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Jane L. d Runkle
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Analytical model of the progression of cracking in fiber-reinforced composites

Runkle, Jane L. D., Ph.D.
The Louisiana State University and Agricultural and Mechanical Col., 1993

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ANALYTICAL MODEL OF THE PROGRESSION
OF CRACKING IN FIBER-REINFORCED COMPOSITES

A Dissertation
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Louisiana State University and
Agricultural and Mechanical College
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Doctor of Philosophy

in

The Interdepartmental Programs in Engineering

by

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DEDICATION

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ABSTRACT

Two analytical shear lag models have been developed for the progressive damage and final failure of epoxy matrix, fiber-reinforced composites under biaxial loading. The first is a three layer model for laminates of the type \([\pm \theta, 90n]_s\) under general biaxial loading. It gives matrix cracking predictions for the central 90° ply group. The second model is a five-layer model for laminates of the type \([90n/0m/90p]_s\) under biaxial loading, but it does not include in-plane shear. The five-layer model predicts matrix cracking in all ply groups. The amount of damage in terms of modulus decrease and number of matrix cracks is determined for each layer under increasing static loads. This is done by assuming that a crack occurs, calculating the energy dissipated due to the crack formation, dividing by the critical crack size, and comparing the result with the critical strain energy release rate. When a layer cracks, the other layers must take additional load. Final failure occurs when the primary load carrying plies reach their ultimate strength. The models incorporate an algorithm for the effect of small, initial local delaminations on matrix cracking. The three-layer model has been verified experimentally using literature data and in-house experimental data. The five-layer model has also been experimentally verified in this work. Experimental verification was performed by statically loading uniaxial glass-epoxy tension specimens and measuring damage accumulation in terms of crack density and the decrease of Young's modulus. The present study shows that the models developed can be used for initial predictions of damage and failure of epoxy matrix, fiber-reinforced composites.
CHAPTER 1
INTRODUCTION

Fiber-reinforced composite materials have found widespread use in the aerospace and automotive industries, among others. They are used in such applications as solid rocket motor cases, aircraft fuselages and wing structures, boats, and automotive leaf springs. In none of these structures are they under uniaxial tension loading only. Since composites are generally used in the form of plates or shells, they typically see biaxial loading. Unfortunately, little analytical modeling has been performed for failure under biaxial loading.

The failure of epoxy-matrix, fiber-reinforced composite materials is usually preceded by a substantial amount of damage. The first event is transverse matrix cracking in the 90° or low angle plies. As the transverse matrix cracks reach the interface between the 90° layer and the neighboring 0° or low angle layer, small delaminations tend to form. The thickness of the 90° layer determines whether delamination occurs. At first, these delaminations grow only a very small amount. Matrix cracks continue to form in the 90° ply layer until the "characteristic damage state" is reached. At this point, substantial delamination may occur. While delamination is occurring, the load carrying plies reach their ultimate strength and failure ensues.

Modeling the damage in composites enables industry to make more accurate decisions regarding the margins of safety of damaged hardware. Predictions can be made of the change in properties as a result of damage, and of the loading states which lead to unacceptable damage. For example, if a pressure vessel is overpressurized, the stress in the vessel can be determined. An estimation of the amount of damage and the probable degradation of properties can be made using models such as those proposed in this work.
While a number of models have been developed for predicting progressive damage in composites, very few are written for biaxial loading. None are written for transverse matrix cracking in composites with more than three layers. Finally, few make any attempt to incorporate the effects of delamination.

The models developed in this work incorporate transverse matrix cracking of fiber-reinforced composites under biaxial tension. The three-layer model is for composites of the type \([\pm \theta/90_n]_s\) under general biaxial tension. Transverse matrix cracking is modeled in the 90° plies and an algorithm for the effect of local delamination is included. In the five-layer model, laminates of the type \([90_n/0_m/90_p]_s\) are studied. Transverse matrix cracking is modeled in all five layers. In-plane biaxial loading, except for in-plane shear, is modeled. In addition, the delamination algorithm is included.

Both models predict the stress and strain at the onset of matrix cracking, the progression of cracking, the effect of local delamination on transverse matrix cracking, and the change in Young’s modulus as damage occurs. They are compared with experimental data.
Microstructural Failure Mechanisms

In order to truly understand fracture of composite materials, one must begin by looking at the microstructural aspects. This section examines matrix microcracking, fiber breakage, matrix-fiber debonding, fiber pull-out, and the microstructural aspects of delamination. The microstructure of composites greatly affects their macroscopic behavior. The observations summarized in this section have led to the development of damage propagation models.

The Pseudoplastic Zone

For stable crack growth, the crack critical energy release rate, $G_c$, is equal to the fracture toughness or resistance to crack growth, $R$. $G_c$ is defined as:

$$ G_c = \frac{1}{2} P_c \left( \frac{dC}{da} \right) $$  \hspace{1cm} (2.1)

where $P_c$ is a critical force, $C$ is the compliance and $da$ is the crack extension [1]. Visconti [2] used $G_c$, the mode I elastic energy release rate, to describe composite fracture. Since the composites examined here are brittle, linear elastic fracture mechanics provides a reasonable measure of fracture toughness. Visconti used a stress intensity factor given by:

$$ K_1 = \frac{\sigma}{2a} $$  \hspace{1cm} (2.2)

where $2a$ is the length of the crack perpendicular to the direction of stress, $\sigma$. The propagation of
the crack initiates when $K_1$ reaches a critical value, $K_{IC}$, which depends on a critical stress, $\sigma_C$.

Visconti proposes that the ability to withstand load in the presence of notches is due to the presence of a "pseudoplastic" zone ahead of the crack tip. In this zone, microcracking and microdelamination (really debonding) dissipate energy. His model is shown in Figure 2.1. Once the zone grows to a certain size, given by $a_0$, cracking proceeds. Thus, $K_{IC}$ can be defined for the case of no initial defects, or the case of initial defects:

$$K_{IC} = \sigma_C \sqrt{(a + a_0)\pi} \quad \text{or} \quad K_{IC} = \sigma^* \sqrt{a_0\pi}$$  \hspace{1cm} (2.3)

where $\sigma^*$ is the strength of the unnotched specimen, $\sigma_C$ is critical stress, and $a_0$ refers to a critical size of unavoidable defects present in the material. Finally, Visconti solves these equations for $a_0$ and states that $a_0$ is a characteristic constant of the material.

$$a_0 = \frac{a}{\left(\frac{\sigma_C}{\sigma^*}\right)^2 - 1}$$  \hspace{1cm} (2.4)

Figure 2.1. Model of the Mechanism Involved in Crack Propagation in a Composite [2]
For maximum fracture toughness, one must maximize the resistance to crack growth, \( R \) or \( G_c \), since they are equal. Visconti realized that contributions to \( G_c \) arise from a number of physical processes, including matrix microcracking, fiber breakage, debonding, and fiber pull-out [1,2].

**Matrix**

The glass transition temperatures of epoxy matrices range from \( 300^\circ \text{F} \) to \( 470^\circ \text{F} \), which means that they are glassy at their use temperatures [3]. In addition, since they are highly crosslinked, they cannot develop any significant degree of crystallinity [2]. They are, therefore, brittle and linearly elastic. It is well known that glassy polymer fracture begins with the formation of low-density regions called crazes [4]. Crazes may initiate at interfaces or voids. They may be caused by post-cure shrinkage [5], straining, or development of pressure pockets due to moisture entering internal voids [3]. They can also be caused by chemical attack. Epoxy matrices, however, are generally quite resistant to chemicals and exhibit low shrinkage during curing [6]. Crazes absorb a certain amount of fracture energy, which helps the composite to fail incrementally rather than catastrophically. On the other hand, crazing reduces the strength of the matrix as shown by Pavan [7], who found the tensile craze yield strength, \( \sigma_{yc} \), of a glassy polymer matrix/particulate composite to be:

\[
\sigma_{yc} = C \left( 2 \left[ 1 + \sqrt{1 + \frac{4D}{C^2}} \right] \right)^{-1}
\]  

(2.5)

Note that \( \sigma_{yc} \) depends on the material properties of the matrix, given by \( C \) and \( D \), so this equation is true regardless of composite type. It also means that matrix cracking is governed by matrix strength, and that matrix properties strongly affect axial and transverse cracking and delamination resistance. In addition, as will be seen later, matrix microcracking usually precedes the other failure mechanisms. Before discussing the other failure modes, one must look at the fibers.
Fibers

Fibers, like graphite, are highly crystalline and oriented, giving them much higher strength and stiffness than epoxy matrices [4]. This is especially true at high rates of strain. Due to their high stiffness, fibers typically strain less than do matrices, especially at high strain rates. In addition, fiber failure is generally defined at the fracture strain. Some fibers, however, like Kevlar (aramid), actually have a higher creep rate than epoxy at low stresses [5], which may contribute to Kevlar-epoxy's relatively high fracture toughness. Graphite and glass fibers do not fall into this category. Individual fibers may break at less than 50% of the ultimate tensile load of the composite [8]. This is a random process due to randomly distributed defects in the fibers. Some researchers have modeled fiber failure due to such defects using a Weibull distribution. This idea is explained more fully in Chapter Two.

Agarwal and Broutman [8] discuss a rule of mixtures approach to define the energy required per unit area of the composite for fracture of fibers in tension, $W_b$, as well as energy release rate caused by fiber breakage during fracture, $G$:

$$W_b = G = \frac{V_f \sigma_f^2 l}{6E_f}$$

(2.6)

where $V_f$ is the volume fraction of fibers, $\sigma_f$ is the fiber ultimate strength, $l$ is the fiber length, and $E_f$ is the fiber Young's modulus. When a fiber fails, other fibers must take up the load, leading to additional fiber failures [6]. This may be an incremental process; however, if it occurs abruptly, it may result in catastrophic failure.

Fiber-Matrix Debonding

Matrix microcracking generally does not lead to catastrophic failure, but rather, dissipates energy. Matrix microcracks are, however, nuclei for further damage. When matrix microcracks
reach fibers, debonding between the fiber and the matrix usually occurs. The purpose of the matrix is to transfer load to the fibers and it does this by a shear mechanism. Since the fibers are stiffer than the matrix, they prevent the matrix from elongating near the fiber. This results in a local strain at the fiber, which is higher than that in the bulk of the matrix [9]. If the corresponding local stress is greater than the local interfacial strength, debonding will occur. Although the matrix-fiber interface is often assumed to be perfect in analytical models, it actually has rough surfaces with corners, which act as stress concentrators. Thus the local stress is even higher than is usually predicted, resulting in premature debonding.

Cracking in a fiber-matrix composite can be modeled using a bimaterial plate, in which one phase is more brittle than the other. The crack propagates in the more ductile portion at, we assume, some constant velocity. As it approaches the second phase, it slows down and stops at the interface in what Theocaris [9] calls the crack-arrest phenomena. The crack then propagates along the interface until the strain energy necessary for it to propagate in phase two is reached. Now there are two independently propagating cracks. While this is a very interesting model, Theocaris, unfortunately, does not develop an equation for the energy of debonding. Kelly [1] does develop such a relation by equating the work of debonding with the strain energy appearing in the filament as a result of debonding. He finds that the work or energy of debonding is:

$$W_d = \frac{2\pi r^2 \sigma_f x}{6E_f}$$  \hspace{1cm} (2.7)

where $r$ is the fiber radius, $E_f$ is the fiber modulus, $\sigma_f$ is the fiber breaking strength, and $x$ is the length of the debond. Debonding contributes less to the fracture toughness than does fiber pull-out, as shall be shown.
Fiber Pull-Out

Fiber pull-out generally occurs after a fiber has broken near, but not in the plane of, a matrix crack. The fiber will debond from the matrix and, since it can no longer carry a load, will pull away from the rest of the fiber, leaving a fiber-shaped hole in the matrix. This phenomenon is shown in Figure 2.2, where the central fiber has broken at a distance, L, from the crack face. Fiber pull-out increases fracture toughness and is enabled by a low interfacial shear strength. Pull-out is most likely to occur when the fiber length is less than the critical fiber length, which is the fiber length needed for the in situ fiber stress to reach its maximum value at the fiber midpoint. However, Piggott [10] has shown that fiber pull-out can occur in continuous fiber composites provided that it is preceded by fiber breakage and/or debonding between fiber and matrix. In a continuous fiber composite, it is usually fiber breakage that precedes pull-out, since it is unlikely that debonding will occur along the entire length of a fiber without fiber breakage occurring, as has been explained by Theocaris [9]. In addition, the fiber will probably break at the point of an inherent flaw.

Figure 2.2. Fibers Bridging a Crack [10]
Kelly [1] defines the critical spacing of flaws as $y_c$.

$$y_c = \left( \frac{\sigma_f - \sigma_f^*}{r} \right) \frac{\Delta \sigma_f}{\tau_i} = \frac{1}{e} \frac{\Delta \sigma_f}{\sigma_f}$$

(2.8)

Here, $\sigma_f$ is the fiber strength, $\sigma_f^*$ is the strength at the flaws, $\Delta \sigma_f$ is $\sigma_f - \sigma_f^*$, $r$ is the fiber radius, $\tau_i$ is the interfacial shear strength, and $l_c$ is the critical fiber length, such that the in situ fiber strength reaches $\sigma_f$ in the middle. If the actual spacing of flaws is $y$ where $y < y_c$, then all fibers will break at flaws lying within a distance $\frac{1}{2} y_c$ from the crack. The average pull-out work per fiber is then:

$$W = \frac{\pi \tau_i y^2}{12}$$

(2.9)

or for the composite per unit area,

$$W = \frac{V \tau_i y^2}{12 r}$$

(2.10)

If, on the other hand, $y > y_c$, a fraction of fibers equal to $\frac{y_c}{y}$ will break at flaws and pull out, while a fraction of fibers equal to $1 - \frac{y_c}{y}$ will not pull out but will break in the plane of the matrix crack. Then the work of fracture for the composite is:

$$W = \frac{V \tau_i y^3}{12 r y_c}$$

(2.11)

By setting $y = y_c$, the dependence of $W$ on $\tau_i$ and $\sigma_f$ can be seen.
If \( y = y_c \),

\[
W = \frac{V \tau \gamma_y^2}{12r} = \frac{V \tau_i \left( \frac{1}{c} \frac{\Delta \sigma}{\sigma_f} \right)^2}{12r} \tag{2.12}
\]

Now,

\[
\frac{\sigma_f \tau}{\tau_i} = c \tag{2.13}
\]

so,

\[
W = \frac{V \tau_i \Delta \sigma_f^2}{12 \tau_i} \tag{2.14}
\]

Since \( \Delta \sigma_f \) is proportional to \( \sigma_f \), \( W \) increases with increasing \( \sigma_f \) and decreasing \( \tau_i \). In addition, to increase fracture toughness, the fiber/matrix interfacial shear strength should be reduced relative to the fiber longitudinal tensile strength. In fact, Piggott [10] has shown experimentally that, when the residual interfacial cure shrinkage stress is decreased, the work of fracture increases. In addition, work of fracture decreases as fiber modulus increases. The fiber stress is small near the break and this loss of stress upon fracture causes the fiber to shrink longitudinally and expand radially. Thus, the fiber wants to pull out and does work during pull-out. If the fiber is brittle, it will exhibit less shrinkage and less expansion, lowering the work of fracture. Piggott also points out that, in order for fiber pull-out to occur, the crack must be very large. This means that even though fiber pull-out is a useful mechanism for increasing fracture toughness, the crack required is often so
large that the material is already in danger of certain failure. Fiber pull-out simply works to slow the failure process.

It can now be shown that the contribution of pull-out energy to fracture toughness is greater than that of debonding. Combining equations 2.8 and 2.10, we see that for a single fiber, the ratio of the pull-out work to the work of debonding, $W_p/W_d$, is proportional to $E/\sigma_f$, which is equal to the reciprocal of the maximum breaking strain. As stated earlier, fibers are very stiff, so the maximum breaking strain is always very small, on the order of a few percent. Therefore, Kelly states that $E/\sigma_f$ is never less than 50 [1]. Hence, $W_p$, the energy associated with fiber pull-out, is substantially greater than $W_d$, the energy associated with debonding. In addition, since a small interfacial bond strength is required for both debonding and pull-out, a small interfacial shear strength will produce debonding followed by fiber pull-out. Therefore, it is worthwhile to look more closely at the interfacial bond strength.

### The Interfacial Bond

The relationship between interfacial bond strength and fracture toughness can be illustrated by comparing graphite-epoxy and aramid-epoxy laminates. The interfacial bond strength in graphite-epoxy laminates is twice as high as the bond strength in aramid-epoxy composites, and this is likely the main reason that aramid-epoxy composites have a higher fracture toughness than graphite-epoxy materials. Penn, et al., [11] investigated the reasons for this phenomena by performing single filament pull-out tests. They suggest three possible reasons for the difference, namely, intermolecular interactions, chemical bonding, and mechanical interference.

Intermolecular interactions and chemical bonding were quickly ruled out, since graphite and aramid have similar forces acting at their surfaces, and they have nearly identical surface functionalities. The difference in modulus was also considered and was ruled out. Penn, et al., [11] found that the most likely cause for the difference in fracture toughness is the radial compression or
tension exerted by the matrix on the fiber because of thermal mismatch between matrix and fiber during cool-down after cure.

The coefficients of thermal expansion for fibers increase as temperature is raised, thus, fibers shrink in length and expand in diameter as the temperature is raised, and the opposite occurs upon cooling. Since graphite's transverse thermal expansion coefficient is less than that of a typical epoxy, the graphite fiber will not shrink upon cooling as much as the annulus of matrix around it will decrease. Therefore, a high compressive stress will be exerted on the fiber by the matrix. Aramid's diameter decreases more upon cooling than does the diameter of graphite, giving a looser fit between matrix and fiber for Aramid. The difference in fit leads to a difference in interfacial shear strength. In fact, the interfacial shear strength between graphite fibers and epoxy is about three times greater than the shear strength between Aramid fibers and epoxy.

**Delamination**

Delamination is a result of failure of the matrix and of the fiber-matrix interface. Saghizadeh and Dharan [13] have made some interesting observations regarding delamination and the relative contributions of matrix cracking and fiber-matrix debonding to delamination fracture toughness. Since the delamination crack must work its way around fibers as it propagates, the local fracture mode is a mixture of Mode I and Mode III, even if the macroscopic fracture mode is Mode I delamination. This is shown in Figure 2.3. They found that delamination fracture toughness depends strongly on the fiber volume fraction; therefore, interfacial fracture toughness is more important than neat resin fracture toughness. Since fiber-matrix debonding, fiber pull-out, and fiber breakage energies also depend on fiber volume fraction and matrix failure does not, one may conclude that debonding, pull-out, and fiber breakage are greater contributors to delamination fracture toughness than is matrix fracture. Specifically, Saghizadeh and Dharan found that for graphite composites, the crack energy release rate decreases with fiber volume fraction. This is because increased fiber volume fraction increases the interfacial surface area, which lowers the crack
energy release rate, in effect, toughening the composite. Therefore, a fiber-reinforced epoxy laminate with a high fiber volume fraction should have a high resistance to delamination.

Figure 2.3. Crack Path Around Fibers During Delamination [13]

Relationship between Microstructure and Fracture Toughness

We have seen that fracture toughness depends on matrix, fiber, and interface properties. In addition, it is evident that fiber pull-out is the greatest contributor to fracture toughness, but it cannot occur without being preceded by matrix microcracking, matrix-fiber debonding and fiber breakage. Often, as is shown in Figure 2.2, fibers will bridge a crack. Fiber bridging is another factor in preventing catastrophic failure, but it leads to complications in analyzing composite fracture. All of the mechanisms discussed are important to the development of fracture toughness in continuous fiber composites.

The fiber-matrix interface has been examined to see the effects of intermolecular and chemical bonds, as well as mechanical interactions between fibers and matrix [11,12]. For graphite- and aramid-epoxy composites, thermal mismatch between fibers and matrix is probably the most important factor in determining the strength of the interfacial bond. Penn, et al. [11] and Piggott [10] looked at aspects of this post-cure phenomenon. A tight fit increases interfacial shear strength and decreases fracture toughness, while a loose fit decreases shear strength and increases fracture
toughness. This helps explain the fact that aramid-epoxy has a higher fracture toughness than graphite-epoxy.

The ideas developed herein show relationships between various microscopic material characteristics and fracture toughness. Fracture toughness depends strongly on fiber pull-out energy, which in turn depends on the interfacial bond strength. Interfacial bond strengths have not been tabulated for many systems, making micromechanical models somewhat impractical at present. Although the models to be described in the next section are not micromechanical in nature, the behaviors they phenomenologically describe are micromechanical.

**Progressive Failure Models**

Progressive failure of a fiber reinforced composite usually begins with matrix microcracking in off-angle plies. In unidirectional laminates loaded parallel to the fiber direction, failure initiates by longitudinal splitting, followed rapidly by fiber failure. In this section, the focus is on laminates with angle plies. As a result of the matrix cracking in transverse or near transverse plies, there is a greater mismatch in local average strains between adjacent plies; therefore, delamination occurs. Final failure is usually preceded by or coincides with fiber failure in the 0° or nearly 0° plies. A critical literature survey is given of theories for transverse matrix cracking, edge and local delamination, and fiber failure.

**Summary of Methods Used to Study Composite Fracture**

Kanninen and Popelar [14] do an excellent job of summarizing composite fracture mechanics research and laying groundrules for such work. Two major types of analysis are those that take a continuum approach and those that use micromechanics. In addition, some researchers have used models that combine the two, using a continuum approach where possible, and integrating it with a micromechanics approach where it is needed to accurately describe the material behavior. Complications arise in any method, because crack growth is not likely to always be self-similar and
where $K_{IC}$ depends on the crack path in composites. Obviously, the simple rule of mixtures is not adequate to describe composite fracture. Anisotropic fracture mechanics is a better macromechanical approach, and the equations derived in detail by Sih and Liebowitz [15] are given below:

\begin{align}
G_1 &= \pi k_1^2 \frac{a_{11} a_{22}}{2} \left[ a_{22}^2 + \frac{2a_{12} + a_{66}}{2a_{11}} \right]^{1/2} \\
G_\| &= \pi k_\| \sqrt{2} \left[ a_{22}^2 + \frac{2a_{12} + a_{66}}{2a_{11}} \right]^{1/2} \\
G_{III} &= \pi k_{III} \frac{\sqrt{a_{44} a_{55}}}{2}
\end{align}

where the $a_{ij}$ are elements of the compliance matrix, and the $k_i$ are given by:

\begin{align}
k_1 &= p \sqrt{a} \\
k_\| &= q \sqrt{a} \\
k_{III} &= s \sqrt{a}
\end{align}

where $p$ is a constant surface pressure on the crack, $q$ is a uniform in-plane shear stress acting on the crack, and $s$ is a uniform antiplane shear stress acting on the crack. These expressions become straightforward for the case of a uniform tensile stress $\sigma$ acting at an angle $\alpha$ to the crack plane in an infinite sheet, where the crack is assumed to be aligned with the planes of material symmetry. In
that case,

\[
G_I = K_{Ic}^2 \frac{a}{11 \frac{a}{22} + \frac{2a + a}{66}} \left( \frac{a}{11} + \frac{2a + a}{66} \right)^\frac{1}{2}
\]

\[
G_{II} = K_{IIc}^2 \frac{a}{11 \frac{a}{22} + \frac{2a + a}{66}} \left( \frac{a}{11} + \frac{2a + a}{66} \right)^\frac{1}{2}
\]

where \( K_I = \sigma \sqrt{\pi a} \sin^2 \alpha \), \( K_{II} = \sigma \sqrt{\pi a} \sin \alpha \cos \alpha \), and \( 2a \) is the crack length [14, 15].

In early fracture studies, it was noted that the reduction in strength for tension specimens containing holes or cracks depended on the size of the hole or crack, and that this effect could not be explained by a classical stress concentration factor approach. Waddoups, et al. [16], hypothesized that an intense energy region near the crack tip or hole, which could be modeled by linear elastic fracture mechanics, would account for the hole size effect. Their model is semi-empirical with \( K_{IC} \) and \( l \), a dimension of a characteristic intense energy region at the crack tip, to be found from experiment. Whitney and Nuismer [17], on the other hand, explained the hole size effect by examining the normal stress distribution ahead of the hole. They found that a sharper stress gradient exists near a smaller hole; thus, a critical defect is more likely to occur in the high stress region around a large hole. This point stress criterion has been generalized to laminates with cracks, but seems to be a poor predictor of cracking behavior [8].

Poe [18] took Whitney and Nuismer's [17] approach one step further by proposing a new fracture toughness parameter, which is independent of laminate layup. This parameter is proportional to the ultimate tensile strain of the fibers and is valid only for through-wall crack growth, not delamination or splitting. Poe's results are better in some cases and worse in other cases than Whitney and Nuismer's results. Potter examines failure only in terms of fiber failure. As fibers fail in the loading direction for uniaxial tension, a damage zone develops. Potter [19] looks at the notch effect in terms of material heterogeneity, showing that it is governed by the properties of
fiber, matrix, and the fiber/matrix interface. He notes that the sequential fiber failure process is the primary factor in the tensile failure of notched composites. The model is useful because it differentiates between a large, blunt notch, which causes brittle failure governed by his initiation criterion and a small defect, which causes propagation of damage between fibers.

The models discussed so far are useful for determining failure due to self-similar propagation of through-wall cracks. They do not, however, incorporate changes in crack direction. Harrison's [20] model allows for non-self-similar crack growth by assigning two different strain energy release rates for growth in the plane of or normal to the crack, but it is only applicable to unidirectional composites. In addition G is hard to calculate for non-self similar crack growth.

The previously discussed models do not include matrix cracking or delamination, but are only used to predict composite failure governed by fiber breakage. Kanninen and coworkers [14] have developed a hybrid model, in which a local heterogeneous region around the crack tip is embedded in an anisotropic elastic continuum. The heterogeneous region contains the fiber, the matrix, and the fiber/matrix interface region; therefore, the constitutive relations of each of these elements must be known up to the point of failure. The model is useful in that it can show the occurrence of fiber breakage, matrix microcracking, crack bridging, matrix/fiber debonding and axial splitting. In addition, it can be used to model any crack orientation. However, it is limited in that it cannot be accurately used for a real material, and, in fact, Kanninen, et al. have obtained qualitative results only. This is because the fiber/matrix interface is very difficult to study, so it's properties are unknown for most materials. The properties of an individual ply can be obtained readily and are available for many materials; therefore, it may be that a better approach is to use a mechanics model of a lamina in transverse tension, which experiences matrix cracking followed by delamination. This idea will be explored later.

Other models have been eliminated from consideration for the present work based on the advice of Reddy [21]. He compares several analytical and numerical methods for studying anisotropic materials with cracks, including classical, classical variational, such as Ritz and Galerkin,
finite difference, finite element, and boundary element formulations. He recommends use of the finite element method. Some very useful work has been done using the finite element method; for example, Lee [22] has developed a three dimensional damage accumulation model, which vividly shows transverse matrix cracking followed by fiber breakage in the load-carrying plies. Failure of an element occurs in a certain mode when the stress appropriate for that failure mode reaches a critical value. Appropriate elements of the stiffness matrix for the failed finite element are then set to zero. Lee’s model also accounts for delamination, but no delamination results are shown. Other finite element models include those of Sandhu, et al. [23], and Murray [24], which do not model delamination. The boundary element method, which is used by Tan and Bigelow [25] gives an approximation only on the boundary of the domain, ignoring the interior of the material. Since damage in a composite often begins with transverse matrix cracking in the interior of the laminate, this method is not useful for development of a general damage propagation model. S. S. Wang’s [26] edge delamination model is also not easily applied to interior cracking, although elements of it are useful as will be seen later.

While finite element models are useful and have given good predictions for matrix cracking and delamination, an analytical approach is used in the present work, with the goal of obtaining closed form expressions for crack density and stiffness change as a load is applied to a laminate. With this goal in mind, the most interesting and useful models for transverse matrix cracking, edge delamination, and local delamination are discussed in the following sections.

Shear Lag Theory and Transverse Matrix Cracking

Shear lag theory was first used to simplify the equilibrium equations for the problem of a broken fiber surrounded by matrix. This type of modeling approach is epitomized by the work of Dharani, Jones and Goree [27]. The shear lag model for laminates with a cracked ply group was first proposed by Bailey and coworkers and expanded upon by Flaggs, Nuismer and coworkers, Lee and Daniel, Daniel and Tsai, and Laws and Dvorak [28-36]. In essence, a shear lag model states
that the interlaminar shear stress is proportional to the difference in the average displacements of the two laminae under an applied load. One type uses an energy criterion for matrix cracking, while the other relies on a strength criterion. The constant of proportionality is called the shear lag parameter. Flaggs' [30] paper, which will be discussed in more detail later, shows that, at least in some instances, 2-D shear lag theory predicts experimental behavior better than the finite element method. Thus, the shear lag model is the type to be developed herein for transverse matrix cracking. There are essentially two kinds of shear lag models, those which use critical strain energy release rate as a criterion for cracking, and those which rely on laminate failure theories to determine the onset of cracking. They are typified by Lee and Daniel [32] and Laws and Dvorak [33]. Both are discussed in this section.

Laws and Dvorak [33] assume that a transverse crack propagates when it is energetically favorable to do so and use a probability density function to define the locations of cracks. Laws and Dvorak's model only considers a [0/90/0]s laminate (Figure 2.6). The model includes residual stresses after cure, which are defined as $\sigma^R_t$ and $\sigma^R_l$ for the residual stresses in the transverse or 90° plies and longitudinal or 0° plies, respectively. It assumes uniform displacements, $u(x)$ and $v(x)$, for the 0° and 90° laminae, respectively.

The fundamental relationship in shear lag theory is:

$$\tau = K (v - u) \quad \text{or} \quad \frac{d\tau}{dx} = K \left[ \frac{dv}{dx} - \frac{du}{dx} \right]$$

(2.23)
where \( \tau \) is the interlaminar shear stress and the constant, \( K \), is one form of the shear lag parameter.

Thus, the differential equations for shear lag theory are:

\[
\frac{d^2 \sigma}{dx^2} - \frac{\xi^2}{\sigma} = -\frac{\xi^2}{\sigma} \left[ \sigma + \frac{1}{\xi \sigma} \right] \quad \text{and} \quad \frac{d^2 \sigma}{dx^2} - \frac{\xi^2}{\sigma} = -\frac{\xi^2}{\sigma} \left[ \sigma + \frac{1}{\xi \sigma} \right]
\]  

(2.24)

where \( \xi \) is the non-dimensional form of the shear lag parameter.

\[
\xi^2 = \frac{K_d(bE_l + dE)}{bE_l E_t}
\]  

(2.25)

Solving the differential equations for \( \sigma_t \) and \( \sigma_l \) with the boundary condition, \( \sigma = 0 \) for \( x = \pm h \) at each crack, gives:

\[
\sigma_t = \left[ \sigma_t + \frac{1}{\xi \sigma} \right] \left[ 1 + \frac{\cosh \xi}{\cosh \frac{h}{d}} \right] \]  

(2.26)

\[
\sigma_l = \frac{E_l}{E_o} \left[ 1 + \frac{1}{\xi \sigma} \right] + \sigma_l \left[ 1 - \frac{\cosh \xi}{\cosh \frac{h}{d}} \right]
\]  

(2.27)

These are the stress distributions in each ply between two cracks a distance \( 2h \) apart. Neglecting the strain due to residual stress, one can calculate the average strain, \( \varepsilon_a \), in the uncracked portion of the laminate in terms of \( \sigma_a \), the applied stress. This leads to:

\[
\frac{\sigma_a}{\varepsilon_a} = E = E \left[ 1 + \frac{\beta dE}{\xi \sigma} \right]^{-1}
\]  

(2.28)
where $\beta = \frac{d}{h}$ and is the crack density parameter. As $\beta \to 0$, $E \to E_0$ and as $\beta \to \infty$, $E \to \frac{bE_1}{(b+d)}$, which is the same as ply discount theory.

The change in strain energy when a third crack is introduced between the first two is given by:

$$\Delta W = \left\{ (b+d)\sigma_a + b\sigma^R_1 \right\} [u_2-u_1]^B_1 + d\sigma^R_1 \left\{ [v_2-v_1]^C_1 + [v_2-v_1]^B_1 \right\}$$

(2.29)

where $\sigma_a$ is the applied stress and the subscripts, 1 and 2, refer to the states before and after the introduction of the next crack. Incidentally, Laws and Dvorak postulate three probability density functions to predict the site of the next crack. The work done by the applied loads is:

$$W_{\text{applied}} = 2(b+d)\sigma_a [u_2-u_1]^B_1$$

(2.30)

Knowing that $G = \beta/2d$, since $G$ is the energy released per unit area of the $90^\circ$ ply, and using $\beta = W_{\text{applied}} - \Delta W$ to get the energy released per unit width of composite, $G$ can be found.

$$G = \frac{d(b+d)E}{\xi bE_1 E_0} \left( \sigma_a^R + \frac{E}{\sigma_a^R} \right)^2 \left[ \frac{\xi h_1}{2d} \tanh \frac{\xi h_1}{2d} + \frac{\xi h_2}{2d} \tanh \frac{\xi h_2}{2d} \right]$$

(2.31)

First ply failure (fpf) occurs when $G = G_c$ and $\sigma_a = \sigma_{a \text{ fpf}}$. If $G_c$ and $\sigma_{a \text{ fpf}}$ are known, $\xi$ can be found; however, $G_c$ is difficult to measure accurately in composites, and the above equation requires an iterative solution. Laws and Dvorak circumvent the latter problem by assuming that the distances between cracks are still large at first ply failure, which causes the term containing tanh’s to go to 1. Note that, under this definition, first ply failure in and of itself has a small effect on the stiffness of the laminate. After first ply failure, increasing crack density causes the laminate stiffness to decrease rather rapidly, as will be shown in the chapter on results. Since $G_c$ is difficult to
measure in composites and has not been documented for many composite materials, many researchers believe it would be wise to develop a theory which does not require determination of \( G_c \) experimentally.

Lee and Daniel [32] have developed a theory without using \( G_c \). It is almost identical to that of Laws and Dvorak with some important exceptions. Assuming a linear variation of the shear stress in the \( z \) direction, Lee and Daniel use general parabolic equations to determine the displacements in each layer as a function of \( z \). Daniel corrected this problem in a more recent paper [34]. The shear stresses in each layer are given by:

\[
\begin{align*}
\tau_{xzl} &= G \frac{du}{12dz} = G_{12} (2C_1 z + C_2) \\
\tau_{xzl} &= G \frac{dv}{23dz} = G_{23} (2C_4 z + C_5)
\end{align*}
\]

where the subscript notation is the same as that of Laws and Dvorak [33]. The \( C_i \) are integration constants. Actually, \( \tau_{xzl} \) is proportional to \( G_{13} \). Lee and Daniel ignore the transverse deformation and assume \( G_{13} = G_{12} \), but this author thinks this assumption, which amounts to saying the material is transversely isotropic with respect to the 2-3 plane, is poor, because the fibers are not necessarily uniformly distributed throughout the cross-section of a lamina. With these relations in mind, Lee and Daniel apply their boundary conditions of zero transverse stress at the crack faces, zero out of plane shear and \( z \) direction normal stress at \( z=0 \) and \( z=d+b \), and equality of out of plane shear and \( z \) direction normal stress in the two plies at \( z=d \). The displacements \( u \) and \( v \) are also assumed equal at \( z=d \), which is a good assumption if bonding is perfect between the layers. Once delamination has occurred, this is not true at the locations of delamination. The resulting average displacements give:

\[
\ddot{u} - \ddot{v} = \frac{\tau}{3G_{12}G_{23}} \left[ bG_{23} + dG_{12} \right]
\]
Using the shear lag equation, \( \tau = H(\nu-u) \), with equation (2.33), one finds:

\[
H = \frac{3G_{12}G_{23}}{bG_{23} + dG_{12}}
\]  

(2.34)

where \( H \) is the shear lag parameter. The shear stress at the interface between layers is:

\[
\tau = -\alpha d \left[ \frac{E}{\sigma} a_1 + \sigma^R \right] \frac{\sinh \alpha x}{\cosh \alpha h}
\]  

(2.35)

where

\[
\alpha^2 = \frac{(d+b)E}{d b E E_t}
\]  

(2.36)

Since the shear stress distribution is assumed to be linear, \( \tau_{xzd} \) and \( \tau_{xzt} \) are related to \( \tau \) by:

\[
\tau_{xzd} = \tau_{xzt} \frac{b+d-z}{b}, \quad d \leq z \leq b + d \quad \text{and} \quad \tau_{xzt} = \tau_{xzt} \frac{z}{d}, \quad z \leq d
\]  

(2.37)

Using equations (2.35) and (2.37), along with equilibrium, \( \frac{\delta \sigma}{\delta z} + \frac{\delta \tau}{\delta x} = 0 \), Lee and Daniel get the following through thickness normal stresses:
\[
\sigma_z = \frac{\alpha E_t (b+d-z)^2 \cosh \alpha x}{2b \cosh \alpha h} \sigma_a
\]

(2.38)

\[
\sigma_z = \frac{\alpha E_t \left( b + \frac{d^2-z^2}{2d} \right) \cosh \alpha x}{2d \cosh \alpha h} \sigma_a
\]

These are, of course, found using the appropriate boundary conditions. These two relations (2.34) are a significant contribution on the part of Lee and Daniel, since they can be used to define the onset of delamination. In addition, the \( \sigma_1 \) and \( \sigma_3 \) distributions are similar to those predicted by the three-dimensional elasticity results of Pagano and coworkers [37, 38]. However, there is a discrepancy between Lee and Daniel’s results for \( \tau_{xz} \) and those of Pagano. Because of the stress free condition at the crack face in the 90° layer, \( \tau_{xz} \) should go to zero there for all \( z<d \), as predicted by Pagano. In Lee and Daniel’s model, \( \tau_{xz} \) is only zero at \( z=0 \). In addition, the correct value for \( \alpha \) in Lee and Daniel’s model is open to question due to the use of \( G_\tau \) rather than \( G_{13} \) in the expression for \( H \).

Lee and Daniel use Case I of Laws and Dvorak [33], which is that the next crack must occur exactly between the originally existing cracks. Rather than using energy considerations involving \( G_c \), Lee and Daniel assume that a new crack forms when \( \sigma_t \) reaches the transverse tensile strength, \( F_{Tt} \). While this assumption works adequately in Lee and Daniel’s model, it is quite controversial. In fact, A.S.D. Wang [39] noted that: "It is the total strain energy trapped in the 90° layer that determines the onset of matrix cracking, not the in-situ tensile stress." The relationship between applied stress and crack density is found by substituting \( F_{Tt} \) for \( \sigma_t \) and 0 for \( x \) in equation (2.27) to get:
This equation is much simpler than Laws and Dvorak's Case I equation for \( \sigma_a \), which involves \( G_c \) in the expression for \( \sigma_{a}^{fp} \). Lee and Daniel's expression for decremented axial modulus is identical to that of Laws and Dvorak given in equation (2.24). The two methods thus give essentially the same result for the decremented stiffness due to cracking.

Nuismer and Tan [31] have developed a more general relationship between laminate properties and crack density. They used the orthotropic constitutive equations for each lamina, so they could model any laminate of type \([\pm \theta/90^\circ]_n\), where ply group 1 refers to the inner 90° plies and ply group 2 refers to the outer ±\( \theta \) plies. They also assumed general in-plane loading. Out of plane shear is given by:

\[
\tau^* = A_{55} \left[ \frac{u^{(2)} - u^{(1)}}{\bar{h}} \right]
\]  

(2.40)

where

\[
A_{55} = \frac{3hC_{55}^{(1)}C_{55}^{(2)}}{h^{(1)}C_{55}^{(2)} + h^{(2)}C_{55}^{(1)}}
\]

(2.41)

and \( C_{55}^{(1)} \) is from the \( i \)th lamina's stiffness matrix, \( h^{(i)} \) is the thickness of the \( i \)th lamina, and \( \bar{h} = h^{(1)} + h^{(2)} \). Note that this expression for \( A_{55} \) is similar to that given for the shear lag parameter, \( K \).

The notation used in this section and throughout this work is the standard notation used for laminates as defined by Jones [38]. The effective damaged laminate compliance relations are given by:
\[
\varepsilon^{(1)} = \left[ 1 + \frac{\beta_1 \beta_2}{\beta'_1} \right] \left[ \frac{S^{(1)} S^{(1)} S^{(1)} S^{(1)}}{S^{(1)} S^{(1)} S^{(1)} S^{(1)}} \right] S^{(1)} \sigma^{(1)} + S^{(1)} \sigma^{(1)} + \varepsilon^{(1)}\]
\[
\varepsilon^{(1)} = S^{(1)} \sigma^{(1)} + S^{(1)} \sigma^{(1)} + \varepsilon^{(1)}
\]
\[
\gamma^{(1)} = \frac{\beta}{\beta} S^{(1)} \gamma^{(1)}
\]

where \(S^{(1)}_{ij}\) are elements of the compliance matrix for ply group 1, \(\varepsilon_{xN}\), and \(\varepsilon_{yN}\) are nonmechanical strains, \(\bar{\varepsilon}\) and \(\bar{\sigma}\) are averaged strains and stresses, and

\[
\beta_1 = \frac{\text{tanh} (\alpha_1 L)}{\alpha_1 L}
\]
\[
\beta_2 = 1 + \frac{h^{(1)} Q^{(1)} \text{tanh} (\alpha_1 L)}{h^{(1)} Q^{(1)} \text{tanh} (\alpha_1 L)}
\]
\[
\beta_3 = 1 + \frac{h^{(2)} Q^{(2)} \text{tanh} (\alpha_2 L)}{h^{(2)} Q^{(2)} \text{tanh} (\alpha_2 L)}
\]

Now \(Q_{ij}\) are elements of the lamina stiffness matrix, \(2L\) is the distance between matrix cracks, and

\[
\alpha_1^2 = \frac{3 C^{(1)}_{55} C^{(2)}_{55}}{h^{(1)} C^{(2)}_{55} + h^{(2)} C^{(1)}_{55}} \left[ \frac{h^{(1)} Q^{(1)}_{11} + h^{(2)} Q^{(2)}_{11}}{h^{(1)} Q^{(1)}_{11} + h^{(2)} Q^{(2)}_{11}} \right]
\]
\[
\alpha_2^2 = \frac{3 C^{(1)}_{44} C^{(2)}_{44}}{h^{(1)} C^{(2)}_{44} + h^{(2)} C^{(1)}_{44}} \left[ \frac{h^{(1)} Q^{(1)}_{66} + h^{(2)} Q^{(2)}_{66}}{h^{(1)} Q^{(1)}_{66} + h^{(2)} Q^{(2)}_{66}} \right]
\]

In addition, Nuismer and Tan [34] used strain, rather than stress, to define first ply failure.
Flaggs [30] uses a two dimensional shear lag model, similar to the one dimensional models described above to study laminates of the type $[\pm \theta, 90_n]$ where $\theta$ refers to a relatively low angle ply, and $[0_2, \pm \phi]_s$, where $\phi$ refers to a high angle ply. Instead of applying the shear lag idea to $\tau_{xz}$ or $\tau_{13}$ only, Flaggs has applied it to $\tau_{23}$ as well. This greatly complicates the model, but it may enhance its accuracy. In addition, Flaggs' model incorporates in-plane shear loading. The derivation follows a line of reasoning similar to Laws and Dvorak's model [33]. Figure 5 shows a comparison of Flaggs' shear lag model with generalized plane strain finite element predictions and experimental data. Flaggs has also shown that shear lag predicts the onset of matrix cracking significantly better than does Tsai-Wu, a failure theory commonly used for composite laminates.

![Figure 2.5](image)

Figure 2.5. Comparison of Laminate Load at Onset of Matrix Cracking to Finite Element and 2-D Shear Lag Predictions for $[\pm 25/90_n]_s$ T300/934 Laminate Family [30]

Daniel and Tsai [35] have also written a model for transverse matrix cracking in the 90° plies of laminates of the type $[0_n/90_m]_n$. It can accommodate general in-plane biaxial loading but is otherwise quite similar to Lee and Daniel's model. Of course, strain energy release rate is not used. Results were shown for 10° off-axis loading only.
Longitudinal splitting is another form of matrix cracking, which is due to the Poisson effect. The strains produced by differential Poisson contractions of the laminae are increased when transverse cracking occurs. Interestingly, Bailey and coworkers [28, 29], did not see longitudinal splitting in carbon/epoxy, but only in glass/epoxy, even though the Poisson's ratio mismatch is much greater in carbon/epoxy. They believe that the reason for this is that the particular CFRP studied had small ultimate failure strains. Bader, et al. [28] have developed an expression for the minimum composite strain at which longitudinal splitting is energetically favored for a crossply laminate.

\[
\varepsilon_{lc} \left( \text{min} \right) = \frac{1}{c} \left[ \frac{E^* (b+d)}{E_1 d} \left[ \frac{-3}{4} \varepsilon + \left[ \frac{1}{16} \varepsilon \right]_{th}^2 + \frac{\gamma_t}{(b+d)E^*E_{lc}} \right] \right] + \nu \varepsilon_{lc} \]  

(2.47)

where \(E^*\) is the Young's modulus in the transverse direction, \(b\), \(d\), \(E_1\), and \(E_t\) are the same as defined by Laws and Dvorak [33], \(\varepsilon_{th}\) is the tensile thermal strain in the transverse ply, \(\varepsilon_c\) is the strain at the first transverse crack, \(\nu_1\) and \(\nu_t\) are the Poisson's ratio of the longitudinal and transverse plies, respectively, \(\gamma_t\) is the fracture surface energy per unit area of the inner ply, and \(\phi\) is given by

\[
\phi = \frac{E G (b+d)}{E_1 E_t b d^2} \]  

(2.48)

According to Swanson [40], matrix cracking reduces strength, and this reduction is more pronounced in tough matrices than in brittle matrices. This is because longitudinal splitting along the fiber/matrix interface in the more brittle matrices is a toughening mechanism. As was seen in Chapter One, debonding between the fiber and matrix contributes to fracture toughness. The contribution of debonding to fracture toughness is greater in brittle matrices, because the matrix has little intrinsic toughness. In composites with tough matrices, however, the loss of toughness due to matrix cracking probably overshadows any toughening effect due to debonding. This toughening
effect should ideally be accounted for in a model of transverse cracking in fiber-reinforced epoxy composites.

Tsai, Daniel, and Lee [36] account for longitudinal cracking in their shear lag model. This model encompasses [0\textdegree/90\textdegree\textdegree]s laminates under biaxial loading; however, like Daniel's previous models, it depends on transverse strength and not on strain energy release rate. It is important to note that shear loading is not included in this model. Tsai, et al., showed that it is impossible under shear lag theory, to incorporate cracking in both directions and shear loading effects. Finally, while the model was written for biaxial loading, it is compared with experimental results for loading in one direction only.

**Delamination**

Delaminations tend to start at free edges, like holes and cutouts, internal flaws, ply drop-offs, or joints. In fiber-reinforced epoxy systems, delamination follows matrix cracking. In a ply with 90° laminae, matrix cracking begins in the 90° plies, followed by delamination at the interfaces bounded by at least one 90° ply. A complete model must include both edge and local delamination. With biaxial testing, local delamination is significantly more important than edge delamination.

**Models for Transverse Cracking and Edge Delamination**

Two edge delamination models will be examined, but first the observations of A.S.D. Wang [39, 41] will be discussed. In composite fracture, the properties of the composite are, of course, very important. Less obvious is the influence of ply thickness on cracking. From a statistical point of view, one assumes that a thicker lamina contains more defects than a thin lamina, so it is likely to fail at a lower stress or strain level. The laminate’s stress field and the distribution of defects are then very important in determining cracking of the material. On the other hand, one may employ a fracture mechanics approach, which uses the strain energy release rate.
A. S. D. Wang [39, 41] used a fracture mechanics approach. Wang assumes that the strain energy trapped in the laminae depends on thickness. He uses finite element calculations to find the strain energy release rate, \( G(A) \), for transverse cracking and edge delamination under thermal and transverse uniaxial mechanical loading, and compares his predictions with uniaxial test results on each layup examined. Assumptions made in his Monte Carlo simulation include self-similar cracking, mode I transverse cracking, and mixed mode edge delamination. He finds that the strain energy release rate for delamination increases as the number of 90° plies increases in a \([\pm 25/90_\alpha]\) laminate, where \( n = 1/2, 1, 2, 3, \text{and} 4 \). Transverse cracking in the 90° plies occurs at a very high strain for small \( n \) (\( n = 1/2 \) and \( n = 1 \)), and it is preceded by edge delamination between the 90° plies.

Wang's results also show that final failure occurs after delaminations from both sides of the specimen grow until 70-80% of the width is delaminated. On the other hand, if \( n \) is greater than 1, transverse cracking precedes edge delamination and occurs at a much smaller strain. In fact, the transverse cracking onset strain in \([\pm 25/90_\alpha]\) laminates with \( 1 < n \leq 4 \) is less than that for a \([90_\alpha]s\) laminate, which proves Wang's assertion that the strain energy in the 90° layer must be used to determine the onset of transverse matrix cracking. The results are somewhat different for crossply laminates [39]. For example, while edge delamination does occur in the \([\pm 25/90_2\alpha]\) laminates, \([0_2/90_\alpha]\) laminates exhibit no edge delamination after matrix cracking. Edge delaminations occur between the central 90° plies for \( n = 1/2, 1 \) and 2, while for \( n = 3 \) and 4, local and edge delaminations are predicted to occur between the -25° and 90° plies. For \( n = 3 \), the prediction is correct: delamination is mixed mode and the onset strain is close to the predicted value.

Failure occurs very rapidly after the onset of edge delamination. For \( n = 4 \), on the other hand, delamination is predicted to be mostly mode II, but in experiments, no noticeable delamination occurs before final failure [39]. In a later paper [41], however, Wang shows that the predicted delamination does occur, and, in addition, finds that the same failure modes occur for \( n = 6 \) and 8, as well. As \( n \) increases above 4, the onset loads for delamination decrease to approach those for
transverse matrix cracking, and for $n = 8$, they are approximately equal to each other and to the failure load.

Wang uses a simple discount method to account for multiple transverse cracks. He assumes that the properties of the $90^\circ$ layers are decreased by 10% when transverse cracking occurs prior to delamination. In addition, his method does not account for delamination occurring at the tips of transverse cracks. He does, however, note that delamination between the $90^\circ$ plies is due primarily to normal stress, delamination between $+25^\circ$ and $-25^\circ$ plies is dominated by shear (Mode II), and delamination between $90^\circ$ and $25^\circ$ plies is mixed mode. These observations are useful for prediction of delamination at matrix crack tips.

Poursartip [42] studied transverse cracking and delamination under static and cyclic tensile loading. He showed that the crack density in the $90^\circ$ plies of an unnotched [45/0/-45/90]$_s$ graphite epoxy laminate was 0.9 cracks/mm. Delamination then occurred first at either the 90/90 interface or the 90/-45 interfaces. These delaminations progressed rapidly in the axial direction and more slowly in the transverse direction, jogging between interfaces by means of matrix cracks in the $90^\circ$ layers. During delamination, transverse crack density grew to 1.8 cracks/mm as the size of the delaminations grew. Transverse cracking occurred to a lesser extent in the $-45^\circ$ plies, where it was initiated at the laminate free edge and was densest in the delaminated region. On the other hand, transverse cracks in the $90^\circ$ plies extended to the area in front of the delamination. Cracks appeared later and crack density was much less in the $45^\circ$ plies than in the $-45^\circ$ plies; however, $45^\circ$ cracks were also found mostly in the delaminated area.

Poursartip [42] has proposed that the matrix cracking required less energy than delamination; therefore, transverse matrix cracking helped to prevent delamination. In other words, as energy is made available for the formation of new surfaces, it is used for additional matrix cracking, rather than for delamination. He has found that the delamination crack resistance, $G_{R'}$, increases as delamination progresses, partly due to the diminishing supply of weaker crack paths, but also due to an increasing use of energy for matrix cracking. This is supported by the observation of a great deal
of matrix cracking during delamination. Poursartip's observations are very useful, and he has achieved good results with a rather simple model.

Poursartip used O'Brien's [42, 43] stiffness reduction equation for edge delamination:

$$E = (E_* - E_{LAM}) \frac{A}{A_*} + E_{LAM}$$  \hspace{1cm} (2.49)

where $E_{LAM}$ is the laminate longitudinal tangent modulus before delamination, $A/A_*$ is the ratio of delaminated area to total interfacial area, and $E_*$ is the modulus of the laminate completely delaminated along one or more interfaces. This delaminated modulus can be found from:

$$E_* = \frac{\sum_{i=1}^{m} E_i t_i}{t}$$  \hspace{1cm} (2.50)

where $t_i$ and $E_i$ are the thickness and modulus of the $i^{th}$ sublaminate and $m$ is the total number of sublaminates. O'Brien has verified these relations by experiments using graphite-epoxy laminates.

Transverse matrix cracking combined with edge delamination has been studied for [0/90]s and [±45/0/90]s glass/epoxy laminates by Caslini, Zanotti and O'Brien [45], who used shear lag analysis and fracture mechanics to characterize damage in a manner similar to that of Laws and Dvorak [33]. They extended the cross ply results to laminates where the 90° ply is constrained by 0° plies, but may have other plies surrounding the 0/90 combination. They did not, however, study cracking in the outer off-angle plies. They base their model on the experimental observations of glass/epoxy under fatigue loading. The number of matrix cracks reaches a saturation value, which is a function of laminate characteristics, and matrix cracking in the 90° plies precedes all other damage. They also mention that damage accumulation behavior depends on test and load levels. The fact that matrix crack density reaches a saturation value is somewhat problematical in their
model, since the closed form solution gives $G$ as a function of $dE/dA$, which continually increases as stress increases, and never goes to zero as it should.

Caslini et al., like Poursartip, used O'Brien's [43] sublaminate method for modeling free edge delamination. (See equations 2.42 and 2.43 above). The strain energy release rate for a body of volume, $V$, under a constant strain, $\epsilon$, is:

$$G = -V\frac{\epsilon^2}{2} \frac{dE}{dA} = \frac{\epsilon^2}{2}(E_{\text{LAM}} - E^*)$$

(2.51)

O'Brien used edge delamination experiments to find the strain at which delamination was first detected. He then used equation (2.44) to determine $G_C$, which was subsequently used to predict delamination in other laminates.

O'Brien's approach for edge delamination works well for graphite-epoxy, but not for glass-epoxy, because glass epoxy is more likely to experience extensive matrix cracking before the onset of delamination, and because edge delaminations grow only a small amount in quasi-static conditions before final failure due to a lower Poisson's ratio mismatch in glass-epoxy. Thus, Caslini, et al., are forced to use a linear regression analysis of experimental data to model stiffness loss due to cracking in glass/epoxy laminates.

Valisetty and Rehfield [46] also used a sublaminate method to model edge delamination, but, unlike Caslini, et al., they found the interlaminar stresses. They used homogeneous plate theory to solve the finite-width free edge delamination problem. They applied the theory on a ply-by-ply basis and were able to satisfy equilibrium and compatibility. They did this for uniaxial tension only and compared their results with a finite element analysis and not with experimental data.

Although some elements of S. S. Wang's [26] approach are not useful in the type of model proposed here, one useful concept is his evaluation of $G$ using Irwin's virtual crack extension concept.
\[ G = G_1 + G_\| + G_\|_3 \]  

\[
\lim_{\delta \beta \to 0} \frac{1}{2 \delta \beta} \int_0^{\delta \beta} \left\{ \sigma_{x}^{(k)} \left[ \begin{array}{c}
\delta \beta - \tau_{r,\phi}
\end{array} \right] - w^{(k+1)} \left[ \begin{array}{c}
\delta \beta - r, \pi
\end{array} \right] \right\} \\
+ \tau_{y}^{(k)} \left[ v^{(k)} \left[ \begin{array}{c}
\delta \beta - r, \pi
\end{array} \right] - v^{(k+1)} \left[ \begin{array}{c}
\delta \beta - r, -\pi
\end{array} \right] \right] \\
+ \tau_{x}^{(k)} \left[ u^{(k)} \left[ \begin{array}{c}
\delta \beta - r, \pi
\end{array} \right] - u^{(k+1)} \left[ \begin{array}{c}
\delta \beta - r, -\pi
\end{array} \right] \right] \right\} \, dr
\]

where polar coordinates \((r, \phi)\) are used, \(\delta \beta\) is the length of the virtual crack extension; \(\sigma_{x}, \tau_{y}\), and \(\tau_{z}\) are the interlaminar stresses; and \(u, v, w\) are the displacements of the \(k\)th lamina in the \(x, y, z\) directions. Notice that \(G_1\) is proportional to \(\sigma_{z}\), \(G_\|\) is proportional to \(\tau_{y}\), and \(G_\|_3\) is proportional to \(\tau_{x}\) in Wang's formulation, which is solved for a general symmetric laminate subjected to a uniform axial strain, \(\epsilon_x\). Note that this author has rotated Wang's axes to make this equation correspond to the other equations given here.

**Models of Transverse Matrix Cracking and Local Delamination**

Armanios [47] has developed a model similar to that of Caslini, et al., but Armanios' model is for local delamination rather than edge delamination. Talreja [48] has proposed a very different model, in which he describes interlaminar and intralaminar failure modes according to associated damage vectors. In addition, O'Brien [44] has developed a sublaminate approach to local delamination.

Armanios [47] has developed a shear deformation model with the sublaminate approach similar to that described above, along with fracture mechanics, to predict local delamination at transverse crack tips. Transverse cracks terminate where the ply orientation changes, and, at the crack tips, local or transverse crack tip delaminations grow in a direction normal to that of the transverse crack. For the purpose of modeling local delaminations, Armanios treats the transverse cracks as free edges. He assumes plane strain conditions in the \(x-z\) plane, neglecting through
thickness strain, \( \varepsilon_{zz} \). This assumption means that, with his method, one cannot correctly estimate the interlaminar normal stress. Thus, Armanios calculates \( G_{II} \) as

\[
G_{II} = \lim_{\delta \to 0} \frac{1}{2\delta} \int_0^\delta T_1(x-\delta) \Delta u(x) \, dx
\]

(2.53)

where \( T_1 \) is the interlaminar shear along sublaminate 1 (Figure 2.6), \( \Delta u \) is the relative sliding displacement, and taking the limit as \( \delta \to 0 \) gives \( G_{II} \) at the delamination crack tip. Note the similarity between this equation and equation (2.45). In fact, S.S. Wang's equation can be adapted for transverse crack tip delamination by showing that \( G_{II} \) is proportional to \( \tau_{xz} \) and \( G_{III} \) is proportional to \( \tau_{yz} \).

In Armanios' model, taking \( \delta = 0 \) in equation (2.46) gives a trivial result, so the limit is actually calculated as \( \delta \) approaches an appropriate decay length, or length within which the presence of the crack significantly alters the response of the material in comparison with the corresponding far field response. Armanios estimates the total strain energy release rate from the following equation:

![Figure 2.6. Modeled Region and Sublamine Scheme [47]](image_url)
\[
G_c = \frac{P^2}{2b} \frac{dC}{da} = \frac{P^2}{2b^2} \left[ \frac{1}{A_{11(1)}} - \frac{1}{A_{11(1)} + A_{11(2)}} \right] + I_1 I_2
\]

(2.54)

where \( P \) is the uniform axial force applied to the specimen, \( b \) is the specimen width, and \( dC/da \) is the change in compliance with crack growth. \( A_{11} \) is the classical laminate theory axial stiffness in the 11 direction, \( (i) \) refers to the sublaminate, and \( I_1 \) and \( I_2 \) are associated with delamination length.

Armanios has fully predicted critical delamination growth stresses for \([\pm 25/90]_n\) laminates with some success and his results are reasonably correlated with experimental values of critical strain for large numbers of 90° plies \((n \geq 4)\).

O’Brien [44] has developed a sublaminate approach for local delamination, as well as for edge delamination. The difference between the two is that in local delamination, the 90° ply becomes totally isolated. O’Brien assumes that the delamination starts at the matrix crack tips and progresses down the length of the laminate. Thus, the equation for the modulus of a locally delaminated laminate, \( E_{LD}^* \) is:

\[
E_{LD}^* = \frac{\sum_{i=1}^{m} E_i t_i}{t}
\]

(2.55)

where \( i \) refers to plies with \( \theta \neq 90 \) only and the other variables have the same meaning as given above. Now the modulus of the locally delaminated cross-section, \( E_{LD} \), is simply the modulus of the remaining plies, and is given by:

\[
E_{LD} = E_{LD}^* \left( \frac{t}{t_{LD}} \right)
\]

(2.56)

where \( t_{LD} \) is the thickness of the laminate less the thickness of the 90° plies that have been isolated.
Finally, the strain at which local delamination occurs is:

\[ \epsilon_c = \frac{1}{E_{\text{LAM}}} \left[ \frac{2mG_C}{E_{\text{LAM}}^t} - \frac{1}{E_{\text{LD}}} - \frac{1}{E_{\text{LAM}}^j} \right] \]  

(2.57)

where \( E_{\text{LAM}} \) is the modulus of the laminate before delamination and \( G_C \) is the critical strain energy release rate of the laminate. Equations 2.50 and 2.47 are essentially the same, except for the term \( I-I^* \) in Armanios' equation. This term indicates that the applied load for additional delamination depends on delamination size, but it is not highly significant. Armanios compares his model with O’Brien’s and the predicted delamination onset strains are the same.

Talreja [48] takes a different, albeit no less interesting, approach to the study of transverse matrix cracking and local delamination growth. He looks at a general orientation of matrix cracks in an initially orthotropic laminate of the type \([0/\pm \theta n]_s\). Specifically, he studies laminates with \( \theta=90^\circ \) and \( n=1/2 \) and 3 and laminates with \( \theta=45^\circ \) and \( n=1 \). A major difference between Talreja’s model and the others is that he assumes a general crack orientation and shows that, after cracking, the laminate loses its orthotropy if the crack is not oriented along the material axes. He groups what he calls “failure entities” into two damage modes, according to their orientation and growth characteristics. The first is the intralaminar damage mode, which is matrix cracking, and the second is the interlaminar damage mode, which is local delamination. His model, based on experimental observations, begins with matrix cracking in plies not oriented along the principal tensile loading direction. These cracks increase monotonically in number until a saturation level is reached, whereupon cracks initiate in adjacent plies transverse to these primary matrix cracks. Obviously, this scenario is correct for a laminate with \( \theta=45^\circ \), but if \( \theta=90 \), the transverse cracks will simply extend through the 90° plies with the same orientation in all of them. Talreja asserts that these intralaminar cracks initiate interlaminar cracks, which appear initially as isolated delamination regions, but grow
and merge into strip-like delamination zones. Eventually, crack interactions increase to a point where fiber failures begin, followed rapidly by final failure of the material.

By assuming that damage modes do not interact substantially, which only holds until delamination begins, Talreja used a form of superposition of damage modes to develop his polynomial expression for the elastic potential which must be invariant to transformations expressing the orthotropic symmetry initially present in the laminate. Unfortunately, the polynomial expression for the elastic potential also contains phenomenological constants, which must be determined experimentally. Talreja has obtained a good comparison of his model with the results of fatigue testing.

Yang and Boehler [49] developed a micromechanical model for matrix cracking and local delamination, which describes in detail the interaction between the two damage modes. Their observations echo those of Talreja. They observed that a small amount of local delamination occurs as soon as transverse matrix cracks reach the bounding region. Theoretically, this is due to a singularity in the interlaminar shear stress at the transverse crack tip. This initial delamination soon arrests due to its stable nature. The size of the initial delamination depends on material properties and ply thickness. A greater 90° ply thickness implies a larger initial delamination. In addition, the likelihood of such initial delaminations decreases as crack spacing decreases. As the crack spacing approaches the shear lag distance, which is related to the thickness of the 90° layer, no further delamination occurs until the characteristic damage state is reached. This characteristic damage state refers to the final matrix crack density. After the characteristic damage state is reached, delamination begins again, because no further energy can be dissipated by the formation of matrix cracks.

Models for transverse matrix cracking, edge delamination, and local delamination have been examined. Before final failure can occur, one additional fracture process must occur. That process is fiber failure in the primary load-carrying plies.
Fiber Failure

Fiber failure in load carrying plies precedes or coincides with final failure, but random fiber breaks can also lead to matrix cracks transverse to and along the direction of the fibers [24]. The rule of mixtures has been used to define the strength of the plies with fibers oriented in the direction of the load, but it overestimates the strength of composites with poor fiber/matrix interface strength [14]. On the other hand, statistical methods have been used, which say that longer fibers have more flaws than shorter fibers, so they tend to break at lower stresses. A commonly used statistical model is the Weibull distribution, which will be discussed in more detail.

Once a fiber breaks, the load is transferred by shear in the matrix back to the fiber a short distance along its length from the break. The distance between the fiber break and the location where load can again be transferred from matrix to fiber is termed the ineffective length, since the fiber does not carry the applied load over that distance. Fiber failure propagates through a ply as fibers must take additional load due to previous fiber failures. In addition, ineffective or debonded fibers aligned at the boundary of voids have a strong effect on transverse tensile strength.

Since the rule of mixtures approach is inaccurate and the statistical approach is difficult to use, many researchers use the experimentally determined failure stress or strain of a unidirectional lamina as the criterion for failure in the fiber direction. For example, fiber failure has been defined by Lee [22] and by Murray [24] to occur at the unidirectional lamina tensile strength in the fiber direction. Approaches of this nature rely on experimental measurement of the unidirectional tensile strength, but the value obtained will include all fiber-related failure micromechanisms, fiber-matrix debonding, pull-out, and breakage, since all of these contribute to unidirectional tensile failure. Poe [18] used an interesting approach based on fiber failure strain. While his model was written for the laminate as a whole, it may be useful for examination of the fiber-dominated failure modes only. He found that:
\[ K_{IC} = \frac{E_y \epsilon_{IC} \sqrt{2\pi x}}{1 - \nu_{xy} \left( \frac{E_x}{E_y} \right)^{1/2} \cos^2 \alpha + \left( \frac{E_y}{E_x} \right)^{1/2} \sin^2 \alpha} \] (2.58)

where \( \epsilon_{IC} \) is the critical fiber strain, \( x \) is the distance from the crack tip, \( E_x, E_y, \) and \( \nu_{xy} \) are the elastic constants of the ply containing the crack with \( x \) parallel to the crack and \( y \) normal to the crack, and \( \alpha \) is the fiber orientation angle of the load-carrying ply with respect to the \( y \)-axis. The problem with this theory is that it will not hold if extensive delamination or longitudinal splitting occurs. It remains to be seen if it will hold if it is used only for fiber failure, rather than laminate failure.

Other interesting approaches are those of Rosen [50] and Phoenix and Wu [51]. Rosen uses the idea of fiber ineffective length to represent the stress field in a unidirectional fiber composite with distributed fractures. By neglecting the stress concentrations at the fiber breaks, the tensile strength can be expressed as a statistical function of ineffective length. Phoenix and Wu use a Weibull distribution function,

\[ F(\sigma) = 1 - \exp \left( -\frac{\sigma}{\sigma_1(R)} \right)^{\beta(\rho+1)} \] (2.59)

where \( \beta(\rho+1) \) is the shape parameter and \( \sigma_1(R) \) is the scale parameter with \( R \) equal to the rate of stress increase. Rosen's distribution is probably a Weibull distribution, but his equation is unclear. It turns out that use of the unidirectional tensile strength or critical fiber strain is adequate for a shear lag model [25, 28].

The approximate analytical models discussed have primarily been developed for and compared with uniaxial tension data only. Obviously, there is a need for a model capable of predicting damage and failure under biaxial loading conditions, since few actual structures see uniaxial loading only.
Failure Theories

The only methods commonly used to predict failure are failure theories and curve fits [52-58] including maximum strain, maximum stress, Tsai-Wu, and curve fits in the form of Tsai-Wu.

Maximum strain is the most commonly used failure criterion and is useful, because the mode of failure is predicted. This criterion says that failure occurs when the strain component in any one of the principal material directions of the lamina exceeds its corresponding ultimate strain. This is similar to the maximum stress criterion, which also can predict failure mode. Both criteria take the following form, which is the explicit form of the maximum stress criterion [52].

\[
\begin{align*}
\sigma_{11} & > X_T \text{ for } \sigma_{11} > 0 \\
\sigma_{22} & > Y_T \text{ for } \sigma_{22} > 0 \\
\left| \sigma_{11} \right| & > X_C \text{ for } \sigma_{11} < 0 \\
\left| \sigma_{22} \right| & > Y_C \text{ for } \sigma_{22} < 0 \\
\left| \sigma_{12} \right| & > S \text{ for all } \sigma_{12}
\end{align*}
\]

(2.60)

where $X_T$ and $X_C$ are the tensile and compressive strengths, respectively, in the longitudinal fiber direction, $Y_T$ and $Y_C$ are the tensile and compressive strengths, respectively, in the transverse fiber direction, and $S$ is the in-plane shear strength.

The Tsai-Wu and Tsai-Hill theories are similar, but Tsai-Wu predicts fiber-reinforced composite failure more accurately than does Tsai-Hill [37]. The Tsai-Wu [53] criterion is:

\[
F_{\sigma_1} + F_{\sigma_2} \sigma \geq 1
\]

(2.61)

or, for specially orthotropic materials in two dimensions,

\[
F_{11} \sigma_{11}^2 + F_{22} \sigma_{22}^2 + F_{12} \sigma_{12}^2 + 2F_{21} \sigma_{11} \sigma_{22} + 2F_{1} \sigma_{12}^2 + F_6 \sigma_{12}^2 \geq 1
\]

(2.62)
where:

\[
\begin{align*}
F_1 &= \frac{1}{X_T} - \frac{1}{X_C} \\
F_{11} &= \frac{1}{X_T X_C} \\
F_{66} &= \frac{1}{T_C Y_Y} \\
F_2 &= \frac{1}{Y_T} - \frac{1}{Y_C} \\
F_{22} &= \frac{1}{Y_T Y_C} \\
F_{12} &= -\frac{1}{2p^2} \left[ 1 - p \left( \frac{1}{X_T X_C} + \frac{1}{Y_T Y_C} \right) - p^2 \left( \frac{1}{X_T X_C} + \frac{1}{Y_T Y_C} \right) \right]
\end{align*}
\]

(2.63)

where \( S_T \) and \( S_C \) are positive and negative shear strengths, respectively, and \( P \) is the strength determined from a biaxial test with \( \sigma_1 = \sigma_2 = P \) and \( \sigma_1 = 0 \).

Hashin's [24] failure theory is three-dimensional in nature, and takes the form:

\[
A_1 I_1 + B_1 I_1^2 + A_2 I_2 + B_2 I_2^2 + C_1 I_1 + A_3 I_3 + A_4 I_4 \geq 1
\]

(2.64)

where \( I_1 \) are the stress invariants,

\[
\begin{align*}
I_1 &= \sigma_1 \\
I_2 &= \sigma_2 \pm \sigma_3 \\
I_3 &= \sigma_2^{1/2} - \sigma_2^{1/2} \sigma_3^{1/2} \\
I_4 &= \sigma_2^{1/2} + \sigma_3^{1/2}
\end{align*}
\]

(2.65)

Hashin correctly argues that failure is due to normal and shear stresses acting on the failure plane. He points out that \( \sigma_{22} \) does not contribute to fiber failure and \( \sigma_{11} \) does not contribute to matrix failure. Additionally, he stated that \( \sigma_{11} \) and \( \sigma_{12} \) acting on the 2-3 plane are responsible for fiber failure, while \( \sigma_{22} \) and \( \sigma_{12} \) acting on the 1-3 plane cause matrix failure.

Feng's [54] model is also very similar to that of Hashin, but he allows large deformations to occur by using Cauchy strain invariants. The Feng criterion, decoupled for fiber and matrix
dominated failure is:

\[ A_1 (I_1 - 3) + A_4 (I_2 - 3)^2 + A_2 (I_2 - 3) - 1 = 0 \quad (2.66) \]

for matrix dominated failure and

\[ A_5 (I_5 - 3) + A_{55} (I_5 - 3)^2 + A_4 (I_4 - 3) - 1 = 0 \quad (2.67) \]

for fiber dominated failure, where the \( I_i \) are the Cauchy strain invariants and the \( A_i \) are constants determined empirically for a given material.

Many researchers have tried to develop failure theories for composites, however simple curve fitting of biaxial test data is still used to create failure surfaces, because the data do not correlate well with any common failure theories [55]. Methods for predicting failure include Tsai-Wu failure theory, linear laminated plate theory with maximum strain criteria, and progressive failure modeling with a maximum strain criterion. These theories have been used to try to predict failure of tubular and cruciform specimens made of several different composite materials.

### Modeling of Biaxial Failure

Swanson and Christoforou [56] used tubular specimens to test AS4/3501-6 carbon epoxy quasi-isotropic \([90/\pm45/0]_s\) laminates. Their stress-strain data shows good agreement with linear laminated plate theory (LPT) until the stress reaches about 90 ksi, when the stiffness of the samples decreases. This is due partly to a nonlinear shear response in the 45 degree plies and partly to matrix microcracking. The reduction in slope would be greater were it not for the increased stiffness at strains greater than 1% noticed in tests of uniaxial tensile coupons. These effects seem to cancel one another to some extent, giving a small net reduction in slope. The data were compared with three failure models. They used both LPT and a progressive failure model with a maximum fiber...
strain failure criterion and Tsai-Wu. The progressive failure model includes a criterion for matrix cracking, a model of stiffness changes with further straining after matrix cracking, a criterion for ultimate fiber failure, and a nonlinear matrix shear response. The two maximum fiber strain criteria compare well with the experimental data, but the Tsai-Wu method does not correlate well at all, presumably because it predicts failure in laminates in which matrix failure coincides with or leads to laminate failure. Fiber dominated failure, in which the matrix simply redistributes stresses as the fibers fail, is not well modeled by the quadratic Tsai-Wu equation [50].

Zimmerman and Adams [57], have used cruciform specimens for biaxial testing and a least squares curve fit of the data to plot an elliptical failure surface for Rynite, which is an injection molded glass reinforced polyethylene terephthalate. The failure surface curve fit is similar in form to the Tsai-Wu equation, as described by Owen [58]. The Tsai-Wu equation is:

\[
F_{xx} \sigma_{xx}^2 + 2F_{xy} \sigma_{xx} \sigma_{xy} + F_{yy} \sigma_{yy}^2 + F_{ss} \sigma_{ss}^2 + F_{xx} \sigma_{xx} + F_{yy} \sigma_{yy} = 1
\]

(2.68)

and the curve fit is

\[
A \sigma_{xx}^2 + C \sigma_{xx} \sigma_{xy} + B \sigma_{yy}^2 + D \sigma_{yy} + E \sigma_{xy} = 1
\]

(2.69)

Although the curve fit equation does not contain an \( F_{ss} \sigma_{ss}^2 \) term, the values given for \( A, B, D, \) and \( E \) are approximately equal to their corresponding Tsai-Wu parameters, \( F_{xx}, F_{yy}, F_x, \) and \( F_y \), as calculated from the curve fit data. The value for \( C \) is of the same order of magnitude as, but is not equal to \( F_{xy} \).

Zimmerman, Walrath and Adams [59] have also studied unidirectional, continuous fiber graphite/aluminum composites. Again, they used cruciform biaxial specimens to study the failure modes of the composite and calculated the elliptical failure surface using a least squares fit. The curve fit parameters corresponded with the Tsai-Wu equation in the same way as the Rynite
parameters did. Of course, since one term is missing, neither comparison of curve-fits with Tsai-Wu theory tells us anything more than that there is some similarity. It is hard to tell whether Tsai-Wu would be too conservative, as it was for the tubular samples discussed above. However, failure is more matrix-dependent in the materials used by Adams' group, than it is in epoxy matrix composites.

Failure models are useful predictors of failure, and one can use them to ascertain some information about material behavior. However, the present failure theories cannot, in and of themselves, predict progressive cracking or the change in stiffness that accompanies such cracking. They do not account for delamination. Therefore, it is important to use a progressive failure model. Two currently accepted best techniques are shear lag theory and finite element analysis. Results shown in Figure 2.5 indicate that, in some cases, shear lag gives better predictions. The development of an analytical model is preferred in this case due to the ability to derive closed form solutions.
CHAPTER 3
DEVELOPMENT OF MODEL

Three-Layer Model

The three-layer model incorporates transverse matrix cracking in the 90° plies of fiber reinforced composites of the type [±θ/90°n]s under biaxial tension. It also includes an estimation of local delamination effects. Transverse matrix cracking typically occurs first in the constrained 90° plies when a load is applied transverse to the ply fiber direction. This is shown in Figure 3.1. Matrix cracking in the 90° plies is modeled using a two-dimensional shear lag theory along with the Griffith energy criterion for crack propagation. Although Flaggs' [30] shear lag model gives the onset of matrix cracking only, it includes through thickness shear terms explicitly in the energy equations. The three-layer model makes extensive use of Flaggs' derivation. The details are incorporated in this work in order to point out the differences between the present model and that of

Figure 3.1. The Onset of Transverse Matrix Cracking
Flaggs. In order to incorporate the progression of cracking, Flaggs' boundary conditions are changed in the present model to allow for more than one crack. After describing the derivation of the governing equation, it is solved for the appropriate boundary conditions. The progressive failure model may then be described.

**Explanation and Solution of the Governing Equation**

The governing equation is developed using linear laminated plate theory constitutive equations for each lamina group (±θ and 90°), equilibrium relations which incorporate transverse shear stress continuity, and shear lag theory. See Figure 3.2. It is assumed that ply group 1 (±θ)

![Figure 3.2. Transverse Matrix Cracking](image)
of laminae having the same orientation, i.e. all +θ or all -θ. The final equations are simplified for the case of ply group one being symmetric about its midplane, or, in other words, being of the type ±θ. The symbols used in the derivation are summarized in Table 3.1.

Table 3.1. Symbols Used in the Model Equations

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>N_j</td>
<td>Integrated laminate force resultant in the j direction</td>
</tr>
<tr>
<td>A_{ij}</td>
<td>Element of extensional stiffness matrix defined by linear laminated plate theory with the assumption of ε_2=0</td>
</tr>
<tr>
<td>ε_j</td>
<td>Laminate midplane strain in j direction</td>
</tr>
<tr>
<td>N_j^T</td>
<td>Integrated laminate thermal force resultant in the j direction</td>
</tr>
<tr>
<td>Q_i</td>
<td>Integrated laminate through-thickness shear resultant in the i direction</td>
</tr>
<tr>
<td>γ_i</td>
<td>Laminate through-thickness shear strain</td>
</tr>
<tr>
<td>n_j^{(k)}</td>
<td>Integrated stress resultant in the j direction for the k^{th} ply group</td>
</tr>
<tr>
<td>n_j^{t(k)}</td>
<td>Integrated thermal force resultant for the k^{th} ply group</td>
</tr>
<tr>
<td>q_i^{(k)}</td>
<td>Integrated through-thickness shear resultant for the k^{th} ply group</td>
</tr>
<tr>
<td>U_i</td>
<td>Total displacement in the i direction</td>
</tr>
<tr>
<td>u_i</td>
<td>Laminate midplane displacement in the i direction</td>
</tr>
<tr>
<td>ε_j^{(k)}</td>
<td>Midplane strain in k^{th} ply group</td>
</tr>
<tr>
<td>τ_{ij}^{(k)}</td>
<td>Shear stress in the ij direction for the k^{th} ply group</td>
</tr>
<tr>
<td>τ_{ij}^{th}</td>
<td>Interlaminar shear stress between ply groups 1 and 2</td>
</tr>
<tr>
<td>Δn_i^{(k)}</td>
<td>Interlaminar shear stress between ply groups 2 and 3</td>
</tr>
<tr>
<td>Δn_i^{th}</td>
<td>Change in integrated force resultant due to cracking</td>
</tr>
<tr>
<td>H_k</td>
<td>Thickness of k^{th} ply group</td>
</tr>
<tr>
<td>c</td>
<td>Distance between two cracks for three-layer model</td>
</tr>
<tr>
<td>L_t</td>
<td>Distance between two cracks in ply group 1 for five-layer model</td>
</tr>
<tr>
<td>L_m</td>
<td>Distance between two cracks in ply group 2 for five-layer model</td>
</tr>
<tr>
<td>L_b</td>
<td>Distance between two cracks in ply group 3 for five-layer model</td>
</tr>
</tbody>
</table>
The laminate material properties are derived from linear laminated plate theory.

\[ A_{ij}^{(k)} = \sum_{k=1}^{n} Q_{1j}^{(k)} H_k \quad \text{for } i,j = 1,3 \]

\[ A_{ij}^{(k)} = \sum_{k=1}^{n} \frac{3}{4} Q_{ij}^{(k)} H_k \quad \text{for } i,j = 4,5 \]

[52, 30] where

\[ Q_{11} = Q_{11} \cos^4 \theta + 2(Q_{12} + 2Q_{66}) \cos^2 \theta \sin^2 \theta + Q_{22} \sin^4 \theta \]

\[ Q_{22} = Q_{22} \cos^4 \theta + 2(Q_{12} + 2Q_{66}) \cos^2 \theta \sin^2 \theta + Q_{11} \sin^4 \theta \]

\[ Q_{12} = (Q_{11} + Q_{22} - 4Q_{66}) \cos^2 \theta \sin^2 \theta + Q_{12} (\cos^4 \theta + \sin^4 \theta) \]

\[ Q_{16} = -\cos \theta \sin ^3 \theta Q_{22} + \sin \theta \cos ^3 \theta Q_{11} - \cos \theta \sin \theta (\cos^2 \theta - \sin^2 \theta) (Q_{12} + 2Q_{66}) \]

\[ Q_{26} = -\sin \theta \cos ^3 \theta Q_{22} + \cos \theta \sin ^3 \theta Q_{11} + \cos \theta \sin \theta (\cos^2 \theta - \sin^2 \theta) (Q_{12} + 2Q_{66}) \]

\[ Q_{66} = (Q_{11} + Q_{22} - 2Q_{12}) \cos^2 \theta \sin^2 \theta + Q_{66} (\cos^2 \theta - \sin^2 \theta)^2 \]
\[ Q_{44} = Q_{44} \cos^2 \theta + Q_{55} \sin^2 \theta \]

\[ Q_{55} = Q_{55} \cos^2 \theta + Q_{44} \sin^2 \theta \]

\[ Q_{45} = (Q_{55} - Q_{44}) \cos \theta \sin \theta \]

The \( Q_{ij} \) are defined in the present model using the assumption, \( \epsilon_x = 0 \). Jones [38] has derived the appropriate expressions for the \( Q_{ij} \), which are given below.

\[
Q_{11} = \frac{(\nu_{13} + \nu_{12}\nu_{23})^2 E_{33}}{E_{11}(1-\nu_{23}\nu_{32}) - 1 - \nu_{12}\nu_{21}}
\]

\[
Q_{22} = \frac{(\nu_{23} + \nu_{21}\nu_{13})^2 E_{33}}{E_{22}(1-\nu_{31}\nu_{13}) - 1 - \nu_{12}\nu_{21}}
\]

\[
Q_{12} = \frac{(\nu_{13} + \nu_{12}\nu_{23})(\nu_{23} + \nu_{21}\nu_{13})E_{33}}{E_{22}(\nu_{12} + \nu_{13}\nu_{32}) - 1 - \nu_{12}\nu_{21}}
\]

\[ Q_{66} = G_{12} \]

\[ Q_{44} = G_{23} \]

\[ Q_{55} = G_{13} \]

where the \( E_{ij} \) and the \( G_{ij} \) are the usual Young's moduli and shear moduli.
Flaggs [30] defines the strain state in the lamina groups as:

\[
\begin{align*}
\epsilon_1^{(k)} &= \frac{\delta U_1^{(k)}}{\delta x_1} = \epsilon^o \\
\epsilon_2^{(k)} &= \frac{\delta U_2^{(k)}}{\delta x_2} = \frac{\delta u_2^{(k)}}{\delta x_2} + z \frac{\delta \gamma_4^{(k)}}{\delta x_2} \\
\epsilon_6^{(k)} &= \frac{\delta U_1^{(k)}}{\delta x_2} = \frac{\delta u_1^{(k)}}{\delta x_2} + z \frac{\delta \gamma_5^{(k)}}{\delta x_2} \\
\gamma_4^{(k)} &= \frac{\delta U_2^{(k)}}{\delta z} \\
\gamma_5^{(k)} &= \frac{\delta U_1^{(k)}}{\delta z}
\end{align*}
\]

(3.4)

where \( k \) refers to the lamina group, the \( U_1^{(k)} \) are total displacements, the \( u_1^{(k)} \) are in-plane displacements, and the strain terms are the lamina group midplane strains, as defined, for example, in reference [52]. Notice that, since the \( x_1 \) direction is not affected by transverse matrix cracking, the strain in the \( x_1 \) direction is always the same in both laminate groups. In addition, \( \epsilon_6 \) is the same as \( \gamma_{12} \) here; that is, \( \epsilon_6 = 2 \epsilon_{12} \). Flaggs gives the in-plane force and moment equilibrium equations for each ply group as:

\[
\begin{align*}
\frac{dn_6^{(k)}}{dx_2} + \tau_{13}^{(k)} \left[ \frac{H_k}{2} \right] - \tau_{13}^{(k)} \left[ -\frac{H_k}{2} \right] &= 0 \\
\frac{dn_2^{(k)}}{dx_2} + \tau_{23}^{(k)} \left[ \frac{H_k}{2} \right] - \tau_{23}^{(k)} \left[ -\frac{H_k}{2} \right] &= 0 \\
q_5^{(k)} - \frac{H_k}{2} \left[ \tau_{13}^{(k)} \left[ \frac{H_k}{2} \right] + \tau_{13}^{(k)} \left[ -\frac{H_k}{2} \right] \right] &= 0 \\
q_4^{(k)} - \frac{H_k}{2} \left[ \tau_{23}^{(k)} \left[ \frac{H_k}{2} \right] + \tau_{23}^{(k)} \left[ -\frac{H_k}{2} \right] \right] &= 0
\end{align*}
\]

(3.5)

where the \( n_1^{(k)} \) are the in-plane stress resultants for lamina group \( k \), the \( q_5^{(k)} \) are the
through-thickness shear resultants for lamina group k, the \( H_k \) are lamina group thicknesses, and the \( \tau_{ij}^{(k)} \) are surface shear stresses, defined at the top and bottom of each lamina group as shown.

The stress resultants are given by linear laminated plate theory, where each lamina group is treated as a laminate in equilibrium with the other lamina group, as is shown below.

\[
\begin{pmatrix}
 n_1^{(1)} \\
 n_2^{(1)} \\
 n_6^{(1)}
\end{pmatrix} = \begin{bmatrix}
 A_{11} & A_{12} & A_{16} \\
 A_{12} & A_{22} & A_{26} \\
 A_{16} & A_{26} & A_{66}
\end{bmatrix} \begin{pmatrix}
 \epsilon_1^{(1)} \\
 \epsilon_2^{(1)} \\
 \epsilon_6^{(1)}
\end{pmatrix} - \begin{pmatrix}
 n_1^{(1)} \\
 n_2^{(1)} \\
 n_6^{(1)}
\end{pmatrix} (3.6)
\]

where the \( n_i^{(1)} \) are the residual integrated thermal loads due to curing. Likewise,

\[
\begin{pmatrix}
 n_1^{(2)} \\
 n_2^{(2)} \\
 n_6^{(2)}
\end{pmatrix} = \begin{bmatrix}
 A_{11} & A_{12} & 0 \\
 A_{12} & A_{22} & 0 \\
 0 & 0 & A_{66}
\end{bmatrix} \begin{pmatrix}
 \epsilon_1^{(2)} \\
 \epsilon_2^{(2)} \\
 \epsilon_6^{(2)}
\end{pmatrix} - \begin{pmatrix}
 n_1^{(2)} \\
 n_2^{(2)} \\
 n_6^{(2)}
\end{pmatrix} (3.7)
\]

In addition,

\[
\begin{pmatrix}
 q_4^{(k)} \\
 q_5^{(k)}
\end{pmatrix} = \begin{bmatrix}
 A_{44} & A_{45} \\
 A_{45} & A_{55}
\end{bmatrix} \begin{pmatrix}
 \gamma_4^{(k)} \\
 \gamma_5^{(k)}
\end{pmatrix} (3.8)
\]

Since there are no applied surface shear stresses and, since the laminate is symmetric, the equilibrium equations can be simplified to:

\[
\begin{align*}
\frac{dn_6^{(1)}}{dx_2} - \tau_{13} &= 0 \\
\frac{dn_6^{(1)}}{dx_2} - \tau_{23} &= 0 \\
q_5^{(1)} - \frac{H_1}{2} \tau_{13} &= 0 \\
q_4^{(1)} - \frac{H_1}{2} \tau_{23} &= 0
\end{align*} (3.9)
\]
Before cracking, the state of strain is the same in all lamina groups, but after cracking, it is different. Thus, after cracking, according to Flaggs [30],

\[
\begin{align*}
\left[ \Delta n_1 \right]^{(1)} &= \left[ n_1 \right]^{(1)} - \left[ A_{11} A_{12} A_{16} \right]^{(1)} \left[ \epsilon_1 \right]^{(1)} + \left[ n_1^\dagger \right]^{(1)} \\
\left[ \Delta n_2 \right]^{(1)} &= \left[ n_2 \right]^{(1)} - \left[ A_{12} A_{22} A_{26} \right]^{(1)} \left[ \epsilon_1 \right]^{(1)} + \left[ n_2^\dagger \right]^{(1)} \\
\left[ \Delta n_6 \right]^{(1)} &= \left[ n_6 \right]^{(1)} - \left[ A_{16} A_{26} A_{66} \right]^{(1)} \left[ \epsilon_1 \right]^{(1)} + \left[ n_6^\dagger \right]^{(1)} \\
\end{align*}
\]

(3.11)

where \( \epsilon_i \) are the strains before cracking and \( n_j \) are the resultant stresses after cracking. Recalling that \( \epsilon_1 \) is not affected by matrix cracking in ply group 2, the relationships between \( \Delta n_1, \Delta n_2, \) and \( \Delta n_6 \) can be found. This author has verified Flaggs' result, shown below.

\[
\Delta n_1^{(1)} = a_2^{(1)} \Delta n_2^{(1)} - a_6^{(1)} \Delta n_6^{(1)}
\]

(3.12)
where

\[
a_2^{(1)} = \left[ \frac{A_{12}A_{66} - A_{16}A_{26}}{A_{22}A_{66} - A_{26}^2} \right]^{(1)}
\]

\[
a_6^{(1)} = \left[ \frac{A_{12}A_{26} - A_{16}A_{22}}{A_{22}A_{66} - A_{26}^2} \right]^{(1)}
\]

Likewise, for ply group 2,

\[
\Delta n_1^{(2)} = a_2^{(2)} \Delta n_2^{(2)} - a_6^{(2)} \Delta n_6^{(2)}
\]

where

\[
a_2^{(2)} = \left[ \frac{A_{12}A_{66} - A_{16}A_{26}}{A_{22}A_{66} - A_{26}^2} \right]^{(2)}
\]

\[
a_6^{(2)} = \left[ \frac{A_{12}A_{26} - A_{16}A_{22}}{A_{22}A_{66} - A_{26}^2} \right]^{(2)}
\]

Differentiating the in-plane equilibrium equations (3.9) with respect to \(x_2\) gives (from Flaggs):

\[
\frac{d^2 n_6^{(1)}}{dx_2^2} - \frac{dr_{13}}{dx_2} = 0
\]

\[
\frac{d^2 n_2^{(1)}}{dx_2^2} - \frac{dr_{23}}{dx_2} = 0
\]
Shear lag theory says:

\[ \gamma_5^{(2)} = \frac{1}{H_2} \left( U_1^{(1)} - U_1^{(2)} \right) \]  
\[ \gamma_4^{(2)} = \frac{1}{H_2} \left( U_2^{(1)} - U_2^{(2)} \right) \]  

(3.17)

where the \( U_i^{(k)} \) are evaluated at \(-\frac{H_k}{2}\) [30]. These equations can be derived from equations (3.1), since, for example:

\[ \gamma_4^{(k)} = \frac{\delta U_2^{(k)}}{\delta z} \Rightarrow \Delta U_2^{(2)} = \gamma_4^{(2)} \Delta z \]  

(3.18)

\[ \Rightarrow U_2^{(2)} \left[ \frac{H_2}{2} \right] - U_2^{(2)} \left[ -\frac{H_2}{2} \right] = H_2\gamma_4^{(2)} \]

Since continuity implies that \( U_2^{(2)} \) at \( \frac{H_2}{2} \) is equal to \( U_1^{(1)} \) at \( \frac{H_1}{2} \), where \( \frac{H_2}{2} \) defines the top surface of lamina group 2 and \( \frac{H_1}{2} \) defines the bottom surface of lamina group 1, equation (3.18) is identical to equation (3.17). Substituting (3.17) into (3.8) then (3.10), and differentiating with respect to \( x_2 \) gives, from Flaggs [30]:

\[ \frac{dr_{13}}{dx_2} = \frac{2A_{55}^{(2)}}{H_2^2} \left[ \frac{\delta U_1^{(1)}}{\delta x_2} - \frac{\delta U_1^{(2)}}{\delta x_2} \right] \]  
\[ \frac{dr_{23}}{dx_2} = \frac{2A_{44}^{(2)}}{H_2^2} \left[ \frac{\delta U_2^{(1)}}{\delta x_2} - \frac{\delta U_2^{(2)}}{\delta x_2} \right] \]  

(3.19)

Substituting for \( \frac{\delta U_1^{(k)}}{\delta x_2} \) using equation (3.4) and remembering that \( U_i^{(k)} \) are evaluated at \(-\frac{H_k}{2}\)
Flaggs [30] finds:

\[
\frac{d\tau_{13}}{dx_2} = \frac{2A_{55}(2)}{H_2^2} \left[ \frac{\epsilon_1(1)}{6} - \frac{H_1 \delta \gamma_5(1)}{2 \delta x_2} - \frac{\epsilon_2(2)}{6} + \frac{H_2 \delta \gamma_5(2)}{2 \delta x_2} \right]
\]

(3.20)

\[
\frac{d\tau_{23}}{dx_2} = \frac{2A_{44}(2)}{H_2^2} \left[ \frac{\epsilon_2(1)}{2} - \frac{H_1 \delta \gamma_4(1)}{2 \delta x_2} - \frac{\epsilon_2(2)}{2} + \frac{H_2 \delta \gamma_4(2)}{2 \delta x_2} \right]
\]

Flaggs shows that by inverting equation (3.11) and noting that \( \epsilon_1(1) - \epsilon_1(1) = 0 \), one obtains the following relation:

\[
\begin{bmatrix} \dot{\epsilon}_2(1) \\ \dot{\epsilon}_6(1) \end{bmatrix} = \begin{bmatrix} \dot{\epsilon}_2 \\ \dot{\epsilon}_6 \end{bmatrix} + \begin{bmatrix} F_{12} & F_{22} & F_{26} \\ F_{16} & F_{26} & F_{66} \end{bmatrix} \begin{bmatrix} \Delta n_1(1) \\ \Delta n_2 \\ \Delta n_6 \end{bmatrix}
\]

(3.21)

At this point, the present derivation deviates from that of Flaggs. Before cracking, a force balance on the 1-z plane gives:

\[
\begin{bmatrix} N_2 \\ N_6 \end{bmatrix} = \begin{bmatrix} n_2(1) \\ n_6 \end{bmatrix} + \begin{bmatrix} n_2(2) \\ n_6 \end{bmatrix}
\]

(3.22)

Flaggs describes the force balance in terms of different locations along x_2. The method used here is chosen because it clarifies the progressive cracking portion of the model. After cracking, the force balance on the 1-z plane is:

\[
\begin{bmatrix} N_2 \\ N_6 \end{bmatrix} = \begin{bmatrix} n_2(1) \\ n_6 \end{bmatrix} + \begin{bmatrix} n_2(2) \\ n_6 \end{bmatrix}
\]

(3.23)
The applied load does not change during cracking, so the total laminate stress resultants do not change; however, the load is redistributed among the ply groups. Thus,

\[
\begin{bmatrix}
N_2 \\
N_6
\end{bmatrix}_i = \begin{bmatrix}
N_2 \\
N_6
\end{bmatrix}_f
\]  
(3.24)

Using equations (3.22), (3.6), and (3.7),

\[
\begin{bmatrix}
N_2 \\
N_6
\end{bmatrix}_i = \begin{bmatrix}
A_{12} & A_{22} & A_{26} \\
A_{16} & A_{26} & A_{66}
\end{bmatrix} \begin{bmatrix}
\hat{\epsilon}_1 \\
\hat{\epsilon}_2 \\
\hat{\epsilon}_6
\end{bmatrix}^{(1)} - \begin{bmatrix}
n_2^{(1)} \\
n_6^{(1)}
\end{bmatrix}
\]
\[
+ \begin{bmatrix}
A_{12} & A_{22} & 0 \\
0 & 0 & A_{66}
\end{bmatrix} \begin{bmatrix}
\hat{\epsilon}_1 \\
\hat{\epsilon}_2 \\
\hat{\epsilon}_6
\end{bmatrix}^{(2)} - \begin{bmatrix}
n_2^{(2)} \\
n_6^{(2)}
\end{bmatrix}
\]  
(3.25)

Using equations (3.23) and (3.11),

\[
\begin{bmatrix}
N_2 \\
N_6
\end{bmatrix}_f = \begin{bmatrix}
A_{12} & A_{22} & A_{26} \\
A_{16} & A_{26} & A_{66}
\end{bmatrix} \begin{bmatrix}
\hat{\epsilon}_1 \\
\hat{\epsilon}_2 \\
\hat{\epsilon}_6
\end{bmatrix}^{(1)} - \begin{bmatrix}
n_2^{(1)} \\
n_6^{(1)}
\end{bmatrix}
\]
\[
+ \begin{bmatrix}
\Delta n_2^{(1)} \\
\Delta n_6
\end{bmatrix} + \begin{bmatrix}
n_2^{(2)} \\
n_6^{(2)}
\end{bmatrix}
\]  
(3.26)
Combining equations (3.24) through (3.26)

\[
\begin{align*}
\begin{bmatrix} \Delta n_2 \\ \Delta n_6 \end{bmatrix}^{(1)} + \begin{bmatrix} n_2 \\ n_6 \end{bmatrix}_f^{(2)} &= \begin{bmatrix} A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}^{(2)} \begin{bmatrix} \hat{\epsilon}_1 \\ \hat{\epsilon}_2 \\ \hat{\epsilon}_6 \end{bmatrix} \\
&= -\begin{bmatrix} n_2^{(2)} \\ n_6^{(2)} \end{bmatrix}
\end{align*}
\]

(3.27)

Substituting for \( \{n\}_f^{(2)} \),

\[
\begin{align*}
\begin{bmatrix} \Delta n_2 \\ \Delta n_6 \end{bmatrix}^{(1)} &= \begin{bmatrix} A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}^{(2)} \begin{bmatrix} \hat{\epsilon}_1 \\ \hat{\epsilon}_2 \\ \hat{\epsilon}_6 \end{bmatrix} - \begin{bmatrix} \hat{\epsilon}_1^{(2)} \\ \hat{\epsilon}_2^{(2)} \\ \hat{\epsilon}_6^{(2)} \end{bmatrix} \\
&= \begin{bmatrix} A_{22} & 0 \\ 0 & A_{66} \end{bmatrix}^{(2)} \begin{bmatrix} \hat{\epsilon}_2 \\ \hat{\epsilon}_6 \end{bmatrix} - \begin{bmatrix} \hat{\epsilon}_2^{(2)} \end{bmatrix}
\end{align*}
\]

(3.28)

Recalling that \( \epsilon_1 - \epsilon_1^{(2)} = 0 \), (3.28) may be simplified to:

\[
\begin{align*}
\begin{bmatrix} \Delta n_2 \\ \Delta n_6 \end{bmatrix}^{(1)} &= \begin{bmatrix} A_{22} & 0 \\ 0 & A_{66} \end{bmatrix}^{(2)} \begin{bmatrix} \hat{\epsilon}_2 \\ \hat{\epsilon}_6 \end{bmatrix} - \begin{bmatrix} \hat{\epsilon}_2^{(2)} \end{bmatrix} \\
&= \begin{bmatrix} \Delta n_2^{(1)} \\ \Delta n_6^{(1)} \end{bmatrix}
\end{align*}
\]

(3.29)

This result differs substantially from Flaggs' result. He found that:

\[
\hat{\epsilon}_2 - \epsilon_2^{(2)} = F_{12} \Delta n_1 + F_{22} \Delta n_2
\]

(3.30)

In order to obtain his result, Flaggs probably did a force balance on the 2-z plane. Since strain in the 1 direction cannot change, however, this force balance is inappropriate. The result shown in
equation (3.30) seems to contradict the results given in equations (3.12) through (3.15). For ply
group 1 symmetric and ply group 2 made of all 90 plies, (3.12) through (3.15) imply that:

$$\frac{\Delta n_1}{A_{22}} = (2) A_{12} \Delta n_2$$

and

$$\Delta n_1 = (1) A_{12} \Delta n_2$$

(3.31)

Since equation 3.8 must also hold for ply group 2, as is implied in equation (3.28),

$$\epsilon_2 - \epsilon_2 = F_{12} \Delta n_1 + F_{22} \Delta n_2$$

(3.32)

Because $\Delta n_2 = -\Delta n_2$, equations (3.30) and (3.32) imply that

$$\Delta n_1 = -\Delta n_1$$

However, this can be true, according to equations (3.31), only if $A_{12} / A_{22} = A_{12} / A_{22}$, which is
obviously not true if ply groups one and two have different orientations. Thus, since the change,
$\Delta n_1$, occurs during cracking due to the Poisson's interaction in each ply group, the constraint that

$$\Delta n_1 = -\Delta n_1$$

cannot be applied.

Equation (3.28) may therefore be combined with equation (3.18) to obtain:

$$\left\{ \begin{array}{c} \epsilon_2 \\ \epsilon_6 \end{array} \right\}^{(1)} - \left\{ \begin{array}{c} \epsilon_2 \\ \epsilon_6 \end{array} \right\}^{(2)} = \left[ \begin{array}{ccc} F_{12} & F_{22} & F_{26} \\ F_{16} & F_{26} & F_{66} \end{array} \right]^{(1)} +$$

$$\left[ \begin{array}{c} 0 & F_{22} & 0 \\ 0 & 0 & F_{66} \end{array} \right]^{(2)} \left\{ \begin{array}{c} \Delta n_1 \\ \Delta n_2 \\ \Delta n_6 \end{array} \right\}^{(1)}$$

(3.33)

Using equations (3.8), (3.9), and (3.20), the following expression is found:
\[
\begin{bmatrix}
\frac{dr_{23}}{dx_2} \\
\frac{dr_{13}}{dx_2}
\end{bmatrix} = \begin{bmatrix} L_{22} & L_{26} \\
L_{62} & L_{66} \end{bmatrix} \begin{bmatrix} \Delta n_2 \\
\Delta n_6 \end{bmatrix}^{(1)}
\]

(3.34)

where the L matrix is given below.

\[
\begin{bmatrix}
L_{22} & L_{26} \\
L_{62} & L_{66}
\end{bmatrix} = \frac{2}{H_2^2} \begin{bmatrix} t_{44} & t_{45} \\
t_{54} & t_{55} \end{bmatrix}^{-1} \begin{bmatrix} A_{44} & 0 \\
0 & A_{55} \end{bmatrix} \begin{bmatrix} F_{12} & F_{22} + \frac{1}{(2)} A_{22} \\
F_{16} & F_{26} \end{bmatrix}^{(1)}
\]

(3.35)

where the \( A_{ij} \) are elements of the laminated plate theory ABD matrix for ply group (1) or (2) as shown, the \( F_{ij} \) terms are elements of the inverse \( A \) matrix, \( H_2 \) is half of the thickness of the 90° ply group and the \( a \) terms are given in equations (3.13) and (3.15). The \( t \) terms are given below:

\[
t_{44} = 1 - \frac{A_{44}}{H_2^2} \left[ \frac{H_2^2}{2} F_{44}^{(2)} - \frac{H_1^2}{2} F_{44}^{(1)} \right]
\]

\[
t_{55} = 1 - \frac{A_{55}}{H_2^2} \left[ \frac{H_2^2}{2} F_{55}^{(2)} - \frac{H_1^2}{2} F_{55}^{(1)} \right]
\]

(3.36)

\[
t_{45} = \frac{A_{44}}{2H_2^2} H_1^2 F_{45}^{(1)}
\]

\[
t_{54} = \frac{A_{55}}{2H_2^2} H_1^2 F_{45}^{(1)}
\]
where the $F_{ij}$ terms are elements of the inverse $A$ matrix defined in equation (3.8). Finally, substituting, equation (3.34) into equation (3.16), the equations for matrix cracking are found.

$$
\begin{align*}
\frac{d^2 \Delta n_2}{dx^2} & + \begin{bmatrix} L_{22} & L_{26} \\ L_{62} & L_{66} \end{bmatrix} \begin{bmatrix} \Delta n_2 \\ \Delta n_6 \end{bmatrix} = 0 \\
\frac{d^2 \Delta n_6}{dx^2} & - L_{26} \begin{bmatrix} \Delta n_2 \\ \Delta n_6 \end{bmatrix} = 0
\end{align*}
$$

Equations (3.34) and (3.37) are thus derived by the same method as Flaggs used and are essentially the same. The only difference is found in the $L$ matrix, which is defined above. The reader should note that this governing equation is defined for ply group 1 symmetric or nonsymmetric. The coupled differential equations can be solved by premultiplying both sides by a matrix with a constant determinant [60].

$$
\begin{align*}
\begin{bmatrix} L_{62} D^2 - L_{22} & D^2 L_{22} - L_{26} \\ 1 & D^2 - L_{62} \end{bmatrix} \begin{bmatrix} D^2 - L_{22} & -L_{26} \\ -L_{62} & D^2 - L_{66} \end{bmatrix} \begin{bmatrix} \Delta n_2 \\ \Delta n_6 \end{bmatrix} & = 0 \\
\begin{bmatrix} \Delta n_2 \\ \Delta n_6 \end{bmatrix} & = 0
\end{align*}
$$

where $D$ is the differential operator. Multiplying through leads to:

$$
\begin{align*}
\begin{bmatrix} 0 & -L_{62} L_{26} + (D^2 - L_{22})(D^2 - L_{66}) \endif{D^2 - L_{22} + L_{26}} D_6^2 (D^2 - L_{66}) \\ -L_{22} & -L_{26} + \frac{D^2}{L_{62}} (D^2 - L_{66}) \end{bmatrix} \begin{bmatrix} \Delta n_2 \\ \Delta n_6 \end{bmatrix} & = 0 \\
\begin{bmatrix} \Delta n_2 \\ \Delta n_6 \end{bmatrix} & = 0
\end{align*}
$$

One of the equations is now expressed in terms of $\Delta n_5$ only. Its auxiliary equation can be solved to
yield the roots:

\[
\lambda = \pm \frac{1}{2} \sqrt{L_{22} + L_{66}} \pm \frac{1}{2} \sqrt{(L_{22} - L_{66})^2 + 4L_{26}L_{62}}
\]  

(3.40)

Then, the assumed expression for \( \Delta n_6 \) can be written as is shown below [60].

\[
\Delta n_6 = Ae^{\lambda_1 x_2} + Be^{\lambda_2 x_2} + Ce^{\lambda_3 x_2} + De^{\lambda_4 x_2}
\]  

(3.41)

At this point, the present three layer model deviates more significantly from Flaggs derivation. Flaggs assumes one crack occurs at \( x_2 = 0 \) and that the original stress field is recovered far from the crack at \( x_2 \rightarrow \infty \). This method does not allow the prediction of the progression of cracking. Flaggs' boundary conditions, which imply exponential decay, allow the positive roots to be discarded. In the present three layer model, two cracks are assumed initially in order to determine the in-plane stress distribution between them. See Figure 3.2. At the locations of cracking, \( x_2 = \pm c/2 \),

\[
\begin{bmatrix}
    n_2 \\
    n_6
\end{bmatrix}^{(2)} = \begin{bmatrix}
    0 \\
    0
\end{bmatrix} \quad \text{so} \quad \begin{bmatrix}
    n_2 \\
    n_6
\end{bmatrix}^{(1)} = \begin{bmatrix}
    N_2 \\
    N_6
\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}
    \Delta n_2 \\
    \Delta n_6
\end{bmatrix}^{(1)} = \begin{bmatrix}
    n_2 \\
    n_6
\end{bmatrix}^{(2)}
\]  

(3.42)

At the cracks, the change in ply group 1 loads are \( \Delta n_2^* \) and \( \Delta n_6^* \), which are the loads in ply group 2 at the crack locations before cracking. The present boundary conditions require that all four roots be kept, so the assumed exponential solution can be written in terms of hyperbolic functions. Substituting (3.41) into (3.39) to get \( \Delta n_2 \) and substituting \( \Delta n_2 \) and \( \Delta n_6 \) into (3.42) gives the solution to the governing equations.
\[ \Delta n_2 = \frac{\left[ \frac{1}{2}(L_{66} - L_{22} + \lambda_3^2 - \lambda_1^2) \Delta n_2^* - L_{26} \Delta n_6^*}{(\lambda_3^2 - \lambda_1^2)} \right] \cosh(\lambda_1 x_2)}{\cosh(\lambda_1 c/2)} + \left[ \frac{1}{2}(L_{22} - L_{66} + \lambda_3^2 - \lambda_1^2) \Delta n_2^* + L_{26} \Delta n_6^*}{(\lambda_3^2 - \lambda_1^2)} \right] \cosh(\lambda_3 x_2)}{\cosh(\lambda_3 c/2)} \] (3.43)

\[ \Delta n_6 = \frac{\left[ \frac{1}{2}(L_{66} - L_{22} + \lambda_3^2 - \lambda_1^2) \Delta n_6^* - L_{62} \Delta n_2^*}{(\lambda_3^2 - \lambda_1^2)} \right] \cosh(\lambda_1 x_2)}{\cosh(\lambda_1 c/2)} + \left[ \frac{1}{2}(L_{66} - L_{22} + \lambda_3^2 - \lambda_1^2) \Delta n_6^* + L_{62} \Delta n_2^*}{(\lambda_3^2 - \lambda_1^2)} \right] \cosh(\lambda_3 x_2)}{\cosh(\lambda_3 c/2)} \] (3.44)

where the \( \Delta n_i^* \) are initially given by the \( n_i \) given in equation (3.7) above. The \( \lambda_i \) are the roots of the auxiliary equation, and are given by:

\[
\lambda_{1,3} = \pm \frac{1}{2} \sqrt{\left( \frac{1}{2} - \frac{1}{2} \left( L_{22} + L_{66} \right) \right)^2 + 4L_{26}L_{62}}
\] (3.45)

Progressive Cracking Model

The Griffith energy criterion for cracking is \( d(\Delta W_{ext} - \Delta W_{int})/da \geq G_c \) where \( \Delta W_{ext} \) is the external work done on the laminate and \( \Delta W_{int} \) is the change in strain energy of the laminate. In the current state there are two cracks, as discussed above. In the new damage state, there will be an additional crack located somewhere between the first two. If probability density based on the stress distribution between the two cracks is used, one finds that the expected average location of the third crack is halfway between the first two. However, flaws in the material are randomly distributed, and the probability density function for the location of the new crack should ideally take that into account. Therefore, the location of the new crack is unknown at present.

In order to get from crack state one to crack state two, one assumes that the applied load must be increased. This is due to the assumption that crack state one is the equilibrium crack state.
for the applied loads which caused it. The derivation of the strain energy density, which is not given explicitly by Flaggs, is shown below. The strain energy density for crack state $j$ is given by the sum of the in-plane or $ip$ and through-thickness or $tt$.

$$
dW^{int}_{int j} = (\Delta W^{int}_{ip j} + \Delta W^{int}_{tt j})dx_2
$$

(3.46)

The in-plane term is given by the following standard strain energy expression.

$$
\Delta W^{int}_{ip j} = 2 \sum_{k=1}^{n} \int_{z_{bot}}^{z_{top}} \sum_{i,j} \epsilon^{ijiso} \sigma^{ij}(k) d\epsilon^{ij}(k) dz
$$

(3.47)

where $z_{bot}$ and $z_{top}$ are the bottom and top of ply group $k$ defined relative the laminate midplane and the factor of 2 is needed since $n$ refers to half of the ply groups. The strain terms, $\epsilon^{ijiso}$ refer to isothermal or, in other words, mechanical strains. Integrating with respect to $z$ gives:

$$
\Delta W^{int}_{ip j} = 2 \sum_{k=1}^{n} \int \epsilon^{ijiso} n^{ij}(k) d\epsilon^{ij}(k)
$$

(3.48)

For each lamina group:

$$
\Delta W^{int}_{ip (k)} = \int \epsilon^{ijiso} (n(\epsilon^{(k)})) d(\epsilon^{(k)})
$$

(3.49)
where the \( k \) should be assumed throughout the remainder of the derivation. Continuing, then,

\[
\Delta W_{\text{int}} \int p^{(k)} = \frac{1}{2} [A] \left\{ (\epsilon - \alpha \Delta T)^2 - [\epsilon - \alpha \Delta T]^2 \right\} \quad (3.50)
\]

where \( \epsilon \) and \( \hat{\epsilon} \) are total strain. The term in braces can be expanded and simplified to get:

\[
\Delta W_{\text{int}} \int p^{(k)} = \frac{1}{2} [A] \left\{ (\Delta \epsilon)^T (\Delta \epsilon) + 2(\Delta \epsilon)^T \{\hat{\epsilon}\} - 2(\Delta \epsilon)^T a \Delta T \right\}
\]

\[
= \frac{1}{2} [A] (\Delta \epsilon)^T (\Delta \epsilon) + 2(\Delta \epsilon)^T \{\hat{\epsilon}\} - 2[\{\hat{\epsilon}\}] \quad (3.51)
\]

Substituting \( (\Delta \epsilon)^T = (\Delta n)^T [F] \) gives:

\[
\Delta W_{\text{int}} \int p^{(k)} = \frac{1}{2} (\Delta n)^T (\Delta n)_{j}^{(k)} [F] (\Delta n)_{j}^{(k)} + 2(\Delta \epsilon)_{j}^{(k)} - 2[F]_{j}^{(k)} \{n_{j}^{(k)}\} \quad (3.52)
\]

The derivation of the through-thickness strain energy density term begins with the z-integrated standard expression,

\[
dW_{\text{int tt}} j^{(k)} = \{q\}^{(k)} d\{\gamma\}^{(k)} \quad (3.53)
\]

Using the equilibrium expression given above,

\[
dW_{\text{int tt}} j^{(k)} = \frac{Hk}{2} \left\{ \frac{\tau_{23}(\frac{Hk}{2}) + \tau_{23}(\frac{Hk}{2})}{\tau_{13}(\frac{Hk}{2}) + \tau_{13}(\frac{Hk}{2})} \right\}^{T} [F_s]^{(k)} d\{q\}^{(k)} \quad (3.54)
\]
and, using the assumptions of zero shear on the outer laminate surface and at the midplane and equal shear at the interface between ply groups, this equation reduces to:

\[
\frac{1}{2} \left[ \frac{Hk^2}{4} \begin{bmatrix} \tau_{23} \\ \tau_{13} \end{bmatrix}^T \right]_{j} \left[ F_s \right]_{j}^{(k)} \begin{bmatrix} \tau_{23} \\ \tau_{13} \end{bmatrix}_{j}
\]

(3.55)

Thus, the strain energy density for state j is, including both in-plane and through-thickness shear terms:

\[
dW_{inj} = \frac{1}{2} \sum_{k=1}^{n} \left\{ \Delta n \right\} \begin{bmatrix} \hat{F}^T \{ \Delta n \} \{ u \}^{(k)} \{ \epsilon \} + 2 \{ \epsilon \} - 2 \{ F \} \{ n \} \{ \epsilon \} \right\} dx_2
\]

(3.56)

\[
+ \frac{1}{2} \sum_{k=1}^{n} \left[ \frac{Hk^2}{4} \begin{bmatrix} \tau_{23} \\ \tau_{13} \end{bmatrix}^T \right]_{j} \left[ F_s \right]_{j}^{(k)} \begin{bmatrix} \tau_{23} \\ \tau_{13} \end{bmatrix}_{j} dx_2
\]

where T refers to transpose, k indicates the ply group, and \( F_s \) is the inverse shear matrix. Both matrix cracking states are referred to the initial strain at the onset of cracking. Integrating this equation between \( x_2 = -c/2 \) and \( x_2 = c/2 \) gives \( \Delta W_{int} \) for state j. Then, \( \Delta W_{int} \) is \( \Delta W_{int2} - \Delta W_{int1} \).

The work done on the total laminate by external loads during crack formation is found from the applied loads and the ply group 1 mechanical strains. The quantity \( dW_{ext} \) is integrated from \( -c/2 \) to \( c/2 \), and is given by:

\[
dW_{ext} = 2N_2 \{ \Delta \epsilon_{2j} \} (\Delta \epsilon_{2j} - \Delta \epsilon_{2j} - 1) dx_2 + 2N_6 \{ \Delta \epsilon_{6j} \} (\Delta \epsilon_{6j} - \Delta \epsilon_{6j} - 1) dx_2
\]

(3.57)
where the one direction term is zero due to the assumption that strain in the one direction does not change, and the coefficient of 2 is used because $N_2$ and $N_6$ are defined as the loads applied to half of the laminate only.

For a laminate in which ply group one is symmetric, the resulting equation for total energy released by the introduction of the first two cracks in the new three-layer model is:

$$
\Delta W_{ext} - \Delta W_{int} = 4 \left[ \frac{N_2}{A_{22}} \begin{pmatrix} A_{12}^{(2)} & A_{12}^{(1)} \\ A_{22} & A_{22} \end{pmatrix} \epsilon_1 + \frac{N_2}{A_{22}} \begin{pmatrix} n_2^{(1)} & n_2^{(2)} \\ A_{22} & A_{22} \end{pmatrix} \left\{ \frac{A_{11} \tanh \frac{\lambda_1}{2}}{\lambda_1} + \frac{B_{11} \tanh \frac{\lambda_3}{2}}{\lambda_3} \right\} \right]
$$

$$
+ \left[ \frac{N_6}{A_{66}} \begin{pmatrix} A_{12}^{(2)} & A_{12}^{(1)} \\ A_{66} & A_{66} \end{pmatrix} \epsilon_1 + \frac{N_6}{A_{66}} \begin{pmatrix} n_6^{(1)} & n_6^{(2)} \\ A_{66} & A_{66} \end{pmatrix} \left\{ \frac{C_{11} \tanh \frac{\lambda_{1c}}{2}}{\lambda_1} + \frac{D_{11} \tanh \frac{\lambda_{3c}}{2}}{\lambda_3} \right\} \right]
$$

$$
- \left[ \frac{1}{A_{22}} \begin{pmatrix} A_{12}^{(1)} & A_{12}^{(2)} \\ A_{22} & A_{22} \end{pmatrix} \right] A^2 \left\{ \frac{1}{2} \tanh \frac{\lambda_{1c}}{2} + \frac{1}{2 \cosh^2(\lambda_{1c}/2)} \right\}
$$

$$
+ B^2 \left\{ \frac{1}{\lambda_3} \tanh \frac{\lambda_{3c}}{2} + \frac{1}{2 \cosh^2(\lambda_{3c}/2)} \right\}
$$

$$
+ 4AB \left\{ \frac{1}{\lambda_1} \tanh \frac{\lambda_{1c}}{2} + \frac{1}{2 \cosh^2(\lambda_{1c}/2)} \right\}
$$

$$
+ D^2 \left\{ \frac{1}{\lambda_3} \tanh \frac{\lambda_{3c}}{2} + \frac{1}{2 \cosh^2(\lambda_{3c}/2)} \right\}
$$

$$
+ 4CD \left\{ \frac{1}{\lambda_1} \tanh \frac{\lambda_{1c}}{2} + \frac{1}{2 \cosh^2(\lambda_{1c}/2)} \right\}
$$

$$
+ 4AB \left\{ \frac{1}{\lambda_3} \tanh \frac{\lambda_{3c}}{2} + \frac{1}{2 \cosh^2(\lambda_{3c}/2)} \right\}
$$

$$
+ 4CD \left\{ \frac{1}{\lambda_1} \tanh \frac{\lambda_{1c}}{2} + \frac{1}{2 \cosh^2(\lambda_{1c}/2)} \right\}
$$

(3.58)
where

\[
A = \frac{\frac{1}{2}[(L_{66}-L_{22}+\lambda_{\delta}^{3}-\lambda_{\delta}^{4})\Delta n_{\delta}^{2}-L_{26}\Delta n_{\delta}^{6}]}{(\lambda_{\delta}^{3}-\lambda_{\delta}^{4})} \\
B = \frac{\frac{1}{2}[(L_{22}-L_{66}+\lambda_{\delta}^{3}-\lambda_{\delta}^{4})\Delta n_{\delta}^{2}+L_{26}\Delta n_{\delta}^{6}]}{(\lambda_{\delta}^{3}-\lambda_{\delta}^{4})} \\
C = \frac{\frac{1}{2}[(L_{22}-L_{66}+\lambda_{\delta}^{3}-\lambda_{\delta}^{4})\Delta n_{\delta}^{2}-L_{62}\Delta n_{\delta}^{6}]}{(\lambda_{\delta}^{3}-\lambda_{\delta}^{4})} \\
D = \frac{\frac{1}{2}[(L_{66}-L_{22}+\lambda_{\delta}^{3}-\lambda_{\delta}^{4})\Delta n_{\delta}^{2}+L_{62}\Delta n_{\delta}^{6}]}{(\lambda_{\delta}^{3}-\lambda_{\delta}^{4})}
\]  

(3.59)

When the energy released described in equation (3.60) is divided by a characteristic crack size, the result can be compared with \( G_c \) and the applied load required to cause cracking can be iteratively determined. The characteristic crack size used is \( 2H_2 \) for \( H_2 \) less than 2 1/2 times the ply thickness, and it is equal to 2 1/2 times the ply thickness if \( H_2 \) is greater than that amount [30]. This is the appropriate size for edge notch flaws, which are more likely than internal flaws in tensile test coupons. Such tensile coupons were used in the experiments, which were done for verification of the current work. In addition, this author has chosen to use the same method as Flaggs [30] for estimating \( G_c \) under biaxial loading conditions. Matrix cracking is essentially Mode I, unless in-plane shear stress is applied, in which case Mode III terms must be taken into account. Flaggs assumes:

\[
\frac{G_{III}}{G_{I}} \approx \frac{G_{II}}{G_{I}}
\]  

(3.60)
An estimate of the ratio of $G_{III}$ to $G_1$ can be found using

$$\frac{G_{III}}{G_1} = \frac{\tau_{12}^2 F_{44} F_{55}}{\sigma_2^2 \sqrt{F_{22}} \left(2 \sqrt{F_{11} F_{22} + 2 F_{12} + F_{55}}\right)^{1/2}}$$

(3.61)

Finally, then the following expression, where $G_{III}$ has been substituted for $G_{II}$ based on the assumption of equation (3.60), can be used to estimate total strain energy release rate.

$$G_C = G_{IC} + 1.98 \left[\frac{G_{III}}{G_1}\right]$$

(3.62)

For the one biaxial model examined in this work, $G_{III}$ was negligible, since $\tau_{12}$ was very small compared with $\sigma_2$. This is discussed in Chapter 5.

At this point, the loads at which subsequent cracks are formed can be determined. The current model assumes incremental increases in load at constant crack density, then checks for cracking at the next crack density by comparing $G$ with $G_C$. The increase of load in each ply must be determined in the present model, because $\Delta n_2^*$ and $\Delta n_6^*$ change as the load changes. As a first approximation for the three-layer model, these quantities are based on the fraction of the load carried by each ply group after the previous cracks have formed. The load in a ply group at any given time is equal to the load it carried before the last crack occurred plus the change in load due to cracking. The proportion of load carried is the load in the ply group divided by the applied load. When the applied load is increased, the proportion of load taken by a ply group is equal to the change in applied load times the proportion of load carried. The resulting ply group loads become the initial loads (before cracking) defined in this theory.

The location of the next crack must also be determined and is not straightforward. As mentioned in Chapter 2, Wang [39] assumes the randomly located flaws follow a normal distribution,
Laws and Dvorak [33] assume a probability density function based on the stress in ply group 2, and Lee and Daniel [32] and Tsai, Daniel, and Lee [36] assume that the next crack must occur exactly half-way between the previous two cracks. The last assumption is used in the present model for simplicity, since both Wang’s [39] and Laws and Dvorak’s [33] probability functions predict that, on the average, the next crack will indeed occur halfway between the previous two.

A FORTRAN program has been written for progressive cracking in the three-layer model. The data which must be inputted to the program include material properties, ply thickness, layup, and applied load ratios for proportional loading. The program outputs crack density versus applied load and crack density versus Young’s modulus. The stresses and strains in the laminate are also given as a function of crack density.

**Five-Layer Model**

The five-layer model is for laminates of the type shown in Figure 3.4. The major difference between the three-layer model and the five-layer model is that the five-layer model incorporates an additional 90° layer. In addition, unlike the three-layer model, the five-layer model incorporates cracking in all layers; however, the five-layer model does not incorporate applied shear forces. At present, the five-layer model does not include angle plies, although the eventual goal of this researcher is to develop a model for laminates of the type [±0/90s], which explicitly includes matrix cracking in angle plies. The system of equations required for laminates with cracking in angle plies cannot be solved directly using the method outlined here; therefore, this researcher has decided to model laminates of the type [90n/0m/90p]s as a first step toward the final goal. By modeling five total layers, two ply group interfaces and corresponding shear stresses must be modeled. In contrast with the five-layer model, the three-layer model requires consideration of only one ply group interface, and, thus, only one set of interlaminar shear stresses. This seemingly small step significantly complicates the model.
Explanation and Solution of the Governing Equations

The approach to modeling progressive cracking is essentially the same as was shown in the three-layer model section with some exceptions. The outermost $90^\circ$ layer will be designated the top layer, the $0^\circ$ layer is the middle, and the innermost $90^\circ$ layer will be called the bottom layer. This makes sense, since only half of the laminate is modeled due to the inherent laminate symmetry.

Since there are three layers to consider, the load lost at the point of cracking of one layer must now be taken by the two layers, which are not cracked at that point. The development of the governing equations is discussed in detail below. Figure 3.4 shows the coordinate system and some of the parameters used in the derivation and Table 3.1 gives symbol definitions for reference.
For a crossply laminate with no applied in-plane shear force, the constitutive relations can be written as:

\[
\begin{bmatrix}
N_1 \\
N_2 \\
Q_4 \\
Q_5
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} \\
A_{12} & A_{22}
\end{bmatrix}
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2
\end{bmatrix} -
\begin{bmatrix}
N_1^T \\
N_2^T
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
Q_4 \\
Q_5
\end{bmatrix} =
\begin{bmatrix}
A_{44} & 0 \\
0 & A_{55}
\end{bmatrix}
\begin{bmatrix}
\gamma_4 \\
\gamma_5
\end{bmatrix}
\]

Each lamina group or layer is treated as a Mindlin plate element. This approach is similar to that taken by Flaggs [30]. The constitutive relations for each ply group take the form:

\[
\begin{bmatrix}
\gamma_1^{(k)} \\
\gamma_2^{(k)}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} \\
A_{12} & A_{22}
\end{bmatrix}
\begin{bmatrix}
\epsilon_1^{(k)} \\
\epsilon_2^{(k)}
\end{bmatrix} -
\begin{bmatrix}
\gamma_1^{(k)} \\
\gamma_2^{(k)}
\end{bmatrix}
\]

(3.64)
where the $A_{ij}$ are the elements of the stiffness matrices for each ply group. Note that for simplicity, we assume that in-plane shear is zero throughout the laminate. Tsai, Daniel and Lee [36] have made this assumption with little loss of accuracy. The displacements are assumed to be:

$$
U_i = u_i(x_1, x_2) + z_{75}(x_1, x_2)
$$

$$
U_2 = u_2(x_1, x_2) + z_{74}(x_1, x_2)
$$

$$
U_3 = 0
$$

The strain in the laminate when the first crack occurs is given by:

$$
\varepsilon = \begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2
\end{bmatrix}
$$

For in-plane loading, the strain is equal in all lamina groups prior to cracking. After cracking, of course, it will no longer be equal in all lamina groups. The strain state for each lamina group before or after cracking is defined as follows:
The equations of force and moment equilibrium are:

\[
\begin{align*}
\epsilon_1 &= \frac{\delta U_1}{\delta x_1} = \frac{\delta u_1}{\delta x_1} + \delta \gamma_k \\
\epsilon_2 &= \frac{\delta U_2}{\delta x_2} = \frac{\delta u_2}{\delta x_2} + \delta \gamma_k \\
\gamma_4 &= \frac{\delta U_2}{\delta z} \\
\gamma_5 &= \frac{\delta U_1}{\delta z}
\end{align*}
\]

(3.68)

Now, symmetry and the sample configuration dictate that the through-thickness shear stresses are zero at the laminate midplane and at the surface of the laminate. In addition, continuity implies that \( \tau_{23} \) and \( \tau_{13} \) are equal at the interfaces between lamina groups. The interlaminar shear stresses are called \( \tau_{ij}^{th} \) between ply groups 1 and 2 and \( \tau_{ij}^i \) between ply groups 2 and 3.

After cracking occurs in a ply group, the load taken by that ply group changes according to the following relation:
Substituting equation (3.70) and the through thickness boundary conditions discussed above into equations (3.69), one finds for ply group one:

\[
\frac{\delta}{\delta x_1} \left[ \Delta n_1^{(1)} + A_{11} \varepsilon_1 + A_{12} \varepsilon_2 \right] - \tau_{13}^{th} = 0
\]

\[
(3.71)
\]

Similar expressions can be obtained for ply groups two and three. Differentiating with respect to \(x_1\) or \(x_2\) as appropriate gives:

\[
\frac{\delta^2 \Delta n_1^{(1)}}{\delta x_1^2} - \frac{\delta \tau_{13}^{th}}{\delta x_1} = 0
\]

\[
(3.72)
\]

\[
\frac{\delta^2 \Delta n_2^{(1)}}{\delta x_2^2} - \frac{\delta \tau_{23}^{th}}{\delta x_2} = 0
\]
Likewise, for group 2,

\[
\frac{\delta^2 \Delta n_1^{(2)}}{\delta x_1^2} + \frac{\delta r_{13}^{th}}{\delta x_1} - \frac{\delta r_{13}^i}{\delta x_1} = 0
\]

and, for group 3,

\[
\frac{\delta^2 \Delta n_2^{(2)}}{\delta x_2^2} + \frac{\delta r_{23}^{th}}{\delta x_2} - \frac{\delta r_{23}^i}{\delta x_2} = 0
\]

The basis of shear lag theory is that

\[
\gamma = \frac{\delta u}{\delta z} \quad \text{and} \quad \gamma = \frac{\Delta u}{\Delta z} = \frac{u(\text{top of layer}) - u(\text{bottom of layer})}{\text{thickness of layer}}
\]
so,

\[ \gamma_{s}^{(1)} = \frac{1}{H_1} \begin{bmatrix} \frac{H_1}{2} & -u_1 \frac{H_2}{2} \\ \end{bmatrix} \]

\[ \gamma_{s}^{(2)} = \frac{1}{H_1} \begin{bmatrix} \frac{H_1}{2} & -u_2 \frac{H_2}{2} \\ \end{bmatrix} \]

\[ \gamma_{s}^{(3)} = \frac{1}{H_2} \begin{bmatrix} \frac{H_1}{2} & -u_1 \frac{H_2}{2} \\ \end{bmatrix} \]

\[ \gamma_{s}^{(4)} = \frac{1}{H_2} \begin{bmatrix} \frac{H_1}{2} & -u_2 \frac{H_2}{2} \\ \end{bmatrix} \]

\[ \gamma_{s}^{(5)} = \frac{1}{H_3} \begin{bmatrix} \frac{H_1}{2} & -u_1 \frac{H_2}{2} \\ \end{bmatrix} \]

\[ \gamma_{s}^{(6)} = \frac{1}{H_3} \begin{bmatrix} \frac{H_1}{2} & -u_2 \frac{H_2}{2} \\ \end{bmatrix} \]

Substituting equations (3.76) into the moment equilibrium equations (3.69),

\[ \frac{A_{55}}{H_1} \begin{bmatrix} \frac{H_1}{2} & -u_1 \frac{H_2}{2} \\ \end{bmatrix} \cdot \frac{H_1}{2} \tau_{18}^{th} = 0 \]

\( (3.77) \)

\[ \frac{A_{44}}{H_1} \begin{bmatrix} \frac{H_1}{2} & -u_2 \frac{H_2}{2} \\ \end{bmatrix} \cdot \frac{H_1}{2} \tau_{23}^{th} = 0 \]

\( (3.78) \)
\[
\frac{A_{55}}{H_3} \begin{bmatrix} u_1 \end{bmatrix} \begin{bmatrix} -H_2/2 \\ -H_2/2 \end{bmatrix} \begin{bmatrix} u_1 \end{bmatrix} - \frac{H_3}{2} \tau_{13}^i = 0
\]

(3.79)

\[
\frac{A_{44}}{H_3} \begin{bmatrix} u_2 \end{bmatrix} \begin{bmatrix} -H_2/2 \\ -H_2/2 \end{bmatrix} \begin{bmatrix} u_2 \end{bmatrix} - \frac{H_3}{2} \tau_{23}^i = 0
\]

Differentiating (3.77)

\[
\frac{A_{55}}{H_1} \begin{bmatrix} \delta u_1 \\ \delta x_1 \end{bmatrix} - \frac{A_{44}}{H_1} \begin{bmatrix} \delta u_2 \\ \delta x_2 \end{bmatrix} - \frac{H_1}{2} \delta \tau_{13}^\text{th} = 0
\]

(3.80)

\[
\frac{A_{44}}{H_1} \begin{bmatrix} \delta u_2 \\ \delta x_2 \end{bmatrix} - \frac{H_1}{2} \delta \tau_{23}^\text{th} = 0
\]

The midplane lamina strains, \( \epsilon_i \) are determined at \( z=0 \), so equations (3.63) become:

\[
\epsilon_1^{(k)} = \frac{\delta u_1}{\delta x_1} \quad \text{and} \quad \epsilon_2^{(k)} = \frac{\delta u_2}{\delta x_2}
\]

(3.81)

Equations (3.80) are then

\[
\frac{A_{55}}{H_1} \begin{bmatrix} \epsilon_1^{(1)} + \frac{H_1}{2} \delta \tau_{13}^\text{th} \\ \epsilon_2^{(1)} - \frac{H_1}{2} \delta \tau_{13}^\text{th} \end{bmatrix} = 0
\]

(3.82)

\[
\frac{A_{44}}{H_1} \begin{bmatrix} \epsilon_2^{(1)} + \frac{H_1}{2} \delta \tau_{23}^\text{th} \\ \epsilon_2^{(1)} - \frac{H_1}{2} \delta \tau_{23}^\text{th} \end{bmatrix} = 0
\]
Likewise, for ply group 2,

\[
A_{55}^{(2)} \left[ \frac{-\epsilon_1^{(1)}}{2} - \frac{H_1 \delta_{75}}{2} \right] - H_2 \left[ \frac{\delta_{13}^{th}}{\delta x_1} + \frac{\delta_{13}^i}{\delta x_1} \right] = 0
\]

\[
A_{44}^{(2)} \left[ \frac{-\epsilon_2^{(1)}}{2} - \frac{H_1 \delta_{74}}{2} \right] - H_2 \left[ \frac{\delta_{23}^{th}}{\delta x_2} + \frac{\delta_{23}^i}{\delta x_2} \right] = 0
\]  

(3.83)

and, for ply group 3,

\[
A_{55}^{(3)} \left[ \frac{-\epsilon_1^{(2)}}{2} - \frac{H_2 \delta_{75}}{2} \right] - H_3 \left[ \frac{\delta_{13}^{th}}{\delta x_1} \right] = 0
\]

\[
A_{44}^{(3)} \left[ \frac{-\epsilon_2^{(2)}}{2} - \frac{H_2 \delta_{74}}{2} \right] - H_3 \left[ \frac{\delta_{23}^{th}}{\delta x_2} \right] = 0
\]

(3.84)

Substituting for \( \gamma_4 \) and \( \gamma_5 \), the following expressions are obtained:

\[
A_{55}^{(1)} \left[ \frac{-\epsilon_1^{(1)}}{4} + \frac{H_1 \delta_{13}^{th}}{4} \right] - H_1 \left[ \frac{\delta_{13}^{th}}{\delta x_1} \right] = 0
\]

\[
A_{44}^{(1)} \left[ \frac{-\epsilon_2^{(1)}}{4} + \frac{H_1 \delta_{23}^{th}}{4} \right] - H_1 \left[ \frac{\delta_{23}^{th}}{\delta x_2} \right] = 0
\]

(3.85)
When a crack occurs in a given ply group, it can no longer carry a load at that point. The
other laminate ply groups must carry the load at the crack locations. Thus, we can give the
boundary conditions on the crack faces:

Ply group 1 or top layer:  \( x_2 = \pm \frac{L_t}{2}, \quad n_2^{(1)} = 0 \)

Ply group 2 or middle layer:  \( x_1 = \pm \frac{L_m}{2}, \quad n_1^{(2)} = 0 \)  \( (3.88) \)

Ply group 1 or top layer:  \( x_2 = \pm \frac{L_b}{2}, \quad n_2^{(3)} = 0 \)
In other words,

\[ n_2^{(2)} + n_2^{(3)} = N_2 \text{ at } x_2 = \pm \frac{L_1}{2} \]
\[ n_2^{(1)} + n_2^{(2)} = N_2 \text{ at } x_2' = \pm \frac{L_b}{2} \]  (3.89)
\[ n_1^{(1)} + n_1^{(3)} = N_1 \text{ at } x_1 = \pm \frac{L_m}{2} \]

which implies that

\[ \Delta n_2^{(2)} + \Delta n_2^{(3)} = n_2^{(1)} \text{ at } x_2 = \pm \frac{L_1}{2} \]
\[ \Delta n_2^{(1)} + \Delta n_2^{(2)} = n_2^{(3)} \text{ at } x_2' = \pm \frac{L_b}{2} \]  (3.90)
\[ \Delta n_1^{(1)} + \Delta n_1^{(3)} = n_1^{(2)} \text{ at } x_1 = \pm \frac{L_m}{2} \]

Now, before cracking, a force balance on the 1z plane gives:

\[ N_{2i} = n_{2i}^{(1)} + n_{2i}^{(2)} + n_{2i}^{(3)} \]  (3.91)

and on the 2z plane,

\[ N_{1i} = n_{1i}^{(1)} + n_{1i}^{(2)} + n_{1i}^{(3)} \]  (3.92)

After cracking,

\[ N_{2f} = n_{2f}^{(1)} + n_{2f}^{(2)} + n_{2f}^{(3)} \]  (3.93)

and

\[ N_{1f} = n_{1f}^{(1)} + n_{1f}^{(2)} + n_{1f}^{(3)} \]  (3.94)
Before cracking occurs, equation (3.64) can be written as:

\[
\{n\}^{(k)} = [A]^{(k)} \{\epsilon\} - \{n^t\}^{(k)}
\]  

(3.95)

so, the initial integrated force resultants can be expressed as:

\[
\{N\}^i = [A]^{(1)} \{\epsilon\} - \{n^t\}^{(1)} + [A]^{(2)} \{\epsilon\} - \{n^t\}^{(2)} + [A]^{(3)} \{\epsilon\} - \{n^t\}^{(3)}
\]  

(3.96)

After cracking, using equation (3.70), the final integrated force resultants are:

\[
\{N\}^f = [A]^{(1)} \{\epsilon\} - \{n^t\}^{(1)} + \{\Delta n\}^{(1)} + [A]^{(2)} \{\epsilon\} - \{n^t\}^{(2)} + \{\Delta n\}^{(2)} + \{n\}^f(3)
\]  

(3.97)

Combining equations (3.96) and (3.97),

\[
\{\Delta n\}^{(1)} + \{\Delta n\}^{(2)} = [A]^{(3)} \{(\hat{\epsilon}\} - \{\bar{\epsilon}\}^{(3)}
\]  

(3.98)

\[
\{\Delta n\}^{(2)} + \{\Delta n\}^{(3)} = [A]^{(1)} \{(\hat{\epsilon}\} - \{\bar{\epsilon}\}^{(1)}
\]  

(3.99)

\[
\{\Delta n\}^{(1)} + \{\Delta n\}^{(3)} = [A]^{(2)} \{(\hat{\epsilon}\} - \{\bar{\epsilon}\}^{(2)}
\]  

(3.100)

Relations (3.93) through (3.95) can be summarized.

\[
\{\bar{\epsilon}\}^{(k)} = \{\epsilon\} + [F]^{(k)} \{\Delta n\}^{(k)}
\]  

(3.101)
where [F]=[A]^{-1}. This leads to the relations,

\[
\{\epsilon\}_{(2)} - \{\tilde{\epsilon}\}_{(3)} = [F]_{(2)} \{\Delta n\}_{(2)} - [F]_{(3)} \{\Delta n\}_{(3)}
\]

\[
\{\tilde{\epsilon}\}_{(1)} - \{\epsilon\}_{(2)} = [F]_{(1)} \{\Delta n\}_{(1)} - [F]_{(2)} \{\Delta n\}_{(2)}
\]

\[
\{\tilde{\epsilon}\}_{(1)} - \{\epsilon\}_{(3)} = [F]_{(1)} \{\Delta n\}_{(1)} - [F]_{(3)} \{\Delta n\}_{(3)}
\]

Finally, by substituting equations (3.102) and the equilibrium relations, equations (3.67-3.69), into equations (3.85-3.87), one obtains the governing equations for matrix cracking:

\[
\frac{1}{H_1} \begin{bmatrix} A_{55} & 0 \\ 0 & A_{44} \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{12} & F_{22} \end{bmatrix} \begin{bmatrix} \Delta n_{1} \\ \Delta n_{2} \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{12} & F_{22} \end{bmatrix} \begin{bmatrix} \Delta n_{1} \\ \Delta n_{2} \end{bmatrix}
\]

\[
\frac{H_2^2}{4} \begin{bmatrix} F_{55} & 0 \\ 0 & F_{44} \end{bmatrix} \begin{bmatrix} 2\frac{\delta^2 \Delta n_{1}}{\delta x_1^2} + \frac{\delta^2 \Delta n_{1}}{\delta x_2^2} \\ 2\frac{\delta^2 \Delta n_{2}}{\delta x_1^2} + \frac{\delta^2 \Delta n_{2}}{\delta x_2^2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

\[
\frac{1}{H_2} \begin{bmatrix} A_{55} & 0 \\ 0 & A_{44} \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{12} & F_{22} \end{bmatrix} \begin{bmatrix} \Delta n_{1} \\ \Delta n_{2} \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{12} & F_{22} \end{bmatrix} \begin{bmatrix} \Delta n_{1} \\ \Delta n_{2} \end{bmatrix}
\]

\[
\frac{H_1^2}{4} \begin{bmatrix} F_{55} & 0 \\ 0 & F_{44} \end{bmatrix} \begin{bmatrix} 2\frac{\delta^2 \Delta n_{1}}{\delta x_1^2} + \frac{\delta^2 \Delta n_{1}}{\delta x_2^2} \\ 2\frac{\delta^2 \Delta n_{2}}{\delta x_1^2} + \frac{\delta^2 \Delta n_{2}}{\delta x_2^2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
The solution of the governing equations with the appropriate boundary conditions is given in detail in Appendix A. Using the solutions for ply group loads from the three-layer model, the forms for the changes in ply group loads due to cracking can be assumed for the five-layer model. These loads are substituted into the governing equations to solve for the constant terms. The boundary conditions of zero stress at the crack locations are then applied. Thus, there are twelve governing equations, three boundary condition equations, six total load equilibrium equations and twenty-one unknowns. In order to solve the equations more efficiently, the Maple symbolic processor in Mathcad 4.0 by Mathsoft was used. The change in stress resultants for the various lamina groups are:

\[
\Delta n_1^{(1)} = A \frac{\cosh(\lambda m x_1)}{\cosh(\lambda m L_m/2)} + B \frac{\cosh(\lambda t x_2)}{\cosh(\lambda t L_t/2)} + C \frac{\cosh(\lambda b x_2)}{\cosh(\lambda b L_b/2)}
\]

(3.106)

\[
\Delta n_2^{(1)} = D \frac{\cosh(\lambda t x_2)}{\cosh(\lambda t L_t/2)} + E \frac{\cosh(\lambda b x_2)}{\cosh(\lambda b L_b/2)} + F \frac{\cosh(\lambda m x_1)}{\cosh(\lambda m L_m/2)}
\]

(3.107)

\[
\Delta n_1^{(2)} = G \frac{\cosh(\lambda m x_1)}{\cosh(\lambda m L_m/2)} + H \frac{\cosh(\lambda t x_2)}{\cosh(\lambda t L_t/2)} + I \frac{\cosh(\lambda b x_2)}{\cosh(\lambda b L_b/2)}
\]

(3.108)
\[ \Delta n_2^{(2)} = J \frac{\cosh(\lambda t x_2)}{\cosh(\lambda t \frac{L_t}{2})} + K \frac{\cosh(\lambda b x_2)}{\cosh(\lambda b \frac{L_b}{2})} + L \frac{\cosh(\lambda m x_1)}{\cosh(\lambda m \frac{L_m}{2})} \]  

\[ \Delta n_1^{(3)} = M \frac{\cosh(\lambda m x_1)}{\cosh(\lambda m \frac{L_m}{2})} + N \frac{\cosh(\lambda t x_2)}{\cosh(\lambda t \frac{L_t}{2})} + O \frac{\cosh(\lambda b x_2)}{\cosh(\lambda b \frac{L_b}{2})} \]  

\[ \Delta n_2^{(3)} = P \frac{\cosh(\lambda t x_2)}{\cosh(\lambda t \frac{L_t}{2})} + Q \frac{\cosh(\lambda b x_2)}{\cosh(\lambda b \frac{L_b}{2})} + R \frac{\cosh(\lambda m x_1)}{\cosh(\lambda m \frac{L_m}{2})} \]  

where the constant terms, A through R, are defined in Appendix A. The \( \lambda_i \) are the assumed roots of the auxiliary equation as described above for the three-layer model. The constants and the \( \lambda_i \) are found by substituting equations (3.106) through (3.111) into the governing equations, (3.103) through (3.105), and the boundary conditions, equations (3.90). See Appendix A for the details.

The form of the change in ply group load terms does not allow for the boundary conditions to be satisfied everywhere, so they are satisfied by an averaging scheme, as described in Appendix A. The averaging scheme used is similar to the method used by Tsai, Daniel, and Lee [36]. In addition, the coordinate systems of the top and bottom ply groups are assumed to be independent, since the coordinate systems are based on the crack locations in the individual ply groups.

**Modeling of Progressive Cracking in the Laminate**

Progressive cracking is again modeled using basic fracture mechanics principles, and a FORTRAN program has been written for ease of calculation. The development of the equations for work done by external loads and strain energy in the individual ply groups is summarized here. Additional details are shown in Appendix B. Once again, the change in strain energy for each ply group includes the in-plane and through thickness shear terms.
The strain energy density for state \( j \) is, including both in-plane and through-thickness shear terms:

\[
\begin{align*}
\text{d}W_{\text{int}}^j &= \frac{1}{2} \sum_{k=1}^{n} \left\{ \Delta n_1 \Delta n_2 \right\} \left[ \begin{bmatrix} F_{11} & F_{12} \\ F_{12} & F_{22} \end{bmatrix}^{(k)} \begin{bmatrix} \Delta n_1 \\ \Delta n_2 \end{bmatrix}^{(k)} + 2 \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} \right] \\
&- 2 \begin{bmatrix} F_{11} & F_{12} \\ F_{12} & F_{22} \end{bmatrix}^{(k)} \begin{bmatrix} \Delta n_1 \\ \Delta n_2 \end{bmatrix}^{(k)} \left( k \right) dx_1 dx_2 + \frac{1}{2} \sum_{k=1}^{n} \left[ \frac{H_k}{4} \right] \begin{bmatrix} \tau_{23} \\ \tau_{13} \end{bmatrix}^{T} \begin{bmatrix} \tau_{23} \\ \tau_{13} \end{bmatrix} dx_1 dx_2
\end{align*}
\]

(3.112)

where \( k \) indicates the ply group and \( F_s \) is the inverse shear matrix. Matrix cracking state \( j \) is referred to the initial strain at the onset of cracking. Then, \( \Delta W_{\text{int}} \) is \( \Delta W_{\text{int}}^2 - \Delta W_{\text{int}}^1 \). The work done on the total laminate by external loads during crack formation is found from the applied loads and the ply group 1 mechanical strains. The \( \Delta n_1 \) and \( \Delta n_1^2 \) for each lamina group are found from:

\[
\Delta n_1 = \frac{\int_{-A/2}^{A/2} \Delta n_1 \, dA}{A} \quad (3.113)
\]

\[
\Delta n_1^2 = \frac{\int_{-A/2}^{A/2} \Delta n_1^2 \, dA}{A} \quad (3.114)
\]

where \( A \) refers to the area defined by the distance between the two cracks in the current crack state and the distance between two previous cracks running perpendicular to the current two. For example, if the current crack occurs in the top ply group, then area is \( L_t L_m \), where, if no crack exists in the middle layer, \( L_m \) may approach infinity. The \( x_2 \) coordinate system is actually not the same in the top and bottom ply groups; however, a linear relationship does exist between the two.
This relationship changes whenever a crack occurs in the top or bottom layers and cannot be known precisely. Because of this, the averaging of the bottom and top layer terms must be done separately. For example, to get the average disturbance in the stress state in the bottom layer over a distance $L_b$, which is the distance between cracks in the top layer, the integration for the bottom layer disturbance term must be done over distance $L_b$. See Appendix B for additional details.

The work done by external loads is given by:

$$\Delta W_{ext, j} = 2N_{1j} \left[ F_{11} \Delta n_{1j} + F_{12} \Delta n_{2j} \right] + 2N_{2j} \left[ F_{12} \Delta n_{2j} + F_{22} \Delta n_{2j} \right] \text{Area}$$

(3.115)

where the Area term is defined above, $j$ refers to current crack state, and the coefficient of 2 is used because $N_2$ and $N_6$ are defined as the loads applied to half of the laminate only.

A FORTRAN program has been written to perform all calculations, since many of them are iterative. For example, the load at which cracking starts is found by assuming a single crack separated from any other crack by an infinite distance occurs in a given ply group. The load in the laminate is increased until the strain energy release rate for the laminate is equal to the critical strain energy release rate for cracking to occur. This procedure is repeated for each ply group. Cracking begins in the ply group in which the strain energy release rate is reached at the smallest applied load. The actual crack densities of the three ply groups are then fed into the progressive cracking portion of the model. In this section, additional cracking is assumed to occur in each ply group respectively. The applied load is increased until the critical strain energy release rate is reached by the laminate for cracking in one of the ply groups. The crack densities are then updated. Thus, the program returns crack density versus applied load. In addition, strains and moduli at each crack density are calculated. Finally, local delamination is modeled in the same manner as in the three-layer model,
except that the shear lag modification term is applied to the shear lag parameters of the top and bottom ply groups. This is necessary, because the through thickness shear stiffness of the bottom layer is included in both shear lag parameters.

**Effect of Local Delamination**

Local delamination occurs as a result of transverse matrix cracking. When the transverse matrix cracks reach the boundary between the layers, the interlaminar shear stress at the crack tip becomes singular, and delamination occurs. See Figure 3.5. In this model, the $\lambda_i$ terms fill the function of the shear lag parameter defined in similar models [28-36]. Since the shear lag parameter is used to describe the interaction between the ply groups, and this interaction will change should damage develop between plies, the shear lag parameter must be sensitive to this change [32, 33].

![Figure 3.5. Transverse Crack Tip Delamination](image)

Figure 3.5. Transverse Crack Tip Delamination
When local delamination occurs, the interlaminar shear stiffness change, $G_{23}$, can be approximated, using the rule of mixtures, by

$$G_{23} = \frac{\sum G_{23j}t_j}{t}$$  \hspace{1cm} (3.116)

where $t_j$ is the length of the segment over which $G_{23}=G_{23j}$ and $t$ is the total length (sum of all $t_j$).

Refer to Figure 3.6. The segment, $c$, between two cracks may be divided into segments equal in length to the shear lag distance, $H_2$, which is assumed to be one-half the thickness of the cracked ply group. It is assumed that the shear lag distance, the distance from the transverse crack over which the transverse shear stress decays to zero, is the distance over which the transverse shear stiffness is zero or very small. The shear stiffness returns to its original value of $G_{23i}$ at a distance of $3H_2$ from the transverse crack. Using the rule of mixtures,

$$G_{23} = \frac{1}{2H_2} \left[ \frac{1}{c} \left( 2G_{23i} \right) + \frac{1}{c} \left( xG_{23i} \right) \right]$$

$$2H_2x = c - 6H_2 \quad \text{so} \quad x = \frac{c}{2H_2} - 3$$  \hspace{1cm} (3.118)

Substituting for $x$ in equation 3.112 gives:

$$G_{23} = G_{23i} \left( \frac{1 - 4H_2}{c} \right) = G_{23i} \left( 1 - 4\beta H_2 \right)$$  \hspace{1cm} (3.119)

Recall that $L_{22}$ and $L_{66}$ are proportional to $A_{44}$ and $A_{55}$, and the $\lambda_i$ are proportional to the square root of $L_{22}$ and $L_{66}$. Thus, the shear lag parameter is proportional to square root of through thickness shear modulus.
It is interesting that Yang and Boehler [49] also developed an expression for the change in interlaminar shear modulus due to delamination at about the same time as the expression shown above was developed [61]. Yang and Boehler developed a micromechanics damage model for crossply laminates. They used a continuous dislocation distribution method for transverse matrix cracking. To model delamination, they detached the bonding interface and replaced it with a periodic shear interaction stress. Delamination induces an elastic shrinkback, which occurs at the intralaminar crack tip and this shrinkback is given by:

$$\delta_e = 2a \bar{\epsilon} h\left(\frac{a}{\lambda t_1}, \frac{b}{\lambda t_1}\right)$$  \hspace{1cm} (3.120)

where $a$ is the distance between two matrix cracks, $b$ is the length of the delamination (approximately equal to twice the shear lag distance), $\lambda$ is a function of material constants, $\bar{\epsilon}$ is the macroscopic damage strain produced by the delamination, and $t_1$ is the thickness of the cracked ply group. The function, $h$, is given by:

$$h\left(\frac{a}{\lambda t_1}, \frac{b}{\lambda t_1}\right) = \frac{b}{a} + \frac{\lambda t_1}{2ma} \sum_{k=1}^{m} \frac{\ln \left[ \eta(1-s_k) \right]}{\sin[\eta(1-s_k)]} B(s_k)$$  \hspace{1cm} (3.121)
where \( \eta = \frac{\pi}{2} (1 - \frac{b}{a}) \), the \( s_k \) depend on the material properties of the laminae, \( B \) is related to the interfacial shear stress and \( k \) and \( m \) refer to the points at which \( B \) is calculated. Finally, Yang and Boehler show that:

\[
\bar{M}_{55}^{-1} = G_2 \left[ 1 - h \left( \frac{a}{\lambda t_1}, \frac{b}{\lambda t_1} \right) \right] \tag{3.122}
\]

where \( \bar{M}_{55}^{-1} \) is the damaged \( G_z \) and \( G_2 \) is \( G_{23} = G_{13} \). Since Yang and Boehler did not compare their interlaminar cracking model with experimental results, the accuracy of this equation is unknown.

Recall that \( L_{22} \) and \( L_{66} \) are proportional to \( A_{44} \) and \( A_{55} \), and the \( \lambda_1 \) are proportional to the square root of \( L_{22} \) and \( L_{66} \) defined in equation 3. Thus, the shear lag parameter is proportional to square root of through thickness shear modulus.

\[
\lambda_{1,3} = \frac{1}{4} \left\{ \frac{1}{2} (L_{22} + L_{66}) \pm \frac{1}{2} \sqrt{(L_{22} - L_{66})^2 + 4L_{26}L_{62}} \right\} \left[ \sqrt{1 - 4\beta H} \right] \tag{3.123}
\]

This shear lag modification term is similar to the interlaminar damage vector developed by Talreja [48]. Note that this equation is only for delamination between ply groups 1 and 2. Edge delamination between the 90° plies cannot be modeled in this manner.

The modification factor quickly reaches a lower limit, since transverse matrix crack tip delamination growth ceases after the first few matrix cracks. This has been explained by Yang and Boehler [49]. They show that, as matrix cracks reach interlaminar boundaries, delamination is initiated due to a high interfacial shear stress at the matrix crack tip. The delamination is soon terminated due to a large decrease in the driving force. In addition, their results indicate that these small delaminations grow less as matrix cracking proceeds, until the characteristic damage state is formed. In other words, for the first several transverse matrix cracks, the strain energy release rate
for local delamination is initially greater than \( G \) for the next matrix crack, but, as the local delamination grows, the resistance to crack growth increases; that is, \( G \) becomes smaller. The reader should refer to equation 2.1. Thus, the delamination only grows a short distance from the matrix crack tip, which, according to Yang and Boehler's [49] analysis, is approximately equal to the shear lag distance. Once the delamination has been arrested, additional matrix cracks may form, since the strain energy release rate for matrix cracking would then be larger than that for continued delamination.

As the distance between matrix cracks approaches the shear lag distance, delamination no longer occurs at each matrix crack. Yang and Boehler [49] show that \( \gamma_4 \) is proportional to \( h \), so \( \tau_1 \) is proportional to \( h(1-h) \), and, using equation 2.1, it can then be shown that the strain energy release rate for delaminations of constant size increases and then decreases as \( G_{23} \) decreases. Thus, after the first few matrix cracks, local delamination is no longer energetically favorable and matrix cracking occurs without associated local delaminations. At this point, the decrease in modulus for local delamination has reached its limiting value.

The point at which the modulus decrease reaches the limiting value can be estimated using O'Brien's [44] equation for local delamination. A more general form of equation (2.50) is:

\[
P = \sqrt{\frac{2mG}{E_{\text{LD}}t_{\text{LD}} - E_{\text{LAM}}t}}
\]

where \( P \) is the applied load, \( G \) is the strain energy release rate for delamination, \( E_{\text{LD}} \) and \( t_{\text{LD}} \) are the modulus and thickness, respectively, of the delaminated cross-section, and \( E_{\text{LAM}} \) and \( t \) are the modulus and thickness, respectively, of the laminate before delamination. At the transition between matrix cracking plus delamination and matrix cracking only, the strain energy release rate for delamination must be equal to that of matrix cracking. By using the strain energy release rate for matrix cracking in O'Brien's [44] equation, the load at which the transition occurs can be determined. O'Brien's equation is used as a rule, then, for the limit on the decrease in shear
modulus as small delaminations occur. This prediction correlates very well with the point at which the limiting distance between matrix cracks is reached.

A check on the consistency of the present method for modeling transverse crack tip delamination is given in Table 3.2. The results are from the model. The first two laminates given in the table were tested in-house, while the other two are from the literature. The crack densities at the end of initial local delamination are close to those seen in experiments. Ideally, according to the present model as shown in Figure 3.6, initial local delamination should cease when two cracks are no closer than to each other than $4H^2$ in. Table 3.2 shows that this criterion is satisfied. Finally, the last column shows the crack spacing at the end of matrix cracking as predicted by the model. This distance represents the characteristic damage state [62]. After the characteristic damage state is formed, the small delaminations already formed grow and form large delaminations. These large delaminations were seen near the location of failure in the specimens tested.

<table>
<thead>
<tr>
<th>Laminate</th>
<th>$1/L_h$ or $L_c$</th>
<th>$1/H_2$ or $H_c$</th>
<th>$L_b$ or $c$</th>
<th>$H_2$ or $H_c$</th>
<th>End $L_b$ or $c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[90/0/90]_s</td>
<td>5.75</td>
<td>.174 in</td>
<td>.0276 in</td>
<td>6.3</td>
<td>.0645 in</td>
</tr>
<tr>
<td>E-glass epoxy</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[0/90]_s</td>
<td>12.0</td>
<td>.0833 in</td>
<td>.0204 in</td>
<td>4.1</td>
<td>.0563 in</td>
</tr>
<tr>
<td>E-glass epoxy</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[0/90]_s</td>
<td>10.0</td>
<td>.100 in</td>
<td>.0237 in</td>
<td>4.2</td>
<td>.0563 in</td>
</tr>
<tr>
<td>E-glass epoxy</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[0/90]_s</td>
<td>23.75</td>
<td>.042 in</td>
<td>.0104 in</td>
<td>4.0</td>
<td>.042 in</td>
</tr>
<tr>
<td>Graphite epoxy</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The three-layer and five-layer models are obviously similar, especially with regard to the basic methods used to derive the governing equations for matrix cracking. They are different in several important respects. Although the three-layer model is written for cracking in the central $90^\circ$ ply group only, it incorporates angle plies in the outer ply group. The five-layer model is for
cracking in all five ply groups of crossply laminates. The three-layer model is for general in-plane loading, while the five-layer model is for loading in the principal in-plane directions only. Both models contain an algorithm for local delamination. The models provide crack density as a function of applied load and modulus decrease as a function of crack density. They are compared with data in the literature and with in-house experimental data in Chapter 5.
CHAPTER 4

EXPERIMENTAL METHODS

Four E-glass epoxy laminates were tested under monotonic tensile loading and the progression of damage was measured. The results were compared with the progressive cracking model previously described.

Specimen Preparation

The materials tested in this research were symmetrical lay-ups of 3M Scotchply 1003 pre-impregnated glass epoxy laminae. Table 4.1 gives the material properties of the cured Scotchply 1003 samples used. The layup used to verify the initial model was [0/90]s. The layups used to verify the final model were [90/0/90]s and [90/0/2/90]s. The layups used were chosen to test

Table 4.1. Material Properties of Cured Scotchply 1003

<table>
<thead>
<tr>
<th></th>
<th>Tensile Modulus (Msi)</th>
<th>Shear Modulus (Msi)</th>
<th>Coefficients of Thermal Expansion</th>
<th>Critical Strain Energy Release Rate (in-lb/in²)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0°, E₁</td>
<td>90°, E₂</td>
<td>G₁₂</td>
<td>G₁₃</td>
</tr>
<tr>
<td></td>
<td>5.7</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
</tr>
<tr>
<td>E₃</td>
<td>1.4</td>
<td>G₂₃</td>
<td>0.63</td>
<td></td>
</tr>
<tr>
<td>Tensile Strength (ksi)</td>
<td>0°, σₚ₁</td>
<td>108</td>
<td>α₁</td>
<td></td>
</tr>
<tr>
<td></td>
<td>90°, σₚ₂</td>
<td>2.9</td>
<td>α₂</td>
<td></td>
</tr>
<tr>
<td>Poisson's Ratios</td>
<td>ν₁₂</td>
<td>0.3</td>
<td>Gₖ</td>
<td>Gₖ</td>
</tr>
<tr>
<td></td>
<td>ν₁₃</td>
<td>0.3</td>
<td>0.227</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ν₂₃</td>
<td>0.49</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
modeling extremes, as well as to investigate the effect of relative thicknesses of constrained and unconstrained 90° plies.

Referring to Table 4.1, the 0° tensile modulus, the in-plane Poisson's ratio, and the 0° tensile strength were verified experimentally. The critical strain energy release rate was estimated using equation 4.1 [39].

\[
G_c = \frac{\pi \sigma_{ul}t^2 a_c}{2E_2}
\]

where \(2a_c\) is the critical or effective flaw size and is equal to 5.0 times the total thickness of the 90° ply group [30]. This method correlates well with the estimates of other researchers [28-30, 32]. The other values were taken from the manufacturer or were found using the assumption of transverse isotropy. The only significant variation from the manufacturer's data was in tensile strength. The manufacturer's value was 140 ksi. The discrepancy is probably due to flaws in the specimens manufactured in-house. A flaw can initiate a local crack or debond that may lead to premature failure. The other material properties would not be substantially affected by such flaws.

The composite laminate specimens were manufactured as 10 x 10 inch lay-ups in a 12-ton simple heated (maximum of 500° F), vacuum bag press. Pre-impregnated Scotchply 1003 was cut into 10 x 10 in squares and stacked in layers of appropriate orientation. After the tacky laminate was layed up and smoothed out, it was placed between two sheets of Airtech Release Ease 234TFP-1 teflon coated, woven glass fiber release film. Three 11 x 11 inch sheets of Richmond, E-5555 #116 high temperature release fabric were placed on top of the sandwich and three layers of coarser high temperature woven fiberglass were placed underneath the sandwich. These fabrics act as bleeder cloths for the curing process. They act to facilitate outgassing and resin pull off during the manufacturing process.

Once the laminate sandwich is laid on the bottom plate of the press, Schnee-Morehead vacuum bag one inch (S-M 5126-2) sealing tape is placed around the perimeter of the sandwich. The tape acts as a sealant between the base of the press and the vacuum bag. Once in position,
the backing of the sealing tape is removed and a 14 x 14 inch layer of Richmond, Vac-Pak UHT-750 ultra high temperature bagging film is placed over the top of the composite sandwich and smoothed against the sealing tape. A vacuum is then drawn through a hole in the base plate at the edge of the laminate.

The press is then closed and brought to a jack pressure of 1,000 psi. At this pressure, the laminate is cured for 20 min at a stabilized 250° F temperature. At the end of this cycle, the jack is pumped up to 1,500 psi and the thermostat is increased to 330° F. Once the temperature reaches 330°, the laminate is cured for 25 minutes. At this point, the laminate is removed from the press, peeled away from the release film and left to cool.

After the laminate has been prepared, the 10 x 10 inch specimen is cut into nine, 1 x 8 inch samples. The top and bottom 1 inch is cut off and discarded, to remove any edge tapering, and the same is done to sample 1 and sample 9. All of the testing in this research involved the use of the best three samples from the center of the 10 x 10 specimen. The machining of the samples was performed using an Everett 3450 rpm cutting machine with an Everett #1410, 10 inch abrasive cut-off wheel.

**Experimental Setup and Procedures**

An Instron 8500 Series Servohydraulic Testing System with a 50,000 lb load cell and standard tension grips was used to apply a uniaxial tensile load to each specimen. For most experiments, strain was measured using Measurements Group CEA-06-125UW-350 350 Ω strain gages, which all had a gage factor of 2.12. A Measurements Group Model 2160 strain amplifier was used to amplify and condition the signal from the strain gages. An excitation voltage of 3.5 volts and an amplification factor of 400 were used. For the experiments done to verify the initial model, an Instron Dynamic Extensometer, with a 0.5 inch gage length and a ± 0.2 inch travel was used. The output from the strain amplifier or extensometer and the output from the 50,000 lb Instron load cell
were fed into a Hewlett Packard 8090A Measurement Plotting System. Damage progression was measured via both the decrease in Young's modulus and by the density of transverse cracks formed.

The samples were loaded in increments of 400 to 1000 lbs, depending on the expected failure load. The idea was to obtain at least six measurements. Loading was done at a rate of 500 lbs/min. This relatively slow rate was chosen to allow the damage state in the laminate to reach equilibrium for each measurement.

In order to obtain stress information from the load data, the average cross sectional area was measured for each specimen. Five measurements were taken of the width and thickness of the specimen in the region where the strain gage was to be placed. Using the information described, Young's modulus was calculated for each loading step.

In order to measure the progression of cracking, a laminate edge was dyed and photographed at each load increment to reveal cracks. The load increments varied according to the specimen being tested, with the goal being to obtain about eight data points per specimen. Thus, the load increments used varied from 250 to 1000 lbs. The specimen cross-sectional edge was inked using stamp pad ink according to a procedure developed by Doucet [63]. The ink was allowed to soak into the cracks for approximately 5 minutes; whereupon, the excess ink was wiped off, and the sample was carefully and lightly polished with 400 grit sandpaper to remove surface ink. A Konica 35mm camera with an Izumanon Close-up lens attachment and black and white film was positioned approximately 4 inches from the inked side of the laminate. Good contrast was obtained for most pictures. The photographs were viewed at 8x magnification with a Hama Lupe, and cracks were counted over at least three different inch-long segments. The three measurements were averaged to get the crack density at a given load. This procedure minimized error due to localized flaws. As a result, a photographic history of the damage was recorded from 0 lbs to failure via this dye penetration technique. Figure 4.1 shows the appearance of cracks and delaminations in the photos.
Error Analysis

Dimensional analysis was performed for the following relationships between modulus of elasticity, $E$, applied stress, $\sigma$, Poisson's ratio, $\nu$, and the measured variables.

$$\sigma = \frac{N}{db}, \quad E = \frac{\Delta N \cdot 1}{\Delta \epsilon_1 \cdot db}, \quad \nu = \frac{\Delta \epsilon_2}{\Delta \epsilon_1}$$

(4.2)

where $N$ = applied load, $\epsilon_1$ = measured longitudinal strain, $\epsilon_2$ = measured transverse strain, $b$ = width of specimen, and $d$ = thickness of cross section. Tables 4.1 through 4.3 outline the sources of experimental error, along with their estimated effect on the modulus values. All error analysis calculations used standard dimensional analysis calculus and root mean squared values for combined error. Values for strain variance for the extensometer were obtained from the Instron - Guide to Advanced Materials Testing [64], and values for strain variance for the strain gages were found from various Measurements Group Tech Notes and Catalogs [65-69]. The modulus experimental error with the extensometer was $\Delta E = \pm 0.09$. The modulus experimental error with the strain gages was $\Delta E = \pm 0.059$. The error in the Poisson's ratio was $-0.007$ and was due to the effect of strain gage
transverse sensitivity on the y direction strain. The Poisson’s ratio used was corrected for this error.

The error in the stress measurements taken with the extensometer is $\Delta \sigma = \pm .35$ ksi. The error in the stress measurement taken with the strain gages is $\Delta \sigma = \pm .23$ ksi.

Table 4.2 Sources of Experimental Error Due to Extensometer

<table>
<thead>
<tr>
<th>Source of Error</th>
<th>$\epsilon$ Var. (%of range)</th>
<th>E Var. (MSI)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Misalignment of Extensometer</td>
<td></td>
<td>$\pm .05$</td>
</tr>
<tr>
<td>Blade slip in crack/smooth surface</td>
<td></td>
<td>$\pm .05$</td>
</tr>
<tr>
<td>Combined Instrumental error</td>
<td>$.25%$</td>
<td>$\pm .012$</td>
</tr>
<tr>
<td>Linearity</td>
<td>$.15%$</td>
<td></td>
</tr>
<tr>
<td>Hysteresis</td>
<td>$.1%$</td>
<td></td>
</tr>
<tr>
<td>Creep</td>
<td>$.05%$</td>
<td></td>
</tr>
<tr>
<td>Repeatability</td>
<td></td>
<td>*</td>
</tr>
</tbody>
</table>

* Error in repeatability is included in the combined instrumental error.

Table 4.3 Sources of Experimental Error Due to Strain Gage

<table>
<thead>
<tr>
<th>Source of Error</th>
<th>$\epsilon$ Var. (%of range)</th>
<th>E Var. (MSI)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Misalignment of Strain Gages</td>
<td></td>
<td>-0-</td>
</tr>
<tr>
<td>Wheatstone Bridge Nonlinearity</td>
<td></td>
<td>$+.028$</td>
</tr>
<tr>
<td>Combined Instrumental error</td>
<td>$.04%$</td>
<td>$\pm .001$</td>
</tr>
<tr>
<td>Transverse sensitivity</td>
<td>$.01%$</td>
<td></td>
</tr>
<tr>
<td>Filament Linearity</td>
<td>$.01%$</td>
<td></td>
</tr>
<tr>
<td>Hysteresis</td>
<td>$.01%$</td>
<td></td>
</tr>
<tr>
<td>Creep</td>
<td>$.01%$</td>
<td></td>
</tr>
<tr>
<td>Repeatability</td>
<td></td>
<td>*</td>
</tr>
</tbody>
</table>

* Error in repeatability is included in the combined instrumental error.
Table 4.4 Combined Sources of Experimental Error with Extensometer

<table>
<thead>
<tr>
<th>Source of Error</th>
<th>E Var. (MSI)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMS Extensometer Error</td>
<td>±.07</td>
</tr>
<tr>
<td>Load Cell Error</td>
<td>±.012</td>
</tr>
<tr>
<td>Strain hardening effects at higher loads</td>
<td>±.03</td>
</tr>
<tr>
<td>Measurement of slope off 10 x 10 graph paper</td>
<td>±.04</td>
</tr>
<tr>
<td>Measurement of thickness of specimen</td>
<td>±.005</td>
</tr>
<tr>
<td>Measurement of width of specimen</td>
<td>±.005</td>
</tr>
<tr>
<td>Temperature variation effects (negligible)</td>
<td>-0-</td>
</tr>
<tr>
<td>TOTAL EXPERIMENTAL ERROR</td>
<td>±.09</td>
</tr>
</tbody>
</table>

Note: Load Cell calibration (21 Aug 90), better than .1% cell rated output, or .5% indicated load; trac National Physics Laboratory.

Table 4.5 Combined Sources of Experimental Error with Strain Gages

<table>
<thead>
<tr>
<th>Source of Error</th>
<th>E Var. (MSI)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMS Strain Gage Error</td>
<td>±.028</td>
</tr>
<tr>
<td>Load Cell Error</td>
<td>±.012</td>
</tr>
<tr>
<td>Strain hardening effects at higher loads</td>
<td>±.03</td>
</tr>
<tr>
<td>Measurement of slope off 10 x 10 graph paper</td>
<td>±.04</td>
</tr>
<tr>
<td>Measurement of thickness of specimen</td>
<td>±.005</td>
</tr>
<tr>
<td>Measurement of width of specimen</td>
<td>±.005</td>
</tr>
<tr>
<td>Temperature variation effects (negligible)</td>
<td>-0-</td>
</tr>
<tr>
<td>TOTAL EXPERIMENTAL ERROR</td>
<td>±.059</td>
</tr>
</tbody>
</table>

Note: Load Cell calibration (21 Aug 90), better than .1% cell rated output, or .5% indicated load; trac National Physics Laboratory.
CHAPTER 5
RESULTS AND DISCUSSION

The progressive cracking model is compared with the results of several in-house experiments and with models and experimental data from the literature. A note on the use of the FORTRAN program is in order. Each iteration must be checked for accuracy. It is possible to overshoot $G_C$ if the crack iteration step size and the load iteration step size are not optimized. When delamination is included, however, the criterion of reaching $G_C$ cannot be strictly enforced.

When delamination is incorporated into the model, i.e. for laminates where the central 90° ply has four or more layers, the strain energy release rate is higher than $G_C$ during the delamination portion. This is expected, since delamination increases the strain energy dissipated due to cracking. Ideally, $G$ should be required to be equal to $G_C$ for matrix cracking plus $G_C$ for local delamination as a criterion for their occurrence. However, since the delamination size for each crack or load increment, as well as the modulus decrease associated with delamination, have been determined a priori, the conditions for delamination would be overconstrained were $G_C$ to be specified as well. Thus, the strain energy release rate increases to a value of almost twice the $G_C$ for matrix cracking and then decreases to the $G_C$ for matrix cracking as delamination ends. Modeling delamination in this manner is not entirely certain, so, in addition to showing the actual model predictions, the model stress versus crack density curves for samples with a large amount of delamination are plotted using the initiation of cracking and the crack state at the end of the initial local delamination as the first two data points. The actual model predictions are shown with a solid line and the model predictions corrected for the uncertainty in the delamination algorithm are plotted in some cases with a dotted line. The results that follow are thus based on engineering judgment and not just a blind usage of the progressive cracking model.
Three-Layer Model

The three-layer model is compared with the results of one set of published experiment and with another shear lag model in the literature. It is then compared with the results of in-house experimentation. The crack density increment used in the model predictions was .25.

\[0/90/3\]_E-Glass Epoxy

The model is first compared with the uniaxial tensile testing results of \[0/90/3\]_E-glass epoxy reported by Highsmith and Reifsnider [62] and the predictions of Laws and Dvorak [33]. The experimental results, their model predictions, and the present model predictions are shown in Figures 5.1 and 5.2. Since transverse crack tip delamination is expected to occur between the \(0^\circ\) and \(90^\circ\) plies, the factor described in equation (3.123) was included in the evaluation of the \(\lambda_i\). The present model shows reasonably good agreement with the experimental data in the region of higher crack densities. Laws and Dvorak's model shows better agreement in the middle crack density range, but their model does not predict transverse matrix crack saturation as well as does the present model.

![Graph](image)

Figure 5.1. Comparison of Model Predictions with Experimental Data for Matrix Cracking in \[0/90/3\]_E-Glass Epoxy [33, 62]
In the model, transverse crack tip delamination ends when the crack density reaches ten cracks/in. The effect of transverse crack tip delamination is to increase the total amount of energy required for matrix cracks to form. In the model, local delamination size is the same, regardless of matrix crack density. In actuality, delaminations are larger initially and decrease in size as matrix cracking continues. Were this change in size included in the model, it is likely that the model predictions would be more accurate in the region of the plot where delamination occurs.

Figure 5.2 shows the modulus decrease for the same laminate. The model predictions are good, with the final modulus being accurately predicted.

![Figure 5.2](image)

Figure 5.2. Comparison of Model Predictions with Experimental Data for Modulus Decrease in [0/902]s E-Glass Epoxy [32, 62]

[0/902]s E-Glass Epoxy Scotchply

Next the three-layer model is compared with the in-house experimental results of uniaxial tension tests on [0/902]s E-glass epoxy Scotchply. Two samples were tested and data from both are plotted in Figures 5.3 and 5.4. An accurate data average could not be taken, since there are not
enough data points. The experimental error is too small to show. The variation between the two samples is likely due to differences in flaw contents, which are due to the manufacturing methods used. The data are given in tabular form in Appendix C. A small amount of delamination is expected in this laminate and is taken into account by the model. Observations verified that prediction, since small delaminations were noticed emanating from the transverse crack tips. Notice that Figures 5.3 and 5.4 are similar to Figures 5.1 and 5.2. In addition, as discussed in the first paragraph of this chapter, Figure 5.3 includes a straight line drawn from the onset of delamination to the end of delamination.

![Figure 5.3. Comparison of Model Predictions with Experimental Data for Matrix Cracking in [0/90]s E-Glass Epoxy](image)

There is a discrepancy between the experimental results and the model predictions of final failure, which is due primarily to the fact that both specimens tested failed at the tabs, indicating a stress concentration not accounted for in the model. The predicted failure stress is based on the stress in the 0° plies of the crossply laminate reaching the ultimate tensile strength found experimentally and is 37.6 ksi. The experimental failure stress was 24.4 ksi for one sample and
25.0 ksi for the other. The modulus predicted using the material properties given in Chapter 4 is 3.16 Msi and the actual moduli measured were 3.14 Msi and 2.80 Msi. This difference is not great, thus, the true composite laminate failure stress probably matches that used in the prediction. Figure 5.4 shows that the model accurately predicts the modulus decrease.

![Figure 5.4. Comparison of Model Predictions with Experimental Data for Modulus Decrease in [0/902]s E-Glass Epoxy](image)

Figure 5.4. Comparison of Model Predictions with Experimental Data for Modulus Decrease in [0/902]s E-Glass Epoxy

[02/902]s AS Graphite-Epoxy

Figures 5.5 and 5.6 show a comparison of the three-layer model predictions with Wang's [39] experimental results for tensile tests of [02/902]s AS graphite-epoxy. The shear lag parameter modification term was not used to model delamination, since delamination is not expected for this laminate. The results of the present model would be improved in this case by the incorporation of a statistical theory for crack location. In fact, Wang [39] and Laws and Dvorak [33] obtained excellent agreement using statistical methods to determine the location of the next crack. Such
statistical methods are needed more when delamination does not occur than when it does, since delamination tends to limit the locations where matrix cracking can occur.

Figure 5.5. Comparison of Model Predictions with Experimental Data for Matrix Cracking in [02/902] AS Graphite-Epoxy [39]

The three-layer model is also compared with experimental data reported by Lee and Daniel [32]. This is shown in Figure 5.6. Although the stress versus crack density predictions are not very good, the model prediction of modulus decrease is excellent. The experimental results shown in Figures 5.5 and 5.6 are obviously from two different researchers' work [39,32]. Thus, although the same properties were assumed for both laminates for the purposes of modeling here, they were probably not exactly the same in reality. Although this may explain the discrepancy in the results, it is possible that the present three-layer model predicts the decrease in modulus associated with matrix cracking better than it predicts the load required to reach a given crack density.
Figure 5.6. Comparison of Model Predictions with Experimental Data for Modulus Decrease in [02/902]s AS Graphite-Epoxy [32]

[0/902]s AS Graphite-Epoxy Under Biaxial Loading

Daniel and Tsai [35] tested a [02/902]s AS/3501-6 graphite-epoxy specimen under uniaxial tension at 10° off-axis in order to obtain a biaxial state of stress in the specimen. A comparison of the present three-layer model with Tsai and Daniel's model and experimental work is shown in Figures 5.7 and 5.8. Delamination is modeled in this specimen. Although delamination would not be expected to occur under simple uniaxial tension loading, the presence of other in-plane stresses can make delamination more likely to occur. Mode III cracking is possible in addition to the Mode I delamination, which occurs under uniaxial tension. Using equations (3.61) and (3.62), \( G_C \) is found to be .001324 in-lb/in\(^2\), since \( G_{IC} \) if .0013 in-lb/in\(^2\). The modulus decrease prediction is quite good, although, once again, it appears that a statistical model may be important for graphite-epoxy, even when delamination occurs.
Figure 5.7. Comparison of Model Predictions with Experimental Data for Matrix Cracking in [0/902]_s AS4/3501-6 Graphite-Epoxy [35]

Figure 5.8. Comparison of Model Predictions with Experimental Data for Modulus Decrease in [0/902]_s AS4/3501-6 Graphite-Epoxy [35]
Five-Layer Model

The five-layer model is compared with the in-house experimental results for \([902/02/90]_6\) and \([90/0/903]_6\) laminates under loading in the \(x\) direction, where \(x\) is the longitudinal material direction of the \(0^\circ\) plies. The experimental data of three different specimens is averaged in the Case 3 plots. The Case 1 plots include averaging of data from three specimens at failure; however, data from two specimens are averaged for crack progression due to poor photographic quality of one of the specimens. The error bars show the variation between samples. It is assumed that the experimental error is included in this variation. See Appendix C for more information on data averages and standard deviations for these specimens. The crack density increment used in the model predictions was \(0.75\).

Case 1: \([902/02/90]_6\) E-Glass Epoxy Scotchply Under \(N_x\)

Agreement between the model and the experimental data is quite good, as is shown in Figures 5.9 and 5.10. Recall that, according to Wang [41], a central ply group must have at least four total plies for delamination to occur. Since the ply groups subject to cracking are thin, no delamination is expected; therefore, delamination is not modeled. The change in slope in the inner ply group stress vs. crack density curve corresponds approximately with the onset of additional matrix cracking in the outer ply group. The prediction of modulus decrease is excellent as is shown in Figure 5.9.

In this case, matrix cracking is predicted to begin first in the inner ply group and this prediction was verified by the experiments. The inner ply group and outer ply group are of equal thickness, but the inner ply group is more constrained by the surrounding \(0^\circ\) plies; therefore, it makes sense for it to crack first. The inner ply group is also predicted to develop more cracks than the outer ply group. Again, this was verified experimentally.

The failure strength is overestimated by the maximum stress criterion used in the model. The model predicted failure at 45 ksi, while the experimental failure occurred at 32 ksi for both
Figure 5.9. Comparison of Model Predictions with Experimental Data for Matrix Cracking in [902/02/90]_s E-Glass Epoxy Under N_x

Figure 5.10. Comparison of Model Predictions with Experimental Data for Modulus Decrease in [902/02/90]_s E-Glass Epoxy Under N_x
samples. Both of the samples reported, like the ones discussed above, failed at the tabs. One sample showed no delamination, while the other showed only a thin three inch long edge delamination along the back side between the top and middle ply groups. Since edge delamination does not interact with matrix cracking to the extent that local delamination interacts with it, edge delamination was not incorporated into the model. The samples failed with no longitudinal splitting.

Case 3: [90/0/903]s E-Glass Epoxy Scotchply Under $N_x$

Again, we see very good agreement between prediction and experiment in Figure 5.11. Figure 5.12 shows good agreement with the initial and final changes in modulus, which were found experimentally. In addition, the prediction of failure stress was excellent. The predicted failure stress was 21.7 ksi and the three samples tested failed at 21.7 ksi, 19.3 ksi and 18.3 ksi. The excellent agreement is likely due to the fact that all samples failed away from the tabs. One sample failed nicely in the gage section with substantial local delamination between the middle and bottom ply groups and no splitting. The other two failed near, but not at, the tabs, and they also showed substantial local delamination with no splitting.

Local delamination was included in the model for this case and, indeed, small local delaminations were observed for the first five to six cracks, as predicted. No further delamination occurred until just before failure. It should be noted that the model predictions for modulus decrease were obviously skewed by the delamination algorithm in this case; therefore, the initial and final values predicted by the model are plotted with a dotted line.

The model predicted that the inner ply group would crack first, followed closely by the outer ply group. The experimental results were unclear in this case, since cracking in the outer layer increased rapidly once it began. In addition, the first few cracks formed in the outer layer evidently appeared sometime between the first and second photographs taken. As predicted, the final crack density is larger in the outer layer than in the inner layer, due to the fact that the outer layer is thinner than the inner layer.
Figure 5.11. Comparison of Model Predictions with Experimental Data for Matrix Cracking in [90/0/903]s E-Glass Epoxy Under $N_X$.

Figure 5.12. Comparison of Model Predictions with Experimental Data for Modulus Decrease in [90/0/903]s E-Glass Epoxy Under $N_X$. 
Cases 2 and 4: Loading Under $N_Y$

The model was designed to predict progressive cracking in the $0^\circ$ plies of the above laminates under loading in the $y$ direction; however, the results were poor. This is most likely due to the additional complication of two different interlaminar shear stresses. As modeled, the outer and central layers of these laminates experience interlaminar shear stress due to cracking on one surface only. The interlaminar shear stress is zero on the outside surface and at the midplane of the laminate. The complex asymmetry of the through thickness boundary conditions on the $0^\circ$ plies is a likely explanation for the problem. The experimental results are shown in Figures 5.13 through 5.16 for future reference, since this is an area to be explored in future work.

Figure 5.13. Experimental Results for Matrix Cracking in [902/02/90]s E-Glass Epoxy Under $N_Y$
Figure 5.14. Experimental Results for Modulus Decrease in [90/0/90]_s E-Glass Epoxy Under $N_y$

Figure 5.15. Experimental Results for Matrix Cracking in [90/0/90]_s E-Glass Epoxy Under $N_y$
Figure 5.16. Experimental Results for Modulus Decrease in [90/0/903]s E-Glass Epoxy Under $N_Y$
Two models have been developed, which predict transverse matrix cracking in fiber-reinforced composites under static biaxial loading. They utilize fundamental fracture mechanics principles and include the effects of local delamination. The three-layer model predicts transverse matrix cracking in the 90° ply group of laminates of the type [±θ/90m]s or [0n/90m]s under in-plane biaxial loading, including in-plane shear. The five-layer model is written to predict transverse matrix cracking in all ply groups of laminates of the type [90n/0m/90p]s under in-plane biaxial loading but not including in-plane shear. The inputs to the models are material properties including critical strain energy release rate. In addition, the degree of delamination expected, in terms of delamination at one or both transverse matrix crack tips, must be supplied. The models give stress and strain at the onset of transverse matrix cracking, predictions of progressive cracking as load is increased, and the decrease in modulus as cracking occurs. They also show the effect of transverse crack tip delamination on transverse matrix cracking. In addition, the laminate ultimate failure stress is predicted. Specific conclusions and contributions are summarized here.

1. The three-layer model is a shear lag model developed to predict progressive matrix cracking under biaxial loading using fundamental fracture mechanics principles. In addition, it incorporates the interaction between ply groups in terms of local delamination.

2. The five-layer model is written for cracking in all ply groups of laminates with five ply groups. It is also written for in-plane biaxial loading and it also incorporates the effect of local delamination on matrix cracking.

3. The models were compared with in-house experimental data for E-glass epoxy composites, as well as, with data from the literature, and excellent agreement was obtained in most
cases. In cases where agreement was not perfect throughout the progression of cracking, excellent agreement with the initial and final damage states was still obtained.

The five-layer model lays the foundation for a more rigorous predictive method for transverse matrix cracking and local delamination in all ply groups of laminates with angle plies; however, additional work remains. First the problem with loading in the Ny direction (Cases 2 and 4) must be solved. Then, the model should be expanded to include transverse matrix cracking in angle plies. A more rigorous method for evaluating the effects of local delamination should be developed. A statistical method for predicting the site of each new crack would enhance accuracy of the model, particularly for the initial damage development, since crack spacing becomes more uniform as cracking progresses. Verification under biaxial loading should also be done.

The success of the model predictions and the experimental observations lead to some important generalizations. Small delaminations occur at transverse crack tips and they account for a decrease in the shear modulus of the laminate. Local delamination depends strongly on the thickness of the central 90° layer, and, when it occurs, it has a substantial effect on transverse matrix cracking. Final failure is accurately predicted by applying the maximum stress criterion to the uncracked plies, provided the samples fail in the gage section.
REFERENCES


APPENDIX A

SOLUTION OF THE GOVERNING EQUATIONS FOR THE FIVE-LAYER MODEL

The governing equations are:

\[
\begin{align*}
\frac{1}{H_1} \begin{pmatrix} A_{55} & 0 \\ 0 & A_{44} \end{pmatrix} & \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{22} \end{pmatrix} & \begin{pmatrix} \Delta n_1 \\ \Delta n_2 \end{pmatrix} = & \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{22} \end{pmatrix} & \begin{pmatrix} \Delta n_1 \\ \Delta n_2 \end{pmatrix} \\
+ & \frac{H_2^2}{4} & \begin{pmatrix} F_{55} & 0 \\ 0 & F_{44} \end{pmatrix} & \begin{pmatrix} \frac{d}{dx_1} & \frac{d}{dx_2} \\ \frac{d}{dx_2} & \frac{d}{dx_1} \end{pmatrix} & \begin{pmatrix} \Delta n_1 \\ \Delta n_2 \end{pmatrix} & = & \begin{pmatrix} H_1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \Delta n_1 \\ \Delta n_2 \end{pmatrix} & = & \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{align*}
\]

(A.1)
The assumed forms of the change in load terms ($nmi$, $nti$, $nbi$) are deduced from the initial model and are given by:

\[
\begin{align*}
\Delta n_1^{(1)} &= A \frac{\cosh(\lambda m x_1)}{\cosh(\lambda m Lm/2)} + B \frac{\cosh(\lambda t x_2)}{\cosh(\lambda t Lt/2)} + C \frac{\cosh(\lambda b x_2)}{\cosh(\lambda b Lb/2)} \\
\Delta n_2^{(1)} &= D \frac{\cosh(\lambda t x_2)}{\cosh(\lambda t Lt/2)} + E \frac{\cosh(\lambda b x_2)}{\cosh(\lambda b Lb/2)} + F \frac{\cosh(\lambda m x_1)}{\cosh(\lambda m Lm/2)} \\
\Delta n_1^{(2)} &= G \frac{\cosh(\lambda m x_1)}{\cosh(\lambda m Lm/2)} + H \frac{\cosh(\lambda t x_2)}{\cosh(\lambda t Lt/2)} + I \frac{\cosh(\lambda b x_2)}{\cosh(\lambda b Lb/2)} \\
\Delta n_2^{(2)} &= J \frac{\cosh(\lambda t x_2)}{\cosh(\lambda t Lt/2)} + K \frac{\cosh(\lambda b x_2)}{\cosh(\lambda b Lb/2)} + L \frac{\cosh(\lambda m x_1)}{\cosh(\lambda m Lm/2)} \\
\Delta n_1^{(3)} &= M \frac{\cosh(\lambda m x_1)}{\cosh(\lambda m Lm/2)} + N \frac{\cosh(\lambda t x_2)}{\cosh(\lambda t Lt/2)} + O \frac{\cosh(\lambda b x_2)}{\cosh(\lambda b Lb/2)} \\
\Delta n_2^{(3)} &= P \frac{\cosh(\lambda t x_2)}{\cosh(\lambda t Lt/2)} + Q \frac{\cosh(\lambda b x_2)}{\cosh(\lambda b Lb/2)} + R \frac{\cosh(\lambda m x_1)}{\cosh(\lambda m Lm/2)}
\end{align*}
\]
where the \( L_i \) are the distances between the cracks in the top, middle, and bottom layers, respectively. The \( \lambda_i \) act as the shear lag parameters discussed earlier for the top, middle, and bottom layers, respectively.

For all \( x_1 \) and all \( x_2 \), the total load carried by the laminate must equal the applied load; therefore,

\[
\Delta n_1^{(1)} + \Delta n_1^{(2)} + \Delta n_1^{(3)} = 0 \quad \text{and} \quad \Delta n_2^{(1)} + \Delta n_2^{(2)} + \Delta n_2^{(3)} = 0
\]

for all \( x_1 \) and \( x_2 \).

Thus,

\[
\begin{align*}
\mathbf{n}_1^{(3)} &= (-G - A)\frac{\cosh(\lambda m \cdot x_1)}{\cosh(\frac{1}{2} \lambda m \cdot L_m)} + (-H - B)\frac{\cosh(\lambda t \cdot x_2)}{\cosh(\frac{1}{2} \lambda t \cdot L_t)} + (-C)\frac{\cosh(\lambda b \cdot x_2)}{\cosh(\frac{1}{2} \lambda b \cdot L_b)} \\
\mathbf{n}_2^{(3)} &= (-E - K)\frac{\cosh(\lambda b \cdot x_2)}{\cosh(\frac{1}{2} \lambda b \cdot L_b)} + (-J - D)\frac{\cosh(\lambda t \cdot x_2)}{\cosh(\frac{1}{2} \lambda t \cdot L_t)} + (-L - F)\frac{\cosh(\lambda m \cdot x_1)}{\cosh(\frac{1}{2} \lambda m \cdot L_m)}
\end{align*}
\]

(A.10)

(A.11)

Substituting into the governing equations:

\[
\begin{align*}
\frac{-F_{12}^{(1)} \cdot E + F_{11}^{(2)} \cdot I + F_{12}^{(2)} \cdot K}{F_{11}^{(1)}} &= C \\
\frac{-F_{12}^{(1)} \cdot D - F_{11}^{(2)} \cdot H - F_{12}^{(2)} \cdot J}{F_{11}^{(1)}} &= B
\end{align*}
\]

(A.12)

(A.13)

\[
\begin{align*}
2 \sqrt{A_{55}^{(1)}} - F_{11}^{(1)} - H_{55}^{(2)} - A + A_{55}^{(1)} \cdot H_{55}^{(2)} + H_{1}^{2} \cdot A &= \lambda m \\
\sqrt{A_{55}^{(1)}} \cdot H_{55}^{(2)} \cdot A + A_{55}^{(1)} \cdot H_{55}^{(2)} \cdot G + H_{1}^{2} \cdot A &= \lambda m^2
\end{align*}
\]

(A.14)

Second equation

\[
\begin{align*}
\frac{-F_{22}^{(1)} \cdot F_{12}^{(2)} \cdot G - F_{22}^{(2)} \cdot L}{F_{12}^{(1)}} &= A
\end{align*}
\]

(A.15)
The third and fourth governing equations are the same as the first two.

Fifth equation

\[- \left[ F_{12}^{(2)} \cdot H + F_{11}^{(2)} \cdot I + F_{11}^{(3)} \cdot J + F_{12}^{(3)} \cdot (E + K) \right] \frac{1}{F_{11}} = \lambda b \]  

(A.18)

Sixth equation

\[- \left[ F_{12}^{(2)} \cdot G + F_{22}^{(2)} \cdot L + F_{22}^{(3)} \cdot (L + F) + F_{12}^{(3)} \cdot G \right] \frac{1}{F_{12}^{(3)}} = \lambda m^2 \]  

(A.20)
Now we can combine these expressions with those given above in order to find more of the constants, noting that, for example, \( F_{11} F_{12} = F_{12} F_{11} \) for a crossply laminate of the type \([90^\circ/0^\circ/90^\circ]s\) only.

Summarizing the constants found,

\[ a = \begin{cases} \frac{F_{12}^{(2)} F_{11}^{(3)} + F_{12}^{(2)} F_{11}^{(1)} + F_{12}^{(3)} F_{11}^{(1)}}{-F_{11}^{(2)} F_{11}^{(3)} - F_{11}^{(2)} F_{11}^{(1)} - F_{11}^{(1)} F_{11}^{(3)}} \\ \frac{F_{22}^{(3)} (J + D) + F_{12}^{(3)} (H + B) + F_{12}^{(2)} H + F_{22}^{(2)} J}{F_{12}^{(2)} H + F_{22}^{(2)} J} \end{cases} \]  

\[ b = \begin{cases} \frac{F_{22}^{(2)} F_{12}^{(3)} + F_{22}^{(2)} F_{12}^{(1)} + F_{22}^{(3)} F_{12}^{(1)}}{-F_{12}^{(2)} F_{12}^{(3)} - F_{12}^{(2)} F_{12}^{(1)} - F_{12}^{(1)} F_{12}^{(3)}} \\ \frac{F_{11}^{(2)} a + F_{12}^{(2)}}{F_{11}^{(1)}} \end{cases} \]

\[ a = \begin{cases} \frac{F_{11}^{(2)} a + F_{12}^{(2)}}{F_{11}^{(1)}} - F_{12}^{(1)} \frac{E}{F_{11}^{(1)}} = C \\ \frac{F_{11}^{(2)} a + F_{12}^{(2)}}{F_{11}^{(1)}} - F_{12}^{(1)} \frac{D}{F_{11}^{(1)}} = B \end{cases} \]

\[ b = \begin{cases} \frac{F_{12}^{(2)} b + F_{22}^{(2)}}{F_{12}^{(1)}} - F_{22}^{(1)} \frac{F}{F_{12}^{(1)}} = A \\ \frac{F_{11}^{(2)} a + F_{12}^{(2)}}{F_{11}^{(1)}} - F_{12}^{(1)} \frac{D}{F_{11}^{(1)}} = B \end{cases} \]

where

\[ c = \begin{cases} \frac{F_{11}^{(2)} a + F_{12}^{(2)}}{F_{11}^{(1)}} \end{cases} \]
\[ d = \frac{F_{12}^{(2)} \cdot b + F_{22}^{(2)}}{F_{12}^{(1)}} \]

\[ e = (b + d) \left[ -d F_{11}^{(1)} + F_{11}^{(2)} \cdot b + F_{12}^{(2)} \right] A_{55}^{(1)} H_{3}^{2} + (-b - 2 \cdot d) \left[ (b + d) F_{11}^{(3)} + 2 F_{11}^{(2)} \cdot b - d F_{11}^{(1)} \right] A_{55}^{(3)} H_{2}^{2} A_{55}^{(1)} F_{55}^{(2)} + d \left[ (b + d) F_{11}^{(3)} + F_{12}^{(2)} + F_{11}^{(2)} \cdot b + F_{12}^{(3)} \right] A_{55}^{(3)} H_{1}^{2} \]

\[ f = \left[ \frac{F_{12}^{(1)} \cdot b + F_{22}^{(2)} + F_{12}^{(2)} \cdot b + \left[ (-b - 2 \cdot d) F_{11}^{(1)} + F_{11}^{(2)} \cdot b + F_{12}^{(2)} \right] F_{22}^{(1)}}{F_{12}^{(1)}} \right] A_{55}^{(1)} H_{3}^{2} + \left[ \frac{F_{12}^{(3)} \cdot d + \left[ (2 \cdot d + b) F_{11}^{(3)} + F_{12}^{(2)} + F_{11}^{(2)} \cdot b + F_{12}^{(3)} \right] F_{22}^{(1)}}{F_{12}^{(1)}} \right] A_{55}^{(3)} H_{1}^{2} + \left[ (2 \cdot d + b) F_{12}^{(1)} + (-b - 2 \cdot d) F_{12}^{(3)} \right] A_{55}^{(3)} F_{55}^{(2)} H_{2}^{2} A_{55}^{(3)} + \left[ 4 F_{11}^{(2)} \cdot b + (3 \cdot b + 4 \cdot d) F_{11}^{(3)} + 2 F_{12}^{(3)} + (-4 \cdot d - b) F_{11}^{(1)} + 4 F_{12}^{(2)} + \right] \frac{F_{22}^{(1)}}{F_{12}^{(1)}} \]

\[ g = 2 \left[ \frac{F_{11}^{(3)} + F_{11}^{(1)}}{F_{12}^{(1)}} F_{22}^{(1)} + \left[ \frac{-F_{12}^{(1)} + F_{12}^{(3)}}{F_{12}^{(1)}} \right] H_{2}^{2} F_{55}^{(2)} A_{55}^{(1)} A_{55}^{(3)} \frac{F_{22}^{(1)}}{F_{12}^{(1)}} \right] \frac{F_{12}^{(1)}}{F_{12}^{(1)}} H_{2}^{2} + \left[ \frac{F_{11}^{(3)}}{F_{12}^{(1)}} \right] H_{1}^{2} F_{22}^{(1)} + \left[ \frac{-F_{11}^{(3)}}{F_{12}^{(1)}} \right] H_{1}^{2} F_{12}^{(3)} \cdot H_{1}^{2} A_{55}^{(3)} \frac{F_{22}^{(1)}}{F_{12}^{(1)}} + \left[ \frac{F_{11}^{(1)} F_{22}^{(1)}}{F_{12}^{(1)}} - F_{12}^{(1)} \right] A_{55}^{(1)} H_{3}^{2} \frac{F_{22}^{(1)}}{F_{12}^{(1)}} \]

\[ h = -2 F_{22}^{(2)} + F_{12}^{(1)} c - b F_{22}^{(3)} - F_{12}^{(3)} c - a \left[ 2 F_{12}^{(2)} + F_{12}^{(3)} \right] A_{44}^{(1)} A_{44}^{(3)} H_{2}^{2} F_{44}^{(2)} + F_{12}^{(1)} c - b F_{22}^{(2)} - a F_{12}^{(2)} ] A_{44}^{(1)} H_{3}^{2} \]
\[ p = -4F_{22}^{(2)} + 2F_{12}^{(1)}c + F_{22}^{(1)} - 2F_{12}^{(3)}c - 3F_{22}^{(3)} \cdot A_{44}^{(1)}A_{44}^{(3)}H_2^2F_{44}^{(2)} \]
\[ + a\left[ 4F_{12}^{(2)} + 2F_{12}^{(3)} \right] + \frac{F_{12}^{(3)} - F_{12}^{(1)}}{F_{11}^{(1)}} \cdot H_1^2 - a\left[ F_{12}^{(3)} + F_{12}^{(2)} \right]H_1^2 \cdot A_{44}^{(3)} \]
\[ + \left[ -F_{22}^{(2)} - F_{12}^{(3)}c - F_{22}^{(3)} \right]H_1^2 - a\left[ F_{12}^{(3)} + F_{12}^{(2)} \right]H_1^2 \cdot A_{44}^{(1)} \]

\[ q = \left[ F_{22}^{(1)} - \frac{F_{12}^{(1)}}{F_{11}^{(1)}} \right]A_{44}^{(1)}H_3^2 \]
\[ + \left[ F_{22}^{(1)} - F_{22}^{(3)} \right]2 + \frac{2F_{12}^{(3)} - 2F_{12}^{(1)}}{F_{11}^{(1)}} \cdot A_{44}^{(1)}A_{44}^{(3)}H_2^2F_{44}^{(2)} \]
\[ + \left[ -F_{22}^{(3)} + F_{12}^{(1)} \right]A_{44}^{(3)}H_1^2 \]

In addition,
\[ e \cdot L^2 + f \cdot F \cdot L + g F^2 = 0 \quad \quad q \cdot E^2 + p \cdot K \cdot E + h \cdot K^2 = 0 \quad \quad h \cdot J^2 + p \cdot D \cdot J + q \cdot D^2 = 0 \]
so

\[ \frac{1}{(2 \cdot g)} \cdot \left( -fL + L \cdot \sqrt{L^2 - 4 \cdot g \cdot e} \right) \quad = F \quad \text{(A.30)} \]
\[ \frac{1}{(2 \cdot g)} \cdot \left( -fL - L \cdot \sqrt{L^2 - 4 \cdot g \cdot e} \right) \quad = E \quad \text{(A.31)} \]

\[ \frac{1}{(2 \cdot q)} \cdot \left( -pJ + J \cdot \sqrt{J^2 - 4 \cdot q \cdot h} \right) \quad = D \quad \text{(A.32)} \]
Summarizing the constants found,

\[ A = \frac{d - F_{22}^{(1)}}{F_{12}^{(1)}} \cdot L \quad B = ra \cdot J \quad C = rb \cdot K \quad D = sa \cdot J \quad E = sb \cdot K \]  
(A.33)

where

\[ sa = \frac{1}{2 - q} \left( -p + \sqrt{p^2 - 4 \cdot q \cdot h} \right) \quad sb = \frac{1}{2 - q} \left( -p - \sqrt{p^2 - 4 \cdot q \cdot h} \right) \]

\[ ra = \frac{1}{F_{12}^{(1)}} \cdot \frac{F_{11}^{(1)}}{sa} \quad rb = \frac{1}{F_{12}^{(1)}} \cdot \frac{F_{11}^{(1)}}{sb} \]

We don’t know apriori which values the above parameters will take for a given laminate. It turns out that, indeed, they must be different for different laminates. For the laminates discussed in this work,

\[ t = \frac{1}{2 - g} \left( f + \sqrt{f^2 - 4 \cdot g \cdot e} \right) \]

and the values taken by sa and sb have opposite signs preceding the square root term for each laminate.

Thus, for the \([90/0/90^3]_s\) laminate,

\[ sa = \frac{1}{2 - q} \left( -p - \sqrt{p^2 - 4 \cdot q \cdot h} \right) \quad sb = \frac{1}{2 - q} \left( p + \sqrt{p^2 - 4 \cdot q \cdot h} \right) \]

and for the \([90_2/0_2/90]_s\) laminate,

\[ sa = \frac{1}{2 - q} \left( p + \sqrt{p^2 - 4 \cdot q \cdot h} \right) \quad sb = \frac{1}{2 - q} \left( -p - \sqrt{p^2 - 4 \cdot q \cdot h} \right) \]

Now, the boundary conditions must be satisfied. They are:

\[ \Delta n_1^{(1)} + \Delta n_1^{(3)} = N_{11}^{(2)} \] at \( x_1 = \frac{L_m}{2} \)  
(A.34)

\[ \Delta n_2^{(3)} + \Delta n_2^{(2)} = N_{21}^{(1)} \] at \( x_2 = \frac{L_t}{2} \)  
(A.35)

\[ \Delta n_2^{(1)} + \Delta n_2^{(2)} = N_{21}^{(3)} \] at \( x_2 = \frac{L_b}{2} \)  
(A.36)
where \( N_{11}^{(2)} \), \( N_{21}^{(1)} \), and \( N_{21}^{(3)} \) are the initial loads taken by the three layers in the 1 and 2 directions. They are found from the constitutive relations given in Appendix A. These cannot be satisfied everywhere, so each disturbance to the change in load in each layer is averaged in order to get an average solution to the boundary conditions.

\[
\begin{pmatrix}
\tanh \left( \frac{\lambda t}{2} \right) \\
-2 - \frac{H - 1 \cdot G}{2} - \frac{L}{2} - 1
\end{pmatrix}
= N_{11}^{(2)}
\]

(A.37)

\[
\begin{pmatrix}
\cosh \left( \frac{\lambda t}{2} \right) \\
- \frac{2 \cdot \tanh \left( \frac{\lambda b}{2} \right)}{L_b}
\end{pmatrix}
- \frac{F \cdot 2 \cdot \tanh \left( \frac{\lambda m}{2} \right)}{L_m}
= N_{21}^{(1)}
\]

(A.38)

\[
\begin{pmatrix}
\cosh \left( \frac{\lambda t}{2} \right) \\
\cosh \left( \frac{\lambda b}{2} \right)
\end{pmatrix}
+ \frac{2 \cdot \tanh \left( \frac{\lambda m}{2} \right)}{L_m}
= N_{21}^{(3)}
\]

(A.39)

Solving the first BC (equation 33) for \( L \),

\[
L = \left[ \begin{pmatrix}
\tanh \left( \frac{\lambda t}{2} \right) \\
-2 - \frac{H - 1 \cdot G}{2} - \frac{L}{2} - 1
\end{pmatrix}
= N_{11}^{(2)}
\right] 
\]

\[
= \left[ \begin{pmatrix}
\tanh \left( \frac{\lambda b}{2} \right)
\end{pmatrix}
- \frac{2 \cdot \tanh \left( \frac{\lambda m}{2} \right)}{L_m}
\right] \cdot a - N_{11}^{(2)}
\]

\[
= b
\]
Substituting for $L$ in the third BC (equation 35) and solving for $K$

\[
K = \left[ N_{2i}^{(3)} + \frac{2(t+1)}{b} N_{1i}^{(2)} \right] \frac{\tanh \left( \frac{1}{2} \lambda m Lm \right)}{Lm}
\]

\[
1 + sb - 4a \frac{(t+1)}{(bLb)} \tanh \left( \frac{1}{2} \lambda b Lb \right) \frac{\tanh \left( \frac{1}{2} \lambda m Lm \right)}{Lm}
\]

\[
2 \left[ 2a \frac{(t+1)}{(bLb)} \tanh \left( \frac{1}{2} \lambda m Lm \right) + 2(-sa-1) \right] \tan \left( \frac{1}{2} \lambda t^2 \right) \frac{J}{Lt}
\]

\[
+ \left[ 1 + sb - 4a \frac{(t+1)}{(bLb)} \tanh \left( \frac{1}{2} \lambda b Lb \right) \frac{\tanh \left( \frac{1}{2} \lambda m Lm \right)}{Lm} \right]
\]

Substituting for $L$ and $K$ in the second BC (equation 34) and solving for $J$ \[J = \left[ N_{2i}^{(1)} \left( \tanh \left( \frac{1}{2} \lambda b Lb \right) \frac{\tanh \left( \frac{1}{2} \lambda m Lm \right)}{Lb} \right) - \frac{2}{2} \right. \frac{\tanh \left( \frac{1}{2} \lambda b Lb \right)}{Lb} \left. \right] - \frac{1}{2} \frac{\tanh \left( \frac{1}{2} \lambda m Lm \right)}{Lm} \]

\[
N_{2i}^{(1)} \left( \tanh \left( \frac{1}{2} \lambda b Lb \right) \frac{\tanh \left( \frac{1}{2} \lambda m Lm \right)}{Lb} \right) - \frac{2}{2} \frac{\tanh \left( \frac{1}{2} \lambda b Lb \right)}{Lb} \left. \right] - \frac{1}{2} \frac{\tanh \left( \frac{1}{2} \lambda m Lm \right)}{Lm} \]

\[
+ \frac{bb + aa}{\frac{2}{2} \frac{\tanh \left( \frac{1}{2} \lambda m Lm \right)}{Lm} \left( \frac{1}{2} \lambda b Lb \right)} \frac{\tanh \left( \frac{1}{2} \lambda m Lm \right)}{Lb} \frac{\tanh \left( \frac{1}{2} \lambda m Lm \right)}{Lm} + \frac{dd}{cc - ee} \frac{\tanh \left( \frac{1}{2} \lambda m Lm \right)}{Lb} \frac{\tanh \left( \frac{1}{2} \lambda m Lm \right)}{Lm}
\]

\[
- sa \left( \tanh \left( \frac{1}{2} \lambda b Lb \right) \frac{\tanh \left( \frac{1}{2} \lambda m Lm \right)}{Lb} \right) - \frac{2}{2} \frac{\tanh \left( \frac{1}{2} \lambda b Lb \right)}{Lb} \left( \frac{1}{2} \frac{\tanh \left( \frac{1}{2} \lambda m Lm \right)}{Lm} - 2 \cdot hh \right)
\[
+ \frac{2 \frac{gg}{(LbLm)} \tanh \left( \frac{1}{2} \lambda m Lm \right) \left( \frac{1}{2} \lambda b Lb \right) \left( \frac{1}{2} \frac{\tanh \left( \frac{1}{2} \lambda m Lm \right)}{Lm} - 2 \cdot hh \right) \frac{\tanh \left( \frac{1}{2} \lambda t^2 \right)}{Lt}
\]

\[
+ \frac{2}{2} \frac{\tanh \left( \frac{1}{2} \lambda m Lm \right)}{Lb} \left( \frac{1}{2} \frac{\tanh \left( \frac{1}{2} \lambda m Lm \right)}{Lm} \right)
\]

\[
+ \frac{ff}{cc - ee} \frac{\tanh \left( \frac{1}{2} \lambda m Lm \right)}{Lb} \frac{\tanh \left( \frac{1}{2} \lambda m Lm \right)}{Lm}
\]
where
\[
\begin{align*}
aa &= 4 \frac{dd}{b} (t + 1) \frac{a}{b}, \\
bb &= -4 \frac{sb}{b} - \frac{t + 1}{b}, \\
c &= 1 + sb, \\
rr &= 2 - \frac{t}{b}, \\
cc &= 1 + sb, \\
cc &= 1 + sb, \\
dd &= 2 - \frac{t}{b}, \\
\end{align*}
\]

and the other constants are given in terms of J, K, and L above. With the crack density terms,

\[
N_{2i}^{(1)} \left[ cc - ee \left( \frac{1}{2} \lambda b Lb \right) \beta b \right] \tanh \left( \frac{1}{2} \lambda m Lm \right) \beta m \\
+ 2 \cdot 2 \cdot sb + ff \tanh \left( \frac{1}{2} \lambda m Lm \right) \beta m \cdot 1 \cdot \tanh \left( \frac{1}{2} \lambda b Lb \right) \beta b \cdot N_{2i}^{(3)} \\
+ \left\{ bb + aa \cdot \tanh \left( \frac{1}{2} \lambda m Lm \right) \beta m \cdot \tanh \left( \frac{1}{2} \lambda b Lb \right) \beta b \right\} - 1 \cdot \tanh \left( \frac{1}{2} \lambda m Lm \right) \beta m \cdot N_{1i}^{(2)} \\
+ dd \cdot \left[ cc - ee \cdot \tanh \left( \frac{1}{2} \lambda b Lb \right) \tanh \left( \frac{1}{2} \lambda m Lm \right) \beta b \right]
\]

where
\[
\beta i = \frac{1}{Li}
\]

\[
J = \left[ N_{2i}^{(1)} + 2 \cdot (t + 1) \cdot N_{1i}^{(2)} \cdot tanh \left( \frac{1}{2} \lambda m Lm \right) \beta m \cdot \frac{1}{2} \right]
\]

\[
K = \left[ 1 + sb - 4 \cdot a \cdot (t + 1) \cdot tanh \left( \frac{1}{2} \lambda b Lb \right) \tanh \left( \frac{1}{2} \lambda m Lm \right) \beta m \beta b \right] \tanh \left( \frac{1}{2} \lambda m Lm \right) \beta m \beta b
\]

\[
L = \left[ -2 \cdot \tanh \left( \frac{1}{2} \lambda t Lt \right) \beta b \cdot a \cdot J - 2 \cdot \tanh \left( \frac{1}{2} \lambda b Lb \right) \beta b \cdot a \cdot K - N_{1i}^{(2)} \right]
\]
The constants can be found for the case of initial cracking. In order to simplify for the case of zero cracks, all the tanh terms are approximately equal to one for Li large.

\[
J = \frac{N_{2i}^{(1)} (cc - ee \cdot \beta b \cdot \beta m) - (2 \cdot sb + ff \cdot \beta m) \cdot \beta b \cdot N_{2i}^{(3)}}{(-2 \cdot sb + ff \cdot \beta m) + ((bb + aa \cdot \beta m) \cdot \beta b + dd \cdot (cc - ee \cdot \beta b \cdot \beta m)) \cdot 1 \cdot \beta m \cdot N_{1i}^{(2)}}
\]

\[
K = \frac{N_{2i}^{(3)} + 2 \cdot \frac{(t+1)}{b} \cdot N_{1i}^{(2)} \cdot \beta m}{1 + sb - 4 \cdot a \cdot \frac{(t+1)}{b} \cdot \beta m \cdot \beta b} + \frac{4 \cdot a \cdot \frac{(t+1)}{b} \cdot \beta m + 2 \cdot (-sa - 1)}{1 + sb - 4 \cdot a \cdot \frac{(t+1)}{b} \cdot \beta m \cdot \beta b} \cdot \beta t \cdot J
\]

\[
L = \frac{-2 \cdot \beta t \cdot a \cdot J - 2 \cdot \beta b \cdot a \cdot K - N_{1i}^{(2)}}{b}
\]

Then the crack density terms are set equal to zero, and

\[
J = -\frac{N_{2i}^{(1)}}{sa} \quad \text{(A.42)}
\]

\[
K = -\frac{N_{2i}^{(3)}}{1 + sb} \quad \text{(A.43)}
\]

\[
L = -\frac{N_{1i}^{(2)}}{b} \quad \text{(A.44)}
\]
APPENDIX B

DETERMINATION OF STRAIN ENERGY RELEASED DUE TO CRACKING

For a given crack density, the change in work due to external applied loads is equal to the work needed for the previous crack density minus the work needed to achieve the current crack density. The applied loading is the same. The external work for state $j$ is thus,

$$
\Delta W_{extj} = 2 \cdot N_1 \cdot F_{11}^{(3)} \cdot \Delta n_1^{(3)} + F_{12}^{(3)} \cdot \Delta n_2^{(3)} + 2 \cdot N_2 \cdot \left[ F_{12}^{(2)} \cdot \Delta n_1^{(2)} + F_{22}^{(2)} \cdot \Delta n_2^{(2)} \right] \cdot \text{Area}
$$  \hspace{1cm} (B.1)

where $N_1$ and $N_2$ are applied loads in the 1 and 2 directions. As stated in Appendix A, the coordinate system is referenced to the 90 degree layer. A factor of 2 is used because $N_1$ and $N_2$ are half laminate loads.

$$
\text{Area} = L_m \cdot L_i
$$  \hspace{1cm} (B.2)

where $L_i = L_b$ or $L_t$ depending upon which lamina is cracked at the current state, or, if the middle lamina is cracked, $i$ refers to the ply group previously cracked.

The in-plane strain energy for ply group $k$ is given by:

$$
\Delta W_{intpj}^{(k)} = \frac{1}{2} \left( \Delta n_1^{(k)} \cdot \Delta n_2^{(k)} \right) \cdot \left[ \begin{array}{cc}
F_{11} & F_{12} \\
F_{12} & F_{22}
\end{array} \right] \cdot \left( \begin{array}{c}
\Delta n_1^{(k)} \\
\Delta n_2^{(k)}
\end{array} \right) + 2 \cdot \left( \begin{array}{c}
\varepsilon_1 \\
\varepsilon_2
\end{array} \right) - 2 \cdot \left( \begin{array}{c}
F_{11} \cdot \varepsilon_1 + F_{12} \cdot \varepsilon_2 \\
F_{12} \cdot \varepsilon_1 + F_{22} \cdot \varepsilon_2
\end{array} \right) \\
\left( \begin{array}{c}
\Delta n_1^{(k)} \\
\Delta n_2^{(k)}
\end{array} \right)
$$  \hspace{1cm} (B.3)

where $\varepsilon_1$ and $\varepsilon_2$ are the strains in the laminate at the time of cracking. They are given with a caret over them in Appendix A. The change in load terms are defined for crack state $j$. The change in load terms are actually area averages. They are integrated over area and divided by area.

For example,

$$
\frac{\Delta n_1}{2} = \int \frac{A}{2} \cdot \Delta n_1 \, dA \hspace{1cm} \text{and} \hspace{1cm} \int \frac{\Delta n_1^2}{2} \, dA
$$  \hspace{1cm} (B.4)

This operation is straightforward for the $\Delta n_1$ terms but not for the $\Delta n_1^2$ terms. Both are described more fully below.
For the change in load in the top layer,

\[ \Delta n_1^{(1)} = A\cdot 2 \cdot \tanh \left( \frac{\lambda m \cdot Lm}{2} \right) + B\cdot 2 \cdot \tanh \left( \frac{\lambda t \cdotLt}{2} \right) + C\cdot 2 \cdot \tanh \left( \frac{\lambda b \cdot Lb}{2} \right) \]

\[ \Delta n_2^{(1)} = D\cdot 2 \cdot \tanh \left( \frac{\lambda t \cdot Lt}{2} \right) + E\cdot 2 \cdot \tanh \left( \frac{\lambda b \cdot Lb}{2} \right) + F\cdot 2 \cdot \tanh \left( \frac{\lambda m \cdot Lm}{2} \right) \] (B.6)

For the change in load in the middle layer,

\[ \Delta n_1^{(2)} = G\cdot 2 \cdot \tanh \left( \frac{\lambda m \cdot Lm}{2} \right) + H\cdot 2 \cdot \tanh \left( \frac{\lambda t \cdot Lt}{2} \right) + I\cdot 2 \cdot \tanh \left( \frac{\lambda b \cdot Lb}{2} \right) \]

\[ \Delta n_2^{(2)} = J\cdot 2 \cdot \tanh \left( \frac{\lambda t \cdot Lt}{2} \right) + K\cdot 2 \cdot \tanh \left( \frac{\lambda b \cdot Lb}{2} \right) + L\cdot 2 \cdot \tanh \left( \frac{\lambda m \cdot Lm}{2} \right) \] (B.7)

For the change in load in the bottom layer,

\[ \Delta n_1^{(3)} = (- \cdot G - A)\cdot 2 \cdot \tanh \left( \frac{\lambda m \cdot Lm}{2} \right) + (- \cdot H - B)\cdot 2 \cdot \tanh \left( \frac{\lambda t \cdot Lt}{2} \right) + (- \cdot I - C)\cdot 2 \cdot \tanh \left( \frac{\lambda b \cdot Lb}{2} \right) \]

\[ \Delta n_2^{(3)} = (- \cdot E - K)\cdot 2 \cdot \tanh \left( \frac{\lambda b \cdot Lb}{2} \right) + (- \cdot J - D)\cdot 2 \cdot \tanh \left( \frac{\lambda t \cdot Lt}{2} \right) + (- \cdot L - F)\cdot 2 \cdot \tanh \left( \frac{\lambda m \cdot Lm}{2} \right) \] (B.8)

These terms have the same form regardless of where cracking has occurred. For example, cracking in the top lamina group only means that the effects of the middle and bottom are averaged, while the top layer term is integrated between cracks.

For the \( \Delta n \) terms this is the same as averaging all three terms, but for the \( \Delta n^2 \) terms the effect is different.

For the example of cracking in the top lamina group, before integration,

\[ \Delta n_1^{(1)} = \frac{A\cdot 2 \cdot \tanh \left( \frac{\lambda m \cdot Lm}{2} \right)}{\lambda m \cdot Lm} + \frac{B}{\cosh(\lambda t \cdot Lt / 2)} + \frac{C\cdot 2 \cdot \tanh \left( \frac{\lambda b \cdot Lb}{2} \right)}{\lambda b \cdot Lb} \] (B.9)
so,

\[ [\Delta n_1^{(1)}]^2 = 4A^2 \frac{\tanh\left(\frac{1}{2} \beta t \cdot L_m\right)^2}{(\beta t^2 \cdot L_m^2)} + 4A \frac{\tanh\left(\frac{1}{2} \beta t \cdot L_m\right)}{(\beta t \cdot L_m)} B \frac{\cosh(\beta t \cdot x^2)}{\cosh\left(\frac{1}{2} \beta t \cdot L_t\right)} + 8A \frac{\tanh\left(\frac{1}{2} \beta t \cdot L_m\right)}{(\beta t \cdot L_m)} C \frac{\tanh\left(\frac{1}{2} \beta b \cdot L_b\right)}{(\beta b \cdot L_b)} + \ldots \]

(B.10)

\[ \int \frac{L_t}{2} \left[ \Delta n_1^{(1)} \right]^2 dx^2 = \left[ \frac{\tanh\left(\frac{1}{2} \beta t \cdot L_t\right)}{\beta t} + \frac{1}{\cosh\left(\frac{1}{2} \beta t \cdot L_t\right)^2} \right] B^2 + \ldots \]

(B.11)

The other terms are given by essentially the same expression, but with the constants switched appropriately.

For \( \Delta n_1^{(1)} \), for example, \( B^2 \) becomes \( B \cdot D \), \( A^2 \) becomes \( A \cdot F \), and \( C^2 \) becomes \( C \cdot E \). In addition, \( A \cdot B \) becomes \( A \cdot D \), \( A \cdot C \) becomes \( A \cdot E \) and \( B \cdot C \) becomes \( B \cdot E \). Shown explicitly, using equation B.9 and equation B.12 shown below,

\[ \Delta n_2^{(1)} = D \frac{\cosh(\beta t \cdot x^2)}{\cosh\left(\frac{1}{2} \beta t \cdot L_t\right)} + E \frac{\tanh\left(\frac{1}{2} \beta b \cdot L_b\right)}{\beta b \cdot L_b} + F \frac{\tanh\left(\frac{1}{2} \beta \cdot L_m\right)}{\beta \cdot L_m} \]

(B.12)
the result is:

$$\int \frac{\Delta n_1^{(1)} \cdot \Delta n_2^{(1)}}{L_{t/2}} \, dx = B \cdot D \cdot \left[ \frac{\tanh \left( \frac{1}{2} \cdot \lambda t \cdot L_t \right)}{\lambda t} + \frac{1}{2} \frac{L_t}{\cosh \left( \frac{1}{2} \cdot \lambda t \cdot L_t \right)^2} \right]$$

\begin{align*}
&+ \left[ \frac{\tanh \left( \frac{1}{2} \cdot \lambda m \cdot L_m \right)}{(\lambda m \cdot L_m)} \cdot \frac{\tanh \left( \frac{1}{2} \cdot \lambda b \cdot L_b \right)}{(\lambda b \cdot L_b)} \right] \\
&+ \left[ \frac{\tanh \left( \frac{1}{2} \cdot \lambda m \cdot L_m \right)}{(\lambda m \cdot L_m)^2} \cdot \frac{\tanh \left( \frac{1}{2} \cdot \lambda b \cdot L_b \right)}{(\lambda b \cdot L_b)^2} \right] \\
&+ A \cdot F \cdot \frac{\tanh \left( \frac{1}{2} \cdot \lambda m \cdot L_m \right)}{(\lambda m \cdot L_m)} \\
&+ (C \cdot F + A \cdot E) \cdot \frac{\tanh \left( \frac{1}{2} \cdot \lambda b \cdot L_b \right)}{(\lambda b \cdot L_b)^2} \\
&+ C \cdot E \cdot \frac{\tanh \left( \frac{1}{2} \cdot \lambda b \cdot L_b \right)}{(\lambda b \cdot L_b)} \\
&+ 4 \cdot Lt \\
&+ (A \cdot D + B \cdot F) \cdot \frac{\tanh \left( \frac{1}{2} \cdot \lambda m \cdot L_m \right)}{(\lambda m \cdot L_m)} \\
&+ (C \cdot D + B \cdot E) \cdot \frac{\tanh \left( \frac{1}{2} \cdot \lambda b \cdot L_b \right)}{(\lambda b \cdot L_b)} \\
&+ 4 \cdot Lt
\end{align*}

Similarly, cracking in the middle lamina group means

$$n_1^{(1)} = A \cdot \cosh(\lambda m \cdot x_1) \cdot \frac{\tanh \left( \frac{1}{2} \cdot \lambda t \cdot L_t \right)}{\cosh \left( \frac{1}{2} \cdot \lambda t \cdot L_t \right) \cdot \cosh \left( \frac{1}{2} \cdot \lambda m \cdot L_m \right) \cdot \cosh \left( \frac{1}{2} \cdot \lambda b \cdot L_b \right)} + B \cdot 2 \cdot \frac{\tanh \left( \frac{1}{2} \cdot \lambda b \cdot L_b \right)}{\lambda b \cdot L_b} \quad (B.14)$$

so

\begin{align*}
\left[ n_1^{(1)} \right]^2 &= A^2 \cdot \frac{\cosh(\lambda m \cdot x_1)^2}{\cosh(\lambda m \cdot L_m)^2} + 4 \cdot A^2 \cdot \frac{\cosh(\lambda m \cdot x_1)}{\cosh(\lambda m \cdot L_m)} \cdot \frac{\tanh \left( \frac{1}{2} \cdot \lambda t \cdot L_t \right)}{(\lambda t \cdot L_t)} \\
&+ 4 \cdot A \cdot \frac{\cosh(\lambda m \cdot x_1)}{\cosh(\lambda m \cdot L_m)} \cdot \frac{\tanh \left( \frac{1}{2} \cdot \lambda b \cdot L_b \right)}{(\lambda b \cdot L_b)} \\
&+ 4 \cdot A \cdot \frac{\tanh \left( \frac{1}{2} \cdot \lambda t \cdot L_t \right)}{(\lambda t \cdot L_t)} \cdot \frac{\tanh \left( \frac{1}{2} \cdot \lambda b \cdot L_b \right)}{(\lambda b \cdot L_b)} \\
&+ 4 \cdot B^2 \cdot \frac{\tanh \left( \frac{1}{2} \cdot \lambda t \cdot L_t \right)}{(\lambda t \cdot L_t^2)} + 8 \cdot B \cdot \frac{\tanh \left( \frac{1}{2} \cdot \lambda t \cdot L_t \right)}{(\lambda t \cdot L_t) \cdot \cosh \left( \frac{1}{2} \cdot \lambda m \cdot L_m \right) \cdot \cosh \left( \frac{1}{2} \cdot \lambda b \cdot L_b \right)} \cdot 4 \cdot C^2 \cdot \frac{\tanh \left( \frac{1}{2} \cdot \lambda b \cdot L_b \right)}{(\lambda b^2 \cdot L_b^2)} \\
&+ 4 \cdot C^2 \cdot \frac{\tanh \left( \frac{1}{2} \cdot \lambda b \cdot L_b \right)}{(\lambda b^2 \cdot L_b^2)}
\end{align*}

\( (B.15) \)
and
\[
\left(\frac{L_m}{2}\right)^2 \int 2^2 \mathrm{d}x_1 = A^2 \cdot \left[ \frac{\tanh\left(\frac{1}{2} \lambda_i - L_m\right)}{\lambda_i} + \frac{L_m}{2} \cdot \cosh\left(\frac{1}{2} \lambda_i L_m\right) \right] \left[ \frac{\tanh\left(\frac{1}{2} \lambda_i - L_m\right)}{\lambda_i} \cdot \tanh\left(\frac{1}{2} \lambda_i - L_i\right) \right] \cdot \frac{\lambda_i}{(\lambda_i - L_i)^2(\lambda_i - L_i)^2} + 4L_m \cdot \tanh\left(\frac{1}{2} \lambda_i - L_i\right) \cdot \tanh\left(\frac{1}{2} \lambda_i - L_i\right) \cdot \frac{\lambda_i}{(\lambda_i - L_i)^2(\lambda_i - L_i)^2} \right]
\]

For example, for initial cracking in the top laminate layer,
\[
\Delta W_{\text{ext}} = 2 \cdot N_1 \left[ F_{11}^{(3)} \cdot \frac{2}{\lambda t} \cdot \frac{B + H}{\lambda t} + F_{12}^{(3)} \cdot \frac{2}{\lambda t} \cdot \frac{J + D}{\lambda t} \right] + 2 \cdot N_2 \left[ F_{12}^{(3)} \cdot \frac{2}{\lambda t} \cdot \frac{B + H}{\lambda t} + F_{22}^{(3)} \cdot \frac{2}{\lambda t} \cdot \frac{J}{\lambda t} \right]
\]

\[
\Delta W_{\text{int}} = F_{11}^{(1)} \cdot \frac{B^2}{\lambda t} + 2F_{12}^{(1)} \cdot \frac{B \cdot D}{\lambda t} + 4F_{22}^{(1)} \cdot \frac{D^2}{\lambda t} \left[ \frac{\varepsilon_i}{2} - \frac{1}{\lambda t} \cdot \varepsilon_i \right] + 2 \cdot F_{12}^{(1)} \cdot \frac{B + H}{\lambda t} \cdot \frac{J}{\lambda t} \left[ \frac{\varepsilon_i}{2} - \frac{1}{\lambda t} \cdot \varepsilon_i \right] + 2 \cdot F_{22}^{(1)} \cdot \frac{B + H}{\lambda t} \cdot \frac{J}{\lambda t} \left[ \frac{\varepsilon_i}{2} - \frac{1}{\lambda t} \cdot \varepsilon_i \right]
\]

Of course, the expressions for subsequent cracking are substantially more complicated.
The transverse shear terms must also be found for each ply group \( k \) and are given by:

\[
\frac{dW_{ts}^{(k)}}{d\text{Area}} = \frac{1}{2} \frac{h_k}{4} \left( F_{44}^{(k)} \alpha_{23}^2 + 2 F_{45}^{(k)} \alpha_{13}^2 \alpha_{23} + F_{55}^{(k)} \alpha_{13}^2 \right) \cdot d\text{Area} \tag{B.19}
\]

where

\[
\tau_{23}^{(k)} = \frac{\delta n_{(k)}^2}{\delta x_{2}} \quad \text{and} \quad \tau_{13}^{(k)} = \frac{\delta n_{1}^{(k)}}{\delta x_{1}} \tag{B.20}
\]

Again, an averaging scheme is used. For the first ply group,

\[
\tau_{13}^{(1)} = A \cdot \lambda m \cdot \frac{\sinh(\lambda \cdot x_{1})}{\cosh(\lambda m \cdot \frac{L_m}{2})} \tag{B.21}
\]

\[
\tau_{23}^{(1)} = D \cdot \lambda t \cdot \frac{\sinh(\lambda t \cdot x_{2})}{\cosh(\lambda t \cdot \frac{L_t}{2})} + E \cdot \lambda b \cdot \frac{\sinh(\lambda b \cdot x_{2})}{\sinh(\lambda b \cdot \frac{L_b}{2})} \tag{B.22}
\]

\[
\left[ \tau_{13}^{(1)} \right]^2 = A^2 \cdot \lambda m^2 \cdot \frac{\sinh(\lambda m \cdot x_{1})^2}{\cosh(\lambda m \cdot \frac{L_m}{2})^2} \tag{B.23}
\]

\[
\left[ \tau_{23}^{(1)} \right]^2 = D^2 \cdot \lambda t^2 \cdot \frac{\sinh(\lambda t \cdot x_{2})^2}{\cosh(\lambda t \cdot \frac{L_t}{2})^2} + E^2 \cdot \lambda b^2 \cdot \frac{\sinh(\lambda b \cdot x_{2})^2}{\sinh(\lambda b \cdot \frac{L_b}{2})^2} + D \cdot \lambda t \cdot \frac{\sinh(\lambda t \cdot x_{2})}{\cosh(\lambda t \cdot \frac{L_t}{2})} \cdot E \cdot \lambda b \cdot \frac{\sinh(\lambda b \cdot x_{2})}{\sinh(\lambda b \cdot \frac{L_b}{2})} \tag{B.24}
\]

For cracking in the top layer only, the effects of the middle and bottom layers are averaged. Thus, when the derivatives are taken of the \( \Delta n_{1}^{(k)} \) terms, the middle and bottom layer terms are constant. The interlaminar shear terms are:

\[
\tau_{13}^{(1)} = 0 \quad \text{and} \quad \left[ \tau_{13}^{(1)} \right]^2 = 0 \tag{B.25}
\]
Note that if a crack has previously formed in the middle ply group, the 13 shear stress may not be zero. In this case the appropriate value is the previous value for each lamina group.
Finally, for the above example of the first crack occurring in the top ply group,

\[ \Delta W_{bs} = \frac{1}{4} H_1^2 \left[ F_{44}^{(1)} D^2 \lambda t \right] + \frac{1}{4} H_2^2 \left[ F_{44}^{(2)} J^2 \lambda t \right] + \frac{1}{4} H_3^2 \left[ F_{44}^{(3)} (D + J)^2 \lambda t \right] \]

(B.31)
APPENDIX C
DATA TABLES

Table C.1. Modulus Measurements for Samples α6 and α7 ([0/902]s E-Glass Epoxy)

<table>
<thead>
<tr>
<th>Crack Density</th>
<th>Sample α6 % of Initial</th>
<th>Sample α7 % of Initial</th>
</tr>
</thead>
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<tr>
<td>1.5</td>
<td>100.</td>
<td></td>
</tr>
<tr>
<td>9.0</td>
<td></td>
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<td>10.3</td>
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<td>71.0</td>
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<td>59.4</td>
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<tr>
<td>16</td>
<td>57.0</td>
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Table C.2. Crack Density Measurements for Samples α6 and α7 ([0/90]s E-Glass Epoxy)

<table>
<thead>
<tr>
<th>σ</th>
<th>Sample α6 Crack Density</th>
<th>Sample α7 Crack Density</th>
</tr>
</thead>
<tbody>
<tr>
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<td>---</td>
</tr>
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<td>---</td>
<td>9.0</td>
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<tr>
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<td>24.4</td>
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<td>15.0</td>
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<tr>
<td>25.0</td>
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Table C.3. Standard Averages and Standard Deviations of Modulus Measurements for Samples 16 and 17 (Case 1 Samples)

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>Average ( % ) of ( E_{\text{initial}} )</th>
<th>Std. Dev. of ( % ) of ( E_{\text{initial}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.5</td>
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<td>0.4</td>
</tr>
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<td>9</td>
<td>93.4</td>
<td>0.6</td>
</tr>
<tr>
<td>13.6</td>
<td>87.5</td>
<td>3.5</td>
</tr>
<tr>
<td>18.2</td>
<td>81.6</td>
<td>4.0</td>
</tr>
<tr>
<td>22.7</td>
<td>78.0</td>
<td>7.8</td>
</tr>
<tr>
<td>27.3</td>
<td>73.5</td>
<td>2.1 *</td>
</tr>
<tr>
<td>31.8</td>
<td>72.0</td>
<td>0 *</td>
</tr>
</tbody>
</table>

* Data also include sample 14.

Table C.4. Standard Averages and Standard Deviations of Crack Density Measurements for Samples 16 and 17 (Case 1 Samples)

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>Ave. ( \beta_{\text{out}} )</th>
<th>Std Dev. ( \beta_{\text{out}} )</th>
<th>Ave. ( \beta_{\text{in}} )</th>
<th>Std Dev. ( \beta_{\text{in}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.5</td>
<td>0.</td>
<td>0.</td>
<td>2.8</td>
<td>.64</td>
</tr>
<tr>
<td>9</td>
<td>1.0</td>
<td>0.</td>
<td>7.2</td>
<td>.21</td>
</tr>
<tr>
<td>13.6</td>
<td>3.0</td>
<td>0.</td>
<td>11.4</td>
<td>.49</td>
</tr>
<tr>
<td>18.2</td>
<td>5.8</td>
<td>4.0</td>
<td>16.0</td>
<td>1.4</td>
</tr>
<tr>
<td>22.7</td>
<td>7.0</td>
<td>4.2</td>
<td>19.3</td>
<td>3.2</td>
</tr>
<tr>
<td>27.3</td>
<td>10.3</td>
<td>2.8</td>
<td>23.3</td>
<td>5.8 *</td>
</tr>
<tr>
<td>31.8</td>
<td>10.9</td>
<td>2.7</td>
<td>27.6</td>
<td>2.6 *</td>
</tr>
</tbody>
</table>

* Data also include sample 14.

Case 1 Samples are \[90/0/90]s under \( N_x \).
Table C.5. Standard Averages and Standard Deviations of Modulus Measurements for Samples 34, 35, and 36 (Case 3 Samples)

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>Average % of $E_{\text{initial}}$</th>
<th>Std. Dev. of % of $E_{\text{initial}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.4</td>
<td>92.2</td>
<td>2.3</td>
</tr>
<tr>
<td>8.6</td>
<td>81.0</td>
<td>1.3</td>
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<tr>
<td>10.9</td>
<td>73.2</td>
<td>3.4</td>
</tr>
<tr>
<td>12.4</td>
<td>71.3</td>
<td>4.3</td>
</tr>
<tr>
<td>13.6</td>
<td>71.0</td>
<td>4.3</td>
</tr>
<tr>
<td>15.8</td>
<td>67.0</td>
<td>5.7</td>
</tr>
<tr>
<td>18.3</td>
<td>60.2</td>
<td>4.4</td>
</tr>
<tr>
<td>19.3</td>
<td>58.2</td>
<td>4.6</td>
</tr>
</tbody>
</table>

Table C.6. Standard Averages and Standard Deviations of Crack Density Measurements for Samples 34, 35, and 36 (Case 3 Samples)

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>Ave. $\theta_{\text{out}}$</th>
<th>Std Dev. $\theta_{\text{out}}$</th>
<th>Ave. $\theta_{\text{in}}$</th>
<th>Std Dev. $\theta_{\text{in}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.4</td>
<td>9.65</td>
<td>1.9</td>
<td>0.8</td>
<td>.80</td>
</tr>
<tr>
<td>8.6</td>
<td>13.1</td>
<td>.78</td>
<td>2.2</td>
<td>.80</td>
</tr>
<tr>
<td>10.9</td>
<td>16.0</td>
<td>2.0</td>
<td>3.3</td>
<td>.60</td>
</tr>
<tr>
<td>12.4</td>
<td>17.4</td>
<td>1.1</td>
<td>4.6</td>
<td>.07</td>
</tr>
<tr>
<td>13.6</td>
<td>18.9</td>
<td>1.3</td>
<td>6.0</td>
<td>.07</td>
</tr>
<tr>
<td>15.8</td>
<td>18.9</td>
<td>1.6</td>
<td>10.2</td>
<td>1.2</td>
</tr>
<tr>
<td>18.3</td>
<td>20.2</td>
<td>2.0</td>
<td>12.0</td>
<td>.35</td>
</tr>
<tr>
<td>19.3</td>
<td>22.2</td>
<td>1.6</td>
<td>13.0</td>
<td>1.1</td>
</tr>
</tbody>
</table>

Case 3 Samples are [90/0/903]s under $N_x$. 
Table C.7. Standard Averages and Standard Deviations of Modulus Measurements for Samples 24, 25, and 26 (Case 2 Samples)

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>Average $%$ of $E_{\text{initial}}$</th>
<th>Std. Dev. of $%$ of $E_{\text{initial}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.6</td>
<td>98.9</td>
<td>1.9</td>
</tr>
<tr>
<td>11.1</td>
<td>98.3</td>
<td>2.2</td>
</tr>
<tr>
<td>16.7</td>
<td>92.2</td>
<td>0.7</td>
</tr>
<tr>
<td>22.2</td>
<td>92.0</td>
<td>3.8</td>
</tr>
<tr>
<td>27.8</td>
<td>87.7</td>
<td>4.6</td>
</tr>
<tr>
<td>33.4</td>
<td>89.1</td>
<td>3.3</td>
</tr>
<tr>
<td>38.9</td>
<td>89.4</td>
<td>3.0</td>
</tr>
<tr>
<td>44.5</td>
<td>89.1</td>
<td>3.3</td>
</tr>
<tr>
<td>50.3</td>
<td>86.2</td>
<td>1.8</td>
</tr>
</tbody>
</table>

Table C.8. Standard Averages and Standard Deviations of Crack Density Measurements for Samples 24, 25, and 26 (Case 2 Samples)

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>Ave $\beta$</th>
<th>Std Dev. $\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.6</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>11.1</td>
<td>9.7</td>
<td>3.2</td>
</tr>
<tr>
<td>16.7</td>
<td>16.3</td>
<td>2.9</td>
</tr>
<tr>
<td>22.2</td>
<td>21.0</td>
<td>1.7</td>
</tr>
<tr>
<td>27.8</td>
<td>22.7</td>
<td>2.3</td>
</tr>
<tr>
<td>33.4</td>
<td>25.2</td>
<td>1.2</td>
</tr>
<tr>
<td>38.9</td>
<td>26.7</td>
<td>2.3</td>
</tr>
<tr>
<td>44.5</td>
<td>29.2</td>
<td>2.9</td>
</tr>
<tr>
<td>50.3</td>
<td>31.0</td>
<td>1.4</td>
</tr>
</tbody>
</table>

Case 2 Samples are [90/0/90]s under $N_y$. 
### Table C.9. Standard Averages and Standard Deviations of Modulus Measurements for Samples 45, 46, and 47 (Case 4 Samples)

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>Average $%$ of $E_{\text{Initial}}$</th>
<th>Std. Dev. of $%$ of $E_{\text{Initial}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.2</td>
<td>99.7</td>
<td>0.6</td>
</tr>
<tr>
<td>22.5</td>
<td>97.8</td>
<td>3.3</td>
</tr>
<tr>
<td>33.7</td>
<td>95.6</td>
<td>2.3</td>
</tr>
<tr>
<td>44.9</td>
<td>94.7</td>
<td>2.4</td>
</tr>
<tr>
<td>56.2</td>
<td>95.0</td>
<td>2.7</td>
</tr>
<tr>
<td>67.4</td>
<td>95.2</td>
<td>3.0</td>
</tr>
<tr>
<td>73.0</td>
<td>96.8</td>
<td>1.6</td>
</tr>
</tbody>
</table>

### Table C.10. Standard Averages and Standard Deviations of Crack Density Measurements for Samples 45, 46, and 47 (Case 4 Samples)

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>Ave $\beta$</th>
<th>Std Dev. $\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.2</td>
<td>0.7</td>
<td>0.6</td>
</tr>
<tr>
<td>22.5</td>
<td>4.3</td>
<td>1.5</td>
</tr>
<tr>
<td>33.7</td>
<td>7.6</td>
<td>3.1</td>
</tr>
<tr>
<td>44.9</td>
<td>21.7</td>
<td>1.2</td>
</tr>
<tr>
<td>56.2</td>
<td>32.8</td>
<td>7.1</td>
</tr>
<tr>
<td>67.4</td>
<td>44.0</td>
<td>7.6</td>
</tr>
<tr>
<td>73.0</td>
<td>44.5</td>
<td>0.7</td>
</tr>
<tr>
<td>78.6</td>
<td>47.6</td>
<td>0.9</td>
</tr>
</tbody>
</table>

Case 4 Samples are $[90/0/903]$s under $N_y$. 
Jane Lynette Daugherty Runkle was born May 29, 1963 in Park Ridge, Illinois. She earned her Bachelor's Degree in Engineering from Purdue University on May 19, 1985. At the same time, she was commissioned an officer in the United States Air Force and was stationed at Los Angeles Air Force Base. While in the Air Force Space Division Office for Manned Spaceflight, she managed contractor efforts on the Space Assembly, Maintenance, and Servicing Study and on the Spacecraft Partitioning and Interface Standards Study. She then joined the Medium Launch Vehicle System Program Office, where she managed the development, qualification and acceptance of solid rocket motor and ordnance systems for the McDonnell Douglas Delta II launch vehicle. Jane earned her M.S. in Engineering Science from Louisiana State University in May of 1991.
DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Jane L. D. Runkle

Major Field: Engineering Science

Title of Dissertation: Analytical Model of the Progression of Cracking in Fiber-Reinforced Composites

Approved:

[Signatures]

Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

[Signatures]

Co-Chair

Date of Examination: October 1, 1993