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## RLD Fisher information bound for multiparameter estimation of quantum channels

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## PAPER

## RLD Fisher information bound for multiparameter estimation of quantum channels

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E-mail: [mwilde@lsu.edu](mailto:mwilde@lsu.edu)**Keywords:** quantum information, quantum estimation theory, RLD Fisher information, quantum channel estimation

## Abstract

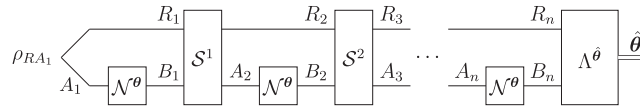
One of the fundamental tasks in quantum metrology is to estimate multiple parameters embedded in a noisy process, i.e. a quantum channel. In this paper, we study fundamental limits to quantum channel estimation via the concept of amortization and the right logarithmic derivative (RLD) Fisher information value. Our key technical result is the proof of a chain-rule inequality for the RLD Fisher information value, which implies that amortization, i.e. access to a catalyst state family, does not increase the RLD Fisher information value of quantum channels. This technical result leads to a fundamental and efficiently computable limitation for multiparameter channel estimation in the sequential setting, in terms of the RLD Fisher information value. As a consequence, we conclude that if the RLD Fisher information value is finite, then Heisenberg scaling is unattainable in the multiparameter setting.

## 1. Introduction

Parameter estimation is an information-processing task in which quantum technologies can provide an improvement in performance over the best known classical technologies. On one hand, classical parameter estimation is fundamentally limited by the shot-noise limit, which means that the error scales no better than  $1/n$ . Here,  $n$  refers either to the number of channel uses or to the total probing time allowed in the estimation task. On the other hand, quantum parameter estimation can offer a superclassical ‘Heisenberg’ scaling in estimation error, in principle [1]. Heisenberg scaling refers to the mean-squared error of an estimator scaling as  $1/n^2$ . One of the fundamental questions in quantum estimation theory is to identify estimation tasks for which Heisenberg scaling is possible and attainable.

One of the most important mathematical tools in estimation theory, both classical and quantum, is the Cramer–Rao bound (CRB) [2–4]. It is used to place lower bounds on the mean squared error of estimators and involves an information quantity known as the Fisher information. The latter captures the amount of information carried by a distribution, quantum state, or quantum channel regarding the unknown parameter(s). When using quantum resources, there is an infinite number of noncommutative generalizations of the Fisher information.

In this paper, we focus on one particular quantum generalization of the Fisher information, known as the right logarithmic derivative (RLD) Fisher information [5], and we study its properties and application in multiparameter quantum channel estimation. In particular, we prove that the RLD Fisher information of quantum channels is a single-letter and efficiently computable multiparameter CRB for all quantum channels. Our bound applies to all quantum channels in the most general setting of sequential channel estimation, depicted in figure 1, which encompasses all possible noisy dynamics. This means that our bound is universally applicable. Moreover, it is ‘single-letter’, a term from information theory, which means that the RLD Fisher information is evaluated with respect to a single copy of the channel only (see theorem 8 for



**Figure 1.** Processing  $n$  uses of channel  $\mathcal{N}^\theta$  in a sequential or adaptive manner is the most general approach to channel parameter estimation. The  $n$  uses of the channel are interleaved with  $n - 1$  quantum channels  $S^1$  through  $S^{n-1}$ , which can also share memory systems with each other. The final measurement's outcome is then used to obtain an estimate of the unknown parameter vector  $\theta$ .

a precise statement of our result). Our bound thus has a number of desirable properties as well as numerical amenability, while being applicable to sequential,  $n$ -round estimation strategies.

We approach this problem by introducing the amortized RLD Fisher information value, which quantifies the net increase in RLD Fisher information that one obtains by sampling from a quantum channel. We prove a chain-rule property for the RLD Fisher information value of a quantum channel, which we use to conclude our single-letter multiparameter CRB. This bound has the implication that Heisenberg scaling is unattainable for channel estimation in the multiparameter setting, whenever the RLD Fisher information value is finite. Finally, we apply our bound to a concrete example of physical interest: estimating the parameters of a generalized amplitude damping channel (GADC).

## 2. Parameter estimation

The goal of classical parameter estimation is to obtain an estimate  $\hat{\theta}$  of an unknown real parameter  $\theta$  embedded in a probability distribution  $p_\theta(x)$ , corresponding to a random variable  $X$ , where we have suppressed the dependence of  $X$  on  $\theta$  in our notation. That is, we guess the value of  $\theta$  from a realization  $x$  of the random variable  $X$ . The estimate  $\hat{\theta}(X)$  is itself a random variable, being a function of  $X$ . It is common to assume unbiased estimators, i.e.  $\mathbb{E}[\hat{\theta}(X)] = \theta$ . The unbiasedness condition means that the estimator is accurate and has no systematic error. Hence we focus on studying the precision of unbiased estimators.

A natural metric to benchmark the performance of an estimator is the mean-squared error  $\mathbb{E}[(\hat{\theta}(X) - \theta)^2]$ , abbreviated as MSE. For an unbiased estimator, the MSE is equal to the variance  $\text{Var}(\hat{\theta}(X))$ . The CRB places a lower bound on the variance of an unbiased estimator [2–4]:

$$\text{Var}(\hat{\theta}(X)) \geq \frac{1}{I_F(\theta; \{p_\theta\}_\theta)}, \quad (1)$$

where  $I_F(\theta; \{p_\theta\}_\theta)$  is the classical Fisher information, defined as

$$I_F(\theta; \{p_\theta\}_\theta) := \mathbb{E}[(\partial_\theta \ln p_\theta(X))^2]. \quad (2)$$

If one has  $n$  independent samples  $x^n$  described by the random sequence  $X^n \equiv X_1, \dots, X_n$ , then the corresponding CRB is

$$\text{Var}(\hat{\theta}(X^n)) \geq \frac{1}{n I_F(\theta; \{p_\theta\}_\theta)}. \quad (3)$$

At this stage, we would like to clarify that our approach adopts the frequentist approach to parameter estimation. In general, the MSE and CRBs may depend on the value of the unknown parameter, in contrast with the more general paradigm of Bayesian parameter estimation [6]. This parameter dependence is alleviated by enforcing the unbiasedness condition.

Estimation theory has been generalized to quantum systems, where quantum probes and quantum measurements are allowed (see, e.g. [7, 8] for recent reviews). We then have an infinite number of logarithmic derivative operators that each reduce to the logarithmic derivative  $\partial_\theta \ln p_\theta(x)$  in the classical case in (2). One such example is the RLD operator, defined implicitly via the equation  $\partial_\theta \rho_\theta = \rho_\theta R_\theta$ , where  $\{\rho_\theta\}_\theta$  is a differentiable family of quantum states. The RLD Fisher information will be formally defined in the next section.

Consider the task of estimating a real parameter  $\theta$  encoded in a quantum channel  $\mathcal{N}_{A \rightarrow B}^\theta$  (see [9–24] for extensive literature on this problem). The most general setting for this problem, given  $n$  uses of the channel, consists of performing the estimation via a sequential strategy [1, 18, 24–27] as depicted in figure 1. Parallel estimation strategies, wherein the  $n$  calls to the channel are made simultaneously, are contained within the larger family of sequential strategies. Assuming an unbiased estimator, the following quantum Cramer–Rao

bound holds in this general setting of channel estimation [28]:

$$\text{Var}(\hat{\theta}) \geq \frac{1}{n\hat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta)}, \quad (4)$$

where  $\hat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta)$  is the RLD Fisher information for the parameter  $\theta$  encoded in the channel family  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  [15], and we define it formally later in (11). The result in (4) is presented in reference [28] and is a special case of the more general result reported here in (152). A restricted subset of the most general sequential strategies consists of parallel strategies, where the  $n$  uses of the channel are made simultaneously using an entangled probe state, and the bound in (4) was previously established in [15] for this special case. In a parallel strategy, the optimal probe state may have an undesired dependence on the unknown parameter  $\theta$ . This dependence, however, can be circumvented in a sequential strategy due to the ability to perform adaptive control (channels  $\mathcal{S}^1$  through  $\mathcal{S}^{n-1}$  in figure 1) between uses of the unknown channel.

The task of simultaneously estimating multiple parameters is a much more involved task than estimating a single parameter; however, CRBs can still be constructed. This problem, too, has an extensive literature and a number of important recent results [5, 29–45]. See [46, 47] for recent reviews on multiparameter estimation. In the quantum case, an additional complication is that the optimal measurements for each parameter may not be compatible. Consider that  $D$  parameters need to be estimated and are encoded in a vector  $\boldsymbol{\theta} := [\theta_1 \ \theta_2 \ \dots \ \theta_D]^T$ . For a differentiable family  $\{\rho_\theta\}_\theta$  of quantum states, one defines  $D$  RLD operators using the equations  $\partial_{\theta_j} \rho_\theta = \rho_\theta R_{\theta_j}$  for  $j \in \{1, 2, \dots, D\}$ . The RLD Fisher information, instead of being a scalar, takes the form of a  $D \times D$  matrix with elements  $\hat{I}_F(\boldsymbol{\theta}; \{\rho_\theta\}_\theta)_{j,k} := \text{Tr}[R_{\theta_j}^\dagger \rho_\theta R_{\theta_k}]$ . This leads to the following matrix CRB [5]:

$$\text{Cov}(\boldsymbol{\theta}) \geq \hat{I}_F(\boldsymbol{\theta}; \{\rho_\theta\}_\theta)^{-1}. \quad (5)$$

In the above,  $\text{Cov}(\boldsymbol{\theta})$  is a covariance matrix with matrix elements defined as

$$[\text{Cov}(\boldsymbol{\theta})]_{j,k} = \sum_l \text{Tr}[M_l \rho_\theta] (\hat{\theta}_j(l) - \theta_j) (\hat{\theta}_k(l) - \theta_k), \quad (6)$$

where  $M_l \geq 0$  are measurement operators satisfying  $\sum_l M_l = \mathbb{I}$ , and

$$\hat{\boldsymbol{\theta}}(l) := [\hat{\theta}_1(l) \ \hat{\theta}_2(l) \ \dots \ \hat{\theta}_D(l)]^T \quad (7)$$

is a function that maps the measurement result  $l$  to an estimate of the parameters  $\boldsymbol{\theta}$ .

### 3. Right logarithmic derivative

The reason that there are an infinite number of noncommutative Fisher information quantities is that in quantum estimation theory, the logarithmic derivative takes the form of an operator, rather than a scalar, as it does in the classical case. Each noncommutative Fisher information collapses to the scalar Fisher information in the classical case, i.e. when only diagonal density operators are involved. One particular noncommutative generalization arises from the RLD operator: for a family of states  $\{\rho_\theta\}_\theta$ , the RLD operator  $R_\theta$  is implicitly defined by the equation  $\partial_\theta \rho_\theta = \rho_\theta R_\theta$ .

The RLD operator introduced above leads to one quantum generalization of the classical logarithmic derivative [5]. This is the one we focus on in this paper. We begin by defining it for the case of single parameter estimation. Consider a single unknown parameter  $\theta$  embedded in a quantum state  $\rho_\theta$ . The RLD Fisher information of a differentiable family  $\{\rho_\theta\}_\theta$  of states is defined as

$$\hat{I}_F(\theta; \{\rho_\theta\}_\theta) = \text{Tr}[(\partial_\theta \rho_\theta)^2 \rho_\theta^{-1}], \quad (8)$$

if  $\text{supp}(\partial_\theta \rho_\theta) \subseteq \text{supp}(\rho_\theta)$ , and it is set to  $+\infty$  otherwise. The inverse  $\rho_\theta^{-1}$  is taken on the support of  $\rho_\theta$ . Alternatively, if the support condition  $\text{supp}(\partial_\theta \rho_\theta) \subseteq \text{supp}(\rho_\theta)$  is satisfied, then the RLD Fisher information can also be defined using the RLD operator  $R_\theta$  as follows:

$$\hat{I}_F(\theta; \{\rho_\theta\}_\theta) = \text{Tr}[R_\theta^2 \rho_\theta]. \quad (9)$$

The RLD Fisher information is the largest noncommutative Fisher information [48]. This mirrors the fact that the geometric Rényi relative entropy is the largest Rényi relative entropy that satisfies the data-processing inequality [49]. We have explored both the qualitative and quantitative connections between the RLD Fisher information and geometric Rényi relative entropy (and hence, between quantum channel estimation and discrimination) in detail in [28].

If, instead of a state family  $\{\rho_\theta\}_\theta$ , we have a differentiable channel family  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$ , then the RLD Fisher information of this channel family is defined as

$$\widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) := \sup_{\rho_{RA}} \widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA})\}_\theta) \quad (10)$$

$$= \left\| \text{Tr}_B[(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1}(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta})] \right\|_\infty, \quad (11)$$

if  $\text{supp}(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \subseteq \text{supp}(\Gamma_{RB}^{\mathcal{N}^\theta})$ , and it is equal to  $+\infty$  otherwise, where

$$\Gamma_{RB}^{\mathcal{N}^\theta} := \sum_{i,j} |i\rangle\langle j|_R \otimes \mathcal{N}_{A \rightarrow B}^\theta(|i\rangle\langle j|_A) \quad (12)$$

is the Choi operator of the channel  $\mathcal{N}_{A \rightarrow B}^\theta$ . The optimization in (10) is with respect to every bipartite input state  $\rho_{RA}$  that has no dependence on the parameter  $\theta$ , and the equality in (11) was established in [15]. Note that the optimal value in (10) is achieved by a pure bipartite state with system  $R$  isomorphic to system  $A$ . The RLD Fisher information of a channel family  $\widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta)$  is efficiently computable via a semi-definite program [28]. We also note here that what we define as the RLD Fisher information in (10) and (11) is also known as the conditional Fisher information [50–52], since we consider known probe states.

The RLD Fisher information of isometric or unitary channels is an uninteresting information measure. This is because a differentiable family  $\{\mathcal{U}_\theta\}_\theta$  of isometric or unitary channels induces a differentiable family of pure states. The RLD Fisher information in this case is either infinite or zero, neither of which lead to a useful CRB.

In multiparameter estimation, we have a differentiable family  $\{\rho_\theta\}_\theta$  of quantum states. Let  $\Pi_{\rho_\theta}^\perp$  denote the projection onto the kernel of  $\rho_\theta$ . In the case that the following finiteness conditions hold

$$(\partial_{\theta_j} \rho_\theta)(\partial_{\theta_k} \rho_\theta) \Pi_{\rho_\theta}^\perp = 0 \quad \forall j, k \in \{1, \dots, D\}, \quad (13)$$

the matrix elements of the multiparameter RLD Fisher information matrix are defined as follows:

$$[\widehat{I}_F(\theta; \{\rho_\theta\}_\theta)]_{j,k} := \text{Tr}[(\partial_{\theta_j} \rho_\theta) \rho_\theta^{-1} (\partial_{\theta_k} \rho_\theta)]. \quad (14)$$

We also note here that the finiteness conditions in (13) can be equivalently written as

$$(\partial_{\theta_k} \rho_\theta) \Pi_{\rho_\theta}^\perp = 0 \quad \forall k \in \{1, \dots, D\}. \quad (15)$$

To see that (13) and (15) are equivalent, suppose in (13) that  $j = k$ . We then have

$$(\partial_{\theta_k} \rho_\theta)(\partial_{\theta_k} \rho_\theta) \Pi_{\rho_\theta}^\perp = 0 \quad \forall k \in \{1, \dots, D\} \quad (16)$$

which implies that

$$(\partial_{\theta_k} \rho_\theta) \Pi_{\rho_\theta}^\perp = 0 \quad \forall k \in \{1, \dots, D\}. \quad (17)$$

Thus we have that (13)  $\Rightarrow$  (15). Next, we show that (15)  $\Rightarrow$  (13). Suppose for all  $k \in \{1, \dots, D\}$  that

$$(\partial_{\theta_k} \rho_\theta) \Pi_{\rho_\theta}^\perp = 0. \quad (18)$$

Now we multiply on both sides by  $(\partial_{\theta_j} \rho_\theta)$  to yield

$$(\partial_{\theta_j} \rho_\theta)(\partial_{\theta_k} \rho_\theta) \Pi_{\rho_\theta}^\perp = 0 \quad \forall j, k \in \{1, \dots, D\} \quad (19)$$

and thus we conclude that (15)  $\iff$  (13).

The RLD Fisher information matrix is then defined as follows:

$$\widehat{I}_F(\theta; \rho_\theta) := \sum_{j,k=1}^D \text{Tr}[(\partial_{\theta_j} \rho_\theta) \rho_\theta^{-1} (\partial_{\theta_k} \rho_\theta)] |j\rangle\langle k| \quad (20)$$

$$= \text{Tr}_2 \left[ \sum_{j,k=1}^D |j\rangle\langle k| \otimes (\partial_{\theta_j} \rho_\theta) \rho_\theta^{-1} (\partial_{\theta_k} \rho_\theta) \right], \quad (21)$$

where  $\text{Tr}_2[\dots]$  refers to tracing over the second subsystem. If the finiteness conditions in (13) do not hold, then some elements of the RLD Fisher information matrix are infinite.

As stated in the previous section, the RLD Fisher information matrix is featured in the matrix quantum Cramér–Rao inequality in (5). To obtain scalar CRBs from this matrix inequality, we define a  $D \times D$  positive semi-definite, unit trace weight matrix  $W$  (also known as the risk matrix). The weight matrix  $W$  can be chosen in accordance with the goal of the parameter estimation experiment. Particular choices of  $W$  can lead to changes in the optimal estimation strategy.

**Remark 1.** Note that, throughout our paper, a weight matrix  $W$  is defined to be a positive semi-definite, unit trace matrix. In general, it may have complex entries.

The RLD Fisher information value is then defined as

$$\widehat{I}_F(\boldsymbol{\theta}, W; \rho_{\boldsymbol{\theta}}) := \text{Tr}[W \widehat{I}_F(\boldsymbol{\theta}; \{\rho_{\boldsymbol{\theta}}\}_{\boldsymbol{\theta}})] \quad (22)$$

$$= \text{Tr} \left[ (W \otimes I_d) \left( \sum_{j,k=1}^D |j\rangle\langle k| \otimes (\partial_{\theta_j} \rho_{\boldsymbol{\theta}}) \rho_{\boldsymbol{\theta}}^{-1} (\partial_{\theta_k} \rho_{\boldsymbol{\theta}}) \right) \right] \quad (23)$$

$$= \sum_{j,k=1}^D \langle k|W|j\rangle \text{Tr} \left[ (\partial_{\theta_j} \rho_{\boldsymbol{\theta}}) \rho_{\boldsymbol{\theta}}^{-1} (\partial_{\theta_k} \rho_{\boldsymbol{\theta}}) \right]. \quad (24)$$

The finiteness condition for the RLD Fisher information value defined above is

$$\left[ \sum_{j,k=1}^D \langle k|W|j\rangle (\partial_{\theta_k} \rho_{\boldsymbol{\theta}}) (\partial_{\theta_j} \rho_{\boldsymbol{\theta}}) \right] \Pi_{\rho_{\boldsymbol{\theta}}}^{\perp} = 0, \quad (25)$$

where  $\Pi_{\rho_{\boldsymbol{\theta}}}^{\perp}$  is the projection onto the kernel of  $\rho_{\boldsymbol{\theta}}$ .

**Theorem 1.** The following scalar CRB holds for estimating multiple parameters  $\boldsymbol{\theta}$  encoded in a family of quantum states  $\{\rho_{\boldsymbol{\theta}}\}_{\boldsymbol{\theta}}$ :

$$\text{Tr}[W \text{Cov}(\boldsymbol{\theta})] \geq \frac{1}{\widehat{I}_F(\boldsymbol{\theta}, W; \rho_{\boldsymbol{\theta}})}, \quad (26)$$

where the weight matrix  $W$  satisfies  $\text{Tr}[W] = 1$ .

**Proof.** We start with the following matrix inequality [7, 53, 54], as recalled in (5):

$$\text{Cov}(\boldsymbol{\theta}) \geq \widehat{I}_F(\boldsymbol{\theta}; \{\rho_{\boldsymbol{\theta}}\}_{\boldsymbol{\theta}})^{-1}. \quad (27)$$

Let us take  $W$  as a non-zero, positive semi-definite matrix, and then define the normalized operator  $W'$  as

$$W' := \frac{W}{\text{Tr}[W]}. \quad (28)$$

The matrix inequality in (27) implies the following:

$$\text{Tr}[W] \text{Tr}[W' \text{Cov}(\boldsymbol{\theta})] \geq \text{Tr}[W] \text{Tr}[W' \widehat{I}_F(\boldsymbol{\theta}; \{\rho_{\boldsymbol{\theta}}\}_{\boldsymbol{\theta}})^{-1}] \quad (29)$$

$$= \text{Tr}[W] \text{Tr}[W'^{1/2} \widehat{I}_F(\boldsymbol{\theta}; \{\rho_{\boldsymbol{\theta}}\}_{\boldsymbol{\theta}})^{-1} W'^{1/2}] \quad (30)$$

$$= \text{Tr}[W] \sum_k \langle k|W'^{1/2} \widehat{I}_F(\boldsymbol{\theta}; \{\rho_{\boldsymbol{\theta}}\}_{\boldsymbol{\theta}})^{-1} W'^{1/2}|k\rangle \quad (31)$$

$$\geq \text{Tr}[W] \left[ \sum_k \langle k|W'^{1/2} \widehat{I}_F(\boldsymbol{\theta}; \{\rho_{\boldsymbol{\theta}}\}_{\boldsymbol{\theta}})^{-1} W'^{1/2}|k\rangle \right]^{-1} \quad (32)$$

$$= \frac{\text{Tr}[W]}{\text{Tr}[W'^{1/2} \widehat{I}_F(\boldsymbol{\theta}; \{\rho_{\boldsymbol{\theta}}\}_{\boldsymbol{\theta}})^{-1} W'^{1/2}]} \quad (33)$$

$$= \frac{\text{Tr}[W]}{\text{Tr}[W' \widehat{I}_F(\boldsymbol{\theta}; \{\rho_{\boldsymbol{\theta}}\}_{\boldsymbol{\theta}})]} \quad (34)$$

$$= \frac{\text{Tr}[W]^2}{\text{Tr}[W \widehat{I}_F(\boldsymbol{\theta}; \{\rho_{\boldsymbol{\theta}}\}_{\boldsymbol{\theta}})]}. \quad (35)$$

The first inequality is a consequence of (27). The first equality follows from cyclicity of trace. The second inequality uses the operator Jensen inequality [55] for the operator convex function  $f(x) = x^{-1}$ . This

inequality is saturated if and only if  $\hat{I}_F(\boldsymbol{\theta}; \rho_{\boldsymbol{\theta}\boldsymbol{\theta}})$  is diagonal in the eigenbasis of  $W$ , i.e.  $[\hat{I}_F(\boldsymbol{\theta}; \rho_{\boldsymbol{\theta}\boldsymbol{\theta}}), W] = 0$ . The next equality comes again from cyclicity of trace, and the last equality comes from the definition of  $W'$ . The reasoning above leads us to

$$\text{Tr}[W \text{Cov}(\boldsymbol{\theta})] \geq \frac{(\text{Tr}[W])^2}{\text{Tr}[W \hat{I}_F(\boldsymbol{\theta}; \{\rho_{\boldsymbol{\theta}\boldsymbol{\theta}}\}_{\boldsymbol{\theta}})]} \quad (36)$$

for every positive semi-definite matrix  $W$ , which implies that

$$\text{Tr}[W' \text{Cov}(\boldsymbol{\theta})] \geq \frac{1}{\text{Tr}[W' \hat{I}_F(\boldsymbol{\theta}; \rho_{\boldsymbol{\theta}\boldsymbol{\theta}})]} \quad (37)$$

for every positive semi-definite matrix  $W'$  such that  $\text{Tr}[W'] = 1$ .  $\blacksquare$

We note here that an alternate scalar CRB involving the RLD Fisher information is the following [5, 33]:

$$\text{Tr}[W \text{Cov}(\boldsymbol{\theta})] \geq \text{Tr}\left[W \text{Re}(\hat{I}_F(\boldsymbol{\theta}; \rho_{\boldsymbol{\theta}\boldsymbol{\theta}})^{-1})\right] + \text{Tr}\left[|W^{1/2} \text{Im}(\hat{I}_F(\boldsymbol{\theta}; \rho_{\boldsymbol{\theta}\boldsymbol{\theta}})^{-1}) W^{1/2}|\right]. \quad (38)$$

For a matrix  $X$ , here we define

$$\text{Re}(X) := \frac{1}{2}(X + \bar{X}), \quad \text{Im}(X) := \frac{1}{2i}(X - \bar{X}), \quad (39)$$

where  $\bar{X}$  is the matrix with entries that are the complex conjugates of the entries of  $X$ . This bound, similar to our bound in (26), arises from the matrix inequality in (27). Our bound in (26) is, in general, looser than (38). Our bound in (26) is explicitly written as

$$\text{Tr}[W \hat{I}_F(\boldsymbol{\theta}; \rho_{\boldsymbol{\theta}\boldsymbol{\theta}})] = \text{Tr}\left[W \text{Re}(\hat{I}_F(\boldsymbol{\theta}; \rho_{\boldsymbol{\theta}\boldsymbol{\theta}}))\right] + i \text{Tr}\left[W \text{Im}(\hat{I}_F(\boldsymbol{\theta}; \rho_{\boldsymbol{\theta}\boldsymbol{\theta}}))\right], \quad (40)$$

wherein it is clear that the major difference between it and (38) is the absence of any matrix inverses. Our bound's relative looseness arises because of the use of the operator Jensen inequality (whose details are in the proof of theorem 1). Despite this, our approach has the advantage that it can incorporate the chain rule and amortization collapse for the RLD Fisher information, which are discussed in detail in section 4. This is directly responsible for the single-letter nature of our bound in (152) and its applicability to general sequential estimation strategies. For state estimation tasks in which the statistical model obeys a condition called  $D$ -invariance [33], we note here that the RLD-based bound (38) is both achievable and the most informative CRB [56].

The RLD Fisher information value of quantum states, as defined in (22), is computable via a semi-definite program.

**Proposition 2.** *Let  $\{\rho_A^\theta\}_\theta$  be a differentiable family of quantum states, and let  $W$  be a  $D \times D$  weight matrix. Suppose that (25) holds. Then the RLD Fisher information value of quantum states can be calculated via the following semi-definite program:*

$$\hat{I}_F(\boldsymbol{\theta}, W; \{\rho_A^\theta\}) = \inf \left\{ \text{Tr}[(W_F \otimes I_A) M_{FA}] : M_{FA} \geq 0, \left[ \begin{array}{cc} M_{FA} & \sum_{j=1}^D |j\rangle_F \otimes (\partial_{\theta_j} \rho_A^\theta) \\ \sum_{j=1}^D \langle j|_F \otimes (\partial_{\theta_j} \rho_A^\theta) & \rho_A^\theta \end{array} \right] \geq 0 \right\}. \quad (41)$$

The dual program is given by

$$\sup_{P_{FA}, Q_{FA \rightarrow A}, R_A} 2 \left( \sum_{j=1}^D \text{Re}[\text{Tr}[Q_{FA \rightarrow A}(|j\rangle_F \otimes (\partial_{\theta_j} \rho_A^\theta))]] \right) - \text{Tr}[R_A \rho_A^\theta], \quad (42)$$

subject to

$$P_{FA} \leq (W_F \otimes I_A), \quad \begin{bmatrix} P_{FA} & (Q_{FA \rightarrow A})^\dagger \\ Q_{FA \rightarrow A} & R_A \end{bmatrix} \geq 0, \quad (43)$$

where  $P_{FA}$  and  $R_A$  are Hermitian, and  $Q_{FA \rightarrow A}$  is a linear operator.



**Proof.** We begin with the formula

$$\widehat{I}(\boldsymbol{\theta}, W; \rho_{A\boldsymbol{\theta}}^\theta) := \text{Tr} \left[ (W_F \otimes I_A) \left( \sum_{j,k=1}^D |j\rangle\langle k|_F \otimes (\partial_{\theta_j} \rho_A^\theta)(\rho_A^\theta)^{-1}(\partial_{\theta_k} \rho_A^\theta) \right) \right] \quad (44)$$

$$= \sum_{j,k=1}^D \langle k|W|j\rangle \text{Tr} \left[ (\partial_{\theta_j} \rho_A^\theta)(\rho_A^\theta)^{-1}(\partial_{\theta_k} \rho_A^\theta) \right]. \quad (45)$$

The above can be written as

$$\text{Tr} \left[ (W_F \otimes I_A)(X^\dagger Y^{-1} X) \right] \quad (46)$$

where  $X = \sum_{j=1}^D |j\rangle_F \otimes (\partial_{\theta_j} \rho_A^\theta)$  and  $Y = \rho_A^\theta$ .

We next use the Schur complement lemma,

$$\begin{bmatrix} M & X^\dagger \\ X & Y \end{bmatrix} \geq 0 \quad \Longleftrightarrow \quad Y \geq 0, \quad M \geq X^\dagger Y^{-1} X \quad (47)$$

which lets us write

$$X^\dagger Y^{-1} X = \min \left\{ M : \begin{bmatrix} M & X^\dagger \\ X & Y \end{bmatrix} \geq 0 \right\}, \quad (48)$$

where the ordering is understood in the Löwner sense.

Combining the above, we obtain the desired primal form in (41).

To obtain the dual program in (42), we apply lemma 9, which we provide in the [appendix](#). ■

### 3.1. RLD Fisher information value of quantum channels

The RLD Fisher information value for a differentiable family  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  of quantum channels is defined as

$$\widehat{I}_F(\boldsymbol{\theta}, W; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) := \sup_{\rho_{RA}} \widehat{I}_F(\boldsymbol{\theta}, W; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA})\}_\theta), \quad (49)$$

where the optimization is with respect to every bipartite state  $\rho_{RA}$  with system  $R$  arbitrarily large. However, note that, by a standard argument, it suffices to optimize over pure states  $\psi_{RA}$  with system  $R$  isomorphic to the channel input system  $A$ .

The finiteness condition for the channel RLD Fisher information value is

$$\left[ \sum_{j,k=1}^D \langle k|W|j\rangle (\partial_{\theta_k} \Gamma_{RB}^{\mathcal{N}^\theta})(\partial_{\theta_j} \Gamma_{RB}^{\mathcal{N}^\theta}) \right] \Pi_{\Gamma_{\mathcal{N}^\theta}}^\perp = 0, \quad (50)$$

where  $\Pi_{\Gamma_{\mathcal{N}^\theta}}^\perp$  is the projection onto the kernel of  $\Gamma_{RB}^{\mathcal{N}^\theta}$ , with  $\Gamma_{RB}^{\mathcal{N}^\theta}$  the Choi operator of the channel  $\mathcal{N}_{A \rightarrow B}^\theta$ .

**Proposition 3.** Let  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  be a differentiable family of quantum channels, and let  $W$  be a  $D \times D$  weight matrix. Suppose that (50) holds. Then the RLD Fisher information value of quantum channels has the following explicit form:

$$\widehat{I}_F(\boldsymbol{\theta}, W; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) = \left\| \sum_{j,k=1}^D \langle k|W|j\rangle \text{Tr}_B [(\partial_{\theta_j} \Gamma_{RB}^{\mathcal{N}^\theta})(\Gamma_{RB}^{\mathcal{N}^\theta})^{-1}(\partial_{\theta_k} \Gamma_{RB}^{\mathcal{N}^\theta})] \right\|_\infty. \quad (51)$$

**Proof.** Recall that every pure state  $\psi_{RA}$  can be written as

$$\psi_{RA} = Z_R \Gamma_{RA} Z_R^\dagger, \quad (52)$$

where  $Z_R$  is a square operator satisfying  $\text{Tr}[Z_R^\dagger Z_R] = 1$ . This implies that

$$\mathcal{N}_{A \rightarrow B}^\theta(\psi_{RA}) = \mathcal{N}_{A \rightarrow B}^\theta(Z_R \Gamma_{RA} Z_R^\dagger) \quad (53)$$

$$= Z_R \mathcal{N}_{A \rightarrow B}^\theta(\Gamma_{RA}) Z_R^\dagger \quad (54)$$

$$= Z_R \Gamma_{RB}^{\mathcal{N}^\theta} Z_R^\dagger. \quad (55)$$

It suffices to optimize over pure states  $\psi_{RA}$  such that  $\psi_A > 0$  because these states are dense in the set of all

pure bipartite states. Then consider that

$$\sup_{\psi_{RA}} \widehat{I}_F(\boldsymbol{\theta}, W; \mathcal{N}_{A \rightarrow B}^{\boldsymbol{\theta}}(\psi_{RA})_{\boldsymbol{\theta}}) = \sup_{\psi_{RA}} \sum_{j,k=1}^D \langle k|W|j \rangle \text{Tr} \left[ (\partial_{\theta_j} \mathcal{N}_{A \rightarrow B}^{\boldsymbol{\theta}}(\psi_{RA})) (\mathcal{N}_{A \rightarrow B}^{\boldsymbol{\theta}}(\psi_{RA}))^{-1} (\partial_{\theta_k} \mathcal{N}_{A \rightarrow B}^{\boldsymbol{\theta}}(\psi_{RA})) \right] \quad (56)$$

$$= \sup_{Z_R: \text{Tr}[Z_R^\dagger Z_R] = 1} \sum_{j,k=1}^D \langle k|W|j \rangle \text{Tr} \left[ (\partial_{\theta_j} Z_R \Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}} Z_R^\dagger) (Z_R \Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}} Z_R^\dagger)^{-1} (\partial_{\theta_k} Z_R \Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}} Z_R^\dagger) \right] \quad (57)$$

$$= \sup_{Z_R: \text{Tr}[Z_R^\dagger Z_R] = 1} \sum_{j,k=1}^D \langle k|W|j \rangle \text{Tr} \left[ Z_R (\partial_{\theta_j} \Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}}) Z_R^\dagger Z_R^{-1} (\Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}})^{-1} Z_R^{-1} Z_R (\partial_{\theta_k} \Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}}) Z_R^\dagger \right] \quad (58)$$

$$= \sup_{Z_R: \text{Tr}[Z_R^\dagger Z_R] = 1} \sum_{j,k=1}^D \langle k|W|j \rangle \text{Tr} \left[ Z_R^\dagger Z_R (\partial_{\theta_j} \Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}}) (\Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}})^{-1} (\partial_{\theta_k} \Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}}) \right] \quad (59)$$

$$= \sup_{Z_R: \text{Tr}[Z_R^\dagger Z_R] = 1} \text{Tr} \left[ Z_R^\dagger Z_R \sum_{j,k=1}^D \langle k|W|j \rangle \text{Tr}_B \left[ (\partial_{\theta_j} \Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}}) (\Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}})^{-1} (\partial_{\theta_k} \Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}}) \right] \right] \quad (60)$$

$$= \left\| \sum_{j,k=1}^D \langle k|W|j \rangle \text{Tr}_B \left[ (\partial_{\theta_j} \Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}}) (\Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}})^{-1} (\partial_{\theta_k} \Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}}) \right] \right\|_{\infty}. \quad (61)$$

The last equality is a consequence of the characterization of the infinity norm of a positive semi-definite operator  $Y$  as  $\|Y\|_{\infty} = \sup_{\rho \geq 0, \text{Tr}[\rho] = 1} \text{Tr}[Y\rho]$ .  $\blacksquare$

As in the case of the RLD Fisher information value for quantum states, the analogous quantity for channels can also be computed via a semi-definite program.

**Proposition 4.** Let  $\{\mathcal{N}_{A \rightarrow B}^{\boldsymbol{\theta}}\}_{\boldsymbol{\theta}}$  be a differentiable family of quantum channels, and let  $W$  be a  $D \times D$  weight matrix. Suppose that (50) holds. Then the RLD Fisher information value of quantum channels can be calculated via the following semi-definite program:

$$\widehat{I}_F(\boldsymbol{\theta}, W; \{\mathcal{N}_{A \rightarrow B}^{\boldsymbol{\theta}}\}_{\boldsymbol{\theta}}) = \inf \lambda \in \mathbb{R}^+, \quad (62)$$

subject to

$$\lambda I_R \geq \text{Tr}_{FB}[(W_F \otimes I_{RB}) M_{FRB}], \quad \begin{bmatrix} M_{FRB} & \sum_j |j\rangle_F \otimes (\partial_{\theta_j} \Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}}) \\ \sum_j \langle j|_F \otimes (\partial_{\theta_j} \Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}}) & \Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}} \end{bmatrix} \geq 0. \quad (63)$$

The dual program is given by

$$\sup_{\rho_R \geq 0, P_{FRB}, Z_{FRB \rightarrow RB}, Q_{RB}} 2 \left( \sum_{j=1}^D \text{Re}[\text{Tr}[Z_{FRB \rightarrow RB}(|j\rangle_F \otimes (\partial_{\theta_j} \Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}})]] \right) - \text{Tr}[Q_{RB} \Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}}], \quad (64)$$

subject to

$$\text{Tr}[\rho_R] \leq 1, \quad \begin{bmatrix} P_{FRB} & (Z_{FRB \rightarrow RB})^\dagger \\ Z_{FRB \rightarrow RB} & Q_{RB} \end{bmatrix} \geq 0, \quad P_{FRB} \leq \rho_R \otimes W_F \otimes I_B. \quad (65)$$

**Proof.** The form of the primal program relies on the combination of a few facts. First, we use the following characterization of the infinity norm of a positive semi-definite operator  $A$ :

$$\|A\|_{\infty} = \inf \{ \lambda \geq 0 : A \leq \lambda I \}. \quad (66)$$

Next we observe that

$$\sum_{j,k=1}^D \langle k|W|j \rangle \text{Tr}_B [(\partial_{\theta_j} \Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}}) (\Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}})^{-1} (\partial_{\theta_k} \Gamma_{RB}^{\mathcal{N}^{\boldsymbol{\theta}}})] \quad (67)$$

can be written as

$$\text{Tr}_{FB}[(W_F \otimes I_{RB})(X^\dagger Y^{-1} X)] \quad (68)$$

where  $X = \sum_{j=1}^D \langle j|_F \otimes (\partial_{\theta_j} \Gamma_{RB}^{\mathcal{N}^\theta})$  and  $Y = \Gamma_{RB}^{\mathcal{N}^\theta}$ .

We next use the Schur complement lemma,

$$\begin{bmatrix} M & X^\dagger \\ X & Y \end{bmatrix} \geq 0 \iff Y \geq 0, \quad M \geq X^\dagger Y^{-1} X \quad (69)$$

which lets us write

$$X^\dagger Y^{-1} X = \min \left\{ M : \begin{bmatrix} M & X^\dagger \\ X & Y \end{bmatrix} \geq 0 \right\}. \quad (70)$$

Combining the above with (51), we obtain the desired primal form in (62).

To arrive at the dual program, we use the standard forms of primal and dual semi-definite programs for Hermitian operators  $A$  and  $B$  and a Hermiticity-preserving map  $\Phi$  [57]:

$$\sup_{X \geq 0} \{ \text{Tr}[AX] : \Phi(X) \leq B \}, \quad \inf_{Y \geq 0} \{ \text{Tr}[BY] : \Phi^\dagger(Y) \geq A \}. \quad (71)$$

From (62) and (63), we identify

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} \lambda & 0 \\ 0 & M_{FRB} \end{bmatrix}, \quad \Phi^\dagger(Y) = \begin{bmatrix} \lambda I_R - \text{Tr}_{FB}[(W_F \otimes I_{RB})M_{FRB}] & 0 & 0 \\ 0 & M_{FRB} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (72)$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sum_{j=1}^D |j\rangle_F \otimes (\partial_{\theta_j} \Gamma_{RB}^{\mathcal{N}^\theta}) \\ 0 & -\sum_{j=1}^D \langle j|_F \otimes (\partial_{\theta_j} \Gamma_{RB}^{\mathcal{N}^\theta}) & -\Gamma_{RB}^{\mathcal{N}^\theta} \end{bmatrix}. \quad (73)$$

Upon setting

$$X = \begin{bmatrix} \rho_R & 0 & 0 \\ 0 & P_{FRB} & (Z_{FRB \rightarrow RB})^\dagger \\ 0 & Z_{FRB \rightarrow RB} & Q_{RB} \end{bmatrix}, \quad (74)$$

we find that

$$\begin{aligned} \text{Tr}[X\Phi^\dagger(Y)] &= \text{Tr} \left[ \begin{bmatrix} \rho_R & 0 & 0 \\ 0 & P_{FRB} & (Z_{FRB \rightarrow RB})^\dagger \\ 0 & Z_{FRB \rightarrow RB} & Q_{RB} \end{bmatrix} \right. \\ &\quad \times \left. \begin{bmatrix} \lambda I_R - \text{Tr}_{FB}[(W_F \otimes I_{RB})M_{FRB}] & 0 & 0 \\ 0 & M_{FRB} & 0 \\ 0 & 0 & 0 \end{bmatrix} \right] \end{aligned} \quad (75)$$

$$= \text{Tr}[\rho_R(\lambda I_R - \text{Tr}_{FB}[(W_F \otimes I_{RB})M_{FRB}])] + \text{Tr}[P_{FRB}M_{FRB}] \quad (76)$$

$$= \lambda \text{Tr}[\rho_R] + \text{Tr}[(P_{FRB} - \rho_R \otimes W_F \otimes I_B)M_{FRB}] \quad (77)$$

$$= \text{Tr} \left[ \begin{bmatrix} \lambda & 0 \\ 0 & M_{FRB} \end{bmatrix} \begin{bmatrix} \text{Tr}[\rho_R] & 0 \\ 0 & P_{FRB} - \rho_R \otimes W_F \otimes I_B \end{bmatrix} \right] \quad (78)$$

$$= \text{Tr}[Y\Phi(X)], \quad (79)$$

to find that the dual is given by

$$\sup_{\rho_R, P_{FRB}, Z_{FRB \rightarrow RB}, Q_{RB}} \text{Tr}[AX], \quad (80)$$

subject to

$$\begin{bmatrix} \rho_R & 0 & 0 \\ 0 & P_{FRB} & (Z_{FRB \rightarrow RB})^\dagger \\ 0 & Z_{FRB \rightarrow RB} & Q_{RB} \end{bmatrix} \geq 0, \quad \begin{bmatrix} \text{Tr}[\rho_R] & 0 \\ 0 & P_{FRB} - \rho_R \otimes W_F \otimes I_B \end{bmatrix} \leq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (81)$$

We can swap  $Z_{FRB \rightarrow RB} \rightarrow -Z_{FRB \rightarrow RB}$  with no change to the optimal value. This leads to the following simplified form of the dual program:

$$\sup_{\rho_R \geq 0, P_{FRB}, Z_{FRB}, Q_{RB}} 2 \sum_{j=1}^D \text{Re}[\text{Tr}[Z_{FRB \rightarrow RB}(|j\rangle_F \otimes \partial_{\theta_j} \Gamma_{RB}^{\mathcal{N}^\theta})]] - \text{Tr}[Q_{RB} \Gamma_{RB}^{\mathcal{N}^\theta}], \quad (82)$$

subject to

$$\text{Tr}[\rho_R] \leq 1, \quad \begin{bmatrix} P_{FRB} & -(Z_{FRB \rightarrow RB})^\dagger \\ -Z_{FRB \rightarrow RB} & Q_{RB} \end{bmatrix} \geq 0, \quad P_{FRB} \leq \rho_R \otimes W_F \otimes I_B. \quad (83)$$

Then we note that

$$\begin{bmatrix} P_{FRB} & -(Z_{FRB \rightarrow RB})^\dagger \\ -Z_{FRB \rightarrow RB} & Q_{RB} \end{bmatrix} \geq 0 \iff \begin{bmatrix} P_{FRB} & (Z_{FRB \rightarrow RB})^\dagger \\ Z_{FRB \rightarrow RB} & Q_{RB} \end{bmatrix} \geq 0 \quad (84)$$

This concludes the proof. ■

#### 4. Amortized Fisher information

Amortized channel divergences were defined in [58] to provide a mathematical framework for studying the power of sequential strategies over parallel ones in quantum channel discrimination. With the view of performing a similar comparison for quantum channel estimation, we define the amortized Fisher information for quantum channel families [28]. For the channel family  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$ , the amortized RLD Fisher information is defined as

$$\hat{I}_F^A(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) := \sup_{\{\rho_{RA}^\theta\}_\theta} [\hat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) - \hat{I}_F(\theta; \{\rho_{RA}^\theta\}_\theta)]. \quad (85)$$

For the case of multiparameter estimation, we define the amortized RLD Fisher information value as follows:

$$\hat{I}_F^A(\boldsymbol{\theta}, W; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) := \sup_{\{\rho_{RA}^\theta\}_\theta} \hat{I}_F(\boldsymbol{\theta}, W; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) - \hat{I}_F(\boldsymbol{\theta}, W; \{\rho_{RA}^\theta\}_\theta). \quad (86)$$

The framework of amortized Fisher information can be applied more generally beyond RLD Fisher information, as discussed in [28], and it is helpful for assessing the power of sequential strategies when estimating a parameter encoded in a quantum channel. In the sequential strategies described in figure 1, one can qualitatively say that the goal is to ‘accumulate’ as much information about  $\boldsymbol{\theta}$  into the state being carried forward from one channel use to another. The amortized Fisher information captures the marginal increase in Fisher information per channel use in such a scenario.

A basic inequality obeyed by the amortized RLD Fisher information, as a direct consequence of definitions in (10) and (85), is that

$$\hat{I}_F^A(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) \geq \hat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta), \quad (87)$$

which also holds for the multiparameter case as follows:

$$\hat{I}_F^A(\boldsymbol{\theta}, W; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) \geq \hat{I}_F(\boldsymbol{\theta}, W; \mathcal{N}_{A \rightarrow B}^\theta). \quad (88)$$

This can be understood by considering the right-hand side to arise from restricting to input states with no parameter dependence. Qualitatively, it means that the marginal increase in Fisher information value can only be improved by using a catalyst state family as input. That is,

$$\hat{I}_F^A(\boldsymbol{\theta}, W; \mathcal{N}_{A \rightarrow B}^{\boldsymbol{\theta}, W}) := \sup_{\{\rho_{RA}^\theta\}_\theta} [\hat{I}_F(\boldsymbol{\theta}, W; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) - \hat{I}_F(\boldsymbol{\theta}, W; \rho_{RA}^\theta)] \quad (89)$$

$$\geq \sup_{\{\rho_{RA}\}_\theta} [\hat{I}_F(\boldsymbol{\theta}, W; \mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA})) - \hat{I}_F(\boldsymbol{\theta}, W; \rho_{RA})] \quad (90)$$

$$= \sup_{\{\rho_{RA}\}_\theta} \left[ \hat{I}_F(\theta, W; \mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA})_\theta) \right] \quad (91)$$

$$= \hat{I}_F(\theta, W; \mathcal{N}_{A \rightarrow B}^\theta). \quad (92)$$

For some amortized channel quantities, the reverse inequality also holds. In fact, we prove in our paper that it holds for the RLD Fisher information value of quantum channels, i.e.

$$\hat{I}_F^A(\theta, W; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) \leq \hat{I}_F(\theta, W; \mathcal{N}_{A \rightarrow B}^\theta). \quad (93)$$

This arises as a direct consequence of the chain rule for the RLD Fisher information value, which we state and prove in proposition 6.

A consequence of the reverse inequality (93) is a so-called amortization collapse, wherein,

$$\hat{I}_F^A(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) = \hat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta), \text{ and} \quad (94)$$

$$\hat{I}_F^A(\theta, W; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) = \hat{I}_F(\theta, W; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta). \quad (95)$$

The meaning of an amortization collapse is that the RLD Fisher information cannot be increased by using a catalyst. We first made the simple observation using (87) that a catalyst cannot decrease the RLD Fisher information. The reverse inequality (93) then implies the opposite, which is that one can do no better with a catalyst state than with one. That is, a catalyst state family  $\{\rho_{RA}^\theta\}_\theta$  at the channel input does not increase the RLD Fisher information  $\hat{I}_F(\theta, W; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta)$  by any more than its own RLD Fisher information  $\hat{I}_F(\theta, W; \{\rho_{RA}^\theta\}_\theta)$ . Continuing the earlier discussion of amortized Fisher information capturing marginal increment in resource, the amortization collapse in (94) and (95) means that the RLD Fisher information gained during each of the  $n$  sequential uses of the channel  $\mathcal{N}_{A \rightarrow B}^\theta$  is the same. It also means that the maximum attainable RLD Fisher information value for a channel family is the same when using sequential or parallel strategies.

#### 4.1. Chain rule for RLD Fisher information value

Let  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  be a differentiable family of quantum channels, and let  $\{\rho_{RA}^\theta\}_\theta$  be a differentiable family of quantum states on systems  $RA$ , where the system  $R$  can be of arbitrary size. Then the following chain rule holds [28]:

$$\hat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) \leq \hat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) + \hat{I}_F(\theta; \{\rho_{RA}^\theta\}_\theta), \quad (96)$$

analogous to one that holds for a different distinguishability measure [59]. The chain rule in (96) can be generalized to the case of multiparameter estimation. To do so, we first state it in the form of an operator inequality. Then we will show that (96) arises as a simple consequence of it.

**Proposition 5.** *Let  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  be a differentiable family of quantum channels, and let  $\{\rho_{RA}^\theta\}_\theta$  be a differentiable family of quantum states. Then the following chain-rule operator inequality holds*

$$\hat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) \leq \sum_{j,k=1}^D |j\rangle\langle k| \text{Tr}[\rho_S^\theta]^T \text{Tr}_B[(\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1}(\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^\theta})] + \hat{I}_F(\theta; \{\rho_{RA}^\theta\}_\theta), \quad (97)$$

where  $\rho_S^\theta$  is equal to the reduced state of  $\rho_{RA}^\theta$  on system  $A$  and system  $S$  is isomorphic to system  $A$ .

**Proof.** In the proof, we make use of the following identity:

$$\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta) = \langle \Gamma |_{AS} \rho_{RA}^\theta \otimes \Gamma_{SB}^{\mathcal{N}^\theta} | \Gamma \rangle_{AS}, \quad (98)$$

where

$$| \Gamma \rangle_{AS} := \sum_i |i\rangle_A |i\rangle_S. \quad (99)$$

Consider that

$$\hat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) = \text{Tr}_2 \left[ \sum_{j,k=1}^D |j\rangle\langle k| \otimes (\partial_{\theta_j} \mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)) (\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta))^{-1} (\partial_{\theta_k} \mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)) \right]. \quad (100)$$

We can, with a series of manipulations, show that

$$\begin{aligned} & \sum_{j,k=1}^D |j\rangle\langle k| \otimes (\partial_{\theta_j} \mathcal{N}_{A \rightarrow B}^{\theta}(\rho_{RA}^{\theta})) (\mathcal{N}_{A \rightarrow B}^{\theta}(\rho_{RA}^{\theta}))^{-1} (\partial_{\theta_k} \mathcal{N}_{A \rightarrow B}^{\theta}(\rho_{RA}^{\theta})) \\ &= \sum_{j,k=1}^D |j\rangle\langle k| \otimes (\partial_{\theta_j} \langle \Gamma|_{AS} \rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}} |\Gamma\rangle_{AS}) (\langle \Gamma|_{AS} \rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}} |\Gamma\rangle_{AS})^{-1} \\ & \quad \times (\partial_{\theta_k} \langle \Gamma|_{AS} \rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}} |\Gamma\rangle_{AS}) \end{aligned} \quad (101)$$

$$\begin{aligned} &= \langle \Gamma|_{AS} \left( \sum_{j=1}^D |j\rangle\langle 0| \otimes \partial_{\theta_j} (\rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}}) \right) |\Gamma\rangle_{AS} (\langle \Gamma|_{AS} (|0\rangle\langle 0| \otimes \rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}}) |\Gamma\rangle_{AS})^{-1} \\ & \quad \times \langle \Gamma|_{AS} \left( \sum_{k=1}^D |0\rangle\langle k| \otimes \partial_{\theta_k} (\rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}}) \right) |\Gamma\rangle_{AS}. \end{aligned} \quad (102)$$

Then identifying

$$X = \sum_{k=1}^D |0\rangle\langle k| \otimes \partial_{\theta_k} (\rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}}), \quad (103)$$

$$Y = |0\rangle\langle 0| \otimes \rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}} \quad (104)$$

$$L = I_D \otimes \langle \Gamma|_{AS} \otimes I_{RB}, \quad (105)$$

we can apply the well known transformer inequality  $LX^{\dagger}L^{\dagger}(LYL^{\dagger})^{-1}LXL^{\dagger} \leq LX^{\dagger}Y^{-1}XL^{\dagger}$  (see, e.g. [28, lemma 54] as well as [59]) to find that the last line above is not larger than the following one in the operator-inequality sense:

$$\begin{aligned} & \langle \Gamma|_{AS} \left( \sum_{j=1}^D |j\rangle\langle 0| \otimes \partial_{\theta_j} (\rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}}) \right) (|0\rangle\langle 0| \otimes \rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}})^{-1} \left( \sum_{k=1}^D |0\rangle\langle k| \otimes \partial_{\theta_k} (\rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}}) \right) |\Gamma\rangle_{AS} \\ &= \langle \Gamma|_{AS} \sum_{j,k=1}^D |j\rangle\langle k| \otimes \partial_{\theta_j} (\rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}}) (\rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}})^{-1} \partial_{\theta_k} (\rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}}) |\Gamma\rangle_{AS} \end{aligned} \quad (106)$$

$$= \sum_{j,k=1}^D |j\rangle\langle k| \otimes \langle \Gamma|_{AS} \partial_{\theta_j} (\rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}}) (\rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}})^{-1} \partial_{\theta_k} (\rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}}) |\Gamma\rangle_{AS}. \quad (107)$$

Consider that

$$\partial_{\theta_j} (\rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}}) = (\partial_{\theta_j} \rho_{RA}^{\theta}) \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}} + \rho_{RA}^{\theta} \otimes (\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}}). \quad (108)$$

Then we find that

$$\begin{aligned} & \partial_{\theta_j} (\rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}}) (\rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}})^{-1} \\ &= ((\partial_{\theta_j} \rho_{RA}^{\theta}) \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}} + \rho_{RA}^{\theta} \otimes (\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}})) (\rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}})^{-1} \end{aligned} \quad (109)$$

$$= ((\partial_{\theta_j} \rho_{RA}^{\theta}) \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}} + \rho_{RA}^{\theta} \otimes (\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}})) ((\rho_{RA}^{\theta})^{-1} \otimes (\Gamma_{SB}^{\mathcal{N}^{\theta}})^{-1}) \quad (110)$$

$$= (\partial_{\theta_j} \rho_{RA}^{\theta}) (\rho_{RA}^{\theta})^{-1} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}} (\Gamma_{SB}^{\mathcal{N}^{\theta}})^{-1} + \rho_{RA}^{\theta} (\rho_{RA}^{\theta})^{-1} \otimes (\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}}) (\Gamma_{SB}^{\mathcal{N}^{\theta}})^{-1} \quad (111)$$

$$= (\partial_{\theta_j} \rho_{RA}^{\theta}) (\rho_{RA}^{\theta})^{-1} \otimes \Pi_{\Gamma^{\mathcal{N}^{\theta}}} + \Pi_{\rho^{\theta}} \otimes (\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}}) (\Gamma_{SB}^{\mathcal{N}^{\theta}})^{-1}. \quad (112)$$

Right multiplying this last line by  $\partial_{\theta_k} (\rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}})$  gives

$$\begin{aligned} & ((\partial_{\theta_j} \rho_{RA}^{\theta}) (\rho_{RA}^{\theta})^{-1} \otimes \Pi_{\Gamma^{\mathcal{N}^{\theta}}} + \Pi_{\rho^{\theta}} \otimes (\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}}) (\Gamma_{SB}^{\mathcal{N}^{\theta}})^{-1}) \partial_{\theta_k} (\rho_{RA}^{\theta} \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}}) \\ &= (\partial_{\theta_j} \rho_{RA}^{\theta}) (\rho_{RA}^{\theta})^{-1} (\partial_{\theta_k} \rho_{RA}^{\theta}) \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}} + (\partial_{\theta_j} \rho_{RA}^{\theta}) \Pi_{\rho^{\theta}} \otimes \Pi_{\Gamma^{\mathcal{N}^{\theta}}} (\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^{\theta}}) \\ & \quad + \Pi_{\rho^{\theta}} (\partial_{\theta_k} \rho_{RA}^{\theta}) \otimes (\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}}) \Pi_{\Gamma^{\mathcal{N}^{\theta}}} + \rho_{RA}^{\theta} \otimes (\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}}) (\Gamma_{SB}^{\mathcal{N}^{\theta}})^{-1} (\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^{\theta}}). \end{aligned} \quad (113)$$

Adding in the terms  $(\partial_{\theta_j} \rho_{RA}^{\theta}) \Pi_{\rho^{\theta}}^{\perp} = 0$ ,  $(\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}}) \Pi_{\Gamma^{\mathcal{N}^{\theta}}}^{\perp} = 0$ ,  $\Pi_{\rho^{\theta}}^{\perp} (\partial_{\theta_k} \rho_{RA}^{\theta}) = 0$ , and  $\Pi_{\Gamma^{\mathcal{N}^{\theta}}}^{\perp} (\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^{\theta}}) = 0$ , which follow from the support conditions for finite RLD Fisher information, the last line above becomes as follows:

$$= (\partial_{\theta_j} \rho_{RA}^{\theta}) (\rho_{RA}^{\theta})^{-1} (\partial_{\theta_k} \rho_{RA}^{\theta}) \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}} + (\partial_{\theta_j} \rho_{RA}^{\theta}) \otimes (\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^{\theta}}) \\ + (\partial_{\theta_k} \rho_{RA}^{\theta}) \otimes (\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}}) + \rho_{RA}^{\theta} \otimes (\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}}) (\Gamma_{SB}^{\mathcal{N}^{\theta}})^{-1} (\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^{\theta}}). \quad (114)$$

So then the relevant matrix is simplified as follows:

$$\sum_{j,k=1}^D |j\rangle\langle k| \otimes \langle \Gamma |_{AS} \left[ (\partial_{\theta_j} \rho_{RA}^{\theta}) (\rho_{RA}^{\theta})^{-1} (\partial_{\theta_k} \rho_{RA}^{\theta}) \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}} + (\partial_{\theta_j} \rho_{RA}^{\theta}) \otimes (\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^{\theta}}) \right. \\ \left. + (\partial_{\theta_k} \rho_{RA}^{\theta}) \otimes (\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}}) + \rho_{RA}^{\theta} \otimes (\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}}) (\Gamma_{SB}^{\mathcal{N}^{\theta}})^{-1} (\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^{\theta}}) \right] | \Gamma \rangle_{AS}. \quad (115)$$

Thus, we have established the following operator inequality:

$$\sum_{j,k=1}^D |j\rangle\langle k| \otimes (\partial_{\theta_j} \mathcal{N}_{A \rightarrow B}^{\theta}(\rho_{RA}^{\theta})) (\mathcal{N}_{A \rightarrow B}^{\theta}(\rho_{RA}^{\theta}))^{-1} (\partial_{\theta_k} \mathcal{N}_{A \rightarrow B}^{\theta}(\rho_{RA}^{\theta})) \\ \leq \sum_{j,k=1}^D |j\rangle\langle k| \otimes \langle \Gamma |_{AS} \left[ (\partial_{\theta_j} \rho_{RA}^{\theta}) (\rho_{RA}^{\theta})^{-1} (\partial_{\theta_k} \rho_{RA}^{\theta}) \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}} + (\partial_{\theta_j} \rho_{RA}^{\theta}) \otimes (\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^{\theta}}) \right. \\ \left. + (\partial_{\theta_k} \rho_{RA}^{\theta}) \otimes (\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}}) + \rho_{RA}^{\theta} \otimes (\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}}) (\Gamma_{SB}^{\mathcal{N}^{\theta}})^{-1} (\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^{\theta}}) \right] | \Gamma \rangle_{AS}. \quad (116)$$

We can then take a partial trace over the  $RB$  systems and arrive at the following operator inequality:

$$\widehat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^{\theta}(\rho_{RA}^{\theta})\}_{\theta}) \\ \leq \sum_{j,k=1}^D |j\rangle\langle k| \text{Tr}_{RB} \left[ \langle \Gamma |_{AS} \left[ (\partial_{\theta_j} \rho_{RA}^{\theta}) (\rho_{RA}^{\theta})^{-1} (\partial_{\theta_k} \rho_{RA}^{\theta}) \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}} + (\partial_{\theta_j} \rho_{RA}^{\theta}) \otimes (\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^{\theta}}) \right. \right. \\ \left. \left. + (\partial_{\theta_k} \rho_{RA}^{\theta}) \otimes (\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}}) + \rho_{RA}^{\theta} \otimes (\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}}) (\Gamma_{SB}^{\mathcal{N}^{\theta}})^{-1} (\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^{\theta}}) \right] | \Gamma \rangle_{AS} \right]. \quad (117)$$

Now we evaluate each term on the right:

$$\langle \Gamma |_{AS} \text{Tr}_{RB} [(\partial_{\theta_j} \rho_{RA}^{\theta}) (\rho_{RA}^{\theta})^{-1} (\partial_{\theta_k} \rho_{RA}^{\theta}) \otimes \Gamma_{SB}^{\mathcal{N}^{\theta}}] | \Gamma \rangle_{AS} \\ = \langle \Gamma |_{AS} \text{Tr}_R [(\partial_{\theta_j} \rho_{RA}^{\theta}) (\rho_{RA}^{\theta})^{-1} (\partial_{\theta_k} \rho_{RA}^{\theta})] \otimes \text{Tr}_B [\Gamma_{SB}^{\mathcal{N}^{\theta}}] | \Gamma \rangle_{AS} \quad (118)$$

$$= \langle \Gamma |_{AS} \text{Tr}_R [(\partial_{\theta_j} \rho_{RA}^{\theta}) (\rho_{RA}^{\theta})^{-1} (\partial_{\theta_k} \rho_{RA}^{\theta})] \otimes I_S | \Gamma \rangle_{AS} \quad (119)$$

$$= \text{Tr}[(\partial_{\theta_j} \rho_{RA}^{\theta}) (\rho_{RA}^{\theta})^{-1} (\partial_{\theta_k} \rho_{RA}^{\theta})], \quad (120)$$

$$\langle \Gamma |_{AS} \text{Tr}_{RB} [(\partial_{\theta_k} \rho_{RA}^{\theta}) \otimes (\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}})] | \Gamma \rangle_{AS} \\ = \langle \Gamma |_{AS} \text{Tr}_R [\partial_{\theta_k} \rho_{RA}^{\theta}] \otimes \text{Tr}_B [\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}}] | \Gamma \rangle_{AS} \quad (121)$$

$$= \langle \Gamma |_{AS} \text{Tr}_R [\partial_{\theta_k} \rho_{RA}^{\theta}] \otimes \partial_{\theta_j} \text{Tr}_B [\Gamma_{SB}^{\mathcal{N}^{\theta}}] | \Gamma \rangle_{AS} \quad (122)$$

$$= \langle \Gamma |_{AS} \text{Tr}_R [\partial_{\theta_k} \rho_{RA}^{\theta}] \otimes (\partial_{\theta_j} I_S) | \Gamma \rangle_{AS} \quad (123)$$

$$= 0, \quad (124)$$

$$\langle \Gamma |_{AS} \text{Tr}_{RB} [(\partial_{\theta_j} \rho_{RA}^{\theta}) \otimes (\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^{\theta}})] | \Gamma \rangle_{AS} = 0, \text{ and} \quad (125)$$

$$\langle \Gamma |_{AS} \text{Tr}_{RB} [\rho_{RA}^{\theta} \otimes (\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}}) (\Gamma_{SB}^{\mathcal{N}^{\theta}})^{-1} (\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^{\theta}})] | \Gamma \rangle_{AS} \\ = \langle \Gamma |_{AS} \rho_{RA}^{\theta} \otimes \text{Tr}_B [(\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}}) (\Gamma_{SB}^{\mathcal{N}^{\theta}})^{-1} (\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^{\theta}})] | \Gamma \rangle_{AS} \quad (126)$$

$$= \text{Tr}[\rho_{RA}^{\theta} \text{Tr}_B [(\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^{\theta}}) (\Gamma_{SB}^{\mathcal{N}^{\theta}})^{-1} (\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^{\theta}})]]. \quad (127)$$

Substituting back above, we find that

$$\begin{aligned} \widehat{I}_F(\boldsymbol{\theta}; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) &\leq \sum_{j,k=1}^D |j\rangle\langle k| \operatorname{Tr}[(\partial_{\theta_j} \rho_{RA}^\theta)(\rho_{RA}^\theta)^{-1}(\partial_{\theta_k} \rho_{RA}^\theta)] \\ &\quad + \sum_{j,k=1}^D |j\rangle\langle k| \operatorname{Tr}[(\rho_S^\theta)^T \operatorname{Tr}_B[(\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1}(\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^\theta})]] \end{aligned} \quad (128)$$

$$= \widehat{I}_F(\boldsymbol{\theta}; \{\rho_{RA}^\theta\}_\theta) + \sum_{j,k=1}^D |j\rangle\langle k| \operatorname{Tr}[(\rho_S^\theta)^T \operatorname{Tr}_B[(\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1}(\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^\theta})]]. \quad (129)$$

This concludes the proof.  $\blacksquare$

Now we show how the operator-inequality chain rule leads us to the desired result in (130).

**Proposition 6.** *With  $W$  a weight matrix as defined previously, let  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  be a differentiable family of quantum channels, and let  $\{\rho_{RA}^\theta\}_\theta$  be a differentiable family of quantum states. Then the following chain-rule inequality holds*

$$\widehat{I}_F(\boldsymbol{\theta}, W; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) \leq \widehat{I}_F(\boldsymbol{\theta}, W; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) + \widehat{I}_F(\boldsymbol{\theta}, W; \{\rho_{RA}^\theta\}_\theta). \quad (130)$$

**Proof.** We start by restating the operator-inequality chain rule in (97):

$$\widehat{I}_F(\boldsymbol{\theta}; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) \leq \sum_{j,k=1}^D |j\rangle\langle k| \operatorname{Tr}[(\rho_S^\theta)^T \operatorname{Tr}_B[(\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1}(\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^\theta})]] + \widehat{I}_F(\boldsymbol{\theta}; \{\rho_{RA}^\theta\}_\theta). \quad (131)$$

We sandwich the above with  $W^{1/2}$  on both sides and take the trace (a positive map overall) to preserve the inequality and obtain

$$\begin{aligned} \operatorname{Tr}[W \widehat{I}_F(\boldsymbol{\theta}; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta)] &\leq \operatorname{Tr} \left[ W \sum_{j,k=1}^D |j\rangle\langle k| \operatorname{Tr}[(\rho_S^\theta)^T \operatorname{Tr}_B[(\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1}(\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^\theta})]] \right] \\ &\quad + \operatorname{Tr}[W \widehat{I}_F(\boldsymbol{\theta}; \{\rho_{RA}^\theta\}_\theta)]. \end{aligned} \quad (132)$$

Using the definition of the RLD Fisher information value, the above simplifies to

$$\begin{aligned} \widehat{I}_F(\boldsymbol{\theta}, W; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) &\leq \operatorname{Tr} \left[ W \sum_{j,k=1}^D |j\rangle\langle k| \operatorname{Tr}[(\rho_S^\theta)^T \operatorname{Tr}_B[(\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1}(\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^\theta})]] \right] \\ &\quad + \widehat{I}_F(\boldsymbol{\theta}, W; \{\rho_{RA}^\theta\}_\theta). \end{aligned} \quad (133)$$

Now we consider that

$$\begin{aligned} \operatorname{Tr} \left[ W \sum_{j,k=1}^D |j\rangle\langle k| \operatorname{Tr}[(\rho_S^\theta)^T \operatorname{Tr}_B[(\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1}(\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^\theta})]] \right] \\ = \sum_{j,k=1}^D \langle k|W|j\rangle \operatorname{Tr}[(\rho_S^\theta)^T \operatorname{Tr}_B[(\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1}(\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^\theta})]] \end{aligned} \quad (134)$$

$$= \operatorname{Tr} \left[ (\rho_S^\theta)^T \sum_{j,k=1}^D \langle k|W|j\rangle \operatorname{Tr}_B[(\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1}(\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^\theta})] \right] \quad (135)$$

$$\leq \sup_{\rho_S^\theta} \operatorname{Tr} \left[ (\rho_S^\theta)^T \sum_{j,k=1}^D \langle k|W|j\rangle \operatorname{Tr}_B[(\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1}(\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^\theta})] \right] \quad (136)$$

$$= \left\| \sum_{j,k=1}^D \langle k|W|j\rangle \operatorname{Tr}_B[(\partial_{\theta_j} \Gamma_{SB}^{\mathcal{N}^\theta})(\Gamma_{SB}^{\mathcal{N}^\theta})^{-1}(\partial_{\theta_k} \Gamma_{SB}^{\mathcal{N}^\theta})] \right\|_\infty \quad (137)$$



$$= \hat{I}_F(\boldsymbol{\theta}, W; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta). \quad (138)$$

Using the above and (133), we conclude that

$$\hat{I}_F(\boldsymbol{\theta}, W; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) \leq \hat{I}_F(\boldsymbol{\theta}, W; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) + \hat{I}_F(\boldsymbol{\theta}, W; \{\rho_{RA}^\theta\}_\theta). \quad (139)$$

This concludes the proof.  $\blacksquare$

We now show how the chain rule results in an amortization collapse. The chain rule in (130) can be rewritten as

$$\hat{I}_F(\boldsymbol{\theta}, W; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) - \hat{I}_F(\boldsymbol{\theta}, W; \{\rho_{RA}^\theta\}_\theta) \leq \hat{I}_F(\boldsymbol{\theta}, W; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta). \quad (140)$$

Taking the supremum over input-state families on the left-hand side gives

$$\hat{I}_F^A(\boldsymbol{\theta}, W; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta) \leq \hat{I}_F(\boldsymbol{\theta}, W; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta). \quad (141)$$

When the above is combined with (87), we obtain the desired amortization collapse in (95).

We thus see that the chain rule for the multiparameter RLD Fisher information (130) leads to the amortization collapse in (95). The amortization collapse implies that the  $n$ -sequential-use RLD Fisher information is simply  $n$  times the single-use RLD Fisher information.

## 5. Single-letter quantum Cramer–Rao bound

The following single-letter CRB for estimating parameter  $\theta$  embedded in  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  holds both in the parallel setting [15] and the more general sequential setting [28] of channel estimation. In the sequential setting, it arises from the combination of the chain rule, the ensuing amortization collapse for the RLD Fisher information, and theorem 18 of [28]:

$$\text{Var}(\hat{\theta}) \geq \frac{1}{n \hat{I}_F(\theta; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta)}. \quad (142)$$

To prove an analogous result for the multiparameter case, we need the following converse theorem, which is a generalization of [28, theorem 18]:

**Theorem 7.** Consider a general sequential estimation protocol of the form depicted in figure 1. The following inequality holds:

$$\hat{I}_F(\boldsymbol{\theta}, W; \{\omega_{R_n B_n}^\theta\}_\theta) \leq n \cdot \hat{I}_F^A(\boldsymbol{\theta}, W; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta), \quad (143)$$

where  $\omega_{R_n B_n}^\theta$  is the final state of an  $n$ -round sequential estimation protocol, and  $\mathcal{S}^1$  through  $\mathcal{S}^{n-1}$  are interleaving quantum channels in the protocol, both as in figure 1.

**Proof.** Consider that

$$\hat{I}_F(\boldsymbol{\theta}, W; \{\omega_{R_n B_n}^\theta\}_\theta) = \hat{I}_F(\boldsymbol{\theta}, W; \{\omega_{R_n B_n}^\theta\}_\theta) - \hat{I}_F(\boldsymbol{\theta}, W; \{\rho_{R_1 A_1}\}_\theta) \quad (144)$$

$$\begin{aligned} &= \hat{I}_F(\boldsymbol{\theta}, W; \{\omega_{R_n B_n}^\theta\}_\theta) - \hat{I}_F(\boldsymbol{\theta}, W; \{\rho_{R_1 A_1}\}_\theta) \\ &\quad + \sum_{i=2}^n \left( \hat{I}_F(\boldsymbol{\theta}, W; \{\rho_{R_i A_i}^\theta\}_\theta) - \hat{I}_F(\boldsymbol{\theta}, W; \{\rho_{R_i A_i}^\theta\}_\theta) \right) \end{aligned} \quad (145)$$

$$\begin{aligned} &= \hat{I}_F(\boldsymbol{\theta}, W; \{\omega_{R_n B_n}^\theta\}_\theta) - \hat{I}_F(\boldsymbol{\theta}, W; \{\rho_{R_1 A_1}\}_\theta) \\ &\quad + \sum_{i=2}^n \left( \hat{I}_F(\boldsymbol{\theta}, W; \{\mathcal{S}_{R_{i-1} B_{i-1} \rightarrow R_i A_i}^{i-1}(\rho_{R_{i-1} B_{i-1}}^\theta)\}_\theta) - \hat{I}_F(\boldsymbol{\theta}, W; \{\rho_{R_i A_i}^\theta\}_\theta) \right) \end{aligned} \quad (146)$$

$$\begin{aligned} &\leq \hat{I}_F(\boldsymbol{\theta}, W; \{\omega_{R_n B_n}^\theta\}_\theta) - \hat{I}_F(\boldsymbol{\theta}, W; \{\rho_{R_1 A_1}\}_\theta) \\ &\quad + \sum_{i=2}^n \left( \hat{I}_F(\boldsymbol{\theta}, W; \{\rho_{R_{i-1} B_{i-1}}^\theta\}_\theta) - \hat{I}_F(\boldsymbol{\theta}, W; \{\rho_{R_i A_i}^\theta\}_\theta) \right) \end{aligned} \quad (147)$$

$$= \sum_{i=1}^n \left( \hat{I}_F(\boldsymbol{\theta}, W; \{\rho_{R_i B_i}^\theta\}_\theta) - \hat{I}_F(\boldsymbol{\theta}, W; \{\rho_{R_i A_i}^\theta\}_\theta) \right) \quad (148)$$

$$= \sum_{i=1}^n \left( \hat{I}_F(\boldsymbol{\theta}, W; \{\mathcal{N}_{A_i \rightarrow B_i}^\theta(\rho_{R_i A_i}^\theta)\}_\theta) - \hat{I}_F(\boldsymbol{\theta}, W; \{\rho_{R_i A_i}^\theta\}_\theta) \right) \quad (149)$$

$$\leq n \cdot \sup_{\{\rho_{RA}^\theta\}_\theta} \left[ \hat{I}_F(\theta, W; \{\mathcal{N}_{A \rightarrow B}^\theta(\rho_{RA}^\theta)\}_\theta) - \hat{I}_F(\theta, W; \{\rho_{RA}^\theta\}_\theta) \right] \quad (150)$$

$$= n \cdot \hat{I}_F^A(\theta, W; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta). \quad (151)$$

The first equality follows because the initial state  $\rho_{R_1 A_1}$  has no dependence on any of the parameters in  $\theta$ . The first inequality arises due to the data-processing inequality for the RLD Fisher information. The other steps are straightforward manipulations. ■

We have now assembled all the necessary components needed to establish our main result, a multiparameter CRB for sequential channel estimation.

**Theorem 8.** *For a differentiable channel family  $\{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta$  and a positive semidefinite weight matrix  $W$  with  $\text{Tr}[W] = 1$ , the following multiparameter CRB holds:*

$$\text{Tr}[W \text{Cov}(\theta)] \geq \frac{1}{n \hat{I}_F^A(\theta, W; \{\mathcal{N}_{A \rightarrow B}^\theta\}_\theta)}. \quad (152)$$

**Proof.** The multiparameter CRB follows as a direct consequence of the chain rule (130), ensuing amortization collapse (95), as well as the converse theorem (7). ■

The bounds in (142) and (152) have an important implication: if the RLD Fisher information is finite, then Heisenberg scaling is impossible. That is, if  $\text{supp}(\partial_\theta \Gamma_{RB}^{\mathcal{N}^\theta}) \subseteq \text{supp}(\Gamma_{RB}^{\mathcal{N}^\theta})$  holds for the single-parameter case or if (50) holds for the multiparameter case, then all estimation strategies are limited by the shot-noise limit. A simple corollary of this is that for every full-rank quantum channel family, Heisenberg scaling is unattainable. Our no-go condition applies generally, i.e., for all quantum channels and in the sequential estimation setting. It builds upon and generalizes results on this problem from other works, which apply to the parallel estimation setting [13, 15], to sequential channel estimation with Markovian noise [20, 21, 24], and to the general sequential channel estimation setting [14].

## 6. Evaluating bounds for the generalized amplitude damping channel

Here, we apply the CRB (152) to the task of estimating the parameters of a GADC [60], a qubit-to-qubit channel that is used to model noise in, e.g. superconducting quantum circuits. A GADC is defined in terms of a loss parameter  $\gamma \in (0, 1)$  and a noise parameter  $N \in (0, 1)$ . The GADC has been studied previously in the context of quantum estimation theory. In reference [10], the SLD Fisher information matrix for a generalized Pauli channel was evaluated, and reference [11] concerns optimal strategies for the estimation of the loss parameter  $\gamma$  of a GADC.

The Choi operator of a GADC  $\mathcal{A}_{\gamma, N}$  with loss parameter  $\gamma$  and noise parameter  $N$  is given by

$$\Gamma_{RB}^{\mathcal{A}_{\gamma, N}} := \begin{bmatrix} 1 - \gamma N & 0 & 0 & \sqrt{1 - \gamma} \\ 0 & \gamma N & 0 & 0 \\ 0 & 0 & \gamma(1 - N) & 0 \\ \sqrt{1 - \gamma} & 0 & 0 & 1 - \gamma(1 - N) \end{bmatrix}. \quad (153)$$

Suppose that we wish to estimate both parameters  $\gamma$  and  $N$  simultaneously. Let us choose

$$W = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, \quad (154)$$

which satisfies the requirements for a valid weight matrix (positive semi-definite with unit trace). To compute the RLD bound in (152), we calculate

$$\begin{aligned} \hat{I}_F(\{\gamma, N\}, W; \{\mathcal{A}_{\gamma, N}\}_{\gamma, N}) &= \frac{1}{4} \left\| \text{Tr}_B \left[ (\partial_\gamma \Gamma^A)(\Gamma^A)^{-1}(\partial_\gamma \Gamma^A) + (\partial_\gamma \Gamma^A)(\Gamma^A)^{-1}(\partial_N \Gamma^A) \right. \right. \\ &\quad \left. \left. + (\partial_N \Gamma^A)(\Gamma^A)^{-1}(\partial_\gamma \Gamma^A) + 3(\partial_N \Gamma^A)(\Gamma^A)^{-1}(\partial_N \Gamma^A) \right] \right\|_\infty. \end{aligned} \quad (155)$$

where  $\mathcal{A}$  is used as shorthand for  $\mathcal{A}_{\gamma, N}$  and the system labels for  $RB$  are omitted. As a consequence of (152), the inverse of (155) is a lower bound on  $\text{Tr}[W \text{Cov}(\{\gamma, N\})]$ .

We have

$$\partial_\gamma \Gamma_{RB}^{\mathcal{A}_{\gamma,N}} = \begin{bmatrix} -N & 0 & 0 & -\frac{1}{2\sqrt{1-\gamma}} \\ 0 & N & 0 & 0 \\ 0 & 0 & 1-N & 0 \\ -\frac{1}{2\sqrt{1-\gamma}} & 0 & 0 & -(1-N) \end{bmatrix}, \text{ and} \quad (156)$$

$$\partial_N \Gamma_{RB}^{\mathcal{A}_{\gamma,N}} = -\gamma (I_2 \otimes \sigma_Z). \quad (157)$$

We then have

$$\text{Tr}_B[(\partial_\gamma \Gamma^{\mathcal{A}_{\gamma,N}})(\Gamma^{\mathcal{A}_{\gamma,N}})^{-1}(\partial_\gamma \Gamma^{\mathcal{A}_{\gamma,N}})] = \begin{bmatrix} \frac{\frac{1}{N-\gamma N} + \frac{1}{1-N} - 4}{4\gamma^2} & 0 \\ 0 & \frac{\frac{1}{(\gamma-1)(N-1)} + \frac{1}{N} - 4}{4\gamma^2} \end{bmatrix}, \quad (158)$$

$$\text{Tr}_B[(\partial_\gamma \Gamma^{\mathcal{A}_{\gamma,N}})(\Gamma^{\mathcal{A}_{\gamma,N}})^{-1}(\partial_N \Gamma^{\mathcal{A}_{\gamma,N}})] = \begin{bmatrix} -\frac{1-2N}{2\gamma N(1-N)} & 0 \\ 0 & -\frac{1-2N}{2\gamma N(1-N)} \end{bmatrix}, \quad (159)$$

$$\text{Tr}_B[(\partial_N \Gamma^{\mathcal{A}_{\gamma,N}})(\Gamma^{\mathcal{A}_{\gamma,N}})^{-1}(\partial_\gamma \Gamma^{\mathcal{A}_{\gamma,N}})] = \begin{bmatrix} -\frac{1-2N}{2\gamma N(1-N)} & 0 \\ 0 & -\frac{1-2N}{2\gamma N(1-N)} \end{bmatrix}, \quad (160)$$

$$\text{Tr}_B[(\partial_N \Gamma^{\mathcal{A}_{\gamma,N}})(\Gamma^{\mathcal{A}_{\gamma,N}})^{-1}(\partial_N \Gamma^{\mathcal{A}_{\gamma,N}})] = \begin{bmatrix} \frac{1}{N(1-N)} & 0 \\ 0 & \frac{1}{N(1-N)} \end{bmatrix}, \quad (161)$$

with which the expression in (155) is directly calculated. Furthermore, we verify our direct calculation by explicitly calculating the SDP in proposition 4 and find agreement between the two approaches with up to eight digits of precision.

We compare the RLD Fisher information bound to the generalized Helstrom CRB [42], which in this case reduces to the SLD Fisher information bound because it is being applied for the case of parametric, rather than semiparametric, estimation. This lower bound can be achieved asymptotically up to a constant prefactor [40, 61–64].

To calculate the SLD Fisher information, we choose input probe state  $|\psi(p)\rangle := \sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle$ , with  $W$  chosen as in the RLD calculation. It suffices to optimize the single-copy SLD Fisher information over such states due to the  $\sigma_Z$  covariance of the channel  $\mathcal{A}_{\gamma,N}$ . This SLD bound for estimating just the loss parameter  $\gamma$  of a GADC was calculated in reference [11].

With our particular choice of input state, we have

$$\rho_{\gamma,N} := \mathcal{A}_{\gamma,N}(\psi_p) = \begin{bmatrix} p(1-\gamma N) & 0 & 0 & \sqrt{p(1-p)(1-\gamma)} \\ 0 & p\gamma N & 0 & 0 \\ 0 & 0 & (1-p)\gamma(1-N) & 0 \\ \sqrt{p(1-p)(1-\gamma)} & 0 & 0 & (1-p)(1-(1-N)\gamma) \end{bmatrix}. \quad (162)$$

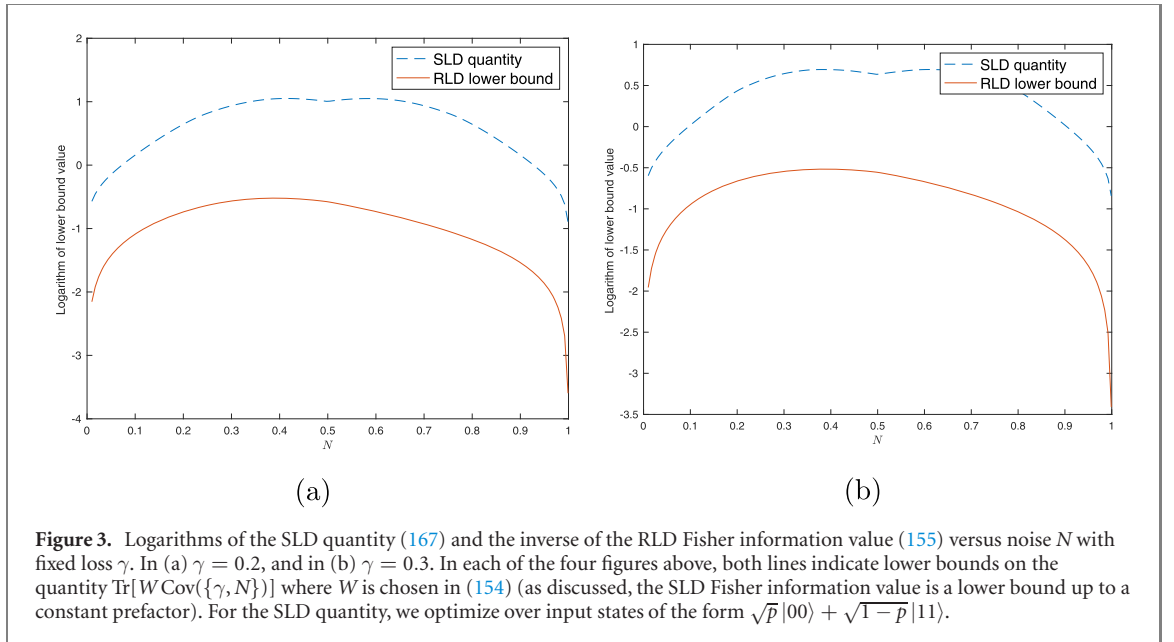
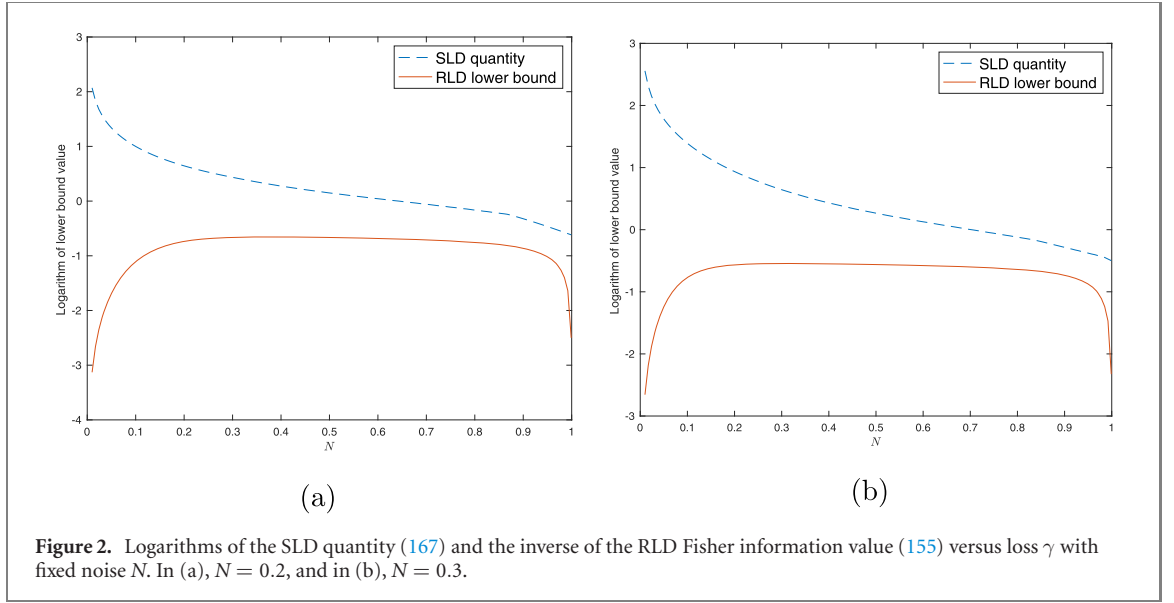
The SLD Fisher information takes the form of a  $2 \times 2$  matrix:

$$I_F(\gamma, N; \{\rho_{\gamma,N}\}_{\gamma,N})_{jk} = \text{Tr}[\rho_{\gamma,N} L_j L_k], \quad (163)$$

where  $j$  and  $k$  each take values 1 or 2 that correspond to either  $\gamma$  or  $N$ . We have  $L_j$  defined implicitly via  $\partial_j \rho_{\gamma,N} = \frac{1}{2}(\rho_{\gamma,N} L_j + L_j \rho_{\gamma,N})$ .

Consider that  $\rho_{\gamma,N}$  has spectral decomposition

$$\rho_{\gamma,N} = \sum_j \lambda_{\gamma,N}^j |\psi_{\gamma,N}^j\rangle \langle \psi_{\gamma,N}^j|. \quad (164)$$



We then have

$$L_\gamma = 2 \sum_{j,k: \lambda_{\gamma,N}^j + \lambda_{\gamma,N}^k > 0} \frac{\langle \psi_{\gamma,N}^j | (\partial_\gamma \rho_{\gamma,N}) | \psi_{\gamma,N}^k \rangle}{\lambda_{\gamma,N}^j + \lambda_{\gamma,N}^k} |\psi_{\gamma,N}^j\rangle \langle \psi_{\gamma,N}^k| \text{ and} \quad (165)$$

$$L_N = 2 \sum_{j,k: \lambda_{\gamma,N}^j + \lambda_{\gamma,N}^k > 0} \frac{\langle \psi_{\gamma,N}^j | (\partial_N \rho_{\gamma,N}) | \psi_{\gamma,N}^k \rangle}{\lambda_{\gamma,N}^j + \lambda_{\gamma,N}^k} |\psi_{\gamma,N}^j\rangle \langle \psi_{\gamma,N}^k|, \quad (166)$$

which enables us to calculate the elements of the SLD Fisher information matrix given in (163).

We optimize over the parameter  $p$  to obtain the SLD Fisher information bound. To be clear, we minimize

$$\text{Tr}[W[I_F(\{\gamma, N\}; \{\mathcal{A}_{\gamma,N}(\psi(p))\})_{\{\gamma, N\}}]^{-1}] \quad (167)$$

with respect to  $p$ . We compare the SLD quantity (167) to our RLD lower bound (155) in figures 2 and 3. In figure 2, we keep  $N$  fixed and vary  $\gamma$  from 0 to 1. In figure 3, instead we keep  $\gamma$  fixed and vary  $N$  from 0 to 1. We demonstrate that the RLD lower bound is within one to two orders of magnitude of the SLD Fisher information.

## 7. Summary and conclusion

In this paper, we provided a single-letter, efficiently computable CRB for the task of multiparameter estimation. We did so by introducing the amortized RLD Fisher information and proving an amortization collapse for it. The single-letter CRB also leads to a simple yet rigorous no-go condition for Heisenberg scaling in quantum multiparameter estimation. Finally, we evaluated our bound when estimating the two parameters of a GADC.

Lately, multiparameter estimation has been a fruitful area of study in quantum information theory. We believe that our results both offer a new perspective and extend the current body of work, and so they should have metrological applications theoretically and experimentally.

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## Data availability statement

No new data were created or analysed in this study.

## Appendix. Technical lemma

**Lemma 9** *Let  $K$  and  $Z$  be Hermitian operators, and let  $W$  be a linear operator. Then the dual of the following semi-definite program*

$$\inf_M \left\{ \text{Tr}[KM] : \begin{bmatrix} M & W^\dagger \\ W & Z \end{bmatrix} \geq 0 \right\}, \quad (168)$$

*with  $M$  Hermitian, is given by*

$$\sup_{P, Q, R} \left\{ 2 \text{Re}(\text{Tr}[W^\dagger Q]) - \text{Tr}[ZR] : P \leq K, \begin{bmatrix} P & Q^\dagger \\ Q & R \end{bmatrix} \geq 0 \right\}, \quad (169)$$

*where  $Q$  is a linear operator and  $P$  and  $R$  are Hermitian.*

**Proof.** The standard forms of a primal and dual semi-definite program, for  $A$  and  $B$  Hermitian and  $\Phi$  a Hermiticity-preserving map, are respectively as follows [57]:

$$\inf_{Y \geq 0} \{ \text{Tr}[BY] : \Phi^\dagger(Y) \geq A \}, \quad (170)$$

$$\sup_{X \geq 0} \{ \text{Tr}[AX] : \Phi(X) \leq B \}, \quad (171)$$

where  $\Phi^\dagger$  is the Hilbert–Schmidt adjoint of  $\Phi$ . Noting that

$$\begin{bmatrix} M & W^\dagger \\ W & Z \end{bmatrix} \geq 0 \iff \begin{bmatrix} M & -W^\dagger \\ -W & Z \end{bmatrix} \geq 0 \iff \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \geq \begin{bmatrix} 0 & W^\dagger \\ W & -Z \end{bmatrix}, \quad (172)$$

we conclude the statement of the lemma after making the following identifications:

$$B = K, \quad Y = M, \quad \Phi^\dagger(M) = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}, \quad (173)$$

$$A = \begin{bmatrix} 0 & W^\dagger \\ W & -Z \end{bmatrix}, \quad X = \begin{bmatrix} P & Q^\dagger \\ Q & R \end{bmatrix}, \quad \Phi(X) = P. \quad (174)$$

This concludes the proof. ■

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## References

- [1] Giovannetti V, Lloyd S and Maccone L 2006 Quantum metrology *Phys. Rev. Lett.* **96** 010401
- [2] Cramér H 1946 *Mathematical Methods of Statistics* (Princeton, NJ: Princeton University Press)
- [3] Radakrishna Rao C 1945 Information and the accuracy attainable in the estimation of statistical parameters *Bull. Calcutta Math. Soc.* **37** 81–9
- [4] Kay S M 1993 *Fundamentals of Statistical Signal Processing (Estimation Theory vol 1)* (Englewood Cliffs, NJ: Prentice-Hall)
- [5] Yuen H and Lax M 1973 Multiple-parameter quantum estimation and measurement of nonselfadjoint observables *IEEE Trans. Inf. Theory* **19** 740–50
- [6] Li Y, Pezzè L, Gessner M, Ren Z, Li W and Smerzi A 2018 Frequentist and Bayesian quantum phase estimation *Entropy* **20** 628
- [7] Sidhu J S and Kok P 2020 Geometric perspective on quantum parameter estimation *AVS Quantum Sci.* **2** 014701
- [8] Dowling J P and Seshadreesan K P 2015 Quantum optical technologies for metrology, sensing, and imaging *J. Lightwave Technol.* **33** 2359–70
- [9] Sasaki M, Ban M and Barnett S M 2002 Optimal parameter estimation of a depolarizing channel *Phys. Rev. A* **66** 022308
- [10] Fujiwara A and Imai H 2003 Quantum parameter estimation of a generalized Pauli channel *J. Phys. A: Math. Gen.* **36** 8093–103
- [11] Fujiwara A 2004 Estimation of a generalized amplitude-damping channel *Phys. Rev. A* **70** 012317
- [12] Ji Z, Wang G, Duan R, Feng Y and Ying M 2008 Parameter estimation of quantum channels *IEEE Trans. Inf. Theory* **54** 5172–85
- [13] Fujiwara A and Imai H 2008 A fibre bundle over manifolds of quantum channels and its application to quantum statistics *J. Phys. A: Math. Theor.* **41** 255304
- [14] Matsumoto K 2010 On metric of quantum channel spaces (arXiv:1006.0300)
- [15] Hayashi M 2011 Comparison between the Cramér–Rao and the mini-max approaches in quantum channel estimation *Commun. Math. Phys.* **304** 689–709
- [16] Demkowicz-Dobrzanski R, Kolodyński J and Guta M 2012 The elusive Heisenberg limit in quantum enhanced metrology *Nat. Commun.* **3** 1063
- [17] Kolodyński J and Demkowicz-Dobrzański R 2013 Efficient tools for quantum metrology with uncorrelated noise *New J. Phys.* **15** 073043
- [18] Demkowicz-Dobrzanski R and Maccone L 2014 Using entanglement against noise in quantum metrology *Phys. Rev. Lett.* **113** 250801
- [19] Sekatski P, Skotiniotis M, Kolodyński J and Dür W 2017 Quantum metrology with full and fast quantum control *Quantum* **1** 27
- [20] Demkowicz-Dobrzanski R, Jan C and Pavel Sekatski 2017 Adaptive quantum metrology under general Markovian noise *Phys. Rev. X* **7** 041009
- [21] Zhou S, Zhang M, Preskill J and Jiang L 2018 Achieving the Heisenberg limit in quantum metrology using quantum error correction *Nat. Commun.* **9** 78
- [22] Zhou S and Jiang L 2020 Optimal approximate quantum error correction for quantum metrology *Phys. Rev. Res.* **2** 013235
- [23] Yang Y, Chiribella G and Hayashi M 2020 Communication cost of quantum channels *IEEE Journal on Selected Areas in Information Theory* **1** 387–400
- [24] Zhou S and Jiang L 2021 Asymptotic theory of quantum channel estimation *PRX Quantum* **2** 010343
- [25] van Dam W, Mauro D’Ariano G, Ekert A, Macchiavello C and Mosca M 2007 Optimal quantum circuits for general phase estimation *Phys. Rev. Lett.* **98** 090501
- [26] Yuan H 2016 Sequential feedback scheme outperforms the parallel scheme for Hamiltonian parameter estimation *Phys. Rev. Lett.* **117** 160801
- [27] Yuan H and Fung C-H F 2017 Fidelity and Fisher information on quantum channels *New J. Phys.* **19** 113039
- [28] Katariya V and Wilde M M 2021 Geometric distinguishability measures limit quantum channel estimation and discrimination *Quantum Inf. Process.* **20** 78
- [29] Helstrom C W 1967 Minimum mean-squared error of estimates in quantum statistics *Phys. Lett. A* **25** 101–2
- [30] Holevo A S 1972 An analogue of statistical decision theory and noncommutative probability theory *Trudy Moskovskogo Matematicheskogo Obshchestva* **26** 133–49
- [31] Belavkin V P 1976 Generalized uncertainty relations and efficient measurements in quantum systems *Theor. Math. Phys.* **26** 213–22
- [32] Bagan E, Ballester M A, Gill R D, Monras A and Muñoz-Tapia R 2006 Optimal full estimation of qubit mixed states *Phys. Rev. A* **73** 032301
- [33] Holevo A S 2011 *Probabilistic and Statistical Aspects of Quantum Theory* vol 1 (Berlin: Springer)
- [34] Monras A and Illuminati F 2011 Measurement of damping and temperature: precision bounds in Gaussian dissipative channels *Phys. Rev. A* **83** 012315
- [35] Humphreys P C, Barbieri M, Datta A and Walmsley I A 2013 Quantum enhanced multiple phase estimation *Phys. Rev. Lett.* **111** 070403
- [36] Yue J-D, Zhang Y-R and Fan H 2015 Quantum-enhanced metrology for multiple phase estimation with noise *Sci. Rep.* **4** 5933
- [37] Ragy S, Jarzyna M and Demkowicz-Dobrzanski R 2016 Compatibility in multiparameter quantum metrology *Phys. Rev. A* **94** 052108
- [38] Sidhu J S, Ouyang Y, Campbell E T and Kok P 2021 Tight bounds on the simultaneous estimation of incompatible parameters *Phys. Rev. X* **11** 011028
- [39] Albarelli F, Friel J F and Datta A 2019 Evaluating the Holevo Cramér–Rao bound for multiparameter quantum metrology *Phys. Rev. Lett.* **123** 200503
- [40] Tsang M 2019 The Holevo Cramér–Rao bound is at most thrice the Helstrom version (arXiv:1911.08359)
- [41] Yang J, Pang S, Zhou Y and Jordan A N 2019 Optimal measurements for quantum multiparameter estimation with general states *Phys. Rev. A* **100** 032104

- [42] Albarelli F, Tsang M and Datta A 2020 Upper bounds on the Holevo Cramer–Rao bound for multiparameter quantum parametric and semiparametric estimation *Phys. Rev. X* **10** 031023
- [43] Demkowicz-Dobrzanski R, Gorecki W and Guta M 2020 Multi-parameter estimation beyond quantum Fisher information *J. Phys. A: Math. Theor.* **53** 363001
- [44] Friel J, Palittapongarnpim P, Albarelli F and Datta A 2020 Attainability of the Holevo–Cramer–Rao bound for two-qubit 3D magnetometry (arXiv:2008.01502)
- [45] Gorecki W, Zhou S, Jiang L and Demkowicz-Dobrzanski R 2020 Optimal probes and error-correction schemes in multi-parameter quantum metrology *Quantum* **4** 288
- [46] Szczykulska M, Baumgratz T and Datta A 2016 Multi-parameter quantum metrology *Adv. Phys. X* **1** 621–39
- [47] Albarelli F, Barbieri M, Genoni M G and Gianani I 2020 A perspective on multiparameter quantum metrology: from theoretical tools to applications in quantum imaging *Phys. Lett. A* **384** 126311
- [48] Petz D and Ghinea C 2011 Introduction to quantum Fisher information *Quantum Probab. Relat. Top.* **261**–81
- [49] Matsumoto K 2013 A new quantum version of  $f$ -divergence (arXiv:1311.4722)
- [50] Heyde C C 1975 Remarks on efficiency in estimation for branching processes *Biometrika* **62** 49–55
- [51] Koenig R 2015 The conditional entropy power inequality for Gaussian quantum states *J. Math. Phys.* **56** 022201
- [52] De Palma G and Trevisan D 2018 The conditional entropy power inequality for Bosonic quantum systems *Commun. Math. Phys.* **360** 639–62
- [53] Helstrom C W 1976 *Quantum Detection and Estimation Theory* (New York: Academic)
- [54] Helstrom C W 1973 Cramer–Rao inequalities for operator-valued measures in quantum mechanics *Int. J. Theor. Phys.* **8** 361–76
- [55] Hansen F and Pedersen G K 2003 Jensen’s operator inequality *Bull. Lond. Math. Soc.* **35** 553–64
- [56] Suzuki J 2016 Explicit formula for the Holevo bound for two-parameter qubit-state estimation problem *J. Math. Phys.* **57** 042201
- [57] Watrous J 2018 *The Theory of Quantum Information* (Cambridge: Cambridge University Press)
- [58] Wilde M M, Berta M, Hirche C and Kaur E 2020 Amortized channel divergence for asymptotic quantum channel discrimination *Lett. Math. Phys.* **110** 2277–336
- [59] Fang K and Fawzi H 2021 Geometric Rényi divergence and its applications in quantum channel capacities *Commun. Math. Phys.* **384** 1615
- [60] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
- [61] Guta M and Kahn J 2006 Local asymptotic normality for qubit states *Phys. Rev. A* **73** 052108
- [62] Hayashi M and Matsumoto K 2008 Asymptotic performance of optimal state estimation in qubit system *J. Math. Phys.* **49** 102101
- [63] Yamagata K, Fujiwara A and Gill R D 2013 Quantum local asymptotic normality based on a new quantum likelihood ratio *Ann. Stat.* **41** 2197–217
- [64] Yang Y, Chiribella G and Hayashi M 2019 Attaining the ultimate precision limit in quantum state estimation *Commun. Math. Phys.* **368** 223–93