}_F(\theta; \{\rho_\theta^\varepsilon\}_\theta) = \lim_{\varepsilon \rightarrow 0} (1 - \varepsilon)^2 \text{Tr} \left[ (\partial_\theta \rho_\theta)^2 \left( \sum_{j \in \mathcal{S}} \frac{1}{(1 - \varepsilon) \lambda_\theta^j + \frac{\varepsilon}{d}} |\psi_\theta^j\rangle\langle\psi_\theta^j| \right) \right] \quad (\text{C.43})$$

$$= \text{Tr} \left[ (\partial_\theta \rho_\theta)^2 \left( \sum_{j \in \mathcal{S}} \frac{1}{\lambda_\theta^j} |\psi_\theta^j\rangle\langle\psi_\theta^j| \right) \right] \quad (\text{C.44})$$

$$= \text{Tr}[(\partial_\theta \rho_\theta)^2 \rho_\theta^{-1}]. \quad (\text{C.45})$$

On the other hand, if  $\text{supp}(\partial_\theta \rho_\theta) \not\subseteq \text{supp}(\rho_\theta)$ , then  $\text{Tr}[(\partial_\theta \rho_\theta)^2 \Pi_{\rho_\theta}^\perp] > 0$ , and  $\lim_{\varepsilon \rightarrow 0} \widehat{I}_F(\theta; \{\rho_\theta^\varepsilon\}_\theta) = +\infty$ . ■



## C.1 Pure-state family examples

**Proposition 63** *Let  $\{|\phi_\theta\rangle\langle\phi_\theta|\}_\theta$  be a differentiable family of pure states. Then the SLD Fisher information is as follows:*

$$I_F(\theta; \{|\phi_\theta\rangle\langle\phi_\theta|\}_\theta) = 4 [\langle\partial_\theta\phi_\theta|\partial_\theta\phi_\theta\rangle - |\langle\partial_\theta\phi_\theta|\phi_\theta\rangle|^2]. \quad (\text{C.46})$$

**Proof.** First, observe that

$$\partial_\theta(|\phi_\theta\rangle\langle\phi_\theta|) = |\partial_\theta\phi_\theta\rangle\langle\phi_\theta| + |\phi_\theta\rangle\langle\partial_\theta\phi_\theta|, \quad (\text{C.47})$$

which, when combined with  $\text{Tr}[\partial_\theta(|\phi_\theta\rangle\langle\phi_\theta|)] = \partial_\theta(\text{Tr}[|\phi_\theta\rangle\langle\phi_\theta|]) = 0$ , implies that

$$0 = \langle\phi_\theta|\partial_\theta\phi_\theta\rangle + \langle\partial_\theta\phi_\theta|\phi_\theta\rangle = 2 \text{Re}[\langle\partial_\theta\phi_\theta|\phi_\theta\rangle]. \quad (\text{C.48})$$

Now consider that the finiteness condition  $\Pi_{\phi_\theta}^\perp(\partial_\theta|\phi_\theta\rangle\langle\phi_\theta|)\Pi_{\phi_\theta}^\perp = 0$  holds for all differentiable pure-state families, where  $\Pi_{\phi_\theta}^\perp = I - |\phi_\theta\rangle\langle\phi_\theta|$ . This is because  $|\phi_\theta\rangle\langle\phi_\theta|\Pi_{\phi_\theta}^\perp = \Pi_{\phi_\theta}^\perp|\phi_\theta\rangle\langle\phi_\theta| = 0$ , so that

$$\Pi_{\phi_\theta}^\perp(\partial_\theta|\phi_\theta\rangle\langle\phi_\theta|)\Pi_{\phi_\theta}^\perp = \Pi_{\phi_\theta}^\perp(|\partial_\theta\phi_\theta\rangle\langle\phi_\theta| + |\phi_\theta\rangle\langle\partial_\theta\phi_\theta|)\Pi_{\phi_\theta}^\perp \quad (\text{C.49})$$

$$= \Pi_{\phi_\theta}^\perp|\partial_\theta\phi_\theta\rangle\langle\phi_\theta|\Pi_{\phi_\theta}^\perp + \Pi_{\phi_\theta}^\perp|\phi_\theta\rangle\langle\partial_\theta\phi_\theta|\Pi_{\phi_\theta}^\perp \quad (\text{C.50})$$

$$= 0. \quad (\text{C.51})$$

Then we can apply the general expression for the SLD Fisher information in (5.15):

$$\begin{aligned} I_F(\theta; \{|\phi_\theta\rangle\langle\phi_\theta|\}_\theta) &= |\langle\phi_\theta|(\partial_\theta(|\phi_\theta\rangle\langle\phi_\theta|))|\phi_\theta\rangle|^2 + 4\langle\phi_\theta|(\partial_\theta(|\phi_\theta\rangle\langle\phi_\theta|))\Pi_{\phi_\theta}^\perp(\partial_\theta(|\phi_\theta\rangle\langle\phi_\theta|))|\phi_\theta\rangle \\ &= |\langle\phi_\theta|(\partial_\theta(|\phi_\theta\rangle\langle\phi_\theta|))|\phi_\theta\rangle|^2 + 4\langle\phi_\theta|(\partial_\theta(|\phi_\theta\rangle\langle\phi_\theta|))(I - |\phi_\theta\rangle\langle\phi_\theta|)(\partial_\theta(|\phi_\theta\rangle\langle\phi_\theta|))|\phi_\theta\rangle \end{aligned} \quad (\text{C.52})$$

$$= |\langle\phi_\theta|(\partial_\theta(|\phi_\theta\rangle\langle\phi_\theta|))|\phi_\theta\rangle|^2 + 4\langle\phi_\theta|(\partial_\theta(|\phi_\theta\rangle\langle\phi_\theta|))(I - |\phi_\theta\rangle\langle\phi_\theta|)(\partial_\theta(|\phi_\theta\rangle\langle\phi_\theta|))|\phi_\theta\rangle \quad (\text{C.53})$$

$$= 4\langle\phi_\theta|(\partial_\theta(|\phi_\theta\rangle\langle\phi_\theta|))^2|\phi_\theta\rangle - 3|\langle\phi_\theta|(\partial_\theta(|\phi_\theta\rangle\langle\phi_\theta|))|\phi_\theta\rangle|^2. \quad (\text{C.54})$$

Then we find that

$$\langle\phi_\theta|(\partial_\theta(|\phi_\theta\rangle\langle\phi_\theta|))|\phi_\theta\rangle = \langle\phi_\theta|(|\partial_\theta\phi_\theta\rangle\langle\phi_\theta| + |\phi_\theta\rangle\langle\partial_\theta\phi_\theta|)|\phi_\theta\rangle \quad (\text{C.55})$$

$$= \langle\phi_\theta|\partial_\theta\phi_\theta\rangle + \langle\partial_\theta\phi_\theta|\phi_\theta\rangle \quad (\text{C.56})$$

$$= 0, \quad (\text{C.57})$$

where we applied (C.48) to get the last line. This implies that

$$I_F(\theta; \{|\phi_\theta\rangle\langle\phi_\theta|\}_\theta) = 4\langle\phi_\theta|(\partial_\theta(|\phi_\theta\rangle\langle\phi_\theta|))^2|\phi_\theta\rangle. \quad (\text{C.58})$$

Now consider that

$$\langle\phi_\theta|(\partial_\theta(|\phi_\theta\rangle\langle\phi_\theta|))^2|\phi_\theta\rangle$$

$$= \langle \phi_\theta | (|\partial_\theta \phi_\theta\rangle\langle \phi_\theta| + |\phi_\theta\rangle\langle \partial_\theta \phi_\theta|) (|\partial_\theta \phi_\theta\rangle\langle \phi_\theta| + |\phi_\theta\rangle\langle \partial_\theta \phi_\theta|) | \phi_\theta \rangle \quad (\text{C.59})$$

$$= \langle \phi_\theta | \partial_\theta \phi_\theta \rangle \langle \phi_\theta | \partial_\theta \phi_\theta \rangle \langle \phi_\theta | \phi_\theta \rangle + \langle \phi_\theta | \partial_\theta \phi_\theta \rangle \langle \phi_\theta | \phi_\theta \rangle \langle \partial_\theta \phi_\theta | \phi_\theta \rangle \\ + \langle \phi_\theta | \phi_\theta \rangle \langle \partial_\theta \phi_\theta | \partial_\theta \phi_\theta \rangle \langle \phi_\theta | \phi_\theta \rangle + \langle \phi_\theta | \phi_\theta \rangle \langle \partial_\theta \phi_\theta | \phi_\theta \rangle \langle \partial_\theta \phi_\theta | \phi_\theta \rangle \quad (\text{C.60})$$

$$= (\langle \phi_\theta | \partial_\theta \phi_\theta \rangle)^2 + |\langle \phi_\theta | \partial_\theta \phi_\theta \rangle|^2 + \langle \partial_\theta \phi_\theta | \partial_\theta \phi_\theta \rangle + (\langle \partial_\theta \phi_\theta | \phi_\theta \rangle)^2 \quad (\text{C.61})$$

$$= (\langle \phi_\theta | \partial_\theta \phi_\theta \rangle)^2 + 2 |\langle \phi_\theta | \partial_\theta \phi_\theta \rangle|^2 + (\langle \partial_\theta \phi_\theta | \phi_\theta \rangle)^2 + \langle \partial_\theta \phi_\theta | \partial_\theta \phi_\theta \rangle - |\langle \phi_\theta | \partial_\theta \phi_\theta \rangle|^2 \quad (\text{C.62})$$

$$= |\langle \phi_\theta | \partial_\theta \phi_\theta \rangle + \langle \partial_\theta \phi_\theta | \phi_\theta \rangle|^2 + \langle \partial_\theta \phi_\theta | \partial_\theta \phi_\theta \rangle - |\langle \phi_\theta | \partial_\theta \phi_\theta \rangle|^2 \quad (\text{C.63})$$

$$= \langle \partial_\theta \phi_\theta | \partial_\theta \phi_\theta \rangle - |\langle \phi_\theta | \partial_\theta \phi_\theta \rangle|^2, \quad (\text{C.64})$$

where we again applied (C.48) to get the last line. Substituting into (C.58), we arrive at the statement of the proposition. ■

**Proposition 64** *Let  $\{|\phi_\theta\rangle\langle \phi_\theta|\}_\theta$  be a differentiable family of pure states. If the family is constant, so that  $|\varphi_\theta\rangle = |\varphi\rangle$  for all  $\theta$ , then the RLD Fisher information is equal to zero. Otherwise, the RLD Fisher information is infinite.*

**Proof.** The RLD Fisher information is finite if and only if the finiteness condition in (5.28) is satisfied. This condition is equivalent to the following:  $0 = \text{Tr}[\Pi_{\phi_\theta}^\perp (\partial_\theta (|\phi_\theta\rangle\langle \phi_\theta|))^2]$ . Now consider that

$$\text{Tr}[\Pi_{\phi_\theta}^\perp (\partial_\theta (|\phi_\theta\rangle\langle \phi_\theta|))^2] \\ = \text{Tr}[\Pi_{\phi_\theta}^\perp (|\partial_\theta \phi_\theta\rangle\langle \phi_\theta| + |\phi_\theta\rangle\langle \partial_\theta \phi_\theta|) (|\partial_\theta \phi_\theta\rangle\langle \phi_\theta| + |\phi_\theta\rangle\langle \partial_\theta \phi_\theta|)] \quad (\text{C.65})$$

$$= \text{Tr}[\Pi_{\phi_\theta}^\perp |\partial_\theta \phi_\theta\rangle\langle \phi_\theta| \partial_\theta \phi_\theta \langle \phi_\theta|] + \text{Tr}[\Pi_{\phi_\theta}^\perp |\partial_\theta \phi_\theta\rangle\langle \phi_\theta| \phi_\theta \langle \partial_\theta \phi_\theta|] \\ + \text{Tr}[\Pi_{\phi_\theta}^\perp |\phi_\theta\rangle\langle \partial_\theta \phi_\theta| \partial_\theta \phi_\theta \langle \phi_\theta|] + \text{Tr}[\Pi_{\phi_\theta}^\perp |\phi_\theta\rangle\langle \partial_\theta \phi_\theta| \phi_\theta \langle \partial_\theta \phi_\theta|] \quad (\text{C.66})$$

$$= \langle \partial_\theta \phi_\theta | \Pi_{\phi_\theta}^\perp | \partial_\theta \phi_\theta \rangle \quad (\text{C.67})$$

$$= \langle \partial_\theta \phi_\theta | \partial_\theta \phi_\theta \rangle - |\langle \partial_\theta \phi_\theta | \phi_\theta \rangle|^2. \quad (\text{C.68})$$

From Proposition 63, it follows that  $\langle \partial_\theta \phi_\theta | \partial_\theta \phi_\theta \rangle - |\langle \partial_\theta \phi_\theta | \phi_\theta \rangle|^2 = I_F(\theta; \{|\phi_\theta\rangle\langle \phi_\theta|\}_\theta)$ . Then, by the faithfulness of SLD Fisher information from Proposition 5, it follows that  $\{|\phi_\theta\rangle\langle \phi_\theta|\}_\theta$  is a constant family. ■

## D Additivity of SLD and RLD Fisher informations

**Proof of Proposition 6.** Let us begin with the SLD Fisher information. We are trying to prove the following statement: Let  $\{\rho_A^\theta\}_\theta$  and  $\{\sigma_B^\theta\}_\theta$  be differentiable families of quantum states. The SLD Fisher information is additive in the following sense:

$$I_F(\theta; \{\rho_A^\theta \otimes \sigma_B^\theta\}_\theta) = I_F(\theta; \{\rho_A^\theta\}_\theta) + I_F(\theta; \{\sigma_B^\theta\}_\theta). \quad (\text{D.1})$$

Let us first consider the finiteness condition in (5.11). For the quantities on the right-hand side of (D.1), the finiteness conditions are

$$\Pi_{\rho_A^\theta}^\perp (\partial_\theta \rho_A^\theta) \Pi_{\rho_A^\theta}^\perp = 0 \quad \wedge \quad \Pi_{\sigma_B^\theta}^\perp (\partial_\theta \sigma_B^\theta) \Pi_{\sigma_B^\theta}^\perp = 0. \quad (\text{D.2})$$

For the quantity on the left-hand side of (D.1), the finiteness condition is

$$\Pi_{\rho_A^\theta \otimes \sigma_B^\theta}^\perp (\partial_\theta (\rho_A^\theta \otimes \sigma_B^\theta)) \Pi_{\rho_A^\theta \otimes \sigma_B^\theta}^\perp = 0. \quad (\text{D.3})$$

We now show that these conditions are equivalent. Consider that

$$\Pi_{\rho_A^\theta \otimes \sigma_B^\theta} = \Pi_{\rho_A^\theta} \otimes \Pi_{\sigma_B^\theta}. \quad (\text{D.4})$$

This implies that

$$\Pi_{\rho_A^\theta \otimes \sigma_B^\theta}^\perp = I_{AB} - \Pi_{\rho_A^\theta} \otimes \Pi_{\sigma_B^\theta} \quad (\text{D.5})$$

$$= \Pi_{\rho_A^\theta}^\perp \otimes \Pi_{\sigma_B^\theta}^\perp + \Pi_{\rho_A^\theta}^\perp \otimes \Pi_{\sigma_B^\theta} + \Pi_{\rho_A^\theta} \otimes \Pi_{\sigma_B^\theta}^\perp. \quad (\text{D.6})$$

Consider that

$$\partial_\theta (\rho_A^\theta \otimes \sigma_B^\theta) = (\partial_\theta \rho_A^\theta) \otimes \sigma_B^\theta + \rho_A^\theta \otimes (\partial_\theta \sigma_B^\theta). \quad (\text{D.7})$$

Then

$$\begin{aligned} & \Pi_{\rho_A^\theta \otimes \sigma_B^\theta}^\perp (\partial_\theta (\rho_A^\theta \otimes \sigma_B^\theta)) \Pi_{\rho_A^\theta \otimes \sigma_B^\theta}^\perp \\ &= \left( \Pi_{\rho_A^\theta}^\perp \otimes \Pi_{\sigma_B^\theta}^\perp + \Pi_{\rho_A^\theta}^\perp \otimes \Pi_{\sigma_B^\theta} + \Pi_{\rho_A^\theta} \otimes \Pi_{\sigma_B^\theta}^\perp \right) \left( (\partial_\theta \rho_A^\theta) \otimes \sigma_B^\theta + \rho_A^\theta \otimes (\partial_\theta \sigma_B^\theta) \right) \\ & \quad \times \left( \Pi_{\rho_A^\theta}^\perp \otimes \Pi_{\sigma_B^\theta}^\perp + \Pi_{\rho_A^\theta}^\perp \otimes \Pi_{\sigma_B^\theta} + \Pi_{\rho_A^\theta} \otimes \Pi_{\sigma_B^\theta}^\perp \right) \end{aligned} \quad (\text{D.8})$$

$$\begin{aligned} &= \left( \Pi_{\rho_A^\theta}^\perp \otimes \Pi_{\sigma_B^\theta}^\perp \right) \left( (\partial_\theta \rho_A^\theta) \otimes \sigma_B^\theta \right) \left( \Pi_{\rho_A^\theta}^\perp \otimes \Pi_{\sigma_B^\theta}^\perp \right) \\ & \quad + \left( \Pi_{\rho_A^\theta} \otimes \Pi_{\sigma_B^\theta}^\perp \right) \left( \rho_A^\theta \otimes (\partial_\theta \sigma_B^\theta) \right) \left( \Pi_{\rho_A^\theta} \otimes \Pi_{\sigma_B^\theta}^\perp \right) \end{aligned} \quad (\text{D.9})$$

$$= \Pi_{\rho_A^\theta}^\perp (\partial_\theta \rho_A^\theta) \Pi_{\rho_A^\theta}^\perp \otimes \sigma_B^\theta + \rho_A^\theta \otimes \Pi_{\sigma_B^\theta}^\perp (\partial_\theta \sigma_B^\theta) \Pi_{\sigma_B^\theta}^\perp. \quad (\text{D.10})$$

From this we see that  $\Pi_{\rho_A^\theta \otimes \sigma_B^\theta}^\perp (\partial_\theta (\rho_A^\theta \otimes \sigma_B^\theta)) \Pi_{\rho_A^\theta \otimes \sigma_B^\theta}^\perp = 0$  if (D.2) holds. Now suppose that  $\Pi_{\rho_A^\theta \otimes \sigma_B^\theta}^\perp (\partial_\theta (\rho_A^\theta \otimes \sigma_B^\theta)) \Pi_{\rho_A^\theta \otimes \sigma_B^\theta}^\perp = 0$  holds. Then we can sandwich this equation by  $I_A \otimes \Pi_{\sigma_B^\theta}$  and perform a partial trace over  $B$  to conclude that  $\Pi_{\rho_A^\theta}^\perp (\partial_\theta \rho_A^\theta) \Pi_{\rho_A^\theta}^\perp = 0$ , i.e.,

$$\left( I \otimes \Pi_{\sigma_B^\theta} \right) \left[ \Pi_{\rho_A^\theta \otimes \sigma_B^\theta}^\perp (\partial_\theta (\rho_A^\theta \otimes \sigma_B^\theta)) \Pi_{\rho_A^\theta \otimes \sigma_B^\theta}^\perp \right] \left( I \otimes \Pi_{\sigma_B^\theta} \right) = \Pi_{\rho_A^\theta}^\perp (\partial_\theta \rho_A^\theta) \Pi_{\rho_A^\theta}^\perp \otimes \sigma_B^\theta. \quad (\text{D.11})$$

Similarly, we can sandwich by  $\Pi_{\rho_A^\theta} \otimes I_B$  and perform a partial trace over  $A$  to conclude that  $\Pi_{\sigma_B^\theta}^\perp (\partial_\theta \sigma_B^\theta) \Pi_{\sigma_B^\theta}^\perp = 0$ .

Due to the equivalence of the conditions in (D.2) and (D.3), it follows that the left-hand side of (D.1) is infinite if and only if the right-hand side of (D.1) is infinite. So we can analyze the case in which the quantities are finite by making use of the explicit formula in (5.10).

Consider the following spectral decompositions of  $\rho_A^\theta$  and  $\sigma_B^\theta$ :

$$\rho_A^\theta = \sum_x \lambda_x^\theta |\psi_x^\theta\rangle \langle \psi_x^\theta|, \quad \sigma_B^\theta = \sum_y \mu_y^\theta |\varphi_y^\theta\rangle \langle \varphi_y^\theta|. \quad (\text{D.12})$$

Plugging into the formula for SLD Fisher information from (5.10), while observing that

$$\partial_\theta(\rho_A^\theta \otimes \sigma_B^\theta) = (\partial_\theta \rho_A^\theta) \otimes \sigma_B^\theta + \rho_A^\theta \otimes (\partial_\theta \sigma_B^\theta), \quad (\text{D.13})$$

we find that

$$\begin{aligned} I_F(\theta; \{\rho_A^\theta \otimes \sigma_B^\theta\}_\theta) \\ = 2 \sum_{\substack{x,y,x',y': \\ \lambda_x^\theta \mu_y^\theta + \lambda_{x'}^\theta \mu_{y'}^\theta > 0}} \frac{|\langle \psi_x^\theta |_A \langle \varphi_y^\theta |_B (\partial_\theta(\rho_A^\theta \otimes \sigma_B^\theta)) | \psi_{x'}^\theta \rangle | \varphi_{y'}^\theta \rangle|^2}{\lambda_x^\theta \mu_y^\theta + \lambda_{x'}^\theta \mu_{y'}^\theta} \end{aligned} \quad (\text{D.14})$$

$$= 2 \sum_{\substack{x,y,x',y': \\ \lambda_x^\theta \mu_y^\theta + \lambda_{x'}^\theta \mu_{y'}^\theta > 0}} \frac{|\langle \psi_x^\theta |_A \langle \varphi_y^\theta |_B ((\partial_\theta \rho_A^\theta) \otimes \sigma_B^\theta + \rho_A^\theta \otimes (\partial_\theta \sigma_B^\theta)) | \psi_{x'}^\theta \rangle | \varphi_{y'}^\theta \rangle|^2}{\lambda_x^\theta \mu_y^\theta + \lambda_{x'}^\theta \mu_{y'}^\theta} \quad (\text{D.15})$$

Then consider that

$$\begin{aligned} & |\langle \psi_x^\theta |_A \langle \varphi_y^\theta |_B ((\partial_\theta \rho_A^\theta) \otimes \sigma_B^\theta + \rho_A^\theta \otimes (\partial_\theta \sigma_B^\theta)) | \psi_{x'}^\theta \rangle | \varphi_{y'}^\theta \rangle|^2 \\ &= |\mu_y^\theta \delta_{y,y'} \langle \psi_x^\theta |_A (\partial_\theta \rho_A^\theta) | \psi_{x'}^\theta \rangle + \lambda_x^\theta \delta_{x,x'} \langle \varphi_y^\theta |_B (\partial_\theta \sigma_B^\theta) | \varphi_{y'}^\theta \rangle|^2 \end{aligned} \quad (\text{D.16})$$

$$\begin{aligned} &= (\mu_y^\theta)^2 \delta_{y,y'} |\langle \psi_x^\theta |_A (\partial_\theta \rho_A^\theta) | \psi_{x'}^\theta \rangle|^2 \\ &\quad + \mu_y^\theta \lambda_x^\theta \delta_{y,y'} \delta_{x,x'} 2 \operatorname{Re} [\langle \psi_x^\theta |_A (\partial_\theta \rho_A^\theta) | \psi_{x'}^\theta \rangle \langle \varphi_y^\theta |_B (\partial_\theta \sigma_B^\theta) | \varphi_{y'}^\theta \rangle] \\ &\quad + (\lambda_x^\theta)^2 \delta_{x,x'} |\langle \varphi_y^\theta |_B (\partial_\theta \sigma_B^\theta) | \varphi_{y'}^\theta \rangle|^2. \end{aligned} \quad (\text{D.17})$$

Plugging back into (D.15) and evaluating each of the three terms separately, we find that

$$\begin{aligned} & 2 \sum_{\substack{x,y,x',y': \\ \lambda_x^\theta \mu_y^\theta + \lambda_{x'}^\theta \mu_{y'}^\theta > 0}} \frac{(\mu_y^\theta)^2 \delta_{y,y'} |\langle \psi_x^\theta |_A (\partial_\theta \rho_A^\theta) | \psi_{x'}^\theta \rangle|^2}{\lambda_x^\theta \mu_y^\theta + \lambda_{x'}^\theta \mu_{y'}^\theta} \\ &= 2 \sum_{\substack{x,y,x',y': \\ \mu_y^\theta (\lambda_x^\theta + \lambda_{x'}^\theta) > 0}} \frac{(\mu_y^\theta)^2 |\langle \psi_x^\theta |_A (\partial_\theta \rho_A^\theta) | \psi_{x'}^\theta \rangle|^2}{\mu_y^\theta (\lambda_x^\theta + \lambda_{x'}^\theta)} \end{aligned} \quad (\text{D.18})$$

$$= 2 \sum_{\substack{x,y,x',y': \\ \lambda_x^\theta + \lambda_{x'}^\theta > 0, \mu_y^\theta > 0}} \frac{(\mu_y^\theta)^2 |\langle \psi_x^\theta |_A (\partial_\theta \rho_A^\theta) | \psi_{x'}^\theta \rangle|^2}{\mu_y^\theta (\lambda_x^\theta + \lambda_{x'}^\theta)} \quad (\text{D.19})$$

$$= 2 \sum_{\substack{x,y,x': \lambda_x^\theta + \lambda_{x'}^\theta > 0, \mu_y^\theta > 0}} \frac{\mu_y^\theta |\langle \psi_x^\theta |_A (\partial_\theta \rho_A^\theta) | \psi_{x'}^\theta \rangle|^2}{\lambda_x^\theta + \lambda_{x'}^\theta} \quad (\text{D.20})$$

$$= 2 \sum_{x,x': \lambda_x^\theta + \lambda_{x'}^\theta > 0} \frac{|\langle \psi_x^\theta |_A (\partial_\theta \rho_A^\theta) | \psi_{x'}^\theta \rangle|^2}{\lambda_x^\theta + \lambda_{x'}^\theta} \sum_{y: \mu_y^\theta > 0} \mu_y^\theta \quad (\text{D.21})$$

$$= 2 \sum_{x,x': \lambda_x^\theta + \lambda_{x'}^\theta > 0} \frac{|\langle \psi_x^\theta |_A (\partial_\theta \rho_A^\theta) | \psi_{x'}^\theta \rangle|^2}{\lambda_x^\theta + \lambda_{x'}^\theta} \quad (\text{D.22})$$

$$= I_F(\theta; \{\rho_A^\theta\}_\theta). \quad (\text{D.23})$$

For the second term:

$$\begin{aligned} & 2 \sum_{\substack{x,y,x',y': \\ \lambda_x^\theta \mu_y^\theta + \lambda_{x'}^\theta \mu_{y'}^\theta > 0}} \frac{\mu_y^\theta \lambda_x^\theta \delta_{y,y'} \delta_{x,x'} 2 \operatorname{Re} [\langle \psi_x^\theta |_A (\partial_\theta \rho_A^\theta) | \psi_{x'}^\theta \rangle \langle \varphi_y^\theta |_B (\partial_\theta \sigma_B^\theta) | \varphi_{y'}^\theta \rangle]}{\lambda_x^\theta \mu_y^\theta + \lambda_{x'}^\theta \mu_{y'}^\theta} \\ &= \sum_{\substack{x,y: \\ \lambda_x^\theta \mu_y^\theta > 0}} \frac{\mu_y^\theta \lambda_x^\theta 2 \operatorname{Re} [\langle \psi_x^\theta |_A (\partial_\theta \rho_A^\theta) | \psi_x^\theta \rangle \langle \varphi_y^\theta |_B (\partial_\theta \sigma_B^\theta) | \varphi_y^\theta \rangle]}{\lambda_x^\theta \mu_y^\theta} \end{aligned} \quad (\text{D.24})$$

$$= \sum_{\substack{x,y: \\ \lambda_x^\theta > 0, \mu_y^\theta > 0}} \frac{\mu_y^\theta \lambda_x^\theta 2 \operatorname{Re} [\langle \psi_x^\theta |_A (\partial_\theta \rho_A^\theta) | \psi_x^\theta \rangle \langle \varphi_y^\theta |_B (\partial_\theta \sigma_B^\theta) | \varphi_y^\theta \rangle]}{\lambda_x^\theta \mu_y^\theta} \quad (\text{D.25})$$

$$= 2 \sum_{\substack{x,y: \\ \lambda_x^\theta > 0, \mu_y^\theta > 0}} \operatorname{Re} [\langle \psi_x^\theta |_A (\partial_\theta \rho_A^\theta) | \psi_x^\theta \rangle \langle \varphi_y^\theta |_B (\partial_\theta \sigma_B^\theta) | \varphi_y^\theta \rangle] \quad (\text{D.26})$$

$$= 2 \operatorname{Re} \left[ \sum_{x: \lambda_x^\theta > 0} \langle \psi_x^\theta |_A (\partial_\theta \rho_A^\theta) | \psi_x^\theta \rangle \sum_{y: \mu_y^\theta > 0} \langle \varphi_y^\theta |_B (\partial_\theta \sigma_B^\theta) | \varphi_y^\theta \rangle \right] \quad (\text{D.27})$$

$$= 2 \operatorname{Re} \left[ \sum_x \langle \psi_x^\theta |_A (\partial_\theta \rho_A^\theta) | \psi_x^\theta \rangle \sum_y \langle \varphi_y^\theta |_B (\partial_\theta \sigma_B^\theta) | \varphi_y^\theta \rangle \right] \quad (\text{D.28})$$

$$= 2 \operatorname{Re} [\operatorname{Tr}[\partial_\theta \rho_A^\theta] \operatorname{Tr}[\partial_\theta \sigma_B^\theta]] \quad (\text{D.29})$$

$$= 0. \quad (\text{D.30})$$

The third-to-last equality follows because (D.2) holds, so that we can add these to the sums to complete the basis for the trace. The last equality follows because  $\operatorname{Tr}[\partial_\theta \rho_A^\theta] = \operatorname{Tr}[\partial_\theta \sigma_B^\theta] = 0$ . The analysis involving the last term  $(\lambda_x^\theta)^2 \delta_{x,x'} |\langle \varphi_y^\theta |_B (\partial_\theta \sigma_B^\theta) | \varphi_{y'}^\theta \rangle|^2$  is similar to that of the first term, and it evaluates to  $I_F(\theta; \{\sigma_B^\theta\}_\theta)$ .

Now let us turn to the RLD Fisher information. We are trying to prove the following statement: Let  $\{\rho_A^\theta\}_\theta$  and  $\{\sigma_B^\theta\}_\theta$  be differentiable families of quantum states. The RLD Fisher information is additive in the following sense:

$$\widehat{I}_F(\theta; \{\rho_A^\theta \otimes \sigma_B^\theta\}_\theta) = \widehat{I}_F(\theta; \{\rho_A^\theta\}_\theta) + \widehat{I}_F(\theta; \{\sigma_B^\theta\}_\theta). \quad (\text{D.31})$$

Let us begin by considering the finiteness condition in (5.28) for RLD Fisher information. For the quantities on the right-hand side of (D.31), the finiteness conditions are

$$\Pi_{\rho_A^\theta}^\perp (\partial_\theta \rho_A^\theta) = 0 \quad \wedge \quad \Pi_{\sigma_B^\theta}^\perp (\partial_\theta \sigma_B^\theta) = 0. \quad (\text{D.32})$$

For the quantity on the left-hand side of (D.31), the finiteness condition is

$$\Pi_{\rho_A^\theta \otimes \sigma_B^\theta}^\perp (\partial_\theta (\rho_A^\theta \otimes \sigma_B^\theta)) = 0. \quad (\text{D.33})$$

We now show that these conditions are equivalent. Consider that

$$\Pi_{\rho_A^\theta \otimes \sigma_B^\theta} = \Pi_{\rho_A^\theta} \otimes \Pi_{\sigma_B^\theta}. \quad (\text{D.34})$$

This implies that

$$\Pi_{\rho_A^\theta \otimes \sigma_B^\theta}^\perp = I_{AB} - \Pi_{\rho_A^\theta} \otimes \Pi_{\sigma_B^\theta} \quad (\text{D.35})$$

$$= \Pi_{\rho_A^\theta}^\perp \otimes \Pi_{\sigma_B^\theta}^\perp + \Pi_{\rho_A^\theta}^\perp \otimes \Pi_{\sigma_B^\theta} + \Pi_{\rho_A^\theta} \otimes \Pi_{\sigma_B^\theta}^\perp. \quad (\text{D.36})$$

Consider that

$$\partial_\theta (\rho_A^\theta \otimes \sigma_B^\theta) = (\partial_\theta \rho_A^\theta) \otimes \sigma_B^\theta + \rho_A^\theta \otimes (\partial_\theta \sigma_B^\theta). \quad (\text{D.37})$$

Then we find that

$$\begin{aligned} & \Pi_{\rho_A^\theta \otimes \sigma_B^\theta}^\perp \partial_\theta (\rho_A^\theta \otimes \sigma_B^\theta) \\ &= \left( \Pi_{\rho_A^\theta}^\perp \otimes \Pi_{\sigma_B^\theta}^\perp + \Pi_{\rho_A^\theta}^\perp \otimes \Pi_{\sigma_B^\theta} + \Pi_{\rho_A^\theta} \otimes \Pi_{\sigma_B^\theta}^\perp \right) ((\partial_\theta \rho_A^\theta) \otimes \sigma_B^\theta + \rho_A^\theta \otimes (\partial_\theta \sigma_B^\theta)) \end{aligned} \quad (\text{D.38})$$

$$= \left( \Pi_{\rho_A^\theta}^\perp \otimes \Pi_{\sigma_B^\theta}^\perp \right) ((\partial_\theta \rho_A^\theta) \otimes \sigma_B^\theta) + \left( \Pi_{\rho_A^\theta} \otimes \Pi_{\sigma_B^\theta}^\perp \right) (\rho_A^\theta \otimes (\partial_\theta \sigma_B^\theta)) \quad (\text{D.39})$$

$$= \Pi_{\rho_A^\theta}^\perp (\partial_\theta \rho_A^\theta) \otimes \sigma_B^\theta + \rho_A^\theta \otimes \Pi_{\sigma_B^\theta}^\perp (\partial_\theta \sigma_B^\theta). \quad (\text{D.40})$$

From this we see that  $\Pi_{\rho_A^\theta \otimes \sigma_B^\theta}^\perp (\partial_\theta (\rho_A^\theta \otimes \sigma_B^\theta)) = 0$  if (D.32) holds. Now suppose that  $\Pi_{\rho_A^\theta \otimes \sigma_B^\theta}^\perp (\partial_\theta (\rho_A^\theta \otimes \sigma_B^\theta)) = 0$  holds. Then we can left-multiply this equation by  $I_A \otimes \Pi_{\sigma_B^\theta}$  and perform a partial trace over  $B$  to conclude that  $\Pi_{\rho_A^\theta}^\perp (\partial_\theta \rho_A^\theta) = 0$ , i.e.,

$$\left( I \otimes \Pi_{\sigma_B^\theta} \right) \left[ \Pi_{\rho_A^\theta \otimes \sigma_B^\theta}^\perp (\partial_\theta (\rho_A^\theta \otimes \sigma_B^\theta)) \right] = \Pi_{\rho_A^\theta}^\perp (\partial_\theta \rho_A^\theta) \otimes \sigma_B^\theta. \quad (\text{D.41})$$

Similarly, we can left-multiply by  $\Pi_{\rho_A^\theta} \otimes I_B$  and perform a partial trace over  $A$  to conclude that  $\Pi_{\sigma_B^\theta}^\perp (\partial_\theta \sigma_B^\theta) = 0$ .

Due to the equivalence of the conditions in (D.32) and (D.33), it follows that the left-hand side of (D.31) is infinite if and only if the right-hand side of (D.31) is infinite. So we can analyze the case in which the quantities are finite by making use of the explicit formula in Definition 3.

Observe that

$$(\partial_\theta (\rho_A^\theta \otimes \sigma_B^\theta))^2 = ((\partial_\theta \rho_A^\theta) \otimes \sigma_B^\theta + \rho_A^\theta \otimes (\partial_\theta \sigma_B^\theta))^2 \quad (\text{D.42})$$

$$\begin{aligned} &= (\partial_\theta \rho_A^\theta)^2 \otimes (\sigma_B^\theta)^2 + (\partial_\theta \rho_A^\theta) \rho_A^\theta \otimes \sigma_B^\theta (\partial_\theta \sigma_B^\theta) \\ &\quad + \rho_A^\theta (\partial_\theta \rho_A^\theta) \otimes (\partial_\theta \sigma_B^\theta) \sigma_B^\theta + (\rho_A^\theta)^2 \otimes (\partial_\theta \sigma_B^\theta)^2. \end{aligned} \quad (\text{D.43})$$

Then consider that

$$\begin{aligned} \widehat{I}_F(\theta; \{\rho_A^\theta \otimes \sigma_B^\theta\}_\theta) \\ = \text{Tr}[(\partial_\theta(\rho_A^\theta \otimes \sigma_B^\theta))^2(\rho_A^\theta \otimes \sigma_B^\theta)^{-1}] \end{aligned} \quad (\text{D.44})$$

$$\begin{aligned} &= \text{Tr}[(\partial_\theta \rho_A^\theta)^2 \otimes (\sigma_B^\theta)^2((\rho_A^\theta)^{-1} \otimes (\sigma_B^\theta)^{-1})] \\ &\quad + \text{Tr}[(\partial_\theta \rho_A^\theta) \rho_A^\theta \otimes \sigma_B^\theta (\partial_\theta \sigma_B^\theta)((\rho_A^\theta)^{-1} \otimes (\sigma_B^\theta)^{-1})] \\ &\quad + \text{Tr}[(\rho_A^\theta (\partial_\theta \rho_A^\theta) \otimes (\partial_\theta \sigma_B^\theta) \sigma_B^\theta)((\rho_A^\theta)^{-1} \otimes (\sigma_B^\theta)^{-1})] \\ &\quad + \text{Tr}[(\rho_A^\theta)^2 \otimes (\partial_\theta \sigma_B^\theta)^2((\rho_A^\theta)^{-1} \otimes (\sigma_B^\theta)^{-1})] \end{aligned} \quad (\text{D.45})$$

$$\begin{aligned} &= \text{Tr}[(\partial_\theta \rho_A^\theta)^2(\rho_A^\theta)^{-1}] \text{Tr}[\sigma_B^\theta] + 2 \text{Tr}[\Pi_{\rho_A^\theta}(\partial_\theta \rho_A^\theta)] \text{Tr}[\Pi_{\sigma_B^\theta}(\partial_\theta \sigma_B^\theta)] \\ &\quad + \text{Tr}[\rho_A^\theta] \text{Tr}[(\partial_\theta \sigma_B^\theta)^2(\sigma_B^\theta)^{-1}] \end{aligned} \quad (\text{D.46})$$

$$= \text{Tr}[(\partial_\theta \rho_A^\theta)^2(\rho_A^\theta)^{-1}] + 2 \text{Tr}[(\partial_\theta \rho_A^\theta)] \text{Tr}[(\partial_\theta \sigma_B^\theta)] + \text{Tr}[(\partial_\theta \sigma_B^\theta)^2(\sigma_B^\theta)^{-1}] \quad (\text{D.47})$$

$$= \widehat{I}_F(\theta; \{\rho_A^\theta\}_\theta) + \widehat{I}_F(\theta; \{\sigma_B^\theta\}_\theta). \quad (\text{D.48})$$

The second-to-last equality follows because  $\Pi_{\rho_A^\theta}^\perp(\partial_\theta \rho_A^\theta) = 0$  and  $\Pi_{\sigma_B^\theta}^\perp(\partial_\theta \sigma_B^\theta) = 0$ , so that  $\text{Tr}[\Pi_{\rho_A^\theta}(\partial_\theta \rho_A^\theta)] = \text{Tr}[(\partial_\theta \rho_A^\theta)]$  and  $\text{Tr}[\Pi_{\sigma_B^\theta}(\partial_\theta \sigma_B^\theta)] = \text{Tr}[(\partial_\theta \sigma_B^\theta)]$ . The final equality follows because  $\text{Tr}[(\partial_\theta \rho_A^\theta)] = \text{Tr}[(\partial_\theta \sigma_B^\theta)] = 0$ . ■

## E SLD and RLD Fisher informations for classical–quantum states

**Proof of Proposition 7.** We begin with the SLD Fisher information, with the goal being to prove the following statement: For a differentiable family of classical–quantum states:

$$\left\{ \sum_x p_\theta(x) |x\rangle\langle x|_X \otimes \rho_\theta^x \right\}_\theta, \quad (\text{E.1})$$

the SLD Fisher information can be evaluated as follows:

$$I_F\left(\theta; \left\{ \sum_x p_\theta(x) |x\rangle\langle x|_X \otimes \rho_\theta^x \right\}_\theta\right) = I_F(\theta; \{p_\theta\}_\theta) + \sum_{x: p_\theta(x) > 0} p_\theta(x) I_F(\theta; \{\rho_\theta^x\}_\theta). \quad (\text{E.2})$$

We first consider the finiteness conditions for the left- and right-hand sides of (E.2) and show that they are equivalent. For the right-hand side, the finiteness conditions are

$$\text{supp}(\partial_\theta p_\theta) \subseteq \text{supp}(p_\theta) \quad \wedge \quad \Pi_{\rho_\theta^x}^\perp(\partial_\theta \rho_\theta^x) \Pi_{\rho_\theta^x}^\perp = 0 \quad \forall x : p_\theta(x) > 0, \quad (\text{E.3})$$

while for the left-hand side, the finiteness condition is

$$\Pi_{\rho_{XB}^\theta}^\perp(\partial_\theta \rho_{XB}^\theta) \Pi_{\rho_{XB}^\theta}^\perp = 0, \quad (\text{E.4})$$

where

$$\rho_{XB}^\theta := \sum_x p_\theta(x) |x\rangle\langle x|_X \otimes \rho_\theta^x. \quad (\text{E.5})$$

Consider that

$$\Pi_{\rho_{XB}^\theta} = \sum_{x:p_\theta(x)>0} |x\rangle\langle x|_X \otimes \Pi_{\rho_\theta^x}, \quad (\text{E.6})$$

which implies that

$$\Pi_{\rho_{XB}^\theta}^\perp = I_{XB} - \Pi_{\rho_{XB}^\theta} \quad (\text{E.7})$$

$$= \sum_{x:p_\theta(x)=0} |x\rangle\langle x|_X \otimes I + \sum_{x:p_\theta(x)>0} |x\rangle\langle x|_X \otimes \Pi_{\rho_\theta^x}^\perp. \quad (\text{E.8})$$

Also, we have that

$$\partial_\theta \rho_{XB}^\theta = \partial_\theta \left( \sum_x p_\theta(x) |x\rangle\langle x|_X \otimes \rho_\theta^x \right) \quad (\text{E.9})$$

$$= \partial_\theta \left( \sum_x |x\rangle\langle x|_X \otimes p_\theta(x) \rho_\theta^x \right) \quad (\text{E.10})$$

$$= \sum_x |x\rangle\langle x|_X \otimes \partial_\theta(p_\theta(x) \rho_\theta^x) \quad (\text{E.11})$$

$$= \sum_x |x\rangle\langle x|_X \otimes [\partial_\theta(p_\theta(x)) \rho_\theta^x + p_\theta(x) (\partial_\theta \rho_\theta^x)]. \quad (\text{E.12})$$

$$= \sum_x \partial_\theta(p_\theta(x)) |x\rangle\langle x|_X \otimes \rho_\theta^x + \sum_x p_\theta(x) |x\rangle\langle x|_X \otimes (\partial_\theta \rho_\theta^x). \quad (\text{E.13})$$

We then find that

$$\begin{aligned} 0 &= \Pi_{\rho_{XB}^\theta}^\perp (\partial_\theta \rho_{XB}^\theta) \Pi_{\rho_{XB}^\theta}^\perp \\ &= \left( \sum_{x:p_\theta(x)=0} |x\rangle\langle x|_X \otimes I + \sum_{x:p_\theta(x)>0} |x\rangle\langle x|_X \otimes \Pi_{\rho_\theta^x}^\perp \right) \times \\ &\quad \left( \sum_x \partial_\theta(p_\theta(x)) |x\rangle\langle x|_X \otimes \rho_\theta^x + \sum_x p_\theta(x) |x\rangle\langle x|_X \otimes (\partial_\theta \rho_\theta^x) \right) \times \\ &\quad \left( \sum_{x:p_\theta(x)=0} |x\rangle\langle x|_X \otimes I + \sum_{x:p_\theta(x)>0} |x\rangle\langle x|_X \otimes \Pi_{\rho_\theta^x}^\perp \right) \end{aligned} \quad (\text{E.14})$$

$$= \sum_{x:p_\theta(x)=0} \partial_\theta(p_\theta(x)) |x\rangle\langle x|_X \otimes \rho_\theta^x + \sum_{x:p_\theta(x)>0} p_\theta(x) |x\rangle\langle x|_X \otimes \Pi_{\rho_\theta^x}^\perp (\partial_\theta \rho_\theta^x) \Pi_{\rho_\theta^x}^\perp. \quad (\text{E.15})$$



Now sandwiching by  $\sum_{x:p_\theta(x)=0} |x\rangle\langle x|_X \otimes I$  on both sides (which projects out the second sum above) and tracing over the second system, we conclude that

$$\sum_{x:p_\theta(x)=0} \partial_\theta(p_\theta(x)) |x\rangle\langle x|_X = 0. \quad (\text{E.16})$$

This is the same as  $\text{supp}(\partial_\theta p_\theta) \subseteq \text{supp}(p_\theta)$ . Instead sandwiching by  $\sum_{x:p_\theta(x)>0} |x\rangle\langle x|_X \otimes I$ , we are left with the following conditions:

$$\Pi_{\rho_\theta^x}^\perp (\partial_\theta \rho_\theta^x) \Pi_{\rho_\theta^x}^\perp = 0 \quad \forall x : p_\theta(x) > 0. \quad (\text{E.17})$$

Thus, the finiteness condition in (E.4) implies the finiteness condition in (E.3). The other implication follows from plugging (E.3) into (E.15).

Since the finiteness of the left-hand side of (E.2) is equivalent to the finiteness of the right-hand side of (E.2), we can now focus on establishing the equality under these conditions. For the state

$$\sum_x p_\theta(x) |x\rangle\langle x|_X \otimes \rho_\theta^x, \quad (\text{E.18})$$

let its spectral decomposition be as follows:

$$\sum_x p_\theta(x) |x\rangle\langle x|_X \otimes \sum_y \lambda_\theta^{x,y} |\psi_\theta^{x,y}\rangle\langle\psi_\theta^{x,y}| = \sum_{x,y} p_\theta(x) \lambda_\theta^{x,y} |x\rangle\langle x|_X \otimes |\psi_\theta^{x,y}\rangle\langle\psi_\theta^{x,y}|. \quad (\text{E.19})$$

Plugging into the SLD Fisher information formula in (5.10), we find that

$$\begin{aligned} I_F\left(\theta; \left\{ \sum_x p_\theta(x) |x\rangle\langle x|_X \otimes \rho_\theta^x \right\}_\theta\right) \\ = 2 \sum_{\substack{x,y,x',y': \\ p_\theta(x)\lambda_\theta^{x,y} + p_\theta(x')\lambda_\theta^{x',y'} > 0}} \frac{|\langle x|_X \langle \psi_\theta^{x,y} | (\partial_\theta \sum_{x''} p_\theta(x'') |x''\rangle\langle x''|_X \otimes \rho_\theta^{x''}) |x'\rangle_X |\psi_\theta^{x',y'} \rangle|^2}{p_\theta(x)\lambda_\theta^{x,y} + p_\theta(x')\lambda_\theta^{x',y'}} \end{aligned} \quad (\text{E.20})$$

$$\begin{aligned} = 2 \sum_{\substack{x,y,x',y': \\ p_\theta(x)\lambda_\theta^{x,y} + p_\theta(x')\lambda_\theta^{x',y'} > 0}} \frac{|\langle x|_X \langle \psi_\theta^{x,y} | (\sum_{x''} |x''\rangle\langle x''|_X \otimes \partial_\theta(p_\theta(x'')\rho_\theta^{x''})) |x'\rangle_X |\psi_\theta^{x',y'} \rangle|^2}{p_\theta(x)\lambda_\theta^{x,y} + p_\theta(x')\lambda_\theta^{x',y'}} \end{aligned} \quad (\text{E.21})$$

$$\begin{aligned} = 2 \sum_{\substack{x,y,y': \\ p_\theta(x)(\lambda_\theta^{x,y} + \lambda_\theta^{x,y'}) > 0}} \frac{|\langle \psi_\theta^{x,y} | \partial_\theta(p_\theta(x)\rho_\theta^x) | \psi_\theta^{x,y'} \rangle|^2}{p_\theta(x) (\lambda_\theta^{x,y} + \lambda_\theta^{x,y'})} \end{aligned} \quad (\text{E.22})$$

$$\begin{aligned} = 2 \sum_{\substack{x,y,y': \\ p_\theta(x) > 0, \lambda_\theta^{x,y} + \lambda_\theta^{x,y'} > 0}} \frac{|\langle \psi_\theta^{x,y} | \partial_\theta(p_\theta(x)\rho_\theta^x) | \psi_\theta^{x,y'} \rangle|^2}{p_\theta(x) (\lambda_\theta^{x,y} + \lambda_\theta^{x,y'})}. \end{aligned} \quad (\text{E.23})$$

Now consider that

$$\partial_\theta(p_\theta(x)\rho_\theta^x) = (\partial_\theta p_\theta(x))\rho_\theta^x + p_\theta(x)(\partial_\theta \rho_\theta^x). \quad (\text{E.24})$$

Plugging into the numerator in (E.23), we find that

$$\begin{aligned} & \langle \psi_\theta^{x,y} | \partial_\theta(p_\theta(x)\rho_\theta^x) | \psi_\theta^{x,y'} \rangle \\ &= \langle \psi_\theta^{x,y} | (\partial_\theta p_\theta(x))\rho_\theta^x | \psi_\theta^{x,y'} \rangle + \langle \psi_\theta^{x,y} | p_\theta(x)(\partial_\theta \rho_\theta^x) | \psi_\theta^{x,y'} \rangle \end{aligned} \quad (\text{E.25})$$

$$= \delta_{y,y'} \lambda_\theta^{x,y} (\partial_\theta p_\theta(x)) + p_\theta(x) \langle \psi_\theta^{x,y} | (\partial_\theta \rho_\theta^x) | \psi_\theta^{x,y'} \rangle. \quad (\text{E.26})$$

Then we can evaluate the numerator in (E.23) as follows:

$$\begin{aligned} & |\langle \psi_\theta^{x,y} | \partial_\theta(p_\theta(x)\rho_\theta^x) | \psi_\theta^{x,y'} \rangle|^2 \\ &= \left| \delta_{y,y'} \lambda_\theta^{x,y} (\partial_\theta p_\theta(x)) + p_\theta(x) \langle \psi_\theta^{x,y} | (\partial_\theta \rho_\theta^x) | \psi_\theta^{x,y'} \rangle \right|^2 \end{aligned} \quad (\text{E.27})$$

$$\begin{aligned} &= \delta_{y,y'} (\lambda_\theta^{x,y})^2 (\partial_\theta p_\theta(x))^2 + 2\delta_{y,y'} \lambda_\theta^{x,y} p_\theta(x) (\partial_\theta p_\theta(x)) \text{Re}[\langle \psi_\theta^{x,y} | (\partial_\theta \rho_\theta^x) | \psi_\theta^{x,y'} \rangle] \\ &\quad + [p_\theta(x)]^2 \left| \langle \psi_\theta^{x,y} | (\partial_\theta \rho_\theta^x) | \psi_\theta^{x,y'} \rangle \right|^2. \end{aligned} \quad (\text{E.28})$$

We can then evaluate the sum in (E.23) for each of the three terms above, starting with the first one:

$$\begin{aligned} & 2 \sum_{\substack{x,y,y': \\ p_\theta(x) > 0, \lambda_\theta^{x,y} + \lambda_\theta^{x,y'} > 0}} \frac{[\partial_\theta p_\theta(x)]^2 \delta_{y,y'} (\lambda_\theta^{x,y})^2}{p_\theta(x) (\lambda_\theta^{x,y} + \lambda_\theta^{x,y'})} \\ &= \sum_{x,y: p_\theta(x) > 0, \lambda_\theta^{x,y} > 0} \frac{[\partial_\theta p_\theta(x)]^2 (\lambda_\theta^{x,y})^2}{p_\theta(x) \lambda_\theta^{x,y}} \end{aligned} \quad (\text{E.29})$$

$$= \sum_{x: p_\theta(x) > 0} \frac{[\partial_\theta p_\theta(x)]^2}{p_\theta(x)} \sum_{y: \lambda_\theta^{x,y} > 0} (\lambda_\theta^{x,y}) \quad (\text{E.30})$$

$$= \sum_{x: p_\theta(x) > 0} \frac{[\partial_\theta p_\theta(x)]^2}{p_\theta(x)} \quad (\text{E.31})$$

$$= I_F(\theta; \{p_\theta\}_\theta). \quad (\text{E.32})$$

Consider the next term:

$$\begin{aligned} & 2 \sum_{\substack{x,y,y': \\ p_\theta(x) > 0, \lambda_\theta^{x,y} + \lambda_\theta^{x,y'} > 0}} \frac{2\delta_{y,y'} \lambda_\theta^{x,y} p_\theta(x) (\partial_\theta p_\theta(x)) \text{Re}[\langle \psi_\theta^{x,y} | (\partial_\theta \rho_\theta^x) | \psi_\theta^{x,y'} \rangle]}{p_\theta(x) (\lambda_\theta^{x,y} + \lambda_\theta^{x,y'})} \\ &= 2 \sum_{\substack{x,y: \\ p_\theta(x) > 0, \lambda_\theta^{x,y} > 0}} \frac{\lambda_\theta^{x,y} p_\theta(x) (\partial_\theta p_\theta(x)) \text{Re}[\langle \psi_\theta^{x,y} | (\partial_\theta \rho_\theta^x) | \psi_\theta^{x,y} \rangle]}{p_\theta(x) \lambda_\theta^{x,y}} \end{aligned} \quad (\text{E.33})$$

$$= 2 \sum_{x,y:p_\theta(x)>0, \lambda_\theta^{x,y}>0} (\partial_\theta p_\theta(x)) \operatorname{Re} [\langle \psi_\theta^{x,y} | (\partial_\theta \rho_\theta^x) | \psi_\theta^{x,y} \rangle] \quad (\text{E.34})$$

$$= 2 \sum_{x:p_\theta(x)>0} (\partial_\theta p_\theta(x)) \operatorname{Re} \left[ \sum_{y:\lambda_\theta^{x,y}>0} \langle \psi_\theta^{x,y} | (\partial_\theta \rho_\theta^x) | \psi_\theta^{x,y} \rangle \right] \quad (\text{E.35})$$

$$= 2 \sum_{x:p_\theta(x)>0} (\partial_\theta p_\theta(x)) \operatorname{Re} \left[ \sum_y \langle \psi_\theta^{x,y} | (\partial_\theta \rho_\theta^x) | \psi_\theta^{x,y} \rangle \right] \quad (\text{E.36})$$

$$= 2 \sum_{x:p_\theta(x)>0} (\partial_\theta p_\theta(x)) \operatorname{Re} [\operatorname{Tr} [\partial_\theta \rho_\theta^x]] \quad (\text{E.37})$$

$$= 0. \quad (\text{E.38})$$

The third-to-last equality holds because  $\Pi_{\rho_\theta^x}^\perp (\partial_\theta \rho_\theta^x) \Pi_{\rho_\theta^x}^\perp = 0 \quad \forall x : p_\theta(x) > 0$ , implying that we can add these terms to the sum to get the full trace in the next line. The last line follows because  $\operatorname{Tr} [\partial_\theta \rho_\theta^x] = \partial_\theta \operatorname{Tr} [\rho_\theta^x] = 0$ . Now consider the final term:

$$\begin{aligned} & 2 \sum_{\substack{x,y,y': \\ p_\theta(x)>0, \lambda_\theta^{x,y} + \lambda_\theta^{x,y'} > 0}} \frac{[p_\theta(x)]^2 \left| \langle \psi_\theta^{x,y} | (\partial_\theta \rho_\theta^x) | \psi_\theta^{x,y'} \rangle \right|^2}{p_\theta(x) (\lambda_\theta^{x,y} + \lambda_\theta^{x,y'})} \\ &= 2 \sum_{\substack{x,y,y': \\ p_\theta(x)>0, \lambda_\theta^{x,y} + \lambda_\theta^{x,y'} > 0}} \frac{p_\theta(x) \left| \langle \psi_\theta^{x,y} | (\partial_\theta \rho_\theta^x) | \psi_\theta^{x,y'} \rangle \right|^2}{\lambda_\theta^{x,y} + \lambda_\theta^{x,y'}} \end{aligned} \quad (\text{E.39})$$

$$= \sum_{x:p_\theta(x)>0} p_\theta(x) \left( 2 \sum_{y,y': \lambda_\theta^{x,y} + \lambda_\theta^{x,y'} > 0} \frac{\left| \langle \psi_\theta^{x,y} | (\partial_\theta \rho_\theta^x) | \psi_\theta^{x,y'} \rangle \right|^2}{\lambda_\theta^{x,y} + \lambda_\theta^{x,y'}} \right) \quad (\text{E.40})$$

$$= \sum_{x:p_\theta(x)>0} p_\theta(x) I_F(\theta; \{\rho_\theta^x\}_\theta). \quad (\text{E.41})$$

So we conclude the formula in (E.2) after putting all of the above together.

We now turn to the RLD Fisher information, with the goal being to prove the following statement: For a differentiable family of classical–quantum states:

$$\left\{ \sum_x p_\theta(x) |x\rangle\langle x|_X \otimes \rho_\theta^x \right\}_\theta, \quad (\text{E.42})$$

the RLD Fisher information can be evaluated as follows:

$$\widehat{I}_F \left( \theta; \left\{ \sum_x p_\theta(x) |x\rangle\langle x|_X \otimes \rho_\theta^x \right\}_\theta \right) = I_F(\theta; \{p_\theta\}_\theta) + \sum_{x:p_\theta(x)>0} p_\theta(x) \widehat{I}_F(\theta; \{\rho_\theta^x\}_\theta). \quad (\text{E.43})$$

The beginning of the proof is similar to the previous proof for SLD Fisher information, and so we use the same notation used there. We first consider the finiteness conditions for the left- and right-hand sides of (E.43) and show that they are equivalent. For the right-hand side, the finiteness conditions are

$$\text{supp}(\partial_\theta p_\theta) \subseteq \text{supp}(p_\theta) \quad \wedge \quad (\partial_\theta \rho_\theta^x) \Pi_{\rho_\theta^x}^\perp = 0 \quad \forall x : p_\theta(x) > 0, \quad (\text{E.44})$$

while for the left-hand side, the finiteness condition is

$$(\partial_\theta \rho_{XB}^\theta) \Pi_{\rho_{XB}^\theta}^\perp = 0. \quad (\text{E.45})$$

We find that

$$\begin{aligned} 0 &= (\partial_\theta \rho_{XB}^\theta) \Pi_{\rho_{XB}^\theta}^\perp \\ &= \left( \sum_x \partial_\theta(p_\theta(x)) |x\rangle\langle x|_X \otimes \rho_\theta^x + \sum_x p_\theta(x) |x\rangle\langle x|_X \otimes (\partial_\theta \rho_\theta^x) \right) \times \\ &\quad \left( \sum_{x:p_\theta(x)=0} |x\rangle\langle x|_X \otimes I + \sum_{x:p_\theta(x)>0} |x\rangle\langle x|_X \otimes \Pi_{\rho_\theta^x}^\perp \right) \end{aligned} \quad (\text{E.46})$$

$$= \sum_{x:p_\theta(x)=0} \partial_\theta(p_\theta(x)) |x\rangle\langle x|_X \otimes \rho_\theta^x + \sum_{x:p_\theta(x)>0} p_\theta(x) |x\rangle\langle x|_X \otimes (\partial_\theta \rho_\theta^x) \Pi_{\rho_\theta^x}^\perp. \quad (\text{E.47})$$

Now sandwiching by  $\sum_{x:p_\theta(x)=0} |x\rangle\langle x|_X \otimes I$  on both sides (which projects out the second sum above) and tracing over the second system, we conclude that

$$\sum_{x:p_\theta(x)=0} \partial_\theta(p_\theta(x)) |x\rangle\langle x|_X = 0. \quad (\text{E.48})$$

This is the same as  $\text{supp}(\partial_\theta p_\theta) \subseteq \text{supp}(p_\theta)$ . Instead sandwiching by  $\sum_{x:p_\theta(x)>0} |x\rangle\langle x|_X \otimes I$ , we are left with the following conditions:

$$(\partial_\theta \rho_\theta^x) \Pi_{\rho_\theta^x}^\perp = 0 \quad \forall x : p_\theta(x) > 0. \quad (\text{E.49})$$

Thus, the finiteness condition in (E.45) implies the finiteness condition in (E.44). The other implication follows from plugging (E.44) into (E.47).

Since the finiteness of the left-hand side of (E.43) is equivalent to the finiteness of the right-hand side of (E.43), we can now focus on establishing the equality under these conditions. Consider that

$$\left( \partial_\theta \left( \sum_x p_\theta(x) |x\rangle\langle x|_X \otimes \rho_\theta^x \right) \right)^2 \quad (\text{E.50})$$

$$= \left( \sum_x |x\rangle\langle x|_X \otimes [\partial_\theta(p_\theta(x)) \rho_\theta^x + p_\theta(x) (\partial_\theta \rho_\theta^x)] \right)^2 \quad (\text{E.51})$$

$$= \left( \sum_x |x\rangle\langle x|_X \otimes [\partial_\theta(p_\theta(x))]^2 [\rho_\theta^x]^2 + p_\theta(x) \partial_\theta(p_\theta(x)) \{ \rho_\theta^x, (\partial_\theta \rho_\theta^x) \} + [p_\theta(x)]^2 (\partial_\theta \rho_\theta^x)^2 \right). \quad (\text{E.52})$$

Then we find that

$$\begin{aligned} & \hat{I}_F \left( \theta; \left\{ \sum_x p_\theta(x) |x\rangle\langle x|_X \otimes \rho_\theta^x \right\}_\theta \right) \\ &= \text{Tr} \left[ \left( \partial_\theta \left( \sum_x p_\theta(x) |x\rangle\langle x|_X \otimes \rho_\theta^x \right) \right)^2 \left( \sum_{x:p_\theta(x)>0} p_\theta(x) |x\rangle\langle x|_X \otimes \rho_\theta^x \right)^{-1} \right] \end{aligned} \quad (\text{E.53})$$

$$= \text{Tr} \left[ \left( \partial_\theta \left( \sum_x p_\theta(x) |x\rangle\langle x|_X \otimes \rho_\theta^x \right) \right)^2 \left( \sum_{x:p_\theta(x)>0} |x\rangle\langle x|_X \otimes [p_\theta(x)]^{-1} [\rho_\theta^x]^{-1} \right) \right] \quad (\text{E.54})$$

$$\begin{aligned} &= \sum_{x:p_\theta(x)>0} \text{Tr} [ [\partial_\theta(p_\theta(x))]^2 [\rho_\theta^x]^2 [p_\theta(x)]^{-1} [\rho_\theta^x]^{-1} ] \\ &\quad + \sum_{x:p_\theta(x)>0} \text{Tr} [ p_\theta(x) \partial_\theta(p_\theta(x)) \{ \rho_\theta^x, (\partial_\theta \rho_\theta^x) \} [p_\theta(x)]^{-1} [\rho_\theta^x]^{-1} ] \\ &\quad + \sum_{x:p_\theta(x)>0} \text{Tr} [ [p_\theta(x)]^2 (\partial_\theta \rho_\theta^x)^2 [p_\theta(x)]^{-1} [\rho_\theta^x]^{-1} ] \end{aligned} \quad (\text{E.55})$$

$$= \sum_{x:p_\theta(x)>0} \left[ \frac{[\partial_\theta(p_\theta(x))]^2}{p_\theta(x)} \text{Tr} [\rho_\theta^x] + 2 \partial_\theta(p_\theta(x)) \text{Tr} [ (\partial_\theta \rho_\theta^x) \Pi_{\rho_\theta^x}^\perp ] + p_\theta(x) \text{Tr} [ (\partial_\theta \rho_\theta^x)^2 [\rho_\theta^x]^{-1} ] \right] \quad (\text{E.56})$$

$$= I_F(\theta; \{p_\theta\}_\theta) + 2 \sum_{x:p_\theta(x)>0} \partial_\theta(p_\theta(x)) \text{Tr} [\partial_\theta \rho_\theta^x] + \sum_{x:p_\theta(x)>0} p_\theta(x) \hat{I}_F(\theta; \{\rho_\theta^x\}_\theta) \quad (\text{E.57})$$

$$= I_F(\theta; \{p_\theta\}_\theta) + \sum_{x:p_\theta(x)>0} p_\theta(x) \hat{I}_F(\theta; \{\rho_\theta^x\}_\theta). \quad (\text{E.58})$$

The second-to-last equality follows because  $\text{Tr} [ (\partial_\theta \rho_\theta^x) \Pi_{\rho_\theta^x}^\perp ] = 0$  and so we can add this term to the sum. The last equality follows because  $\text{Tr} [\partial_\theta \rho_\theta^x] = \partial_\theta \text{Tr} [\rho_\theta^x] = 0$ . ■

## F Proof of Proposition 27 (Bilinear program for SLD Fisher information of quantum channels)

Recall that the Fisher information of channels is defined as the following optimization over pure state inputs:

$$I_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}) = \sup_{\psi_{RA}} I_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\psi_{RA})\}). \quad (\text{F.1})$$

It suffices to optimize over pure state inputs  $\psi_{RA}$  such that the reduced state  $\psi_R > 0$ , because this set is dense in the set of all pure bipartite states. Now consider a fixed input state  $\psi_{RA}$ , and recall that it can be written as follows:

$$\psi_{RA} = Z_R \Gamma_{RA} Z_R^\dagger, \quad (\text{F.2})$$

where  $Z_R$  is an invertible operator satisfying  $\text{Tr}[Z_R^\dagger Z_R] = 1$ . Then the output state is as follows:

$$\omega_{RB}^\theta := \mathcal{N}_{A \rightarrow B}^\theta(\psi_{RA}) = Z_R \Gamma_{RB}^{\mathcal{N}^\theta} Z_R^\dagger, \quad (\text{F.3})$$

and we find that

$$\begin{aligned} & \frac{1}{2} I_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\psi_{RA})\}) \\ &= \inf \left\{ \mu : \begin{bmatrix} \mu & \langle \Gamma |_{RR'BB'} (\partial_\theta \omega_{RB}^\theta \otimes I_{R'B'}) \\ (\partial_\theta \omega_{RB}^\theta \otimes I_{R'B'}) | \Gamma \rangle_{RR'BB'} & \omega_{RB}^\theta \otimes I_{R'B'} + I_{RB} \otimes (\omega_{R'B'}^\theta)^T \end{bmatrix} \geq 0 \right\}, \quad (\text{F.4}) \end{aligned}$$

by applying Proposition 24. Now consider that

$$\begin{aligned} & \begin{bmatrix} \mu & \langle \Gamma |_{RR'BB'} (\partial_\theta \omega_{RB}^\theta \otimes I_{R'B'}) \\ (\partial_\theta \omega_{RB}^\theta \otimes I_{R'B'}) | \Gamma \rangle_{RR'BB'} & \omega_{RB}^\theta \otimes I_{R'B'} + I_{RB} \otimes (\omega_{R'B'}^\theta)^T \end{bmatrix} \\ &= \begin{bmatrix} \mu & \langle \Gamma |_{RR'BB'} (Z_R (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) Z_R^\dagger \otimes I_{R'B'}) \\ (Z_R (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) Z_R^\dagger \otimes I_{R'B'}) | \Gamma \rangle_{RR'BB'} & Z_R \Gamma_{RB}^{\mathcal{N}^\theta} Z_R^\dagger \otimes I_{R'B'} + I_{RB} \otimes \bar{Z}_{R'} (\Gamma_{R'B'}^{\mathcal{N}^\theta})^T Z_{R'}^T \end{bmatrix} \quad (\text{F.5}) \end{aligned}$$

$$\begin{aligned} &= \begin{bmatrix} 1 & 0 \\ 0 & Z_R \otimes I_B \otimes \bar{Z}_{R'} \otimes I_{B'} \end{bmatrix} \times \\ & \quad \begin{bmatrix} \mu & \langle \Gamma |_{RR'BB'} ((\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \otimes I_{R'B'}) \\ ((\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \otimes I_{R'B'}) | \Gamma \rangle_{RR'BB'} & \Gamma_{RB}^{\mathcal{N}^\theta} \otimes \sigma_{R'}^{-T} \otimes I_{B'} + \sigma_R^{-1} \otimes I_B \otimes (\Gamma_{R'B'}^{\mathcal{N}^\theta})^T \end{bmatrix} \times \\ & \quad \begin{bmatrix} 1 & 0 \\ 0 & Z_R \otimes I_B \otimes \bar{Z}_{R'} \otimes I_{B'} \end{bmatrix}^\dagger, \quad (\text{F.6}) \end{aligned}$$

where we define

$$\sigma_R := Z_R^\dagger Z_R, \quad (\text{F.7})$$

and we applied the following observations:

$$\begin{aligned} & (Z_R (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) Z_R^\dagger \otimes I_{R'B'}) | \Gamma \rangle_{RR'BB'} \\ &= (Z_R (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \otimes \bar{Z}_{R'} \otimes I_{B'}) | \Gamma \rangle_{RR'BB'} \quad (\text{F.8}) \end{aligned}$$

$$= (Z_R \otimes \bar{Z}_{R'}) ((\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \otimes I_{R'B'}) | \Gamma \rangle_{RR'BB'}, \quad (\text{F.9})$$

$$\begin{aligned}
& Z_R \Gamma_{RB}^{\mathcal{N}^\theta} Z_R^\dagger \otimes I_{R'B'} + I_{RB} \otimes \bar{Z}_{R'} (\Gamma_{R'B'}^{\mathcal{N}^\theta})^T Z_{R'}^T \\
&= Z_R \Gamma_{RB}^{\mathcal{N}^\theta} Z_R^\dagger \otimes \bar{Z}_{R'} (\bar{Z}_{R'})^{-1} (Z_{R'}^T)^{-1} Z_{R'}^T \otimes I_{B'} \\
&\quad + Z_R (Z_R)^{-1} (Z_R^\dagger)^{-1} Z_R^\dagger \otimes I_B \otimes \bar{Z}_{R'} (\Gamma_{R'B'}^{\mathcal{N}^\theta})^T Z_{R'}^T \quad (\text{F.10})
\end{aligned}$$

$$= Z_R \Gamma_{RB}^{\mathcal{N}^\theta} Z_R^\dagger \otimes \bar{Z}_{R'} \sigma_R^{-T} Z_{R'}^T \otimes I_{B'} + Z_R \sigma_R^{-1} Z_R^\dagger \otimes I_B \otimes \bar{Z}_{R'} (\Gamma_{R'B'}^{\mathcal{N}^\theta})^T Z_{R'}^T \quad (\text{F.11})$$

$$\begin{aligned}
&= (Z_R \otimes I_B \otimes \bar{Z}_{R'} \otimes I_{B'}) \left( \Gamma_{RB}^{\mathcal{N}^\theta} \otimes \sigma_R^{-T} \otimes I_{B'} \right) (Z_R \otimes I_B \otimes \bar{Z}_{R'} \otimes I_{B'})^\dagger \\
&\quad + (Z_R \otimes I_B \otimes \bar{Z}_{R'} \otimes I_{B'}) \left( \sigma_R^{-1} \otimes I_B \otimes (\Gamma_{R'B'}^{\mathcal{N}^\theta})^T \right) (Z_R \otimes I_B \otimes \bar{Z}_{R'} \otimes I_{B'})^\dagger \quad (\text{F.12})
\end{aligned}$$

$$\begin{aligned}
&= (Z_R \otimes I_B \otimes \bar{Z}_{R'} \otimes I_{B'}) \left( \Gamma_{RB}^{\mathcal{N}^\theta} \otimes \sigma_R^{-T} \otimes I_{B'} + \sigma_R^{-1} \otimes I_B \otimes (\Gamma_{R'B'}^{\mathcal{N}^\theta})^T \right) \\
&\quad \times (Z_R \otimes I_B \otimes \bar{Z}_{R'} \otimes I_{B'})^\dagger. \quad (\text{F.13})
\end{aligned}$$

Since the first matrix in (F.5)–(F.6) above is positive semi-definite if and only if the last one is, the semi-definite program in (F.4) becomes as follows:

$$\inf \left\{ \mu : \begin{bmatrix} \mu & \langle \Gamma |_{RR'BB'} \left( (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \otimes I_{R'B'} \right) \\ \left( (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \otimes I_{R'B'} \right) | \Gamma \rangle_{RR'BB'} & \Gamma_{RB}^{\mathcal{N}^\theta} \otimes \sigma_{R'}^{-T} \otimes I_{B'} + \sigma_R^{-1} \otimes I_B \otimes (\Gamma_{R'B'}^{\mathcal{N}^\theta})^T \end{bmatrix} \geq 0 \right\}. \quad (\text{F.14})$$

By invoking Lemma 58, the dual of this program is given by

$$\begin{aligned}
&\sup_{\lambda, |\varphi\rangle_{RBR'B'}, W_{RBR'B'}} 2 \operatorname{Re}[\langle \varphi |_{RBR'B'} (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) | \Gamma \rangle_{RR'BB'}] \\
&\quad - \operatorname{Tr}[(\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \sigma_{R'}^{-T} \otimes I_{B'} + \sigma_R^{-1} \otimes I_B \otimes (\Gamma_{R'B'}^{\mathcal{N}^\theta})^T) W_{RBR'B'}] \quad (\text{F.15})
\end{aligned}$$

subject to

$$\lambda \leq 1, \quad \begin{bmatrix} \lambda & \langle \varphi |_{RBR'B'} \\ |\varphi\rangle_{RBR'B'} & W_{RBR'B'} \end{bmatrix} \geq 0. \quad (\text{F.16})$$

Strong duality holds, so that (F.15) is equal to (F.14), because we are free to choose values  $\lambda$ ,  $|\varphi\rangle_{RBR'B'}$ , and  $W_{RBR'B'}$  such that the constraints in (F.16) are strict. Employing the unitary swap operators  $F_{RR'}$  and  $F_{BB'}$ , we can rewrite the second term in the objective function as follows:

$$\begin{aligned}
&\operatorname{Tr}[(\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \sigma_{R'}^{-T} \otimes I_{B'} + \sigma_R^{-1} \otimes I_B \otimes (\Gamma_{R'B'}^{\mathcal{N}^\theta})^T) W_{RBR'B'}] \\
&= \operatorname{Tr}[(\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \sigma_{R'}^{-T} \otimes I_{B'}) W_{RBR'B'}] + \operatorname{Tr}[(\sigma_R^{-1} \otimes I_B \otimes (\Gamma_{R'B'}^{\mathcal{N}^\theta})^T) W_{RBR'B'}] \quad (\text{F.17})
\end{aligned}$$

$$\begin{aligned}
&= \operatorname{Tr}[(F_{RR'} \otimes F_{BB'}) (\sigma_R^{-T} \otimes I_B \otimes \Gamma_{R'B'}^{\mathcal{N}^\theta}) (F_{RR'} \otimes F_{BB'}) W_{RBR'B'}] \\
&\quad + \operatorname{Tr}[\sigma_R^{-1} \operatorname{Tr}_{BR'B'}[(\Gamma_{R'B'}^{\mathcal{N}^\theta})^T W_{RBR'B'}]] \quad (\text{F.18})
\end{aligned}$$

$$\begin{aligned}
&= \operatorname{Tr}[(\sigma_R^{-T} \otimes I_B \otimes \Gamma_{R'B'}^{\mathcal{N}^\theta}) (F_{RR'} \otimes F_{BB'}) W_{RBR'B'} (F_{RR'} \otimes F_{BB'})] \\
&\quad + \operatorname{Tr}[\sigma_R^{-1} \operatorname{Tr}_{BR'B'}[(\Gamma_{R'B'}^{\mathcal{N}^\theta})^T W_{RBR'B'}]] \quad (\text{F.19})
\end{aligned}$$

$$\begin{aligned}
&= \text{Tr}[\sigma_R^{-T} \text{Tr}_{BB'B'}[\Gamma_{R'B'}^{\mathcal{N}^\theta} (F_{RR'} \otimes F_{BB'}) W_{RBR'B'} (F_{RR'} \otimes F_{BB'})]] \\
&\quad + \text{Tr}[\sigma_R^{-1} \text{Tr}_{BB'B'}[(\Gamma_{R'B'}^{\mathcal{N}^\theta})^T W_{RBR'B'}]]
\end{aligned} \tag{F.20}$$

$$\begin{aligned}
&= \text{Tr}[\sigma_R^{-1} (\text{Tr}_{BB'B'}[\Gamma_{R'B'}^{\mathcal{N}^\theta} (F_{RR'} \otimes F_{BB'}) W_{RBR'B'} (F_{RR'} \otimes F_{BB'})])^T] \\
&\quad + \text{Tr}[\sigma_R^{-1} \text{Tr}_{BB'B'}[(\Gamma_{R'B'}^{\mathcal{N}^\theta})^T W_{RBR'B'}]]
\end{aligned} \tag{F.21}$$

$$= \text{Tr}[\sigma_R^{-1} K_R], \tag{F.22}$$

where

$$\begin{aligned}
K_R &= (\text{Tr}_{BB'B'}[\Gamma_{R'B'}^{\mathcal{N}^\theta} (F_{RR'} \otimes F_{BB'}) W_{RBR'B'} (F_{RR'} \otimes F_{BB'})])^T \\
&\quad + \text{Tr}_{BB'B'}[(\Gamma_{R'B'}^{\mathcal{N}^\theta})^T W_{RBR'B'}].
\end{aligned} \tag{F.23}$$

So the SDP in (F.15) can be written as

$$\sup_{\lambda, |\varphi\rangle_{RBR'B'}, W_{RBR'B'}} 2 \text{Re}[\langle \varphi |_{RBR'B'} (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) | \Gamma \rangle_{RR'BB'}] - \text{Tr}[\sigma_R^{-1} K_R] \tag{F.24}$$

subject to

$$\lambda \leq 1, \quad \begin{bmatrix} \lambda & \langle \varphi |_{RBR'B'} \\ | \varphi \rangle_{RBR'B'} & W_{RBR'B'} \end{bmatrix} \geq 0. \tag{F.25}$$

Now noting from Lemma 57 that

$$\sigma_R^{-1} = \inf \left\{ Y_R : \begin{bmatrix} \sigma_R & I_R \\ I_R & Y_R \end{bmatrix} \geq 0 \right\}, \tag{F.26}$$

and that  $\sigma_R^{-1}$  and  $K_R$  are positive semi-definite, we can rewrite the SDP in (F.24) as

$$\begin{aligned}
&\sup_{\lambda, |\varphi\rangle_{RBR'B'}, W_{RBR'B'}} \left( 2 \text{Re}[\langle \varphi |_{RBR'B'} (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) | \Gamma \rangle_{RR'BB'}] - \inf_{Y_R} \text{Tr}[Y_R K_R] \right) \\
&= \sup_{\lambda, |\varphi\rangle_{RBR'B'}, W_{RBR'B'}, Y_R} \left( 2 \text{Re}[\langle \varphi |_{RBR'B'} (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) | \Gamma \rangle_{RR'BB'}] - \text{Tr}[Y_R K_R] \right)
\end{aligned} \tag{F.27}$$

subject to

$$\lambda \leq 1, \quad \begin{bmatrix} \lambda & \langle \varphi |_{RBR'B'} \\ | \varphi \rangle_{RBR'B'} & W_{RBR'B'} \end{bmatrix} \geq 0, \quad \begin{bmatrix} \sigma_R & I_R \\ I_R & Y_R \end{bmatrix} \geq 0. \tag{F.28}$$

Then we can finally include the maximization over input states  $\sigma_R$  (satisfying  $\sigma_R \geq 0$  and  $\text{Tr}[\sigma_R] = 1$ ) to arrive at the form given in (5.144).

## G Proof of Propositions 29 and 37 (Formula for RLD Fisher information of quantum channels and its additivity)

**Proof of Proposition 29.** From (5.73), the finiteness condition for the RLD Fisher information  $I_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta)$  of the family  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  of channels is that  $\Pi_{\Gamma^{\mathcal{N}^\theta}}^\perp(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) = 0$ ,



where  $\Gamma_{RB}^{\mathcal{N}^\theta}$  is the Choi state of the channel  $\mathcal{N}_{A \rightarrow B}^\theta$ . So we suppose that this condition holds. This condition implies that  $(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1}(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})$  is a well-defined operator with the inverse taken on the support of  $(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1}$ . Recall that any pure state  $\psi_{RA}$  can be written as

$$\psi_{RA} = Z_R \Gamma_{RA} Z_R^\dagger, \quad (\text{G.1})$$

where

$$\Gamma_{RA} = |\Gamma\rangle\langle\Gamma|_{RA}, \quad (\text{G.2})$$

$$|\Gamma\rangle_{RA} = \sum_i |i\rangle_R |i\rangle_A, \quad (\text{G.3})$$

and  $Z_R$  is a square operator satisfying  $\text{Tr}[Z_R^\dagger Z_R] = 1$ . This implies that

$$\mathcal{N}_{A \rightarrow B}^\theta(\psi_{RA}) = \mathcal{N}_{A \rightarrow B}^\theta(Z_R \Gamma_{RA} Z_R^\dagger) = Z_R \mathcal{N}_{A \rightarrow B}^\theta(\Gamma_{RA}) Z_R^\dagger = Z_R \Gamma_{RB}^{\mathcal{N}^\theta} Z_R^\dagger. \quad (\text{G.4})$$

It suffices to optimize over pure states  $\psi_{RA}$  such that  $\psi_A > 0$  because these states are dense in the set of all pure bipartite states. Then consider that

$$\begin{aligned} & \sup_{\psi_{RA}} \widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\psi_{RA})\}_\theta) \\ &= \sup_{\psi_{RA}} \text{Tr}[(\partial_\theta \mathcal{N}_{A \rightarrow B}^\theta(\psi_{RA}))^2 (\mathcal{N}_{A \rightarrow B}^\theta(\psi_{RA}))^{-1}] \end{aligned} \quad (\text{G.5})$$

$$= \sup_{Z_R: \text{Tr}[Z_R^\dagger Z_R]=1} \text{Tr}[(\partial_\theta Z_R \Gamma_{RB}^{\mathcal{N}^\theta} Z_R^\dagger)^2 (Z_R \Gamma_{RB}^{\mathcal{N}^\theta} Z_R^\dagger)^{-1}] \quad (\text{G.6})$$

$$= \sup_{Z_R: \text{Tr}[Z_R^\dagger Z_R]=1} \text{Tr}[(Z_R (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) Z_R^\dagger)^2 (Z_R \Gamma_{RB}^{\mathcal{N}^\theta} Z_R^\dagger)^{-1}] \quad (\text{G.7})$$

$$= \sup_{Z_R: \text{Tr}[Z_R^\dagger Z_R]=1} \text{Tr}[(Z_R (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) Z_R^\dagger) (Z_R \Gamma_{RB}^{\mathcal{N}^\theta} Z_R^\dagger)^{-1} (Z_R (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) Z_R^\dagger)] \quad (\text{G.8})$$

$$= \sup_{Z_R: \text{Tr}[Z_R^\dagger Z_R]=1} \text{Tr}[Z_R (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) (\Gamma_{RB}^{\mathcal{N}^\theta})^{-1} (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) Z_R^\dagger] \quad (\text{G.9})$$

$$= \sup_{Z_R: \text{Tr}[Z_R^\dagger Z_R]=1} \text{Tr}[Z_R^\dagger Z_R \text{Tr}_B[(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) (\Gamma_{RB}^{\mathcal{N}^\theta})^{-1} (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})]] \quad (\text{G.10})$$

$$= \left\| \text{Tr}_B[(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) (\Gamma_{RB}^{\mathcal{N}^\theta})^{-1} (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})] \right\|_\infty. \quad (\text{G.11})$$

The fifth equality is a consequence of the transformer equality in Lemma 59, with  $L = Z_R$ ,  $X = \partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}$ , and  $Y = \Gamma_{RB}^{\mathcal{N}^\theta}$ . The last equality is a consequence of the characterization of the infinity norm of a positive semi-definite operator  $Y$  as  $\|Y\|_\infty = \sup_{\rho>0, \text{Tr}[\rho]=1} \text{Tr}[Y\rho]$ .  $\blacksquare$

**Proof of Proposition 37.** The proof begins by considering the finiteness condition in (5.73) and showing that finiteness of the left-hand side is equivalent to finiteness of the right-hand side. The manipulations are the same as given in the proof of Proposition 6, and so we omit

showing them again. So we can focus on the case when the quantities are finite and exploit the explicit formula from Proposition 29 to evaluate the left-hand side directly. Consider that

$$\begin{aligned} \widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta \otimes \mathcal{M}_{C \rightarrow D}^\theta\}_\theta) \\ = \left\| \text{Tr}_{BD}[(\partial_\theta(\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \Gamma_{SD}^{\mathcal{M}^\theta}))(\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \Gamma_{SD}^{\mathcal{M}^\theta})^{-1}(\partial_\theta(\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \Gamma_{SD}^{\mathcal{M}^\theta}))] \right\|_\infty, \end{aligned} \quad (\text{G.12})$$

because the Choi operator of the tensor-product channel  $\mathcal{N}_{A \rightarrow B}^\theta \otimes \mathcal{M}_{C \rightarrow D}^\theta$  is  $\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \Gamma_{SD}^{\mathcal{M}^\theta}$ . Then

$$\partial_\theta(\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \Gamma_{SD}^{\mathcal{M}^\theta}) = (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \otimes \Gamma_{SD}^{\mathcal{M}^\theta} + \Gamma_{RB}^{\mathcal{N}^\theta} \otimes \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta}), \quad (\text{G.13})$$

and right multiplying by  $(\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \Gamma_{SD}^{\mathcal{M}^\theta})^{-1}$  gives

$$\begin{aligned} & (\partial_\theta(\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \Gamma_{SD}^{\mathcal{M}^\theta}))(\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \Gamma_{SD}^{\mathcal{M}^\theta})^{-1} \\ &= \left[ (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \otimes \Gamma_{SD}^{\mathcal{M}^\theta} + \Gamma_{RB}^{\mathcal{N}^\theta} \otimes \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta}) \right] (\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \Gamma_{SD}^{\mathcal{M}^\theta})^{-1} \end{aligned} \quad (\text{G.14})$$

$$= (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1} \otimes \Gamma_{SD}^{\mathcal{M}^\theta}(\Gamma_{SD}^{\mathcal{M}^\theta})^{-1} + \Gamma_{RB}^{\mathcal{N}^\theta}(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1} \otimes \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})(\Gamma_{SD}^{\mathcal{M}^\theta})^{-1} \quad (\text{G.15})$$

$$= (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1} \otimes \Pi_{\Gamma^{\mathcal{M}^\theta}} + \Pi_{\Gamma^{\mathcal{N}^\theta}} \otimes \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})(\Gamma_{SD}^{\mathcal{M}^\theta})^{-1}. \quad (\text{G.16})$$

Right multiplying again by  $(\partial_\theta(\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \Gamma_{SD}^{\mathcal{M}^\theta}))$  gives

$$\begin{aligned} & \left[ (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1} \otimes \Pi_{\Gamma^{\mathcal{M}^\theta}} + \Pi_{\Gamma^{\mathcal{N}^\theta}} \otimes \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})(\Gamma_{SD}^{\mathcal{M}^\theta})^{-1} \right] (\partial_\theta(\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \Gamma_{SD}^{\mathcal{M}^\theta})) \\ &= \left[ (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1} \otimes \Pi_{\Gamma^{\mathcal{M}^\theta}} + \Pi_{\Gamma^{\mathcal{N}^\theta}} \otimes \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})(\Gamma_{SD}^{\mathcal{M}^\theta})^{-1} \right] \\ & \quad \times \left[ (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \otimes \Gamma_{SD}^{\mathcal{M}^\theta} + \Gamma_{RB}^{\mathcal{N}^\theta} \otimes \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta}) \right] \end{aligned} \quad (\text{G.17})$$

$$\begin{aligned} &= (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1} (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \otimes \Gamma_{SD}^{\mathcal{M}^\theta} + (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1} \Gamma_{RB}^{\mathcal{N}^\theta} \otimes \Pi_{\Gamma^{\mathcal{M}^\theta}} \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta}) \\ & \quad + \Pi_{\Gamma^{\mathcal{N}^\theta}} (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \otimes \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})(\Gamma_{SD}^{\mathcal{M}^\theta})^{-1} \Gamma_{SD}^{\mathcal{M}^\theta} + \Gamma_{RB}^{\mathcal{N}^\theta} \otimes \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})(\Gamma_{SD}^{\mathcal{M}^\theta})^{-1} \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta}) \end{aligned} \quad (\text{G.18})$$

$$\begin{aligned} &= (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1} (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \otimes \Gamma_{SD}^{\mathcal{M}^\theta} + (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \Pi_{\Gamma^{\mathcal{N}^\theta}} \otimes \Pi_{\Gamma^{\mathcal{M}^\theta}} \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta}) \\ & \quad + \Pi_{\Gamma^{\mathcal{N}^\theta}} (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \otimes \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta}) \Pi_{\Gamma^{\mathcal{M}^\theta}} + \Gamma_{RB}^{\mathcal{N}^\theta} \otimes \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})(\Gamma_{SD}^{\mathcal{M}^\theta})^{-1} \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta}) \end{aligned} \quad (\text{G.19})$$

$$\begin{aligned} &= (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1} (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \otimes \Gamma_{SD}^{\mathcal{M}^\theta} + 2(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \otimes \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta}) \\ & \quad + \Gamma_{RB}^{\mathcal{N}^\theta} \otimes \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})(\Gamma_{SD}^{\mathcal{M}^\theta})^{-1} \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta}), \end{aligned} \quad (\text{G.20})$$

where the last line follows because we can “add in” zero-valued terms like  $(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \Pi_{\Gamma^{\mathcal{N}^\theta}}^\perp = \Pi_{\Gamma^{\mathcal{N}^\theta}}^\perp (\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) = 0$  and  $\Pi_{\Gamma^{\mathcal{M}^\theta}}^\perp \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta}) = \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta}) \Pi_{\Gamma^{\mathcal{M}^\theta}}^\perp = 0$ , due to the finiteness condition in (5.73) holding. Now taking the trace over systems  $BD$  for each term, we find that

$$\text{Tr}_{BD}[(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1}(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \otimes \Gamma_{SD}^{\mathcal{M}^\theta}]$$

$$= \text{Tr}_B[(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1}(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})] \otimes I_S, \quad (\text{G.21})$$

$$\text{Tr}_{BD}[2(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \otimes \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})] = 2 \text{Tr}_B[(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})] \otimes \text{Tr}_D[\partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})] \quad (\text{G.22})$$

$$= 2(\partial_\theta \text{Tr}_B[\Gamma_{RB}^{\mathcal{N}^\theta}]) \otimes \partial_\theta(\text{Tr}_D[\Gamma_{SD}^{\mathcal{M}^\theta}]) \quad (\text{G.23})$$

$$= 2(\partial_\theta I_R) \otimes (\partial_\theta I_S) \quad (\text{G.24})$$

$$= 0, \quad (\text{G.25})$$

$$\begin{aligned} \text{Tr}_{BD}[\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})(\Gamma_{SD}^{\mathcal{M}^\theta})^{-1} \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})] \\ = I_R \otimes \text{Tr}_D[\partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})(\Gamma_{SD}^{\mathcal{M}^\theta})^{-1} \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})]. \end{aligned} \quad (\text{G.26})$$

So we conclude that

$$\begin{aligned} \text{Tr}_{BD}[(\partial_\theta(\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \Gamma_{SD}^{\mathcal{M}^\theta}))(\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \Gamma_{SD}^{\mathcal{M}^\theta})^{-1}(\partial_\theta(\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \Gamma_{SD}^{\mathcal{M}^\theta}))] \\ = \text{Tr}_B[(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1}(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})] \otimes I_S \\ + I_R \otimes \text{Tr}_D[\partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})(\Gamma_{SD}^{\mathcal{M}^\theta})^{-1} \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})] \end{aligned} \quad (\text{G.27})$$

Consider now from Lemma 60 that

$$\|X \otimes I + I \otimes Y\|_\infty = \|X\|_\infty + \|Y\|_\infty, \quad (\text{G.28})$$

for positive semi-definite operators  $X$  and  $Y$ . Now applying (G.28), we find that

$$\widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta \otimes \mathcal{M}_{C \rightarrow D}^\theta\}_\theta) \quad (\text{G.29})$$

$$= \left\| \text{Tr}_{BD}[(\partial_\theta(\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \Gamma_{SD}^{\mathcal{M}^\theta}))(\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \Gamma_{SD}^{\mathcal{M}^\theta})^{-1}(\partial_\theta(\Gamma_{RB}^{\mathcal{N}^\theta} \otimes \Gamma_{SD}^{\mathcal{M}^\theta}))] \right\|_\infty \quad (\text{G.30})$$

$$= \left\| \text{Tr}_B[(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1}(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})] \otimes I_S + I_R \otimes \text{Tr}_D[\partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})(\Gamma_{SD}^{\mathcal{M}^\theta})^{-1} \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})] \right\|_\infty \quad (\text{G.31})$$

$$= \left\| \text{Tr}_B[(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1}(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})] \right\|_\infty + \left\| \text{Tr}_D[\partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})(\Gamma_{SD}^{\mathcal{M}^\theta})^{-1} \partial_\theta(\Gamma_{SD}^{\mathcal{M}^\theta})] \right\|_\infty \quad (\text{G.32})$$

$$= \widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) + \widehat{I}_F(\theta; \{\mathcal{M}_{C \rightarrow D}^\theta\}_\theta). \quad (\text{G.33})$$

This concludes the proof. ■

## H Geometric Rényi relative entropy and its properties

Before going into detail for the geometric Rényi relative entropy, we first briefly recall some quantum Rényi relative entropies.

The Petz–Rényi relative entropy [71, 121] is defined as follows for a state  $\rho$ , a positive semi-definite operator  $\sigma$ , and  $\alpha \in (0, 1) \cup (1, \infty)$ :

$$D_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \ln Q_\alpha(\rho\|\sigma), \quad (\text{H.1})$$

where the Petz–Rényi relative quasi-entropy is defined as

$$Q_\alpha(\rho\|\sigma) := \begin{cases} \text{Tr}[\rho^\alpha \sigma^{1-\alpha}] & \text{if } \alpha \in (0, 1) \text{ or} \\ & \text{supp}(\rho) \subseteq \text{supp}(\sigma) \text{ and } \alpha \in (1, \infty) \\ +\infty & \text{otherwise} \end{cases}. \quad (\text{H.2})$$

The full definition with the support condition was given in [72]. The Petz–Rényi relative entropy obeys the data-processing inequality for  $\alpha \in (0, 1) \cup (1, 2]$ :

$$D_\alpha(\rho\|\sigma) \geq D_\alpha(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)), \quad (\text{H.3})$$

where  $\mathcal{N}$  is a quantum channel [71, 121]. Note that the following limit holds [136]

$$D_\alpha(\rho\|\sigma) = \lim_{\varepsilon \rightarrow 0^+} D_\alpha(\rho\|\sigma_\varepsilon), \quad (\text{H.4})$$

where  $\sigma_\varepsilon := \sigma + \varepsilon I$ .

The sandwiched Rényi relative entropy [81, 119] is defined as follows for a state  $\rho$ , a positive semi-definite operator  $\sigma$ , and  $\alpha \in (0, 1) \cup (1, \infty)$ :

$$\tilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \ln \tilde{Q}_\alpha(\rho\|\sigma), \quad (\text{H.5})$$

where the sandwiched Rényi relative quasi-entropy is defined as

$$\tilde{Q}_\alpha(\rho\|\sigma) := \begin{cases} \text{Tr} \left[ \left( \sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha \right] & \text{if } \alpha \in (0, 1) \text{ or} \\ & \text{supp}(\rho) \subseteq \text{supp}(\sigma) \text{ and } \alpha \in (1, \infty) \\ +\infty & \text{otherwise} \end{cases}. \quad (\text{H.6})$$

Note that the following limit holds [119]

$$\tilde{D}_\alpha(\rho\|\sigma) = \lim_{\varepsilon \rightarrow 0^+} \tilde{D}_\alpha(\rho\|\sigma_\varepsilon). \quad (\text{H.7})$$

Let us also recall the quantum relative entropy [137]:

$$D(\rho\|\sigma) := \begin{cases} \text{Tr}[\rho(\ln \rho - \ln \sigma)] & \text{if } \text{supp}(\rho) \subseteq \text{supp}(\sigma) \\ +\infty & \text{otherwise} \end{cases}, \quad (\text{H.8})$$

and note that the following limit holds (see, e.g., [44])

$$D(\rho\|\sigma) = \lim_{\varepsilon \rightarrow 0^+} D(\rho\|\sigma_\varepsilon). \quad (\text{H.9})$$

It is known that the Petz– [71, 121] and sandwiched [81, 119] Rényi relative entropies converge to the quantum relative entropy in the limit  $\alpha \rightarrow 1$ :

$$\lim_{\alpha \rightarrow 1} \tilde{D}_\alpha(\rho \parallel \sigma) = \lim_{\alpha \rightarrow 1} D_\alpha(\rho \parallel \sigma) = D(\rho \parallel \sigma). \quad (\text{H.10})$$

The max-relative entropy is defined as [118]

$$D_{\max}(\rho \parallel \sigma) := \inf \{ \lambda \geq 0 : \rho \leq e^\lambda \sigma \}, \quad (\text{H.11})$$

and the following limit is known [119]

$$\lim_{\alpha \rightarrow \infty} \tilde{D}_\alpha(\rho \parallel \sigma) = D_{\max}(\rho \parallel \sigma). \quad (\text{H.12})$$

We now recall the definition of the geometric Rényi relative entropy:

**Definition 65 (Geometric Rényi relative entropy)** *Let  $\rho$  be a state,  $\sigma$  a positive semi-definite operator, and  $\alpha \in (0, 1) \cup (1, \infty)$ . The geometric Rényi relative quasi-entropy is defined as*

$$\hat{Q}_\alpha(\rho \parallel \sigma) := \lim_{\varepsilon \rightarrow 0^+} \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right], \quad (\text{H.13})$$

where  $\sigma_\varepsilon := \sigma + \varepsilon I$ , and the geometric Rényi relative entropy is then defined as

$$\hat{D}_\alpha(\rho \parallel \sigma) := \frac{1}{\alpha - 1} \ln \hat{Q}_\alpha(\rho \parallel \sigma). \quad (\text{H.14})$$

In Definition 65, we have defined the geometric Rényi relative entropy as a limit, in contrast to how the Petz–Rényi relative entropy and the sandwiched Rényi relative entropy are usually defined (see, e.g., [33]). The geometric Rényi relative entropy is a bit more complicated than these other Rényi relative entropies for  $\alpha \in (0, 1)$ , and so defining it as such gives us a more compact expression to work with. Proposition 66 below gives explicit formulas to work with in all cases for which the geometric Rényi relative entropy is defined.

**Proposition 66** *For any state  $\rho$ , positive semi-definite operator  $\sigma$ , and  $\alpha \in (0, 1) \cup (1, \infty)$ , the following equality holds*

$$\hat{Q}_\alpha(\rho \parallel \sigma) = \begin{cases} \text{Tr} \left[ \sigma \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right)^\alpha \right] & \text{if } \alpha \in (0, 1) \cup (1, \infty) \\ & \text{and } \text{supp}(\rho) \subseteq \text{supp}(\sigma) \\ \text{Tr} \left[ \sigma \left( \sigma^{-\frac{1}{2}} \tilde{\rho} \sigma^{-\frac{1}{2}} \right)^\alpha \right] & \text{if } \alpha \in (0, 1) \\ & \text{and } \text{supp}(\rho) \not\subseteq \text{supp}(\sigma) \\ +\infty & \text{if } \alpha \in (1, \infty) \text{ and} \\ & \text{supp}(\rho) \not\subseteq \text{supp}(\sigma). \end{cases}, \quad (\text{H.15})$$

where

$$\tilde{\rho} := \rho_{0,0} - \rho_{0,1}\rho_{1,1}^{-1}\rho_{0,1}^\dagger, \quad \rho = \begin{bmatrix} \rho_{0,0} & \rho_{0,1} \\ \rho_{0,1}^\dagger & \rho_{1,1} \end{bmatrix}, \quad (\text{H.16})$$

$$\rho_{0,0} := \Pi_\sigma \rho \Pi_\sigma, \quad \rho_{0,1} := \Pi_\sigma \rho \Pi_\sigma^\perp, \quad \rho_{1,1} := \Pi_\sigma^\perp \rho \Pi_\sigma^\perp, \quad (\text{H.17})$$

$\Pi_\sigma$  is the projection onto the support of  $\sigma$ ,  $\Pi_\sigma^\perp$  is the projection onto the kernel of  $\sigma$ , and the inverses  $\sigma^{-\frac{1}{2}}$  and  $\rho_{1,1}^{-1}$  are generalized inverses (taken on the support of  $\sigma$  and  $\rho_{1,1}$ , respectively). We also have the alternative expressions below for certain cases:

$$\widehat{Q}_\alpha(\rho||\sigma) = \begin{cases} \text{Tr} \left[ \rho \left( \rho^{-\frac{1}{2}} \sigma \rho^{-\frac{1}{2}} \right)^{1-\alpha} \right] & \text{if } \alpha \in (0, 1) \\ & \text{and } \text{supp}(\sigma) \subseteq \text{supp}(\rho) \\ \text{Tr} \left[ \rho \left( \rho^{\frac{1}{2}} \sigma^{-1} \rho^{\frac{1}{2}} \right)^{\alpha-1} \right] & \text{if } \alpha \in (1, \infty) \\ & \text{and } \text{supp}(\rho) \subseteq \text{supp}(\sigma) \end{cases}, \quad (\text{H.18})$$

where the inverses  $\rho^{-\frac{1}{2}}$  and  $\sigma^{-1}$  are generalized inverses.

One should observe that when  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$  and  $\alpha \in (0, 1)$ , the expression  $\text{Tr} \left[ \sigma \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right)^\alpha \right]$  is actually a special case of  $\text{Tr} \left[ \sigma \left( \sigma^{-\frac{1}{2}} \tilde{\rho} \sigma^{-\frac{1}{2}} \right)^\alpha \right]$ , because the operators  $\rho_{0,1}$  and  $\rho_{1,1}$  are both equal to zero in this case, so that  $\Pi_\sigma \rho = \rho \Pi_\sigma = \rho$  and  $\tilde{\rho} = \rho_{0,0}$ . The expression  $\text{Tr} \left[ \sigma \left( \sigma^{-\frac{1}{2}} \tilde{\rho} \sigma^{-\frac{1}{2}} \right)^\alpha \right]$  for  $\alpha = 1/2$  and  $\text{supp}(\rho) \not\subseteq \text{supp}(\sigma)$  was identified in [113, Section 3] and later generalized to all  $\alpha \in (0, 1)$  in [114, Section 2].

The main intuition behind some of the formulas in Proposition 66 is as follows. If  $\rho$  and  $\sigma$  are positive definite, then the following equalities hold

$$\text{Tr} \left[ \sigma \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right)^\alpha \right] = \text{Tr} \left[ \rho \left( \rho^{-\frac{1}{2}} \sigma \rho^{-\frac{1}{2}} \right)^{1-\alpha} \right] \quad (\text{H.19})$$

$$= \text{Tr} \left[ \rho \left( \rho^{\frac{1}{2}} \sigma^{-1} \rho^{\frac{1}{2}} \right)^{\alpha-1} \right], \quad (\text{H.20})$$

for all  $\alpha \in (0, 1) \cup (1, \infty)$ , as shown in Proposition 67 below. If the support condition  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$  holds, then we can think of  $\text{supp}(\sigma)$  as being the whole Hilbert space and  $\sigma$  being invertible on the whole space. So then generalized inverses like  $\sigma^{-\frac{1}{2}}$  or  $\sigma^{-1}$  are true inverses on  $\text{supp}(\sigma)$ , and the expression  $\text{Tr}[\sigma(\sigma^{-1/2}\rho\sigma^{-1/2})^\alpha]$  is sensible for  $\alpha \in (0, 1) \cup (1, \infty)$ , with the only inverse in the expression being  $\sigma^{-\frac{1}{2}}$ . Similarly, the expression  $\text{Tr}[\rho(\rho^{1/2}\sigma^{-1}\rho^{1/2})^{\alpha-1}]$  is sensible for  $\alpha \in (1, \infty)$ , with the only inverse in the expression being  $\sigma^{-1}$ . On the other hand, if the support condition  $\text{supp}(\sigma) \subseteq \text{supp}(\rho)$  holds, then we can think of  $\text{supp}(\rho)$  as being the whole Hilbert space and  $\rho$  being invertible on the whole space. So then the generalized inverse  $\rho^{-\frac{1}{2}}$  is a true inverse on  $\text{supp}(\rho)$ , and the expression  $\text{Tr}[\rho(\rho^{-1/2}\sigma\rho^{-1/2})^{1-\alpha}]$  is sensible for  $\alpha \in (0, 1)$ , with the only inverse in the expression being  $\rho^{-\frac{1}{2}}$ . After developing a few properties of the geometric Rényi relative entropy, we prove Proposition 66.

Due to the fact that Definition 65 does not involve an inverse of the state  $\rho$ , the following equality holds for all  $\alpha \in (0, 1) \cup (1, \infty)$ :

$$\widehat{Q}_\alpha(\rho||\sigma) = \lim_{\varepsilon \rightarrow 0^+} \lim_{\delta \rightarrow 0^+} \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right], \quad (\text{H.21})$$

where

$$\rho_\delta := (1 - \delta) \rho + \delta \pi, \quad (\text{H.22})$$

and  $\pi$  is the maximally mixed state. The equality in (H.21) is useful for establishing the data-processing inequality for the geometric Rényi relative entropy (Theorem 73 below), as well as its monotonicity with respect to  $\alpha$  (Proposition 72 below). Note that we can exchange the order of the limits in (H.21) for  $\alpha \in (0, 1)$ , which we show later on in Lemma 69.

The geometric Rényi relative entropy is named as such because it can be written in terms of the *weighted operator geometric mean*. The weighted operator geometric mean of two positive definite operators  $X$  and  $Y$  is defined as follows:

$$G_\beta(X, Y) := X^{\frac{1}{2}} \left( X^{-\frac{1}{2}} Y X^{-\frac{1}{2}} \right)^\beta X^{\frac{1}{2}}, \quad (\text{H.23})$$

where  $\beta \in \mathbb{R}$  is the weight parameter. We recover the standard operator geometric mean by setting  $\beta = 1/2$ . By using the definition in (H.23), we see that the geometric Rényi relative quasi-entropy can be written in terms of the weighted operator geometric mean as

$$\widehat{Q}_\alpha(\rho||\sigma) = \text{Tr} \left[ \sigma^{\frac{1}{2}} \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right)^\alpha \sigma^{\frac{1}{2}} \right] \quad (\text{H.24})$$

$$= \text{Tr}[G_\alpha(\sigma, \rho)], \quad (\text{H.25})$$

whenever  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ .

Whenever  $\rho$  and  $\sigma$  are positive definite, an alternative way of writing the geometric Rényi relative quasi-entropy is given by the following proposition:

**Proposition 67** *Let  $\rho$  be a positive definite state and  $\sigma$  a positive definite operator. For all  $\alpha \in (0, 1) \cup (1, \infty)$ , the following equalities hold*

$$\widehat{Q}_\alpha(\rho||\sigma) = \text{Tr} \left[ \rho \left( \rho^{-\frac{1}{2}} \sigma \rho^{-\frac{1}{2}} \right)^{1-\alpha} \right] \quad (\text{H.26})$$

$$= \text{Tr}[G_{1-\alpha}(\rho, \sigma)] \quad (\text{H.27})$$

$$= \text{Tr} \left[ \rho \left( \rho^{\frac{1}{2}} \sigma^{-1} \rho^{\frac{1}{2}} \right)^{\alpha-1} \right]. \quad (\text{H.28})$$

**Proof.** The first two equalities follow from a fundamental property of the weighted operator geometric mean given in Lemma 68 below. The last equality follows because  $(\rho^{-1/2} \sigma \rho^{-1/2})^{1-\alpha} = (\rho^{1/2} \sigma^{-1} \rho^{1/2})^{\alpha-1}$  whenever  $\rho$  and  $\sigma$  are positive definite. ■

**Lemma 68** *Let  $X$  and  $Y$  be positive definite operators and  $\beta \in \mathbb{R}$ . Then the following equality holds*

$$G_\beta(X, Y) = G_{1-\beta}(Y, X), \quad (\text{H.29})$$

with  $G_\beta(X, Y)$  defined in (H.23).

**Proof.** To see (H.29), consider that

$$G_{1-\beta}(Y, X) = Y^{\frac{1}{2}} \left( Y^{-\frac{1}{2}} X Y^{-\frac{1}{2}} \right)^{1-\beta} Y^{\frac{1}{2}} \quad (\text{H.30})$$

$$= Y^{\frac{1}{2}} \left( Y^{-\frac{1}{2}} X Y^{-\frac{1}{2}} \right) \left( Y^{-\frac{1}{2}} X Y^{-\frac{1}{2}} \right)^{-\beta} Y^{\frac{1}{2}} \quad (\text{H.31})$$

$$= X^{\frac{1}{2}} X^{\frac{1}{2}} Y^{-\frac{1}{2}} \left( Y^{-\frac{1}{2}} X^{\frac{1}{2}} X^{\frac{1}{2}} Y^{-\frac{1}{2}} \right)^{-\beta} Y^{\frac{1}{2}} \quad (\text{H.32})$$

$$= X^{\frac{1}{2}} \left( X^{\frac{1}{2}} Y^{-\frac{1}{2}} Y^{-\frac{1}{2}} X^{\frac{1}{2}} \right)^{-\beta} X^{\frac{1}{2}} Y^{-\frac{1}{2}} Y^{\frac{1}{2}} \quad (\text{H.33})$$

$$= X^{\frac{1}{2}} \left( X^{-\frac{1}{2}} Y X^{-\frac{1}{2}} \right)^{\beta} X^{\frac{1}{2}} \quad (\text{H.34})$$

$$= G_\beta(X, Y). \quad (\text{H.35})$$

The fourth equality follows from Lemma 61, by setting  $L = X^{\frac{1}{2}} Y^{-\frac{1}{2}}$  and  $f(x) = x^{-\beta}$  therein. ■

We now show that the order of limits in (H.21) does not matter when  $\alpha \in (0, 1)$ :

**Lemma 69** *Let  $\rho$  be a state and  $\sigma$  a positive semi-definite operator. For  $\alpha \in (0, 1)$ , the following equality holds*

$$\widehat{Q}_\alpha(\rho \parallel \sigma) = \lim_{\varepsilon \rightarrow 0^+} \lim_{\delta \rightarrow 0^+} \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] \quad (\text{H.36})$$

$$= \inf_{\varepsilon, \delta > 0} \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] \quad (\text{H.37})$$

$$= \lim_{\delta \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right], \quad (\text{H.38})$$

where  $\rho_\delta := (1 - \delta) \rho + \delta \pi$ ,  $\delta \in (0, 1)$ ,  $\pi$  is the maximally mixed state,  $\sigma_\varepsilon := \sigma + \varepsilon I$ , and  $\varepsilon > 0$ .

**Proof.** First consider that

$$(1 - \delta) \rho'_\delta \leq \rho_\delta \leq \rho'_\delta, \quad (\text{H.39})$$

where

$$\rho'_\delta := \rho + \delta \pi. \quad (\text{H.40})$$

By operator monotonicity of  $x^\alpha$  for  $\alpha \in (0, 1)$ , we conclude that

$$(1 - \delta)^\alpha \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho'_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] \leq \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] \quad (\text{H.41})$$



$$\leq \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho'_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right]. \quad (\text{H.42})$$

These bounds are uniform and independent of  $\varepsilon$ , and so it follows that

$$\lim_{\varepsilon \rightarrow 0^+} \lim_{\delta \rightarrow 0^+} \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] = \lim_{\varepsilon \rightarrow 0^+} \lim_{\delta \rightarrow 0^+} \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho'_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right], \quad (\text{H.43})$$

$$\lim_{\delta \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] = \lim_{\delta \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho'_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right]. \quad (\text{H.44})$$

Again from the operator monotonicity of  $x^\alpha$  for  $\alpha \in (0, 1)$ , we conclude for fixed  $\varepsilon > 0$  that

$$\delta_1 \leq \delta_2 \quad \Rightarrow \quad \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho'_{\delta_1} \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] \leq \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho'_{\delta_2} \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right], \quad (\text{H.45})$$

where  $\delta_1 > 0$ . By exploiting the identity

$$\text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho'_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] = \text{Tr} \left[ \rho'_\delta \left( (\rho'_\delta)^{-\frac{1}{2}} \sigma_\varepsilon (\rho'_\delta)^{-\frac{1}{2}} \right)^{1-\alpha} \right] \quad (\text{H.46})$$

from Proposition 67 and operator monotonicity of  $x^{1-\alpha}$  for  $\alpha \in (0, 1)$ , we conclude for fixed  $\delta > 0$  that

$$\varepsilon_1 \leq \varepsilon_2 \quad \Rightarrow \quad \text{Tr} \left[ \sigma_{\varepsilon_1} \left( \sigma_{\varepsilon_1}^{-\frac{1}{2}} \rho_\delta \sigma_{\varepsilon_1}^{-\frac{1}{2}} \right)^\alpha \right] \leq \text{Tr} \left[ \sigma_{\varepsilon_2} \left( \sigma_{\varepsilon_2}^{-\frac{1}{2}} \rho_\delta \sigma_{\varepsilon_2}^{-\frac{1}{2}} \right)^\alpha \right], \quad (\text{H.47})$$

where  $\varepsilon_1 > 0$ . Thus, we find that

$$\lim_{\varepsilon \rightarrow 0^+} \lim_{\delta \rightarrow 0^+} \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho'_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] = \inf_{\varepsilon > 0} \inf_{\delta > 0} \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho'_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right], \quad (\text{H.48})$$

$$\lim_{\delta \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho'_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] = \inf_{\delta > 0} \inf_{\varepsilon > 0} \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho'_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right]. \quad (\text{H.49})$$

Since infima can be exchanged, we conclude the statement of the proposition. ■

A first property of the geometric Rényi relative entropy that we recall is its relation to the sandwiched Rényi relative entropy [81, 119]. The inequality below was established for the interval  $\alpha \in (0, 1) \cup (1, 2]$  in [37] (by making use of a general result in [35, 36]) and for the full interval  $\alpha \in (1, \infty)$  in [89]. Below we follow the approach of [89] and offer a unified proof in terms of the Araki–Lieb–Thirring inequality [138, 139].

**Proposition 70** *Let  $\rho$  be a state and  $\sigma$  a positive semi-definite operator. The geometric Rényi relative entropy is not smaller than the sandwiched Rényi relative entropy for all  $\alpha \in (0, 1) \cup (1, \infty)$ :*

$$\tilde{D}_\alpha(\rho \parallel \sigma) \leq \hat{D}_\alpha(\rho \parallel \sigma). \quad (\text{H.50})$$

**Proof.** This is a direct consequence of the Araki–Lieb–Thirring inequality [138, 139]. For positive semi-definite operators  $X$  and  $Y$ ,  $q \geq 0$ , and  $r \in [0, 1]$ , the following inequality holds

$$\mathrm{Tr} \left[ \left( Y^{\frac{1}{2}} X Y^{\frac{1}{2}} \right)^{rq} \right] \geq \mathrm{Tr} \left[ \left( Y^{\frac{r}{2}} X^r Y^{\frac{r}{2}} \right)^q \right]. \quad (\text{H.51})$$

For  $r \geq 1$ , the following inequality holds

$$\mathrm{Tr} \left[ \left( Y^{\frac{1}{2}} X Y^{\frac{1}{2}} \right)^{rq} \right] \leq \mathrm{Tr} \left[ \left( Y^{\frac{r}{2}} X^r Y^{\frac{r}{2}} \right)^q \right]. \quad (\text{H.52})$$

By employing it with  $q = 1$ ,  $r = \alpha \in (0, 1)$ ,  $Y = \sigma_\varepsilon^{-\frac{1}{2}}$ , and  $X = \sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}}$ , and recalling that  $\sigma_\varepsilon := \sigma + \varepsilon I$ , we find that

$$\widehat{Q}_\alpha(\rho \| \sigma_\varepsilon) = \mathrm{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] \quad (\text{H.53})$$

$$= \mathrm{Tr} \left[ \left( \sigma_\varepsilon^{\frac{1}{2\alpha}} \right)^\alpha \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \left( \sigma_\varepsilon^{\frac{1}{2\alpha}} \right)^\alpha \right] \quad (\text{H.54})$$

$$\leq \mathrm{Tr} \left[ \left( \sigma_\varepsilon^{\frac{1}{2\alpha}} \sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} \sigma_\varepsilon^{\frac{1}{2\alpha}} \right)^\alpha \right] \quad (\text{H.55})$$

$$= \mathrm{Tr} \left[ \left( \sigma_\varepsilon^{\frac{1-\alpha}{2\alpha}} \rho \sigma_\varepsilon^{\frac{1-\alpha}{2\alpha}} \right)^\alpha \right] \quad (\text{H.56})$$

$$= \widetilde{Q}_\alpha(\rho \| \sigma_\varepsilon), \quad (\text{H.57})$$

which implies for  $\alpha \in (0, 1)$ , by using definitions, that

$$\widetilde{D}_\alpha(\rho \| \sigma_\varepsilon) \leq \widehat{D}_\alpha(\rho \| \sigma_\varepsilon). \quad (\text{H.58})$$

Now taking the limit as  $\varepsilon \rightarrow 0^+$ , employing (H.7) and Definition 65, we arrive at the inequality in (H.50).

Since the Araki–Lieb–Thirring inequality is reversed for  $r = \alpha \in (1, \infty)$ , we can employ similar reasoning as above and definitions to arrive at (H.50) for  $\alpha \in (1, \infty)$ . ■

We are now ready to provide a proof of Proposition 66.

**Proof of Proposition 66.** First suppose that  $\alpha \in (1, \infty)$  and  $\mathrm{supp}(\rho) \not\subseteq \mathrm{supp}(\sigma)$ . Then from (H.7) and Proposition 70 and the fact that the sandwiched Rényi relative quasi-entropy  $\widetilde{Q}_\alpha(\rho \| \sigma) = +\infty$  in this case, it follows that  $\widehat{Q}_\alpha(\rho \| \sigma) = +\infty$ , thus establishing the third expression in (H.15).

Now suppose that  $\alpha \in (0, 1) \cup (1, \infty)$  and  $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$ . Let us employ the decomposition of the Hilbert space  $\mathcal{H}$  as  $\mathcal{H} = \mathrm{supp}(\sigma) \oplus \ker(\sigma)$ . Then we can write  $\rho$  as

$$\rho = \begin{pmatrix} \rho_{0,0} & \rho_{0,1} \\ \rho_{0,1}^\dagger & \rho_{1,1} \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}. \quad (\text{H.59})$$

Writing  $I = \Pi_\sigma + \Pi_\sigma^\perp$ , where  $\Pi_\sigma$  is the projection onto the support of  $\sigma$  and  $\Pi_\sigma^\perp$  is the projection onto the orthogonal complement of  $\mathrm{supp}(\sigma)$ , we find that

$$\sigma_\varepsilon = \begin{pmatrix} \sigma + \varepsilon \Pi_\sigma & 0 \\ 0 & \varepsilon \Pi_\sigma^\perp \end{pmatrix}, \quad (\text{H.60})$$

which implies that

$$\sigma_\varepsilon^{-\frac{1}{2}} = \begin{pmatrix} (\sigma + \varepsilon \Pi_\sigma)^{-\frac{1}{2}} & 0 \\ 0 & \varepsilon^{-\frac{1}{2}} \Pi_\sigma^\perp \end{pmatrix}. \quad (\text{H.61})$$

The condition  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$  implies that  $\rho_{0,1} = 0$  and  $\rho_{1,1} = 0$ . Then

$$\sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} = \begin{pmatrix} (\sigma + \varepsilon \Pi_\sigma)^{-\frac{1}{2}} \rho_{0,0} (\sigma + \varepsilon \Pi_\sigma)^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix}, \quad (\text{H.62})$$

so that

$$\begin{aligned} & \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] \\ &= \text{Tr} \left[ \begin{pmatrix} \sigma + \varepsilon \Pi_\sigma & 0 \\ 0 & \varepsilon \Pi_\sigma^\perp \end{pmatrix} \begin{pmatrix} [(\sigma + \varepsilon \Pi_\sigma)^{-\frac{1}{2}} \rho_{0,0} (\sigma + \varepsilon \Pi_\sigma)^{-\frac{1}{2}}]^\alpha & 0 \\ 0 & 0 \end{pmatrix} \right] \end{aligned} \quad (\text{H.63})$$

$$= \text{Tr} \left[ (\sigma + \varepsilon \Pi_\sigma) \left[ (\sigma + \varepsilon \Pi_\sigma)^{-\frac{1}{2}} \rho_{0,0} (\sigma + \varepsilon \Pi_\sigma)^{-\frac{1}{2}} \right]^\alpha \right]. \quad (\text{H.64})$$

Taking the limit  $\varepsilon \rightarrow 0^+$  then leads to

$$\lim_{\varepsilon \rightarrow 0^+} \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] = \text{Tr} \left[ \sigma \left( \sigma^{-\frac{1}{2}} \rho_{0,0} \sigma^{-\frac{1}{2}} \right)^\alpha \right] \quad (\text{H.65})$$

$$= \text{Tr} \left[ \sigma \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right)^\alpha \right], \quad (\text{H.66})$$

thus establishing the first expression in (H.15).

We now establish (H.18). For  $\alpha \in (1, \infty)$  and  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ , the same analysis implies that

$$\text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] = \text{Tr} \left[ \hat{\sigma}_\varepsilon \left( \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,0} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right], \quad (\text{H.67})$$

where

$$\hat{\sigma}_\varepsilon := \sigma + \varepsilon \Pi_\sigma. \quad (\text{H.68})$$

Since

$$\left( \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,0} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \right)^\alpha = \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,0} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \left( \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,0} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \right)^{\alpha-1} \quad (\text{H.69})$$

for  $\alpha > 1$ , we have that

$$\begin{aligned} & \text{Tr} \left[ \hat{\sigma}_\varepsilon \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,0} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \left( \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,0} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \right)^{\alpha-1} \right] \\ &= \text{Tr} \left[ \hat{\sigma}_\varepsilon^{\frac{1}{2}} \rho_{0,0}^{\frac{1}{2}} \rho_{0,0}^{\frac{1}{2}} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \left( \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,0}^{\frac{1}{2}} \rho_{0,0}^{\frac{1}{2}} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \right)^{\alpha-1} \right] \end{aligned} \quad (\text{H.70})$$

$$= \text{Tr} \left[ \hat{\sigma}_\varepsilon^{\frac{1}{2}} \rho_{0,0}^{\frac{1}{2}} \left( \rho_{0,0}^{\frac{1}{2}} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,0}^{\frac{1}{2}} \right)^{\alpha-1} \rho_{0,0}^{\frac{1}{2}} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \right] \quad (\text{H.71})$$

$$= \text{Tr} \left[ \rho_{0,0}^{\frac{1}{2}} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \hat{\sigma}_\varepsilon^{\frac{1}{2}} \rho_{0,0}^{\frac{1}{2}} \left( \rho_{0,0}^{\frac{1}{2}} \hat{\sigma}_\varepsilon^{-1} \rho_{0,0}^{\frac{1}{2}} \right)^{\alpha-1} \right] \quad (\text{H.72})$$

$$= \text{Tr} \left[ \rho_{0,0} \left( \rho_{0,0}^{\frac{1}{2}} \hat{\sigma}_\varepsilon^{-1} \rho_{0,0}^{\frac{1}{2}} \right)^{\alpha-1} \right], \quad (\text{H.73})$$

where we applied Lemma 61 with  $f(x) = x^{\alpha-1}$  and  $L = \rho_{0,0}^{\frac{1}{2}} \hat{\sigma}_\varepsilon^{-\frac{1}{2}}$ . Now taking the limit  $\varepsilon \rightarrow 0^+$ , we conclude that

$$\lim_{\varepsilon \rightarrow 0^+} \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] = \lim_{\varepsilon \rightarrow 0^+} \text{Tr} \left[ \rho_{0,0} \left( \rho_{0,0}^{\frac{1}{2}} \hat{\sigma}_\varepsilon^{-1} \rho_{0,0}^{\frac{1}{2}} \right)^{\alpha-1} \right] \quad (\text{H.74})$$

$$= \text{Tr} \left[ \rho_{0,0} \left( \rho_{0,0}^{\frac{1}{2}} \sigma^{-1} \rho_{0,0}^{\frac{1}{2}} \right)^{\alpha-1} \right] \quad (\text{H.75})$$

$$= \text{Tr} \left[ \rho \left( \rho^{\frac{1}{2}} \sigma^{-1} \rho^{\frac{1}{2}} \right)^{\alpha-1} \right], \quad (\text{H.76})$$

for the case  $\alpha \in (1, \infty)$  and  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ , thus establishing (H.18).

For the case that  $\alpha \in (0, 1)$  and  $\text{supp}(\sigma) \subseteq \text{supp}(\rho)$ , we can employ the limit exchange from Lemma 69 and a similar argument as in (H.59)–(H.66), but with respect to the decomposition  $\mathcal{H} = \text{supp}(\rho) \oplus \ker(\rho)$ , to conclude that

$$\hat{Q}_\alpha(\rho \parallel \sigma) = \text{Tr} \left[ \rho \left( \rho^{-\frac{1}{2}} \sigma \rho^{-\frac{1}{2}} \right)^{1-\alpha} \right], \quad (\text{H.77})$$

thus establishing the second expression in (H.15). This case amounts to the exchange  $\rho \leftrightarrow \sigma$  and  $\alpha \leftrightarrow 1 - \alpha$ .

We finally consider the case  $\alpha \in (0, 1)$  and  $\text{supp}(\rho) \not\subseteq \text{supp}(\sigma)$ , which is the most involved case. Consider that

$$\sigma_\varepsilon := \sigma + \varepsilon I = \begin{bmatrix} \hat{\sigma}_\varepsilon & 0 \\ 0 & \varepsilon \Pi_\sigma^\perp \end{bmatrix}, \quad (\text{H.78})$$

where  $\hat{\sigma}_\varepsilon := \sigma + \varepsilon \Pi_\sigma$ . Let us define

$$\rho_\delta := (1 - \delta) \rho + \delta \pi, \quad (\text{H.79})$$

with  $\delta \in (0, 1)$  and  $\pi$  the maximally mixed state. By invoking Lemma 69, we conclude that the following exchange of limits is possible for  $\alpha \in (0, 1)$ :

$$\lim_{\varepsilon \rightarrow 0^+} D_\alpha(\rho \parallel \sigma_\varepsilon) = \lim_{\varepsilon \rightarrow 0^+} \lim_{\delta \rightarrow 0^+} D_\alpha(\rho_\delta \parallel \sigma_\varepsilon) = \lim_{\delta \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} D_\alpha(\rho_\delta \parallel \sigma_\varepsilon). \quad (\text{H.80})$$

Now define

$$\rho_{0,0}^\delta := \Pi_\sigma \rho_\delta \Pi_\sigma, \quad \rho_{0,1}^\delta := \Pi_\sigma \rho_\delta \Pi_\sigma^\perp, \quad \rho_{1,1}^\delta := \Pi_\sigma^\perp \rho_\delta \Pi_\sigma^\perp, \quad (\text{H.81})$$

so that

$$\rho_\delta = \begin{bmatrix} \rho_{0,0}^\delta & \rho_{0,1}^\delta \\ (\rho_{0,1}^\delta)^\dagger & \rho_{1,1}^\delta \end{bmatrix}. \quad (\text{H.82})$$

Then

$$D_\alpha(\rho_\delta \| \sigma_\varepsilon) = \frac{1}{\alpha - 1} \ln \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right]. \quad (\text{H.83})$$

Consider that

$$\sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} = \begin{bmatrix} \hat{\sigma}_\varepsilon & 0 \\ 0 & \varepsilon \Pi_\sigma^\perp \end{bmatrix}^{-\frac{1}{2}} \begin{bmatrix} \rho_{0,0}^\delta & \rho_{0,1}^\delta \\ (\rho_{0,1}^\delta)^\dagger & \rho_{1,1}^\delta \end{bmatrix} \begin{bmatrix} \hat{\sigma}_\varepsilon & 0 \\ 0 & \varepsilon \Pi_\sigma^\perp \end{bmatrix}^{-\frac{1}{2}} \quad (\text{H.84})$$

$$= \begin{bmatrix} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} & 0 \\ 0 & \varepsilon^{-\frac{1}{2}} \Pi_\sigma^\perp \end{bmatrix} \begin{bmatrix} \rho_{0,0}^\delta & \rho_{0,1}^\delta \\ (\rho_{0,1}^\delta)^\dagger & \rho_{1,1}^\delta \end{bmatrix} \begin{bmatrix} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} & 0 \\ 0 & \varepsilon^{-\frac{1}{2}} \Pi_\sigma^\perp \end{bmatrix} \quad (\text{H.85})$$

$$= \begin{bmatrix} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,0}^\delta \hat{\sigma}_\varepsilon^{-\frac{1}{2}} & \varepsilon^{-\frac{1}{2}} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,1}^\delta \Pi_\sigma^\perp \\ \varepsilon^{-\frac{1}{2}} \Pi_\sigma^\perp (\rho_{0,1}^\delta)^\dagger \hat{\sigma}_\varepsilon^{-\frac{1}{2}} & \varepsilon^{-1} \Pi_\sigma^\perp \rho_{1,1}^\delta \Pi_\sigma^\perp \end{bmatrix} \quad (\text{H.86})$$

$$= \begin{bmatrix} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,0}^\delta \hat{\sigma}_\varepsilon^{-\frac{1}{2}} & \varepsilon^{-\frac{1}{2}} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,1}^\delta \\ \varepsilon^{-\frac{1}{2}} (\rho_{0,1}^\delta)^\dagger \hat{\sigma}_\varepsilon^{-\frac{1}{2}} & \varepsilon^{-1} \rho_{1,1}^\delta \end{bmatrix}. \quad (\text{H.87})$$

So then

$$\begin{aligned} & \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] \\ &= \text{Tr} \left[ \begin{bmatrix} \hat{\sigma}_\varepsilon & 0 \\ 0 & \varepsilon \Pi_\sigma^\perp \end{bmatrix} \left( \begin{bmatrix} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,0}^\delta \hat{\sigma}_\varepsilon^{-\frac{1}{2}} & \varepsilon^{-\frac{1}{2}} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,1}^\delta \\ \varepsilon^{-\frac{1}{2}} (\rho_{0,1}^\delta)^\dagger \hat{\sigma}_\varepsilon^{-\frac{1}{2}} & \varepsilon^{-1} \rho_{1,1}^\delta \end{bmatrix} \right)^\alpha \right] \end{aligned} \quad (\text{H.88})$$

$$= \text{Tr} \left[ \begin{bmatrix} \hat{\sigma}_\varepsilon & 0 \\ 0 & \varepsilon \Pi_\sigma^\perp \end{bmatrix} \left( \varepsilon^{-1} \begin{bmatrix} \varepsilon \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,0}^\delta \hat{\sigma}_\varepsilon^{-\frac{1}{2}} & \varepsilon^{\frac{1}{2}} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,1}^\delta \\ \varepsilon^{\frac{1}{2}} (\rho_{0,1}^\delta)^\dagger \hat{\sigma}_\varepsilon^{-\frac{1}{2}} & \rho_{1,1}^\delta \end{bmatrix} \right)^\alpha \right] \quad (\text{H.89})$$

$$= \text{Tr} \left[ \begin{bmatrix} \varepsilon^{-\alpha} \hat{\sigma}_\varepsilon & 0 \\ 0 & \varepsilon^{1-\alpha} \Pi_\sigma^\perp \end{bmatrix} \begin{bmatrix} \varepsilon \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,0}^\delta \hat{\sigma}_\varepsilon^{-\frac{1}{2}} & \varepsilon^{\frac{1}{2}} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,1}^\delta \\ \varepsilon^{\frac{1}{2}} (\rho_{0,1}^\delta)^\dagger \hat{\sigma}_\varepsilon^{-\frac{1}{2}} & \rho_{1,1}^\delta \end{bmatrix} \right] \quad (\text{H.90})$$

Let us define

$$K(\varepsilon) := \begin{bmatrix} \varepsilon \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,0}^\delta \hat{\sigma}_\varepsilon^{-\frac{1}{2}} & \varepsilon^{\frac{1}{2}} \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,1}^\delta \\ \varepsilon^{\frac{1}{2}} (\rho_{0,1}^\delta)^\dagger \hat{\sigma}_\varepsilon^{-\frac{1}{2}} & \rho_{1,1}^\delta \end{bmatrix}, \quad (\text{H.91})$$

so that we can write

$$\text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] = \text{Tr} \left[ \begin{bmatrix} \varepsilon^{-\alpha} \hat{\sigma}_\varepsilon & 0 \\ 0 & \varepsilon^{1-\alpha} \Pi_\sigma^\perp \end{bmatrix} (K(\varepsilon))^\alpha \right]. \quad (\text{H.92})$$

Now let us invoke Lemma 62 with the substitutions

$$A \leftrightarrow \rho_{1,1}^\delta, \quad (\text{H.93})$$

$$B \leftrightarrow (\rho_{0,1}^\delta)^\dagger \hat{\sigma}_\varepsilon^{-\frac{1}{2}}, \quad (\text{H.94})$$

$$C \leftrightarrow \hat{\sigma}_\varepsilon^{-\frac{1}{2}} \rho_{0,0}^\delta \hat{\sigma}_\varepsilon^{-\frac{1}{2}}, \quad (\text{H.95})$$

$$\varepsilon \leftrightarrow \varepsilon^{\frac{1}{2}}. \quad (\text{H.96})$$

Defining

$$L(\varepsilon) := \begin{bmatrix} \varepsilon S(\rho^\delta, \hat{\sigma}_\varepsilon) & 0 \\ 0 & \rho_{1,1}^\delta + \varepsilon R \end{bmatrix}, \quad (\text{H.97})$$

$$S(\rho^\delta, \hat{\sigma}_\varepsilon) := \hat{\sigma}_\varepsilon^{-\frac{1}{2}} (\rho_{0,0}^\delta - \rho_{0,1}^\delta (\rho_{1,1}^\delta)^{-1} (\rho_{0,1}^\delta)^\dagger) \hat{\sigma}_\varepsilon^{-\frac{1}{2}}, \quad (\text{H.98})$$

$$R := \text{Re}[(\rho_{1,1}^\delta)^{-1} (\rho_{0,1}^\delta)^\dagger (\hat{\sigma}_\varepsilon)^{-1} (\rho_{0,1}^\delta)], \quad (\text{H.99})$$

we conclude from Lemma 62 that

$$\left\| K(\varepsilon) - e^{-i\sqrt{\varepsilon}G} L(\varepsilon) e^{i\sqrt{\varepsilon}G} \right\|_\infty \leq o(\varepsilon), \quad (\text{H.100})$$

where  $G$  in Lemma 62 is defined from  $A$  and  $B$  above. The inequality in (H.100) in turn implies the following operator inequalities:

$$e^{-i\sqrt{\varepsilon}G} L(\varepsilon) e^{i\sqrt{\varepsilon}G} - o(\varepsilon)I \leq K(\varepsilon) \leq e^{-i\sqrt{\varepsilon}G} L(\varepsilon) e^{i\sqrt{\varepsilon}G} + o(\varepsilon)I. \quad (\text{H.101})$$

Observe that

$$e^{-i\sqrt{\varepsilon}G} L(\varepsilon) e^{i\sqrt{\varepsilon}G} + o(\varepsilon)I = e^{-i\sqrt{\varepsilon}G} [L(\varepsilon) + o(\varepsilon)I] e^{i\sqrt{\varepsilon}G}. \quad (\text{H.102})$$

Now invoking these and the operator monotonicity of the function  $x^\alpha$  for  $\alpha \in (0, 1)$ , we find that

$$\text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] \quad (\text{H.103})$$

$$= \text{Tr} \left[ \begin{bmatrix} \varepsilon^{-\alpha} \hat{\sigma}_\varepsilon & 0 \\ 0 & \varepsilon^{1-\alpha} \Pi_\sigma^\perp \end{bmatrix} (K(\varepsilon))^\alpha \right] \quad (\text{H.104})$$

$$\leq \text{Tr} \left[ \begin{bmatrix} \varepsilon^{-\alpha} \hat{\sigma}_\varepsilon & 0 \\ 0 & \varepsilon^{1-\alpha} \Pi_\sigma^\perp \end{bmatrix} \left( e^{-i\sqrt{\varepsilon}G} [L(\varepsilon) + o(\varepsilon)I] e^{i\sqrt{\varepsilon}G} \right)^\alpha \right] \quad (\text{H.105})$$

$$= \text{Tr} \left[ \begin{bmatrix} \varepsilon^{-\alpha} \hat{\sigma}_\varepsilon & 0 \\ 0 & \varepsilon^{1-\alpha} \Pi_\sigma^\perp \end{bmatrix} e^{-i\sqrt{\varepsilon}G} (L(\varepsilon) + o(\varepsilon)I)^\alpha e^{i\sqrt{\varepsilon}G} \right]. \quad (\text{H.106})$$

Consider that

$$\begin{aligned} & (L(\varepsilon) + o(\varepsilon)I)^\alpha \\ &= \begin{bmatrix} \varepsilon S(\rho^\delta, \hat{\sigma}_\varepsilon) + o(\varepsilon)I & 0 \\ 0 & \rho_{1,1}^\delta + \varepsilon R + o(\varepsilon)I \end{bmatrix}^\alpha \end{aligned} \quad (\text{H.107})$$

$$= \begin{bmatrix} (\varepsilon S(\rho^\delta, \hat{\sigma}_\varepsilon) + o(\varepsilon)I)^\alpha & 0 \\ 0 & (\rho_{1,1}^\delta + \varepsilon R + o(\varepsilon)I)^\alpha \end{bmatrix} \quad (\text{H.108})$$

$$= \begin{bmatrix} \varepsilon^\alpha (S(\rho^\delta, \hat{\sigma}_\varepsilon) + o(1)I)^\alpha & 0 \\ 0 & (\rho_{1,1}^\delta + \varepsilon R + o(\varepsilon)I)^\alpha \end{bmatrix}. \quad (\text{H.109})$$

Now expanding  $e^{i\sqrt{\varepsilon}G}$  to first order in order to evaluate (H.106) (higher order terms will end up being irrelevant), we find that

$$\begin{aligned}
& \text{Tr} \left[ \begin{bmatrix} \varepsilon^{-\alpha} \hat{\sigma}_\varepsilon & 0 \\ 0 & \varepsilon^{1-\alpha} \Pi_\sigma^\perp \end{bmatrix} e^{-i\sqrt{\varepsilon}G} (L(\varepsilon) + o(\varepsilon)I)^\alpha e^{i\sqrt{\varepsilon}G} \right] \\
&= \text{Tr} \left[ \begin{bmatrix} \varepsilon^{-\alpha} \hat{\sigma}_\varepsilon & 0 \\ 0 & \varepsilon^{1-\alpha} \Pi_\sigma^\perp \end{bmatrix} (L(\varepsilon) + o(\varepsilon)I)^\alpha \right] \\
&+ \text{Tr} \left[ \begin{bmatrix} \varepsilon^{-\alpha} \hat{\sigma}_\varepsilon & 0 \\ 0 & \varepsilon^{1-\alpha} \Pi_\sigma^\perp \end{bmatrix} (-i\sqrt{\varepsilon}G) (L(\varepsilon) + o(\varepsilon)I)^\alpha \right] \\
&+ \text{Tr} \left[ \begin{bmatrix} \varepsilon^{-\alpha} \hat{\sigma}_\varepsilon & 0 \\ 0 & \varepsilon^{1-\alpha} \Pi_\sigma^\perp \end{bmatrix} (L(\varepsilon) + o(\varepsilon)I)^\alpha (i\sqrt{\varepsilon}G) \right] + o(1) \tag{H.110}
\end{aligned}$$

$$\begin{aligned}
&= \text{Tr} \left[ \begin{bmatrix} \hat{\sigma}_\varepsilon (S(\rho^\delta, \hat{\sigma}_\varepsilon) + o(1)I)^\alpha & 0 \\ 0 & \varepsilon^{1-\alpha} \Pi_\sigma^\perp (\rho_{1,1}^\delta + \varepsilon R + o(\varepsilon)I)^\alpha \end{bmatrix} \right] \\
&- i\sqrt{\varepsilon} \text{Tr} \left[ \begin{bmatrix} (S(\rho^\delta, \hat{\sigma}_\varepsilon) + o(1)I)^\alpha \hat{\sigma}_\varepsilon & 0 \\ 0 & \varepsilon^{1-\alpha} (\rho_{1,1}^\delta + \varepsilon R + o(\varepsilon)I)^\alpha \Pi_\sigma^\perp \end{bmatrix} G \right] \\
&+ i\sqrt{\varepsilon} \text{Tr} \left[ \begin{bmatrix} \hat{\sigma}_\varepsilon (S(\rho^\delta, \hat{\sigma}_\varepsilon) + o(1)I)^\alpha & 0 \\ 0 & \varepsilon^{1-\alpha} \Pi_\sigma^\perp (\rho_{1,1}^\delta + \varepsilon R + o(\varepsilon)I)^\alpha \end{bmatrix} G \right] + o(1) \tag{H.111}
\end{aligned}$$

$$= \text{Tr} \left[ \begin{bmatrix} \hat{\sigma}_\varepsilon (S(\rho^\delta, \hat{\sigma}_\varepsilon) + o(1)I)^\alpha & 0 \\ 0 & \varepsilon^{1-\alpha} \Pi_\sigma^\perp (\rho_{1,1}^\delta + \varepsilon R + o(\varepsilon)I)^\alpha \end{bmatrix} \right] + o(1) \tag{H.112}$$

$$= \text{Tr} [\hat{\sigma}_\varepsilon (S(\rho^\delta, \hat{\sigma}_\varepsilon) + o(1)I)^\alpha] + \varepsilon^{1-\alpha} \text{Tr} [\Pi_\sigma^\perp (\rho_{1,1}^\delta + \varepsilon R + o(\varepsilon)I)^\alpha] + o(1). \tag{H.113}$$

By observing the last line, we see that higher order terms for  $e^{i\sqrt{\varepsilon}G}$  include prefactors of  $\varepsilon$  (or higher powers), which vanish in the  $\varepsilon \rightarrow 0^+$  limit. Now taking the limit  $\varepsilon \rightarrow 0^+$ , we find that

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0^+} \text{Tr} \left[ \begin{bmatrix} \varepsilon^{-\alpha} \hat{\sigma}_\varepsilon & 0 \\ 0 & \varepsilon^{1-\alpha} \Pi_\sigma^\perp \end{bmatrix} e^{-i\sqrt{\varepsilon}G} (L(\varepsilon) + o(\varepsilon)I)^\alpha e^{i\sqrt{\varepsilon}G} \right] \\
&= \text{Tr} \left[ \sigma \left( \sigma^{-\frac{1}{2}} (\rho_{0,0}^\delta - \rho_{0,1}^\delta (\rho_{1,1}^\delta)^{-1} (\rho_{0,1}^\delta)^\dagger) \sigma^{-\frac{1}{2}} \right)^\alpha \right], \tag{H.114}
\end{aligned}$$

where the inverses are taken on the support of  $\sigma$ . By proceeding in a similar way, but using the lower bound in (H.101), we find the following lower bound on (H.103):

$$\text{Tr} \left[ \begin{bmatrix} \varepsilon^{-\alpha} \hat{\sigma}_\varepsilon & 0 \\ 0 & \varepsilon^{1-\alpha} \Pi_\sigma^\perp \end{bmatrix} e^{-i\sqrt{\varepsilon}G} (L(\varepsilon) - o(\varepsilon)I)^\alpha e^{i\sqrt{\varepsilon}G} \right]. \tag{H.115}$$

Then by the same argument above, the lower bound on (H.103) after taking the limit  $\varepsilon \rightarrow 0^+$  is the same as in (H.114). So we conclude that

$$\lim_{\varepsilon \rightarrow 0^+} \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] = \text{Tr} \left[ \sigma \left( \sigma^{-\frac{1}{2}} (\rho_{0,0}^\delta - \rho_{0,1}^\delta (\rho_{1,1}^\delta)^{-1} (\rho_{0,1}^\delta)^\dagger) \sigma^{-\frac{1}{2}} \right)^\alpha \right]. \tag{H.116}$$

Now consider that

$$\lim_{\delta \rightarrow 0^+} \rho_{0,0}^\delta - \rho_{0,1}^\delta (\rho_{1,1}^\delta)^{-1} (\rho_{0,1}^\delta)^\dagger = \rho_{0,0} - \rho_{0,1} \rho_{1,1}^{-1} \rho_{0,1}^\dagger, \quad (\text{H.117})$$

where the inverse on the right is taken on the support of  $\rho_{1,1}$ . This follows because the image of  $\rho_{0,1}^\dagger$  is contained in the support of  $\rho_{1,1}$ . Thus, we take the limit  $\delta \rightarrow 0^+$ , and find that

$$\lim_{\delta \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] = \text{Tr} \left[ \sigma \left( \sigma^{-\frac{1}{2}} \left( \rho_{0,0} - \rho_{0,1} \rho_{1,1}^{-1} \rho_{0,1}^\dagger \right) \sigma^{-\frac{1}{2}} \right)^\alpha \right], \quad (\text{H.118})$$

where all inverses are taken on the support. This concludes the proof. ■

If the state  $\rho$  is pure, then the geometric Rényi relative entropy simplifies as follows, such that it is independent of  $\alpha$ :

**Proposition 71** *Let  $\rho = |\psi\rangle\langle\psi|$  be a pure state and  $\sigma$  a positive semi-definite operator. Then the following equality holds for all  $\alpha \in (0, 1) \cup (1, \infty)$ :*

$$\widehat{D}_\alpha(\rho||\sigma) = \begin{cases} \ln \langle \psi | \sigma^{-1} | \psi \rangle & \text{if } \text{supp}(|\psi\rangle\langle\psi|) \subseteq \text{supp}(\sigma) \\ +\infty & \text{otherwise} \end{cases}, \quad (\text{H.119})$$

where  $\sigma^{-1}$  is understood as a generalized inverse. If  $\sigma$  is also a rank-one operator, so that  $\sigma = |\phi\rangle\langle\phi|$  and  $\|\phi\|_2 > 0$ , then the following equality holds for all  $\alpha \in (0, 1) \cup (1, \infty)$ :

$$\widehat{D}_\alpha(\rho||\sigma) = \begin{cases} -\ln \|\phi\|_2^2 & \text{if } \exists c \in \mathbb{C} \text{ such that } |\psi\rangle = c|\phi\rangle \\ +\infty & \text{otherwise} \end{cases}. \quad (\text{H.120})$$

In particular, if  $\sigma = |\phi\rangle\langle\phi|$  is a state so that  $\|\phi\|_2^2 = 1$ , then

$$\widehat{D}_\alpha(\rho||\sigma) = \begin{cases} 0 & \text{if } |\psi\rangle = |\phi\rangle \\ +\infty & \text{otherwise} \end{cases}. \quad (\text{H.121})$$

**Proof.** Defining  $\sigma_\varepsilon := \sigma + \varepsilon I$ , consider that

$$\text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] = \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} |\psi\rangle\langle\psi| \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] \quad (\text{H.122})$$

$$= \left( \left\| \sigma_\varepsilon^{-\frac{1}{2}} |\psi\rangle \right\|_2^2 \right)^\alpha \text{Tr} \left[ \sigma_\varepsilon \left( \frac{\sigma_\varepsilon^{-\frac{1}{2}} |\psi\rangle\langle\psi| \sigma_\varepsilon^{-\frac{1}{2}}}{\left\| \sigma_\varepsilon^{-\frac{1}{2}} |\psi\rangle \right\|_2^2} \right)^\alpha \right] \quad (\text{H.123})$$

$$= \left( \left\| \sigma_\varepsilon^{-\frac{1}{2}} |\psi\rangle \right\|_2^2 \right)^\alpha \text{Tr} \left[ \sigma_\varepsilon \frac{\sigma_\varepsilon^{-\frac{1}{2}} |\psi\rangle\langle\psi| \sigma_\varepsilon^{-\frac{1}{2}}}{\left\| \sigma_\varepsilon^{-\frac{1}{2}} |\psi\rangle \right\|_2^2} \right] \quad (\text{H.124})$$



$$= \left( \left\| \sigma_\varepsilon^{-\frac{1}{2}} |\psi\rangle \right\|_2^2 \right)^{\alpha-1} \text{Tr} \left[ \sigma_\varepsilon \sigma_\varepsilon^{-\frac{1}{2}} |\psi\rangle \langle \psi| \sigma_\varepsilon^{-\frac{1}{2}} \right] \quad (\text{H.125})$$

$$= \left( \left\| \sigma_\varepsilon^{-\frac{1}{2}} |\psi\rangle \right\|_2^2 \right)^{\alpha-1} \text{Tr} [|\psi\rangle \langle \psi|] \quad (\text{H.126})$$

$$= [\langle \psi | \sigma_\varepsilon^{-1} | \psi \rangle]^{\alpha-1}. \quad (\text{H.127})$$

The third equality follows because  $|\varphi\rangle \langle \varphi|^\alpha = |\varphi\rangle \langle \varphi|$  for all  $\alpha \in (0, 1) \cup (1, \infty)$  when  $\| |\varphi\rangle \|_2 = 1$ . Applying the above chain of equalities, we find that

$$\frac{1}{\alpha-1} \ln \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] = \frac{1}{\alpha-1} \log_2 [\langle \psi | \sigma_\varepsilon^{-1} | \psi \rangle]^{\alpha-1} \quad (\text{H.128})$$

$$= \ln \langle \psi | \sigma_\varepsilon^{-1} | \psi \rangle. \quad (\text{H.129})$$

Now let a spectral decomposition of  $\sigma$  be given by

$$\sigma = \sum_y \mu_y Q_y, \quad (\text{H.130})$$

where  $\mu_y$  are the non-negative eigenvalues and  $Q_y$  are the eigenprojections. In this decomposition, we are including values of  $\mu_y$  for which  $\mu_y = 0$ . Then it follows that

$$\sigma_\varepsilon = \sigma + \varepsilon I = \sum_y (\mu_y + \varepsilon) Q_y, \quad (\text{H.131})$$

and we find that

$$\sigma_\varepsilon^{-1} = \sum_y (\mu_y + \varepsilon)^{-1} Q_y. \quad (\text{H.132})$$

We can then conclude that

$$\ln \langle \psi | \sigma_\varepsilon^{-1} | \psi \rangle = \ln \left[ \langle \psi | \sum_y (\mu_y + \varepsilon)^{-1} Q_y | \psi \rangle \right] \quad (\text{H.133})$$

$$= \ln \left[ \sum_y (\mu_y + \varepsilon)^{-1} \langle \psi | Q_y | \psi \rangle \right] \quad (\text{H.134})$$

$$= \ln \left[ \sum_{y: \mu_y \neq 0} (\mu_y + \varepsilon)^{-1} \langle \psi | Q_y | \psi \rangle + \varepsilon^{-1} \langle \psi | Q_{y_0} | \psi \rangle \right], \quad (\text{H.135})$$

where  $y_0$  is the value of  $y$  for which  $\mu_y = 0$  (if no such value of  $y$  exists, then  $Q_{y_0}$  is equal to the zero operator). Thus, if  $\langle \psi | Q_{y_0} | \psi \rangle \neq 0$  (equivalent to  $|\psi\rangle$  being outside the support of  $\sigma$ ), then it follows that

$$\lim_{\varepsilon \rightarrow 0^+} \ln \langle \psi | \sigma_\varepsilon^{-1} | \psi \rangle = +\infty. \quad (\text{H.136})$$

Otherwise the expression converges as claimed.

Now suppose that  $\sigma$  is a rank-one operator, so that  $\sigma = |\phi\rangle\langle\phi|$  and  $\|\phi\|_2 > 0$ . By defining

$$|\phi'\rangle := \frac{|\phi\rangle}{\sqrt{\|\phi\|_2}}, \quad (\text{H.137})$$

$$N := \|\phi\|_2^2, \quad (\text{H.138})$$

we find that

$$\sigma_\varepsilon = |\phi\rangle\langle\phi| + \varepsilon I \quad (\text{H.139})$$

$$= N |\phi'\rangle\langle\phi'| + \varepsilon (I - |\phi'\rangle\langle\phi'| + |\phi'\rangle\langle\phi'|) \quad (\text{H.140})$$

$$= (N + \varepsilon) |\phi'\rangle\langle\phi'| + \varepsilon (I - |\phi'\rangle\langle\phi'|), \quad (\text{H.141})$$

so that

$$\sigma_\varepsilon^{-1} = (N + \varepsilon)^{-1} |\phi'\rangle\langle\phi'| + \varepsilon^{-1} (I - |\phi'\rangle\langle\phi'|) \quad (\text{H.142})$$

$$= ((N + \varepsilon)^{-1} - \varepsilon^{-1}) |\phi'\rangle\langle\phi'| + \varepsilon^{-1} I \quad (\text{H.143})$$

and then

$$\ln [\langle\psi|\sigma_\varepsilon^{-1}|\psi\rangle] = \ln [\langle\psi| ((N + \varepsilon)^{-1} - \varepsilon^{-1}) |\phi'\rangle\langle\phi'| + \varepsilon^{-1} I |\psi\rangle] \quad (\text{H.144})$$

$$= \ln \left[ ((N + \varepsilon)^{-1} - \varepsilon^{-1}) |\langle\psi|\phi'\rangle|^2 + \varepsilon^{-1} \right] \quad (\text{H.145})$$

$$= \ln \left[ \frac{|\langle\psi|\phi'\rangle|^2}{N + \varepsilon} + \frac{1 - |\langle\psi|\phi'\rangle|^2}{\varepsilon} \right]. \quad (\text{H.146})$$

Note that we always have  $|\langle\psi|\phi'\rangle|^2 \in [0, 1]$  because  $|\psi\rangle$  and  $|\phi'\rangle$  are unit vectors. In the case that  $|\langle\psi|\phi'\rangle|^2 \in [0, 1)$ , then we find that

$$\lim_{\varepsilon \rightarrow 0^+} \ln \left[ \frac{|\langle\psi|\phi'\rangle|^2}{N + \varepsilon} + \frac{1 - |\langle\psi|\phi'\rangle|^2}{\varepsilon} \right] = +\infty. \quad (\text{H.147})$$

Otherwise, if  $|\langle\psi|\phi'\rangle|^2 = 1$ , then

$$\lim_{\varepsilon \rightarrow 0^+} \ln \left[ \frac{|\langle\psi|\phi'\rangle|^2}{N + \varepsilon} + \frac{1 - |\langle\psi|\phi'\rangle|^2}{\varepsilon} \right] = \lim_{\varepsilon \rightarrow 0^+} \ln \left[ \frac{1}{N + \varepsilon} \right] \quad (\text{H.148})$$

$$= -\ln N, \quad (\text{H.149})$$

concluding the proof. ■

We note here that, for pure states  $\rho$  and  $\sigma$ , the geometric Rényi relative entropy is either equal to zero or  $+\infty$ , depending on whether  $\rho = \sigma$ . This behavior of the geometric Rényi

relative entropy for pure states  $\rho$  and  $\sigma$  is very different from that of the Petz–Rényi and sandwiched Rényi relative entropies. The latter quantities always evaluate to a finite value if the pure states are non-orthogonal.

The geometric Rényi relative entropy possesses a number of useful properties, which we list in the proposition below.

**Proposition 72 (Properties of the geometric Rényi relative entropy)** *For all states  $\rho, \rho_1, \rho_2$  and positive semi-definite operators  $\sigma, \sigma_1, \sigma_2$ , the geometric Rényi relative entropy satisfies the following properties.*

1. *Isometric invariance: For all  $\alpha \in (0, 1) \cup (1, \infty)$  and for all isometries  $V$ ,*

$$\widehat{D}_\alpha(\rho\|\sigma) = \widehat{D}_\alpha(V\rho V^\dagger\|V\sigma V^\dagger). \quad (\text{H.150})$$

2. *Monotonicity in  $\alpha$ : For all  $\alpha \in (0, 1) \cup (1, \infty)$ , the geometric Rényi relative entropy  $\widehat{D}_\alpha$  is monotonically increasing in  $\alpha$ ; i.e.,  $\alpha < \beta$  implies  $\widehat{D}_\alpha(\rho\|\sigma) \leq \widehat{D}_\beta(\rho\|\sigma)$ .*
3. *Additivity: For all  $\alpha \in (0, 1) \cup (1, \infty)$ ,*

$$\widehat{D}_\alpha(\rho_1 \otimes \rho_2\|\sigma_1 \otimes \sigma_2) = \widehat{D}_\alpha(\rho_1\|\sigma_1) + \widehat{D}_\alpha(\rho_2\|\sigma_2). \quad (\text{H.151})$$

4. *Direct-sum property: Let  $p : \mathcal{X} \rightarrow [0, 1]$  be a probability distribution over a finite alphabet  $\mathcal{X}$  with associated  $|\mathcal{X}|$ -dimensional system  $X$ , and let  $q : \mathcal{X} \rightarrow (0, \infty)$  be a positive function on  $\mathcal{X}$ . Let  $\{\rho_A^x : x \in \mathcal{X}\}$  be a set of states on a system  $A$ , and let  $\{\sigma_A^x : x \in \mathcal{X}\}$  be a set of positive semi-definite operators on  $A$ . Then,*

$$\widehat{Q}_\alpha(\rho_{XA}\|\sigma_{XA}) = \sum_{x \in \mathcal{X}} p(x)^\alpha q(x)^{1-\alpha} \widehat{Q}_\alpha(\rho_A^x\|\sigma_A^x), \quad (\text{H.152})$$

where

$$\rho_{XA} := \sum_{x \in \mathcal{X}} p(x) |x\rangle\langle x|_X \otimes \rho_A^x, \quad (\text{H.153})$$

$$\sigma_{XA} := \sum_{x \in \mathcal{X}} q(x) |x\rangle\langle x|_X \otimes \sigma_A^x. \quad (\text{H.154})$$

**Proof.**

1. *Proof of isometric invariance:* Let us start by writing  $\widehat{D}_\alpha(\rho\|\sigma)$  as in (H.13)–(H.14):

$$\widehat{D}_\alpha(\rho\|\sigma) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\alpha - 1} \ln \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right]. \quad (\text{H.155})$$

where

$$\sigma_\varepsilon := \sigma + \varepsilon I. \quad (\text{H.156})$$

Let  $V$  be an isometry. Then, defining

$$\omega_\varepsilon := V\sigma V^\dagger + \varepsilon I, \quad (\text{H.157})$$

we find that

$$\widehat{D}_\alpha(V\rho V^\dagger \| V\sigma V^\dagger) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\alpha - 1} \ln \text{Tr} \left[ \omega_\varepsilon^{-\frac{1}{2}} V\rho V^\dagger \omega_\varepsilon^{-\frac{1}{2}} \right]^\alpha. \quad (\text{H.158})$$

Now let  $\Pi := VV^\dagger$  be the projection onto the image of  $V$ , so that  $\Pi V = V$ , and let  $\hat{\Pi} := I - \Pi$ . Then, we can write

$$\omega_\varepsilon = V\sigma V^\dagger + \varepsilon \Pi + \varepsilon \hat{\Pi} = V\sigma_\varepsilon V^\dagger + \varepsilon \hat{\Pi}. \quad (\text{H.159})$$

Since  $V\sigma_\varepsilon V^\dagger$  and  $\varepsilon \hat{\Pi}$  are supported on orthogonal subspaces, we obtain

$$\omega_\varepsilon^{-\frac{1}{2}} = V\sigma_\varepsilon^{-\frac{1}{2}} V^\dagger + \varepsilon^{-\frac{1}{2}} \hat{\Pi}. \quad (\text{H.160})$$

Consider then that

$$\omega_\varepsilon^{-\frac{1}{2}} V\rho V^\dagger \omega_\varepsilon^{-\frac{1}{2}} = \left( V\sigma_\varepsilon^{-\frac{1}{2}} V^\dagger + \varepsilon^{-\frac{1}{2}} \hat{\Pi} \right) \Pi V\rho V^\dagger \Pi \left( V\sigma_\varepsilon^{-\frac{1}{2}} V^\dagger + \varepsilon^{-\frac{1}{2}} \hat{\Pi} \right) \quad (\text{H.161})$$

$$= \left( V\sigma_\varepsilon^{-\frac{1}{2}} V^\dagger \right) \Pi V\rho V^\dagger \Pi \left( V\sigma_\varepsilon^{-\frac{1}{2}} V^\dagger \right) \quad (\text{H.162})$$

$$= V\sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} V^\dagger, \quad (\text{H.163})$$

where the second equality follows because  $\hat{\Pi}\Pi = \Pi\hat{\Pi} = 0$ . Thus,

$$\left( \omega_\varepsilon^{-\frac{1}{2}} V\rho V^\dagger \omega_\varepsilon^{-\frac{1}{2}} \right)^\alpha = V \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha V^\dagger, \quad (\text{H.164})$$

and we find that

$$\text{Tr} \left[ \omega_\varepsilon \left( \omega_\varepsilon^{-\frac{1}{2}} V\rho V^\dagger \omega_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] = \text{Tr} \left[ \left( V\sigma_\varepsilon V^\dagger + \varepsilon \hat{\Pi} \right) V \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha V^\dagger \right] \quad (\text{H.165})$$

$$= \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right]. \quad (\text{H.166})$$

Since the equality

$$\text{Tr} \left[ \omega_\varepsilon \left( \omega_\varepsilon^{-\frac{1}{2}} V\rho V^\dagger \omega_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] = \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho \sigma_\varepsilon^{-\frac{1}{2}} \right)^\alpha \right] \quad (\text{H.167})$$

holds for all  $\varepsilon > 0$ , we conclude the proof of isometric invariance by taking the limit  $\varepsilon \rightarrow 0^+$ .

2. *Proof of monotonicity in  $\alpha$ :* We prove this by showing that the derivative is non-negative for all  $\alpha > 0$ . By applying (H.21), we can consider  $\rho$  and  $\sigma$  to be positive definite without loss of generality. By applying (H.26), consider that

$$\widehat{Q}_\alpha(\rho||\sigma) = \text{Tr} \left[ \rho \left( \rho^{-\frac{1}{2}} \sigma \rho^{-\frac{1}{2}} \right)^{1-\alpha} \right] \quad (\text{H.168})$$

$$= \text{Tr} \left[ \rho \left( \rho^{\frac{1}{2}} \sigma^{-1} \rho^{\frac{1}{2}} \right)^{\alpha-1} \right]. \quad (\text{H.169})$$

Now defining  $|\varphi^\rho\rangle = (\rho^{\frac{1}{2}} \otimes I)|\Gamma\rangle$  as a purification of  $\rho$ , and setting

$$\gamma = \alpha - 1, \quad (\text{H.170})$$

$$X = \rho^{\frac{1}{2}} \sigma^{-1} \rho^{\frac{1}{2}}, \quad (\text{H.171})$$

we can write the geometric Rényi relative entropy as

$$\widehat{D}_\alpha(\rho||\sigma) = \frac{1}{\gamma} \ln \langle \varphi^\rho | X^\gamma \otimes I | \varphi^\rho \rangle, \quad (\text{H.172})$$

where we made use of (H.169). Then  $\frac{\partial}{\partial \alpha} = \frac{\partial}{\partial \gamma} \frac{\partial \gamma}{\partial \alpha} = \frac{\partial}{\partial \gamma}$ , and so we find that

$$\begin{aligned} & \frac{\partial}{\partial \alpha} \widehat{D}_\alpha(\rho||\sigma) \\ &= \frac{\partial}{\partial \gamma} \left[ \frac{1}{\gamma} \ln \langle \varphi^\rho | X^\gamma \otimes I | \varphi^\rho \rangle \right] \end{aligned} \quad (\text{H.173})$$

$$= \left[ -\frac{1}{\gamma^2} \ln \langle \varphi^\rho | X^\gamma \otimes I | \varphi^\rho \rangle + \frac{1}{\gamma} \frac{\partial}{\partial \gamma} \ln \langle \varphi^\rho | X^\gamma \otimes I | \varphi^\rho \rangle \right] \quad (\text{H.174})$$

$$= \left[ -\frac{1}{\gamma^2} \ln \langle \varphi^\rho | X^\gamma \otimes I | \varphi^\rho \rangle + \frac{1}{\gamma} \frac{\langle \varphi^\rho | X^\gamma \ln X \otimes I | \varphi^\rho \rangle}{\langle \varphi^\rho | X^\gamma \otimes I | \varphi^\rho \rangle} \right] \quad (\text{H.175})$$

$$= \left[ \frac{-\langle \varphi^\rho | X^\gamma \otimes I | \varphi^\rho \rangle \ln \langle \varphi^\rho | X^\gamma \otimes I | \varphi^\rho \rangle + \gamma \langle \varphi^\rho | X^\gamma \ln X \otimes I | \varphi^\rho \rangle}{\gamma^2 \langle \varphi^\rho | X^\gamma \otimes I | \varphi^\rho \rangle} \right] \quad (\text{H.176})$$

$$= \left[ \frac{-\langle \varphi^\rho | X^\gamma \otimes I | \varphi^\rho \rangle \ln \langle \varphi^\rho | X^\gamma \otimes I | \varphi^\rho \rangle + \langle \varphi^\rho | X^\gamma \ln X^\gamma \otimes I | \varphi^\rho \rangle}{\gamma^2 \langle \varphi^\rho | X^\gamma \otimes I | \varphi^\rho \rangle} \right]. \quad (\text{H.177})$$

Letting  $g(x) := x \ln x$ , we write

$$\frac{\partial}{\partial \alpha} \widehat{D}_\alpha(\rho||\sigma) = \frac{\langle \varphi^\rho | g(X^\gamma \otimes I) | \varphi^\rho \rangle - g(\langle \varphi^\rho | (X^\gamma \otimes I) | \varphi^\rho \rangle)}{\gamma^2 \langle \varphi^\rho | X^\gamma \otimes I | \varphi^\rho \rangle}. \quad (\text{H.178})$$

Then, since  $g(x)$  is operator convex, by the operator Jensen inequality [140], we conclude that

$$\langle \varphi^\rho | g(X^\gamma \otimes I) | \varphi^\rho \rangle \geq g(\langle \varphi^\rho | (X^\gamma \otimes I) | \varphi^\rho \rangle), \quad (\text{H.179})$$

which means that  $\frac{\partial}{\partial \alpha} \widehat{D}_\alpha(\rho||\sigma) \geq 0$ . Therefore,  $\widehat{D}_\alpha(\rho||\sigma)$  is monotonically increasing in  $\alpha$ , as required.

3. *Proof of additivity*: The proof of (H.151) is found by direct evaluation.
4. *Proof of direct-sum property*: The proof of (H.152) is found by direct evaluation.

■

We now recall the data-processing inequality for the geometric Rényi relative entropy for  $\alpha \in (0, 1) \cup (1, 2]$ . This was established by an operator-theoretic approach in [34] and by an operational method in [35, 36]. The operator-theoretic method has its roots in [141, Proposition 2.5] and was reviewed in [38, Corollary 3.31]. We follow the operator-theoretic approach here.

**Theorem 73 (Data-processing inequality for geometric Rényi relative entropy)** *Let  $\rho$  be a state,  $\sigma$  a positive semi-definite operator, and  $\mathcal{N}$  a quantum channel. Then, for all  $\alpha \in (0, 1) \cup (1, 2]$ , the following inequality holds*

$$\widehat{D}_\alpha(\rho\|\sigma) \geq \widehat{D}_\alpha(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)). \quad (\text{H.180})$$

**Proof.** From Stinespring's dilation theorem [142], we know that the action of a quantum channel  $\mathcal{N}$  on any linear operator  $X$  can be written as

$$\mathcal{N}(X) = \text{Tr}_E[VXV^\dagger], \quad (\text{H.181})$$

where  $V$  is an isometry and  $E$  is an auxiliary system with dimension  $d_E \geq \text{rank}(\Gamma_{AB}^\mathcal{N})$ , with  $\Gamma_{AB}^\mathcal{N}$  the Choi operator for the channel  $\mathcal{N}$ . As stated in Proposition 72, the geometric Rényi relative entropy  $\widehat{D}_\alpha$  is isometrically invariant. Therefore, it suffices to establish the data-processing inequality for  $\widehat{D}_\alpha$  under partial trace; i.e., it suffices to show that for any state  $\rho_{AB}$  and any positive semi-definite operator  $\sigma_{AB}$ ,

$$\widehat{D}_\alpha(\rho_{AB}\|\sigma_{AB}) \geq \widehat{D}_\alpha(\rho_A\|\sigma_A) \quad \forall \alpha \in (0, 1) \cup (1, 2]. \quad (\text{H.182})$$

We now proceed to prove this inequality. We prove it for  $\rho_{AB}$ , and hence  $\rho_A$ , invertible, as well as for  $\sigma_{AB}$  and  $\sigma_A$  invertible. The result follows in the general case of  $\rho_{AB}$  and/or  $\rho_A$  non-invertible, as well as  $\sigma_{AB}$  and/or  $\sigma_A$  non-invertible, by applying the result to the invertible operators  $(1 - \delta)\rho_{AB} + \delta\pi_{AB}$  and  $\sigma_{AB} + \varepsilon I_{AB}$ , with  $\delta \in (0, 1)$  and  $\varepsilon > 0$ , and taking the limit  $\delta \rightarrow 0^+$  followed by  $\varepsilon \rightarrow 0^+$ , because

$$\widehat{D}_\alpha(\rho_{AB}\|\sigma_{AB}) = \lim_{\varepsilon \rightarrow 0^+} \lim_{\delta \rightarrow 0^+} \widehat{D}_\alpha((1 - \delta)\rho_{AB} + \delta\pi_{AB}\|\sigma_{AB} + \varepsilon I_{AB}), \quad (\text{H.183})$$

$$\widehat{D}_\alpha(\rho_A\|\sigma_A) = \lim_{\varepsilon \rightarrow 0^+} \lim_{\delta \rightarrow 0^+} \widehat{D}_\alpha((1 - \delta)\rho_A + \delta\pi_A\|\sigma_A + d_B\varepsilon I_A), \quad (\text{H.184})$$

which follows from (H.21) and the fact that the dimensional factor  $d_B$  does not affect the limit in the second quantity above.

To establish the data-processing inequality, we make use of the Petz recovery channel for partial trace [143, 144], as well as the operator Jensen inequality [140]. Recall that the Petz recovery channel  $\mathcal{P}_{\sigma_{AB}, \text{Tr}_B}$  for partial trace is defined as

$$\mathcal{P}_{\sigma_{AB}, \text{Tr}_B}(X_A) \equiv \mathcal{P}(X_A) := \sigma_{AB}^{\frac{1}{2}} \left( \sigma_A^{-\frac{1}{2}} X_A \sigma_A^{-\frac{1}{2}} \otimes I_B \right) \sigma_{AB}^{\frac{1}{2}}. \quad (\text{H.185})$$

The Petz recovery channel has the following property:

$$\mathcal{P}(\sigma_A) = \sigma_{AB}, \quad (\text{H.186})$$

which can be verified by inspection. Since  $\mathcal{P}_{\sigma_{AB}, \text{Tr}_B}$  is completely positive and trace preserving, it follows that its adjoint

$$\mathcal{P}^\dagger(Y_{AB}) := \sigma_A^{-\frac{1}{2}} \text{Tr}_B[\sigma_{AB}^{\frac{1}{2}} Y_{AB} \sigma_{AB}^{\frac{1}{2}}] \sigma_A^{-\frac{1}{2}}, \quad (\text{H.187})$$

is completely positive and unital. Observe that

$$\mathcal{P}^\dagger(\sigma_{AB}^{-\frac{1}{2}} \rho_{AB} \sigma_{AB}^{-\frac{1}{2}}) = \sigma_A^{-\frac{1}{2}} \rho_A \sigma_A^{-\frac{1}{2}}. \quad (\text{H.188})$$

We then find for  $\alpha \in (1, 2]$  that

$$\widehat{Q}_\alpha(\rho_{AB} \| \sigma_{AB}) = \text{Tr} \left[ \sigma_{AB} \left( \sigma_{AB}^{-\frac{1}{2}} \rho_{AB} \sigma_{AB}^{-\frac{1}{2}} \right)^\alpha \right] \quad (\text{H.189})$$

$$= \text{Tr} \left[ \mathcal{P}(\sigma_A) \left( \sigma_{AB}^{-\frac{1}{2}} \rho_{AB} \sigma_{AB}^{-\frac{1}{2}} \right)^\alpha \right] \quad (\text{H.190})$$

$$= \text{Tr} \left[ \sigma_A \mathcal{P}^\dagger \left( \sigma_{AB}^{-\frac{1}{2}} \rho_{AB} \sigma_{AB}^{-\frac{1}{2}} \right)^\alpha \right] \quad (\text{H.191})$$

$$\geq \text{Tr} \left[ \sigma_A \left( \mathcal{P}^\dagger \left( \sigma_{AB}^{-\frac{1}{2}} \rho_{AB} \sigma_{AB}^{-\frac{1}{2}} \right) \right)^\alpha \right] \quad (\text{H.192})$$

$$= \text{Tr} \left[ \sigma_A \left( \sigma_A^{-\frac{1}{2}} \rho_A \sigma_A^{-\frac{1}{2}} \right)^\alpha \right] \quad (\text{H.193})$$

$$= \widehat{Q}_\alpha(\rho_A \| \sigma_A). \quad (\text{H.194})$$

The second equality follows from (H.186). The sole inequality is a consequence of the operator Jensen inequality and the fact that  $x^\alpha$  is operator convex for  $\alpha \in (1, 2]$ . Indeed, for  $\mathcal{M}$  a completely positive unital map, it follows from the operator Jensen inequality that

$$f(\mathcal{M}(X)) \leq \mathcal{M}(f(X)) \quad (\text{H.195})$$

for Hermitian  $X$  and an operator convex function  $f$ . The second-to-last equality follows from (H.188).

Applying the same reasoning as above, but using the fact that  $x^\alpha$  is operator concave for  $\alpha \in (0, 1)$ , we find for  $\alpha \in (0, 1)$  that

$$\widehat{Q}_\alpha(\rho_A \| \sigma_A) \geq \widehat{Q}_\alpha(\rho_{AB} \| \sigma_{AB}). \quad (\text{H.196})$$

Putting together the above and employing definitions, we find that the following inequality holds for  $\alpha \in (0, 1) \cup (1, 2]$ :

$$\hat{D}_\alpha(\rho_{AB} \parallel \sigma_{AB}) \geq \hat{D}_\alpha(\rho_A \parallel \sigma_A), \quad (\text{H.197})$$

concluding the proof. ■

With the data-processing inequality for the geometric Rényi relative entropy in hand, we can easily establish some additional properties.

**Proposition 74 (Additional Properties of the Geometric Rényi Relative Entropy)** *The geometric Rényi relative entropy  $\hat{D}_\alpha$  satisfies the following properties for all states  $\rho$  and positive semi-definite operators  $\sigma$  for  $\alpha \in (0, 1) \cup (1, 2]$ .*

1. If  $\text{Tr}[\sigma] \leq \text{Tr}[\rho] = 1$ , then  $\hat{D}_\alpha(\rho \parallel \sigma) \geq 0$ .
2. *Faithfulness:* Suppose that  $\text{Tr}[\sigma] \leq \text{Tr}[\rho] = 1$  and let  $\alpha \in (0, 1) \cup (1, \infty)$ . Then  $\hat{D}_\alpha(\rho \parallel \sigma) = 0$  if and only if  $\rho = \sigma$ .
3. If  $\rho \leq \sigma$ , then  $\hat{D}_\alpha(\rho \parallel \sigma) \leq 0$ .
4. For any positive semi-definite operator  $\sigma'$  such that  $\sigma' \geq \sigma$ , the following inequality holds  $\hat{D}_\alpha(\rho \parallel \sigma') \leq \hat{D}_\alpha(\rho \parallel \sigma)$ .

**Proof.**

1. Apply the data processing inequality with the channel being the full trace-out channel:

$$\hat{D}_\alpha(\rho \parallel \sigma) \geq \hat{D}_\alpha(\text{Tr}[\rho] \parallel \text{Tr}[\sigma]) \quad (\text{H.198})$$

$$= \frac{1}{\alpha - 1} \ln [(\text{Tr}[\rho])^\alpha (\text{Tr}[\sigma])^{1-\alpha}] \quad (\text{H.199})$$

$$= -\ln \text{Tr}[\sigma] \quad (\text{H.200})$$

$$\geq 0. \quad (\text{H.201})$$

2. If  $\rho = \sigma$ , then it follows by direct evaluation that  $\hat{D}_\alpha(\rho \parallel \sigma) = 0$ . Suppose first that  $\alpha \in (0, 1) \cup (1, 2]$ . Then  $\hat{D}_\alpha(\rho \parallel \sigma) = 0$  implies that  $\hat{D}_\alpha(\mathcal{M}(\rho) \parallel \mathcal{M}(\sigma)) = 0$  for all measurement channels  $\mathcal{M}$ . This includes informationally complete measurements [145–147]. By applying the faithfulness of the classical Rényi relative entropy and the informationally completeness property, we conclude that  $\rho = \sigma$ . To get the range outside the data-processing interval of  $(0, 1) \cup (1, 2]$ , note that  $\hat{D}_\alpha(\rho \parallel \sigma) = 0$  for  $\alpha > 2$  implies by monotonicity (Property 2 of Proposition 72) that  $\hat{D}_\alpha(\rho \parallel \sigma) = 0$  for  $\alpha \leq 2$ . Then it follows that  $\rho = \sigma$ . The other implication follows for  $\alpha \in (0, 1) \cup (1, \infty)$  by direct evaluation.



3. Consider that  $\rho \leq \sigma$  implies that  $\sigma - \rho \geq 0$ . Then define the following positive semi-definite operators:

$$\hat{\rho} := |0\rangle\langle 0| \otimes \rho, \quad (\text{H.202})$$

$$\hat{\sigma} := |0\rangle\langle 0| \otimes \rho + |1\rangle\langle 1| \otimes (\sigma - \rho). \quad (\text{H.203})$$

By exploiting the direct-sum property of geometric Rényi relative entropy (Proposition 72) and the data-processing inequality (Theorem 73), we find that

$$0 = \hat{D}_\alpha(\rho\|\rho) = \hat{D}_\alpha(\hat{\rho}\|\hat{\sigma}) \geq \hat{D}_\alpha(\rho\|\sigma), \quad (\text{H.204})$$

where the inequality follows from data processing with respect to partial trace over the classical register.

4. Similar to the above proof, the condition  $\sigma' \geq \sigma$  implies that  $\sigma' - \sigma \geq 0$ . Then define the following positive semi-definite operators:

$$\hat{\rho} := |0\rangle\langle 0| \otimes \rho, \quad (\text{H.205})$$

$$\hat{\sigma} := |0\rangle\langle 0| \otimes \sigma + |1\rangle\langle 1| \otimes (\sigma' - \sigma). \quad (\text{H.206})$$

By exploiting the direct-sum property of geometric Rényi relative entropy (Proposition 72) and the data-processing inequality (Theorem 73), we find that

$$\hat{D}_\alpha(\rho\|\sigma) = \hat{D}_\alpha(\hat{\rho}\|\hat{\sigma}) \geq \hat{D}_\alpha(\rho\|\sigma'), \quad (\text{H.207})$$

where the inequality follows from data processing with respect to partial trace over the classical register.

■

The data-processing inequality for the geometric Rényi relative entropy can be written using the geometric Rényi relative quasi-entropy  $\hat{Q}_\alpha(\rho\|\sigma)$  as

$$\frac{1}{\alpha - 1} \ln \hat{Q}_\alpha(\rho\|\sigma) \geq \frac{1}{\alpha - 1} \ln \hat{Q}_\alpha(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)). \quad (\text{H.208})$$

Since  $\alpha - 1$  is negative for  $\alpha \in (0, 1)$ , we can use the monotonicity of the function  $\ln$  to obtain

$$\hat{Q}_\alpha(\rho\|\sigma) \geq \hat{Q}_\alpha(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)), \quad \text{for } \alpha \in (1, 2], \quad (\text{H.209})$$

$$\hat{Q}_\alpha(\rho\|\sigma) \leq \hat{Q}_\alpha(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)), \quad \text{for } \alpha \in (0, 1). \quad (\text{H.210})$$

We can use this to establish some convexity statements for the geometric Rényi relative entropy.

**Proposition 75** Let  $p : \mathcal{X} \rightarrow [0, 1]$  be a probability distribution over a finite alphabet  $\mathcal{X}$  with associated  $|\mathcal{X}|$ -dimensional system  $X$ , let  $\{\rho_A^x : x \in \mathcal{X}\}$  be a set of states on system  $A$ , and let  $\{\sigma_A^x : x \in \mathcal{X}\}$  be a set of positive semi-definite operators on  $A$ . Then, for  $\alpha \in (1, 2]$ ,

$$\widehat{Q}_\alpha \left( \sum_{x \in \mathcal{X}} p(x) \rho_A^x \left\| \sum_{x \in \mathcal{X}} p(x) \sigma_A^x \right. \right) \leq \sum_{x \in \mathcal{X}} p(x) \widehat{Q}_\alpha(\rho_A^x \| \sigma_A^x), \quad (\text{H.211})$$

and for  $\alpha \in (0, 1)$ ,

$$\widehat{Q}_\alpha \left( \sum_{x \in \mathcal{X}} p(x) \rho_A^x \left\| \sum_{x \in \mathcal{X}} p(x) \sigma_A^x \right. \right) \geq \sum_{x \in \mathcal{X}} p(x) \widehat{Q}_\alpha(\rho_A^x \| \sigma_A^x). \quad (\text{H.212})$$

Consequently, the geometric Rényi relative entropy  $\widehat{D}_\alpha$  is jointly convex for  $\alpha \in (0, 1)$ :

$$\widehat{D}_\alpha \left( \sum_{x \in \mathcal{X}} p(x) \rho_A^x \left\| \sum_{x \in \mathcal{X}} p(x) \sigma_A^x \right. \right) \leq \sum_{x \in \mathcal{X}} p(x) \widehat{D}_\alpha(\rho_A^x \| \sigma_A^x). \quad (\text{H.213})$$

**Proof.** The first two inequalities follow directly from the direct-sum property of geometric Rényi relative entropy (Proposition 72) and the data-processing inequality (Theorem 73). The last inequality follows from the first by applying the logarithm and scaling by  $1/(\alpha - 1)$  and taking a maximum. ■

Although the geometric Rényi relative entropy is not jointly convex for  $\alpha \in (1, 2]$ , it is jointly quasi-convex, in the sense that

$$\widehat{D}_\alpha \left( \sum_{x \in \mathcal{X}} p(x) \rho_A^x \left\| \sum_{x \in \mathcal{X}} p(x) \sigma_A^x \right. \right) \leq \max_{x \in \mathcal{X}} \widehat{D}_\alpha(\rho_A^x \| \sigma_A^x), \quad (\text{H.214})$$

for any finite alphabet  $\mathcal{X}$ , probability distribution  $p : \mathcal{X} \rightarrow [0, 1]$ , set  $\{\rho_A^x : x \in \mathcal{X}\}$  of states, and set  $\{\sigma_A^x : x \in \mathcal{X}\}$  of positive semi-definite operators. Indeed, from (H.211), we immediately obtain

$$\widehat{Q}_\alpha \left( \sum_{x \in \mathcal{X}} p(x) \rho_A^x \left\| \sum_{x \in \mathcal{X}} p(x) \sigma_A^x \right. \right) \leq \max_{x \in \mathcal{X}} \widehat{Q}_\alpha(\rho_A^x \| \sigma_A^x). \quad (\text{H.215})$$

Taking the logarithm and multiplying by  $\frac{1}{\alpha-1}$  on both sides of this inequality leads to (H.213).

The geometric Rényi relative entropy has another interpretation, which was discovered in [35, 36] and is worthwhile to mention.

**Proposition 76 (Geometric Rényi relative entropy from classical preparations)** Let  $\rho$  be a state and  $\sigma$  a positive semi-definite operator satisfying  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ . For all

$\alpha \in (0, 1) \cup (1, 2]$ , the geometric Rényi relative entropy is equal to the smallest value that the classical Rényi relative entropy can take by minimizing over classical–quantum channels that realize the state  $\rho$  and the positive semi-definite operator  $\sigma$ . That is, the following equality holds

$$\widehat{D}_\alpha(\rho\|\sigma) = \inf_{\{p,q,\mathcal{P}\}} \{D_\alpha(p\|q) : \mathcal{P}(p) = \rho, \mathcal{P}(q) = \sigma\}, \quad (\text{H.216})$$

where the classical Rényi relative entropy is defined as

$$D_\alpha(p\|q) := \frac{1}{\alpha - 1} \sum_{x \in \mathcal{X}} p(x)^\alpha q(x)^{1-\alpha}, \quad (\text{H.217})$$

the channel  $\mathcal{P}$  is a classical–quantum channel,  $p : \mathcal{X} \rightarrow [0, 1]$  is a probability distribution over a finite alphabet  $\mathcal{X}$ , and  $q : \mathcal{X} \rightarrow (0, \infty)$  is a positive function on  $\mathcal{X}$ .

**Proof.** First, let us define the classical (diagonal) state  $\omega(p)$  and diagonal positive semi-definite operator  $\omega(q)$  as an embedding of the respective probability distribution  $p$  and positive function  $q$ :

$$\omega(p) := \sum_{x \in \mathcal{X}} p(x) |x\rangle\langle x|, \quad \omega(q) := \sum_{x \in \mathcal{X}} q(x) |x\rangle\langle x|, \quad (\text{H.218})$$

and suppose that there exists a quantum channel  $\mathcal{P}$  such that

$$\mathcal{P}(\omega(p)) = \rho, \quad \mathcal{P}(\omega(q)) = \sigma. \quad (\text{H.219})$$

Then consider the following chain of inequalities:

$$D_\alpha(p\|q) = \widehat{D}_\alpha(\omega(p)\|\omega(q)) \quad (\text{H.220})$$

$$\geq \widehat{D}_\alpha(\mathcal{P}(\omega(p))\|\mathcal{P}(\omega(q))) \quad (\text{H.221})$$

$$= \widehat{D}_\alpha(\rho\|\sigma). \quad (\text{H.222})$$

The first equality follows because the geometric Rényi relative entropy reduces to the classical Rényi relative entropy for commuting operators. The inequality is a consequence of the data-processing inequality for the geometric Rényi relative entropy (Theorem 73). The final equality follows from the constraint in (H.219). Since the inequality holds for arbitrary  $p$ ,  $q$ , and  $\mathcal{P}$  satisfying (H.219), we conclude that

$$\inf_{\{p,q,\mathcal{P}\}} \{D_\alpha(p\|q) : \mathcal{P}(p) = \rho, \mathcal{P}(q) = \sigma\} \geq \widehat{D}_\alpha(\rho\|\sigma). \quad (\text{H.223})$$

The equality in (H.216) then follows by demonstrating a specific distribution  $p$ , positive function  $q$ , and preparation channel  $\mathcal{P}$  that saturate the inequality in (H.223). The optimal choices of  $p$ ,  $q$ , and  $\mathcal{P}$  are given by

$$p(x) := \lambda_x q(x), \quad (\text{H.224})$$

$$q(x) := \text{Tr}[\Pi_x \sigma], \quad (\text{H.225})$$

$$\mathcal{P}(\cdot) := \sum_x \langle x | (\cdot) | x \rangle \frac{\sigma^{\frac{1}{2}} \Pi_x \sigma^{\frac{1}{2}}}{q(x)}, \quad (\text{H.226})$$

where the spectral decomposition of the positive semi-definite operator  $\sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}}$  is given by

$$\sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} = \sum_x \lambda_x \Pi_x. \quad (\text{H.227})$$

The choice of  $p(x)$  above is a probability distribution because

$$\sum_x p(x) = \sum_x \lambda_x q(x) = \sum_x \lambda_x \text{Tr}[\Pi_x \sigma] = \text{Tr}[\sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \sigma] = \text{Tr}[\Pi_\sigma \rho] = 1. \quad (\text{H.228})$$

The preparation channel  $\mathcal{P}$  is a classical–quantum channel that measures the input in the basis  $\{|x\rangle\}_x$  and prepares the state  $\frac{\sigma^{\frac{1}{2}} \Pi_x \sigma^{\frac{1}{2}}}{q(x)}$  if the measurement outcome is  $x$ . We find that

$$\mathcal{P}(\omega(p)) = \sum_x \frac{p(x)}{q(x)} \sigma^{\frac{1}{2}} \Pi_x \sigma^{\frac{1}{2}} = \sum_x \frac{\lambda_x q(x)}{q(x)} \sigma^{\frac{1}{2}} \Pi_x \sigma^{\frac{1}{2}} = \sigma^{\frac{1}{2}} \left( \sum_x \lambda_x \Pi_x \right) \sigma^{\frac{1}{2}} \quad (\text{H.229})$$

$$= \sigma^{\frac{1}{2}} \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right) \sigma^{\frac{1}{2}} = \Pi_\sigma \rho \Pi_\sigma = \rho, \quad (\text{H.230})$$

and

$$\mathcal{P}(\omega(q)) = \sum_x \frac{q(x)}{q(x)} \sigma^{\frac{1}{2}} \Pi_x \sigma^{\frac{1}{2}} = \sigma^{\frac{1}{2}} \left( \sum_x \Pi_x \right) \sigma^{\frac{1}{2}} = \sigma. \quad (\text{H.231})$$

Finally, consider the classical Rényi relative quasi-entropy:

$$\sum_x p(x)^\alpha q(x)^{1-\alpha} = \sum_x (\lambda_x q(x))^\alpha q(x)^{1-\alpha} = \sum_x \lambda_x^\alpha q(x) = \sum_x \lambda_x^\alpha \text{Tr}[\Pi_x \sigma] \quad (\text{H.232})$$

$$= \text{Tr} \left[ \sigma \left( \sum_x \lambda_x^\alpha \Pi_x \right) \right] = \text{Tr} \left[ \sigma \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right)^\alpha \right] = \hat{Q}_\alpha(\rho \| \sigma), \quad (\text{H.233})$$

where the second-to-last equality follows from the spectral decomposition in (H.227) and the form of the geometric Rényi relative quasi-entropy from Proposition 66. As a consequence of the equality

$$\sum_x p(x)^\alpha q(x)^{1-\alpha} = \hat{Q}_\alpha(\rho \| \sigma), \quad (\text{H.234})$$

and the fact that these choices of  $p$ ,  $q$ , and  $\mathcal{P}$  satisfy the constraints  $\mathcal{P}(p) = \rho$  and  $\mathcal{P}(q) = \sigma$ , we conclude that

$$D_\alpha(p \| q) = \hat{D}_\alpha(\rho \| \sigma). \quad (\text{H.235})$$

Combining this equality with (H.223), we conclude the equality in (H.216). ■

The following proposition recalls the ordering between the sandwiched, Petz–, and geometric Rényi relative entropies for the interval  $\alpha \in (0, 1) \cup (1, 2]$ . The first inequality in Proposition 77 was established for  $\alpha \in (1, 2]$  in [81] and for  $\alpha \in (0, 1)$  in [148], by employing the Araki–Lieb–Thirring inequality [138, 139]. The second inequality was established by [35, 36] and reviewed in [37]. It follows by applying similar reasoning as in the proof of Proposition 76.

**Proposition 77** *Let  $\rho$  be a state and  $\sigma$  a positive semi-definite operator. For  $\alpha \in (0, 1) \cup (1, 2]$ , the following inequalities hold*

$$\tilde{D}_\alpha(\rho\|\sigma) \leq D_\alpha(\rho\|\sigma) \leq \hat{D}_\alpha(\rho\|\sigma), \quad (\text{H.236})$$

for the sandwiched ( $\tilde{D}_\alpha$ ), Petz ( $D_\alpha$ ), and geometric ( $\hat{D}_\alpha$ ) Rényi relative entropies.

**Proof.** As stated above, the first inequality follows from the Araki–Lieb–Thirring inequalities in (H.51)–(H.52) by picking  $q = 1$ ,  $r = \alpha$ ,  $X = \rho$ , and  $Y = \sigma^{\frac{1-\alpha}{\alpha}}$ . So we recall the proof of the second inequality here. Suppose that  $\mathcal{P}$  is a classical–quantum channel,  $p : \mathcal{X} \rightarrow [0, 1]$  is a probability distribution over a finite alphabet  $\mathcal{X}$ , and  $q : \mathcal{X} \rightarrow (0, \infty)$  is a positive function on  $\mathcal{X}$  satisfying

$$\mathcal{P}(\omega(p)) = \rho, \quad \mathcal{P}(\omega(q)) = \sigma, \quad (\text{H.237})$$

where

$$\omega(p) := \sum_{x \in \mathcal{X}} p(x) |x\rangle\langle x|, \quad \omega(q) := \sum_{x \in \mathcal{X}} q(x) |x\rangle\langle x|. \quad (\text{H.238})$$

Then consider the following chain of inequalities:

$$D_\alpha(p\|q) = D_\alpha(\omega(p)\|\omega(q)) \quad (\text{H.239})$$

$$\geq D_\alpha(\mathcal{P}(\omega(p))\|\mathcal{P}(\omega(q))) \quad (\text{H.240})$$

$$= D_\alpha(\rho\|\sigma). \quad (\text{H.241})$$

The first equality follows because the Petz–Rényi relative entropy reduces to the classical Rényi relative entropy for commuting operators. The inequality follows from the data-processing inequality for the Petz–Rényi relative entropy for  $\alpha \in (0, 1) \cup (1, 2]$  [71, 121]. The final equality follows from the constraint in (H.237). Since the inequality above holds for all  $p$ ,  $q$ , and  $\mathcal{P}$  satisfying (H.237), we conclude that

$$\inf_{\{p, q, \mathcal{P}\}} \{D_\alpha(p\|q) : \mathcal{P}(p) = \rho, \mathcal{P}(q) = \sigma\} \geq D_\alpha(\rho\|\sigma). \quad (\text{H.242})$$

Now applying Proposition 76, we conclude the second inequality in (H.236). ■

## H.1 Belavkin–Staszewski relative entropy

A different quantum generalization of the classical relative entropy is given by the Belavkin–Staszewski<sup>1</sup> relative entropy [117]:

**Definition 78 (Belavkin–Staszewski relative entropy)** *The Belavkin–Staszewski relative entropy of a quantum state  $\rho$  and a positive semi-definite operator  $\sigma$  is defined as*

$$\widehat{D}(\rho\|\sigma) := \begin{cases} \text{Tr}\left[\rho \ln\left(\rho^{\frac{1}{2}}\sigma^{-1}\rho^{\frac{1}{2}}\right)\right] & \text{if } \text{supp}(\rho) \subseteq \text{supp}(\sigma) \\ +\infty & \text{otherwise} \end{cases}, \quad (\text{H.243})$$

where the inverse  $\sigma^{-1}$  is understood in the generalized sense and the logarithm is evaluated on the support of  $\rho$ .

This quantum generalization of classical relative entropy is not known to possess an information-theoretic meaning. However, it is quite useful for obtaining upper bounds on quantum channel capacities and quantum channel discrimination rates [39].

An important property of the Belavkin–Staszewski relative entropy is that it is the limit of the geometric Rényi relative entropy as  $\alpha \rightarrow 1$  [35, 36]. The proposition below was known for positive definite operators, but it is not clear to us whether it has been established in the general case.

**Proposition 79** *Let  $\rho$  be a state and  $\sigma$  a positive semi-definite operator. Then, in the limit  $\alpha \rightarrow 1$ , the geometric Rényi relative entropy converges to the Belavkin–Staszewski relative entropy:*

$$\lim_{\alpha \rightarrow 1} \widehat{D}_\alpha(\rho\|\sigma) = \widehat{D}(\rho\|\sigma). \quad (\text{H.244})$$

**Proof.** Suppose at first that  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ . Then  $\widehat{D}_\alpha(\rho\|\sigma)$  is finite for all  $\alpha \in (0, 1) \cup (1, \infty)$ , and we can write the following explicit formula for the geometric Rényi relative entropy by employing Proposition 66:

$$\widehat{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \ln \widehat{Q}_\alpha(\rho\|\sigma) \quad (\text{H.245})$$

$$= \frac{1}{\alpha - 1} \ln \text{Tr}\left[\sigma \left(\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\right)^\alpha\right]. \quad (\text{H.246})$$

Our assumption implies that  $\text{Tr}[\Pi_\sigma\rho] = 1$ , and we find that

$$\widehat{Q}_1(\rho\|\sigma) = \text{Tr}\left[\sigma \left(\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\right)\right] \quad (\text{H.247})$$

$$= \text{Tr}[\Pi_\sigma\rho] \quad (\text{H.248})$$

$$= 1. \quad (\text{H.249})$$

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<sup>1</sup>The name Staszewski is pronounced Stah-shev-ski, with emphasis on the second syllable.

Since  $\ln 1 = 0$ , we can write

$$\widehat{D}_\alpha(\rho\|\sigma) = \frac{\ln \widehat{Q}_\alpha(\rho\|\sigma) - \ln \widehat{Q}_1(\rho\|\sigma)}{\alpha - 1}, \quad (\text{H.250})$$

so that

$$\lim_{\alpha \rightarrow 1} \widehat{D}_\alpha(\rho\|\sigma) = \lim_{\alpha \rightarrow 1} \frac{\ln \widehat{Q}_\alpha(\rho\|\sigma) - \ln \widehat{Q}_1(\rho\|\sigma)}{\alpha - 1} \quad (\text{H.251})$$

$$= \frac{d}{d\alpha} \ln \widehat{Q}_\alpha(\rho\|\sigma) \Big|_{\alpha=1} \quad (\text{H.252})$$

$$= \frac{\frac{d}{d\alpha} \widehat{Q}_\alpha(\rho\|\sigma) \Big|_{\alpha=1}}{\widehat{Q}_1(\rho\|\sigma)} \quad (\text{H.253})$$

$$= \frac{d}{d\alpha} \widehat{Q}_\alpha(\rho\|\sigma) \Big|_{\alpha=1}. \quad (\text{H.254})$$

Then

$$\begin{aligned} \frac{d}{d\alpha} \widehat{Q}_\alpha(\rho\|\sigma) \Big|_{\alpha=1} &= \frac{d}{d\alpha} \text{Tr} \left[ \sigma \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right)^\alpha \right] \Big|_{\alpha=1} \\ &= \text{Tr} \left[ \sigma \frac{d}{d\alpha} \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right)^\alpha \right] \Big|_{\alpha=1}. \end{aligned}$$

For a positive semi-definite operator  $X$  with spectral decomposition

$$X = \sum_z \nu_z \Pi_z, \quad (\text{H.255})$$

it follows that

$$\frac{d}{d\alpha} X^\alpha \Big|_{\alpha=1} = \frac{d}{d\alpha} \sum_z \nu_z^\alpha \Pi_z \Big|_{\alpha=1} \quad (\text{H.256})$$

$$= \sum_z \left( \frac{d}{d\alpha} \nu_z^\alpha \Big|_{\alpha=1} \right) \Pi_z \quad (\text{H.257})$$

$$= \sum_z (\nu_z^\alpha \ln \nu_z^\alpha \Big|_{\alpha=1}) \Pi_z \quad (\text{H.258})$$

$$= \sum_z (\nu_z \ln \nu_z) \Pi_z \quad (\text{H.259})$$

$$= X \ln_* X, \quad (\text{H.260})$$

where

$$\ln_*(x) := \begin{cases} \ln(x) & x > 0 \\ 0 & x = 0 \end{cases}. \quad (\text{H.261})$$

Thus we find that

$$\begin{aligned} & \left. \text{Tr} \left[ \sigma \frac{d}{d\alpha} \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right)^\alpha \right] \right|_{\alpha=1} \\ &= \text{Tr} \left[ \sigma \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right) \ln_* \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right) \right] \end{aligned} \quad (\text{H.262})$$

$$= \text{Tr} \left[ \sigma^{\frac{1}{2}} \rho^{\frac{1}{2}} \rho^{\frac{1}{2}} \sigma^{-\frac{1}{2}} \ln_* \left( \sigma^{-\frac{1}{2}} \rho^{\frac{1}{2}} \rho^{\frac{1}{2}} \sigma^{-\frac{1}{2}} \right) \right] \quad (\text{H.263})$$

$$= \text{Tr} \left[ \sigma^{\frac{1}{2}} \rho^{\frac{1}{2}} \ln_* \left( \rho^{\frac{1}{2}} \sigma^{-\frac{1}{2}} \sigma^{-\frac{1}{2}} \rho^{\frac{1}{2}} \right) \rho^{\frac{1}{2}} \sigma^{-\frac{1}{2}} \right] \quad (\text{H.264})$$

$$= \text{Tr} \left[ \rho^{\frac{1}{2}} \Pi_\sigma \rho^{\frac{1}{2}} \ln_* \left( \rho^{\frac{1}{2}} \sigma^{-\frac{1}{2}} \sigma^{-\frac{1}{2}} \rho^{\frac{1}{2}} \right) \right] \quad (\text{H.265})$$

$$= \text{Tr} \left[ \rho \ln \left( \rho^{\frac{1}{2}} \sigma^{-1} \rho^{\frac{1}{2}} \right) \right]. \quad (\text{H.266})$$

The third equality follows from Lemma 61. The final equality follows from the assumption  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$  and by applying the interpretation of the logarithm exactly as stated in Definition 78. Then we find that

$$\lim_{\alpha \rightarrow 1} \widehat{D}_\alpha(\rho \| \sigma) = \text{Tr} \left[ \rho \ln \left( \rho^{\frac{1}{2}} \sigma^{-1} \rho^{\frac{1}{2}} \right) \right], \quad (\text{H.267})$$

for the case in which  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ .

Now suppose that  $\alpha \in (1, \infty)$  and  $\text{supp}(\rho) \not\subseteq \text{supp}(\sigma)$ . Then  $\widehat{D}_\alpha(\rho \| \sigma) = +\infty$ , so that  $\lim_{\alpha \rightarrow 1+} \widehat{D}_\alpha(\rho \| \sigma) = +\infty$ , consistent with the definition of the Belavkin–Staszewski relative entropy in this case (see Definition 78).

Suppose that  $\alpha \in (0, 1)$  and  $\text{supp}(\rho) \not\subseteq \text{supp}(\sigma)$ . Employing Proposition 70, we have that  $\widehat{D}_\alpha(\rho \| \sigma) \geq \widetilde{D}_\alpha(\rho \| \sigma)$  for all  $\alpha \in (0, 1)$ . Since  $\lim_{\alpha \rightarrow 1-} \widetilde{D}_\alpha(\rho \| \sigma) = +\infty$  in this case [149, Corollary III.2], it follows that  $\lim_{\alpha \rightarrow 1-} \widehat{D}_\alpha(\rho \| \sigma) = +\infty$ .

Therefore,

$$\begin{aligned} & \lim_{\alpha \rightarrow 1-} \widehat{D}_\alpha(\rho \| \sigma) \\ &= \begin{cases} \text{Tr} \left[ \rho \ln \left( \rho^{\frac{1}{2}} \sigma^{-1} \rho^{\frac{1}{2}} \right) \right] & \text{if } \text{supp}(\rho) \subseteq \text{supp}(\sigma) \\ +\infty & \text{otherwise} \end{cases} \end{aligned} \quad (\text{H.268})$$

$$= \widehat{D}(\rho \| \sigma). \quad (\text{H.269})$$

To conclude, we have established that  $\lim_{\alpha \rightarrow 1+} \widehat{D}_\alpha(\rho \| \sigma) = \lim_{\alpha \rightarrow 1-} \widehat{D}_\alpha(\rho \| \sigma) = \widehat{D}(\rho \| \sigma)$ , which means that

$$\lim_{\alpha \rightarrow 1} \widehat{D}_\alpha(\rho \| \sigma) = \widehat{D}(\rho \| \sigma), \quad (\text{H.270})$$

as required. ■

The following inequality relates the quantum relative entropy to the Belavkin–Staszewski relative entropy [141]:



**Proposition 80** *Let  $\rho$  be a state and  $\sigma$  a positive semi-definite operator. Then the quantum relative entropy is never larger than the Belavkin–Staszewski relative entropy:*

$$D(\rho\|\sigma) \leq \widehat{D}(\rho\|\sigma). \quad (\text{H.271})$$

**Proof.** If  $\text{supp}(\rho) \not\subseteq \text{supp}(\sigma)$ , then there is nothing to prove in this case because both

$$D(\rho\|\sigma) = \widehat{D}(\rho\|\sigma) = +\infty, \quad (\text{H.272})$$

and so the inequality in (H.271) holds trivially in this case. So let us suppose instead that  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ . From Propositions 70 and 66, we conclude for all  $\alpha \in (0, 1) \cup (1, \infty)$  that

$$\widetilde{D}_\alpha(\rho\|\sigma) \leq \widehat{D}_\alpha(\rho\|\sigma). \quad (\text{H.273})$$

From (H.10), we know that

$$\lim_{\alpha \rightarrow 1} \widetilde{D}_\alpha(\rho\|\sigma) = D(\rho\|\sigma). \quad (\text{H.274})$$

While from Proposition 79, we know that

$$\lim_{\alpha \rightarrow 1} \widehat{D}_\alpha(\rho\|\sigma) = \widehat{D}(\rho\|\sigma). \quad (\text{H.275})$$

Thus, applying the limit  $\alpha \rightarrow 1$  to (H.273) and the two equalities above, we conclude (H.271). ■

Similar to (H.9), Definition 78 is consistent with the following limit:

**Proposition 81** *For any state  $\rho$  and positive semi-definite operator  $\sigma$ , the following limit holds*

$$\widehat{D}(\rho\|\sigma) = \lim_{\varepsilon \rightarrow 0^+} \lim_{\delta \rightarrow 0^+} \text{Tr} \left[ \rho_\delta \log_2 \left( \rho_\delta^{\frac{1}{2}} \sigma_\varepsilon^{-1} \rho_\delta^{\frac{1}{2}} \right) \right], \quad (\text{H.276})$$

where  $\delta \in (0, 1)$  and

$$\rho_\delta := (1 - \delta) \rho + \delta \pi, \quad \sigma_\varepsilon := \sigma + \varepsilon I, \quad (\text{H.277})$$

with  $\pi$  the maximally mixed state.

**Proof.** Suppose first that  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ . We follow an approach similar to that given in the proof of Proposition 66. Let us employ the decomposition of the Hilbert space into  $\text{supp}(\sigma) \oplus \ker(\sigma)$ . Then we can write  $\rho$  and  $\sigma$  as in (H.59), so that

$$\sigma_\varepsilon^{-1} = \begin{pmatrix} (\sigma + \varepsilon \Pi_\sigma)^{-1} & 0 \\ 0 & \varepsilon^{-1} \Pi_\sigma^\perp \end{pmatrix}, \quad (\text{H.278})$$

where we have followed the developments in (H.59)–(H.61). The condition  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$  implies that  $\rho_{0,1} = 0$  and  $\rho_{1,1} = 0$ . It thus follows that  $\lim_{\delta \rightarrow 0^+} \rho_\delta = \rho_{0,0}$ . We then find that

$$\text{Tr} \left[ \rho_\delta \ln \left( \rho_\delta^{\frac{1}{2}} \sigma_\varepsilon^{-1} \rho_\delta^{\frac{1}{2}} \right) \right] = \text{Tr} \left[ \rho_\delta^{\frac{1}{2}} \sigma_\varepsilon^{\frac{1}{2}} \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta^{\frac{1}{2}} \ln \left( \rho_\delta^{\frac{1}{2}} \sigma_\varepsilon^{-\frac{1}{2}} \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta^{\frac{1}{2}} \right) \right] \quad (\text{H.279})$$

$$= \text{Tr} \left[ \rho_\delta^{\frac{1}{2}} \sigma_\varepsilon^{\frac{1}{2}} \ln \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta^{\frac{1}{2}} \rho_\delta^{\frac{1}{2}} \sigma_\varepsilon^{-\frac{1}{2}} \right) \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta^{\frac{1}{2}} \right] \quad (\text{H.280})$$

$$= \text{Tr} \left[ \ln \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right) \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right) \sigma_\varepsilon \right] \quad (\text{H.281})$$

$$= \text{Tr} \left[ \sigma_\varepsilon \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right) \ln \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right) \right] \quad (\text{H.282})$$

$$= \text{Tr} \left[ \sigma_\varepsilon \eta \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right) \right], \quad (\text{H.283})$$

where the second equality follows from applying Lemma 61 with  $f = \ln$  and  $L = \rho_\delta^{\frac{1}{2}} \sigma_\varepsilon^{-\frac{1}{2}}$ . The second-to-last equality follows because  $\sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}}$  commutes with  $\ln(\sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}})$ , and by employing cyclicity of trace. In the last line, we made use of the following function:

$$\eta(x) := x \ln x, \quad (\text{H.284})$$

defined for all  $x \in [0, \infty)$  with  $\eta(0) = 0$ . By appealing to the continuity of the function  $\eta(x)$  on  $x \in [0, \infty)$  and the fact that  $\lim_{\delta \rightarrow 0^+} \rho_\delta = \rho_{0,0}$ , we find that

$$\lim_{\delta \rightarrow 0^+} \text{Tr} \left[ \sigma_\varepsilon \eta \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_\delta \sigma_\varepsilon^{-\frac{1}{2}} \right) \right] = \text{Tr} \left[ \sigma_\varepsilon \eta \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_{0,0} \sigma_\varepsilon^{-\frac{1}{2}} \right) \right]. \quad (\text{H.285})$$

Now recall the function  $\ln_*$  defined in (H.261). Using it, we can write

$$\begin{aligned} & \text{Tr} \left[ \sigma_\varepsilon \eta \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_{0,0} \sigma_\varepsilon^{-\frac{1}{2}} \right) \right] \\ &= \text{Tr} \left[ \sigma_\varepsilon \sigma_\varepsilon^{-\frac{1}{2}} \rho_{0,0} \sigma_\varepsilon^{-\frac{1}{2}} \ln_* \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_{0,0} \sigma_\varepsilon^{-\frac{1}{2}} \right) \right] \end{aligned} \quad (\text{H.286})$$

$$= \text{Tr} \left[ \sigma_\varepsilon^{\frac{1}{2}} \rho_{0,0}^{\frac{1}{2}} \rho_{0,0}^{\frac{1}{2}} \sigma_\varepsilon^{-\frac{1}{2}} \ln_* \left( \sigma_\varepsilon^{-\frac{1}{2}} \rho_{0,0}^{\frac{1}{2}} \rho_{0,0}^{\frac{1}{2}} \sigma_\varepsilon^{-\frac{1}{2}} \right) \right] \quad (\text{H.287})$$

$$= \text{Tr} \left[ \sigma_\varepsilon^{\frac{1}{2}} \rho_{0,0}^{\frac{1}{2}} \ln_* \left( \rho_{0,0}^{\frac{1}{2}} \sigma_\varepsilon^{-\frac{1}{2}} \sigma_\varepsilon^{-\frac{1}{2}} \rho_{0,0}^{\frac{1}{2}} \right) \rho_{0,0}^{\frac{1}{2}} \sigma_\varepsilon^{-\frac{1}{2}} \right] \quad (\text{H.288})$$

$$= \text{Tr} \left[ \rho_{0,0} \ln_* \left( \rho_{0,0}^{\frac{1}{2}} \sigma_\varepsilon^{-1} \rho_{0,0}^{\frac{1}{2}} \right) \right] \quad (\text{H.289})$$

$$= \text{Tr} \left[ \rho_{0,0} \ln_* \left( \rho_{0,0}^{\frac{1}{2}} (\sigma + \varepsilon \Pi_\sigma)^{-1} \rho_{0,0}^{\frac{1}{2}} \right) \right], \quad (\text{H.290})$$

where the last line follows because

$$\begin{aligned} & \rho_{0,0}^{\frac{1}{2}} (\sigma + \varepsilon \Pi_\sigma)^{-1} \rho_{0,0}^{\frac{1}{2}} \\ &= \begin{pmatrix} \rho_{0,0}^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (\sigma + \varepsilon \Pi_\sigma)^{-1} & 0 \\ 0 & \varepsilon^{-1} \Pi_\sigma^\perp \end{pmatrix} \begin{pmatrix} \rho_{0,0}^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \quad (\text{H.291})$$

$$= \begin{pmatrix} \rho_{0,0}^{\frac{1}{2}} (\sigma + \varepsilon \Pi_\sigma)^{-1} \rho_{0,0}^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix}. \quad (\text{H.292})$$

Now taking the limit as  $\varepsilon \rightarrow 0^+$ , and appealing to continuity of  $\ln_*(x)$  and  $x^{-1}$  for  $x > 0$ , we find that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \text{Tr} \left[ \rho_{0,0} \ln_* \left( \rho_{0,0}^{\frac{1}{2}} (\sigma + \varepsilon \Pi_\sigma)^{-1} \rho_{0,0}^{\frac{1}{2}} \right) \right] \\ &= \text{Tr} \left[ \rho_{0,0} \ln_* \left( \rho_{0,0}^{\frac{1}{2}} \sigma^{-1} \rho_{0,0}^{\frac{1}{2}} \right) \right] \end{aligned} \quad (\text{H.293})$$

$$= \text{Tr} \left[ \rho \ln \left( \rho^{\frac{1}{2}} \sigma^{-1} \rho^{\frac{1}{2}} \right) \right] \quad (\text{H.294})$$

where the formula in the last line is interpreted exactly as stated in Definition 78. Thus, we conclude that

$$\lim_{\varepsilon \rightarrow 0^+} \lim_{\delta \rightarrow 0^+} \text{Tr} \left[ \rho_\delta \ln \left( \rho_\delta^{\frac{1}{2}} \sigma_\varepsilon^{-1} \rho_\delta^{\frac{1}{2}} \right) \right] = \text{Tr} \left[ \rho \ln \left( \rho^{\frac{1}{2}} \sigma^{-1} \rho^{\frac{1}{2}} \right) \right]. \quad (\text{H.295})$$

Now suppose that  $\text{supp}(\rho) \not\subseteq \text{supp}(\sigma)$ . Then applying Proposition 80, we find that the following inequality holds for all  $\delta \in (0, 1)$  and  $\varepsilon > 0$ :

$$\widehat{D}(\rho_\delta \| \sigma_\varepsilon) \geq D(\rho_\delta \| \sigma_\varepsilon). \quad (\text{H.296})$$

Now taking limits and applying (H.9), we find that

$$\lim_{\varepsilon \rightarrow 0^+} \lim_{\delta \rightarrow 0^+} \widehat{D}(\rho_\delta \| \sigma_\varepsilon) \geq \lim_{\varepsilon \rightarrow 0^+} \lim_{\delta \rightarrow 0^+} D(\rho_\delta \| \sigma_\varepsilon) \quad (\text{H.297})$$

$$= \lim_{\varepsilon \rightarrow 0^+} D(\rho \| \sigma_\varepsilon) \quad (\text{H.298})$$

$$= +\infty. \quad (\text{H.299})$$

This concludes the proof. ■

By taking the limit  $\alpha \rightarrow 1$  in the statement of the data-processing inequality for  $\widehat{D}_\alpha$ , and applying Proposition 79, we immediately obtain the data-processing inequality for the Belavkin–Staszewski relative entropy. This was shown by a different method in [141].

**Corollary 82 (Data-Processing Inequality for Belavkin–Staszewski Relative Entropy)**

*Let  $\rho$  be a state,  $\sigma$  a positive semi-definite operator, and  $\mathcal{N}$  a quantum channel. Then*

$$\widehat{D}(\rho \| \sigma) \geq \widehat{D}(\mathcal{N}(\rho) \| \mathcal{N}(\sigma)). \quad (\text{H.300})$$

Some basic properties of the Belavkin–Staszewski relative entropy are as follows:

**Proposition 83 (Basic Properties of Belavkin–Staszewski Relative Entropy)** *The Belavkin–Staszewski relative entropy satisfies the following properties for states  $\rho, \rho_1, \rho_2$  and positive semi-definite operators  $\sigma, \sigma_1, \sigma_2$ .*

1. *Isometric invariance: For any isometry  $V$ ,*

$$\widehat{D}(V\rho V^\dagger \| V\sigma V^\dagger) = \widehat{D}(\rho \| \sigma). \quad (\text{H.301})$$

2. (a) *If  $\text{Tr}[\sigma] \leq 1$ , then  $\widehat{D}(\rho \| \sigma) \geq 0$ .*

(b) *Faithfulness: Suppose that  $\text{Tr}[\sigma] \leq \text{Tr}[\rho] = 1$ . Then  $\widehat{D}(\rho \| \sigma) = 0$  if and only if  $\rho = \sigma$ .*

(c) *If  $\rho \leq \sigma$ , then  $\widehat{D}(\rho \| \sigma) \leq 0$ .*

(d) *If  $\sigma \leq \sigma'$ , then  $\widehat{D}(\rho \| \sigma) \geq \widehat{D}(\rho \| \sigma')$ .*

3. *Additivity:*

$$\widehat{D}(\rho_1 \otimes \rho_2 \| \sigma_1 \otimes \sigma_2) = \widehat{D}(\rho_1 \| \sigma_1) + \widehat{D}(\rho_2 \| \sigma_2). \quad (\text{H.302})$$

*As a special case, for any  $\beta \in (0, \infty)$ ,*

$$\widehat{D}(\rho \| \beta\sigma) = \widehat{D}(\rho \| \sigma) + \log_2\left(\frac{1}{\beta}\right). \quad (\text{H.303})$$

4. *Direct-sum property: Let  $p : \mathcal{X} \rightarrow [0, 1]$  be a probability distribution over a finite alphabet  $\mathcal{X}$  with associated  $|\mathcal{X}|$ -dimensional system  $X$ , and let  $q : \mathcal{X} \rightarrow [0, \infty)$  be a positive function on  $\mathcal{X}$ . Let  $\{\rho_A^x : x \in \mathcal{X}\}$  be a set of states on a system  $A$ , and let  $\{\sigma_A^x : x \in \mathcal{X}\}$  be a set of positive semi-definite operators on  $A$ . Then,*

$$\widehat{D}(\rho_{XA} \| \sigma_{XA}) = \widehat{D}(p \| q) + \sum_{x \in \mathcal{X}} p(x) \widehat{D}(\rho_A^x \| \sigma_A^x). \quad (\text{H.304})$$

*where*

$$\rho_{XA} := \sum_{x \in \mathcal{X}} p(x) |x\rangle\langle x|_X \otimes \rho_A^x, \quad (\text{H.305})$$

$$\sigma_{XA} := \sum_{x \in \mathcal{X}} q(x) |x\rangle\langle x|_X \otimes \sigma_A^x. \quad (\text{H.306})$$

**Proof.**

1. Isometric invariance is a direct consequence of Propositions 72 and 79.

2. All of the properties in the second item follow from data processing (Corollary 82). Applying the trace-out channel, we find that

$$\widehat{D}(\rho \| \sigma) \geq \widehat{D}(\text{Tr}[\rho] \| \text{Tr}[\sigma]) \quad (\text{H.307})$$

$$= \text{Tr}[\rho] \ln(\text{Tr}[\rho] / \text{Tr}[\sigma]) \quad (\text{H.308})$$

$$= -\ln \text{Tr}[\sigma] \quad (\text{H.309})$$

$$\geq 0. \quad (\text{H.310})$$

If  $\rho = \sigma$ , then it follows by direct evaluation that  $\widehat{D}(\rho\|\sigma) = 0$ . If  $\widehat{D}(\rho\|\sigma) = 0$  and  $\text{Tr}[\sigma] \leq 1$ , then  $D(\rho\|\sigma) = 0$  by Proposition 80 and we conclude that  $\rho = \sigma$  from faithfulness of the quantum relative entropy (see, e.g., [44, Theorem 11.8.2]).

If  $\rho \leq \sigma$ , then  $\sigma - \rho$  is positive semi-definite, and the following operator is positive semi-definite:

$$\hat{\sigma} := |0\rangle\langle 0| \otimes \rho + |1\rangle\langle 1| \otimes (\sigma - \rho). \quad (\text{H.311})$$

Defining  $\hat{\rho} := |0\rangle\langle 0| \otimes \rho$ , we find from the direct-sum property that

$$0 = \widehat{D}(\rho\|\rho) = \widehat{D}(\hat{\rho}\|\hat{\sigma}) \geq \widehat{D}(\rho\|\sigma), \quad (\text{H.312})$$

where the inequality follows from data processing by tracing out the first classical register of  $\hat{\rho}$  and  $\hat{\sigma}$ .

If  $\sigma \leq \sigma'$ , then the operator  $\sigma' - \sigma$  is positive semi-definite and so is the following one:

$$\hat{\sigma} := |0\rangle\langle 0| \otimes \sigma + |1\rangle\langle 1| \otimes (\sigma' - \sigma). \quad (\text{H.313})$$

Defining  $\hat{\rho} := |0\rangle\langle 0| \otimes \rho$ , we find from the direct-sum property that

$$\widehat{D}(\rho\|\sigma) = \widehat{D}(\hat{\rho}\|\hat{\sigma}) \geq \widehat{D}(\rho\|\sigma'), \quad (\text{H.314})$$

where the inequality follows from data processing by tracing out the first classical register of  $\hat{\rho}$  and  $\hat{\sigma}$ .

3. Additivity follows by direct evaluation.
4. The direct-sum property follows by direct evaluation.

■

A statement similar to that made by Proposition 76 holds for the Belavkin–Staszewski relative entropy [35, 36]:

**Proposition 84 (Belavkin–Staszewski Relative Entropy from Classical Preparations)** *Let  $\rho$  be a state and  $\sigma$  a positive semi-definite operator satisfying  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ . The Belavkin–Staszewski relative entropy is equal to the smallest value that the classical relative entropy can take by minimizing over classical–quantum channels that realize the state  $\rho$  and the positive semi-definite operator  $\sigma$ . That is, the following equality holds*

$$\widehat{D}(\rho\|\sigma) = \inf_{\{p,q,\mathcal{P}\}} \{D(p\|q) : \mathcal{P}(p) = \rho, \mathcal{P}(q) = \sigma\}, \quad (\text{H.315})$$

where the classical relative entropy is defined as

$$D(p\|q) := \sum_x p(x) \ln \left( \frac{p(x)}{q(x)} \right), \quad (\text{H.316})$$

the channel  $\mathcal{P}$  is a classical–quantum channel,  $p : \mathcal{X} \rightarrow [0, 1]$  is a probability distribution over a finite alphabet  $\mathcal{X}$ , and  $q : \mathcal{X} \rightarrow (0, \infty)$  is a positive function on  $\mathcal{X}$ .

**Proof.** The proof is very similar to the proof of Proposition 76, and so we use the same notation to provide a brief proof. By following the same reasoning that leads to (H.223), it follows that

$$\inf_{\{p, q, \mathcal{P}\}} \{D(p\|q) : \mathcal{P}(p) = \rho, \mathcal{P}(q) = \sigma\} \geq \widehat{D}(\rho\|\sigma). \quad (\text{H.317})$$

The optimal choices of  $p$ ,  $q$ , and  $\mathcal{P}$  saturating the inequality in (H.317) are again given by (H.224)–(H.226). Consider for those choices that

$$\sum_x p(x) \ln \left( \frac{p(x)}{q(x)} \right) = \sum_x p(x) \ln(\lambda_x) \quad (\text{H.318})$$

$$= \sum_x \lambda_x q(x) \ln(\lambda_x) \quad (\text{H.319})$$

$$= \sum_x \lambda_x \text{Tr}[\Pi_x \sigma] \ln(\lambda_x) \quad (\text{H.320})$$

$$= \text{Tr} \left[ \sigma \left( \sum_x \lambda_x \log_2(\lambda_x) \Pi_x \right) \right] \quad (\text{H.321})$$

$$= \text{Tr} \left[ \sigma \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right) \ln \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right) \right] \quad (\text{H.322})$$

$$= \text{Tr} \left[ \rho \ln \left( \rho^{\frac{1}{2}} \sigma^{-1} \rho^{\frac{1}{2}} \right) \right], \quad (\text{H.323})$$

where the last equality follows from reasoning similar to that used to justify (H.262)–(H.266). Then by following the reasoning at the end of the proof of Proposition 76, we conclude (H.315). ■

## H.2 Convergence of geometric Rényi relative entropy to max-relative entropy

**Proposition 85** *The geometric Rényi relative entropy converges to the max-relative entropy in the limit as  $\alpha \rightarrow \infty$ :*

$$\lim_{\alpha \rightarrow \infty} \widehat{D}_\alpha(\rho\|\sigma) = D_{\max}(\rho\|\sigma). \quad (\text{H.324})$$

**Proof.** We only consider the case in which  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ . Otherwise, we trivially have  $\widehat{D}_\alpha(\rho\|\sigma) = +\infty$  for all  $\alpha > 1$ . In the case that  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ , we can consider, without loss of generality, that  $\text{supp}(\sigma) = \mathcal{H}$ , which implies that  $\lambda_{\min}(\sigma) > 0$ . Since we have that

$$\lambda_{\min}(\sigma)I \leq \sigma \leq \lambda_{\max}(\sigma)I \quad (\text{H.325})$$

it follows that

$$\lambda_{\min}(\sigma) \operatorname{Tr} \left[ \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right)^\alpha \right] \leq \operatorname{Tr} \left[ \sigma \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right)^\alpha \right] \quad (\text{H.326})$$

$$\leq \lambda_{\max}(\sigma) \operatorname{Tr} \left[ \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right)^\alpha \right]. \quad (\text{H.327})$$

Now taking a logarithm, dividing by  $\alpha - 1$ , and applying definitions, we find that the following inequalities hold for  $\alpha > 1$ :

$$\begin{aligned} & \frac{1}{\alpha - 1} \ln \lambda_{\min}(\sigma) + \frac{1}{\alpha - 1} \log_2 \operatorname{Tr} \left[ \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right)^\alpha \right] \\ & \leq \widehat{D}_\alpha(\rho \| \sigma) \end{aligned} \quad (\text{H.328})$$

$$\leq \frac{1}{\alpha - 1} \ln \lambda_{\max}(\sigma) + \frac{1}{\alpha - 1} \ln \operatorname{Tr} \left[ \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right)^\alpha \right]. \quad (\text{H.329})$$

Rewriting

$$\frac{1}{\alpha - 1} \ln \operatorname{Tr} \left[ \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right)^\alpha \right] = \frac{\alpha}{\alpha - 1} \ln \left( \operatorname{Tr} \left[ \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right)^\alpha \right] \right)^{\frac{1}{\alpha}} \quad (\text{H.330})$$

$$= \frac{\alpha}{\alpha - 1} \ln \left\| \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right\|_\alpha. \quad (\text{H.331})$$

Then by applying  $\lim_{\alpha \rightarrow \infty} \|X\|_\alpha = \|X\|_\infty$ , it follows that

$$\lim_{\alpha \rightarrow \infty} \frac{1}{\alpha - 1} \ln \operatorname{Tr} \left[ \left( \sigma^{-\frac{1}{2}} \rho \sigma^{-\frac{1}{2}} \right)^\alpha \right] = D_{\max}(\rho \| \sigma). \quad (\text{H.332})$$

Combining this limit with the inequalities in (H.328) and (H.329), we arrive at the equality in (H.324). ■

## I Geometric Rényi relative entropy of quantum channels

Here we prove the explicit form for the geometric Rényi relative entropy of quantum channels from Proposition 44, as well as the chain rule from Proposition 45. We first begin by recalling the transformer inequality from [150] and [39, Lemma 47].

**Lemma 86** *Let  $X$  and  $Y$  be positive semi-definite such that  $\operatorname{supp}(Y) \subseteq \operatorname{supp}(X)$ , and let  $L$  be a linear operator. Then for  $\alpha \in (1, 2]$ , the following inequality holds*

$$G_\alpha(LXL^\dagger, LYL^\dagger) \leq LG_\alpha(X, Y)L^\dagger, \quad (\text{I.1})$$

where  $G_\alpha$  is defined in (H.23). For  $\alpha \in (0, 1)$ , the following inequality holds

$$LG_\alpha(X, Y)L^\dagger \leq G_\alpha(LXL^\dagger, LYL^\dagger), \quad (\text{I.2})$$

In both of the above inequalities, the inverses  $(LXL^\dagger)^{-1}$  are taken on the support of  $LXL^\dagger$ . If  $L$  is invertible, then the inequalities hold with equality.

**Proof.** For positive definite  $X$  and  $Y$  and  $\alpha \in (1, 2]$ , we have that

$$G_\alpha(X, Y) = G_{1-\alpha}(Y, X), \quad (\text{I.3})$$

$$G_{1-\alpha}(LYL^\dagger, LXL^\dagger) \leq LG_{1-\alpha}(Y, X)L^\dagger, \quad (\text{I.4})$$

where the equality follows from Lemma 68 and the inequality from [39, Lemma 47] (the special case of  $\alpha = 2$  was established in [77, Proposition 4.1]). Then by defining  $Y_\varepsilon = Y + \varepsilon I$  for  $\varepsilon > 0$ , we conclude that

$$G_\alpha(LXL^\dagger, LY_\varepsilon L^\dagger) \leq LG_\alpha(X, Y_\varepsilon)L^\dagger. \quad (\text{I.5})$$

By taking the limit  $\varepsilon \rightarrow 0^+$ , we conclude (I.1), holding for  $X$  and  $Y$  positive semi-definite such that  $\text{supp}(Y) \subseteq \text{supp}(X)$ .

The inequality in (I.2) is known from [150] for positive definite  $X$  and  $Y$ . Then we get (I.2) by employing  $Y_\varepsilon$  again and taking the limit  $\varepsilon \rightarrow 0^+$ .

For invertible  $L$ , the equalities follow by applying the inequality again, as shown in [150] and [39, Lemma 47]. For  $\alpha \in (1, 2]$ , we have the following for invertible  $L$ :

$$G_\alpha(LXL^\dagger, LYL^\dagger) \leq LG_\alpha(X, Y)L^\dagger \quad (\text{I.6})$$

$$= LG_\alpha(L^{-1}LXL^\dagger L^{-\dagger}, L^{-1}LYL^\dagger L^{-\dagger})L^\dagger \quad (\text{I.7})$$

$$\leq LL^{-1}G_\alpha(LXL^\dagger, LYL^\dagger)L^{-\dagger}L^\dagger \quad (\text{I.8})$$

$$= G_\alpha(LXL^\dagger, LYL^\dagger). \quad (\text{I.9})$$

The same argument applies for  $\alpha \in (0, 1)$ , but the inequalities flip. ■

**Proof of Proposition 44.** First, suppose that  $\alpha \in (1, 2]$  and  $\text{supp}(\Gamma_{RB}^\mathcal{N}) \not\subseteq \text{supp}(\Gamma_{RB}^\mathcal{M})$ . Then we can take the maximally entangled state  $\Phi_{RA}$  (normalized version of  $\Gamma_{RA}$ ) as input, and it follows that  $\widehat{D}_\alpha(\mathcal{N}||\mathcal{M}) = +\infty$ .

So let us suppose that  $\alpha \in (1, 2]$  and  $\text{supp}(\Gamma_{RB}^\mathcal{N}) \subseteq \text{supp}(\Gamma_{RB}^\mathcal{M})$ . Let  $\psi_{RA}$  be an arbitrary pure bipartite input state. We can write such a state as follows:

$$\psi_{RA} = Z_R \Gamma_{RA} Z_R^\dagger, \quad (\text{I.10})$$

where  $Z_R$  is an operator satisfying  $\text{Tr}[Z_R^\dagger Z_R] = 1$ . Then it follows that

$$\mathcal{N}_{A \rightarrow B}(\psi_{RA}) = Z_R \Gamma_{RB}^\mathcal{N} Z_R^\dagger. \quad (\text{I.11})$$

Due to the fact that the set of states with  $Z_R$  invertible is dense in the set of all pure bipartite states, it suffices to optimize with respect to this set:

$$\widehat{D}_\alpha(\mathcal{N}||\mathcal{M}) \quad (\text{I.12})$$

$$= \sup_{\psi_{RA}} \widehat{D}_\alpha(\mathcal{N}_{A \rightarrow B}(\psi_{RA})||\mathcal{M}_{A \rightarrow B}(\psi_{RA})) \quad (\text{I.13})$$



$$= \sup_{\substack{Z_R: |Z_R| > 0, \\ \text{Tr}[Z_R^\dagger Z_R] = 1}} \widehat{D}_\alpha(Z_R \Gamma_{RB}^{\mathcal{N}} Z_R^\dagger \| Z_R \Gamma_{RB}^{\mathcal{M}} Z_R^\dagger) \quad (\text{I.14})$$

$$= \sup_{\substack{Z_R: |Z_R| > 0, \\ \text{Tr}[Z_R^\dagger Z_R] = 1}} \frac{1}{\alpha - 1} \ln \text{Tr}[G_\alpha(Z_R \Gamma_{RB}^{\mathcal{M}} Z_R^\dagger, Z_R \Gamma_{RB}^{\mathcal{N}} Z_R^\dagger)] \quad (\text{I.15})$$

$$= \sup_{\substack{Z_R: |Z_R| > 0, \\ \text{Tr}[Z_R^\dagger Z_R] = 1}} \frac{1}{\alpha - 1} \ln \text{Tr}[Z_R G_\alpha(\Gamma_{RB}^{\mathcal{M}}, \Gamma_{RB}^{\mathcal{N}}) Z_R^\dagger] \quad (\text{I.16})$$

$$= \sup_{\substack{Z_R: |Z_R| > 0, \\ \text{Tr}[Z_R^\dagger Z_R] = 1}} \frac{1}{\alpha - 1} \ln \text{Tr}[Z_R^\dagger Z_R G_\alpha(\Gamma_{RB}^{\mathcal{M}}, \Gamma_{RB}^{\mathcal{N}})] \quad (\text{I.17})$$

$$= \frac{1}{\alpha - 1} \ln \sup_{\substack{Z_R: |Z_R| > 0, \\ \text{Tr}[Z_R^\dagger Z_R] = 1}} \text{Tr}[Z_R^\dagger Z_R G_\alpha(\Gamma_{RB}^{\mathcal{M}}, \Gamma_{RB}^{\mathcal{N}})] \quad (\text{I.18})$$

$$= \frac{1}{\alpha - 1} \ln \left\| \text{Tr}_B \left[ [\Gamma_{RB}^{\mathcal{M}}]^{1/2} ([\Gamma_{RB}^{\mathcal{M}}]^{-1/2} \Gamma_{RB}^{\mathcal{N}} [\Gamma_{RB}^{\mathcal{M}}]^{-1/2})^\alpha [\Gamma_{RB}^{\mathcal{M}}]^{1/2} \right] \right\|_\infty. \quad (\text{I.19})$$

The critical equality is the fourth one, which follows from the transformer equality of [39, Lemma 47].

Now suppose that  $\alpha \in (0, 1)$ . Then proceeding by similar reasoning, but taking care with various limits and the sign flip due to the prefactor  $\frac{1}{\alpha-1}$ , we find the following:

$$\begin{aligned} & \widehat{D}_\alpha(\mathcal{N} \| \mathcal{M}) \\ &= \sup_{\psi_{RA}} \widehat{D}_\alpha(\mathcal{N}_{A \rightarrow B}(\psi_{RA}) \| \mathcal{M}_{A \rightarrow B}(\psi_{RA})) \end{aligned} \quad (\text{I.20})$$

$$= \sup_{Z_R: |Z_R| > 0} \widehat{D}_\alpha(Z_R \Gamma_{RB}^{\mathcal{N}} Z_R^\dagger \| Z_R \Gamma_{RB}^{\mathcal{M}} Z_R^\dagger) \quad (\text{I.21})$$

$$= \sup_{Z_R: |Z_R| > 0} \lim_{\varepsilon \rightarrow 0^+} \widehat{D}_\alpha(Z_R \Gamma_{RB}^{\mathcal{N}} Z_R^\dagger \| Z_R \Gamma_{RB}^{\mathcal{M}_\varepsilon} Z_R^\dagger) \quad (\text{I.22})$$

$$= \sup_{Z_R: |Z_R| > 0} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\alpha - 1} \ln \text{Tr}[G_\alpha(Z_R \Gamma_{RB}^{\mathcal{M}_\varepsilon} Z_R^\dagger, Z_R \Gamma_{RB}^{\mathcal{N}} Z_R^\dagger)] \quad (\text{I.23})$$

$$= \frac{1}{\alpha - 1} \ln \inf_{Z_R: |Z_R| > 0} \lim_{\varepsilon \rightarrow 0^+} \text{Tr}[G_\alpha(Z_R \Gamma_{RB}^{\mathcal{M}_\varepsilon} Z_R^\dagger, Z_R \Gamma_{RB}^{\mathcal{N}} Z_R^\dagger)] \quad (\text{I.24})$$

$$= \frac{1}{\alpha - 1} \ln \inf_{Z_R: |Z_R| > 0} \inf_{\varepsilon > 0} \text{Tr}[G_\alpha(Z_R \Gamma_{RB}^{\mathcal{M}_\varepsilon} Z_R^\dagger, Z_R \Gamma_{RB}^{\mathcal{N}} Z_R^\dagger)]. \quad (\text{I.25})$$

The last equality follows from reasoning similar to that in Lemma 69, that the limit  $\varepsilon \rightarrow 0^+$  is the same as the infimum over  $\varepsilon > 0$ . Continuing, we find that

$$\begin{aligned} & \widehat{D}_\alpha(\mathcal{N} \| \mathcal{M}) \\ &= \frac{1}{\alpha - 1} \ln \inf_{\varepsilon > 0} \inf_{Z_R: |Z_R| > 0} \text{Tr}[G_\alpha(Z_R \Gamma_{RB}^{\mathcal{M}_\varepsilon} Z_R^\dagger, Z_R \Gamma_{RB}^{\mathcal{N}} Z_R^\dagger)] \end{aligned} \quad (\text{I.26})$$

$$= \frac{1}{\alpha - 1} \ln \inf_{\varepsilon > 0} \inf_{Z_R: |Z_R| > 0} \text{Tr}[Z_R G_\alpha(\Gamma_{RB}^{\mathcal{M}}, \Gamma_{RB}^{\mathcal{N}}) Z_R^\dagger] \quad (\text{I.27})$$

$$= \frac{1}{\alpha - 1} \ln \inf_{\varepsilon > 0} \lambda_{\min} \left( \text{Tr}_B \left( G_\alpha(\Gamma_{RB}^{\mathcal{M}}, \Gamma_{RB}^{\mathcal{N}}) \right) \right) \quad (\text{I.28})$$

$$= \frac{1}{\alpha - 1} \ln \lim_{\varepsilon \rightarrow 0^+} \lambda_{\min} \left( \text{Tr}_B \left( G_\alpha(\Gamma_{RB}^{\mathcal{M}}, \Gamma_{RB}^{\mathcal{N}}) \right) \right) \quad (\text{I.29})$$

$$= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\alpha - 1} \ln \lambda_{\min} \left( \text{Tr}_B \left( [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2} ([\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2} \Gamma_{RB}^{\mathcal{N}} [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2})^\alpha [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2} \right) \right). \quad (\text{I.30})$$

Now we establish the formula in (6.23) for  $\alpha \in (0, 1)$ . If  $\text{supp}(\Gamma_{RB}^{\mathcal{N}}) \subseteq \text{supp}(\Gamma_{RB}^{\mathcal{M}})$ , then taking the limit  $\varepsilon \rightarrow 0^+$  leads to the formula

$$\widehat{D}_\alpha(\mathcal{N} \parallel \mathcal{M}) = \frac{1}{\alpha - 1} \ln \lambda_{\min} \left( \text{Tr}_B \left( [\Gamma_{RB}^{\mathcal{M}}]^{1/2} ([\Gamma_{RB}^{\mathcal{M}}]^{-1/2} \Gamma_{RB}^{\mathcal{N}} [\Gamma_{RB}^{\mathcal{M}}]^{-1/2})^\alpha [\Gamma_{RB}^{\mathcal{M}}]^{1/2} \right) \right). \quad (\text{I.31})$$

If  $\text{supp}(\Gamma_{RB}^{\mathcal{N}}) \not\subseteq \text{supp}(\Gamma_{RB}^{\mathcal{M}})$ , then the proof is similar to the proof of (6.6), but more involved. We need to evaluate the following limit for  $\alpha \in (0, 1)$ :

$$\lim_{\varepsilon \rightarrow 0^+} \lambda_{\min} \left( \text{Tr}_B \left[ [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2} ([\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2} \Gamma_{RB}^{\mathcal{N}} [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2})^\alpha [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2} \right] \right). \quad (\text{I.32})$$

For  $\varepsilon > 0$  and  $\delta \in (0, 1)$ , let us write

$$\Gamma_{RB}^{\mathcal{M}_\varepsilon} = \begin{bmatrix} \widehat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon} & 0 \\ 0 & \varepsilon \Pi_{\Gamma^{\mathcal{M}}}^\perp \end{bmatrix}, \quad (\text{I.33})$$

$$\widehat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon} := \Gamma_{RB}^{\mathcal{M}} + \varepsilon \Pi_{\Gamma^{\mathcal{M}}}, \quad (\text{I.34})$$

$$\Gamma_{RB}^{\mathcal{N}_\delta} := (1 - \delta) \Gamma_{RB}^{\mathcal{N}} + \delta I_R \otimes \pi_B, \quad (\text{I.35})$$

$$\Gamma_{RB}^{\mathcal{N}_\delta} = \begin{bmatrix} (\Gamma_{RB}^{\mathcal{N}_\delta})_{0,0} & (\Gamma_{RB}^{\mathcal{N}_\delta})_{0,1} \\ (\Gamma_{RB}^{\mathcal{N}_\delta})_{0,1}^\dagger & (\Gamma_{RB}^{\mathcal{N}_\delta})_{1,1} \end{bmatrix}, \quad (\text{I.36})$$

$$(\Gamma_{RB}^{\mathcal{N}_\delta})_{0,0} := \Pi_{\Gamma^{\mathcal{M}}} \Gamma_{RB}^{\mathcal{N}_\delta} \Pi_{\Gamma^{\mathcal{M}}}, \quad (\text{I.37})$$

$$(\Gamma_{RB}^{\mathcal{N}_\delta})_{0,1} := \Pi_{\Gamma^{\mathcal{M}}} \Gamma_{RB}^{\mathcal{N}_\delta} \Pi_{\Gamma^{\mathcal{M}}}^\perp, \quad (\text{I.38})$$

$$(\Gamma_{RB}^{\mathcal{N}_\delta})_{1,1} := \Pi_{\Gamma^{\mathcal{M}}}^\perp \Gamma_{RB}^{\mathcal{N}_\delta} \Pi_{\Gamma^{\mathcal{M}}}^\perp. \quad (\text{I.39})$$

Consider that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \lambda_{\min} \left( \text{Tr}_B \left[ [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2} ([\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2} \Gamma_{RB}^{\mathcal{N}} [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2})^\alpha [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2} \right] \right) \\ &= \lim_{\varepsilon \rightarrow 0^+} \lim_{\delta \rightarrow 0^+} \lambda_{\min} \left( \text{Tr}_B \left[ [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2} ([\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2} \Gamma_{RB}^{\mathcal{N}_\delta} [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2})^\alpha [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2} \right] \right). \end{aligned} \quad (\text{I.40})$$

We note by similar reasoning given to establish Lemma 69, it follows for  $\alpha \in (0, 1)$  that

$$\lim_{\varepsilon \rightarrow 0^+} \lim_{\delta \rightarrow 0^+} \lambda_{\min} \left( \text{Tr}_B \left[ [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2} ([\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2} \Gamma_{RB}^{\mathcal{N}_\delta} [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2})^\alpha [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2} \right] \right)$$



so that we can write

$$\begin{aligned} & [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2} ([\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2} \Gamma_{RB}^{\mathcal{N}} [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2})^\alpha [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2} \\ &= \begin{bmatrix} \varepsilon^{-\frac{\alpha}{2}} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{\frac{1-\alpha}{2}} \Pi_{\Gamma_{\mathcal{M}}}^\perp \end{bmatrix} [K(\varepsilon)]^\alpha \begin{bmatrix} \varepsilon^{-\frac{\alpha}{2}} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{\frac{1-\alpha}{2}} \Pi_{\Gamma_{\mathcal{M}}}^\perp \end{bmatrix}. \end{aligned} \quad (\text{I.51})$$

Now let us invoke Lemma 62 with the substitutions

$$A \leftrightarrow (\Gamma_{RB}^{\mathcal{N}_\delta})_{1,1}, \quad (\text{I.52})$$

$$B \leftrightarrow (\Gamma_{RB}^{\mathcal{N}_\delta})_{0,1}^\dagger (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{-\frac{1}{2}}, \quad (\text{I.53})$$

$$C \leftrightarrow \varepsilon (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{-\frac{1}{2}} (\Gamma_{RB}^{\mathcal{N}_\delta})_{0,0} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{-\frac{1}{2}}, \quad (\text{I.54})$$

$$\varepsilon \leftrightarrow \varepsilon^{\frac{1}{2}}. \quad (\text{I.55})$$

Defining

$$L(\varepsilon) := \begin{bmatrix} \varepsilon S_\delta & 0 \\ 0 & (\Gamma_{RB}^{\mathcal{N}_\delta})_{1,1} + \varepsilon R \end{bmatrix}, \quad (\text{I.56})$$

$$S_\delta := (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{-\frac{1}{2}} \left( (\Gamma_{RB}^{\mathcal{N}_\delta})_{0,0} - (\Gamma_{RB}^{\mathcal{N}_\delta})_{0,1} (\Gamma_{RB}^{\mathcal{N}_\delta})_{1,1}^{-1} (\Gamma_{RB}^{\mathcal{N}_\delta})_{0,1}^\dagger \right) (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{-\frac{1}{2}}, \quad (\text{I.57})$$

$$R := \text{Re}[(\Gamma_{RB}^{\mathcal{N}_\delta})_{1,1}^{-1} (\Gamma_{RB}^{\mathcal{N}_\delta})_{0,1}^\dagger (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{-1} (\Gamma_{RB}^{\mathcal{N}_\delta})_{0,1}], \quad (\text{I.58})$$

we conclude from Lemma 62 that

$$\left\| K(\varepsilon) - e^{-i\sqrt{\varepsilon}G} L(\varepsilon) e^{i\sqrt{\varepsilon}G} \right\|_\infty \leq o(\varepsilon), \quad (\text{I.59})$$

where  $G$  in Lemma 62 is defined from  $A$  and  $B$  above. The inequality in (I.59) in turn implies the following operator inequalities:

$$e^{-i\sqrt{\varepsilon}G} L(\varepsilon) e^{i\sqrt{\varepsilon}G} - o(\varepsilon)I \leq K(\varepsilon) \leq e^{-i\sqrt{\varepsilon}G} L(\varepsilon) e^{i\sqrt{\varepsilon}G} + o(\varepsilon)I. \quad (\text{I.60})$$

Observe that

$$e^{-i\sqrt{\varepsilon}G} L(\varepsilon) e^{i\sqrt{\varepsilon}G} + o(\varepsilon)I = e^{-i\sqrt{\varepsilon}G} [L(\varepsilon) + o(\varepsilon)I] e^{i\sqrt{\varepsilon}G}. \quad (\text{I.61})$$

Now invoking these and the operator monotonicity of the function  $x^\alpha$  for  $\alpha \in (0, 1)$ , we find that

$$\begin{aligned} & [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2} ([\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2} \Gamma_{RB}^{\mathcal{N}} [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2})^\alpha [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2} \\ &= \begin{bmatrix} \varepsilon^{-\frac{\alpha}{2}} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{\frac{1-\alpha}{2}} \Pi_{\Gamma_{\mathcal{M}}}^\perp \end{bmatrix} [K(\varepsilon)]^\alpha \begin{bmatrix} \varepsilon^{-\frac{\alpha}{2}} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{\frac{1-\alpha}{2}} \Pi_{\Gamma_{\mathcal{M}}}^\perp \end{bmatrix} \end{aligned} \quad (\text{I.62})$$

$$\leq \begin{bmatrix} \varepsilon^{-\frac{\alpha}{2}} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{\frac{1-\alpha}{2}} \Pi_{\Gamma_{\mathcal{M}}}^\perp \end{bmatrix} \left[ e^{-i\sqrt{\varepsilon}G} [L(\varepsilon) + o(\varepsilon)I] e^{i\sqrt{\varepsilon}G} \right]^\alpha \begin{bmatrix} \varepsilon^{-\frac{\alpha}{2}} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{\frac{1-\alpha}{2}} \Pi_{\Gamma_{\mathcal{M}}}^\perp \end{bmatrix} \quad (\text{I.63})$$

$$= \begin{bmatrix} \varepsilon^{-\frac{\alpha}{2}} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{\frac{1-\alpha}{2}} \Pi_{\Gamma\mathcal{M}}^\perp \end{bmatrix} e^{-i\sqrt{\varepsilon}G} [L(\varepsilon) + o(\varepsilon)I]^\alpha e^{i\sqrt{\varepsilon}G} \begin{bmatrix} \varepsilon^{-\frac{\alpha}{2}} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{\frac{1-\alpha}{2}} \Pi_{\Gamma\mathcal{M}}^\perp \end{bmatrix}. \quad (\text{I.64})$$

Defining

$$Q(\varepsilon) := (\Gamma_{RB}^{\mathcal{N}_\delta})_{1,1} + \varepsilon R + o(\varepsilon)I, \quad (\text{I.65})$$

consider that

$$[L(\varepsilon) + o(\varepsilon)I]^\alpha = \begin{bmatrix} \varepsilon S_\delta + o(\varepsilon)I & 0 \\ 0 & (\Gamma_{RB}^{\mathcal{N}_\delta})_{1,1} + \varepsilon R + o(\varepsilon)I \end{bmatrix}^\alpha \quad (\text{I.66})$$

$$= \begin{bmatrix} (\varepsilon S_\delta + o(\varepsilon)I)^\alpha & 0 \\ 0 & ((\Gamma_{RB}^{\mathcal{N}_\delta})_{1,1} + \varepsilon R + o(\varepsilon)I)^\alpha \end{bmatrix} \quad (\text{I.67})$$

$$= \begin{bmatrix} \varepsilon^\alpha (S_\delta + o(1)I)^\alpha & 0 \\ 0 & (Q(\varepsilon))^\alpha \end{bmatrix}. \quad (\text{I.68})$$

Now expanding  $e^{-i\sqrt{\varepsilon}G}$  and  $e^{i\sqrt{\varepsilon}G}$  to first order to evaluate (I.64), we find that

$$\begin{aligned} & \begin{bmatrix} \varepsilon^{-\frac{\alpha}{2}} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{\frac{1-\alpha}{2}} \Pi_{\Gamma\mathcal{M}}^\perp \end{bmatrix} e^{-i\sqrt{\varepsilon}G} [L(\varepsilon) + o(\varepsilon)I]^\alpha e^{i\sqrt{\varepsilon}G} \begin{bmatrix} \varepsilon^{-\frac{\alpha}{2}} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{\frac{1-\alpha}{2}} \Pi_{\Gamma\mathcal{M}}^\perp \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon^{-\frac{\alpha}{2}} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{\frac{1-\alpha}{2}} \Pi_{\Gamma\mathcal{M}}^\perp \end{bmatrix} [L(\varepsilon) + o(\varepsilon)I]^\alpha \begin{bmatrix} \varepsilon^{-\frac{\alpha}{2}} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{\frac{1-\alpha}{2}} \Pi_{\Gamma\mathcal{M}}^\perp \end{bmatrix} \\ &\quad - i\varepsilon^{\frac{1}{2}} \begin{bmatrix} \varepsilon^{-\frac{\alpha}{2}} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{\frac{1-\alpha}{2}} \Pi_{\Gamma\mathcal{M}}^\perp \end{bmatrix} G [L(\varepsilon) + o(\varepsilon)I]^\alpha \begin{bmatrix} \varepsilon^{-\frac{\alpha}{2}} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{\frac{1-\alpha}{2}} \Pi_{\Gamma\mathcal{M}}^\perp \end{bmatrix} \\ &\quad + i\varepsilon^{\frac{1}{2}} \begin{bmatrix} \varepsilon^{-\frac{\alpha}{2}} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{\frac{1-\alpha}{2}} \Pi_{\Gamma\mathcal{M}}^\perp \end{bmatrix} [L(\varepsilon) + o(\varepsilon)I]^\alpha G \begin{bmatrix} \varepsilon^{-\frac{\alpha}{2}} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{\frac{1-\alpha}{2}} \Pi_{\Gamma\mathcal{M}}^\perp \end{bmatrix} + o(1) \end{aligned} \quad (\text{I.69})$$

$$\begin{aligned} &= \begin{bmatrix} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} (S_\delta + o(1)I)^\alpha (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{1-\alpha} \Pi_{\Gamma\mathcal{M}}^\perp (Q(\varepsilon))^\alpha \Pi_{\Gamma\mathcal{M}}^\perp \end{bmatrix} \\ &\quad - i \begin{bmatrix} \varepsilon^{\frac{1-\alpha}{2}} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{\frac{2-\alpha}{2}} \Pi_{\Gamma\mathcal{M}}^\perp \end{bmatrix} G \begin{bmatrix} \varepsilon^{\frac{\alpha}{2}} (S_\delta + o(1)I)^\alpha (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{\frac{1-\alpha}{2}} (Q(\varepsilon))^\alpha \Pi_{\Gamma\mathcal{M}}^\perp \end{bmatrix} \\ &\quad + i \begin{bmatrix} \varepsilon^{\frac{\alpha}{2}} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} (S_\delta + o(1)I)^\alpha & 0 \\ 0 & \varepsilon^{\frac{1-\alpha}{2}} \Pi_{\Gamma\mathcal{M}}^\perp (Q(\varepsilon))^\alpha \end{bmatrix} G \begin{bmatrix} \varepsilon^{\frac{1-\alpha}{2}} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{\frac{2-\alpha}{2}} \Pi_{\Gamma\mathcal{M}}^\perp \end{bmatrix} + o(1) \end{aligned} \quad (\text{I.70})$$

$$= \begin{bmatrix} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} (S_\delta + o(1)I)^\alpha (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{1-\alpha} \Pi_{\Gamma\mathcal{M}}^\perp (Q(\varepsilon))^\alpha \Pi_{\Gamma\mathcal{M}}^\perp \end{bmatrix} + o(1). \quad (\text{I.71})$$

Thus, we have established the following operator inequality:

$$\begin{aligned}
& [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2} ([\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2} \Gamma_{RB}^{\mathcal{N}} [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2})^\alpha [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2} \\
& \leq \begin{bmatrix} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} (S_\delta + o(1)I)^\alpha (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{1-\alpha} \Pi_{\Gamma_{\mathcal{M}}}^\perp (Q(\varepsilon))^\alpha \Pi_{\Gamma_{\mathcal{M}}}^\perp \end{bmatrix} + o(1) \quad (\text{I.72})
\end{aligned}$$

By similar reasoning, but applying the lower bound in (I.60), we also establish the following operator inequality lower bound:

$$\begin{aligned}
& \begin{bmatrix} (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} (S_\delta - o(1)I)^\alpha (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{\frac{1}{2}} & 0 \\ 0 & \varepsilon^{1-\alpha} \Pi_{\Gamma_{\mathcal{M}}}^\perp (Q(\varepsilon))^\alpha \Pi_{\Gamma_{\mathcal{M}}}^\perp \end{bmatrix} + o(1) \\
& \leq [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2} ([\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2} \Gamma_{RB}^{\mathcal{N}} [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2})^\alpha [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2}. \quad (\text{I.73})
\end{aligned}$$

Now taking the partial trace, evaluating the minimum eigenvalue, and the limit  $\varepsilon \rightarrow 0^+$ , we conclude that

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0^+} \lambda_{\min} \left( \text{Tr}_B \left[ [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2} ([\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2} \Gamma_{RB}^{\mathcal{N}_\delta} [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{-1/2})^\alpha [\Gamma_{RB}^{\mathcal{M}_\varepsilon}]^{1/2} \right] \right) \\
& = \lambda_{\min} \left( \text{Tr}_B \left[ [\Gamma_{RB}^{\mathcal{M}}]^{1/2} ([\Gamma_{RB}^{\mathcal{M}}]^{-1/2} \tilde{\Gamma}_{RB}^{\mathcal{N}_\delta} [\Gamma_{RB}^{\mathcal{M}}]^{-1/2})^\alpha [\Gamma_{RB}^{\mathcal{M}}]^{1/2} \right] \right), \quad (\text{I.74})
\end{aligned}$$

where

$$\tilde{\Gamma}_{RB}^{\mathcal{N}_\delta} := \left( (\Gamma_{RB}^{\mathcal{N}_\delta})_{0,0} - (\Gamma_{RB}^{\mathcal{N}_\delta})_{0,1} (\Gamma_{RB}^{\mathcal{N}_\delta})_{1,1}^{-1} (\Gamma_{RB}^{\mathcal{N}_\delta})_{0,1}^\dagger \right) (\hat{\Gamma}_{RB}^{\mathcal{M}_\varepsilon})^{-\frac{1}{2}}. \quad (\text{I.75})$$

Noting that

$$\lim_{\delta \rightarrow 0^+} \tilde{\Gamma}_{RB}^{\mathcal{N}_\delta} = \tilde{\Gamma}_{RB}^{\mathcal{N}}, \quad (\text{I.76})$$

where  $\tilde{\Gamma}_{RB}^{\mathcal{N}} := \tilde{\Gamma}_{RB}^{\mathcal{N}_{\delta=0}}$ , because the image of  $(\Gamma_{RB}^{\mathcal{N}})_{0,1}^\dagger$  is contained in the support of  $(\Gamma_{RB}^{\mathcal{N}})_{1,1}$ , we conclude that

$$\begin{aligned}
& \lim_{\delta \rightarrow 0^+} \lambda_{\min} \left( \text{Tr}_B \left[ [\Gamma_{RB}^{\mathcal{M}}]^{1/2} ([\Gamma_{RB}^{\mathcal{M}}]^{-1/2} \tilde{\Gamma}_{RB}^{\mathcal{N}_\delta} [\Gamma_{RB}^{\mathcal{M}}]^{-1/2})^\alpha [\Gamma_{RB}^{\mathcal{M}}]^{1/2} \right] \right) \\
& = \lambda_{\min} \left( \text{Tr}_B \left[ [\Gamma_{RB}^{\mathcal{M}}]^{1/2} ([\Gamma_{RB}^{\mathcal{M}}]^{-1/2} \tilde{\Gamma}_{RB}^{\mathcal{N}} [\Gamma_{RB}^{\mathcal{M}}]^{-1/2})^\alpha [\Gamma_{RB}^{\mathcal{M}}]^{1/2} \right] \right). \quad (\text{I.77})
\end{aligned}$$

Combining with (I.40) and (I.41), this concludes the proof. ■

**Proof of Proposition 45.** Let us first consider the case  $\alpha \in (1, 2]$  and  $\text{supp}(\rho_{RA}) \not\subseteq \text{supp}(\sigma_{RA})$  or  $\text{supp}(\Gamma_{RB}^{\mathcal{N}}) \not\subseteq \text{supp}(\Gamma_{RB}^{\mathcal{M}})$ . In this case, the sum on the right-hand side is equal to  $+\infty$ , so that the inequality trivially holds.

Let us then consider the case  $\alpha \in (1, 2]$  and  $\text{supp}(\rho_{RA}) \subseteq \text{supp}(\sigma_{RA})$  and  $\text{supp}(\Gamma_{RB}^{\mathcal{N}}) \subseteq \text{supp}(\Gamma_{RB}^{\mathcal{M}})$ . The postselected teleportation identity implies that

$$\mathcal{N}_{A \rightarrow B}(\rho_{RA}) = \langle \Gamma |_{AS} \rho_{RA} \otimes \Gamma_{SB}^{\mathcal{N}} | \Gamma \rangle_{AS}, \quad (\text{I.78})$$

$$\mathcal{M}_{A \rightarrow B}(\sigma_{RA}) = \langle \Gamma |_{AS} \sigma_{RA} \otimes \Gamma_{SB}^{\mathcal{M}} | \Gamma \rangle_{AS}. \quad (\text{I.79})$$

Consider that

$$\text{Tr}[G_\alpha(\mathcal{M}_{A \rightarrow B}(\sigma_{RA}), \mathcal{N}_{A \rightarrow B}(\rho_{RA}))]$$

$$= \text{Tr}[G_\alpha(\langle \Gamma|_{AS} \sigma_{RA} \otimes \Gamma_{SB}^{\mathcal{M}} |\Gamma\rangle_{AS}, \langle \Gamma|_{AS} \rho_{RA} \otimes \Gamma_{SB}^{\mathcal{N}} |\Gamma\rangle_{AS})] \quad (\text{I.80})$$

$$\leq \text{Tr}[\langle \Gamma|_{AS} G_\alpha(\sigma_{RA} \otimes \Gamma_{SB}^{\mathcal{M}}, \rho_{RA} \otimes \Gamma_{SB}^{\mathcal{N}}) |\Gamma\rangle_{AS}] \quad (\text{I.81})$$

$$= \text{Tr}[\langle \Gamma|_{AS} G_\alpha(\sigma_{RA}, \rho_{RA}) \otimes G_\alpha(\Gamma_{SB}^{\mathcal{M}}, \Gamma_{SB}^{\mathcal{N}}) |\Gamma\rangle_{AS}] \quad (\text{I.82})$$

$$= \text{Tr}_{RB}[\langle \Gamma|_{AS} G_\alpha(\sigma_{RA}, \rho_{RA}) \otimes G_\alpha(\Gamma_{SB}^{\mathcal{M}}, \Gamma_{SB}^{\mathcal{N}}) |\Gamma\rangle_{AS}] \quad (\text{I.83})$$

$$= \langle \Gamma|_{AS} \text{Tr}_R[G_\alpha(\sigma_{RA}, \rho_{RA})] \otimes \text{Tr}_B[G_\alpha(\Gamma_{SB}^{\mathcal{M}}, \Gamma_{SB}^{\mathcal{N}})] |\Gamma\rangle_{AS} \quad (\text{I.84})$$

$$\leq \|\text{Tr}_B[G_\alpha(\Gamma_{SB}^{\mathcal{M}}, \Gamma_{SB}^{\mathcal{N}})]\|_\infty \cdot \langle \Gamma|_{AS} \text{Tr}_R[G_\alpha(\sigma_{RA}, \rho_{RA})] \otimes I_S |\Gamma\rangle_{AS} \quad (\text{I.85})$$

$$= \|\text{Tr}_B[G_\alpha(\Gamma_{SB}^{\mathcal{M}}, \Gamma_{SB}^{\mathcal{N}})]\|_\infty \cdot \text{Tr}_{RA}[G_\alpha(\sigma_{RA}, \rho_{RA})] \quad (\text{I.86})$$

$$= \|\text{Tr}_B[G_\alpha(\Gamma_{SB}^{\mathcal{M}}, \Gamma_{SB}^{\mathcal{N}})]\|_\infty \cdot \text{Tr}[G_\alpha(\sigma_{RA}, \rho_{RA})]. \quad (\text{I.87})$$

Now applying a logarithm and dividing by  $\alpha - 1$ , we conclude the chain rule:

$$\begin{aligned} \widehat{D}_\alpha(\mathcal{N}_{A \rightarrow B}(\rho_{RA})) \|\mathcal{M}_{A \rightarrow B}(\sigma_{RA})\| &\leq \frac{1}{\alpha - 1} \ln \|\text{Tr}_B[G_\alpha(\Gamma_{SB}^{\mathcal{M}}, \Gamma_{SB}^{\mathcal{N}})]\|_\infty \\ &\quad + \widehat{D}_\alpha(\rho_{RA} \|\sigma_{RA}). \end{aligned} \quad (\text{I.88})$$

The argument for  $\alpha \in (0, 1)$  is similar, but we should be careful with limits and we exploit the minimum eigenvalue instead of the maximum eigenvalue. Fix  $\varepsilon > 0$ ,  $\delta \in (0, 1)$ , and consider that

$$\begin{aligned} &\text{Tr}[G_\alpha(\mathcal{M}_{A \rightarrow B}^\varepsilon(\sigma_{RA}^\varepsilon), \mathcal{N}_{A \rightarrow B}^\delta(\rho_{RA}^\delta))] \\ &= \text{Tr}[G_\alpha(\langle \Gamma|_{AS} \sigma_{RA}^\varepsilon \otimes \Gamma_{SB}^{\mathcal{M}_\varepsilon} |\Gamma\rangle_{AS}, \langle \Gamma|_{AS} \rho_{RA}^\delta \otimes \Gamma_{SB}^{\mathcal{N}_\delta} |\Gamma\rangle_{AS})] \end{aligned} \quad (\text{I.89})$$

$$\geq \text{Tr}[\langle \Gamma|_{AS} G_\alpha(\sigma_{RA}^\varepsilon \otimes \Gamma_{SB}^{\mathcal{M}_\varepsilon}, \rho_{RA}^\delta \otimes \Gamma_{SB}^{\mathcal{N}_\delta}) |\Gamma\rangle_{AS}] \quad (\text{I.90})$$

$$= \text{Tr}[\langle \Gamma|_{AS} G_\alpha(\sigma_{RA}^\varepsilon, \rho_{RA}^\delta) \otimes G_\alpha(\Gamma_{SB}^{\mathcal{M}_\varepsilon}, \Gamma_{SB}^{\mathcal{N}_\delta}) |\Gamma\rangle_{AS}] \quad (\text{I.91})$$

$$= \text{Tr}_{RB}[\langle \Gamma|_{AS} G_\alpha(\sigma_{RA}^\varepsilon, \rho_{RA}^\delta) \otimes G_\alpha(\Gamma_{SB}^{\mathcal{M}_\varepsilon}, \Gamma_{SB}^{\mathcal{N}_\delta}) |\Gamma\rangle_{AS}] \quad (\text{I.92})$$

$$= \langle \Gamma|_{AS} \text{Tr}_R[G_\alpha(\sigma_{RA}^\varepsilon, \rho_{RA}^\delta)] \otimes \text{Tr}_B[G_\alpha(\Gamma_{SB}^{\mathcal{M}_\varepsilon}, \Gamma_{SB}^{\mathcal{N}_\delta})] |\Gamma\rangle_{AS} \quad (\text{I.93})$$

$$\geq \lambda_{\min} \left( \text{Tr}_B[G_\alpha(\Gamma_{SB}^{\mathcal{M}_\varepsilon}, \Gamma_{SB}^{\mathcal{N}_\delta})] \right) \cdot \langle \Gamma|_{AS} \text{Tr}_R[G_\alpha(\sigma_{RA}^\varepsilon, \rho_{RA}^\delta)] \otimes I_S |\Gamma\rangle_{AS} \quad (\text{I.94})$$

$$= \lambda_{\min} \left( \text{Tr}_B[G_\alpha(\Gamma_{SB}^{\mathcal{M}_\varepsilon}, \Gamma_{SB}^{\mathcal{N}_\delta})] \right) \cdot \text{Tr}_{RA}[G_\alpha(\sigma_{RA}^\varepsilon, \rho_{RA}^\delta)] \quad (\text{I.95})$$

$$= \lambda_{\min} \left( \text{Tr}_B[G_\alpha(\Gamma_{SB}^{\mathcal{M}_\varepsilon}, \Gamma_{SB}^{\mathcal{N}_\delta})] \right) \cdot \text{Tr}[G_\alpha(\sigma_{RA}^\varepsilon, \rho_{RA}^\delta)]. \quad (\text{I.96})$$

Now taking a logarithm and dividing by  $\alpha - 1$ , we arrive at the following inequality:

$$\begin{aligned} \widehat{D}_\alpha(\mathcal{N}_{A \rightarrow B}^\delta(\rho_{RA}^\delta)) \|\mathcal{M}_{A \rightarrow B}^\varepsilon(\sigma_{RA}^\varepsilon)\| &\leq \frac{1}{\alpha - 1} \ln \lambda_{\min} \left( \text{Tr}_B[G_\alpha(\Gamma_{SB}^{\mathcal{M}_\varepsilon}, \Gamma_{SB}^{\mathcal{N}_\delta})] \right) \\ &\quad + \widehat{D}_\alpha(\rho_{RA}^\delta \|\sigma_{RA}^\varepsilon). \end{aligned} \quad (\text{I.97})$$

Taking the limit as  $\delta \rightarrow 0^+$ , we find that

$$\begin{aligned} \widehat{D}_\alpha(\mathcal{N}_{A \rightarrow B}(\rho_{RA})) \|\mathcal{M}_{A \rightarrow B}^\varepsilon(\sigma_{RA}^\varepsilon)\| &\leq \frac{1}{\alpha - 1} \ln \lambda_{\min} \left( \text{Tr}_B[G_\alpha(\Gamma_{SB}^{\mathcal{M}_\varepsilon}, \Gamma_{SB}^{\mathcal{N}})] \right) \\ &\quad + \widehat{D}_\alpha(\rho_{RA} \|\sigma_{RA}^\varepsilon). \end{aligned} \quad (\text{I.98})$$

where we used the fact that the operations of evaluating the minimum eigenvalue and the limit  $\delta \rightarrow 0^+$  commute. Then taking the limit as  $\varepsilon \rightarrow 0^+$ , we conclude that

$$\begin{aligned} \widehat{D}_\alpha(\mathcal{N}_{A \rightarrow B}(\rho_{RA})) \|\mathcal{M}_{A \rightarrow B}(\sigma_{RA})\| &\leq \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\alpha - 1} \ln \lambda_{\min} \left( \text{Tr}_B[G_\alpha(\Gamma_{SB}^{\mathcal{M}_\varepsilon}, \Gamma_{SB}^{\mathcal{N}})] \right) \\ &\quad + \widehat{D}_\alpha(\rho_{RA} \|\sigma_{RA}). \end{aligned} \quad (\text{I.99})$$

This concludes the proof. ■

## J SLD and RLD Fisher informations as limits of Rényi relative entropies

**Lemma 87** *For a second-order differentiable family  $\{\rho_\theta\}_\theta$  of quantum states, the expressions in (7.1) and (7.2) are equal.*

**Proof.** This follows from the linear approximation of the logarithm around one. Set

$$I_F \equiv I_F(\theta; \{\mathcal{N}_\theta\}_\theta) := \lim_{\delta \rightarrow 0} \frac{8}{\delta^2} f(\theta, \delta), \quad (\text{J.1})$$

where

$$f(\theta, \delta) := 1 - \sqrt{F}(\rho_\theta \|\rho_{\theta+\delta}), \quad (\text{J.2})$$

and suppose that the limit in (J.1) exists and is a finite number. Then for sufficiently small  $\delta > 0$ , the following inequalities hold

$$\frac{8}{\delta^2} f(\theta, \delta) < |I_F| + 1, \quad |f(\theta, \delta)| < 1/2. \quad (\text{J.3})$$

Using the following expansion for  $x \in [0, 1)$

$$-\ln(1 - x) = \sum_{n=1}^{\infty} \frac{x^n}{n}, \quad (\text{J.4})$$

we find that

$$-\frac{4}{\delta^2} \ln F(\rho_\theta \|\rho_{\theta+\delta}) = -\frac{8}{\delta^2} \ln \sqrt{F}(\rho_\theta \|\rho_{\theta+\delta}) \quad (\text{J.5})$$

$$= -\frac{8}{\delta^2} \ln(1 - f(\theta, \delta)) \quad (\text{J.6})$$



$$= \frac{8}{\delta^2} \sum_{n=1}^{\infty} \frac{f(\theta, \delta)^n}{n} \quad (\text{J.7})$$

$$= \frac{8}{\delta^2} \left[ f(\theta, \delta) + \frac{f(\theta, \delta)^2}{2} + \sum_{n=1}^{\infty} \frac{f(\theta, \delta)^{n+2}}{n+2} \right] \quad (\text{J.8})$$

$$= \frac{8}{\delta^2} \left[ f(\theta, \delta) + f(\theta, \delta)^2 \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{f(\theta, \delta)^n}{n+2} \right) \right] \quad (\text{J.9})$$

$$= \frac{8}{\delta^2} f(\theta, \delta) + \frac{\delta^2}{8} \left[ \frac{8f(\theta, \delta)}{\delta^2} \right]^2 \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{f(\theta, \delta)^n}{n+2} \right) \quad (\text{J.10})$$

$$= \frac{8}{\delta^2} f(\theta, \delta) + \frac{\delta^2}{8} g(\theta, \delta), \quad (\text{J.11})$$

where

$$g(\theta, \delta) := \left[ \frac{8f(\theta, \delta)}{\delta^2} \right]^2 \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{f(\theta, \delta)^n}{n+2} \right). \quad (\text{J.12})$$

For sufficiently small  $\delta$ , it follows from (J.3) that

$$|g(\theta, \delta)| \leq [|I_F| + 1]^2 \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n = 2(|I_F| + 1)^2.$$

Then we find that

$$\lim_{\delta \rightarrow 0} -\frac{4}{\delta^2} \ln F(\rho_\theta \| \rho_{\theta+\delta}) = \lim_{\delta \rightarrow 0} \frac{8}{\delta^2} f(\theta, \delta), \quad (\text{J.13})$$

concluding the proof. ■

**Lemma 88** *For a second-order differentiable family  $\{\rho_\theta\}_\theta$  of quantum states, the expressions in (7.6) and (7.7) are equal.*

**Proof.** This again follows from the linear approximation of the logarithm around one. Suppose  $\alpha \in (0, 1) \cup (1, \infty)$ . Set

$$\widehat{I}_F \equiv \widehat{I}_F(\theta; \{\mathcal{N}_\theta\}_\theta) := \lim_{\delta \rightarrow 0} \frac{2q_\alpha(\theta, \delta)}{\alpha(1-\alpha)\delta^2}, \quad (\text{J.14})$$

where

$$q_\alpha(\theta, \delta) := 1 - \widehat{Q}_\alpha(\rho_{\theta+\delta} \| \rho_\theta), \quad (\text{J.15})$$

and suppose that the limit in (J.1) exists and is a finite number. Then, for sufficiently small  $\delta > 0$ , the following inequalities hold

$$\frac{2}{\alpha(1-\alpha)\delta^2} q_\alpha(\theta, \delta) < \left| \widehat{I}_F \right| + 1, \quad |q_\alpha(\theta, \delta)| < \frac{1}{2}. \quad (\text{J.16})$$

Using the following expansion for  $x \in [0, 1)$

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}, \quad (\text{J.17})$$

and taking sufficiently small  $\delta > 0$  as stated above, we find that

$$\frac{2}{\alpha\delta^2} \widehat{D}_\alpha(\rho_{\theta+\delta} \parallel \rho_\theta) \quad (\text{J.18})$$

$$= -\frac{2}{\alpha(1-\alpha)\delta^2} \ln \widehat{Q}_\alpha(\rho_{\theta+\delta} \parallel \rho_\theta) \quad (\text{J.19})$$

$$= -\frac{2}{\alpha(1-\alpha)\delta^2} \ln(1 - q_\alpha(\theta, \delta)) \quad (\text{J.20})$$

$$= \frac{2}{\alpha(1-\alpha)\delta^2} \sum_{n=1}^{\infty} \frac{q_\alpha(\theta, \delta)^n}{n} \quad (\text{J.21})$$

$$= \frac{2}{\alpha(1-\alpha)\delta^2} \left[ q_\alpha(\theta, \delta) + \frac{q_\alpha(\theta, \delta)^2}{2} + \sum_{n=1}^{\infty} \frac{q_\alpha(\theta, \delta)^{n+2}}{n+2} \right] \quad (\text{J.22})$$

$$= \frac{2}{\alpha(1-\alpha)\delta^2} \left[ q_\alpha(\theta, \delta) + q_\alpha(\theta, \delta)^2 \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{q_\alpha(\theta, \delta)^n}{n+2} \right) \right] \quad (\text{J.23})$$

$$= \frac{2}{\alpha(1-\alpha)\delta^2} q_\alpha(\theta, \delta) + \frac{\alpha(1-\alpha)}{2} \delta^2 \left[ \frac{2q_\alpha(\theta, \delta)}{\alpha(1-\alpha)\delta^2} \right]^2 \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{q_\alpha(\theta, \delta)^n}{n+2} \right) \quad (\text{J.24})$$

$$= \frac{2}{\alpha(1-\alpha)\delta^2} q_\alpha(\theta, \delta) + \frac{\alpha(1-\alpha)}{2} \delta^2 g_\alpha(\theta, \delta), \quad (\text{J.25})$$

where

$$g_\alpha(\theta, \delta) := \left[ \frac{2q_\alpha(\theta, \delta)}{\alpha(1-\alpha)\delta^2} \right]^2 \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{q_\alpha(\theta, \delta)^n}{n+2} \right). \quad (\text{J.26})$$

For sufficiently small  $\delta$ , it follows from (J.3) that

$$|g_\alpha(\theta, \delta)| \leq \left( |\widehat{I}_F| + 1 \right)^2 \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n = 2(|I_F| + 1)^2.$$

Then we find that

$$\lim_{\delta \rightarrow 0} \frac{2}{\alpha\delta^2} \widehat{D}_\alpha(\rho_{\theta+\delta} \parallel \rho_\theta) = \lim_{\delta \rightarrow 0} \frac{2q_\alpha(\theta, \delta)}{\alpha(1-\alpha)\delta^2}, \quad (\text{J.27})$$

concluding the proof. ■

## K RLD Fisher information of quantum channels as a limit of geometric Rényi relative entropy

**Proposition 89** *Let  $\{\mathcal{N}_\theta\}_\theta$  be a second-order differentiable family of channels such that the support condition in (5.73) holds. Then for all  $\alpha \in (0, 1) \cup (1, \infty)$ , the RLD Fisher information of channels can be written as*

$$\widehat{I}_F(\theta; \{\mathcal{N}_\theta\}_\theta) = \lim_{\varepsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \frac{2}{\alpha(1-\alpha)\delta^2} \left(1 - \widehat{Q}_\alpha(\mathcal{N}_{\theta+\delta}^\varepsilon \| \mathcal{N}_\theta^\varepsilon)\right) \quad (\text{K.1})$$

$$= \lim_{\varepsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \frac{2}{\alpha\delta^2} \widehat{D}_\alpha(\mathcal{N}_{\theta+\delta}^\varepsilon \| \mathcal{N}_\theta^\varepsilon), \quad (\text{K.2})$$

where  $\mathcal{N}_\theta^\varepsilon(\rho) := (1-\varepsilon)\mathcal{N}_\theta(\rho) + \varepsilon \text{Tr}[\rho]\pi$ . Additionally, we have that

$$\widehat{I}_F(\theta; \{\mathcal{N}_\theta\}_\theta) = \lim_{\varepsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \frac{2}{\delta^2} \widehat{D}(\mathcal{N}_{\theta+\delta}^\varepsilon \| \mathcal{N}_\theta^\varepsilon). \quad (\text{K.3})$$

**Proof.** We focus on the case when  $\alpha \in (0, 1)$  and for full-rank channels, due to the order of limits given above and the fact that  $\mathcal{N}_\theta^\varepsilon(\rho)$  is a full-rank channel for all  $\varepsilon \in (0, 1)$ . Let  $\Gamma_{RB}^{\mathcal{N}_\theta}$  denote the Choi operator of the channel  $\mathcal{N}_\theta$ , and let  $\Gamma_{RB}^{\mathcal{N}_{\theta+\delta}}$  denote the Choi operator of the channel  $\mathcal{N}_{\theta+\delta}$ . Let us define

$$d\Gamma_{RB}^{\mathcal{N}_\theta} := \Gamma_{RB}^{\mathcal{N}_{\theta+\delta}} - \Gamma_{RB}^{\mathcal{N}_\theta}, \quad (\text{K.4})$$

and observe that

$$\text{Tr}_B[d\Gamma_{RB}^{\mathcal{N}_\theta}] = 0, \quad (\text{K.5})$$

because  $\text{Tr}_B[\Gamma_{RB}^{\mathcal{N}_\theta}] = \text{Tr}_B[\Gamma_{RB}^{\mathcal{N}_{\theta+\delta}}] = I_R$ . Then by plugging into (6.23), we find that

$$\begin{aligned} \widehat{Q}_\alpha(\mathcal{N}_{\theta+\delta} \| \mathcal{N}_\theta) &= \\ \lambda_{\min} \left( \text{Tr}_B \left[ \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{1/2} \left( \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1/2} \Gamma_{RB}^{\mathcal{N}_{\theta+\delta}} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1/2} \right)^\alpha \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{1/2} \right] \right). \end{aligned} \quad (\text{K.6})$$

Now, by using the expansion

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + O(x^3), \quad (\text{K.7})$$

we evaluate the innermost expression of (K.6):

$$\begin{aligned} &\left( \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1/2} \Gamma_{RB}^{\mathcal{N}_{\theta+\delta}} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1/2} \right)^\alpha \\ &= \left( \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1/2} \left( \Gamma_{RB}^{\mathcal{N}_\theta} + d\Gamma_{RB}^{\mathcal{N}_\theta} \right) \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1/2} \right)^\alpha \end{aligned} \quad (\text{K.8})$$

$$= \left( I_{RB} + \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1/2} d\Gamma_{RB}^{\mathcal{N}_\theta} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1/2} \right)^\alpha \quad (\text{K.9})$$

$$= I_{RB} + \alpha \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1/2} d\Gamma_{RB}^{\mathcal{N}_\theta} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1/2} + \frac{\alpha(\alpha-1)}{2} \left( \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1/2} d\Gamma_{RB}^{\mathcal{N}_\theta} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1/2} \right)^2 + O\left( \left( d\Gamma_{RB}^{\mathcal{N}_\theta} \right)^3 \right). \quad (\text{K.10})$$

Sandwiching the last expression by  $\left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{1/2}$  on both sides, we arrive at

$$\begin{aligned} & \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{1/2} \left( \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1/2} \Gamma_{RB}^{\mathcal{N}_{\theta+\delta}} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1/2} \right)^\alpha \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{1/2} \\ &= \Gamma_{RB}^{\mathcal{N}_\theta} + \alpha d\Gamma_{RB}^{\mathcal{N}_\theta} + \frac{\alpha(\alpha-1)}{2} d\Gamma_{RB}^{\mathcal{N}_\theta} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1} d\Gamma_{RB}^{\mathcal{N}_\theta} + O\left( \left( d\Gamma_{RB}^{\mathcal{N}_\theta} \right)^3 \right). \end{aligned} \quad (\text{K.11})$$

Then it follows that the partial trace  $\text{Tr}_B$  is given by

$$\begin{aligned} & \text{Tr}_B \left[ \Gamma_{RB}^{\mathcal{N}_\theta} + \frac{1}{2} d\Gamma_{RB}^{\mathcal{N}_\theta} - \frac{1}{8} d\Gamma_{RB}^{\mathcal{N}_\theta} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1} d\Gamma_{RB}^{\mathcal{N}_\theta} + O\left( \left( d\Gamma_{RB}^{\mathcal{N}_\theta} \right)^3 \right) \right] \\ &= I_R + \frac{\alpha(\alpha-1)}{2} \text{Tr}_B \left[ d\Gamma_{RB}^{\mathcal{N}_\theta} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1} d\Gamma_{RB}^{\mathcal{N}_\theta} \right] + O\left( \text{Tr}_B \left[ \left( d\Gamma_{RB}^{\mathcal{N}_\theta} \right)^3 \right] \right), \end{aligned} \quad (\text{K.12})$$

where we used (K.5). Observe that all higher order terms correspond to a positive semi-definite operator (each term being  $\left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1}$  sandwiched by other operators). Supposing that  $\delta$  is sufficiently small so that

$$I_R + \frac{\alpha(\alpha-1)}{2} \text{Tr}_B \left[ d\Gamma_{RB}^{\mathcal{N}_\theta} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1} d\Gamma_{RB}^{\mathcal{N}_\theta} \right] \quad (\text{K.13})$$

is a positive definite operator, we then have the bounds

$$\begin{aligned} & \lambda_{\min} \left( I_R + \frac{\alpha(\alpha-1)}{2} \text{Tr}_B \left[ d\Gamma_{RB}^{\mathcal{N}_\theta} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1} d\Gamma_{RB}^{\mathcal{N}_\theta} \right] \right) \\ & \leq \lambda_{\min} \left( I_R + \frac{\alpha(\alpha-1)}{2} \text{Tr}_B \left[ d\Gamma_{RB}^{\mathcal{N}_\theta} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1} d\Gamma_{RB}^{\mathcal{N}_\theta} \right] + O\left( \text{Tr}_B \left[ \left( d\Gamma_{RB}^{\mathcal{N}_\theta} \right)^3 \right] \right) \right) \end{aligned} \quad (\text{K.14})$$

$$\leq \lambda_{\min} \left( I_R + \frac{\alpha(\alpha-1)}{2} \text{Tr}_B \left[ d\Gamma_{RB}^{\mathcal{N}_\theta} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1} d\Gamma_{RB}^{\mathcal{N}_\theta} \right] \right) + \lambda_{\max} \left( O\left( \text{Tr}_B \left[ \left( d\Gamma_{RB}^{\mathcal{N}_\theta} \right)^3 \right] \right) \right) \quad (\text{K.15})$$

$$= \lambda_{\min} \left( I_R + \frac{\alpha(\alpha-1)}{2} \text{Tr}_B \left[ d\Gamma_{RB}^{\mathcal{N}_\theta} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1} d\Gamma_{RB}^{\mathcal{N}_\theta} \right] \right) + \left\| O\left( \text{Tr}_B \left[ \left( d\Gamma_{RB}^{\mathcal{N}_\theta} \right)^3 \right] \right) \right\|_\infty \quad (\text{K.16})$$

$$\leq \lambda_{\min} \left( I_R + \frac{\alpha(\alpha-1)}{2} \text{Tr}_B \left[ d\Gamma_{RB}^{\mathcal{N}_\theta} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1} d\Gamma_{RB}^{\mathcal{N}_\theta} \right] \right) + O \left( \left\| \text{Tr}_B \left[ \left( d\Gamma_{RB}^{\mathcal{N}_\theta} \right)^3 \right] \right\|_\infty \right) \quad (\text{K.17})$$

$$= \lambda_{\min} \left( I_R + \frac{\alpha(\alpha-1)}{2} \text{Tr}_B \left[ d\Gamma_{RB}^{\mathcal{N}_\theta} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1} d\Gamma_{RB}^{\mathcal{N}_\theta} \right] \right) + O \left( \left\| d\Gamma_{RB}^{\mathcal{N}_\theta} \right\|_\infty^3 \right). \quad (\text{K.18})$$

The first two inequalities are a consequence of the following inequalities that hold for positive definite operators  $X$  and  $Y$ :

$$\lambda_{\min}(X) \leq \lambda_{\min}(X+Y) \leq \lambda_{\min}(X) + \lambda_{\max}(Y). \quad (\text{K.19})$$

In the second-to-last line we employed the submultiplicativity of the infinity norm, and in the last line the bound

$$\left\| \text{Tr}_B[X_{AB}] \right\|_\infty \leq d_B \left\| X_{AB} \right\|_\infty, \quad (\text{K.20})$$

where  $d_B$  is the dimension of the channel output system  $B$ , as well as the fact that  $d_B < \infty$  is a constant. For a second-order differentiable family, the following limit holds

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta^2} O \left( \left\| d\Gamma_{RB}^{\mathcal{N}_\theta} \right\|_\infty^3 \right) = \lim_{\delta \rightarrow 0} \delta O \left( \left[ \left\| d\Gamma_{RB}^{\mathcal{N}_\theta} / \delta \right\|_\infty \right]^3 \right) = 0. \quad (\text{K.21})$$

This means that we can then focus on the term

$$\lambda_{\min} \left( I_R + \frac{\alpha(\alpha-1)}{2} \text{Tr}_B \left[ d\Gamma_{RB}^{\mathcal{N}_\theta} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1} d\Gamma_{RB}^{\mathcal{N}_\theta} \right] \right), \quad (\text{K.22})$$

because the last term in (K.18) will vanish when we divide by  $\delta^2$  and take the final limit as  $\delta \rightarrow 0$ . For any positive semi-definite operator  $A$  with sufficiently small eigenvalues all strictly less than one, it follows that

$$\lambda_{\min}(I - A) = 1 - \lambda_{\max}(A) = 1 - \|A\|_\infty. \quad (\text{K.23})$$

We can apply this reasoning to the operator

$$- \frac{\alpha(\alpha-1)}{2} \text{Tr}_B \left[ d\Gamma_{RB}^{\mathcal{N}_\theta} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1} d\Gamma_{RB}^{\mathcal{N}_\theta} \right] \quad (\text{K.24})$$

because it is positive semi-definite and its eigenvalues can be made arbitrarily close to zero for  $\delta$  small enough. Then by employing the expression in (K.1), we find that

$$\begin{aligned} & \hat{I}_F(\theta; \{\mathcal{N}_\theta\}_\theta) \\ &= \lim_{\delta \rightarrow 0} \frac{2}{\alpha(\alpha-1)\delta^2} \left( \hat{Q}_\alpha(\mathcal{N}_{\theta+\delta} \| \mathcal{N}_\theta) - 1 \right) \end{aligned} \quad (\text{K.25})$$

$$= \lim_{\delta \rightarrow 0} \frac{2}{\alpha(\alpha-1)\delta^2} \left( \left[ 1 + \frac{\alpha(\alpha-1)}{2} \left\| \text{Tr}_B \left[ d\Gamma_{RB}^{\mathcal{N}_\theta} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1} d\Gamma_{RB}^{\mathcal{N}_\theta} \right] \right\|_\infty \right] - 1 \right) \quad (\text{K.26})$$

$$= \lim_{\delta \rightarrow 0} \frac{1}{\delta^2} \left\| \text{Tr}_B \left[ d\Gamma_{RB}^{\mathcal{N}_\theta} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1} d\Gamma_{RB}^{\mathcal{N}_\theta} \right] \right\|_\infty \quad (\text{K.27})$$

$$= \lim_{\delta \rightarrow 0} \left\| \text{Tr}_B \left[ \frac{d\Gamma_{RB}^{\mathcal{N}_\theta}}{\delta} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1} \frac{d\Gamma_{RB}^{\mathcal{N}_\theta}}{\delta} \right] \right\|_\infty \quad (\text{K.28})$$

$$= \left\| \lim_{\delta \rightarrow 0} \text{Tr}_B \left[ \frac{d\Gamma_{RB}^{\mathcal{N}_\theta}}{\delta} \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1} \frac{d\Gamma_{RB}^{\mathcal{N}_\theta}}{\delta} \right] \right\|_\infty \quad (\text{K.29})$$

$$= \left\| \text{Tr}_B \left[ \left( \partial_\theta \Gamma_{RB}^{\mathcal{N}_\theta} \right) \left( \Gamma_{RB}^{\mathcal{N}_\theta} \right)^{-1} \left( \partial_\theta \Gamma_{RB}^{\mathcal{N}_\theta} \right) \right] \right\|_\infty. \quad (\text{K.30})$$

The third-to-last line follows because  $c\lambda_{\max}(A) = \lambda_{\max}(cA)$  for a positive semi-definite operator  $A$  and scaling parameter  $c > 0$ . The second-to-last line follows because the maximum and limit commute. The last line follows by evaluating the limit.

The proof for  $\alpha \in (1, \infty)$  is similar, except that we work with  $\lambda_{\max}$  instead of  $\lambda_{\min}$ .

The proof that (K.1) is equal to (K.2) is similar to the proof of Lemma 88.

The proof of (K.3) is similar to the proof of (7.9). Consider that

$$\widehat{D}(\mathcal{N}_{\theta+\delta} \| \mathcal{N}_\theta) = \left\| \text{Tr}_B \left[ \left( \Gamma^{\mathcal{N}_{\theta+\delta}} \right)^{\frac{1}{2}} \ln \left( \left( \Gamma^{\mathcal{N}_{\theta+\delta}} \right)^{\frac{1}{2}} \left( \Gamma^{\mathcal{N}_\theta} \right)^{-1} \left( \Gamma^{\mathcal{N}_{\theta+\delta}} \right)^{\frac{1}{2}} \right) \left( \Gamma^{\mathcal{N}_{\theta+\delta}} \right)^{\frac{1}{2}} \right] \right\|_\infty. \quad (\text{K.31})$$

So we focus on the operator in the middle. It suffices to consider a full-rank channel family and consider for  $\eta(x) = x \ln x$  that

$$\begin{aligned} & \left( \Gamma^{\mathcal{N}_{\theta+\delta}} \right)^{\frac{1}{2}} \ln \left( \left( \Gamma^{\mathcal{N}_{\theta+\delta}} \right)^{\frac{1}{2}} \left( \Gamma^{\mathcal{N}_\theta} \right)^{-1} \left( \Gamma^{\mathcal{N}_{\theta+\delta}} \right)^{\frac{1}{2}} \right) \left( \Gamma^{\mathcal{N}_{\theta+\delta}} \right)^{\frac{1}{2}} \\ &= \left( \Gamma^{\mathcal{N}_\theta} \right)^{\frac{1}{2}} \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} \left( \Gamma^{\mathcal{N}_{\theta+\delta}} \right)^{\frac{1}{2}} \ln \left( \left( \Gamma^{\mathcal{N}_{\theta+\delta}} \right)^{\frac{1}{2}} \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} \left( \Gamma^{\mathcal{N}_{\theta+\delta}} \right)^{\frac{1}{2}} \right) \left( \Gamma^{\mathcal{N}_{\theta+\delta}} \right)^{\frac{1}{2}} \end{aligned} \quad (\text{K.32})$$

$$= \left( \Gamma^{\mathcal{N}_\theta} \right)^{\frac{1}{2}} \ln \left( \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} \left( \Gamma^{\mathcal{N}_{\theta+\delta}} \right)^{\frac{1}{2}} \left( \Gamma^{\mathcal{N}_{\theta+\delta}} \right)^{\frac{1}{2}} \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} \right) \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} \left( \Gamma^{\mathcal{N}_{\theta+\delta}} \right)^{\frac{1}{2}} \left( \Gamma^{\mathcal{N}_{\theta+\delta}} \right)^{\frac{1}{2}} \quad (\text{K.33})$$

$$= \left( \Gamma^{\mathcal{N}_\theta} \right)^{\frac{1}{2}} \ln \left( \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} \left( \Gamma^{\mathcal{N}_{\theta+\delta}} \right) \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} \right) \left( \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} \left( \Gamma^{\mathcal{N}_{\theta+\delta}} \right) \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} \right) \left( \Gamma^{\mathcal{N}_\theta} \right)^{\frac{1}{2}} \quad (\text{K.34})$$

$$= \left( \Gamma^{\mathcal{N}_\theta} \right)^{\frac{1}{2}} \eta \left( \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} \left( \Gamma^{\mathcal{N}_{\theta+\delta}} \right) \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} \right) \left( \Gamma^{\mathcal{N}_\theta} \right)^{\frac{1}{2}} \quad (\text{K.35})$$

$$= \left( \Gamma^{\mathcal{N}_\theta} \right)^{\frac{1}{2}} \eta \left( \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} \left( \Gamma^{\mathcal{N}_\theta} + d\Gamma^{\mathcal{N}_\theta} \right) \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} \right) \left( \Gamma^{\mathcal{N}_\theta} \right)^{\frac{1}{2}} \quad (\text{K.36})$$

$$= \left( \Gamma^{\mathcal{N}_\theta} \right)^{\frac{1}{2}} \eta \left( I + \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} d\Gamma^{\mathcal{N}_\theta} \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} \right) \left( \Gamma^{\mathcal{N}_\theta} \right)^{\frac{1}{2}} \quad (\text{K.37})$$

$$= \left( \Gamma^{\mathcal{N}_\theta} \right)^{\frac{1}{2}} \left( \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} d\Gamma^{\mathcal{N}_\theta} \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} + \left[ \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} d\Gamma^{\mathcal{N}_\theta} \left( \Gamma^{\mathcal{N}_\theta} \right)^{-\frac{1}{2}} \right]^2 / 2 + O((d\Gamma^{\mathcal{N}_\theta})^3) \right) \left( \Gamma^{\mathcal{N}_\theta} \right)^{\frac{1}{2}} \quad (\text{K.38})$$

$$= d\Gamma^{\mathcal{N}_\theta} + d\Gamma^{\mathcal{N}_\theta} \left( \Gamma^{\mathcal{N}_\theta} \right)^{-1} d\Gamma^{\mathcal{N}_\theta} / 2 + O((d\Gamma^{\mathcal{N}_\theta})^3). \quad (\text{K.39})$$

The second equality follows from Lemma 61, with  $f = \ln$  and  $L = (\Gamma^{\mathcal{N}_{\theta+\delta}})^{\frac{1}{2}} (\Gamma^{\mathcal{N}_{\theta}})^{-\frac{1}{2}}$ . The second-to-last equality follows because  $\eta(1+x) = x + x^2/2 + O(x^3)$ . Now evaluating the partial trace over  $B$ , we find that

$$\begin{aligned} \text{Tr}_B [d\Gamma^{\mathcal{N}_{\theta}} + d\Gamma^{\mathcal{N}_{\theta}}(\Gamma^{\mathcal{N}_{\theta}})^{-1}d\Gamma^{\mathcal{N}_{\theta}}/2 + O((d\Gamma^{\mathcal{N}_{\theta}})^3)] \\ = \frac{1}{2} \text{Tr}_B [d\Gamma^{\mathcal{N}_{\theta}}(\Gamma^{\mathcal{N}_{\theta}})^{-1}d\Gamma^{\mathcal{N}_{\theta}}] + O((d\Gamma^{\mathcal{N}_{\theta}})^3), \end{aligned} \quad (\text{K.40})$$

which follows because  $\text{Tr}_B [d\Gamma^{\mathcal{N}_{\theta}}] = 0$ . Then finally

$$\begin{aligned} \lim_{\delta \rightarrow 0} \frac{2}{\delta^2} \widehat{D}(\mathcal{N}_{\theta+\delta} \| \mathcal{N}_{\theta}) \\ = \lim_{\delta \rightarrow 0} \frac{2}{\delta^2} \left\| \frac{1}{2} \text{Tr}_B [d\Gamma^{\mathcal{N}_{\theta}}(\Gamma^{\mathcal{N}_{\theta}})^{-1}d\Gamma^{\mathcal{N}_{\theta}}] + O((d\Gamma^{\mathcal{N}_{\theta}})^3) \right\|_{\infty} \end{aligned} \quad (\text{K.41})$$

$$= \lim_{\delta \rightarrow 0} \left\| \text{Tr}_B \left[ \frac{d\Gamma^{\mathcal{N}_{\theta}}}{\delta} (\Gamma^{\mathcal{N}_{\theta}})^{-1} \frac{d\Gamma^{\mathcal{N}_{\theta}}}{\delta} \right] + O(\delta(d\Gamma^{\mathcal{N}_{\theta}}/\delta)^3) \right\|_{\infty} \quad (\text{K.42})$$

$$= \left\| \text{Tr}_B \left[ \left( \partial_{\theta} \Gamma_{RB}^{\mathcal{N}_{\theta}} \right) \left( \Gamma_{RB}^{\mathcal{N}_{\theta}} \right)^{-1} \left( \partial_{\theta} \Gamma_{RB}^{\mathcal{N}_{\theta}} \right) \right] \right\|_{\infty}. \quad (\text{K.43})$$

This concludes the proof. ■

## L Semi-definite program for the root fidelity of quantum channels

**Proof of Proposition 55.** For a pure bipartite state  $\psi_{RA}$ , we use the fact that

$$\psi_{RA} = X_R \Gamma_{RA} X_R^{\dagger}, \quad (\text{L.1})$$

where  $\text{Tr}[X_R^{\dagger} X_R] = 1$  to see that

$$\mathcal{N}_{A \rightarrow B}(\psi_{RA}) = X_R \Gamma_{RB}^{\mathcal{N}} X_R^{\dagger}, \quad \mathcal{M}_{A \rightarrow B}(\psi_{RA}) = X_R \Gamma_{RB}^{\mathcal{M}} X_R^{\dagger}, \quad (\text{L.2})$$

and then plug in to (7.33) to get that

$$\sqrt{F}(\mathcal{N}_{A \rightarrow B}, \mathcal{M}_{A \rightarrow B}) = \frac{1}{2} \inf_{W_{RB}, Z_{RB}} \text{Tr}[X_R \Gamma_{RB}^{\mathcal{N}} X_R^{\dagger} W_{RB}] + \text{Tr}[X_R \Gamma_{RB}^{\mathcal{M}} X_R^{\dagger} Z_{RB}] \quad (\text{L.3})$$

subject to

$$\begin{bmatrix} W_{RB} & I_{RB} \\ I_{RB} & Z_{RB} \end{bmatrix} \geq 0. \quad (\text{L.4})$$

Consider that the objective function can be written as

$$\text{Tr}[\Gamma_{RB}^{\mathcal{N}} W'_{RB}] + \text{Tr}[\Gamma_{RB}^{\mathcal{M}} Z'_{RB}], \quad (\text{L.5})$$

with

$$W'_{RB} := X_R^\dagger W_{RB} X_R, \quad Z'_{RB} := X_R^\dagger Z_{RB} X_R \quad (\text{L.6})$$

Now consider that the inequality in (L.4) is equivalent to

$$\begin{bmatrix} X_R & 0 \\ 0 & X_R \end{bmatrix}^\dagger \begin{bmatrix} W_{RB} & I_{RB} \\ I_{RB} & Z_{RB} \end{bmatrix} \begin{bmatrix} X_R & 0 \\ 0 & X_R \end{bmatrix} \geq 0 \quad (\text{L.7})$$

Multiplying out the last matrix we find that

$$\begin{aligned} & \begin{bmatrix} X_R & 0 \\ 0 & X_R \end{bmatrix}^\dagger \begin{bmatrix} W_{RB} & I_{RB} \\ I_{RB} & Z_{RB} \end{bmatrix} \begin{bmatrix} X_R & 0 \\ 0 & X_R \end{bmatrix} \\ &= \begin{bmatrix} X_R^\dagger W_{RB} X_R & X_R^\dagger X_R \otimes I_B \\ X_R^\dagger X_R \otimes I_B & X_R^\dagger Z_{RB} X_R \end{bmatrix} \end{aligned} \quad (\text{L.8})$$

$$= \begin{bmatrix} W'_{RB} & \rho_R \otimes I_B \\ \rho_R \otimes I_B & Z'_{RB} \end{bmatrix}, \quad (\text{L.9})$$

where we defined  $\rho_R = X_R^\dagger X_R$ . Observing that  $\rho_R \geq 0$  and  $\text{Tr}[\rho_R] = 1$ , we can write the final SDP as follows:

$$\sqrt{F}(\mathcal{N}_{A \rightarrow B}, \mathcal{M}_{A \rightarrow B}) = \frac{1}{2} \inf_{\rho_R, W_{RB}, Z_{RB}} \text{Tr}[\Gamma_{RB}^{\mathcal{N}} W_{RB}] + \text{Tr}[\Gamma_{RB}^{\mathcal{M}} Z_{RB}], \quad (\text{L.10})$$

subject to

$$\rho_R \geq 0, \quad \text{Tr}[\rho_R] = 1, \quad \begin{bmatrix} W_{RB} & \rho_R \otimes I_B \\ \rho_R \otimes I_B & Z_{RB} \end{bmatrix} \geq 0. \quad (\text{L.11})$$

Now let us calculate the dual SDP to this, using the following standard forms for primal and dual SDPs, with Hermitian operators  $A$  and  $B$  and a Hermiticity-preserving map  $\Phi$  [43]:

$$\sup_{X \geq 0} \{ \text{Tr}[AX] : \Phi(X) \leq B \}, \quad \inf_{Y \geq 0} \{ \text{Tr}[BY] : \Phi^\dagger(Y) \geq A \}. \quad (\text{L.12})$$

Consider that the constraint in (L.11) implies  $W_{RB} \geq 0$  and  $Z_{RB} \geq 0$ , so that we can set

$$Y = \begin{bmatrix} W_{RB} & 0 & 0 \\ 0 & Z_{RB} & 0 \\ 0 & 0 & \rho_R \end{bmatrix}, \quad B = \begin{bmatrix} \Gamma_{RB}^{\mathcal{N}} & 0 & 0 \\ 0 & \Gamma_{RB}^{\mathcal{M}} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (\text{L.13})$$

$$\Phi^\dagger(Y) = \begin{bmatrix} W_{RB} & \rho_R \otimes I_B & 0 & 0 \\ \rho_R \otimes I_B & Z_{RB} & 0 & 0 \\ 0 & 0 & \text{Tr}[\rho_R] & 0 \\ 0 & 0 & 0 & -\text{Tr}[\rho_R] \end{bmatrix}, \quad (\text{L.14})$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (\text{L.15})$$



Then with

$$X = \begin{bmatrix} P_{RB} & Q_{RB}^\dagger & 0 & 0 \\ Q_{RB} & S_{RB} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} \quad (\text{L.16})$$

the map  $\Phi$  is given by

$$\begin{aligned} & \text{Tr}[X\Phi^\dagger(Y)] \\ &= \text{Tr} \left[ \begin{bmatrix} P_{RB} & Q_{RB}^\dagger & 0 & 0 \\ Q_{RB} & S_{RB} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} W_{RB} & \rho_R \otimes I_B & 0 & 0 \\ \rho_R \otimes I_B & Z_{RB} & 0 & 0 \\ 0 & 0 & \text{Tr}[\rho_R] & 0 \\ 0 & 0 & 0 & -\text{Tr}[\rho_R] \end{bmatrix} \right] \end{aligned} \quad (\text{L.17})$$

$$\begin{aligned} &= \text{Tr}[P_{RB}W_{RB}] + \text{Tr}[Q_{RB}^\dagger(\rho_R \otimes I_B)] + \text{Tr}[Q_{RB}(\rho_R \otimes I_B)] \\ &\quad + \text{Tr}[S_{RB}Z_{RB}] + (\lambda - \mu) \text{Tr}[\rho_R] \end{aligned} \quad (\text{L.18})$$

$$= \text{Tr}[P_{RB}W_{RB}] + \text{Tr}[S_{RB}Z_{RB}] + \text{Tr}[(\text{Tr}_B[Q_{RB} + Q_{RB}^\dagger] + (\lambda - \mu) I_R)\rho_R] \quad (\text{L.19})$$

$$= \text{Tr} \left[ \begin{bmatrix} P_{RB} & 0 & 0 \\ 0 & S_{RB} & 0 \\ 0 & 0 & \text{Tr}_B[Q_{RB} + Q_{RB}^\dagger] + (\lambda - \mu) I_R \end{bmatrix} \begin{bmatrix} W_{RB} & 0 & 0 \\ 0 & Z_{RB} & 0 \\ 0 & 0 & \rho_R \end{bmatrix} \right]. \quad (\text{L.20})$$

So then

$$\Phi(X) = \begin{bmatrix} P_{RB} & 0 & 0 \\ 0 & S_{RB} & 0 \\ 0 & 0 & \text{Tr}_B[Q_{RB} + Q_{RB}^\dagger] + (\lambda - \mu) I_R \end{bmatrix}. \quad (\text{L.21})$$

The primal is then given by

$$\frac{1}{2} \sup \text{Tr} \left[ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} P_{RB} & Q_{RB}^\dagger & 0 & 0 \\ Q_{RB} & S_{RB} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} \right], \quad (\text{L.22})$$

subject to

$$\begin{bmatrix} P_{RB} & 0 & 0 \\ 0 & S_{RB} & 0 \\ 0 & 0 & \text{Tr}_B[Q_{RB} + Q_{RB}^\dagger] + (\lambda - \mu) I_R \end{bmatrix} \leq \begin{bmatrix} \Gamma_{RB}^{\mathcal{N}} & 0 & 0 \\ 0 & \Gamma_{RB}^{\mathcal{M}} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (\text{L.23})$$

$$\begin{bmatrix} P_{RB} & Q_{RB}^\dagger & 0 & 0 \\ Q_{RB} & S_{RB} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} \geq 0, \quad (\text{L.24})$$

which simplifies to

$$\frac{1}{2} \sup (\lambda - \mu) \quad (\text{L.25})$$

subject to

$$P_{RB} \leq \Gamma_{RB}^{\mathcal{N}}, \quad (\text{L.26})$$

$$S_{RB} \leq \Gamma_{RB}^{\mathcal{M}}, \quad (\text{L.27})$$

$$\text{Tr}_B[Q_{RB} + Q_{RB}^\dagger] + (\lambda - \mu) I_R \leq 0, \quad (\text{L.28})$$

$$\begin{bmatrix} P_{RB} & Q_{RB}^\dagger \\ Q_{RB} & S_{RB} \end{bmatrix} \geq 0, \quad (\text{L.29})$$

$$\lambda, \mu \geq 0. \quad (\text{L.30})$$

We can simplify this even more. We can set  $\lambda' = \lambda - \mu \in \mathbb{R}$ , and we can substitute  $Q_{RB}$  with  $-Q_{RB}$  without changing the value, so then it becomes

$$\frac{1}{2} \sup \lambda' \quad (\text{L.31})$$

subject to

$$P_{RB} \leq \Gamma_{RB}^{\mathcal{N}}, \quad (\text{L.32})$$

$$S_{RB} \leq \Gamma_{RB}^{\mathcal{M}}, \quad (\text{L.33})$$

$$\lambda' I_R \leq \text{Tr}_B[Q_{RB} + Q_{RB}^\dagger], \quad (\text{L.34})$$

$$\begin{bmatrix} P_{RB} & -Q_{RB}^\dagger \\ -Q_{RB} & S_{RB} \end{bmatrix} \geq 0, \quad (\text{L.35})$$

$$\lambda' \in \mathbb{R}. \quad (\text{L.36})$$

We can rewrite

$$\begin{bmatrix} P_{RB} & -Q_{RB}^\dagger \\ -Q_{RB} & S_{RB} \end{bmatrix} \geq 0 \iff \begin{bmatrix} P_{RB} & Q_{RB}^\dagger \\ Q_{RB} & S_{RB} \end{bmatrix} \geq 0 \quad (\text{L.37})$$

$$\iff \begin{bmatrix} P_{RB} & 0 \\ 0 & S_{RB} \end{bmatrix} \geq \begin{bmatrix} 0 & -Q_{RB}^\dagger \\ -Q_{RB} & 0 \end{bmatrix} \quad (\text{L.38})$$

We then have the simplified condition

$$\begin{bmatrix} 0 & -Q_{RB}^\dagger \\ -Q_{RB} & 0 \end{bmatrix} \leq \begin{bmatrix} P_{RB} & 0 \\ 0 & S_{RB} \end{bmatrix} \leq \begin{bmatrix} \Gamma_{RB}^{\mathcal{N}} & 0 \\ 0 & \Gamma_{RB}^{\mathcal{M}} \end{bmatrix}. \quad (\text{L.39})$$

Since  $P_{RB}$  and  $S_{RB}$  do not appear in the objective function, we can set them to their largest value and obtain the following simplification

$$\frac{1}{2} \sup \lambda' \quad (\text{L.40})$$

subject to

$$\lambda' I_R \leq \text{Tr}_B[Q_{RB} + Q_{RB}^\dagger], \quad \begin{bmatrix} \Gamma_{RB}^{\mathcal{N}} & Q_{RB}^\dagger \\ Q_{RB} & \Gamma_{RB}^{\mathcal{M}} \end{bmatrix} \geq 0, \quad \lambda' \in \mathbb{R} \quad (\text{L.41})$$

Since a feasible solution is  $\lambda' = 0$  and  $Q_{RB} = 0$ , it is clear that we can restrict to  $\lambda' \geq 0$ . After a relabeling, this becomes

$$\begin{aligned} \frac{1}{2} \sup_{\lambda \geq 0, Q_{RB}} \left\{ \lambda : \lambda I_R \leq \text{Tr}_B[Q_{RB} + Q_{RB}^\dagger], \quad \begin{bmatrix} \Gamma_{RB}^{\mathcal{N}} & Q_{RB}^\dagger \\ Q_{RB} & \Gamma_{RB}^{\mathcal{M}} \end{bmatrix} \geq 0 \right\} \\ = \sup_{\lambda \geq 0, Q_{RB}} \left\{ \lambda : \lambda I_R \leq \text{Re}[\text{Tr}_B[Q_{RB}]], \quad \begin{bmatrix} \Gamma_{RB}^{\mathcal{N}} & Q_{RB}^\dagger \\ Q_{RB} & \Gamma_{RB}^{\mathcal{M}} \end{bmatrix} \geq 0 \right\}. \quad (\text{L.42}) \end{aligned}$$

This is equivalent to

$$\sup_{Q_{RB}} \left\{ \lambda_{\min}(\text{Re}[\text{Tr}_B[Q_{RB}]]): \begin{bmatrix} \Gamma_{RB}^{\mathcal{N}} & Q_{RB}^\dagger \\ Q_{RB} & \Gamma_{RB}^{\mathcal{M}} \end{bmatrix} \geq 0 \right\}. \quad (\text{L.43})$$

This concludes the proof. ■