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Noncommutative Tensor Triangular Geometry and Its Applications to Representation Theory

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NONCOMMUTATIVE TENSOR TRIANGULAR GEOMETRY AND ITS APPLICATIONS TO REPRESENTATION THEORY

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Abstract

One of the cornerstones of the representation theory of Hopf algebras and finite tensor categories is the theory of support varieties. Balmer introduced tensor triangular geometry for symmetric monoidal triangulated categories, which united various support variety theories coming from disparate areas such as homotopy theory, algebraic geometry, and representation theory. In this thesis a noncommutative version will be introduced and developed. We show that this noncommutative analogue of Balmer's theory can be determined in many concrete situations via the theory of abstract support data, and can be used to classify thick tensor ideals. We prove an analogue of prime ideal contraction, connecting the Balmer spectrum of a stable category of a finite tensor category with the stable category of its Drinfeld center. We classify the Balmer spectra for various examples arising in representation theory, such as Drinfeld doubles of cosemisimple Hopf algebras, the smash coproducts studied by Benson and Witherspoon, and the small quantum Borels. Lastly, we leverage the theory to prove the tensor product property for cohomological support varieties in a family of small quantum Borels, a conjecture of Negron and Pevtsova.

Chapter 1. Introduction

1.1. Support varieties and tensor triangular geometry

Monoidal categories were originally defined by Saunders Mac Lane [70], and triangulated categories by Verdier in his thesis [103]. Monoidal triangulated categories have played important roles ever since in fields which range from algebraic topology, homotopy theory, K-theory, algebraic geometry, and representation theory. The study of such categories via the geometry of support varieties, specifically in the context of modular representation theory for finite groups, goes back to Quillen [93], who described the spectrum of the cohomology ring of a finite group via a stratification theorem. The notion of complexity for modules, a homological invariant based on the growth of a minimal projective resolution, was defined by Alperin and Evens in [2]. Support varieties were defined soon after by Carlson [27], which gave a geometric interpretation of the complexity. A tensor product theorem which gave a compatibility between the tensor product of modules and their support varieties was proven by Avrunin-Scott [5]. See also [1] for a survey of the developement of the theory of support varieties.

One of the foundational results in the area was the proof of the finite generation of the cohomology rings for finite groups by Golod, Venkov, and Evens [47, 102, 37]. Later results on finite generation of cohomology rings pushed the theory of support varieties past finite groups and into the world of Hopf algebras more generally. These include the proofs of the finite generation of cohomology rings of restricted enveloping algebras [40, 3], small quantum groups at an ℓ root of unity [46] (under conditions on ℓ, which were improved upon in [12]), and for all finite group schemes (e.g. finite-dimensional cocommutative Hopf algebras) by Friedlander-Suslin [42]. Very recently, the finite generation of cohomology rings has been proven for all finite-dimensional pointed Hopf algebras with an abelian group of grouplike elements [4]. Finite generation has been proved in many additional specific cases [104, 32, 51, 33, 71, 89, 34, 88, 97, 39, 85]. It was famously conjectured by Etingof-Ostrik that the finite generation condition holds in broad generality– for all finite tensor categories [36, Conjecture 2.18].

One important question for which support varieties have provided one tool is the problem of thick subcategory and thick ideal classifications for monoidal triangulated categories. These problems were initiated in the world of commutative algebra by Hopkins [53] and Neeman [79]. Thick ideal classification problems were brought into the world of modular representation theory by Benson-Carlson-Rickard, who classified the thick ideals for certain stable module categories of finite groups in positive characterstic [15]. This classification involved the use of idempotent functors (constructed in [95]), which were later generalized by localization and colocalization functors [16]. In the process of this classification, it was necessary to construct support variety theories for infinite-dimensional modules [14].

Two further thick ideal classification results which motivated the foundation of tensor triangular geometry were (1) Thomason's classifications of thick tensor ideals for the perfect derived category $D^{\mathrm{perf}}(X)$ of a topologically Noetherian scheme X, considered as a monoidal category under the derived tensor product $- \otimes^L_{\mathcal{O}_X} -$ [99]; and (2) Friedlander-Pevtsova's classifications of thick tensor ideals for the stable module category $\mathsf{stmod}(G)$ of a finite group scheme G over a field $\Bbbk$ of positive characteristic, considered as a monoidal category under the usual tensor product $- \otimes_{\Bbbk} -$ [41]. Both (1) and (2) use geometry to classify the thick ideals of the category in question; in (1), the geometry used is the original scheme X, whereas in (2), the geometry used is $\operatorname{Proj} \mathrm{H}^{\bullet}(G, \Bbbk)$, the Proj of the cohomology ring of G.

The classifications (1) and (2) were united in the work of Paul Balmer [7, 8, 9], who introduced a new geometric space for an arbitrary braided tensor triangulated category $\mathbf{K}$– *the Balmer spectrum* $\operatorname{Spc}(\mathbf{K})$. The braided condition requires there exist natural isomorphisms $A \otimes B \cong B \otimes A$ for all objects A and B of $\mathbf{K}$, and so Balmer's setting is a fundamentally commutative one. Balmer spectra for braided tensor triangulated categories satisfy a universal property [8, Theorem 3.2], and in the special cases of (1) and (2) above, the Balmer spectrum recovers X and $\operatorname{Proj} \mathrm{H}^{\bullet}(G, \Bbbk)$, respectively [8, Theorem 6.3].

Roughly speaking, the Balmer spectrum is a categorification of the construction of prime spectra for commutative rings– in other words, he lifts the notion of a prime ideal from the world of commutative algebra to the categorical setting. Recall that for a commutative ring R, the prime spectrum of R is defined as the collection of prime ideals of R, that is, ideals P such that $ab \in P$ implies either a is in P or b is in P for all elements $a, b \in R$. Balmer defines a thick ideal (that is, a subset of objects in $\mathbf{K}$ closed under summands, cones, and tensoring with arbitrary objects) $\mathbf{P}$ of $\mathbf{K}$ to be *prime* if $A \otimes B \in \mathbf{P}$ implies A or B is in $\mathbf{P}$, over all objects A and B in $\mathbf{K}$.

1.2. Towards a noncommutative theory

Since Balmer's original formulation, many authors have applied Balmer's tensor triangular geometry in various settings. Recently, the results have been achieved in homotopy theory and algebraic topology [11], commutative algebra [72], and representation theory for Lie superalgebras and quantum groups at roots of unity [21, 22]. These represent only a small sample of the applications of Balmer's theory– for a more complete survey, see [10]. In these applications, the monoidal triangulated categories under consideration were braided. On the other hand, many of the monoidal categories which arise in categorification (see [68, 73, 59, 60]), as well as the representation theory of non-quasitriangular Hopf algebras, are not braided. A noncommutative, ideal-theoretic approach to tensor triangular geometry was originally suggested in [26]. However, in representation theory, many results have focused on the fact that support varieties in non-braided tensor categories fail to satisfy the axioms of support data laid out by Balmer, see e.g. [19, 92].

In this thesis, we introduce the theory of noncommutative Balmer spectra. In particular, we develop noncommutative versions of the classification of Balmer spectra, in terms of abstract support data, which was originally achieved in the commutative situation by [21, 22]. If $\mathbf{K}$ is a monoidal triangulated category (which we will abbreviate by MΔC), then $\mathbf{P}$ is a *prime ideal* if it satisfies the property that $\mathbf{I} \otimes \mathbf{J} \subseteq \mathbf{P}$ for two thick ideals $\mathbf{I}$ and $\mathbf{J}$ implies that either $\mathbf{I} \subseteq \mathbf{P}$ or $\mathbf{J} \subseteq \mathbf{P}$. The Balmer spectrum $\mathrm{Spc}(\mathbf{K})$ of $\mathbf{K}$ is the collection of prime ideals of $\mathbf{K}$ as a topological space under the Zariski topology, in which closed sets are defined as $V(\mathcal{S}) = \{\mathbf{P} \in \mathrm{Spc}(\mathbf{K}) : \mathcal{S} \cap \mathbf{P} = \varnothing\}$ for any collection of objects $\mathcal{S}$ of $\mathbf{K}$.

We define several notions of *support data*, using insight from noncommutative ring theory, with support varieties for Hopf algebras as a model. In particular, a *weak support datum* is a map σ which sends objects of $\mathbf{K}$ to subsets of some topological space X, such that $\sigma(0) = \varnothing$, $\sigma(1) = X$, σ is compatible with direct sums, shifts, and triangles, and satisfies a tensor compatibility:

$$\Phi_\sigma(\mathbf{I} \otimes \mathbf{J}) = \Phi_\sigma(\mathbf{I}) \cap \Phi_\sigma(\mathbf{J})$$

over all thick ideals $\mathbf{I}$ and $\mathbf{J}$, where $\Phi_\sigma(\mathcal{S}) := \bigcup_{A \in \mathcal{S}} \sigma(A)$ for any subset $\mathcal{S}$ of $\mathbf{K}$.

The map V defined by the Balmer spectrum is an example of a weak support datum. The cohomological theories of support varieties for finite-dimensional Hopf algebras are also a model for support data, although in general they may not satisfy the tensor product property. For a finite-dimensional Hopf algebra H satisfying (fg) with cohomology ring R, the cohomological support is defined by the map

$$\sigma(M) = \{P \in \operatorname{Proj} R : \operatorname{Ext}^\bullet_H(M, M)_P \neq 0\}$$

for each finite-dimensional H-module M. We show that the Balmer spectrum $\mathrm{Spc}(\mathbf{K})$ of an arbitrary MΔC $\mathbf{K}$ is the universal final weak support datum among support data which satisfy the condition that $\sigma(A)$ is closed for each object A of $\mathbf{K}$. In other words, given a weak support datum $\sigma : \mathbf{K} \to \{\text{closed subsets of } X\}$ there is a unique continuous map $f_\sigma : X \to \mathrm{Spc}(\mathbf{K})$ compatible with the support maps.

Using the theory of support for infinite-dimensional modules developed by Benson-Iyengar-Krause [16], we are able to prove a classification theorem for Balmer spectra of generic MΔCs. In order to state the theorem, one needs to consider two additional properties that a weak support datum σ can satisfy:

(i) Faithfulness Property: $\Phi_\sigma(\langle A \rangle) := \bigcup_{B \in \langle A \rangle} \sigma(B) = \emptyset$ if and only if $A = 0$;

(ii) Realization Property: for each closed set W of X, there exists a compact object C with $\Phi_\sigma(\langle C \rangle) = W$.

Then we are able to prove the following:

Theorem 1.2.1. *Let* $\mathbf{K}$ *be an* $M\Delta C$ *and* $\sigma : \mathbf{K} \to \{$*subsets of* $X\}$ *be a weak support datum satisfying the Faithfulness and Realization Properties and such that* $\sigma(C)$ *is a closed set for all compact objects* C *of* $\mathbf{K}$*. Then there is a homeomorphism* $f : X \to \mathrm{Spc}(\mathbf{K}^c)$*, where* $\mathbf{K}^c$ *is the subcategory of* $\mathbf{K}$ *consisting of compact objects, and there is a bijection between thick ideals of* $\mathbf{K}^c$ *and specialization-closed (i.e. arbitrary unions of closed sets) subsets of* X*.*

This theorem contrasts with classical noncommutative ring theory, as it gives a recipe for direct computation of the Balmer spectrum, whereas for noncommutative rings the space of prime ideals is rarely classifiable; for instance, it is extremely difficult to describe the Zariski topology on the prime spectra of universal enveloping algebras and quantum groups, which were studied by Dixmier [31] and Joseph [58] respectively.

One important construction in the theory of tensor categories is the Drinfeld center. If $\mathbf{C}$ is a finite tensor category, then its Drinfeld center $\mathsf{Z}(\mathbf{C})$ is a braided finite tensor category, and is equipped with a forgetful functor $F : \mathsf{Z}(\mathbf{C}) \to \mathbf{C}$. We consider the relationship between the Balmer spectrum of the stable category of a finite tensor category $\mathbf{C}$ and the Balmer spectrum of the stable category of the Drinfeld center of $\mathbf{C}$. We verify that the forgetful functor F extends to a monoidal triangulated functor $\overline{F} : \mathrm{st}(\mathsf{Z}(\mathbf{C})) \to \mathrm{st}(\mathbf{C})$. We then prove that this functor induces a support datum $W : \mathrm{st}(\mathsf{Z}(\mathbf{C}))$ to subsets of $\mathrm{Spc}(\mathrm{st}(\mathbf{C}))$ and a continuous map between their spectra, $f : \mathrm{Spc}\ \mathrm{st}(\mathbf{C}) \to \mathrm{Spc}\ \mathrm{st}(\mathsf{Z}(\mathbf{C}))$. This is a categorical analogue of prime ideal contraction, and reflects the classical noncommutative ring theory situation: in general, the prime spectrum of noncommutative rings is not functorial; however, there is an induced map between the prime ideals of a ring and its center. In the case that $\mathbf{C}$ is the category of representations of a finite-dimensional Hopf algebra, we are able to describe the image of f, as well as the collection of ideals of

$\mathrm{st}(\mathsf{Z}(\mathbf{C}))$ which can be recovered from their support W. These are done in reference to the kernel of $\overline{F}$, that is, the collection of objects A of $\mathrm{st}(\mathsf{Z}(\mathbf{C}))$ such that $\overline{F}(A) \cong 0$.

Theorem 1.2.2. *Suppose that the kernel of $\overline{F}$ is compactly generated as a localizing subcategory by its finite-dimensional objects. Then*

(a) The image of f is precisely the set of prime ideals containing all finite-dimensional objects in the kernel of $\overline{F}$. This is the complement of a specialization-closed set.

(b) The thick ideals $\mathbf{I}$ which can be recovered from their support $W(\mathbf{I})$ are the ideals which contain the finite-dimensional objects in the kernel of $\overline{F}$.

After developing the general abstract theory for noncommutative tensor triangular geometry, we turn to applications, focusing on the cohomological support varieties for the representation theory of finite-dimensional Hopf algebras. For multiple different families of Hopf algebras H, we are able to use the cohomological support datum, or some modification of it, to determine the Balmer spectrum of the monoidal triangulated category $\mathsf{stmod}(H)$.

Theorem 1.2.3. *(a) Let $\mathfrak{g}$ be a complex simple Lie algebra, ζ a primitive ℓth roof of unity in $\mathbb{C}$ with ℓ greater than the Coxeter number of the root system of $\mathfrak{g}$ and coprime to the size of the weight lattice quotient by the root lattice. Let $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$, and $u_\zeta(\mathfrak{b})$ the small quantum group at ζ. Let $\mathbf{K} = \mathsf{StMod}(u_\zeta(\mathfrak{b}))$, and hence $\mathbf{K}^c = \mathsf{stmod}(u_\zeta(\mathfrak{b}))$, the stable module categories of all modules and finite-dimensional modules, respectively. Then there is a homeomorphism $\mathrm{Spc}(\mathbf{K}^c) \cong \mathrm{Proj}(\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}))$, and the thick ideals of $\mathbf{K}^c$ are in bijection with specialization-closed subsets of $\mathrm{Proj}(\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}))$.*

(b) Let G and H be finite groups with H acting on G by group automorphisms, and

$\Bbbk$ be a field of positive characteristic dividing the order of G. Let A be the Hopf algebra dual to the smash product $\Bbbk[G]\#\Bbbk H$. Let $\mathbf{K} = \mathsf{StMod}(A)$, and hence $\mathbf{K}^c = \mathsf{stmod}(A)$, the stable module categories of A. Then there is a homeomorphism $\mathrm{Spc}(\mathbf{K}^c) \cong H\text{-}\mathrm{Proj}(\mathrm{H}^\bullet(A,\Bbbk))$, and there is a bijection between thick ideals of $\mathbf{K}^c$ and specialization-closed subsets of $H\text{-}\mathrm{Proj}(\mathrm{H}^\bullet(A,\Bbbk))$.

We apply the theory developed for Drinfeld centers to several different Hopf algebras, giving examples where f is surjective and when it fails to be surjective. In particular, this allows us to classify the Balmer spectrum and thick ideals for all finite-dimensional cosemisimple Hopf algebras. We also prove that f is injective for Benson-Witherspoon smash coproduct algebras.

Theorem 1.2.4. *(a) Let H be a finite-dimensional cosemisimple Hopf algebra. Then the map from the Balmer spectrum of the stable category of H to the Balmer spectrum of the stable category of its Drinfeld double $D(H)$ is a homeomorphism, and induces a bijection between the thick ideals of the two categories.*

(b) Let G and H be finite groups with H acting on G by group automorphisms, and $\Bbbk$ be a field of positive characteristic dividing the order of G. Let A be the Hopf algebra dual to the smash product $\Bbbk[G]\#\Bbbk H$. Then prime ideals of $\mathsf{stmod}(A)$ are determined by their image under f in $\mathrm{Spc}(\mathsf{stmod}(D(A)))$, in other words, f is injective.

Lastly, using the technology of Balmer spectra, we answer a conjecture by Negron-Pevtsova. A support datum σ is said to satisfy the tensor product property if $\sigma(A \otimes B) = \sigma(A) \cap \sigma(B)$. Negron-Pevtsova construct in [85] a version of small quantum groups of Borel subalgebras $u_\zeta(\mathfrak{b})$, which are defined according to a choice of lattice Γ in between the root lattice and weight lattice, and they prove that the cohomological support datum on $u_\zeta(\mathfrak{b})$

has the tensor product property in Type A. They conjecture that it holds in arbitrary type, which we prove under some restrictions on ℓ.

Theorem 1.2.5. *The cohomological support datum of the Negron-Pevtsova small quantum group* $u_\zeta(\mathfrak{b})$ *has the tensor product property, in arbitrary type.*

Chapter 2. Background

2.1. Monoidal and finite tensor categories

We will begin by recalling the structure of a monoidal category.

Definition 2.1.1. A *monoidal category* is a collection $(\mathbf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ where $\mathbf{C}$ is a category, $- \otimes -$ is a bifunctor $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$, $\mathbb{1}$ is an object of $\mathbf{C}$ called the *unit*, and α, λ, and ρ are natural isomorphisms with components

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C),$$

$$\rho_A : A \otimes \mathbb{1} \to A,$$

$$\lambda_A : \mathbb{1} \otimes A \to A,$$

satisfying the following axioms:

(a) The diagram

$$\begin{tikzcd}
& ((A \otimes B) \otimes C) \otimes D \arrow[dl, "\alpha_{A \otimes B, C, D}"'] \arrow[dr, "\alpha_{A,B,C} \otimes \mathrm{id}_D"] & \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C \otimes D}"] & & (A \otimes (B \otimes C)) \otimes D \arrow[d, "\alpha_{A, B \otimes C, D}"] \\
A \otimes (B \otimes (C \otimes D)) & & A \otimes ((B \otimes C) \otimes D) \arrow[ll, "\mathrm{id}_A \otimes \alpha_{B,C,D}"]
\end{tikzcd}$$

commutes for all objects A, B, C, and D in $\mathbf{C}$.

(b) The diagram

$$\begin{tikzcd}
(A \otimes \mathbb{1}) \otimes B \arrow[rr, "\alpha_{A,\mathbb{1},B}"] \arrow[dr, "\rho_A \otimes \mathrm{id}_B"] & & A \otimes (\mathbb{1} \otimes B) \arrow[dl, "\mathrm{id}_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

commutes for all objects A and B of $\mathbf{C}$.

If $\mathbf{C}$ and $\mathbf{D}$ are monoidal categories, then a functor $F : \mathbf{C} \to \mathbf{D}$ together with a natural isomorphism η with components $\eta_{A,B} : F(A) \otimes_{\mathbf{D}} F(B) \to F(A \otimes_{\mathbf{C}} B)$ for A and B in $\mathbf{C}$ is called a *monoidal functor* if $F(\mathbb{1}_{\mathbf{C}}) \cong \mathbb{1}_{\mathbf{D}}$, and the diagram

$$\begin{tikzcd}
(F(A) \otimes_{\mathbf{D}} F(B)) \otimes_{\mathbf{D}} F(C) \arrow[r, "\alpha^{\mathbf{D}}_{F(A),F(B),F(C)}"] \arrow[d, "\eta_{A,B} \otimes_{\mathbf{D}} \mathrm{id}_{F(C)}"'] & F(A) \otimes_{\mathbf{D}} (F(B) \otimes_{\mathbf{D}} F(C)) \arrow[d, "\mathrm{id}_{F(A)} \otimes_{\mathbf{D}} \eta_{B,C}"] \\
F(A \otimes_{\mathbf{C}} B) \otimes_{\mathbf{D}} F(C) \arrow[d, "\eta_{A \otimes_{\mathbf{C}} B, C}"'] & F(A) \otimes_{\mathbf{D}} F(B \otimes_{\mathbf{C}} C) \arrow[d, "\eta_{A, B \otimes_{\mathbf{C}} D}"] \\
F((A \otimes_{\mathbf{C}} B) \otimes_{\mathbf{C}} C) \arrow[r, "F(\alpha^{\mathbf{C}}_{A,B,C})"'] & F(A \otimes_{\mathbf{C}} (B \otimes_{\mathbf{C}} C)
\end{tikzcd}$$

commutes for all A, B, and $C \in \mathbf{C}$.

We recall one classical theorem of monoidal categories, proved by Mac Lane in [70], known as the Mac Lane Strictness Theorem. A monoidal category is called *strict* if for all objects A, B, and C, we have equalities $(A \otimes B) \otimes C = A \otimes (B \otimes C)$, $A \otimes \mathbb{1} = A = \mathbb{1} \otimes A$, and the associativity and unit natural isomorphisms are each the identity.

Theorem 2.1.2. *Every monoidal category is monoidally equivalent to a strict monoidal category.*

By Theorem 2.1.2, we typically avoid the complicated bureaucracy of the associativity and unit isomorphisms by omitting them entirely.

Definition 2.1.3. Let $\mathbf{C}$ be a monoidal category, and $A \in \mathbf{C}$. Then a *left dual* of A is an object A^* of $\mathbf{C}$ together with morphisms $\mathrm{ev}_A : A^* \otimes A \to \mathbb{1}$ and $\mathrm{coev}_A : \mathbb{1} \to A \otimes A^*$ such that the compositions

$$A \xrightarrow{\mathrm{coev}_A \otimes \mathrm{id}_A} A \otimes A^* \otimes A \xrightarrow{\mathrm{id}_A \otimes \mathrm{ev}_A} A$$

and

$$A^* \xrightarrow{\mathrm{id}_{A^*} \otimes \mathrm{coev}_A} A^* \otimes A \otimes A^* \xrightarrow{\mathrm{ev}_A \otimes \mathrm{id}_{A^*}} A^*$$

are the identity morphisms. A *right dual* of A is an object ${}^*A$ together with morphisms

$\mathrm{ev}'_A : A \otimes {}^*A \to \mathbb{1}$ and $\mathrm{coev}'_A : \mathbb{1} \to {}^*A \otimes A$ such that the compositions

$$A \xrightarrow{\mathrm{id}_A \otimes \mathrm{coev}'_A} A \otimes {}^*A \otimes A \xrightarrow{\mathrm{ev}'_A \otimes \mathrm{id}_A} A$$

and

$${}^*A \xrightarrow{\mathrm{coev}'_A \otimes \mathrm{id}_{{}^*A}} {}^*A \otimes A \otimes {}^*A \xrightarrow{\mathrm{id}_{{}^*A} \otimes \mathrm{ev}'_A} {}^*A$$

are the identity morphisms. If every object of $\mathbf{C}$ has a left dual then $\mathbf{C}$ is called *left rigid*, and it is called *right rigid* if every object has a right dual. If $\mathbf{C}$ is both left and right rigid, then it is called *rigid*.

Right and left dual objects, if they exist, are unique up to unique isomorphism [35, Proposition 2.10.5]. By [35, Proposition 2.10.8], we have adjunctions arising from dual objects as follows.

Theorem 2.1.4. *If an object B of a monoidal category $\mathbf{C}$ has a left dual B^* then there are natural adjunction isomorphisms*

$$\mathrm{Hom}_{\mathbf{C}}(A \otimes B, C) \cong \mathrm{Hom}_{\mathbf{C}}(A, C \otimes B^*)$$

and

$$\mathrm{Hom}_{\mathbf{C}}(B^* \otimes A, C) \cong \mathrm{Hom}_{\mathbf{C}}(A, B \otimes C),$$

and if B has a right dual ${}^*B$ then there are natural adjunction isomorphisms

$$\mathrm{Hom}_{\mathbf{C}}(A \otimes {}^*B, C) \cong \mathrm{Hom}_{\mathbf{C}}(A, C \otimes B)$$

and

$$\mathrm{Hom}_{\mathbf{C}}(B \otimes A, C) \cong \mathrm{Hom}_{\mathbf{C}}(A, {}^*B \otimes C).$$

While the focus of this thesis will be on monoidal triangulated categories, defined below, many examples arise from finite tensor categories. We follow the definition as given by Etingof and Ostrik in [36]. Here and below, let $\mathbb{k}$ be an algebraically closed field.

Definition 2.1.5. A *finite tensor category* consists of a monoidal category $\mathbf{C}$ such that:

(a) $\mathbf{C}$ is abelian and $\mathbb{k}$-linear;

(b) The monoidal product $-\otimes-$ is bilinear on spaces of morphisms;

(c) Every object of $\mathbf{C}$ has finite length;

(d) $\mathrm{Hom}_{\mathbf{C}}(\mathbb{1},\mathbb{1})\cong\mathbb{k}$;

(e) For any pair of objects A and B, $\mathrm{Hom}_{\mathbf{C}}(A,B)$ is finite-dimensional over $\mathbb{k}$;

(f) $\mathbf{C}$ has enough projectives;

(g) There are finitely many isomorphism classes of simple objects of $\mathbf{C}$;

(h) $\mathbf{C}$ is rigid.

We recall a few elementary facts about finite tensor categories, for reference.

Proposition 2.1.6. *Let $\mathbf{C}$ be a rigid $\mathbb{k}$-linear abelian monoidal category.*

(a) The monoidal product $-\otimes-$ is biexact.

(b) If P is a projective object of $\mathbf{C}$, then both $A\otimes P$ and $P\otimes A$ are projective, for any object A of $\mathbf{C}$.

(c) Every injective object of $\mathbf{C}$ is projective, and vice versa.

Proof. (a) follows from the fact that $A\otimes-$ and $-\otimes A$ have left and right adjoints, via left and right duals. (b) also follows from the rigidity condition, since

$$\mathrm{Hom}_{\mathbf{C}}(P\otimes A,-)\cong\mathrm{Hom}_{\mathbf{C}}(P,-\otimes A^*);$$

the functor on the right is exact by (a) and the projectivity of P, and so the functor on the left is exact as well. For (c), note that if P is projective, then P^* is injective, since

$$\mathrm{Hom}_{\mathbf{C}}(-, P^*) \cong \mathrm{Hom}_{\mathbf{C}}(- \otimes P, \mathbb{1}),$$

and the right hand side is an exact functor since $- \otimes P$ sends a short exact sequence to a split short exact sequence, by (b). But alternatively, if P is projective, then so is P^*, via the adjunction

$$\mathrm{Hom}_{\mathbf{C}}(P^*, -) \cong \mathrm{Hom}_{\mathbf{C}}(\mathbb{1}, P \otimes -).$$

Again by (b), the right hand side is an exact functor, and so the functor on the left is exact as well. We have now seen that P projective implies that P^* is both injective and projective, and symmetric arguments also work for the right dual ${}^*P$. In any rigid monoidal category, ${}^*(P^*) \cong P$, and so P is projective if and only if it is injective. □

Example 2.1.7. Many of the examples of finite tensor categories which will be studied in this thesis arise as module categories of finite-dimensional Hopf algebras. If H is a Hopf algebra over $\Bbbk$, then its category of finite-dimensional modules $\mathsf{mod}(H)$ is an abelian monoidal category with tensor product $- \otimes_{\Bbbk} -$, using the comultiplication of H. The antipode of H gives a module structure on the vector space dual $\mathrm{Hom}_{\Bbbk}(A, \Bbbk)$ for any H-module A, which is a left dual A^* for A. If the antipode is invertible, then via the inverse antipode, there is a second choice of module structure on $\mathrm{Hom}_{\Bbbk}(A, \Bbbk)$ for any object A, giving a right dual ${}^*A$. Explicitly, if S is an antipode for H, then the module structure on A^* is given by

$$(h.f)(a) = f(S(h).a)$$

for $f \in A^*$, $h \in H$, and $a \in A$, and if S^{-1} is the inverse of S, then the module structure for

${}^*A$ is given by

$$(h.f)(a) = f(S^{-1}(h).a)$$

for f, h, and a as before. For both the left and right dual, the evaluation and coevaluation morphisms are the usual evaluation and coevaluation from the category of vector spaces. If H is finite-dimensional, then its antipode S is automatically invertible, by a classical theorem of Larson and Sweedler [67]. Hence, for any finite-dimensional Hopf algebra H, its category of modules $\mathsf{mod}(H)$ is a finite tensor category.

For a comprehensive introduction to monoidal categories and tensor categories, one can see [6, 35]. For background on Hopf algebras, see [75, 62].

2.2. Drinfeld centers

Let $\mathbf{C}$ be a strict monoidal category. Then the *Drinfeld center* or *center* of $\mathbf{C}$, which we will denote by $\mathsf{Z}(\mathbf{C})$, is defined as a braided monoidal category where:

(a) Objects are pairs (A, γ) where A is an object of $\mathbf{T}$ and γ is a natural isomorphism $\gamma_B : B \otimes A \xrightarrow{\cong} A \otimes B$ for all $B \in \mathbf{C}$, satisfying the diagram

$$\begin{tikzcd} B \otimes C \otimes A \arrow[r, "\mathrm{id}_B \otimes \gamma_C"] \arrow[rr, bend left, "\gamma_{B \otimes C}"] & B \otimes A \otimes C \arrow[r, "\gamma_B \otimes \mathrm{id}_C"] & A \otimes B \otimes C \end{tikzcd}$$

for all B and C. Such a natural isomorphism γ is called a *half-braiding* of A.

(b) Morphisms $(A, \gamma) \to (A', \gamma')$ are morphisms $f : A \to A'$ such that for all B, the diagram

$$\begin{tikzcd} B \otimes A \arrow[r, "\mathrm{id}_B \otimes f"] \arrow[d, "\gamma_B"] & B \otimes A' \arrow[d, "\gamma'_B"] \\ A \otimes B \arrow[r, "f \otimes \mathrm{id}_B"] & A' \otimes B \end{tikzcd}$$

commutes.

(c) The monoidal product $(A,\gamma)\otimes(A,\gamma')$ is defined as $(A\otimes A',\tilde{\gamma})$ where $\tilde{\gamma}$ is defined as

$$\begin{tikzcd} B\otimes A\otimes A' \arrow[r, "\gamma_B\otimes\mathrm{id}_{A'}"] \arrow[d, "\tilde{\gamma}_B"] & A\otimes B\otimes A' \arrow[dl, "\mathrm{id}_A\otimes\gamma'_B"] \\ A\otimes A'\otimes B & \end{tikzcd}$$

(d) The braiding $c_{(A,\gamma),(A',\gamma')} : (A,\gamma)\otimes(A',\gamma') \xrightarrow{\cong} (A',\gamma')\otimes(A,\gamma)$ is defined as γ'_A. The map γ'_A being a valid map in $\mathsf{Z}(\mathbf{C})$ amounts to checking the commutativity of the diagram

$$\begin{tikzcd} B\otimes A\otimes A' \arrow[r, "\mathrm{id}_B\otimes\gamma'_A"] \arrow[d, "\gamma_B\otimes\mathrm{id}_{A'}"] & B\otimes A'\otimes A \arrow[d, "\gamma'_B\otimes\mathrm{id}_A"] \\ A\otimes B\otimes A' \arrow[d, "\mathrm{id}_A\otimes\gamma'_B"] & A'\otimes B\otimes A \arrow[d, "\mathrm{id}_{A'}\otimes\gamma_B"] \\ A\otimes A'\otimes B \arrow[r, "\gamma'_A\otimes\mathrm{id}_B"] & A'\otimes A\otimes B \end{tikzcd}$$

This diagram commutes by the naturality of γ', since it can be rewritten, using the defining diagram for γ', as

$$\begin{tikzcd} B\otimes A\otimes A' \arrow[r, "\gamma'_{B\otimes A}"] \arrow[d, "\gamma_B\otimes\mathrm{id}_{A'}"] & A'\otimes B\otimes A \arrow[d, "\mathrm{id}_{A'}\otimes\gamma_B"] \\ A\otimes B\otimes A' \arrow[r, "\gamma'_{A\otimes B}"] & A'\otimes A\otimes B \end{tikzcd}$$

We will denote by $F : \mathsf{Z}(\mathbf{C}) \to \mathbf{C}$ the forgetful functor sending $(A,\gamma)\mapsto A$.

If H is a Hopf algebra and $\mathbf{C}$ is the category of H-modules, it is well-known that the Drinfeld center $\mathsf{Z}(\mathbf{C})$ of $\mathbf{C}$ is equivalent to the category of modules of $D(H)$ the Drinfeld double of H. For the details of Drinfeld doubles, see [75, Section 10.3], [62, Section IX.4], or [35, Section 7.14]. The Drinfeld double $D(H)$ is isomorphic as a vector space to $(H^{\mathrm{op}})^*\otimes H$, and contains both H and $(H^{\mathrm{op}})^*$ as Hopf subalgebras. To be explicit, if H is a Hopf algebra with multiplication μ, unit η, comultiplication Δ, counit ϵ, and antipode S, then $(H^{\mathrm{op}})^*$ is the Hopf algebra with multiplication Δ^*, unit ϵ^*, comultiplication $(\mu^{\mathrm{op}})^*$, counit η^*, and antipode $(S^{-1})^*$.

The following result of Etingof-Ostrik will be important in extending the forgetful functor $\mathsf{Z}(\mathbf{C}) \to \mathbf{C}$ to the stable categories [36].

Proposition 2.2.1. *If* $\mathbf{C}$ *is a finite tensor category, then its Drinfeld center* $\mathsf{Z}(\mathbf{C})$ *is a finite tensor category, and the forgetful functor* F *is exact and sends projective objects to projective objects.*

The fact that F preserves projectivity is a generalization of the classical Nichols-Zoeller theorem for Hopf algebras, which states that a finite-dimensional Hopf algebra is free as a module over any Hopf subalgebra [87].

2.3. Triangulated and monoidal triangulated categories

Having now completed our initial discussion of monoidal and tensor categories, we turn to the structure of triangulated categories, which constitute the second critical ingredient for the monoidal triangulated categories that we will study.

Let $\mathbf{T}$ be a category together with an autoequivalence $\Sigma : \mathbf{T} \to \mathbf{T}$. Then a *triangle* in $\mathbf{T}$ is a sequence of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A.$$

A *morphism of triangles* from the triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

to the triangle

$$A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A'$$

is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"] & B \arrow[r, "g"] \arrow[d, "j"] & C \arrow[r, "h"] \arrow[d, "k"] & \Sigma A \arrow[d, "\Sigma i"] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & \Sigma A'
\end{tikzcd}$$

It is an *isomorphism of triangles* if each of i, j, and k are isomorphisms.

Definition 2.3.1. A *triangulated category* is an additive category $\mathbf{T}$ together with an autoequivalence $\Sigma : \mathbf{T} \to \mathbf{T}$ and a collection of triangles in $\mathbf{T}$ called *distinguished triangles* such that the following axioms hold:

TR1. (i) For any object A, the triangle

$$A \xrightarrow{\mathrm{id}_A} A \to 0 \to \Sigma A$$

is a distinguished triangle;

(ii) For any morphism $f : A \to B$, there is some distinguished triangle of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A.$$

(iii) The set of distinguished triangles is closed under isomorphism.

TR2. The triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is a distinguished triangle if and only if

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

is a distinguished triangle as well.

TR3. If there exists a diagram of the form

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"] & B \arrow[r, "g"] \arrow[d, "j"] & C \arrow[r, "h"] & \Sigma A \arrow[d, "\Sigma i"] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & \Sigma A'
\end{tikzcd}$$

such that both rows are distinguished triangles and the leftmost square commutes, then there exists a morphism $k : C \to C'$ giving a morphism of triangles:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"] & B \arrow[r, "g"] \arrow[d, "j"] & C \arrow[r, "h"] \arrow[d, "k"] & \Sigma A \arrow[d, "\Sigma i"] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & \Sigma A'
\end{tikzcd}$$

TR4. If there exists a morphism of distinguished triangles of the form

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"] & B \arrow[r, "g"] \arrow[d, "j"] & C' \arrow[r, "h"] \arrow[d, "k"] & \Sigma A \arrow[d, "\mathrm{id}_{\Sigma A}"] \\
A \arrow[r, "f'"] & C \arrow[r, "g'"] & B' \arrow[r, "h'"] & \Sigma A
\end{tikzcd}$$

then there exist morphisms which give the following commutative diagram, where the collumns are distinguished triangles as well:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"] & B \arrow[r, "g"] \arrow[d, "j"] & C' \arrow[r, "h"] \arrow[d, "k"] & \Sigma A \arrow[d, "\mathrm{id}_{\Sigma A}"] \\
A \arrow[r, "f'"] & C \arrow[r, "g'"] \arrow[d, "i"] & B' \arrow[r, "h'"] \arrow[d, "i'"] & \Sigma A \\
& A' \arrow[r, "\mathrm{id}_{A'}"] \arrow[d, "l"] & A' \arrow[d, "l'"] & \\
& \Sigma B \arrow[r, "\Sigma g"] & \Sigma C' &
\end{tikzcd}$$

Additionally, it is required that the diagram

$$\begin{tikzcd}
B' \arrow[r, "h'"] \arrow[d, "i'"] & \Sigma A \arrow[d, "\Sigma f"] \\
A' \arrow[r, "l"] & \Sigma B
\end{tikzcd}$$

also commutes.

We will recall a few elementary results on triangulated categories, and establish notation. For in-depth introductions to triangulated categories, see [52, 82, 45].

In a triangulated category $\mathbf{T}$, we will denote by $\mathrm{Hom}^i_{\mathbf{T}}(A, B) := \mathrm{Hom}_{\mathbf{T}}(A, \Sigma^i B)$. Since Σ is an autoequivalence, we have a canonical isomorphism $\mathrm{Hom}^i_{\mathbf{T}}(A, B) = \mathrm{Hom}_{\mathbf{T}}(A, \Sigma^i B) \cong \mathrm{Hom}_{\mathbf{T}}(\Sigma^{-i} A, B)$.

By TR1, given a distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \to \Sigma A,$$

the composition $gf = 0$. Using this fact, together with TR2 and TR3, one obtains the following result.

Lemma 2.3.2. *Let* $\mathbf{T}$ *be a triangulated category, and*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

be a distinguished triangle. Then for any object D*, the sequences*

$$... \to \mathrm{Hom}^i_{\mathbf{T}}(D, A) \xrightarrow{\Sigma^i f \circ -} \mathrm{Hom}^i_{\mathbf{T}}(D, B) \xrightarrow{\Sigma^i g \circ -} \mathrm{Hom}^i_{\mathbf{T}}(D, C) \xrightarrow{\Sigma^i h \circ -} \mathrm{Hom}^{i+1}_{\mathbf{T}}(D, A) \to ...$$

and

$$... \to \mathrm{Hom}^i_{\mathbf{T}}(C, D) \xrightarrow{- \circ \Sigma^i g} \mathrm{Hom}^i_{\mathbf{T}}(B, D) \xrightarrow{- \circ \Sigma^i f} \mathrm{Hom}^i_{\mathbf{T}}(A, D) \xrightarrow{- \circ \Sigma^i h} \mathrm{Hom}^{i-1}_{\mathbf{T}}(C, D) \to ...$$

are long exact sequences.

Using Lemma 2.3.2, it can be shown that if there is a morphism of distinguished triangles of the form

$$\begin{array}{ccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \Sigma A'
\end{array}$$

then if f and g are isomorphisms, h is also an isomorphism. Hence, given a morphism $f : A \to B$ in a triangulated category, there is an object, denoted $\mathrm{cone}(f)$ and called the *cone of* f, which is unique up to isomorphism, such that there is a distinguished triangle of the form

$$A \xrightarrow{f} B \to \mathrm{cone}(f) \to \Sigma A.$$

The cone of f is not necessarily unique up to unique isomorphism, since TR3 does not require the existence of a unique extension morphism.

Finally, we recall the relationship of direct sums to distinguished triangles.

Lemma 2.3.3. *Let* $\mathbf{T}$ *be a triangulated category,* A *and* B *in* $\mathbf{T}$*, and* $A \oplus B$ *the direct sum of* A *and* B *equipped with the usual inclusion and projection maps* $i_A : A \to A \oplus B$ *and* $p_B : A \oplus B \to B$.

(a) *The triangle*

$$A \xrightarrow{i_A} A \oplus B \xrightarrow{p_B} B \xrightarrow{0} \Sigma A$$

is distinguished.

(b) *If*

$$A \xrightarrow{f} C \xrightarrow{g} B \xrightarrow{h} C$$

is a distinguished triangle, then the following are equivalent:

(i) *There exists a map* $r : C \to A$ *with* $r \circ f = \mathrm{id}_A$.

(ii) *There exists a map* $s : B \to C$ *with* $g \circ s = \mathrm{id}_B$.

(iii) *The map* $h = 0$.

(iv) *There is an isomorphism of distinguished triangles*

$$\begin{array}{ccccccc} A & \xrightarrow{f} & C & \xrightarrow{g} & B & \xrightarrow{h} & \Sigma A \\ \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mathrm{id}_B} & & \downarrow{\scriptstyle \mathrm{id}_{\Sigma A}} \\ A & \xrightarrow{i_A} & A \oplus B & \xrightarrow{p_B} & B & \xrightarrow{0} & \Sigma A \end{array}$$

Suppose a triangulated category $\mathbf{T}$ admits arbitrary set-indexed coproducts. Then an object C in $\mathbf{T}$ is called *compact* if $\mathrm{Hom}_{\mathbf{T}}(C, -)$ commutes with all coproducts, and we will denote by $\mathbf{T}^c$ the collection of compact objects in $\mathbf{T}$. We will see below that under reasonable assumptions, in the stable categories arising from representation theory, the compact objects correspond precisely to the finite-dimensional representations. While many of the monoidal triangulated categories we wish to study– e.g. the stable categories of finite-dimensional representations $\mathsf{mod}(H)$ for a finite-dimensional Hopf algebra H– do not admit infinite coproducts, we will need to consider ambient "big" monoidal triangulated categories containing them, in order to apply the machinery of localization and colocalization functors described below.

We now recall the definitions of several important classes of subcategories of triangulated categories, and of compactly generated triangulated categories.

Definition 2.3.4. Let $\mathbf{T}$ be a triangulated category.

(a) A *triangulated subcategory* of $\mathbf{T}$ is a full subcategory $\mathbf{T}'$ such that for any distinguished triangle

$$A \to B \to C \to \Sigma A,$$

if A and B are in $\mathbf{T}'$, then so is C; and if A is in $\mathbf{T}'$, then so are ΣA and $\Sigma^{-1}A$.

(b) A *thick subcategory* of $\mathbf{T}$ is a triangulated subcategory $\mathbf{T}'$ such that if $A \oplus B \in \mathbf{T}'$, then so are A and B.

(c) If $\mathbf{T}$ contains all set-indexed coproducts, then a *localizing subcategory* of $\mathbf{T}$ is a subcategory $\mathbf{L}$ which is triangulated and closed under set-indexed coproducts. If $\mathbf{C} \subseteq \mathbf{T}$ is a subcategory, then $\mathrm{Loc}_{\mathbf{T}}(\mathbf{C})$ denotes the smallest localizing subcategory containing $\mathbf{C}$.

(d) A triangulated category $\mathbf{T}$ is called a *compactly generated triangulated category* if it is closed under set-indexed coproducts, and there exists a subcategory $\mathbf{C} \subseteq \mathbf{T}$ of compact objects such that $\mathbf{T} = \mathrm{Loc}_{\mathbf{T}}(\mathbf{C})$.

Using a version of the Eilenberg swindle, one can deduce that if **L** is a localizing subcategory of a compactly generated triangulated category, then **L** is thick.

Having now established our notation and some of the relevant background for both monoidal and triangulated categories individually, we define the main object of study moving forward– monoidal triangulated categories.

Definition 2.3.5. We call **K** a *monoidal triangulated category*, abbreviated MΔC, if **K** is triangulated and monoidal, such that the monoidal product is biexact. In other words there exist natural isomorphisms $A \otimes \Sigma B \xrightarrow{\beta_{A,B}} \Sigma(A \otimes B)$ and $\Sigma A \otimes B \xrightarrow{\gamma_{A,B}} \Sigma(A \otimes B)$ such that if

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

is a distinguished triangle, then so are

$$D \otimes A \xrightarrow{\mathrm{id}_D \otimes f} D \otimes B \xrightarrow{\mathrm{id}_D \otimes g} D \otimes C \xrightarrow{\beta_{D,A} \circ \mathrm{id}_D \otimes h} \Sigma(D \otimes A)$$

and

$$A \otimes D \xrightarrow{f \otimes \mathrm{id}_D} B \otimes D \xrightarrow{g \otimes \mathrm{id}_D} C \otimes D \xrightarrow{\gamma_{A,D} \circ h \otimes \mathrm{id}_D} \Sigma(A \otimes D).$$

Some authors, e.g. [54, 25], require the structure morphisms in a monoidal triangulated category to satisfy the additional diagrams

$$\begin{tikzcd} & \mathbb{1} \otimes \Sigma A \arrow[dl, "\beta_{\mathbb{1},A}"'] \arrow[dr, "\lambda_{\Sigma A}"] & \\ \Sigma(\mathbb{1} \otimes A) \arrow[rr, "\Sigma(\lambda_A)"'] & & \Sigma A \end{tikzcd}$$

and

$$\begin{tikzcd} & \Sigma A \otimes \mathbb{1} \arrow[dl, "\gamma_{A,\mathbb{1}}"'] \arrow[dr, "\rho_{\Sigma A}"] & \\ \Sigma(A \otimes \mathbb{1}) \arrow[rr, "\Sigma(\rho_A)"'] & & \Sigma A \end{tikzcd}$$

and that the diagram

$$\begin{tikzcd}
\Sigma A \otimes \Sigma B \arrow[r, "\gamma_{A,\Sigma B}"] \arrow[d, "\beta_{\Sigma A,B}"'] & \Sigma(A \otimes \Sigma B) \arrow[d, "\Sigma(\beta_{A,B})"] \\
\Sigma(\Sigma A \otimes B) \arrow[r, "\Sigma(\gamma_{A,B})"'] & \Sigma^2(A \otimes B)
\end{tikzcd}$$

anticommutes. We will call a monoidal triangulated which satisfies these additional diagrams a *coherent*.

As we noted above, we will need to employ the machinery of localization and colocalization functors in the context of categories admitting infinite coproducts. In particular, we will need to study compactly generated monoidal triangulated categories.

Definition 2.3.6. We say that a monoidal triangulated category $\mathbf{K}$ is a *compactly generated monoidal triangulated category* if the monoidal product of $\mathbf{K}$ preserves set indexed coproducts, $\mathbf{K}$ is compactly generated as a triangulated category, the tensor product of compact objects is compact, $\mathbb{1}$ is a compact object, and every compact object has a left and right dual.

In particular, note that the collection of compact objects of $\mathbf{K}$, denoted $\mathbf{K}^c$, is an MΔC on its own.

2.4. Stable categories of finite tensor categories

In this section, we will recall the construction and a few fundamental results on stable categories of finite tensor categories, which will form the primary examples of monoidal triangulated categories whose geometry we will study.

Definition 2.4.1. Let $\mathbf{C}$ be a an abelian category. Then the *stable category of* $\mathbf{C}$, denoted $\mathsf{st}(\mathbf{C})$, is defined to be the category where:

(a) The objects are the same as the objects of $\mathbf{C}$.

(b) The morphisms $\operatorname{Hom}_{\mathsf{st}(\mathbf{C})}(A, B)$ are defined to be $\operatorname{Hom}_{\mathbf{C}}(A, B)/\operatorname{PHom}_{\mathbf{C}}(A, B)$, where $\mathrm{PHom}_{\mathbf{C}}(A,B)$ consists of the collection of morphisms $f : A \to B$ such that there is a projective object P and morphisms $g : A \to P$, $h : P \to B$ with the diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[rr, bend left, "f"] & P \arrow[r, "h"] & B \end{tikzcd}$$

commuting, in other words the collection of morphisms f which factor through a projective object.

(c) Composition of morphisms in $\mathsf{st}(\mathbf{C})$ is defined by the composition of their representatives in $\mathbf{C}$.

Proposition 2.4.2. *If* $\mathbf{C}$ *is an abelian category, then its stable category* $\mathsf{st}(\mathbf{C})$ *is well-defined.*

Proof. First, we check that $\mathrm{PHom}_{\mathbf{C}}(A,B)$ is a subgroup of $\mathrm{Hom}_{\mathbf{C}}(A,B)$. The zero morphism factors through any projective object, and if f and $f' : A \to B$ where both factor through projectives, say via

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[rr, bend left, "f"] & P \arrow[r, "h"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "g'"] \arrow[rr, bend left, "f'"] & P' \arrow[r, "h'"] & B, \end{tikzcd}$$

then $f+f'$ factors as

$$\begin{tikzcd} A \arrow[r, "\Delta_A"] \arrow[rrrr, bend left, "f+g"] & A \oplus A \arrow[r, "g \oplus g'"] & P \oplus P' \arrow[r, "h \oplus h'"] & B \oplus B \arrow[r, "\nabla_B"] & B \end{tikzcd}$$

by the axioms of additive categories, where Δ_A and ∇_B are the standard diagonal and co-diagonal maps; since $P \oplus P'$ is projective, this implies $f + g \in \mathrm{PHom}_{\mathbf{C}}(A,B)$.

Secondly, we check that composition in $\mathsf{st}(\mathbf{C})$ is well-defined. Suppose $f = f'$ in $\mathsf{st}(\mathbf{C})$, for f and $f' : A \to B$ in $\mathbf{C}$. Then $f = f' + p$, where p factors through a projective.

Let g be a morphism $C \to A$. We observe that $fg = (f'+p)g = f'g+pg$ in $\mathbf{C}$, and since pg factors through a projective, this is equal to $f'g$ in the stable category $\mathsf{st}(\mathbf{C})$. Composing with a morphism on the left is similar, and so composition of morphisms is well-defined in $\mathsf{st}(\mathbf{C})$. $\square$

While one can consider (projectively) stable categories of arbitrary abelian categories, as defined above, we will be primarily interested in stable categories of quasi-Frobenius categories.

Definition 2.4.3. An abelian category $\mathbf{C}$ is called *quasi-Frobenius* if it has enough projectives, enough injectives, and projective and injective objects coincide.

Let A be an object of $\mathbf{C}$. Then define $\Omega(A)$ as the kernel of a surjective morphism $P \twoheadrightarrow A$, where P is projective.

Proposition 2.4.4. *Let $\mathbf{C}$ be a quasi-Frobenius category. Then $\Omega(A)$ is well-defined as an object of $\mathsf{st}(\mathbf{C})$. Additionally, Ω is functorial, and has an inverse $\Omega^{-1} : \mathsf{st}(\mathbf{C}) \to \mathsf{st}(\mathbf{C})$.*

Proof. First we show functoriality. Let $f : A \to A'$, and let $P \to A$ and $P' \to A'$ be surjective morphisms from projective objects P and P'. Then we define $\Omega(f)$ via the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega(A) \arrow[r, "i"] \arrow[d, dashed, "h"] & P \arrow[r, "p"] \arrow[d, dashed, "g"] & A \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \Omega(A') \arrow[r, "i'"] & P' \arrow[r, "p'"] & A' \arrow[r] & 0
\end{tikzcd}$$

Here g exists by the projectivity of P (note that it is not necessarily unique). Given g, the morphism h exists by the kernel property of $\Omega(A')$. If g and g' are two different choices of lifts of f to morphisms $P \to P'$, then $g - g'$ factors through $\Omega(A')$ (again using its kernel property), and so if h and h' are the unique maps $\Omega(A) \to \Omega(A')$ corresponding to g and g' respectively, we can see that $h - h'$ factors through P, a projective. In other words, in

$\mathsf{st}(\mathbf{C})$, defining the morphism $\Omega(f) := h$ as in the above diagram does not depend on the choice of morphism $g : P \to P'$ extending f.

Next, we show that $\Omega(A)$ is unique up to unique isomophism; this result is often referred to as Schanuel's Lemma, see e.g. [28, Proposition 2.5.1]. Suppose $P \to A$ and $P' \to A$ are two surjective morphisms from projective objects to A. We have the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K' \arrow[r, equal] \arrow[d] & K' \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & P \times_A P' \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Here all rows and columns are exact. By projectivity of P and P', the two exact sequences involving $P \times_A P'$ both split. Hence $K' \oplus P \cong K \oplus P'$, and so in the stable category $K' \cong K$, by the morphism h which is equal to the composition of the inclusion and projection maps $K' \to K' \oplus P' \cong K \oplus P \to K$. This isomorphism makes the diagram

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d, "-h"] & P' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

commute, by a diagram chase and the axioms of additive categories. By the first part of the proof, $-h$ is therefore unique after moving to the stable category, and so $\Omega(A)$ is unique up to unique isomorphism in $\mathsf{st}(\mathbf{C})$.

The inverse equivalence to Ω is constructed by setting $\Omega^{-1}(A)$ as the cokernel of an monic morphism $A \to I$, where I is an injective object. This gives a well-defined functor by the dual proof to that for Ω. Now, using the fact that injectives and projectives coincide, it is straightforward that $A \cong \Omega(\Omega^{-1}(A)) \cong \Omega^{-1}(\Omega(A))$, since A is the cokernel of the map $\Omega(A) \to P$, and A is the kernel of the map $I \to \Omega(A)$, if P and I are projective-injective objects used to define $\Omega(A)$ and $\Omega^{-1}(A)$, respectively.

□

We define $\Sigma : \mathsf{st}(\mathbf{C}) \to \mathsf{st}(\mathbf{C})$ to be the inverse Ω^{-1} of the syzygy functor Ω. Suppose

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is a short exact sequence in $\mathbf{C}$. Then we define a *standard triangle* in $\mathsf{st}(\mathbf{C})$ of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A,$$

where h is the unique (in $\mathsf{st}(\mathbf{C})$) morphism making the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r] & I \arrow[r] & \Sigma A \arrow[r] & 0
\end{tikzcd}$$

commute, where I is an injective object and the map $B \to I$ is defined by the injectivity of I. The uniqueness of h here follows by the same proof as the proof of the functoriality of Ω on the stable category given above.

The following is a classical result. For the proof, see [52, Theorem 2.6].

Theorem 2.4.5. *Let $\mathbf{C}$ be a quasi-Frobenius category. Then $\mathsf{st}(\mathbf{C})$, equipped with the autoequivalence Σ and set of distinguished triangles equal to those triangles which are isomorphic to standard triangles, is a triangulated category.*

Example 2.4.6. An algebra A is called *self-injective* if it is injective as a module over itself. This is equivalent to the category of all (not necessarily finite-dimensional) A-

modules being quasi-Frobenius, by Theorem 15.9 of [65]; hence, if A is finite-dimensional, then $\mathsf{mod}(A)$ is quasi-Frobenius if and only if $\mathsf{Mod}(A)$ is quasi-Frobenius. We can therefore form the stable categories of both: $\mathsf{stmod}(A) := \mathsf{st}(\mathsf{mod}(A))$, and $\mathsf{StMod}(A) := \mathsf{st}(\mathsf{Mod}(A))$. By [74, Theorem 3], if A is a finite-dimensional self-injective algebra then the compact objects of $\mathsf{StMod}(A)$ are precisely those which are isomorphic to finite-dimensional A-modules. Therefore, $\mathsf{StMod}(A)$ is a compactly generated triangulated category with compact part equal to $\mathsf{StMod}(A)^c \cong \mathsf{stmod}(A)$.

Next, we upgrade $\mathbf{C}$ from being a quasi-Frobenius category to being a finite tensor category. Recall that finite tensor categories are indeed quasi-Frobenius, by Proposition 2.1.6.

Proposition 2.4.7. *Let $\mathbf{C}$ be a finite tensor category. Then $\mathsf{st}(\mathbf{C})$ is a coherent monoidal triangulated category.*

Proof. The monoidal product on $\mathsf{st}(\mathbf{C})$ is defined by extending the monoidal product of $\mathbf{C}$. It is well-defined on objects since the objects of $\mathsf{st}(\mathbf{C})$ and $\mathbf{C}$ coincide. On morphisms, if $f = f'$ in $\mathsf{st}(\mathbf{C})$ as morphisms $A \to B$, then $f - f'$ factors as

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[rr, bend left, "f-f'"] & P \arrow[r, "h"] & B \end{tikzcd}$$

for some projective object P. We must check that for any $k : C \to D$, we have $f \otimes k = f' \otimes k$, i.e. $f \otimes k - f' \otimes k = (f - f') \otimes k$ factors through a projective. By construction, this map factors through $P \otimes C$ (and $P \otimes D$ as well). By Proposition 2.1.6, $P \otimes C$ and $P \otimes D$ are projective, and so $f \otimes k = f' \otimes k$ as morphisms in $\mathsf{st}(\mathbf{C})$.

More generally, note that if F is any additive functor $\mathbf{C} \to \mathbf{D}$, then the same proof shows that F can be extended to a functor $\mathsf{st}(\mathbf{C}) \to \mathbf{D}$ if $F(P) \cong 0$ in $\mathbf{D}$ for all projective objects P.

The associativity and unit morphisms for $-\otimes-$ on $\mathsf{st}(\mathbf{C})$ are defined as the images of the associativity and unit morphisms in $\mathbf{C}$, and it is immediate that the pentagon and triangle axioms are satisfied.

The natural transformations $\beta : -\otimes\Sigma- \to \Sigma(-\otimes-)$ and $\gamma : \Sigma-\otimes- \to \Sigma(-\otimes-)$ are defined via the unique morphisms given Proposition 2.4.4. In more detail, if A and B are two objects, then ΣB is defined as the cokernel of an injective map $B \to I$, where I is injective, and so we have an exact sequence

$$0 \to B \to I \to \Sigma B \to 0.$$

But then

$$0 \to A\otimes B \to A\otimes I \to A\otimes\Sigma B \to 0$$

is also exact, and $A\otimes I$ is injective, by Proposition 2.1.6, and so by Proposition 2.4.4 there is a unique isomorphism $\Sigma(A\otimes B) \cong A\otimes\Sigma B$. We define component $\beta_{A,B} : A\otimes\Sigma B \to \Sigma(A\otimes B)$ of β by this morphism, and γ is defined similarly. Using these structures, one can verify that $\mathsf{st}(\mathbf{C})$ is coherent. $\square$

2.5. Cohomology rings

With the goal of defining support varieties, we begin by recalling a few standard results on the extended endomorphism ring of the unit object of a coherent monoidal triangulated category.

Definition 2.5.1. Let $\mathbf{K}$ be a MΔC. For any object A of $\mathbf{K}$, we have a ring structure on

$$\mathrm{End}^{\bullet}_{\mathbf{K}}(A) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{K}}(A, \Sigma^i A),$$

where the multiplication is defined via composition: if $f \in \mathrm{Hom}_{\mathbf{K}}(A, \Sigma^i A)$ and $g \in \mathrm{Hom}_{\mathbf{K}}(A, \Sigma^j A)$ then $f \cdot g$ is defined to be $(\Sigma^j f) \circ g$, which is a morphism $A \to \Sigma^{i+j} A$. It is straightforward from the construction that this is a graded ring, with degree i piece equal to $\mathrm{Hom}_{\mathbf{K}}(A, \Sigma^i A)$. Given another object B of $\mathbf{K}$, we denote by

$$\mathrm{Hom}^{\bullet}_{\mathbf{K}}(A, B) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathbf{K}}(A, \Sigma^i B).$$

There are natural actions of $\mathrm{End}^{\bullet}_{\mathbf{K}}(A)$ (resp. $\mathrm{End}^{\bullet}_{\mathbf{K}}(B)$) on the right (resp. left) of $\mathrm{Hom}^{\bullet}_{\mathbf{K}}(A, B)$:

(a) If $f : A \to \Sigma^i A$, then the right action of f on $g : A \to \Sigma^j B$ is given by $\Sigma^i(g) \circ f : A \to \Sigma^{i+j}(B)$;

(b) If $f : B \to \Sigma^i B$, then the left action of f on $g : A \to \Sigma^j B$ is given by $\Sigma^j(f) \circ g$.

We define the *extended endomorphism ring of* $\mathbf{K}$ to be $\mathrm{End}^{\bullet}_{\mathbf{K}}(\mathbb{1})$. There are ring homomorphisms

$$\mathrm{End}^{\bullet}_{\mathbf{K}}(\mathbb{1}) \to \mathrm{End}^{\bullet}_{\mathbf{K}}(A)$$

$$f \mapsto f \otimes \mathrm{id}_A$$

and

$$\mathrm{End}^{\bullet}_{\mathbf{K}}(\mathbb{1}) \to \mathrm{End}^{\bullet}_{\mathbf{K}}(A)$$

$$f \mapsto \mathrm{id}_A \otimes f,$$

under the canonical identifications $\mathbb{1} \otimes A \cong A$, $\Sigma^i \mathbb{1} \otimes A \cong \Sigma^i A$, $A \otimes \mathbb{1} \cong A$, and $A \otimes \Sigma^i \mathbb{1} \cong \Sigma^i A$.

The following result in this generality is due to Suarez-Alvarez [98].

Proposition 2.5.2. *The extended endomorphism ring of a coherent monoidal triangulated category is graded commutative.*

Proof. Let $f : \mathbb{1} \to \Sigma^i \mathbb{1}$ and $g : \mathbb{1} \to \Sigma^j \mathbb{1}$ be homogeneous elements of $\mathrm{End}^{\bullet}_{\mathbf{K}}(\mathbb{1})$. We have the following diagram, which commutes by the definition of a coherent MΔC (where we suppress the subscripts of the natural transformations ρ, λ, γ, and β for readability):

$$
\begin{tikzcd}
\mathbb{1} \otimes \mathbb{1} \arrow[r, "f\otimes 1"] \arrow[d, "\rho"] & \Sigma^i \mathbb{1} \otimes \mathbb{1} \arrow[rr, "1\otimes g"] \arrow[d, "\rho"] \arrow[dr, "\gamma"] & & \Sigma^i \mathbb{1} \otimes \Sigma^j \mathbb{1} \arrow[d, "\gamma"] \\
\mathbb{1} \arrow[r, "f"] & \Sigma^i \mathbb{1} \arrow[d, "\Sigma^i g"'] & \Sigma^i(\mathbb{1} \otimes \mathbb{1}) \arrow[l, "\Sigma^i(\rho)=\Sigma^i(\lambda)"'] \arrow[r, "\Sigma^i(1\otimes g)"] & \Sigma^i(\mathbb{1} \otimes \Sigma^j \mathbb{1}) \arrow[d, "\Sigma^i(\beta)"] \arrow[dll, "\Sigma^i(\lambda)"] \\
 & \Sigma^{i+j}(\mathbb{1} & & \Sigma^{i+j}(\mathbb{1} \otimes \mathbb{1}) \arrow[ll, "\Sigma^{i+j}(\lambda)"]
\end{tikzcd}
$$

Additionally, we have the following diagram, where the outside commutes up to a factor of $(-1)^{ij}$:

$$
\begin{tikzcd}
\mathbb{1} \otimes \mathbb{1} \arrow[r, "1\otimes g"] \arrow[d, "\lambda"] & \mathbb{1} \otimes \Sigma^j \mathbb{1} \arrow[rr, "f\otimes 1"] \arrow[d, "\lambda"] \arrow[dr, "\beta"] & & \Sigma^i \mathbb{1} \otimes \Sigma^j \mathbb{1} \arrow[r, "\gamma"] \arrow[d, "\beta"] \arrow[dr, phantom, "(-1)^{ij}"] & \Sigma^i(\mathbb{1} \otimes \Sigma^j \mathbb{1}) \arrow[d, "\Sigma^i(\beta)"] \\
\mathbb{1} \arrow[r, "g"] & \Sigma^j \mathbb{1} \arrow[drr, "\Sigma^j(f)"'] & \Sigma^j(\mathbb{1} \otimes \mathbb{1}) \arrow[l, "\Sigma^j(\lambda)"'] \arrow[r, "\Sigma^j(f\otimes 1)"] & \Sigma^j(\Sigma^i \mathbb{1} \otimes \mathbb{1}) \arrow[r, "\Sigma^j(\gamma)"] \arrow[d, "\Sigma^j(\rho)"] & \Sigma^{i+j}(\mathbb{1} \otimes \mathbb{1}) \arrow[dl, "\Sigma^{i+j}(\rho)"] \\
 & & & \Sigma^{i+j} \mathbb{1} &
\end{tikzcd}
$$

Summarizing the outer boundaries of these two diagrams, we have the (skew) commutative diagrams

$$
\begin{tikzcd}
\mathbb{1} \otimes \mathbb{1} \arrow[r, "f\otimes g"] \arrow[d, "\lambda"] \arrow[dr, phantom, "(-1)^{ij}"] & \Sigma^i \mathbb{1} \otimes \Sigma^j \mathbb{1} \arrow[d, "\Sigma^{i+j}(\rho)\circ\Sigma^i(\beta)\circ\gamma_i"] \\
\mathbb{1} \arrow[r, "f\cdot g"] & \Sigma^{i+j} \mathbb{1}
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
\mathbb{1} \otimes \mathbb{1} \arrow[r, "f\otimes g"] \arrow[d, "\lambda"] & \Sigma^i \mathbb{1} \otimes \Sigma^j \mathbb{1} \arrow[d, "\Sigma^{i+j}(\rho)\circ\Sigma^i(\beta)\circ\gamma_i"] \\
\mathbb{1} \arrow[r, "g\cdot f"] & \Sigma^{i+j} \mathbb{1}
\end{tikzcd}
$$

(using the fact that $\lambda_{\mathbb{1}} = \rho_{\mathbb{1}}$, which follows from Theorem 2.1.2). $\square$

Corollary 2.5.3. *The multiplication* $f \cdot g$ *in* $\mathrm{End}^\bullet_{\mathbf{K}}(\mathbb{1})$ *corresponds to* $f \cdot g = f \otimes g$ *(up to potentially a factor of* -1*), under choices of structure isomorphisms* $\mathbb{1} \otimes \mathbb{1} \cong \mathbb{1}$ *and* $\Sigma^i \mathbb{1} \otimes \Sigma^j \mathbb{1} \cong \Sigma^{i+j} \mathbb{1}$.

To define support varieties, we will use the structure of $\mathrm{Hom}^\bullet_{\mathbf{K}}(A, B)$ as a left and right $\mathrm{End}^\bullet_{\mathbf{K}}(\mathbb{1})$-module via the ring homomorphisms given above $\mathrm{End}^\bullet_{\mathbf{K}}(\mathbb{1}) \to \mathrm{End}^\bullet_{\mathbf{K}}(A)$ and $\mathrm{End}^\bullet_{\mathbf{K}}(\mathbb{1}) \to \mathrm{End}^\bullet_{\mathbf{K}}(B)$. A priori, as we noted above, there are two choices of such homomorphisms; in the interest of defining support varieties, we will fix one.

Convention 2.5.4. *We consider* $\mathrm{Hom}^\bullet_{\mathbf{K}}(A, B)$ *as a left* $\mathrm{End}^\bullet_{\mathbf{K}}(\mathbb{1})$*-module via the ring homomorphism* $\mathrm{End}^\bullet_{\mathbf{K}}(\mathbb{1}) \to \mathrm{End}^\bullet_{\mathbf{K}}(B)$ *that sends* $f \mapsto f \otimes \mathrm{id}_B$*, and as a right* $\mathrm{End}^\bullet_{\mathbf{K}}(\mathbb{1})$*-module via the ring homomorphism* $\mathrm{End}^\bullet_{\mathbf{K}}(\mathbb{1}) \to \mathrm{End}^\bullet_{\mathbf{K}}(A)$ *that sends* $f \mapsto f \otimes \mathrm{id}_A$.

The following proposition follows as in [25, Proposition 2.2].

Proposition 2.5.5. *Let* $f : \mathbb{1} \to \Sigma^i \mathbb{1}$ *and* $g : A \to \Sigma^j B$*. Then under the left and right actions defined above,* $f \cdot g = (-1)^{i+j} g \cdot f$.

While we now have an action of the full extended endomorphism ring on each $\mathrm{Hom}^\bullet_{\mathbf{K}}(A, B)$, for the purpose of defining support varieties we will restrict to the positively-graded parts.

Definition 2.5.6. We define the *cohomology ring* of $\mathbf{K}$ to be

$$\mathrm{H}^\bullet(\mathbf{K}) := \begin{cases} \bigoplus_{i \geq 0} \mathrm{Hom}(\mathbb{1}, \Sigma^i \mathbb{1}) & \mathrm{char}\, \mathbb{k} = 2 \\ \bigoplus_{i \geq 0} \mathrm{Hom}(\mathbb{1}, \Sigma^{2i} \mathbb{1}) & \mathrm{char}\, \mathbb{k} \neq 2 \end{cases}.$$

Note that $\mathrm{H}^\bullet(\mathbf{K})$ is now an $\mathbb{N}$-graded commutative ring. Note also that the left and right actions of $\mathrm{End}^\bullet_{\mathbf{K}}(\mathbb{1})$ on $\mathrm{Hom}^\bullet_{\mathbf{K}}(A, B)$ restrict to actions of $\mathrm{H}^\bullet(\mathbf{K})$ on $\mathrm{Hom}^{\geq 0}_{\mathbf{K}}(A, B)$.

If $\mathbf{K}$ is the stable category $\mathsf{st}(\mathbf{C})$ for a finite tensor category $\mathbf{C}$, then

$$\begin{aligned}\mathrm{Ext}^i_{\mathbf{C}}(A,B) &\cong \mathrm{Hom}^i_{\mathsf{st}(\mathbf{C})}(A,B)\\ &= \mathrm{Hom}_{\mathsf{st}(\mathbf{C})}(A,\Sigma^i B)\end{aligned}$$

for all $i > 0$ ([28, Proposition 2.6.2] gives the argument for finite groups, which extends to arbitrary finite tensor categories), and if $\mathbf{C}$ is not semisimple then

$$\mathrm{H}^\bullet(\mathsf{st}(\mathbf{C})) \cong \mathrm{Ext}^{2\bullet}_{\mathbf{C}}(\mathbb{1},\mathbb{1}). \tag{2.1}$$

This uses the fact that $\mathrm{Hom}_{\mathbf{C}}(\mathbb{1},\mathbb{1}) \cong \Bbbk$ by the axioms of finite tensor categories, and that if $\mathbf{C}$ is not semisimple, then $\mathbb{1}$ is not projective, which in turn implies that $\mathrm{Hom}_{\mathsf{st}(\mathbf{C})}(\mathbb{1},\mathbb{1}) \cong \Bbbk$. In fact, the isomorphism (2.1) is not just an isomorphism of vector spaces, but rather of graded algebras, where the product on $\mathrm{Ext}^\bullet_{\mathbf{C}}(\mathbb{1},\mathbb{1})$ is given by the Yoneda product, also called the cup product [25, Remark 2.4 (ii)]. This example motivates the terminology for calling $\mathrm{H}^\bullet(\mathbf{K})$ the cohomology ring of $\mathbf{K}$.

Definition 2.5.7. Let $\mathbf{K}$ be a coherent monoidal triangulated category. We say that $\mathbf{K}$ satisfies the *(fg) condition* if

(a) The cohomology ring $\mathrm{H}^\bullet(\mathbf{K})$ is a finitely generated algebra.

(b) Given A and B in $\mathbf{K}$, $\mathrm{Hom}^{\geq 0}_{\mathbf{K}}(A,B)$ is a finitely generated $\mathrm{H}^\bullet(\mathbf{K})$-module.

It has been famously conjectured by Etingof and Ostrik that the (fg) condition holds for all finite tensor categories [36, Conjecture 2.18].

The problem of proving the (fg) condition for various Hopf algebras has attracted significant attention since it was initiated by the works of Golod, Venkov, and Evans in the 1950s, which proved (fg) for finite groups [47, 102, 37]. One of the most foundational results in this area is the Friedlander-Suslin proof of the (fg) condition for all finite group schemes [42]. Beyond this general result, the (fg) condition has also been shown for many specific families. In the 80's, it was shown that (fg) holds for certain subalgebras of Steenrod algebras [104] and restricted enveloping algebras [40, 3]. Quantized results soon followed, with Ginzburg-Kumar [46] proving that small quantum groups in characteristic 0 at a root of unity ℓ satisfy (fg) under certain conditions on ℓ. The conditions on ℓ were weakened in [12], the results extended to small quantum groups in positive characteristic in [32], and (fg) proven for the duals of the small quantum groups, e.g. the small quantum function algebras, in [51]. Further results have proven the (fg) condition for finite supergroup schemes [33], for certain families of pointed Hopf algebras with abelian group of grouplikes [71], for certain Hopf algebras of dimension p^3 in a field of characteristic p [89, 34], for certain skew group algebras [88], for the 12-dimensional Fomin-Kirillov algebra and certain bosonizations of them [97], for Drinfeld doubles of infinitesimal group schemes [39], and for integrable Hopf algebras [85]. Very recently, in [4] it is shown that (fg) holds for any finite-dimensional pointed Hopf algebra with an abelian group of grouplikes.

2.6. Cohomological support varieties

Via the actions of the cohomology ring, we are now able to define and state the initial properties of cohomological support varieties.

Definition 2.6.1. Let A and B be objects of $\mathbf{K}$. Denote by $I(A, B)$ the annihilator ideal of $\mathrm{Hom}_{\mathbf{K}}^{\geq 0}(A, B)$ in $\mathrm{H}^{\bullet}(\mathbf{K})$. Then the *(cohomological) support variety* of A and B is defined as

$$W_{\mathbf{K}}(A, B) = \{\mathfrak{p} \in \operatorname{Proj} \mathrm{H}^{\bullet}(\mathbf{K}) : I(A, B) \subseteq \mathfrak{p}\} = Z(I(A, B)).$$

In the case that $A = B$, we denote $I(A, A)$ by $I(A)$, and

$$W_{\mathbf{K}}(A) := W_{\mathbf{K}}(A, A).$$

Versions of cohomological support varieties may also be defined using either the maximal ideal spectrum or the homogeneous prime ideal spectrum as opposed to Proj. Support varieties defined in this way give similar theories.

Note that for $f \in \mathrm{H}^\bullet(\mathbf{K})$, we have $f \in I(A)$ if and only if $f \otimes \mathrm{id}_A = 0$.

We will now prove some elementary results on support varieties.

Lemma 2.6.2. *The following hold for support varieties on a monoidal triangulated category $\mathbf{K}$.*

(a) *Let $A \to B \to C \to \Sigma A$ be a distinguished triangle in $\mathbf{K}$. Then for any object D, we have $W_{\mathbf{K}}(D, B) \subseteq W_{\mathbf{K}}(D, A) \cup W_{\mathbf{K}}(D, C)$ and similarly $W_{\mathbf{K}}(B, D) \subseteq W_{\mathbf{K}}(A, D) \cup W_{\mathbf{K}}(C, D)$.*

(b) *For any objects A and B in $\mathbf{K}$, we have $W_{\mathbf{K}}(A, B) \subseteq W_{\mathbf{K}}(A) \cap W_{\mathbf{K}}(B)$.*

Proof. Assume the situation of (a). We have a long exact sequence by Lemma 2.3.2:

$$\ldots \to \mathrm{Hom}_{\mathbf{K}}(D, \Sigma^i A) \to \mathrm{Hom}_{\mathbf{K}}(D, \Sigma^i B) \to \mathrm{Hom}_{\mathbf{K}}(D, \Sigma^i C) \to \mathrm{Hom}_{\mathbf{K}}(D, \Sigma^{i+1} A) \to \ldots$$

Collected together, these maps give us a sequence of $\mathrm{H}^\bullet(\mathbf{K})$-modules

$$\mathrm{Hom}^{\geq 0}_{\mathbf{K}}(D, A) \to \mathrm{Hom}^{\geq 0}_{\mathbf{K}}(D, B) \to \mathrm{Hom}^{\geq 0}_{\mathbf{K}}(D, C)$$

exact at the middle term. This implies by standard commutative algebra that $I(D, A) \cdot I(D, C) \subseteq I(D, B)$. This implies immediately that

$$W_{\mathbf{K}}(D, B) \subseteq W_{\mathbf{K}}(D, A) \cup W_{\mathbf{K}}(D, C).$$

This proves the first claim; the second is analogous.

(b) is straightforward, since the action of $\mathrm{H}^\bullet(\mathbf{K})$ on $\mathrm{Hom}^{\geq 0}_{\mathbf{K}}(A,B)$ factors through both $\mathrm{End}^{\geq 0}_{\mathbf{K}}(A)$ and $\mathrm{End}^{\geq 0}_{\mathbf{K}}(B)$. Hence if $f \in \mathrm{H}^\bullet(\mathbf{K})$ annihilates either $\mathrm{End}^{\geq 0}_{\mathbf{K}}(A)$ or $\mathrm{End}^{\geq 0}_{\mathbf{K}}(B)$, then it also annihilates $\mathrm{Hom}^{\geq 0}_{\mathbf{K}}(A,B)$, showing that $I(A) \cup I(B) \subseteq I(A,B)$. This immediately implies $W_{\mathbf{K}}(A,B) \subseteq W_{\mathbf{K}}(A) \cap W_{\mathbf{K}}(B)$. □

Proposition 2.6.3. *The following hold for support varieties on a monoidal triangulated category* $\mathbf{K}$.

(a) $W_{\mathbf{K}}(0) = \varnothing$, *and* $W_{\mathbf{K}}(\mathbb{1}) = \operatorname{Proj} \mathrm{H}^\bullet(\mathbf{K})$.

(b) *For all objects A and B in $\mathbf{K}$, we have* $W_{\mathbf{K}}(A \oplus B) = W_{\mathbf{K}}(A) \cup W_{\mathbf{K}}(B)$.

(c) *For all objects $A \in \mathbf{K}$, we have* $W_{\mathbf{K}}(\Sigma A) = W_{\mathbf{K}}(A)$.

(d) *For every distinguished triangle $A \to B \to C \to \Sigma A$, we have* $W_{\mathbf{K}}(B) \subseteq W_{\mathbf{K}}(A) \cup W_{\mathbf{K}}(C)$.

(e) *For all objects A and B in $\mathbf{K}$, we have* $W_{\mathbf{K}}(A \otimes B) \subseteq W_{\mathbf{K}}(A)$.

Proof. (a) is clear, since $I(0) = \mathrm{H}^\bullet(\mathbf{K})$ and so no homogeneous prime in $\operatorname{Proj} \mathrm{H}^\bullet(\mathbf{K})$ contains $I(0)$; additionally, $I(\mathbb{1})$ is $\langle 0 \rangle$ in $\mathrm{H}^\bullet(\mathbf{K})$, since $f \otimes \mathrm{id}_{\mathbb{1}}$ corresponds to f under the isomorphisms $\mathbb{1} \otimes \mathbb{1} \cong \mathbb{1}$ and $\Sigma^i \mathbb{1} \otimes \mathbb{1} \cong \Sigma^i \mathbb{1}$ by the defining equations for a monoidal category, and so f is in $I(\mathbb{1})$ if and only if f is the 0 element of $\mathrm{H}^\bullet(\mathbf{K})$.

For (b), let $f : \mathbb{1} \to \Sigma^i \mathbb{1}$. We note that $f \otimes \mathrm{id}_{A \oplus B}$ corresponds to $(f \otimes \mathrm{id}_A) \oplus (f \otimes \mathrm{id}_B)$ under the isomorphism $\mathbb{1} \otimes (A \oplus B) \cong (\mathbb{1} \otimes A) \oplus (\mathbb{1} \otimes B)$ and similarly for $\Sigma^i \mathbb{1}$. In particular, $f \otimes \mathrm{id}_{A \oplus B} = 0$ if and onlly if $f \otimes \mathrm{id}_A$ and $f \otimes \mathrm{id}_B$ are both 0 as well, in other words $I(A \oplus B) = I(A) \cap I(B)$. Hence $Z(I(A \oplus B)) = Z(I(A) \cap I(B)) = Z(I(A)) \cup Z(I(B))$.

For (c), we will show that $I(A) = I(\Sigma A)$. If $f \in \mathrm{H}^\bullet(\mathbf{K})$ a homogeneous element, then by the naturality of the structure morphisms β, we have a commutative diagram

$$\begin{tikzcd} \Sigma(\mathbb{1} \otimes A) \arrow[r, "\Sigma(f\otimes \mathrm{id}_A)"] \arrow[d, "\beta^{-1}_{\mathbb{1},A}"] & \Sigma(\Sigma^i \mathbb{1} \otimes A) \arrow[d, "\beta^{-1}_{\Sigma^i\mathbb{1},A}"] \\ \mathbb{1} \otimes \Sigma A \arrow[r, "f\otimes \mathrm{id}_{\Sigma A}"] & \Sigma^i \mathbb{1} \otimes \Sigma A \end{tikzcd}$$

where the vertical maps are isomorphisms. Therefore, $f \otimes \mathrm{id}_{\Sigma A} = 0$ if and only if $\Sigma(f \otimes \mathrm{id}_A) = 0$. But since Σ is an autoequivalence of $\mathbf{K}$, we know $\Sigma(f \otimes \mathrm{id}_A) = 0$ if and only if $f \otimes \mathrm{id}_A = 0$. Since $f \otimes \mathrm{id}_{\Sigma A} = 0$ if and only if $f \otimes \mathrm{id}_A = 0$, we have $f \in I(A)$ if and only if $f \in I(\Sigma A)$, in other words $I(A) = I(\Sigma A)$.

Suppose $A \to B \to C \to \Sigma A$ is a distinguished triangle of $\mathbf{K}$. By parts (a) and (b) respectively of Lemma 2.6.2, we have $W_{\mathbf{K}}(B) \subseteq W_{\mathbf{K}}(A, B) \cup W_{\mathbf{K}}(C, B) \subseteq W_{\mathbf{K}}(A) \cup W_{\mathbf{K}}(C)$. This proves (d).

For (e), we note that for $f \in \mathrm{H}^\bullet(\mathbf{K})$ homogeneous of degree i, we have $f \otimes \mathrm{id}_{A\otimes B} = f \otimes \mathrm{id}_A \otimes \mathrm{id}_B$ and so if $f \in I(A)$ then $f \in I(A \otimes B)$. In other words, $I(A) \subseteq I(A \otimes B)$, and so $W_{\mathbf{K}}(A) \supseteq W_{\mathbf{K}}(A \otimes B)$.

□

The collection of objects of the form cone(f), for $f : \mathbb{1} \to \Sigma^i \mathbb{1}$ in $\mathrm{H}^\bullet(\mathbf{K})$, play an important role in the theory of support varieties. They were originally introduced by Carlson in the theory of finite groups, where they were called L_ζ objects (where ζ indicates an element of the cohomology ring), see [13, Section 5.9]. In the context of triangulated categories, these objects are sometimes called *Koszul objects*, see e.g. [25].

The following two results are given in this generality in [25, Proposition 3.6, Proposition 3.7].

Proposition 2.6.4. *Let* $\mathbf{K}$ *be a* $M\Delta C$*, and*

$$\mathbb{1} \xrightarrow{f} \Sigma^m \mathbb{1} \xrightarrow{g} \mathrm{cone}(f) \xrightarrow{h} \Sigma \mathbb{1}$$

be a distinguished triangle. Then $f \otimes f \otimes \mathrm{id}_{\mathrm{cone}(f)} = 0$.

Proof. From the distinguished triangle given in the proposition, we obtain for any object A a long exact sequence

$$... \to \mathrm{Hom}^j_{\mathbf{K}}(A, \mathbb{1}) \xrightarrow{f\cdot} \mathrm{Hom}^j_{\mathbf{K}}(A, \Sigma^i \mathbb{1}) \to \mathrm{Hom}^j_{\mathbf{K}}(A, \mathrm{cone}(f)) \to ...$$

by Lemma 2.3.2, where the maps are obtained by composition with (shifts of) the morphisms f, g, and h. From this long exact sequence, we obtain the short exact sequence

$$0 \to \mathrm{Hom}^\bullet_{\mathbf{K}}(A, \mathbb{1})/(f \cdot \mathrm{Hom}^\bullet_{\mathbf{K}}(A, \mathbb{1})) \to \mathrm{Hom}^\bullet_{\mathbf{K}}(A, \mathrm{cone}(f)) \tag{2.2}$$

$$\to \ker(f \cdot |_{\mathrm{Hom}^\bullet_{\mathbf{K}}(A, \mathbb{1})}) \to 0,$$

where we have used the fact that $\mathrm{Hom}^\bullet_{\mathbf{K}}(A, \mathbb{1}) \cong \mathrm{Hom}^\bullet_{\mathbf{K}}(A, \Sigma^i \mathbb{1})$. In fact, this is a short exact sequence of $\mathrm{End}^\bullet_{\mathbf{K}}(\mathbb{1})$-modules. From this short exact sequence, it is clear that the action of f^2 acting on $\mathrm{Hom}^\bullet_{\mathbf{K}}(A, \mathrm{cone}(f))$ is 0 for any object A, and in particular this holds for $A = \mathrm{cone}(f)$. Hence $f \otimes f \otimes \mathrm{id}_{\mathrm{cone}(f)} = 0$. □

Proposition 2.6.5. *Let* A *be any object of a monoidal triangulated category* $\mathbf{K}$*. Let* $f : \mathbb{1} \to \Sigma^i \mathbb{1}$ *where* $i \geq 0$*, in other words* f *is a homogeneous element of* $\mathrm{H}^\bullet(\mathbf{K})$ *of degree* i*.*

(a) $W_{\mathbf{K}}(\mathrm{cone}(f) \otimes A) \subseteq W_{\mathbf{K}}(A) \cap \{\mathfrak{p} \in \mathrm{Proj}\, \mathrm{H}^\bullet(\mathbf{K}) : f \in \mathfrak{p}\} := W_{\mathbf{K}}(A) \cap Z(f)$.

(b) If $\mathbf{K}$ *satisfies (wfg), then* $W_{\mathbf{K}}(\mathrm{cone}(f) \otimes A) = W_{\mathbf{K}}(A) \cap Z(f)$.

Proof. Since we have a distinguished triangle

$$A \to \Sigma^i A \to \mathrm{cone}(f) \otimes A \to \Sigma A,$$

we know that $W(\mathrm{cone}(f)\otimes A) \subseteq W_{\mathbf{K}}(A)$ by parts (c) and (d) of Proposition 2.6.3. We know additionally that $W_{\mathbf{K}}(\mathrm{cone}(f)\otimes A) \subseteq W_{\mathbf{K}}(\mathrm{cone}(f))$, by part (e) of Proposition 2.6.3. It remains to show that $W_{\mathbf{K}}(\mathrm{cone}(f)) \subseteq Z(f)$. This follows from Proposition 2.6.4, since if $\mathfrak{p}$ contains $I(\mathrm{cone}(f))$ then it must contain f^2, and by primeness then contains f. This proves (a).

For (b), suppose that $\mathfrak{p} \in \operatorname{Proj} \mathrm{H}^\bullet(\mathbf{K})$ contains f and $I(A)$. We must show that $\mathfrak{p} \in W_{\mathbf{K}}(\mathrm{cone}(f)\otimes A)$. By Lemma 2.6.2 (b), it is enough to show that $\mathfrak{p} \in W_{\mathbf{K}}(B, \mathrm{cone}(f)\otimes A)$ for some B. Suppose to the contrary. By using the triangle

$$A \to A \to \mathrm{cone}(f)\otimes A \to \Sigma A,$$

we obtain a short exact sequence of $\mathrm{End}^\bullet_{\mathbf{K}}(\mathbb{1})$-modules by the same argument as for the short exact sequence (2.2), for any object B:

$$0 \to \mathrm{Hom}^\bullet_{\mathbf{K}}(B,A)/(f\cdot \mathrm{Hom}^\bullet_{\mathbf{K}}(B,A)) \to \mathrm{Hom}^\bullet_{\mathbf{K}}(B,\mathrm{cone}(f)\otimes A) \tag{2.3}$$

$$\to \ker(f\cdot|_{\mathrm{Hom}^\bullet_{\mathbf{K}}(B,A)}) \to 0.$$

Restricting to the positively-graded parts, we have a short exact sequence of $\mathrm{H}^\bullet(\mathbf{K})$-modules:

$$0 \to \mathrm{Hom}^{\geq i}_{\mathbf{K}}(B,A)/(f\cdot \mathrm{Hom}^{\geq 0}_{\mathbf{K}}(B,A)) \to \mathrm{Hom}^{\geq 0}_{\mathbf{K}}(B,\mathrm{cone}(f)\otimes A) \tag{2.4}$$

$$\to \ker(f\cdot|_{\mathrm{Hom}^{\geq 0}_{\mathbf{K}}(B,A)}) \to 0.$$

Since $\mathfrak{p}$ does not contain $I(B,\mathrm{cone}(f)\otimes A)$ by assumption, we have that

$$\mathrm{Hom}^{\geq 0}_{\mathbf{K}}(B,\mathrm{cone}(f)\otimes A)_{\mathfrak{p}} = 0.$$

By (2.4), $\mathrm{Hom}_{\mathbf{K}}^{\geq i}(B,A)_{\mathfrak{p}} = f \cdot \mathrm{Hom}_{\mathbf{K}}^{\geq 0}(B,A)_{\mathfrak{p}}$. By commutative algebra, since $\mathfrak{p}$ does not contain the irrelevant ideal of $\mathrm{H}^{\bullet}(\mathbf{K})$, $\mathrm{Hom}_{\mathbf{K}}^{\geq i}(B,A)_{\mathfrak{p}} \cong \mathrm{Hom}_{\mathbf{K}}^{\geq 0}(B,A)_{\mathfrak{p}}$, via the following map. Let t be a homogeneous element of positive degree which is not in $\mathfrak{p}$, which exists because $\mathfrak{p}$ doesn't contain the irrelevant ideal. Then for $x \in \mathrm{Hom}_{\mathbf{K}}^{\geq 0}(B,A)$ and $y \notin \mathfrak{p}$, then the map

$$\mathrm{Hom}_{\mathbf{K}}^{\geq 0}(B,A)_{\mathfrak{p}} \to \mathrm{Hom}_{\mathbf{K}}^{\geq i}(B,A)_{\mathfrak{p}},$$
$$\frac{x}{y} \mapsto \frac{t^i x}{t^i y}$$

is an isomorphism.

Hence, we have now shown that $\mathrm{Hom}_{\mathbf{K}}^{\geq 0}(B,A)_{\mathfrak{p}} = f \cdot \mathrm{Hom}_{\mathbf{K}}^{\geq 0}(B,A)_{\mathfrak{p}}$. Now using the hypothesis that $\mathrm{Hom}_{\mathbf{K}}^{\geq 0}(B,A)_{\mathfrak{p}}$ is a finitely-generated $\mathrm{H}^{\bullet}(\mathbf{K})_{\mathfrak{p}}$-module, and the fact that f is in $\mathfrak{p}\,\mathrm{H}^{\bullet}(\mathbf{K})_{\mathfrak{p}}$, by Nakayama's Lemma $\mathrm{Hom}_{\mathbf{K}}^{\geq 0}(B,A)_{\mathfrak{p}} = 0$. Since $\mathrm{Hom}_{\mathbf{K}}^{\geq 0}(B,A)$ is a finitely-generated $\mathrm{H}^{\bullet}(\mathbf{K})$-module by assumption, this implies that $I(B,A) \not\subseteq \mathfrak{p}$, for any object B. But this is a contradiction, since we know that $I(A,A) \subseteq \mathfrak{p}$ by assumption. □

Corollary 2.6.6. *Suppose a monoidal triangulated category* $\mathbf{K}$ *satisfies the (fg) condition. Then for any closed set* S *of* $\mathrm{Proj}\,\mathrm{H}^{\bullet}(\mathbf{K})$*, there exists an object* A *of* $\mathbf{K}$ *such that* $W_{\mathbf{K}}(A) = S$*.*

Proof. Since $\mathrm{H}^{\bullet}(\mathbf{K})$ is finitely-generated, every closed S set in $\mathrm{Proj}\,\mathrm{H}^{\bullet}(\mathbf{K})$ is the variety defined by a finitely-generated homogeneous ideal, say with homogeneous generators $f_1, ..., f_n$. But then by Proposition 2.6.5, we have

$$W_{\mathbf{K}}(\mathrm{cone}(f_1) \otimes ... \otimes \mathrm{cone}(f_n)) = Z(f_1) \cap ... \cap Z(f_n) = Z(f_1, ..., f_n) = S.$$

□

Recall that by Proposition 2.6.3(e), $W_{\mathbf{K}}(A \otimes B) \subseteq W_{\mathbf{K}}(A)$. On the other hand, for f and g in $H^{\bullet}(\mathbf{K})$, we have the stronger property $W_{\mathbf{K}}(\mathrm{cone}(f) \otimes A) = W_{\mathbf{K}}(A \otimes \mathrm{cone}(f)) = W_{\mathbf{K}}(\mathrm{cone}(f)) \cap W_{\mathbf{K}}(A)$ for any object A, as long as $\mathbf{K}$ satisfies (fg), by Proposition 2.6.5. If $W_{\mathbf{K}}$ satisfies the stronger property

$$W_{\mathbf{K}}(A \otimes B) = W_{\mathbf{K}}(A) \cap W_{\mathbf{K}}(B)$$

for any A and B in $\mathbf{K}$, then we say that $W_{\mathbf{K}}$ satisfies the *tensor product property.* For the cohomological support for modular representations of finite groups this was proved in [27] and for finite group schemes in [41]. There has been a great deal of research on this problem for the cohomological support for the stable module category $\mathsf{stmod}(H)$ of a finite dimensional Hopf algebra H. Negative results were obtained in [19, 92] for certain smash products, and the tensor product property for small quantum groups in type A were proven in [85].

Finally, we mention one result which holds for stable categories of finite tensor categories satisfying (fg); we are not aware of generalizations of this result to arbitrary coherent monoidal triangulated categories, unlike most of the other results listed above. In this generality, see [20, Corollary 4.2]. The statement for finite-dimensional Hopf algebras can be found in [38, Proposition 2.4], and for the corresponding theorem for finite groups is given in [13, Proposition 5.7.2].

Theorem 2.6.7. *Let $\mathbf{C}$ be a finite tensor category satisfying (fg) and $\mathbf{K} = \mathsf{st}(\mathbf{C})$. Then $W_{\mathbf{K}}(P) = \varnothing$ if and only if P is projective in $\mathbf{C}$, in other words, if $P \cong 0$ in $\mathbf{K}$.*

Theorem 2.6.7 is proven by showing that for any object A the dimension of the support variety $W_{\mathbf{K}}(A)$ is equal to the complexity of A, where complexity is defined as the growth of the minimal projective resolution of A.

2.7. The local support of Benson-Iyengar-Krause

Some important technical tools that we will need are the localization and colocalization functors as given in [16, Section 3]. These functors are defined in the setting of compactly generated triangulated categories– in particular, in order to use these functors, we must have access to arbitrary set-indexed coproducts. Localization and colocalization functors are constructed using Brown representability, which originated in homotopy theory [23] and was later generalized by Keller and Neeman [63, 81].

If $\mathbf{T}$ is a compactly generated triangulated category, then a functor $L : \mathbf{T} \to \mathbf{T}$ is called a *localization functor* if there exists a natural transformation $\eta : \mathrm{Id}_{\mathbf{T}} \to L$ such that the natural transformation $L\eta : L \to L^2$ is a natural isomorphism, and $L\eta = \eta L$. A functor $\Gamma : \mathbf{T} \to \mathbf{T}$ is a *colocalization functor* if its opposite functor $\mathbf{T}^{\mathrm{op}} \to \mathbf{T}^{\mathrm{op}}$ is a localization functor. In this case, there is a natural transformation $\Theta : \Gamma \to \mathrm{Id}_{\mathbf{T}}$ such that the natural transformation $\Theta\Gamma : \Gamma \to \Gamma^2$ is a natural isomorphism, and $\Theta\Gamma = \Gamma\Theta$. Localization functors all arise from localizations (justifying the terminology) [17, Lemma 2.14]: for any localization functor $L : \mathbf{T} \to \mathbf{T}$, there exist an adjoint pair (F, G) of functors $F : \mathbf{T} \to \mathbf{T}'$, $G : \mathbf{T}' \to \mathbf{T}$ such that $L = GF$, η is the adjunction morphism, and F induces an equivalence $\mathbf{T}' \cong \mathbf{T}[S^{-1}]$ for some class of morphisms S. Here $\mathbf{T}[S^{-1}]$ is the category obtained by inverting the elements of S, see [82, Chapter 2].

The following theorem, as stated in [21, Theorem 3.1.1, Lemma 3.1.2], summarizes the particular results which we will employ.

Theorem 2.7.1. *Let $\mathbf{T}$ be a compactly generated triangulated category, $\mathbf{C}$ be a thick sub-*

category of $\mathbf{T}^c$ *and* M *an object of* $\mathbf{T}$.

(a) *There exists a functorial triangle in* $\mathbf{T}$,

$$\Gamma_{\mathbf{C}}(M) \to M \to L_{\mathbf{C}}(M) \to$$

which is unique up to isomorphism, such that $\Gamma_{\mathbf{C}}(M)$ *is in* $\mathrm{Loc}(\mathbf{C})$ *and there are no non-zero maps in* $\mathbf{T}$ *from* $\mathbf{C}$ *or, equivalently, from* $\mathrm{Loc}(\mathbf{C})$ *to* $L_{\mathbf{C}}(M)$.

(b) $M \in \mathrm{Loc}(\mathbf{C})$ *if and only if* $\Gamma_{\mathbf{C}}(M) \cong M$.

Using localization and colocalization functors, a version of support theory is constructed in [16] for a compactly generated coherent monoidal triangulated category $\mathbf{K}$. This theory of support builds on the original work on supports for infinite-dimensional representations of finite groups that was given in [14]. In fact, the authors of [16] work in a slightly more general setting: rather than assuming a coherent monoidal triangulated structure, they assume only that there exists a ring homomorphism from some graded-commutative Noetherian ring R to the graded center of $\mathbf{K}$. In other words, they assume that there exists a graded-commutative Noetherian ring R which acts on each $\mathrm{Hom}^\bullet_{\mathbf{K}}(A, B)$. By the results we have mentioned above in Section 2.5, if $\mathbf{K}$ is a compactly generated coherent monoidal triangulated category satisfying (fg), then taking $R := \mathrm{H}^\bullet(\mathbf{K})$ fits the setting of [16]. For symmetric monoidal triangulated categories, an isomorphic version of support was constructed in [54].

We note also that [16] defines support relative to the homogeneous prime ideal spectrum of ring R, rather than the Proj; in other words, they do not exclude the irrelevant ideal. We use the Proj-based versions of their results in order to be compatible with the tensor triangular geometry approach.

This *local support* is defined in the following way. Given a homogeneous prime ideal $\mathfrak{p}$ in $\operatorname{Proj} \mathrm{H}^\bullet(\mathbf{K})$, they construct a certain pair of localization and colocalization functors $L_\mathfrak{p}$ and $\Gamma_\mathfrak{p}$, and then define the local support to be

$$W^{\mathrm{loc}}_{\mathbf{K}}(A) := \{\mathfrak{p} \in \operatorname{Proj} \mathrm{H}^\bullet(\mathbf{K}) : \Gamma_\mathfrak{p}(A) \neq 0\}.$$

The local cohomological support satisfies the following properties [16, Proposition 5.1, Theorem 5.5, Corollary 6.6].

Theorem 2.7.2. *Let* $\mathbf{K}$ *be a compactly generated triangulated category satisfying (fg).*

(a) *For every distinguished triangle*

$$A \to B \to C \to \Sigma A$$

we have $W^{\mathrm{loc}}_{\mathbf{K}}(A) \subseteq W^{\mathrm{loc}}_{\mathbf{K}}(B) \cup W^{\mathrm{loc}}_{\mathbf{K}}(C)$.

(b) $W^{\mathrm{loc}}_{\mathbf{K}}(A) = W^{\mathrm{loc}}_{\mathbf{K}}(\Sigma A)$.

(c) *If* C *is a compact object of* $\mathbf{K}$*, then* $W^{\mathrm{loc}}_{\mathbf{K}}(C) = W_{\mathbf{K}}(C)$.

(d) *Given any set of objects* $\{A_i\}$*, we have*

$$W^{\mathrm{loc}}_{\mathbf{K}}\left(\coprod_i A_i\right) = \bigcup_i W^{\mathrm{loc}}_{\mathbf{K}}(A_i).$$

The important corollary from these results which we will need is the following.

Corollary 2.7.3. *Let* $\mathbf{C}$ *be a subcategory of* $\mathbf{K}^c$*, and let* $\operatorname{Loc}(\mathbf{C})$ *be the localizing subcategory of* $\mathbf{K}$ *generated by* $\mathbf{C}$*. Then for any* $A \in \operatorname{Loc}(\mathbf{C})$*,*

$$W^{\mathrm{loc}}_{\mathbf{K}}(A) \subseteq \bigcup_{C \in \mathbf{C}} W_{\mathbf{K}}(C).$$

Chapter 3. Noncommutative Balmer Spectra

3.1. Thick ideals

We begin by recalling some terminology for various subcategories of monoidal triangulated categories.

Definition 3.1.1. Let $\mathbf{K}$ be a monoidal triangulated category.

(a) A *thick right* (respectively *left*) *ideal* of $\mathbf{K}$ is a thick subcategory $\mathbf{I}$ such that if $A \in \mathbf{I}$ and $B \in \mathbf{K}$, then $A \otimes B$ (respectively $B \otimes A$) is in $\mathbf{I}$.

(b) A *thick ideal* of $\mathbf{K}$ is a thick subcategory $\mathbf{I}$ such that if $A \in \mathbf{I}$, then so are $B \otimes A$ and $A \otimes B$, for any object $B \in \mathbf{K}$.

We will denote by $\langle \mathcal{S} \rangle$ the thick ideal generated by a collection of objects $\mathcal{S}$ in a monoidal triangulated category $\mathbf{K}$.

The following lemma is the primary tool by which we connect classical noncommutative ring theory to the setting of MΔCs.

Lemma 3.1.2. *For every two collections $\mathcal{M}$ and $\mathcal{N}$ of objects of a monoidal triangulated category $\mathbf{K}$,*

$$\langle \mathcal{M} \rangle \otimes \langle \mathcal{N} \rangle \subseteq \langle \mathcal{M} \otimes \mathbf{K} \otimes \mathcal{N} \rangle. \tag{3.1}$$

Proof. First, we will show that

$$\langle \mathcal{M} \rangle \otimes \mathcal{N} \subseteq \langle \mathcal{M} \otimes \mathbf{K} \otimes \mathcal{N} \rangle. \tag{3.2}$$

Let $\mathbf{I}$ denote the collection of all objects A which have the property that for all $N \in \mathcal{N}, B \in \mathbf{K}$, we have $A \otimes B \otimes N \in \langle \mathcal{M} \otimes \mathbf{K} \otimes \mathcal{N} \rangle$. Note that $\mathcal{M} \subseteq \mathbf{I}$. We claim that $\mathbf{I}$ is a thick ideal.

(1) Suppose that we have a distinguished triangle

$$A \to B \to C \to \Sigma A$$

such that two of A, B, and C are in $\mathbf{I}$. Since the monoidal product is an exact functor, for any $D \in \mathbf{K}, N \in \mathcal{N}$, we have

$$A \otimes D \otimes N \to B \otimes D \otimes N \to C \otimes D \otimes N \to \Sigma A \otimes D \otimes N$$

is a distinguished triangle, and by assumption two out of three of its components are in $\langle \mathcal{M} \otimes \mathbf{K} \otimes \mathcal{N} \rangle$. Since it is an ideal, so is the third. Additionally, suppose that $A \in \mathbf{I}$. Then for any $B \in \mathbf{K}$ and $N \in \mathcal{N}$, $(\Sigma A) \otimes B \otimes N \cong \Sigma(A \otimes B \otimes N)$ is in $\langle \mathcal{M} \otimes \mathbf{K} \otimes \mathcal{N} \rangle$, since $A \otimes B \otimes N$ is in $\mathcal{M} \otimes \mathbf{K} \otimes \mathcal{N}$ by assumption, and so $\Sigma(A) \in \mathbf{I}$. Therefore, $\mathbf{I}$ is a triangulated subcategory.

(2) Let $B \in \mathbf{K}$ and $N \in \mathcal{N}$. Suppose $A = C \oplus D$ is in $\mathbf{I}$. Then $A \otimes B \otimes N \cong (C \otimes B \otimes N) \oplus (D \otimes B \otimes N)$ is in $\langle \mathcal{M} \otimes \mathbf{K} \otimes \mathcal{N} \rangle$; by its thickness, $C \otimes B \otimes N$ and $D \otimes B \otimes N$ are in $\langle \mathcal{M} \otimes \mathbf{K} \otimes \mathcal{N} \rangle$. Hence, C and D are both in $\mathbf{I}$. Therefore, we have that $\mathbf{I}$ is a thick subcategory.

(3) Let $A \in \mathbf{I}$, and let $C \in \mathbf{K}$. Then for any $B \in \mathbf{K}, N \in \mathcal{N}$, $C \otimes A \otimes B \otimes N \in \langle \mathcal{M} \otimes \mathbf{K} \otimes \mathcal{N} \rangle$ by the fact that $\langle \mathcal{M} \otimes \mathbf{K} \otimes \mathcal{N} \rangle$ is an ideal and $A \otimes B \otimes N$ is in it; and $A \otimes C \otimes B \otimes N \in \langle \mathcal{M} \otimes \mathbf{K} \otimes \mathcal{N} \rangle$ by the fact that $A \in \mathbf{I}$ and $C \otimes B$ is an object of $\mathbf{K}$. Therefore, $\mathbf{I}$ is a thick ideal of $\mathbf{K}$.

Since $\mathbf{I}$ is a thick ideal containing $\mathcal{M}$, $\langle \mathcal{M} \rangle \subseteq \mathbf{I}$. From this, we obtain (3.2).

By symmetry, we can obtain

$$\mathcal{M} \otimes \langle \mathcal{N} \rangle \subseteq \langle \mathcal{M} \otimes \mathbf{K} \otimes \mathcal{N} \rangle. \tag{3.3}$$

Then, by an identical argument to (1)-(3) but using instead $\mathbf{I}$ to be the set of morphisms A for which $A \otimes B \otimes N \in \langle \mathcal{M} \otimes \mathbf{K} \otimes \mathcal{N} \rangle$ for all $B \in \mathbf{K}, N \in \langle \mathcal{N} \rangle$. This completes the proof. $\square$

Lemma 3.1.3. *Let* $\mathbf{K}$ *be a monoidal triangulated category, and let* $A \in \mathbf{K}$*. If* A *has a left dual* A^* *(recalling Definition 2.1.3), then* $\langle A \rangle = \langle A^* \rangle$*. Similarly, if* A *has a right dual* ${}^*A$*, then* $\langle A \rangle = \langle {}^*A \rangle$*.*

Proof. By Lemma 2.3.3, if a morphism $A \xrightarrow{f} B$ is a retraction, i.e. there exists a map $B \xrightarrow{g} A$ such that $g \circ f = \mathrm{id}_A$, then A is a direct summand of B. By the definition of a dual A^*, there are evaluation and coevaluation maps

$$\mathrm{ev} : A^* \otimes A \to \mathbb{1},$$

$$\mathrm{coev} : \mathbb{1} \to A \otimes A^*,$$

such that the compositions

$$A \xrightarrow{\mathrm{coev} \otimes \mathrm{id}} A \otimes A^* \otimes A \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} A \tag{3.4}$$

and

$$A^* \xrightarrow{\mathrm{id} \otimes \mathrm{coev}} A^* \otimes A \otimes A^* \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} A^* \tag{3.5}$$

are the identity maps on A and A^*, respectively. Hence A is a direct summand of $A \otimes A^* \otimes A$, and A^* is a direct summand of $A^* \otimes A \otimes A^*$. Hence $A \in \langle A^* \rangle$, and $A^* \in \langle A \rangle$. The argument for ${}^*A$ is similar. $\square$

3.2. The Balmer spectrum

Definition 3.2.1. Let $\mathbf{P}$ be a proper thick ideal of $\mathbf{K}$. Then $\mathbf{P}$ is a *prime ideal* of $\mathbf{K}$ if for every pair of thick ideals $\mathbf{I}$ and $\mathbf{J}$ of $\mathbf{K}$, we have

$$\mathbf{I} \otimes \mathbf{J} \subseteq \mathbf{P} \Rightarrow \quad \mathbf{I} \subseteq \mathbf{P} \text{ or } \mathbf{J} \subseteq \mathbf{P}.$$

Theorem 3.2.2. *Suppose $\mathbf{P}$ is a proper thick ideal of a monoidal triangulated category $\mathbf{K}$. Then the following are equivalent:*

(a) $\mathbf{P}$ *is prime.*

(b) *If A and $B \in \mathbf{K}$, then $A \otimes \mathbf{K} \otimes B \subseteq \mathbf{P}$ implies that either A or B is in $\mathbf{P}$.*

(c) *If $\mathbf{I}$ and $\mathbf{J}$ are thick right ideals of $\mathbf{K}$, then $\mathbf{I} \otimes \mathbf{J} \subseteq \mathbf{P}$ implies that either $\mathbf{I}$ or $\mathbf{J}$ is contained in $\mathbf{P}$.*

(d) *If $\mathbf{I}$ and $\mathbf{J}$ are thick left ideals of $\mathbf{K}$, then $\mathbf{I} \otimes \mathbf{J} \subseteq \mathbf{P}$ implies that either $\mathbf{I}$ or $\mathbf{J}$ is contained in $\mathbf{P}$.*

(e) *If $\mathbf{I}$ and $\mathbf{J}$ are thick ideals of $\mathbf{K}$ which properly contain $\mathbf{P}$, then $\mathbf{I} \otimes \mathbf{J} \not\subseteq \mathbf{P}$.*

Proof. By definition, it is clear that $(c) \Rightarrow (a)$, $(d) \Rightarrow (a)$, and $(a) \Rightarrow (e)$.

(a)⇒(b) Suppose $\mathbf{P}$ is prime and $A \otimes \mathbf{K} \otimes B \subseteq \mathbf{P}$. By Lemma 3.1.2, $\langle A \rangle \otimes \langle B \rangle \subseteq \mathbf{P}$, and thus either $\langle A \rangle$ or $\langle B \rangle \subseteq \mathbf{P}$; therefore, either A or B is in $\mathbf{P}$.

(b)⇒(c) Suppose $\mathbf{I}$ and $\mathbf{J}$ are right thick ideals with $\mathbf{I} \otimes \mathbf{J} \subseteq \mathbf{P}$, and suppose that neither $\mathbf{I}$ nor $\mathbf{J}$ is contained in $\mathbf{P}$. Then there exist objects $A \in \mathbf{I}, B \in \mathbf{J}$ such that neither A nor B is in $\mathbf{P}$. Since $\mathbf{I}$ is a right ideal, $A \otimes \mathbf{K} \subseteq \mathbf{I}$, and therefore $A \otimes \mathbf{K} \otimes B \subseteq \mathbf{P}$. Since neither A nor B is in $\mathbf{P}$, we have proved the contrapositive. The direction (b)⇒(d) is analogous.

(e)⇒(b) Let $A \otimes \mathbf{K} \otimes B \subseteq \mathbf{P}$. Suppose neither A nor B is in $\mathbf{P}$. Then by Lemma

3.1.2, we have

$$\langle \mathbf{P} \cup \{A\}\rangle \otimes \langle \mathbf{P} \cup \{B\}\rangle \subseteq \langle (\mathbf{P} \cup \{A\}) \otimes \mathbf{K} \otimes (\mathbf{P} \cup \{B\})\rangle \subseteq \mathbf{P}.$$

However, both $\langle \mathbf{P} \cup \{A\}\rangle$ and $\langle \mathbf{P} \cup \{B\}\rangle$ are thick ideals properly containing $\mathbf{P}$, thus proving the contrapositive. $\square$

We next give a result which guarantees the existence of prime ideals.

Definition 3.2.3. A collection of objects $\mathcal{M}$ in a monoidal triangulated category $\mathbf{K}$ is called *multiplicative* if it is closed under isomorphism, does not contain the object 0, and for each A and B in $\mathcal{M}$, we also have $A \otimes B \in \mathcal{M}$.

Theorem 3.2.4. *Suppose $\mathcal{M}$ is a multiplicative collection of objects of $\mathbf{K}$ for a monoidal triangulated category $\mathbf{K}$, and suppose $\mathbf{I}$ is a proper thick ideal of $\mathbf{K}$ which intersects $\mathcal{M}$ trivially. If $\mathbf{P}$ is maximal element of the (nonempty) set*

$$X(\mathcal{M}, \mathbf{I}) := \{\mathbf{J} \textit{ a thick ideal of } \mathbf{K} : \mathbf{J} \supseteq \mathbf{I}, \mathbf{J} \cap \mathcal{M} = \varnothing\},$$

then $\mathbf{P}$ is a prime ideal of $\mathbf{K}$. Furthermore, $X(\mathcal{M}, \mathbf{I})$ always contains a maximal element.

Proof. Let $\mathbf{P}$ be a maximal element of $X(\mathcal{M}, \mathbf{I})$. If $\mathbf{I}$ and $\mathbf{J}$ are two thick ideals of $\mathbf{K}$ which properly contain $\mathbf{P}$, then both $\mathbf{I}$ and $\mathbf{J}$ contain some element of $\mathcal{M}$, by maximality of $\mathbf{P}$. Say $A \in \mathbf{I}$ and $B \in \mathbf{J}$, with both A and $B \in \mathcal{M}$. Then $A \otimes B \in \mathbf{I} \otimes \mathbf{J}$, and also $A \otimes B \in \mathcal{M}$ by multiplicativity of $\mathcal{M}$. Since $\mathbf{P}$ intersects $\mathcal{M}$ trivially, $A \otimes B \notin \mathbf{P}$, and so $\mathbf{I} \otimes \mathbf{J} \not\subseteq \mathbf{P}$. By property (e) of Theorem 3.2.2, $\mathbf{P}$ is prime.

For the last statement, note that every chain of ideals in $X(\mathcal{M}, \mathbf{I})$ has an upper bound given by the union of these ideals. By Zorn's Lemma, all sets $X(\mathcal{M}, \mathbf{I})$ have maximal elements. $\square$

Corollary 3.2.5. *For every monoidal triangulated category* $\mathbf{K}$*, every proper thick ideal* $\mathbf{I}$ *is contained in a prime ideal of* $\mathbf{K}$*. In particular, the collection of prime ideals is nonempty.*

Proof. Given any thick ideal $\mathbf{I}$, then by taking $\mathcal{M}$ as the multiplicative set generated by the unit object $\mathbb{1}$, we obtain a prime ideal containing $\mathbf{I}$. Since every monoidal triangulated category has at least one proper thick ideal (namely the ideal generated by 0), this implies that there is at least one prime ideal of $\mathbf{K}$. $\square$

We now put a Zariski-type topology on the collection of prime ideals of a monoidal triangulated category $\mathbf{K}$.

Definition 3.2.6. Let $\mathbf{K}$ be a monoidal triangulated category. The *Balmer spectrum* of $\mathbf{K}$, denoted $\operatorname{Spc}\mathbf{K}$, is the collection of prime ideals of $\mathbf{K}$, with closed sets

$$V_{\mathbf{K}}(\mathcal{S}) = \{\mathbf{P} \in \operatorname{Spc}\mathbf{K} : \mathbf{P} \cap \mathcal{S} = \varnothing\},$$

for any collection $\mathcal{S}$ of objects in $\mathbf{K}$.

In cases when $\mathbf{K}$ is clear by context, we will often denote $V_{\mathbf{K}}(\mathcal{S}) = V(\mathcal{S})$. Note that this is the "opposite" topology one might expect by the analogy with rings.

Lemma 3.2.7. *The Balmer spectrum of a monoidal triangulated category* $\mathbf{K}$ *is a topological space.*

Proof. It is straightforward to check that for any collections of objects $\mathcal{S}_i$, we have $V(0) = \varnothing$, $V(\mathbb{1}) = \operatorname{Spc}\mathbf{K}$, $V(\mathcal{S}_1) \cup V(\mathcal{S}_2) = V(\mathcal{S}_1 \oplus \mathcal{S}_2)$, and $\bigcap_{i\in I}(V(\mathcal{S}_i)) = V\left(\bigcup_{i\in I}\mathcal{S}_i\right)$. $\square$

3.3. Semiprime ideals

Definition 3.3.1. A thick ideal of an monoidal triangulated category will be called *semiprime* if it is an intersection of prime ideals, cf. [26].

Theorem 3.3.2. *Suppose* $\mathbf{Q}$ *is a proper thick ideal of a monoidal triangulated category* $\mathbf{K}$*. Then the following are equivalent:*

(a) $\mathbf{Q}$ *is a semiprime ideal;*

(b) *For all* $A \in \mathbf{K}$*, if* $A \otimes \mathbf{K} \otimes A \subseteq \mathbf{Q}$*, then* $A \in \mathbf{Q}$;

(c) *If* $\mathbf{I}$ *is any thick ideal of* $\mathbf{K}$ *such that* $\mathbf{I} \otimes \mathbf{I} \subseteq \mathbf{Q}$*, then* $\mathbf{I} \subseteq \mathbf{Q}$*;*

(d) *If* $\mathbf{I}$ *is any thick ideal properly containing* $\mathbf{Q}$*, then* $\mathbf{I} \otimes \mathbf{I} \not\subseteq \mathbf{Q}$*;*

(e) *If* $\mathbf{I}$ *is any left thick ideal of* $\mathbf{K}$ *such that* $\mathbf{I} \otimes \mathbf{I} \subseteq \mathbf{Q}$*, then* $\mathbf{I} \subseteq \mathbf{Q}$.

(f) *If* $\mathbf{I}$ *is any right thick ideal of* $\mathbf{K}$ *such that* $\mathbf{I} \otimes \mathbf{I} \subseteq \mathbf{Q}$*, then* $\mathbf{I} \subseteq \mathbf{Q}$.

Proof. (a)$\Rightarrow$(b) Suppose $A \otimes \mathbf{K} \otimes A \subseteq \mathbf{Q}$, and let $\mathbf{Q} = \bigcap_\alpha \mathbf{P}_\alpha$ for prime ideals $\mathbf{P}_\alpha$. Then by Theorem 3.2.2, A is in $\mathbf{P}_\alpha$ for each α, and hence $A \in \mathbf{Q}$.

(b)$\Rightarrow$(e) Let $\mathbf{I}$ be a left thick ideal, and suppose $\mathbf{I} \otimes \mathbf{I} \subseteq \mathbf{Q}$, and $\mathbf{I} \not\subseteq \mathbf{Q}$. Then there is $A \in \mathbf{I}$ with $A \notin \mathbf{Q}$. Hence, since $B \otimes A \in \mathbf{I}$ for each $B \in \mathbf{K}$, we have $\mathbf{K} \otimes A \subseteq \mathbf{I}$, and hence $A \otimes \mathbf{K} \otimes A \subseteq \mathbf{Q}$. Since $A \notin \mathbf{Q}$, we see that $\mathbf{Q}$ does not satisfy property (b). The implication (b)$\Rightarrow$(f) is analogous.

The implications (e)$\Rightarrow$(c) and (f)$\Rightarrow$(c) are clear, as is (c)$\Rightarrow$(d).

(d)$\Rightarrow$(a) Let $\mathbf{Q}$ a proper thick ideal satisfying (d), and let $\mathbf{R}$ be the semiprime ideal defined as the intersection of all prime ideals containing $\mathbf{Q}$; there is at least one such prime ideal by Corollary 3.2.5. We will show that $\mathbf{R} = \mathbf{Q}$; to do this, for an arbitrary object A which is not in $\mathbf{Q}$, we will produce a prime ideal which contains $\mathbf{Q}$ and does not contain A. Denote $A := A_1$. Since $A_1 \notin \mathbf{Q}$, we have $\mathbf{Q}^{(1)} := \langle \mathbf{Q} \cup \{A_1\} \rangle$ properly contains $\mathbf{Q}$. Hence, there is some $A_2 \in \mathbf{Q}^{(1)} \otimes \mathbf{Q}^{(1)}$ with $A_2 \notin \mathbf{Q}$. Continue in this manner, defining $\mathbf{Q}^{(i)} := \langle \mathbf{Q} \cup \{A_i\} \rangle$ and then A_{i+1} as an element of $\mathbf{Q}^{(i)} \otimes \mathbf{Q}^{(i)}$ which is not in $\mathbf{Q}$. Note that for any i,

$$\mathbf{Q}^{(i)} \subseteq \mathbf{Q}^{(i-1)} \otimes \mathbf{Q}^{(i-1)} \subseteq \mathbf{Q}^{(i-1)}.$$

Now consider a maximal element of the set of ideals containing $\mathbf{Q}$ and not containing any of the objects A_i. Call this maximal element $\mathbf{P}$. We will demonstrate that $\mathbf{P}$ is prime. Consider $\mathbf{I}$, $\mathbf{J}$ two ideals properly containing $\mathbf{P}$. Let $A_i \in \mathbf{J}$, $A_j \in \mathbf{J}$, which exist by maximality of $\mathbf{P}$. Without loss of generality, let $i \geq j$. Then note that since $\mathbf{J}$ contains both $\mathbf{Q}$ and A_j, we have

$$\mathbf{J} \supseteq \mathbf{Q}^{(j)} \supseteq \mathbf{Q}^{j+1} \supseteq ... \supseteq \mathbf{Q}^{(i)}.$$

Therefore, A_i is in both $\mathbf{I}$ and $\mathbf{J}$. Then by Lemma 3.1.2, we have

$$A_{i+1} \in \mathbf{Q}^{(i)} \otimes \mathbf{Q}^{(i)} \subseteq \langle (\mathbf{Q} \cup \{A_i\}) \otimes \mathbf{K} \otimes (\mathbf{Q} \cup \{A_i\}) \rangle$$

and

$$A_{j+1} \notin \mathbf{P}.$$

Therefore,

$$A_i \otimes \mathbf{K} \otimes A_i \not\subseteq \mathbf{P},$$

which implies

$$\mathbf{I} \otimes \mathbf{J} \not\subseteq \mathbf{P}.$$

Thus, by Theorem 3.2.2, $\mathbf{P}$ is prime. By construction, it contains $\mathbf{Q}$ and not $A_1 = A$, which completes the proof. □

Proposition 3.3.3. *Let* $\mathbf{K}$ *be a left or right rigid monoidal category, in other words, every object has a left dual, or every object has a right dual. Then every ideal of* $\mathbf{K}$ *is semiprime.*

Proof. Suppose every object of $\mathbf{K}$ has a left dual; the right case is similar. Let $\mathbf{I}$ be a thick ideal of $\mathbf{K}$, and suppose $A \otimes \mathbf{K} \otimes A \subseteq \mathbf{I}$. By Theorem 3.3.2, it is enough to show that $A \in \mathbf{I}$. Since A has a left dual A^*, we know that $A \otimes A^* \otimes A$ is in $\mathbf{I}$. As noted in the proof of Lemma 3.1.3, A is a direct summand of $A \otimes A^* \otimes A$; by the thickness of $\mathbf{I}$, this implies $A \in \mathbf{I}$, and we are done. □

3.4. Completely prime ideals

Definition 3.4.1. A thick ideal $\mathbf{P}$ of a monoidal triangulated category $\mathbf{K}$ will be called *completely prime* when it has the property that for all $A, B \in \mathbf{K}$:

$$A \otimes B \in \mathbf{P} \quad \Rightarrow \quad A \in \mathbf{P} \text{ or } B \in \mathbf{P}.$$

It is clear that every completely prime ideal is prime, by Theorem 3.2.2.

Theorem 3.4.2. *For every monoidal triangulated category* $\mathbf{K}$*, the following are equivalent:*

(a) *The map* $V : \mathbf{K} \to \mathcal{X}(\operatorname{Spc} \mathbf{K})$ *has the tensor product property*

$$V(A \otimes B) = V(A) \cap V(B), \quad \forall A, B \in \mathbf{K}.$$

(b) *Every prime ideal of* $\mathbf{K}$ *is completely prime.*

Proof. (a $\Rightarrow$ b) Let $\mathbf{P} \in \operatorname{Spc} \mathbf{K}$ and $A, B \in \mathbf{K}$ be such that $A \otimes B \in \mathbf{P}$. Then

$$\mathbf{P} \notin V(A \otimes B) = V(A) \cap V(B).$$

Hence, either $\mathbf{P} \notin V(A)$ or $\mathbf{P} \notin V(B)$, and thus, either $A \in \mathbf{P}$ or $B \in \mathbf{P}$.

(b $\Rightarrow$ a) For $A, B \in \mathbf{K}$, we have

$$\begin{aligned}\operatorname{Spc}\mathbf{K}\backslash V(A \otimes B) &= \{\mathbf{P} \in \operatorname{Spc}\mathbf{K} \mid A \otimes B \in \mathbf{P}\}\\ &= \{\mathbf{P} \in \operatorname{Spc}\mathbf{K} \mid A \in \mathbf{P}\} \cup \{\mathbf{P} \in \operatorname{Spc}\mathbf{K} \mid B \in \mathbf{P}\}\\ &= (\operatorname{Spc}\mathbf{K}\backslash V(A)) \cup (\operatorname{Spc}\mathbf{K}\backslash V(B)).\end{aligned}$$

Thus $V(A \otimes B) = V(A) \cap V(B)$. □

Theorem 3.4.3. *Let* $\mathbf{K}$ *be a monoidal triangulated category in which every thick right ideal is two-sided. Then every prime ideal of* $\mathbf{K}$ *is completely prime, and as a consequence, the map* $V : \mathbf{K} \to \mathcal{X}(\operatorname{Spc}\mathbf{K})$ *has the tensor product property*

$$V(A \otimes B) = V(A) \cap V(B), \quad \forall A, B \in \mathbf{K}.$$

Proof. First we claim that

$$\langle M \rangle_{\mathrm{r}} = \langle M \rangle, \quad \forall M \in \mathbf{K}. \tag{3.6}$$

The inclusion $\langle M \rangle_{\mathrm{r}} \subseteq \langle M \rangle$ is obvious. The reverse inclusion is proved as follows. The hypothesis states that $\langle M \rangle_{\mathrm{r}}$ is a a two-sided thick ideal and, in particular, it contains $\langle N \rangle$ for all $N \in \langle M \rangle_{\mathrm{r}}$. Applying this for $N = M$ yields $\langle M \rangle_{\mathrm{r}} \supseteq \langle M \rangle$.

Let $\mathbf{P} \in \operatorname{Spc}\mathbf{K}$ and $A, B \in \mathbf{K}$ be such that $A \otimes B \in \mathbf{P}$. Therefore $A \otimes \langle B \rangle_{\mathrm{r}} \subseteq \mathbf{P}$ and, by (3.6), $A \otimes \langle B \rangle \subseteq \mathbf{P}$. This implies that $A \otimes C \otimes B \in \mathbf{P}$ for all $C \in \mathbf{K}$ and, by the primeness of $\mathbf{P}$, $A \in \mathbf{P}$ or $B \in \mathbf{P}$. Therefore, the thick ideal $\mathbf{P}$ is completely prime. The second statement follows from the first and Theorem 3.4.2. □

Given a monoidal tensor category where every object is either left or right dualizable, one can now show that the existence of a nilpotent element insures that the map V associated to the Balmer spectrum does not satisfy the tensor product property.

Theorem 3.4.4. *Let* $\mathbf{K}$ *be a monoidal triangulated category in which every object is either left or right dualizable. If* $\mathbf{K}$ *has a non-zero nilpotent object* M *(i.e.,* $M \ncong 0$ *but* $M^{\otimes n} := M \otimes \cdots \otimes M \cong 0$*, for some* $n > 0$*) then not all prime ideals of* $\mathbf{K}$ *are completely prime. As a consequence, the map* $V : \mathbf{K} \to \mathcal{X}(\operatorname{Spc} \mathbf{K})$ *does not have the tensor product property.*

Proof. By Proposition 3.3.3, $\langle 0 \rangle$ is a semiprime ideal of $\mathbf{K}$. Hence, the prime radical of $\mathbf{K}$ equals $\langle 0 \rangle$.

On the other hand M lies in all completely prime ideals $\mathbf{P}$ of $\mathbf{K}$ because $M^{\otimes n} \cong 0 \in \mathbf{P}$. If all prime ideals of $\mathbf{K}$ are completely prime, this would imply that M belongs to the prime radical of $\mathbf{K}$ (i.e. $M \in \langle 0 \rangle$), which is a contradiction. $\square$

The following corollary follows from Theorem 3.4.4, because all objects of $\mathsf{stmod}(H)$ are rigid for finite dimensional Hopf algebras H.

Corollary 3.4.5. *Assume that* H *is a finite dimensional Hopf algebra which admits a non-projective finite dimensional module* M *such that* $M^{\otimes n}$ *is projective. Then not all prime ideals of the stable module category* $\mathsf{stmod}(H)$ *are completely prime, i.e., the map* $V : \mathbf{K} \to \mathcal{X}(\operatorname{Spc}(\mathsf{stmod}(H)))$ *does not have the tensor product property.*

The following corollary of Theorem 3.4.4 is of independent interest.

Corollary 3.4.6. *If* $\mathbf{K}$ *is a monoidal triangulated category in which every object is either left or right dualizable and* $\mathbf{K}$ *has objects* A *and* B*, such that* $A \otimes B \cong 0$ *but* $B \otimes A \ncong 0$*, then not all prime ideals of* $\mathbf{K}$ *are completely prime, i.e., the map* $V : \mathbf{K} \to \mathcal{X}(\operatorname{Spc} \mathbf{K})$ *does not have the tensor product property.*

This follows from Theorem 3.4.4, because $M := B \otimes A$ is not the zero object in $\mathbf{K}$, but $M \otimes M \cong B \otimes (A \otimes B) \otimes A \cong 0$.

Chapter 4. Classification of Thick Ideals and Balmer Spectra

4.1. Axiomatic support data: from categories to geometry

Throughout this chapter, let X be a topological space. Let $\mathcal{X}$ denote the collection of all subsets of X, $\mathcal{X}_{cl}$ the collection of all closed subsets of X, and $\mathcal{X}_{sp}$ the collection of all specialization closed subsets of X, that is, arbitrary unions of closed sets.

When it is necessary to emphasize the underlying topological space X, we will use the notation $\mathcal{X}_{cl}(X)$ and $\mathcal{X}_{sp}(X)$.

Definition 4.1.1. Let $\mathbf{K}$ be a monoidal triangulated category and σ a map $\mathbf{K} \to \mathcal{X}$. We will say that σ is a (noncommutative) *support datum* if the following hold:

(a) $\sigma(0) = \varnothing$ and $\sigma(\mathbb{1}) = X$;

(b) $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$, $\forall A, B \in \mathbf{K}$;

(c) $\sigma(\Sigma A) = \sigma(A)$, $\forall A \in \mathbf{K}$;

(d) If $A \to B \to C \to \Sigma A$ is a distinguished triangle, then $\sigma(A) \subseteq \sigma(B) \cup \sigma(C)$;

(e) $\bigcup_{C \in \mathbf{K}} \sigma(A \otimes C \otimes B) = \sigma(A) \cap \sigma(B)$, $\forall A, B \in \mathbf{K}$.

It follows from conditions (a) and (d) that σ is constant along the isomorphism classes of objects of $\mathbf{K}$. The same will be true for all other notions of support datum that we consider in this paper. Recall the map $V := V_{\mathbf{K}}$ defined in Definition 3.2.6. The restriction of V to objects of $\mathbf{K}$ will be referred to as the *Balmer support*.

Lemma 4.1.2. *For any $M\Delta C$, $\mathbf{K}$, the Balmer support V is a support datum $\mathbf{K} \to \mathcal{X}_{cl}(\operatorname{Spc}(\mathbf{K}))$.*

Proof. By the definition of the Zariski topology, $V(A)$ is closed for any A. We verify the properties (a)-(e) in Definition 4.1.1 below.

(a) $V(0) = \varnothing$ because 0 is in every prime ideal of $\mathbf{K}$. Since prime ideals are required to be proper, $\mathbb{1}$ is in no prime ideal, and hence $V(\mathbb{1}) = \mathrm{Spc}(\mathbf{K})$.

(b) Prime ideals are closed under sums and summands. Hence, if $\mathbf{P}$ is a prime ideal, then $A \oplus B \in \mathbf{P}$ if and only if both A and B are in $\mathbf{P}$.

(c) Prime ideals are closed under shifts; hence, A is in a prime ideal $\mathbf{P}$ if and only if ΣA is in $\mathbf{P}$.

(d) Since prime ideals are triangulated, if

$$A \to B \to C \to \Sigma A$$

is a distinguished triangle with $\mathbf{P} \in V(A)$, then $A \notin \mathbf{P}$ and hence one of B or C be not be in $\mathbf{P}$. Therefore, $\mathbf{P}$ is in $V(B)$ or $V(C)$.

(e) First, we will show $\subseteq$. Suppose $\mathbf{P}$ is in some $V(A \otimes C \otimes B)$ for some C; in other words, $A \otimes C \otimes B \notin \mathbf{P}$. Then since $\mathbf{P}$ is a thick ideal, neither A nor B can be in $\mathbf{P}$, and hence $\mathbf{P} \in V(A)$ and $V(B)$. For $\supseteq$, suppose $\mathbf{P} \in V(A) \cap V(B)$. Then by the primeness condition, $A \otimes \mathbf{K} \otimes B \nsubseteq \mathbf{P}$, since that would imply either A or B would be in $\mathbf{P}$. Hence, there is some C with $A \otimes C \otimes B \notin \mathbf{P}$, and so $\mathbf{P} \in V(A \otimes C \otimes B)$ for some choice of C. $\square$

Recall the cohomological support $W_{\mathbf{K}}$ of a coherent monoidal triangulated category $\mathbf{K}$ constructed in Section 2.6. This is a map from objects of $\mathbf{K}$ to closed sets in $\operatorname{Proj} \mathrm{H}^{\bullet}(\mathbf{K})$. By Proposition 2.6.3, the cohomological support satisfies conditions (a)-(d) for support data; however, in general, it satisfies a weaker version of (e). However, in many specific cases (for instance, in cases where the cohomological support satisfies the tensor product property), $W_{\mathbf{K}}$ does satisfy (e).

We will proceed by proving that the Balmer spectrum has a universal property in the category of support data. To accomplish this, we first prove a lemma that shows that if we have continuous maps from X to $\operatorname{Spc}\mathbf{K}$ that whose inverse images agree on closed sets then the maps must be equal.

Lemma 4.1.3. *Let X be a set and $f_1, f_2 : X \to \operatorname{Spc}\mathbf{K}$ be two maps such that $f_1^{-1}(V(A)) = f_2^{-1}(V(A))$ for all objects A of $\mathbf{K}$. Then $f_1 = f_2$.*

Proof. By assumption, for all $A \in \mathbf{K}$ and $x \in X$, $f_1(x) \in V(A) \Leftrightarrow f_2(x) \in V(A)$. Hence, for all $x \in X$,

$$\bigcap_{A\in\mathbf{K}, f_1(x)\in V(A)} V(A) = \bigcap_{A\in\mathbf{K}, f_2(x)\in V(A)} V(A),$$

and thus,

$$\begin{aligned} V(\mathbf{K}\backslash f_1(x)) = \overline{\{f_1(x)\}} &= \bigcap_{A\in\mathbf{K}, f_1(x)\in V(A)} V(A) = \bigcap_{A\in\mathbf{K}, f_2(x)\in V(A)} V(A) \\ &= \overline{\{f_2(A)\}} = V(\mathbf{K}\backslash f_2(x)). \end{aligned}$$

Since $f_1(x) \in V(\mathbf{K}\backslash f_1(x))$, the above equality implies that $f_1(x) \in V(\mathbf{K}\backslash f_2(x))$. Therefore, $f_1(x) \subseteq f_2(x)$, and analogously, $f_2(x) \subseteq f_1(x)$. Hence, $f_1 = f_2$. $\square$

With the prior results we can show that there exists a final support datum.

Theorem 4.1.4. *Let $\mathbf{K}$ be a monoidal triangulated category. In the collection of support data σ for $\mathbf{K}$ such that $\sigma(A)$ is closed for each object A, the support V is the final support object: that is, given any other support datum σ as above, there is a unique continuous map $f_\sigma : X \to \operatorname{Spc}\mathbf{K}$ satisfying $\sigma(A) = f_\sigma^{-1}(V(A))$. Explicitly, this map is defined by*

$$f_\sigma(x) = \{A \in \mathbf{K} : x \notin \sigma(A)\}.$$

Proof. The uniqueness of this map follows directly from Lemma 4.1.3. We need to show that the formula given for $f_\sigma(x)$ defines a prime ideal, and that $\sigma(A) = f_\sigma^{-1}(V(A))$, which will then imply that f is continuous.

The subset $f_\sigma(x)$ satisfies the two-out-of-three condition, since if

$$A \to B \to C \to \Sigma A$$

is a distinguished triangle with B and C in $f_\sigma(x)$, this means that x is not in $\sigma(A)$ or $\sigma(B)$, and by condition (d) of support data that implies that $x \notin \sigma(A)$, and so $A \in f_\sigma(x)$. Additionally, $A \in f_\sigma(x)$ if and only if $x \notin \sigma(A)$, which, by condition (c) for support data, happens if and only if $x \notin \sigma(\Sigma A)$, i.e. $\Sigma A \in f_\sigma(x)$. Therefore, $f_\sigma(x)$ is closed under shifts, and so it is triangulated.

The triangulated subcategory $f_\sigma(x)$ is also thick, because if $A \oplus B \in f_\sigma(x)$ then x is not in $\sigma(A \oplus B)$, and by condition (b) of support data x is not in $\sigma(A)$ or $\sigma(B)$. Therefore, A and B are in $f_\sigma(x)$.

Next, we will observe that $f_\sigma(x)$ is a (two-sided) ideal. Suppose that $A \in f_\sigma(x)$. Then $x \notin \sigma(A)$. For any B, since by condition (e) for support data

$$\sigma(A \otimes B) \subseteq \sigma(A) \cap \sigma(B),$$

we have $x \notin \sigma(A \otimes B)$, and therefore $A \otimes B \in f_\sigma(x)$. The same argument shows that $B \otimes A \in f_\sigma(x)$ as well.

Lastly, we verify that $f_\sigma(x)$ is prime. Suppose $A \otimes \mathbf{K} \otimes B \subseteq f_\sigma(x)$. Then for all objects C, $x \notin V(A \otimes C \otimes B)$. Hence, by condition (e) of being a support datum, $x \notin \sigma(A) \cap \sigma(B)$, implying that it is not in $\sigma(A)$ or $\sigma(B)$. Therefore, either A or B is in $f_\sigma(x)$.

Lastly, we just verify the formula $\sigma(A) = f_\sigma^{-1}(V(A))$. We have

$$x \in f_\sigma^{-1}(V(A)) \Leftrightarrow f_\sigma(x) = \{B : x \notin \sigma(B)\} \in V(A),$$

$$\Leftrightarrow A \notin \{B : x \notin \sigma(B)\}$$

$$\Leftrightarrow x \in \sigma(A).$$

This completes the proof. $\square$

For any map $\sigma : \mathbf{K} \to \mathcal{X}$ with a topological space X, we associate a map Φ_σ from subsets of $\mathbf{K}$ to $\mathcal{X}$ given by

$$\Phi_\sigma(\mathcal{S}) := \bigcup_{A \in \mathcal{S}} \sigma(A). \tag{4.1}$$

By definition, the map Φ_σ is order preserving with respect to the inclusion partial order.

If $\sigma : \mathbf{K} \to \mathcal{X}_{cl}$ is a support datum, then $\Phi_\sigma(\mathcal{S})$ is a specialization-closed subset of X for every $\mathcal{S} \subseteq \mathbf{K}$. We can now prove that the map Φ_σ respects the tensor product property on ideals.

Lemma 4.1.5. *Let $\mathbf{K}$ be an $M\Delta C$ and $\sigma : \mathbf{K} \to \mathcal{X}_{cl}$ be a support datum. Then*

$$\Phi_\sigma(\mathbf{I} \otimes \mathbf{J}) = \Phi_\sigma(\mathbf{I}) \cap \Phi_\sigma(\mathbf{J})$$

for every two thick ideals $\mathbf{I}$ and $\mathbf{J}$ of $\mathbf{K}$, recall (4.1).

Proof. We have

$$\begin{aligned}\Phi_\sigma(\mathbf{I} \otimes \mathbf{J}) &= \bigcup_{A \in \mathbf{I}, B \in \mathbf{J}} \sigma(A \otimes B) = \bigcup_{A \in \mathbf{I}, B \in \mathbf{J}, C \in \mathbf{K}} \sigma(A \otimes C \otimes B) \\ &= \bigcup_{A \in \mathbf{I}, B \in \mathbf{J}} \sigma(A) \cap \sigma(B) = \left(\bigcup_{A \in \mathbf{I}} \sigma(A) \right) \cap \left(\bigcup_{B \in \mathbf{J}} \sigma(B) \right) \\ &= \Phi_\sigma(\mathbf{I}) \cap \Phi_\sigma(\mathbf{J}).\end{aligned}$$

$\square$

Lemma 4.1.6. *Let* $\mathbf{K}$ *be an* $M\Delta C$ *and* $\sigma : \mathbf{K} \to \mathcal{X}$ *be a support datum. For any subset* $\mathcal{S}$ *of* $\mathbf{K}$, $\Phi_\sigma(\mathcal{S}) = \Phi_\sigma(\langle \mathcal{S} \rangle)$.

Proof. We will check that by adjoining direct summands, shifts, cones, and tensor products to $\mathcal{S}$ one does not alter $\Phi_\sigma(\mathcal{S})$; this will prove the statement.

Let $M \oplus N \in \mathcal{S}$. By condition (b) of support data, $\sigma(M) \subseteq \sigma(M \oplus N)$, so adjoining each M to $\mathcal{S}$ does not change $\bigcup_{A \in \mathcal{S}} \sigma(A)$.

Let $M \in \mathcal{S}$. Then, by condition (c) for support data, $\sigma(\Sigma^m M) = \sigma(M)$, so adjoining shifts to $\mathcal{S}$ does not alter $\Phi_\sigma(\mathcal{S})$ either.

If $A \to B \to C \to \Sigma A$ is a distinguished triangle with B and C in $\mathcal{S}$ then $\sigma(A) \subseteq \sigma(B) \cup \sigma(C)$ by condition (d) for support data, so adding A to $\mathcal{S}$ does not change $\Phi_\sigma(\mathcal{S})$.

Lastly, if $M \in \mathcal{S}$, then by condition (5) for support data we have $\sigma(M \otimes N) \subseteq \sigma(M) \cap \sigma(N) \subseteq \sigma(M)$. Hence, we can add $M \otimes N$ to $\mathcal{S}$ without affecting $\Phi_\sigma(\mathcal{S})$. Likewise for $N \otimes M$.

Therefore, closing $\mathcal{S}$ under summands, shifts, cones, and tensor product with arbitrary objects of $\mathbf{K}$ does not alter Φ_σ, which proves the lemma. □

The following theorem summarizes our results for support datum.

Theorem 4.1.7. *For an* $M\Delta C$, $\mathbf{K}$, *and a support datum* $\sigma : \mathbf{K} \to \mathcal{X}_{cl}$, *the map* Φ_σ *is a morphism of ordered monoids from the set of thick ideals of* $\mathbf{K}$ *with the operation* $\mathbf{I}, \mathbf{J} \mapsto \langle \mathbf{I} \otimes \mathbf{J} \rangle$ *and the inclusion partial order to* $\mathcal{X}_{sp}$ *with the operation of intersection and the inclusion partial order.*

Proof. Clearly, Φ_σ preserves inclusions. For every two thick ideals ideals $\mathbf{I}$ and $\mathbf{J}$ of $\mathbf{K}$,

$$\Phi_\sigma(\langle \mathbf{I} \otimes \mathbf{J} \rangle) = \Phi_\sigma(\mathbf{I}) \cap \Phi_\sigma(\mathbf{J}),$$

which follows from Lemmas 4.1.5 and 4.1.6. $\square$

Definition 4.1.8. Let $\mathbf{K}$ be a monoidal triangulated category and σ a map $\mathbf{K} \to \mathcal{X}$. We will call σ a (noncommutative) *weak support datum* if

(a) $\sigma(0) = \varnothing$ and $\sigma(\mathbb{1}) = X$;

(b) $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$, $\forall A, B \in \mathbf{K}$;

(c) $\sigma(\Sigma A) = \sigma(A)$, $\forall A \in \mathbf{K}$;

(d) If $A \to B \to C \to \Sigma A$ is a distinguished triangle, then $\sigma(A) \subseteq \sigma(B) \cup \sigma(C)$;

(e) $\Phi_\sigma(\mathbf{I} \otimes \mathbf{J}) = \Phi_\sigma(\mathbf{I}) \cap \Phi_\sigma(\mathbf{J})$ for all thick ideals $\mathbf{I}$ and $\mathbf{J}$ of $\mathbf{K}$

(recall (4.1)).

Note that, for any weak support datum $\sigma : \mathbf{K} \to \mathcal{X}$ satisfying the additional condition that each $\Phi_\sigma(\langle A \rangle)$ is closed for any object A, and all thick ideals $\mathbf{I}$ of $\mathbf{K}$,

$$\Phi_\sigma(\mathbf{I}) = \bigcup_{A \in \mathbf{I}} \Phi_\sigma(\langle A \rangle) \in \mathcal{X}_{sp}.$$

By Lemma 4.1.5, a support datum is automatically a weak support datum.

For categories $\mathbf{K}$ containing set-indexed coproducts, we make a minor modification in the definition by replacing (b) with

(b') $\sigma(\bigoplus_{i \in I} A_i) = \bigcup_{i \in I} \sigma(A_i)$, $\forall A_i \in \mathbf{K}$

With this replacement, we will use the term *extended weak support datum*.

The following two lemmas provide information on ideals generated by objects in the category $\mathbf{K}$.

Lemma 4.1.9. *If $\sigma : \mathbf{K} \to \mathcal{X}$ is a weak support datum for an $M\Delta C$, $\mathbf{K}$, and $\mathbf{I}$ and $\mathbf{J}$ are thick ideals of $\mathbf{K}$, then*

$$\Phi_\sigma(\langle \mathbf{I} \otimes \mathbf{J} \rangle) = \Phi_\sigma(\mathbf{I} \otimes \mathbf{J}) = \Phi_\sigma(\mathbf{I}) \cap \Phi_\sigma(\mathbf{J}).$$

Proof. By assumption, $\Phi_\sigma(\mathbf{I} \otimes \mathbf{J}) = \Phi_\sigma(\mathbf{I}) \cap \Phi_\sigma(\mathbf{J})$. Since every element of the set $\mathbf{I} \otimes \mathbf{J}$ is in $\mathbf{I}$, and in $\mathbf{J}$, we have $\langle \mathbf{I} \otimes \mathbf{J} \rangle \subseteq \mathbf{I} \cap \mathbf{J}$. Hence $\Phi_\sigma(\langle \mathbf{I} \otimes \mathbf{J} \rangle) \subseteq \Phi_\sigma(\mathbf{I}) \cap \Phi_\sigma(\mathbf{J})$. It is also automatic that $\Phi_\sigma(\mathbf{I} \otimes \mathbf{J}) \subseteq \Phi_\sigma(\langle \mathbf{I} \otimes \mathbf{J} \rangle)$. Hence, we have the commutative diagram

$$\begin{tikzcd}
\Phi_\sigma(\mathbf{I} \otimes \mathbf{J}) \arrow[rr, leftrightarrow, "="'] \arrow[dr, hook] & & \Phi_\sigma(\mathbf{I}) \cap \Phi_\sigma(\mathbf{J}) \\
& \Phi_\sigma(\langle \mathbf{I} \otimes \mathbf{J} \rangle) \arrow[ur, hook] &
\end{tikzcd}$$

which gives the statement of the lemma. □

Lemma 4.1.10. *Suppose $A \to B \to C \to \Sigma A$ is a distinguished triangle in an $M\Delta C$, $\mathbf{K}$, and σ a weak support datum. Then $\Phi_\sigma(\langle A \rangle) \subseteq \Phi_\sigma(\langle B \rangle) \cup \Phi_\sigma(\langle C \rangle)$.*

Proof. Define

$$\mathbf{I} = \{M \in \langle A \rangle : \Phi_\sigma(\mathbf{K} \otimes M \otimes \mathbf{K}) \subseteq \Phi_\sigma(\langle B \rangle) \cup \Phi_\sigma(\langle C \rangle)\}.$$

We will show that $\mathbf{I}$ is a thick ideal which contains A; since it is contained in $\langle A \rangle$ by definition, it is therefore equal to $\langle A \rangle$.

Suppose M is in $\mathbf{I}$, and let X and Y two arbitrary objects of $\mathbf{K}$. We have $X \otimes \Sigma M \otimes Y \cong \Sigma(\Sigma^{-1}(X) \otimes M \otimes \Sigma^{-1}(Y))$, and hence $\sigma(X \otimes \Sigma M \otimes Y) = \sigma(\Sigma^{-1}(X) \otimes M \otimes \Sigma^{-1}(Y)) \subseteq \Phi_\sigma(\langle B \rangle) \cup \Phi_\sigma(\langle C \rangle)$, showing that $\Sigma(M) \in \mathbf{I}$.

Let $K \to L \to M \to \Sigma K$ be a distinguished triangle with L and M in $\mathbf{I}$. Then by the exactness of the tensor product, $X \otimes K \otimes Y \to X \otimes L \otimes Y \to X \otimes M \otimes Y \to X \otimes \Sigma K \otimes Y$ is a distinguished triangle. Hence,

$$\sigma(X \otimes K \otimes Y) \subseteq \sigma(X \otimes L \otimes Y) \cup \sigma(X \otimes M \otimes Y) \subseteq \Phi_\sigma(\langle B \rangle) \cup \Phi_\sigma(\langle C \rangle).$$

Therefore, K is in $\mathbf{I}$.

Suppose $M \oplus N$ is in $\mathbf{I}$. Then $\sigma(X \otimes M \otimes Y) \subseteq \sigma(X \otimes (M \oplus N) \otimes Y) \subseteq \Phi_\sigma(\langle B \rangle) \cup \Phi_\sigma(\langle C \rangle)$, and so M is in $\mathbf{I}$ (and likewise, so is N).

It is clear from the definition of $\mathbf{I}$ that it is closed under tensoring on the right and left. By exactness of the tensor product, $\mathbf{I}$ contains A. Thus, $\mathbf{I}$ is a thick subideal of $\langle A \rangle$ which contains A, and hence, $\mathbf{I} = \langle A \rangle$. □

Using the final weak support data one can identify the Balmer spectrum for $\mathbf{K}$.

Theorem 4.1.11. *Suppose that* $\mathbf{K}$ *is an* $M\Delta C$ *and* $\sigma : \mathbf{K} \to \mathcal{X}$ *is a weak support datum satisfying the additional condition that* $\Phi_\sigma(\langle A \rangle)$ *is closed for every object* A *of* $\mathbf{K}$. *Then there is a unique continuous map* $f_\sigma : X \to \operatorname{Spc} \mathbf{K}$ *satisfying* $\Phi_\sigma(\langle A \rangle) = f_\sigma^{-1}(V(A))$, *for all* $A \in \mathbf{K}$. *Explicitly, this map is defined by*

$$f_\sigma(x) = \{A \in \mathbf{K} : x \notin \Phi_\sigma(\langle A \rangle)\} \quad \textit{for} \quad x \in X.$$

Proof. The uniqueness of this map follows directly from Lemma 4.1.3. The continuity will follow from the claimed formula for $f_\sigma^{-1}(V(A))$, since $\Phi_\sigma(\langle A \rangle)$ is closed by definition. We need to verify that $f_\sigma(x)$ is a prime ideal, and that $f_\sigma^{-1}(V(A))$ has the formula that has been claimed.

Since $\langle M \rangle = \langle \Sigma M \rangle$, we clearly have M in $f_\sigma(x)$ if and only if ΣM in $f_\sigma(x)$. If $A \to B \to C \to \Sigma A$ is a distinguished triangle with B and C in $f_\sigma(x)$, then by Lemma 4.1.10 we have A in $f_\sigma(x)$. If $M \oplus N$ is in $f_\sigma(x)$, then since $M \in \langle M \oplus N \rangle$, we have $\Phi_\sigma(\langle M \rangle) \subseteq \Phi_\sigma(\langle M \oplus N \rangle)$, and so $M \in f_\sigma(x)$ (and likewise for N). Suppose $M \in f_\sigma(x)$ and N is any object. Then since $\langle M \otimes N \rangle \subseteq \langle M \rangle$, we have $\Phi_\sigma(\langle M \otimes N \rangle) \subseteq \Phi_\sigma(\langle M \rangle)$ and hence $M \otimes N \in f_\sigma(x)$ (and likewise for $N \otimes M$). Hence, $f_\sigma(x)$ is a thick ideal.

Now, suppose that there are thick ideals $\mathbf{I}$ and $\mathbf{J}$ with $\mathbf{I} \otimes \mathbf{J} \subseteq f_\sigma(x)$. Then for each

$X \in \mathbf{I}$ and $Y \in \mathbf{J}$, $x \notin \Phi_\sigma(\langle X \otimes Y \rangle)$. But now we can observe that

$$\bigcup_{X \in \mathbf{I}, Y \in \mathbf{J}} \Phi_\sigma(\langle X \otimes Y \rangle) \supseteq \Phi_\sigma(\mathbf{I} \otimes \mathbf{J}) = \Phi_\sigma(\mathbf{I}) \cap \Phi_\sigma(\mathbf{J}),$$

and therefore $x \notin \Phi_\sigma(\mathbf{I}) \cap \Phi_\sigma(\mathbf{J})$. Therefore, one of $\mathbf{I}$ and $\mathbf{J}$ must be in $f_\sigma(x)$. Therefore, $f_\sigma(x)$ is a prime ideal. Last, we verify that $f_\sigma^{-1}(V(A)) = \Phi_\sigma(\langle A \rangle)$:

$$\begin{aligned} f_\sigma^{-1}(V(A)) &= \{x : f_\sigma(x) \in V(A)\} = \{x : A \notin f_\sigma(x)\} \\ &= \{x : x \in \Phi_\sigma(\langle A \rangle)\} = \Phi_\sigma(\langle A \rangle). \end{aligned}$$

□

4.2. A noncommutative Hopkins' Theorem

In this section we prove a generalization of Hopkins' Theorem which will be used in next section for our first approach to the explicit description of the (noncommutative) Balmer spectrum of a M∆C as a topological space.

For the remainder of this chapter, we will assume that $\mathbf{K}$ is a compactly-generated M∆C, recall Definition 2.3.6. Recall that this implies that the compact part $\mathbf{K}^c$ of $\mathbf{K}$ is itself a M∆C. Our goal for the remainder of the chapter is giving a classification theorem for the Balmer spectrum of $\mathbf{K}^c$. The reason for assuming that $\mathbf{K}^c$ is the compact part of a compactly-generated monoidal triangulated category is so that we can employ the tools of localization and colocalization functors, recall Section 2.7. In this section, we will use these functors to prove a noncommutative version of Hopkins' Theorem, which is one of the primary tools used in our classification theorem in the following section.

Recall that for an M∆C, $\mathbf{K}$, and a map $\sigma : \mathbf{K} \to \mathcal{X}$, the map Φ_σ from subsets of objects of $\mathbf{K}$ to $\mathcal{X}$ is defined by (4.1). In this context, given any subset $\mathcal{S}$ of $\mathbf{K}^c$, the notation $\langle \mathcal{S} \rangle$ will refer to the thick two-sided ideal of $\mathbf{K}^c$ generated by $\mathcal{S}$, whereas if $\mathcal{S}$ is any subset of $\mathbf{K}$, then the notation $\langle\langle \mathcal{S} \rangle\rangle$ will refer to the thick two-sided ideal of $\mathbf{K}$ generated by $\mathcal{S}$.

Recall that for an MΔC, $\mathbf{K}$, and a map $\sigma : \mathbf{K} \to \mathcal{X}$, the map Φ_σ from subsets of objects of $\mathbf{K}$ to $\mathcal{X}$ is defined by (4.1). At many points in this section and the sequel we will be interested in weak support data which satisfy the following two conditions:

$$\Phi_\sigma(\langle\langle M \rangle\rangle) = \varnothing \text{ if and only if } M = 0,\ \forall M \in \mathbf{K} \text{ (Faithfulness Property);} \tag{4.2}$$

$$\text{For any } W \in \mathcal{X}_{cl},\ \exists M \in \mathbf{K}^c \text{ such that } \Phi_\sigma(\langle M \rangle) = W \text{ (Realization Property).} \tag{4.3}$$

The following result is a generalization of the theorem presented in [21, Theorem 3.3.1].

Theorem 4.2.1. *Let $\mathbf{K}$ be a compactly generated MΔC and $\sigma : \mathbf{K} \to \mathcal{X}$ an extended weak support datum satisfying the Faithfulness Property (4.2) for a Zariski space X.*

Fix an object $M \in \mathbf{K}^c$, and set $Y := \Phi_\sigma(\langle M \rangle)$ (defined in (4.1)). Let $\mathbf{I}_Y = \{N \in \mathbf{K}^c : \Phi_\sigma(\langle N \rangle) \subseteq Y\}$. Then

$$\mathbf{I}_Y = \langle M \rangle.$$

Proof. Let $\mathbf{I} = \mathbf{I}_Y$ and $\mathbf{I}' = \langle M \rangle$. By definition $\mathbf{I}' = \langle M \rangle$ is the smallest thick (two-sided) tensor ideal of $\mathbf{K}^c$ containing M, so it follows that $\mathbf{I} \supseteq \mathbf{I}'$.

For the other containment, let $N \in \mathbf{I}$. Using the localization and colocalization functors associated to $\mathbf{I}'$, we obtain a distinguished triangle:

$$\Gamma_{\mathbf{I}'} N \to N \to L_{\mathbf{I}'} N \to \tag{4.4}$$

Using the fact that σ is compatible with arbitrary set-indexed coproducts, one can conclude that $\sigma(L_{\mathbf{I}'}(N)) \subseteq Y$, since (i) the first term belongs to $\mathrm{Loc}(\mathbf{I}') \subseteq \mathrm{Loc}(\mathbf{I})$ (since $\mathbf{I}' \subseteq \mathbf{I}$) and (ii) N belongs to $\mathbf{I}$.

According to Theorem 2.7.1 there are no non-zero maps from $\mathbf{I}'$ to $L_{\mathbf{I}'}(N)$. Consequently, for any $S, Q \in \mathbf{K}^c$, one can use the duality adjunctions Theorem 2.1.4 to show that

$$0 = \mathrm{Hom}_{\mathbf{K}}(S \otimes M \otimes Q, L_{\mathbf{I}'}(N)) \cong \mathrm{Hom}_{\mathbf{K}}(S, L_{\mathbf{I}'}(N) \otimes Q^* \otimes M^*). \tag{4.5}$$

Since $\mathbf{K}$ is compactly generated it follows that $L_{\mathbf{I}'}(N) \otimes Q^* \otimes M^* = 0$ in $\mathbf{K}$. Hence $L_{\mathbf{I}'}(N) \otimes \mathbf{K}^c \otimes M^* = 0$, and since one can find a set of compact objects $\mathcal{C}$ with $\mathrm{Loc}(\mathcal{C}) = \mathbf{K}$, this implies $L_{\mathbf{I}'}(N) \otimes \mathbf{K} \otimes M^* = 0$. One can now conclude the following:

$$\begin{aligned} \varnothing &= \Phi_\sigma(\langle\!\langle L_{\mathbf{I}'}(N) \otimes \mathbf{K} \otimes M^* \rangle\!\rangle) \\ &= \Phi_\sigma(\langle\!\langle L_{\mathbf{I}'}(N) \rangle\!\rangle \otimes \langle\!\langle M^* \rangle\!\rangle) \\ &= \Phi_\sigma(\langle\!\langle L_{\mathbf{I}'}(N) \rangle\!\rangle) \cap \Phi_\sigma(\langle\!\langle M \rangle\!\rangle) \\ &\supseteq \Phi_\sigma(\langle\!\langle L_{\mathbf{I}'}(N) \rangle\!\rangle) \cap \Phi_\sigma(\langle M \rangle) \\ &= \Phi_\sigma(\langle\!\langle L_{\mathbf{I}'}(N) \rangle\!\rangle) \cap Y \\ &= \Phi_\sigma(\langle\!\langle L_{\mathbf{I}'}(N) \rangle\!\rangle). \end{aligned}$$

The second equality is an application of Lemma 3.1.2. The third equality uses condition (v) in Definition 4.1.8. Therefore, by (4.2), $L_{\mathbf{I}'}(N) = 0$ in $\mathbf{K}$, and it follows that $N \cong \Gamma_{\mathbf{I}'}(N)$ via (4.4) and $N \in \mathrm{Loc}\,(\mathbf{I}')$ by Theorem 2.7.1. Now by [80, Lemma 2.2] we see that in fact $N \in \mathbf{I}'$. Consequently, $\mathbf{I} \subseteq \mathbf{I}'$. $\square$

4.3. Classification of thick two-sided ideals and Balmer spectra

In this section we present a method for the classification of the thick (two-sided) ideals of an MΔC and our first approach towards the explicit description of the Balmer spectrum of an MΔC as a topological space. They are based on the use of a weak support datum having the Faithfulness and Realization Properties (4.2)–(4.3).

Let $\mathbf{K}$ be a compactly generated MΔC with a weak support datum $\sigma : \mathbf{K} \to \mathcal{X}$. Denote by Θ_σ the map from specialization-closed subsets of X to subsets of $\mathbf{K}^c$ given by

$$\Theta_\sigma(W) = \{M \in \mathbf{K}^c : \Phi_\sigma(\langle M \rangle) \subseteq W\}. \tag{4.6}$$

The following result verifies that $\Theta_\sigma(W)$ is a thick tensor ideal.

Proposition 4.3.1. *Let $\sigma : \mathbf{K} \to \mathcal{X}$ be a weak support datum for a compactly generated MΔC, $\mathbf{K}$. For any $W \in \mathcal{X}_{sp}$, $\Theta_\sigma(W)$ is a thick tensor ideal of $\mathbf{K}^c$.*

Proof. Since $\langle M \rangle = \langle \Sigma M \rangle$, we have $M \in \Theta_\sigma(W)$ if and only if $\Sigma M \in \Theta_\sigma(W)$. Suppose $M \oplus N \in \Theta_\sigma(W)$. Then since M and N are in $\langle M \oplus N \rangle$, it follows that M and N are in $\Theta_\sigma(W)$. If $A \to B \to C \to \Sigma A$ is a distinguished triangle with B and C in $\Theta_\sigma(W)$, then by Lemma 4.1.10 we have $\Phi_\sigma(\langle A \rangle) \subseteq \Phi_\sigma(\langle B \rangle) \cup \Phi_\sigma(\langle C \rangle)$ and so $A \in \Theta_\sigma(W)$. If M is in $\Theta_\sigma(W)$, then since $N \otimes M$ and $M \otimes N$ are both in $\langle M \rangle$, we have N and M in $\Theta_\sigma(W)$. $\square$

Suppose S is any subset of the topological space X. Denote by

$$S_{sp} \textit{ the largest specialization-closed set contained in } S. \tag{4.7}$$

That is, S_{sp} is the union of all closed sets contained in S. With this definition, we can describe the image of f_σ.

Proposition 4.3.2. *Suppose* $\mathbf{K}$ *is a compactly generated* $M\Delta C$ *with weak support datum* $\sigma : \mathbf{K} \to \mathcal{X}$ *such that* $\Phi_\sigma(\langle C \rangle)$ *is closed for every compact object* C*. Then the map* $f_\sigma : X \to \operatorname{Spc} \mathbf{K}^c$ *defined in Theorem 4.1.11 associated to the restriction of* σ *to* $\mathbf{K}^c$ *satisfies* $f_\sigma(x) = \Theta_\sigma((X\backslash\{x\})_{sp})$, $\forall x \in X$.

Proof. We have

$$\begin{aligned} f_\sigma(x) &= \{M \in \mathbf{K}^c : x \notin \Phi_\sigma(\langle M \rangle)\} \\ &= \{M \in \mathbf{K}^c : \Phi_\sigma(\langle M \rangle) \subseteq X\backslash\{x\}\} \\ &= \{M \in \mathbf{K}^c : \Phi_\sigma(\langle M \rangle) \subseteq (X\backslash\{x\})_{sp}\} = \Theta_\sigma((X\backslash\{x\})_{sp}). \end{aligned}$$

□

We end this section by recording a useful fact that will be used later.

Lemma 4.3.3. *Suppose* X *is a Zariski space. Then, for all* $x, y \in X$,

$$(X\backslash\{x\})_{sp} = (X\backslash\{y\})_{sp} \quad \Leftrightarrow \quad x = y.$$

Proof. Suppose $(X\backslash\{x\})_{sp} = (X\backslash\{y\})_{sp}$. Then there is no closed set which contains y and not x, and vice versa. Therefore, $\overline{\{x\}} = \overline{\{y\}}$. In a Zariski space, every irreducible set has a unique generic point, but since x and y are generic points of their closures, we have $x = y$ by the assumed uniqueness. □

If $\mathbf{K}$ is a compactly generated MΔC with a weak support datum σ, we have now exhibited maps

$$\operatorname{ThickId}(\mathbf{K}^c) \underset{\Theta_\sigma}{\overset{\Phi_\sigma}{\rightleftarrows}} \mathcal{X}_{sp}.$$

If X is a Zariski space and σ satisfies the additional conditions (4.2) and (4.3), we can now classify thick tensor ideals of $\mathbf{K}^c$ and the Balmer spectrum.

Theorem 4.3.4. *Let* $\mathbf{K}$ *be a compactly generated* $M\Delta C$ *and* $\sigma : \mathbf{K} \to \mathcal{X}$ *be an extended weak support datum for a Zariski space* X *such that* $\Phi_\sigma(\langle C\rangle)$ *is closed for every compact object* C*. Recall the maps* Φ_σ *and* Θ_σ *defined in* (4.1) *and* (4.6)*, and the map* f_σ *from Theorem 4.1.11 and Proposition 4.3.2.*

(a) *If* σ *satisfies the Faithfulness Property (4.2), then* $\Theta_\sigma \circ \Phi_\sigma = \mathrm{id}$.

(b) *If* σ *satisfies the realization property (4.3), then:*

  (i) $\Phi_\sigma \circ \Theta_\sigma = \mathrm{id}$.

  (ii) *The map* f_σ *is injective.*

(c) *If* σ *satisfies both conditions (4.2) and (4.3), then:*

  (i) Φ_σ *and* Θ_σ *are mutually inverse maps. They are isomorphisms of ordered monoids, where the set of thick ideals of* $\mathbf{K}^c$ *is equipped with the operation* $\mathbf{I}, \mathbf{J} \mapsto \langle \mathbf{I} \otimes \mathbf{J}\rangle$ *and the inclusion partial order, and* $\mathcal{X}_{sp}$ *is equipped with the operation of intersection and the inclusion partial order.*

  (ii) *For every prime ideal* $\mathbf{P}$ *of* $\mathbf{K}^c$*, there exists* $x \in X$ *with* $\Phi_\sigma(\mathbf{P}) = (X\backslash\{x\})_{sp}$.

  (iii) *The map* $f_\sigma : X \to \operatorname{Spc} \mathbf{K}^c$ *is a homeomorphism.*

Proof. We first show (a). Given a thick tensor ideal $\mathbf{I}$ of $\mathbf{K}^c$, set $W = \Phi_\sigma(\mathbf{I})$ and $\mathbf{I}_W = \Theta_\sigma(W)$. Then by definition

$$I_W = \Theta_\sigma(W) = \Theta_\sigma(\Phi_\sigma(\mathbf{I})) = \{M : \Phi_\sigma(\langle M\rangle) \subseteq \Phi_\sigma(\mathbf{I})\} \supseteq \mathbf{I}.$$

For the reverse inclusion, let $N \in \mathbf{I}_W$, so $\Phi_\sigma(\langle N\rangle) \subseteq W$. Since X is a Zariski space, $\Phi_\sigma(\langle N\rangle) = W_1 \cup \cdots \cup W_n$, where the W_i are the irreducible components of $\Phi_\sigma(\langle N\rangle)$.

Moreover, each W_i has a generic point x_i with $\overline{\{x_i\}} = W_i$. Since $W_i \subseteq W$ one has $x_i \in W$. By definition of W, there exists $M_i \in \mathbf{I}$ such that $x_i \in \Phi_\sigma(\langle M_i \rangle)$. Since each $\Phi_\sigma(\langle M_i \rangle)$ is closed, it follows that $W_i \subseteq \Phi_\sigma(\langle M_i \rangle)$. Now set $M := \bigoplus_{i=1}^n M_i \in \mathbf{I}$. Then

$$\Phi_\sigma(\langle N \rangle) \subseteq \bigcup_{i=1}^n \Phi_\sigma(\langle M_i \rangle) = \Phi_\sigma(\langle M \rangle) \subseteq W.$$

We claim that $\langle N \rangle \subseteq \langle M \rangle$. Observe that $\mathbf{I}$ is a thick tensor ideal containing $\langle M \rangle$, so $\langle M \rangle \subseteq \mathbf{I}$. This implies that the aforementioned assertion will complete the proof of the inclusion $\mathbf{I}_W \subseteq \mathbf{I}$.

To prove the claim, we employ Hopkins' Theorem (Theorem 4.2.1). By this result one has $\langle M \rangle = \mathbf{I}_{\Phi_\sigma(\langle M \rangle)}$. However, $\Phi_\sigma(\langle N \rangle) \subseteq \Phi_\sigma(\langle M \rangle)$, so $\langle N \rangle \subseteq \mathbf{I}_{\Phi_\sigma(\langle M \rangle)} = \langle M \rangle$.

Next, we show (b)(i). We have automatically that

$$\Phi_\sigma(\Theta_\sigma(W)) = \Phi_\sigma(\mathbf{I}_W) = \bigcup_{M \in \mathbf{I}_W} \Phi_\sigma(\langle M \rangle) \subseteq W.$$

For the reverse inclusion, express $W = \bigcup_{j \in J} W_j$ for some index set J and closed subsets $W_j \in \mathcal{X}$. By the assumption (4.3), there exist objects $N_j \in \mathbf{K}^c$ such that $\Phi_\sigma(\langle N_j \rangle) = W_j$ for $j \in J$. It follows that $N_j \in \mathbf{I}_W$ so $W \subseteq \bigcup_{M \in \mathbf{I}_W} \Phi_\sigma(\langle M \rangle)$. Consequently, $\Phi_\sigma(\Theta_\sigma(W)) = W$.

For (b)(ii), we just note that by (b)(i), Θ_σ is injective. By Lemma 4.3.3, the map sending $x \mapsto (X \backslash \{x\})_{sp}$ is injective. By Proposition 4.3.2, $f_\sigma(x) = \Theta_\sigma((X \backslash \{x\})_{sp})$. Hence, f is injective.

By (a) and (b), (c)(i) is automatic. We now show (c)(ii). Suppose $\mathbf{P}$ is a prime ideal. By (b)(i), we an write arbitrary specialization-closed sets as $\Phi_\sigma(\mathbf{I})$ and $\Phi_\sigma(\mathbf{J})$ for some ideals $\mathbf{I}$ and $\mathbf{J}$. We have

$$\Phi_\sigma(\mathbf{I}) \cap \Phi_\sigma(\mathbf{J}) \subseteq \Phi_\sigma(\mathbf{P})$$

$$\Updownarrow$$

$$\Phi_\sigma(\mathbf{I} \otimes \mathbf{J}) \subseteq \Phi_\sigma(\mathbf{P})$$

$$\Updownarrow$$

$$\mathbf{I} \otimes \mathbf{J} \subseteq \mathbf{P}$$

$$\Updownarrow$$

$$\mathbf{I} \text{ or } \mathbf{J} \subseteq \mathbf{P}$$

$$\Updownarrow$$

$$\Phi_\sigma(\mathbf{I}) \text{ or } \Phi_\sigma(\mathbf{J}) \subseteq \Phi_\sigma(\mathbf{P}).$$

Hence, $\Phi_\sigma(\mathbf{P})$ has the property that for any specialization-closed sets S and T of X, $S \cap T \subseteq \Phi_\sigma(\mathbf{P}) \Rightarrow S$ or $T \subseteq \Phi_\sigma(\mathbf{P})$. We claim that the only sets with this property are sets of the form $(X \backslash \{x\})_{sp}$. Suppose $\Phi_\sigma(\mathbf{P})$ is not a set of this form. Then for every point x in its complement, there exists some closed set V_x which does not contain x and is not contained in $\Phi_\sigma(\mathbf{P})$. We have $\bigcap_{x\in\Phi_\sigma(\mathbf{P})^c} V_x \subseteq \Phi_\sigma(\mathbf{P})$, but for each x we have $V_x \not\subseteq \Phi_\sigma(\mathbf{P})$. By assumption X is Noetherian, so there is a finite subset S of $\Phi_\sigma(\mathbf{P})^c$ with $\bigcap_{x\in\Phi_\sigma(\mathbf{P})^c} V_x = \bigcap_{x\in S} V_x$, and since this is now a finite intersection this shows that there exist closed sets S and T with $S \cap T \subseteq \Phi_\sigma(\mathbf{P})$, but neither S nor T is contained in $\Phi_\sigma(\mathbf{P})$. Since this is a contradiction, we have that $\Phi_\sigma(\mathbf{P})$ has the form $(X\backslash\{x\})_{sp}$ for some x.

Now we will show (c)(iii), that f_σ is a homeomorphism. Given a prime ideal $\mathbf{P}$ of $\mathbf{K}$, there exists $x \in X$ such that $\Phi_\sigma(\mathbf{P}) = (X\backslash\{x\})_{sp}$, and so

$$\mathbf{P} = \Theta_\sigma(\Phi_\sigma(\mathbf{P})) = \Theta_\sigma((X\backslash\{x\})_{sp}) = f_\sigma(x)$$

by Proposition 4.3.2. This shows that f_σ is surjective, and hence bijective by (b)(ii). We now show that f_σ is a closed map. To an arbitrary closed set, which by (4.3) is of the form $\Phi_\sigma(\langle M\rangle)$, we apply f_σ:

$$f_\sigma(\Phi_\sigma(\langle M\rangle)) = f_\sigma(f_\sigma^{-1}(V(M))) = V(M),$$

using the surjectivity of f_σ and the formula for $f_\sigma^{-1}(V(M))$ given in Theorem 4.1.11. Since $V(M)$ is closed, f_σ is a closed and continuous bijection, and hence a homeomorphism. $\square$

4.4. Classification of one-sided ideals

In this section we present a method for the classification of the thick right ideals of an MΔC. We introduce a new concept (quasi support datum) to deal with thick one-sided ideals.

A thick right ideal of an MΔC $\mathbf{K}$ is a full triangulated subcategory of $\mathbf{K}$ that contains all direct summands of its objects and is closed under right tensoring with arbitrary objects of $\mathbf{K}$.

Definition 4.4.1. Let $\mathbf{K}$ be a monoidal triangulated category, X a topological space, and σ a map $\mathbf{K} \to \mathcal{X}$. We call σ a *(noncommutative) quasi support datum* if

(a) $\sigma(0) = \varnothing$ and $\sigma(\mathbb{1}) = X$;

(b) $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$, $\forall A, B \in \mathbf{K}$;

(c) $\sigma(\Sigma A) = \sigma(A)$, $\forall A \in \mathbf{K}$;

(d) If $A \to B \to C \to \Sigma A$ is a distinguished triangle, then $\sigma(A) \subseteq \sigma(B) \cup \sigma(C)$;

(e) $\sigma(A \otimes B) \subseteq \sigma(A)$, $\forall A, B \in \mathbf{K}$.

For an *extended quasi support datum* we replace (b) with

(b') $\sigma(\bigoplus_{i\in I} A_i) = \bigcup_{i\in I} A_i$, $\forall A_i \in \mathbf{K}$.

Similarly to the previous two sections, we will be interested in quasi support data $\sigma : \mathbf{K} \to \mathcal{X}$ that satisfy the following one-sided type assumptions:

$$\Phi_\sigma(\langle\!\langle M \rangle\!\rangle_{\mathrm{r}}) = \varnothing \text{ if and only if } M = 0,\ \forall M \in \mathbf{K} \text{ (Faithfulness Property);} \tag{4.8}$$

For any $W \in \mathcal{X}_{cl}$, $\exists M \in \mathbf{K}^c$ such that $\Phi_\sigma(\langle M \rangle_{\mathrm{r}}) = W$ (Realization Property). (4.9)

Here and below, similarly to the two-sided case, for $M \in \mathbf{K}^c$, $\langle M \rangle_{\mathrm{r}}$ denotes the smallest thick right ideal of $\mathbf{K}^c$ containing M; that is the intersection of all thick right ideals containing M. For $M \in \mathbf{K}$, $\langle\!\langle M \rangle\!\rangle_{\mathrm{r}}$ denotes the smallest thick right ideal of $\mathbf{K}$ containing M.

We first state an assumption that acts as a (one-sided) replacement for condition (e) in the definition of weak support datum. Recall the definition (4.1) of the map Φ_σ. Similarly to the arguments in Section 4.1, one shows that in the presence of the other conditions for quasi support datum, condition (e) is equivalent to

$$\Phi_\sigma(\langle\!\langle M \rangle\!\rangle_{\mathrm{r}}) = \sigma(M), \quad \forall M \in \mathbf{K}. \tag{4.10}$$

Assumption 4.4.2. *Suppose that $M, N \in \mathbf{K}^c$ such that*

$$\Phi_\sigma(\langle N \rangle_{\mathrm{r}}) \subseteq \sigma(M).$$

Set $\mathbf{I}' = \langle M \rangle_{\mathrm{r}}$. If $M^* \otimes L_{\mathbf{I}'}(N) = 0$, then $L_{\mathbf{I}'}(N) = 0$ (for the localization functor as in Section 2.7).

With this assumption, one proves the following one-sided version of Theorem 4.2.1. Similarly to (4.6), for a quasi support datum $\sigma : \mathbf{K} \to \mathcal{X}$ for a compactly generated MΔC

$\mathbf{K}$, denote by Θ_σ the map from specialization-closed subsets of X to subsets of $\mathbf{K}^c$:

$$\Theta_\sigma(W) = \{M \in \mathbf{K}^c : \Phi_\sigma(\langle M\rangle_{\mathrm{r}}) \subseteq W\}, \quad \text{for} \quad W \in \mathcal{X}_{sp}.$$

Theorem 4.4.3. *Let $\mathbf{K}$ be a compactly generated $M\Delta C$ and $\sigma : \mathbf{K} \to \mathcal{X}$ be an assignment to subsets of a Zariski space X that satisfies the conditions (a), (b'), (c), (d) for an extended quasi support datum, such that $\sigma : \mathbf{K}^c \to \mathcal{X}$ is a quasi support datum for a Zariski space X. Assume that Assumption 4.4.2 holds. Then for each object $M \in \mathbf{K}^c$,*

$$\Theta_\sigma(\Phi_\sigma(\langle M\rangle_{\mathrm{r}})) = \langle M\rangle_{\mathrm{r}}.$$

With Theorem 4.4.3, we can state a classification theorem for thick (right) ideals for $\mathbf{K}^c$. The proof follows the same line of reasoning as given in Theorem 4.3.4.

Theorem 4.4.4. *Let $\mathbf{K}$ be a compactly generated $M\Delta C$ and $\sigma : \mathbf{K} \to \mathcal{X}$ be an assignment to subsets of a Zariski space X that satisfies the conditions (a), (b'), (c), (d) for an extended quasi support datum. Suppose that σ restricts to a quasi support datum on $\mathbf{K}^c$ where $\Phi_\sigma(\langle C\rangle)$ is closed for every $C \in \mathbf{K}^c$. Moreover, assume that σ satisfies the realization property (7.1.4.9) and Assumption 4.4.2 holds.*

Then the maps Φ_σ and Θ_σ

$$\{\textit{thick right ideals of } \mathbf{K}^c\} \; \underset{\Theta_\sigma}{\overset{\Phi_\sigma}{\rightleftarrows}} \; \mathcal{X}_{sp}$$

are mutually inverse.

Remark 4.4.5. The set of thick right ideals of $\mathbf{K}^c$ is an ordered monoid with the operation $\mathbf{I}, \mathbf{J} \mapsto \langle \mathbf{I} \otimes \mathbf{J}\rangle_{\mathrm{r}}$ and the inclusion partial order. The set $\mathcal{X}_{sp}$ is is an ordered monoid with the operation of intersection and the inclusion partial order. The maps Φ_σ and Θ_σ preserve inclusions but in general they are not isomorphisms of monoids.

More precisely, Φ_σ *and* Θ_σ *are isomorphisms of ordered monoids if and only if* $\sigma : \mathbf{K} \to \mathcal{X}$ *is a support datum.*

Indeed, Φ_σ is an isomorphism of monoids if and only if $\Phi_\sigma(\langle \mathbf{I} \otimes \mathbf{J} \rangle_{\mathrm{r}}) = \Phi_\sigma(\mathbf{I}) \cap \Phi_\sigma(\mathbf{J})$ for all thick right ideals $\mathbf{I}$ and $\mathbf{J}$ of $\mathbf{K}^c$. This in turn is equivalent to $\Phi_\sigma(\mathbf{I} \otimes \mathbf{J}) = \Phi_\sigma(\mathbf{I}) \cap \Phi_\sigma(\mathbf{J})$ by an argument similar to the proof of Lemma 4.1.6. Since $\mathbf{I} = \cup_{A \in \mathbf{I}} \langle A \rangle_{\mathrm{r}}$, the last property is equivalent to $\Phi_\sigma(\langle A \rangle_{\mathrm{r}} \otimes \langle B \rangle_{\mathrm{r}}) = \Phi_\sigma(\langle A \rangle_{\mathrm{r}}) \cap \Phi_\sigma(\langle B \rangle_{\mathrm{r}})$, $\forall A, B \in \mathbf{K}$. By (4.10) the last property is equivalent to

$$\bigcup_{C \in \mathbf{K}} \sigma(A \otimes C \otimes B) = \sigma(A) \cap \sigma(B), \quad \forall A, B \in \mathbf{K},$$

which is the fifth property in the definition of support data. $\square$

Chapter 5. Balmer Spectra of Drinfeld Centers

5.1. Contraction of primes

Let $\mathbf{C}$ be a finite tensor category, $\mathsf{st}(\mathbf{C})$ its stable category, $\mathsf{Z}(\mathbf{C})$ its Drinfeld center, and $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$ the stable category of its Drinfeld center (which may be formed by Proposition 2.2.1). We have a forgetful functor $F : \mathsf{Z}(\mathbf{C}) \to \mathbf{C}$, and we have functors $G : \mathbf{C} \to \mathsf{st}(\mathbf{C})$ and $H : \mathsf{Z}(\mathbf{C}) \to \mathsf{st}(\mathsf{Z}(\mathbf{C}))$. We have the respective Balmer support data associated to $\mathsf{st}(\mathbf{C})$ and $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$:

$$V_{\mathsf{st}\,\mathbf{C}} : \mathsf{st}(\mathbf{C}) \to \mathcal{X}_{cl}(\operatorname{Spc}\mathsf{st}(\mathbf{C}))$$

and

$$V_{\mathsf{st}(\mathsf{Z}(\mathbf{C}))} : \mathsf{st}(\mathsf{Z}(\mathbf{C})) \to \mathcal{X}_{cl}(\operatorname{Spc}\mathsf{st}(\mathsf{Z}(\mathbf{C}))),$$

defined in their respective categories by sending

$$A \mapsto \{\text{primes not containing } A\}.$$

For readability, we will denote $V_{\mathbf{C}} := V_{\mathsf{st}\,\mathbf{C}}$ and $V_{\mathsf{Z}} := V_{\mathsf{st}(\mathsf{Z}(\mathbf{C}))}$. The corresponding maps Φ (recalling the construction from Section 4.1) associated to these support data will similarly be denoted $\Phi_{\mathbf{C}}$ and Φ_{Z}.

Proposition 5.1.1. *There is a functor $\overline{F} : \mathsf{st}(\mathsf{Z}(\mathbf{C})) \to \mathsf{st}(\mathbf{C})$ which extends the forgetful functor F, i.e. the diagram of functors*

$$\begin{tikzcd} \mathbf{C} \arrow[r, "G"] & \mathsf{st}(\mathbf{C}) \\ \mathsf{Z}(\mathbf{C}) \arrow[u, "F"] \arrow[r, "H"] & \mathsf{st}(\mathsf{Z}(\mathbf{C})) \arrow[u, dashed, "\overline{F}"'] \end{tikzcd}$$

commutes. This functor $\overline{F}$ is monoidal and triangulated.

Proof. Since the objects of $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$ are the in bijection with those of $\mathsf{Z}(\mathbf{C})$, $\overline{F}$ is well-defined on objects, namely by defining

$$\overline{F}(H(X)) := G(F(X)).$$

Let $f : X \to Y$ be a morphism in $\mathsf{Z}(\mathbf{C})$. Then for $\overline{F}(H(f)) := GF(f)$ to be well-defined, we need $GF(g) = 0$ for each morphism g which factors through a projective in $\mathsf{Z}(\mathbf{C})$. In other words, we need $F(g)$ to factor through a projective in $\mathbf{C}$. Hence, to define $\overline{F}$, it is enough to know that $G \circ F$ sends all projective objects of $\mathsf{Z}(\mathbf{C})$ to 0, which is true by Proposition 2.2.1.

Let $H(X) \in \mathsf{st}(\mathsf{Z}(\mathbf{C}))$ an arbitrary object, where $X \in \mathsf{Z}(\mathbf{C})$. Then $\Sigma H(X)$ is defined as $H(Z)$, such that there exists a short exact sequence

$$0 \to X \to P \to Z \to 0$$

in $\mathsf{Z}(\mathbf{C})$, where P is projective. $H(Z)$ is well-defined in $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$, by Schanuel's Lemma. Since F is exact and sends projectives to projectives,

$$0 \to F(X) \to F(P) \to F(Z) \to 0$$

is an exact sequence in $\mathbf{C}$ with $F(P)$ projective; therefore, $\Sigma(GF(X)) \cong GF(Z)$ in $\mathsf{st}(\mathbf{C})$, and so we have $\overline{F}(\Sigma X) \cong \Sigma\overline{F}(X)$.

Now, let $X \to Y \to Z \to \Sigma X$ be a distinguished triangle. Then it is isomorphic to a triangle of the form

$$H(X') \to H(Y') \to H(Z') \to \Sigma H(X')$$

for some short exact sequence

$$0 \to X' \to Y' \to Z' \to 0$$

in $\mathsf{Z}(\mathbf{C})$. Since F is exact, and G sends exact sequences to triangles, we have that the composition GF is exact and hence

$$\overline{F}H(X') \to \overline{F}H(Y') \to \overline{F}H(Z') \to \Sigma\overline{F}H(X')$$

a triangle in $\mathsf{st}(\mathbf{C})$. Therefore,

$$\overline{F}(X) \to \overline{F}(Y) \to \overline{F}(Z) \to \Sigma\overline{F}(X)$$

is a triangle as well, and so $\overline{F}$ is a triangulated functor. □

For braided tensor triangulated categories, the Balmer spectrum Spc is functorial, as Balmer has shown in Proposition 3.6 of [8]. This is a categorical reflection the ring-theoretic fact that Spec is functorial for commutative rings. However, for noncommutative rings, Spec is not a functor (for a more in-depth exploration of the extent of the failure of functoriality of Spec for noncommutative rings, see [94]). It is not suprising, then, that for generic monoidal triangulated categories, the Balmer spectrum is also not functorial; in other words, an exact monoidal functor between monoidal triangulated categories does not necessarily induce a map between their Balmer spectra.

However, reflecting the classical prime ideal contraction for noncommutative rings, the forgetful functor $\overline{F}$ does induce a map between the Balmer spectra of $\mathsf{st}(\mathbf{C})$ and $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$.

Proposition 5.1.2. *$\overline{F}$ induces a continuous map $\operatorname{Spc}\mathsf{st}(\mathbf{C}) \xrightarrow{f} \operatorname{Spc}\mathsf{st}(\mathsf{Z}(\mathbf{C}))$, defined explicitly by*

$$f(\mathbf{P}) := \{X \in \mathsf{st}(\mathsf{Z}(\mathbf{C})) : \overline{F}(X) \in \mathbf{P}\}$$

for each prime ideal $\mathbf{P}$ of $\mathsf{st}(\mathbf{C})$.

Proof. We must first show that $f(\mathbf{P})$ is a prime ideal of $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$.

We first check that $f(\mathbf{P})$ is a thick ideal of $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$. This necessitates checking three properties:

(Triangulated) Suppose $\Sigma X \in f(\mathbf{P})$, in other words, $\overline{F}(\Sigma X) \in \mathbf{P}$. Since $\overline{F}$ is exact, this is true if and only if $\Sigma\overline{F}(X) \in \mathbf{P}$, which is true if and only if $\overline{F}(X) \in \mathbf{P}$, in other words, $X \in f(\mathbf{P})$. Now, suppose

$$X \to Y \to Z \to \Sigma X$$

is a distinguished triangle with X and Y in $f(\mathbf{P})$. This means that $\overline{F}(X)$ and $\overline{F}(Y)$ are in $\mathbf{P}$. Since $\overline{F}$ is triangulated, the triangle

$$\overline{F}(X) \to \overline{F}(Y) \to \overline{F}(Z) \to \Sigma\overline{F}(X)$$

is distinguished in $\mathsf{st}(\mathbf{C})$. Now since the first two objects are in $\mathbf{P}$, so is $\overline{F}(Z)$, and so $Z \in f(\mathbf{P})$.

(Thick) If $X \oplus Y$ is in $f(\mathbf{P})$, then $\overline{F}(X \oplus Y) \in \mathbf{P}$; $\overline{F}$ is an additive functor, and so $\overline{F}(X) \oplus \overline{F}(Y) \in \mathbf{P}$. This implies that both $\overline{F}(X)$ and $\overline{F}(Y)$ are in $\mathbf{P}$, and so X and Y are both in $f(\mathbf{P})$.

(Ideal) Suppose $X \in f(\mathbf{P})$ and $Y \in \mathsf{st}(\mathsf{Z}(\mathbf{C}))$. Since $\overline{F}$ is exact, $\overline{F}(X \otimes Y) \cong \overline{F}(X) \otimes \overline{F}(Y)$. Since $\overline{F}(X) \in \mathbf{P}$, so is $\overline{F}(X) \otimes \overline{F}(Y)$, and thus $\overline{F}(X \otimes Y) \in \mathbf{P}$ as well. Hence $X \otimes Y \in f(\mathbf{P})$. The symmetric argument shows that $Y \otimes X$ is in $f(\mathbf{P})$ as well, so $f(\mathbf{P})$ is a two-sided ideal.

(Prime) Let $A \otimes B \in f(\mathbf{P})$. Then $\overline{F}(A) \otimes \overline{F}(B) \in \mathbf{P}$. But $\overline{F}(A)$ and $\overline{F}(B)$ commute with every object of $\mathsf{st}(\mathbf{C})$: by the ideal property of $\mathbf{P}$, we have

$$\begin{aligned}
&\mathsf{st}(\mathbf{C}) \otimes \overline{F}(A) \otimes \overline{F}(B) \subseteq \mathbf{P} \\
&\Rightarrow \overline{F}(A) \otimes \mathsf{st}(\mathbf{C}) \otimes \overline{F}(B) \subseteq \mathbf{P} \\
&\Rightarrow \overline{F}(A) \text{ or } \overline{F}(B) \in \mathbf{P},
\end{aligned}$$

with the last step following by primeness of $\mathbf{P}$. This implies that either A or B is in $f(\mathbf{P})$, which means that $f(\mathbf{P})$ is prime.

We can also check directly that f is continuous: an arbitrary closed set of $\mathrm{Spc}(\mathsf{st}(\mathsf{Z}(\mathbf{C})))$ is of the form $V_{\mathsf{Z}}(\mathcal{T}) = \{\mathbf{P} \in \mathrm{Spc}(\mathsf{st}(\mathsf{Z}(\mathbf{C}))) : \mathcal{T} \cap \mathbf{P} = \varnothing\}$ for some collection of objects $\mathcal{T}$ of $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$. Then

$$\begin{aligned}
f^{-1}(V_{\mathsf{Z}}(\mathcal{T})) &= \{\mathbf{P} \in \mathrm{Spc}\,\mathsf{st}(\mathbf{C}) : \mathcal{T} \cap \{X \in \mathsf{st}(\mathsf{Z}(\mathbf{C})) : \overline{F}(X) \in \mathbf{P}\} = \varnothing\} \\
&= \{\mathbf{P} \in \mathrm{Spc}\,\mathsf{st}(\mathbf{C}) : \overline{F}(\mathcal{T}) \cap \mathbf{P} = \varnothing\} \\
&= V_{\mathbf{C}}(\overline{F}(\mathcal{T})),
\end{aligned}$$

where by $\overline{F}(\mathcal{T})$ we mean the collection $\{\overline{F}(X) : X \in \mathcal{T}\}$. □

Remark 5.1.3. If H is a finite-dimensional Hopf algebra, then $\mathsf{Z}(\mathsf{mod}(H)) \cong \mathsf{mod}(D(H))$, recall Section 5.1. In this case, $D(H) \cong D((H^{\mathrm{op}})^*)$, and so we have two functors:

$$\begin{tikzcd}
\mathsf{mod}(H) & & \mathsf{mod}((H^{\mathrm{op}})^*) \\
& \mathsf{Z}(\mathsf{mod}(H)) \cong \mathsf{mod}(D(H)) \cong \mathsf{Z}(\mathsf{mod}((H^{\mathrm{op}})^*)) \arrow[ul, "F_H"] \arrow[ur, "F_{(H^{\mathrm{op}})^*}"'] &
\end{tikzcd}$$

which then give two maps between Balmer spectra:

$$\begin{tikzcd}
\mathrm{Spc}(\mathsf{stmod}(H)) \arrow[dr, "f_H"'] & & \mathrm{Spc}(\mathsf{stmod}((H^{\mathrm{op}})^*)) \arrow[dl, "f_{(H^{\mathrm{op}})^*}"] \\
& \mathrm{Spc}(\mathsf{stmod}(D(H))) &
\end{tikzcd}$$

We can interpret the map f in the context of support data (recalling the definition from Section 4.1), by first defining a new support datum given as the composition of the functor $\overline{F}$ with the Balmer support $V_{\mathbf{C}}$ on $\mathsf{st}(\mathbf{C})$.

Proposition 5.1.4. *Define a map* $W : \mathsf{st}(\mathsf{Z}(\mathbf{C})) \to \mathcal{X}_{cl}(\operatorname{Spc} \mathsf{st}(\mathbf{C}))$ *by*

$$W(X) := V_{\mathbf{C}}(\overline{F}(X)) = \{\mathbf{P} \in \operatorname{Spc} \mathsf{st}(\mathbf{C}) : \overline{F}(X) \notin \mathbf{P}\}.$$

This map is a support datum.

Proof. The first four conditions follow directly from the facts that $\overline{F}$ is an exact functor and $V_{\mathbf{C}}$ is itself a support datum, since

(a) $\overline{F}(0_{\mathsf{st}(\mathsf{Z}(\mathbf{C}))}) = 0_{\mathsf{st}(\mathbf{C})}$,

(b) $\overline{F}(X \oplus Y) = \overline{F}(X) \oplus \overline{F}(Y)$,

(c) $\overline{F}(\Sigma X) \cong \Sigma \overline{F}(X)$,

(d) and if $X \to Y \to Z \to \Sigma X$ is a distinguished triangle, then so is $\overline{F}(X) \to \overline{F}(Y) \to \overline{F}(Z) \to \Sigma \overline{F}(X)$.

To check the last condition, we need to show that

$$\bigcup_{Z \in \mathsf{st}(\mathsf{Z}(\mathbf{C}))} W(X \otimes Z \otimes Y) = W(X) \cap W(Y).$$

By the ideal condition, if $\mathbf{P}$ is a prime ideal which does not contain $\overline{F}(X) \otimes \overline{F}(Z) \otimes \overline{F}(Y)$ for some object Z, then it must also not contain $\overline{F}(X)$ or $\overline{F}(Y)$. Hence,

$$\bigcup_{Z \in \mathsf{st}(\mathsf{Z}(\mathbf{C}))} W(X \otimes Z \otimes Y) \subseteq W(X) \cap W(Y)$$

is automatic.

For the reverse containment, suppose $\mathbf{P}$ is a prime ideal which does not contain $\overline{F}(X)$ or $\overline{F}(Y)$. By the prime condition, that means $\mathbf{P}$ does not contain the entire collection of objects $\overline{F}(X) \otimes \mathsf{st}(\mathbf{C}) \otimes \overline{F}(Y)$. But since $\overline{F}(X)$ and $\overline{F}(Y)$ commute up to isomorphism with all elements of $\mathsf{st}(\mathbf{C})$, if $\overline{F}(X) \otimes \overline{F}(Y) \in \mathbf{P}$, that would imply $\overline{F}(X) \otimes \overline{F}(Y) \otimes \mathsf{st}(\mathbf{C}) \subseteq \mathbf{P}$, which would then imply $\overline{F}(X) \otimes \mathsf{st}(\mathbf{C}) \otimes \overline{F}(Y) \subseteq \mathbf{P}$, a contradiction. Hence, $\mathbf{P} \in W(X \otimes Y)$, and we have the claimed equality. $\square$

By the universal property of the Balmer spectrum, the support datum W induces a continuous map $\operatorname{Spc}\mathsf{st}(\mathbf{C}) \to \operatorname{Spc}\mathsf{st}(\mathsf{Z}(\mathbf{C}))$. This map is defined as

$$\mathbf{P} \mapsto \{X \in \mathsf{st}(\mathsf{Z}(\mathbf{C})) : \mathbf{P} \notin W(X)\}.$$

This map is the same as the map defined in Proposition 5.1.2. We have the following diagram, which commutes by definition:

$$\begin{tikzcd}
\mathsf{st}(\mathsf{Z}(\mathbf{C})) \arrow[rr, "W"] \arrow[dr, "\overline{F}"'] & & \mathcal{X}_{cl}(\operatorname{Spc}(\mathsf{st}(\mathbf{C}))) \\
& \mathsf{st}(\mathbf{C}) \arrow[ur, "V_{\mathbf{C}}"'] &
\end{tikzcd}$$

On the level of ideals, this now induces the following maps, recall the maps Φ and Θ as constructed in Sections 4.1 and 4.3:

$$\begin{tikzcd}
\mathbf{I} \arrow[ddrr, mapsto] & \operatorname{ThickId}(\mathsf{st}(\mathsf{Z}(\mathbf{C}))) \arrow[rr, "\Phi_W"] \arrow[dr] & & \mathcal{X}_{sp}(\operatorname{Spc}(\mathsf{st}(\mathbf{C}))) \arrow[ll, bend right, "\Theta_W"'] \arrow[dl, bend left, "\Theta_{\mathbf{C}}"] \\
& & \operatorname{ThickId}(\mathsf{st}(\mathbf{C})) \arrow[ur, "\Phi_{\mathbf{C}}"'] & \\
& & \langle \overline{F}(\mathbf{I}) \rangle &
\end{tikzcd}$$

Here, the middle triangle also commutes, by definition.

5.2. Recovery of ideals

In this section, we prove some general results on recovery of ideals from their W-support. To do this, we will make use of the localization and colocalization functors of

[16], as described in Section 2.7. To do this, we must work in the situation of a compactly generated monoidal triangulated category. For convenience, then, we specialize to the case where $\mathbf{C} = \mathsf{mod}(H)$ for a Hopf algebra H. In that case, we denote $\mathsf{St}(\mathbf{C})$ for the stable module category of (not necessarily finite-dimensional) modules of H. In this situation, then, we have $\mathsf{Z}(\mathbf{C}) \cong \mathsf{mod}(D(H))$ and $\mathsf{st}(\mathsf{Z}(\mathbf{C})) \cong \mathsf{stmod}(D(H))$ is the collection of compact objects in the compactly generated category $\mathsf{St}(\mathsf{Z}(\mathbf{C})) \cong \mathsf{StMod}(D(H))$.

We now introduce some terminology, which will be useful for our reconstruction.

Definition 5.2.1. The *kernel of* $\overline{F}$, which we will denote by $\mathbf{K}$ for the sake of brevity, is defined by

$$\mathbf{K} := \{X \in \mathsf{St}(\mathsf{Z}(\mathbf{C})) : \overline{F}(X) \cong 0 \in \mathsf{St}(\mathbf{C})\}.$$

An equivalent characterization of the kernel of $\overline{F}$ can be given by

$$\mathbf{K} = \{H(X) : X \text{ a } D(H)\text{-module, such that } F(X) \text{ is projective as an } H\text{-module}\}.$$

Lemma 5.2.2. *The kernel of* $\overline{F}$ *is a thick ideal of* $\mathsf{St}(\mathsf{Z}(\mathbf{C}))$.

Proof. Since $\overline{F}$ is a monoidal triangulated functor, it is straightforward to verify that the collection of objects X such that $\overline{F}(X) \cong 0$ is closed under taking cones, shifts, direct summands, and by tensoring on the left or right by arbitrary objects of $\mathsf{St}(\mathsf{Z}(\mathbf{C}))$. □

Lemma 5.2.3. *An object* $A \in \mathsf{st}(\mathsf{Z}(\mathbf{C}))$ *satisfies* $W(A) = \varnothing$ *if and only if* $A \in \mathbf{K}$.

Proof. First, note that if $A \in \mathbf{K}$, then by definition $\overline{F}(A) \cong 0$, and so

$$W(A) = V_{\mathbf{C}}(0) = \{\mathbf{P} \in \operatorname{Spc}(\mathsf{st}(\mathbf{C})) : 0 \notin \mathbf{P}\} = \varnothing.$$

For the other direction, recall that by the rigidity of $\mathbf{C}$, all thick ideals of $\mathsf{st}(\mathbf{C})$ are semiprime, i.e. intersections of prime ideals, Proposition 3.3.3. This implies in particular that the ideal $\langle 0 \rangle$ is semiprime; in other words, the only object contained in all prime ideals of $\mathsf{st}(\mathbf{C})$ is 0. By definition, this means that if B is an object of $\mathsf{st}(\mathbf{C})$ such that $V_{\mathbf{C}}(B) = \varnothing$, then $X \cong 0$. Hence, we have

$$\varnothing = W(A) = V_{\mathbf{C}}(\overline{F}(A)) \Rightarrow \overline{F}(A) \cong 0 \Rightarrow A \in \mathbf{K}.$$

$\square$

Using the localization and colocalization functors, we are now able to prove the following, which is the critical step in determining which ideals can be recovered from their W-support and determining the image of the map $f : \operatorname{Spc} \mathsf{st}(\mathbf{C}) \to \operatorname{Spc} \mathsf{st}(\mathsf{Z}(\mathbf{C}))$.

Theorem 5.2.4. *Let $\mathbf{I}$ be a thick ideal of $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$ such that $\operatorname{Loc}(\mathbf{I})$ contains $\mathbf{K}$. Suppose that X is an object of $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$ such that $\overline{F}(X) \in \langle \overline{F}(\mathbf{I}) \rangle$, that is, the thick ideal of $\mathsf{st}(\mathbf{C})$ generated by all $\overline{F}(Y)$ for $Y \in \mathbf{I}$. Then X is in $\mathbf{I}$.*

Proof. By Theorem 2.7.1, we have a distinguished triangle

$$\Gamma_{\mathbf{I}}(X) \to X \to L_{\mathbf{I}}(X) \to \Sigma\Gamma_{\mathbf{I}}(X)$$

in $\mathsf{St}(\mathsf{Z}(\mathbf{C}))$, using the localization and colocalization functors associated to the thick ideal $\mathbf{I}$. We know that there are no morphisms from $\mathbf{I}$ to $L_{\mathbf{I}}(X)$; in other words, if $Y \in \mathbf{I}$ and Z is any compact object in $\mathsf{St}(\mathsf{Z}(\mathbf{C}))$, then

$$\begin{aligned} 0 &= \operatorname{Hom}(Z \otimes Y, L_{\mathbf{I}}(X)) \\ &\cong \operatorname{Hom}(Z, L_{\mathbf{I}}(X) \otimes Y^*). \end{aligned}$$

Since this holds for all compact objects Z, this implies that $L_{\mathbf{I}}(X) \otimes Y^* \cong 0$. Since all compact objects are rigid, and by Lemma 3.1.3 all thick ideals are closed under taking duals, we have $L_{\mathbf{I}}(X) \otimes Y \cong 0$ for all $Y \in \mathbf{I}$. Since $\overline{F}$ is a monoidal functor, this additionally implies that

$$\overline{F}(L_{\mathbf{I}}(X)) \otimes \overline{F}(Y) \cong 0$$

in $\mathsf{St}(\mathbf{C})$, for all $Y \in \mathbf{I}$.

Now, consider the thick ideal $\langle \overline{F}(\mathbf{I}) \rangle$. This is formed successively by taking shifts, cones, direct summands, and tensor products with arbitrary elements of $\mathsf{st}(\mathbf{C})$, starting from the collection of objects of the form $\overline{F}(Y)$ for $Y \in \mathbf{I}$. This allows us to conclude inductively that $\overline{F}(L_{\mathbf{I}}(X)) \otimes A \cong 0$ for all A in $\langle \overline{F}(\mathbf{I}) \rangle$, since inductively each step by which we construct $\langle \overline{F}(\mathbf{I}) \rangle$ preserves the property that tensoring with $\overline{F}(L_{\mathbf{I}}(X))$ gives 0. To be more explicit, if

$$A \to B \to C \to \Sigma A$$

is a distinguished triangle in $\mathsf{st}(\mathbf{C})$ such that $A \otimes \overline{F}(L_{\mathbf{I}}(X)) \cong B \otimes \overline{F}(L_{\mathbf{I}}(X)) \cong 0$, then it is straightforward that additionally $C \otimes \overline{F}(L_{\mathbf{I}}(X)) \cong 0$ as well. Similarly, if $A \otimes \overline{F}(L_{\mathbf{I}}(X)) \cong 0$, then $\Sigma(A) \otimes \overline{F}(L_{\mathbf{I}}(X)) \cong \Sigma(A \otimes \overline{F}(L_{\mathbf{I}}(X))) \cong \Sigma 0 \cong 0$. Furthermore, if we have $(A \oplus B) \otimes \overline{F}(L_{\mathbf{I}}(X)) \cong 0$, then we also have $A \otimes \overline{F}(L_{\mathbf{I}}(X)) \cong 0 \cong B \otimes \overline{F}(L_{\mathbf{I}}(X))$. Lastly, if $A \otimes \overline{F}(L_{\mathbf{I}}(X)) \cong 0$ and B is an arbitrary object in $\mathsf{st}(\mathbf{C})$, then $A \otimes B \otimes \overline{F}(L_{\mathbf{I}}(X)) \cong A \otimes \overline{F}(L_{\mathbf{I}}(X)) \otimes B \cong 0$ as well, using the commutativity of $\overline{F}(L_{\mathbf{I}}(X))$.

To reiterate, the upshot of all this is that we have $A \otimes \overline{F}(L_{\mathbf{I}}(X)) \cong 0$ for all $A \in \langle \overline{F}(\mathbf{I}) \rangle$. But by assumption, we have $\overline{F}(X) \in \langle \overline{F}(\mathbf{I}) \rangle$. Hence,

$$\overline{F}(X \otimes L_{\mathbf{I}}(X)) \cong \overline{F}(X) \otimes \overline{F}(L_{\mathbf{I}}(X)) \cong 0.$$

Therefore, $X \otimes L_{\mathbf{I}}(X)$ is an object in $\mathbf{K}$, the collection of objects of $\mathsf{St}(\mathsf{Z}(\mathbf{C}))$ mapped to 0 by $\overline{F}$. By assumption, $\mathrm{Loc}(\mathbf{I})$ contains $\mathbf{K}$, and so $X \otimes L_{\mathbf{I}}(X) \in \mathrm{Loc}(\mathbf{I})$.

Now, consider the distinguished triangle obtained by tensoring the triangle

$$\Gamma_{\mathbf{I}}(X) \to X \to L_{\mathbf{I}}(X) \to \Sigma\Gamma_{\mathbf{I}}(X)$$

by X: this gives us

$$X \otimes \Gamma_{\mathbf{I}}(X) \to X \otimes X \to X \otimes L_{\mathbf{I}}(X) \to \Sigma X \otimes \Gamma_{\mathbf{I}}(X).$$

We have just finished showing that the third object of this triangle is in $\mathrm{Loc}(\mathbf{I})$. The first object is in $\mathrm{Loc}(\mathbf{I})$ as well, by Theorem 2.7.1. Since $\mathrm{Loc}(\mathbf{I})$ is triangulated, this implies X is in $\mathrm{Loc}(\mathbf{I})$. But by [82, Lemma 2.2], since $\mathbf{I}$ is a thick subcategory of compact objects, the compact objects in $\mathrm{Loc}(\mathbf{I})$ are precisely the objects of $\mathbf{I}$. Thus, $X \in \mathbf{I}$, and we are done. □

We can now give a condition under which an ideal $\mathbf{I}$ can be recovered from its support $\Phi_W(\mathbf{I})$.

Corollary 5.2.5. *Let $\mathbf{I}$ be an ideal such that $\mathrm{Loc}(\mathbf{I})$ contains $\mathbf{K}$. Then $\Theta_W \circ \Phi_W(\mathbf{I}) = \mathbf{I}$.*

Proof. By definition,

$$\begin{aligned}
\Theta_W \circ \Phi_W(\mathbf{I}) &= \Theta_W(\Phi_{\mathbf{C}}(\overline{F}(I))) \\
&= \{X \in \mathsf{st}(\mathsf{Z}(\mathbf{C})) : W(X) \subseteq \Phi_{\mathbf{C}}(\overline{F}(I))\} \\
&= \{X \in \mathsf{st}(\mathsf{Z}(\mathbf{C})) : V_{\mathbf{C}}(\overline{F}(X)) \subseteq \Phi_{\mathbf{C}}(\langle \overline{F}(I)\rangle)\} \\
&= \{X \in \mathsf{st}(\mathsf{Z}(\mathbf{C})) : \ \forall\, \mathbf{P} \in \mathrm{Spc}\,\mathsf{st}(\mathbf{C}) \text{ with } \overline{F}(X) \notin \mathbf{P}, \langle \overline{F}(\mathbf{I})\rangle \not\subseteq \mathbf{P}\} \\
&= \{X \in \mathsf{st}(\mathsf{Z}(\mathbf{C})) : \ \forall\, \mathbf{P} \in \mathrm{Spc}\,\mathsf{st}(\mathbf{C}) \text{ with } \langle \overline{F}(\mathbf{I})\rangle \subseteq \mathbf{P}, \overline{F}(X) \in \mathbf{P}\} \\
&= \left\{ X \in \mathsf{st}(\mathsf{Z}(\mathbf{C})) : \overline{F}(X) \in \bigcap_{\mathbf{P} \in \mathrm{Spc}\,\mathsf{st}(\mathbf{C}), \langle \overline{F}(\mathbf{I})\rangle \subseteq \mathbf{P}} \mathbf{P} \right\} \\
&= \{X \in \mathsf{st}(\mathsf{Z}(\mathbf{C})) : \overline{F}(X) \in \langle \overline{F}(\mathbf{I})\rangle\}.
\end{aligned}$$

The last equality follows from Proposition 3.3.3. The corollary now follows direclty from Theorem 5.2.4. $\square$

5.3. The image of contraction

We now describe the relationship of the image of the map f to the kernel $\mathbf{K}$ of $\overline{F}$. As in the previous section, we continue our assumption in this section that we are working with the category of modules for a finite-dimensional Hopf algebra, allowing us to apply Theorem 5.2.4.

Proposition 5.3.1. *Let* $\mathbf{C}$ *be the category of modules of a finite-dimensional Hopf algebra* H*.*

(a) *If* $\mathbf{P}$ *is in the image of the map* $f : \operatorname{Spc} \mathsf{st}(\mathbf{C}) \to \operatorname{Spc} \mathsf{st}(\mathsf{Z}(\mathbf{C}))$*, then* $\mathbf{P}$ *contains* $\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))$*, the kernel of* $\overline{F}$ *restricted to compact objects.*

(b) *If* $\mathbf{P}$ *is a prime ideal of* $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$ *such that* $\operatorname{Loc}(\mathbf{P})$ *contains* $\mathbf{K}$*, then* $\mathbf{P}$ *is in the image of* f*.*

Proof. For (a), if $\mathbf{Q}$ is a prime ideal of $\mathsf{st}(\mathbf{C})$, then $f(\mathbf{Q})$ contains $\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))$, which are by definition the finite-dimensional objects X such that $\overline{F}(X) \cong 0$: if X is in $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$ and $\overline{F}(X) \cong 0$, then $X \in \{Y : \overline{F}(Y) \in \mathbf{Q}\} = f(\mathbf{Q})$, since 0 is in every prime ideal of $\mathsf{st}(\mathbf{C})$.

Part (b) is an application of both Theorem 5.2.4 and Theorem 3.2.4. Let $\mathbf{P}$ be a prime ideal of $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$ such that $\operatorname{Loc}(\mathbf{P})$ contains $\mathbf{K}$. Consider the following two collections of objects in $\mathsf{st}(\mathbf{C})$:

(i) The ideal $\mathbf{I} := \langle \overline{F}(X) : X \in \mathbf{P} \rangle$ of $\mathsf{st}(\mathbf{C})$;

(ii) The collection $\mathcal{M} := \{\overline{F}(Y) : Y \notin \mathbf{P}\}$ of objects in $\mathsf{st}(\mathbf{C})$.

We first claim that these two collections of objects are disjoint. If $\overline{F}(Y) \in \mathbf{I}$ then

$Y \in \Theta_W(\Phi_W(\mathbf{P}))$, implying that $Y \in \mathbf{P}$ by Corollary 5.2.5. This means that in particular, if $\overline{F}(X) \cong \overline{F}(Y)$, then either both X and Y are in $\mathbf{P}$, or neither are, and so $\mathbf{I}$ and $\mathcal{M}$ are indeed disjoint.

Since $\mathbf{P}$ is a proper ideal of $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$, $\mathcal{M}$ is nonempty, and thus $\mathbf{I}$ is a proper ideal of $\mathsf{st}(\mathbf{C})$. We claim that $\mathcal{M}$ is a multiplicative subset. Suppose $\overline{F}(X)$ and $\overline{F}(Y)$ are in $\mathcal{M}$. Then if $\overline{F}(X) \otimes \overline{F}(Y) \cong \overline{F}(X \otimes Y)$ was not in $\mathcal{M}$, this would imply that $X \otimes Y \in \mathbf{P}$; by the prime condition of $\mathbf{P}$, either X or Y would then be in $\mathbf{P}$; without loss of generality, say $Y \in \mathbf{P}$. This is a contradiction, since $\overline{F}(Y) \in \mathcal{M}$ implies $Y \notin \mathbf{P}$, which is a consequence of the observation above that $\mathbf{I}$ and $\mathcal{M}$ are disjoint.

By Theorem 3.2.4, given a disjoint pair consisting of a multiplicative subset and a proper ideal of any monoidal triangulated category (in this case, $\mathsf{st}(\mathbf{C})$), there exists a prime ideal $\mathbf{Q}$ of $\mathsf{st}(\mathbf{C})$ such that $\mathbf{Q} \cap \mathcal{M} = \varnothing$ and $\mathbf{I} \subseteq \mathbf{Q}$. We have

$$f(\mathbf{Q}) = \{X \in \mathsf{st}(\mathsf{Z}(\mathbf{C})) : \overline{F}(X) \in \mathbf{Q}\},$$

and then since $\mathbf{I} \subseteq \mathbf{Q}$, it is automatic that $\mathbf{P} \subseteq f(\mathbf{Q})$; and since $\mathbf{Q}$ is disjoint from $\mathcal{M}$, in fact $\mathbf{P} = f(\mathbf{Q})$. Thus, f surjects onto the collection of prime ideals $\mathbf{P}$ such that $\mathrm{Loc}(\mathbf{P})$ contains $\mathbf{K}$, which completes the proof. □

By Proposition 5.3.1, we have inclusions of the following subsets of $\operatorname{Spc} \mathsf{st}(\mathsf{Z}(\mathbf{C}))$:

$$\{\mathbf{P} : \mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C})) \subseteq \mathbf{P}\} \supseteq \operatorname{im} f \supseteq \{\mathbf{P} : \mathbf{K} \subseteq \mathrm{Loc}(\mathbf{P})\}. \tag{5.1}$$

We recall that the following conditions are equivalent, which can be observed as a direct consequence of Theorem 2.7.1.

Lemma 5.3.2. *The following are equivalent.*

(a) $\mathbf{K}$ *is generated as a localizing category by the set* $\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))$.

(b) *For every nonzero* X *in* $\mathbf{K}$*, there exists a finite-dimensional module* Y *in* $\mathbf{K}$ *which has some nonzero map* $Y \to X$ *in* $\mathsf{St}(\mathsf{Z}(\mathbf{C}))$.

If these conditions are satisfied, then we can sharpen (5.1), as well as Corollary 5.2.5.

Corollary 5.3.3. *Suppose the kernel* $\mathbf{K}$ *of* $\overline{F}$ *satisfies the equivalent conditions of Lemma 5.3.2.*

(a) *The image of* f *is precisely the collection of prime ideals of* $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$ *which contain* $\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))$*, that is, the collection of finite-dimensional* $D(H)$*-modules* X *such that* $\overline{F}(X) \cong 0$.

(b) *A thick ideal* $\mathbf{I}$ *of* $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$ *satisfies* $\Theta_W \circ \Phi_W(\mathbf{I}) = \mathbf{I}$ *if and only if* $\mathbf{I}$ *contains* $\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))$.

Proof. Suppose $\mathbf{K}$ satsfies the conditions of Lemma 5.3.2, in other words, $\mathrm{Loc}(\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))) = \mathbf{K}$.

For (a), let $\mathbf{P}$ be a prime ideal of $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$ containing $\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))$. Then $\mathrm{Loc}(\mathbf{P})$ contains $\mathrm{Loc}(\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))) = \mathbf{K}$. Hence the collection of inequalities of (5.1) becomes an equality, and we are done.

For (b), similarly, we have by Corollary 5.2.5 that if $\mathrm{Loc}(\mathbf{I})$ contains $\mathbf{K}$, then $\Theta_W \circ \Phi_W(\mathbf{I}) = \mathbf{I}$. Since $\mathbf{K} = \mathrm{Loc}(\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C})))$, we have $\mathbf{K} \subseteq \mathrm{Loc}(\mathbf{I})$ if and only if $\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C})) \subseteq \mathbf{I}$. For the other direction, we note that for any ideal $\mathbf{I}$, we have $\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C})) \subseteq \Theta_W \circ \Phi_W(\mathbf{I})$, and so any thick ideal satisfying $\Theta_W \circ \Phi_W(\mathbf{I}) = \mathbf{I}$ must have $\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C})) \subseteq \mathbf{I}$ as well. $\square$

Remark 5.3.4. Corollary 5.3.3 implies that if H satisfies the conditions of Lemma 5.3.2, then the image of $f : \operatorname{Spc} \mathsf{st}(\mathbf{C}) \to \operatorname{Spc} \mathsf{st}(\mathsf{Z}(\mathbf{C}))$ is automatically the complement of a specialization-closed set, since we have

$$\begin{aligned} \operatorname{im}(f) &= \{\mathbf{P} \in \operatorname{Spc} \mathsf{st}(\mathsf{Z}(\mathbf{C})) : \mathbf{P} \supseteq \mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))\} \\ &= (\Phi_{\mathsf{Z}}(\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))))^c. \end{aligned}$$

In other words, the image of f can be written as an intersection of open sets. If $\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))$ is generated (as a thick ideal) by a finite collection of objects, say $\{X_i\}_{i=1}^n$, then it follows that $\operatorname{im}(f)$ is in fact an open subset of $\operatorname{Spc} \mathsf{st}(\mathsf{Z}(\mathbf{C}))$, namely

$$\operatorname{im}(f) = (V_{\mathsf{Z}}(X_1 \oplus ... \oplus X_n))^c.$$

Remark 5.3.5. In the situation of Corollary 5.3.3 (2), we have Corollary 5.2.5 sharpened from a one-way implication to a two-way implication. We note on the other hand that if the conditions of Lemma 5.3.2 are not satisfied, then Corollary 5.2.5 can never be an if-and-only-if, for the following reason. The collection of objects $\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))$ is itself a thick ideal of $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$, since it is in particular the kernel of the monoidal triangulated functor $\overline{F}$ restricted to compact objects. But now note that

$$\begin{aligned} \Theta_W \circ \Phi_W(\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))) &= \{X \in \mathsf{st}(\mathsf{Z}(\mathbf{C})) : W(X) \subseteq \Phi_W(\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C})))\} \\ &= \{X \in \mathsf{st}(\mathsf{Z}(\mathbf{C})) : W(X) \subseteq \varnothing\} \\ &= \{X \in \mathsf{st}(\mathsf{Z}(\mathbf{C})) : \overline{F}(X) \cong 0\} \\ &= \mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C})), \end{aligned}$$

recall Lemma 5.2.3. In other words, the thick ideal $\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))$ can be recovered from its support. But plainly, since we are assuming the conditions of Lemma 5.3.2 are not satisfied, we have

$$\mathrm{Loc}(\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))) \not\supseteq \mathbf{K},$$

and so Corollary 5.2.5 cannot be sharpened to an if-and-only-if statement.

We now give conditions under which Φ_W and Θ_W are inverses, and f is surjective, injective, and a homeomorphism.

Theorem 5.3.6. *Suppose* $\mathbf{C} = \mathsf{mod}(H)$ *for a finite-dimensional Hopf algebra* H.

(a) *The following conditions are equivalent:*

- *(i)* $X \cong 0$ *in* $\mathsf{St}(\mathsf{Z}(\mathbf{C}))$ *for all* $X \in \mathbf{K}$.
- *(ii)* *The map* f *is surjective and* $\mathbf{K}$ *is generated as a localizing category by* $\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))$.
- *(iii)* $\Theta_W \circ \Phi_W = \mathrm{id}$.

(b) *If* $\mathbf{C}$ *is braided, then the following conditions hold:*

- *(i)* *The map* f *is injective.*
- *(ii)* *If additionally* $\mathrm{Spc}\,\mathsf{st}(\mathbf{C})$ *is topologically Noetherian, then* $\Phi_W \circ \Theta_W = \mathrm{id}$.

(c) *If* $X \cong 0$ *in* $\mathsf{St}(\mathsf{Z}(\mathbf{C}))$ *for all* $X \in \mathbf{K}$ *and* $\mathbf{C}$ *is braided, then the following conditions hold:*

- *(i)* f *is a homeomorphism.*
- *(ii)* *If additionally* $\mathrm{Spc}\,\mathsf{st}(\mathbf{C})$ *is topologically Noetherian, then* Φ_W *and* Θ_W *are mutually inverse maps.*

Proof. Suppose (a)(i) holds, and so $\mathbf{K}$ consists only of objects isomorphic to 0, in other words, for all $D(H)$-modules X,

$$F(X) \text{ is projective as an } H\text{-module} \Leftrightarrow X \text{ is projective as a } D(H)\text{-module}.$$

In particular this means that $\mathbf{K}$ is generated by $\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))$, since all objects of $\mathbf{K}$ are isomorphic to 0. Then the conditions (a)(ii) and (a)(iii) follow directly from Corollary 5.3.3.

Now, suppose (a)(ii) is satisfied. By Proposition 5.3.1, this means that every prime ideal of $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$ contains $\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))$. But since every ideal is semiprime, this means that the 0-ideal is equal to the intersection of all primes of $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$, and so $\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))$ is contained in the zero ideal. Since $\mathbf{K}$ is generated by $\mathbf{K} \cap \mathsf{st}(\mathsf{Z}(\mathbf{C}))$, i.e. the zero ideal, this implies that (a)(i) holds.

For the third implication, suppose condition (a)(iii) holds. This implies by Corollary 5.2.5 that $\mathbf{K} \subseteq \mathrm{Loc}(\mathbf{I})$ for every thick ideal $\mathbf{I}$; in particular, this means that $\mathbf{K}$ is contained in the localizing category generated by 0, which consists only of objects isomorphic to 0. Hence, (a)(i) holds.

To show (b), first note that if $\mathbf{C}$ is braided with a braiding γ, then $\overline{F}$ is essentially surjective, since for any object X in $\mathbf{C}$, the pair (X, γ_X) is an object of $\mathsf{Z}(\mathbf{C})$ and $\overline{F}$ sends

$H(X, \gamma_X)$ to $G(X)$. Now, we note that if there exist prime ideals $\mathbf{P}$ and $\mathbf{Q}$ of $\mathsf{st}(\mathbf{C})$, then:

$$f(\mathbf{P}) = f(\mathbf{Q})$$

$$\Updownarrow$$

$$\{X : \overline{F}(X) \in \mathbf{P}\} = \{X : \overline{F}(X) \in \mathbf{Q}\}$$

$$\Updownarrow$$

$$\forall\, X \in \mathsf{st}(\mathsf{Z}(\mathbf{C})),\ \overline{F}(X) \in \mathbf{P} \Leftrightarrow \overline{F}(X) \in \mathbf{Q}$$

$$\Updownarrow$$

$$\forall\, Y \in \mathsf{st}(\mathbf{C}),\ Y \in \mathbf{P} \Leftrightarrow Y \in \mathbf{Q}$$

$$\Updownarrow$$

$$\mathbf{P} = \mathbf{Q}.$$

Hence, if $\mathbf{C}$ is braided then (b)(i) follows.

For (b)(ii), note that by [8, Corollary 2.17], $\mathrm{Spc}(\mathsf{st}(\mathbf{C}))$ is Noetherian if and only if every closed set is of the form $V_{\mathbf{C}}(A)$ for some object $A \in \mathsf{st}(\mathbf{C})$. If S is a specialization-closed set in $\mathrm{Spc}(\mathsf{st}(\mathbf{C}))$, then by definition

$$\begin{aligned}\Phi_W(\Theta_W(S)) &= \Phi_W(\{X \in \mathsf{st}(\mathsf{Z}(\mathbf{C})) : W(X) \subseteq S\}) \\ &= \bigcup_{X \in \Theta_W(S)} W(X) \\ &\subseteq S.\end{aligned}$$

For the other direction, we can write S as a union of closed sets, say $S = \bigcup_{i \in I} S_i$, and by the Noetherianity of $\mathrm{Spc}(\mathsf{st}(\mathbf{C}))$, there exist objects A_i of $\mathsf{st}(\mathbf{C})$ such that $S_i = V_{\mathbf{C}}(A_i)$.

Since $\overline{F}$ is essentially surjective, we can pick $X_i \in \mathsf{st}(\mathsf{Z}(\mathbf{C}))$ with $\overline{F}(X_i) = A_i$. Since

$$W(X_i) = V_{\mathbf{C}}(A_i) = S_i \subseteq S,$$

we have by definition each X_i is in $\Theta_W(S)$. Therefore,

$$\begin{aligned}\Phi_W(\Theta_W(S)) &\supseteq \bigcup_{i\in I} W(X_i) \\ &= \bigcup_{i\in I} S_i \\ &= S.\end{aligned}$$

Thus $S = \Phi_W(\Theta_W(S))$.

Suppose the assumptions of (c). Then (c)(ii) follows immediately from parts (a) and (b). To show (c)(i), it is enough to show that f is a closed map, by (a)(i) and (b)(i). Take an arbitrary closed set $V_{\mathbf{C}}(\mathcal{T})$ in $\operatorname{Spc}\mathsf{st}(\mathbf{C})$. We claim that the image of $V_{\mathbf{C}}(\mathcal{T})$ under f is precisely $V_{\mathsf{Z}}(\hat{\mathcal{T}})$, where $\hat{\mathcal{T}} = \{X \in \mathsf{st}(\mathsf{Z}(\mathbf{C})) : \overline{F}(X) \in \mathcal{T}\}$.

For the first direction, suppose $\mathbf{P} \in V_{\mathbf{C}}(\mathcal{T})$, in other words, $\mathbf{P} \cap \mathcal{T} = \varnothing$. Since $f(\mathbf{P}) = \{X : \overline{F}(X) \in \mathbf{P}\}$, this implies that for all $X \in f(\mathbf{P})$, we have $X \notin \hat{\mathcal{T}}$. Therefore $f(\mathbf{P}) \cap \hat{\mathcal{T}} = \varnothing$, and so $f(\mathbf{P}) \in V_{\mathsf{Z}}(\hat{\mathcal{T}})$. This shows $f(V_{\mathbf{C}}(\mathcal{T})) \subseteq V_{\mathsf{Z}}(\hat{\mathcal{T}})$.

For the other containment, suppose $\mathbf{Q}$ is a prime ideal of $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$ in $V(\hat{\mathcal{T}})$. Then $\overline{F}(X) \notin \mathcal{T}$ for all $X \in \mathbf{Q}$. Since f is surjective, we can pick $\mathbf{P} \in \operatorname{Spc}\mathsf{st}(\mathbf{C})$ with $f(\mathbf{P}) = \mathbf{Q}$, and for all $\overline{F}(X) \in \mathbf{P}$, we must have $\overline{F}(X) \notin \mathcal{T}$. Since $\overline{F}$ is essentially surjective, this implies $A \notin \mathcal{T}$ for all $A \in \mathbf{P}$, and so $\mathbf{P} \cap \mathcal{T} = \varnothing$, i.e. $\mathbf{P} \in V(\mathcal{T})$. This shows the other containment $f(V_{\mathbf{C}}(\mathcal{T})) \supseteq V_{\mathsf{Z}}(\hat{\mathcal{T}})$, and so we have equality.

Hence, f sends the closed set $V_{\mathbf{C}}(\mathcal{T})$ to the closed set $V_{\mathsf{Z}}(\hat{\mathcal{T}})$, and so it is a continuous, bijective, closed map, and therefore a homeomorphism. $\square$

Chapter 6. Drinfeld Doubles of Cosemisimple Hopf Algebras

6.1. Drinfeld doubles of finite groups

Let G be a finite group and $\Bbbk$ be a field of characteristic p which divides the order of G, and $\Bbbk G$ the group algebra of G over $\Bbbk$. Let $\mathbf{C} = \mathsf{mod}(\Bbbk G)$, a finite tensor category. The Drinfeld double $D(\Bbbk G)$ is a Hopf algebra containing $\Bbbk G$ and $(\Bbbk G^{\mathrm{op}})^*$ as Hopf subalgbras. We will denote the dual of the group algebra by $\Bbbk[G]$, and in that case we can write $(\Bbbk G^{\mathrm{op}})^* = \Bbbk[G]^{\mathrm{cop}}$. The collection

$$\{e_g h : g, h \in G\}$$

is a $\Bbbk$-basis, where the elements $\{e_g : g \in G\}$ refer to the dual basis of $\Bbbk[G]^{\mathrm{cop}}$. The multiplication is determined by the relations

$$he_g = e_{hgh^{-1}}h,$$

see for instance [62, Section IX.4.3].

Lemma 6.1.1. *Let G be a finite group and $\Bbbk$ a field of characteristic p dividing the order of G, and let $F : \mathsf{Mod}(D(\Bbbk G)) \to \mathsf{Mod}(\Bbbk G)$ be the forgetful functor. Then F has the property that $F(P)$ is projective in as a $\Bbbk G$-module implies P is projective as a $D(\Bbbk G)$-module.*

Proof. A module for $D(\Bbbk G)$ is a $\Bbbk G$ module M which is also a G-graded vector space, such that if $m \in M$ is a homogeneous element of degree g, then $h.m$ is homogeneous of degree hgh^{-1}. Suppose we have a short exact sequence

$$0 \to A \to B \xrightarrow{t} C \to 0$$

of $D(\Bbbk G)$-modules such that

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is a split short exact sequence of G-modules. We claim that the original sequence splits as $D(\Bbbk G)$-modules. Pick a homogeneous basis $\{c_i\}$ of C under the G-grading, where c_i has degree g_i. Now pick a splitting $s : C \to B$. Define $\hat{s}(c_i) = e_{g_i}s(c_i)$. This map is homogeneous with respect to the G-grading, and it is still a G-module map:

$$\begin{aligned} g\hat{s}(c_i) &= ge_{g_i}s(c_i) \\ &= e_{gg_ig^{-1}}gs(c_i) \\ &= e_{gg_ig^{-1}}s(gc_i) \\ &= \hat{s}(gc_i). \end{aligned}$$

Since on the basis $\{c_i\}$ we have

$$\begin{aligned} t \circ \hat{s}(c_i) &= t(e_{g_i}.s(c_i)) \\ &= e_{g_i}.ts(c_i) \\ &= e_{g_i}.c_i \\ &= c_i, \end{aligned}$$

we have that $\hat{s}$ is a splitting of $D(\Bbbk G)$-modules.

Now, to prove the original claim, suppose $F(P)$ is projective as a G-module. Since F is exact, this means that for every short exact sequence

$$0 \to A \to B \to P \to 0$$

in $D(H)$-modules, the sequence

$$0 \to F(A) \to F(B) \to F(P) \to 0$$

is split as G-modules. Therefore, the original sequences are all split, and so P is projective. $\square$

We recall that by [8, Corollary 5.10], $\operatorname{Spc} \mathsf{stmod}(\Bbbk G) \cong \operatorname{Proj} \mathrm{H}^\bullet(G, \Bbbk)$

Theorem 6.1.2. *Let G be a finite group and $\Bbbk$ a field of characteristic p dividing the order of G. Let $H^\bullet(G, \Bbbk)$ be the cohomology ring of G.*

(a) *The map $f : \operatorname{Spc} \mathsf{stmod}(\Bbbk G) \to \operatorname{Spc} \mathsf{stmod}(D(\Bbbk G))$ gives a homeomorphism, and so* $\operatorname{Spc} \mathsf{stmod}(D(\Bbbk G)) \cong \operatorname{Spc} \mathsf{stmod}(\Bbbk G) \cong \operatorname{Proj} \mathrm{H}^\bullet(G, \Bbbk)$.

(b) *Thick ideals of $\mathsf{stmod}(D(\Bbbk G))$ are in bijection with specialization-closed sets in* $\operatorname{Proj} \mathrm{H}^\bullet(G, \Bbbk)$*, which are in bijection with thick ideals of $\mathsf{stmod}(\Bbbk G)$, via the maps*

$$\operatorname{ThickId}(\mathsf{stmod}(D(\Bbbk G)) \underset{\Theta_W}{\overset{\Phi_W}{\rightleftarrows}} \mathcal{X}_{sp}(\operatorname{Proj} H^\bullet(G, \Bbbk)) \underset{\Phi_{\Bbbk G}}{\overset{\Theta_{\Bbbk G}}{\rightleftarrows}} \operatorname{ThickId}(\mathsf{stmod}(\Bbbk G)).$$

Proof. Since $\Bbbk G$ is cocommutative, $\mathsf{mod}(\Bbbk G)$ is braided. By Lemma 6.1.1, we have $X \cong 0$ in $\mathsf{StMod}(D(H))$ for all $X \in \mathbf{K}$, and so we are in the situation given of Theorem 5.3.6(3). Additionally, since cohomology rings of groups are finitely generated (see [42], in which finite generation of cohomology rings for finite-dimensional cocommutative Hopf algebras in positive characteristic was proven), we know that $\operatorname{Proj} \mathrm{H}^\bullet(G, \Bbbk)$ is a Noetherian topological space. Using Balmer's classification of thick ideals [8, Theorem 4.10], the thick ideals of $\mathsf{stmod}(\Bbbk G)$ are in bijection with specialization-closed sets in $\operatorname{Spc} \mathsf{stmod}(\Bbbk G)$. The rest of the theorem now follows directly as an application of Theorem 5.3.6. $\square$

Now, note that since $\Bbbk[G]^{\text{cop}}$ is a semisimple algebra, $\mathsf{stmod}(\Bbbk[G]^{\text{cop}})$ consists only of the zero object, up to isomorphism, and so $\mathrm{Spc}(\mathsf{stmod}(\Bbbk[G]^{\text{cop}}))$ is the empty set. Thus, the diagram from Remark 5.1.3 becomes

$$\begin{tikzcd}
\operatorname{Spc}\mathsf{stmod}(\Bbbk G) \arrow[r, leftrightarrow, "\cong"] \arrow[dr, "\cong"'] & \operatorname{Proj} \mathrm{H}^\bullet(G,\Bbbk) & \operatorname{Spc}\mathsf{stmod}(\Bbbk[G]^{\text{cop}}) = \varnothing \arrow[dl] \\
& \operatorname{Spc}\mathsf{stmod}(D(\Bbbk G)) &
\end{tikzcd}$$

6.2. Drinfeld doubles of cosemisimple Hopf algebras

In fact, we are able to generalize Lemma 6.1.1 from the group algebra case to the case of all finite-dimensional cosemisimple Hopf algebras.

Lemma 6.2.1. *Let H be a finite-dimensional cosemisimple Hopf algebra with Drinfeld double $D(H)$ and $F : \mathsf{Mod}(D(H)) \to \mathsf{Mod}(H)$ be the forgetful functor. Then F has the property that $F(P)$ is projective in as a H-module implies P is projective as a $D(H)$-module.*

Proof. We will utilize the proof of [35, Proposition 7.18.15]. In the course of this proof, it is shown that if H is cosemisimple, then $\mathbb{1}_{D(H)}$ is a direct summand of $D(H) \otimes_H \mathbb{1}_H$ as $D(H)$-modules (note that here, we are reversing the roles of H and H^* given in their proof). We note that although the proof of [35, Proposition 7.18.15] assumes a stronger condition– that H itself is also semisimple– this assumption is not used for the part of the proof by which $D(H) \otimes_H \mathbb{1}_H$ has $\mathbb{1}_{D(H)}$ as a summand.

The functor $D(H) \otimes_H -$ is a left adjoint to the forgetful functor F. Since F is exact, if Q is a projective H-module then

$$\mathrm{Hom}_H(Q, F(-)) \cong \mathrm{Hom}_{D(H)}(D(H) \otimes_H Q, -)$$

is an exact functor (recalling that projectives are also injective), and so $D(H) \otimes_H -$ preserves projectivity. Therefore, if P is a $D(H)$-module such that $F(P)$ is projective, then $D(H) \otimes_H F(P)$ is a projective $D(H)$-module. But then, we have

$$D(H) \otimes_H F(P) \cong D(H) \otimes_H (\mathbb{1}_H \otimes_{\Bbbk} F(P)) \cong (D(H) \otimes_H \mathbb{1}_H) \otimes_{\Bbbk} P,$$

where the last isomorphism here can be seen from, for example, [44, Proposition 1.7] and the remark following it, which notes that although the proposition is stated for certain universal enveloping algebras, in fact the proof uses only the Hopf algebra structure, and so the result holds for arbitrary Hopf algebras. Note that it holds not just for finite-dimensional modules, but for arbitrary modules, which we need since in this case P may be infinite-dimensional.

Now, since $\mathbb{1}_{D(H)}$ is a summand of $D(H) \otimes_H \mathbb{1}_H$, we have that $P \cong \mathbb{1}_{D(H)} \otimes_{\Bbbk} P$ is a direct summand of $(D(H) \otimes_H \mathbb{1}_H) \otimes_{\Bbbk} P$, which is a projective $D(H)$-module, and hence P is projective as well, and the claim is proven.

$\square$

Lemma 6.2.1 now implies the following, by Theorem 5.3.6.

Theorem 6.2.2. *Let H be a finite-dimensional cosemisimple Hopf algebra.*

(a) *The map $f : \operatorname{Spc} \mathsf{stmod}(H) \to \operatorname{Spc} \mathsf{stmod}(D(H))$ constructed in Section 5.2 is surjective, and the map $\Theta_W \circ \Phi_W$ constructed in Section 5.2 is the identity, as a map from the collection of thick ideals of $\mathsf{stmod}(D(H))$ to itself.*

(b) *If H is additionally quasitriangular, then f is a homeomorphism.*

(c) *If H is both quasitriangular and $\operatorname{Spc} \mathsf{stmod}(H)$ is topologically Noetherian, then Φ_W*

and Θ_W *are inverse maps, and so we have the following bijections of thick ideals:*

$$\mathrm{ThickId}(\mathsf{stmod}(D(H)) \underset{\Theta_W}{\overset{\Phi_W}{\rightleftarrows}} \mathcal{X}_{sp}(\mathrm{Spc}(\mathsf{stmod}(H))) \underset{\Phi_H}{\overset{\Theta_H}{\rightleftarrows}} \mathrm{ThickId}(\mathsf{stmod}(H)).$$

Of course, if H itself is also semisimple, then Theorem 6.2.2 is not interesting, since this implies that $D(H)$ is also semisimple, and the Balmer spectra of $\mathsf{stmod}(H)$ and $\mathsf{stmod}(D(H))$ are both $\varnothing$. It is a classical theorem of Larson-Radford [66] that in characteristic 0, all cosemisimple finite-dimensional Hopf algebras are also semisimple. Hence, Theorem 6.2.2 only provides interesting examples in positive characteristic.

Chapter 7. Benson-Witherspoon Smash Coproducts

7.1. Benson-Witherspoon smash coproducts

In this section, we use the method of Section 4.3, to give an explicit description of the Balmer spectra of the stable module categories of the Benson–Witherspoon Hopf algebras [19] and a classification of their thick two-sided ideals.

The Benson–Witherspoon Hopf algebras are the Hopf duals of smash products of a group algebra and a coordinate ring of a group. They were studied in [19].

In more detail, let G and H be finite groups, with H acting on G by group automorphisms. Let $\Bbbk$ be a field of positive characteristic dividing the order of G. Define A as the Hopf algebra dual to the smash product $\Bbbk[G]\#\Bbbk H$, where $\Bbbk[G]$ is the dual of the group algebra of G, and $\Bbbk H$ is the group algebra of H. Denote by p_g the dual basis element of $\Bbbk[G]$ corresponding to $g \in G$. By definition, this smash product is $\Bbbk[G] \otimes \Bbbk H$ as a vector space, and multiplication is given by

$$(p_g \otimes x)(p_h \otimes y) = p_g(x_{(1)}.p_h) \otimes x_{(2)}y = p_g p_{x.h} \otimes xy = \delta_{g,x.h} p_g \otimes xy$$

for all $g \in G$ and $x, y \in H$. Now define

$$A = \mathrm{Hom}_{\Bbbk}(\Bbbk[G]\#\Bbbk H, \Bbbk).$$

As an algebra, $A = \Bbbk G \otimes \Bbbk[H]$. The comultiplication in A is given by

$$\Delta(g \otimes p_x) = \sum_{y \in H} (g \otimes p_y) \otimes (y^{-1}.g \otimes p_{y^{-1}x}).$$

The counit and antipode are given by

$$\epsilon(g \otimes p_x) = \delta_{x,1} \quad \text{and} \quad S(g \otimes p_x) = x^{-1}.(g^{-1}) \otimes p_{x^{-1}}.$$

Note that while G is a subalgebra of A, via the map $g \mapsto g \otimes 1$, it is not a Hopf subalgebra, since

$$\Delta_G(g) = g \otimes g$$

and

$$\Delta_A(g \otimes 1) = \sum_{x \in H} \Delta_A(g \otimes p_x) = \sum_{x,y \in H} (g \otimes p_y) \otimes (y^{-1}.g \otimes p_{y^{-1}x}) \neq (g \otimes 1) \otimes (g \otimes 1).$$

An A-module is the same as an H-graded $\Bbbk G$-module. We may write any A-module M in the form

$$M = \bigoplus_{x \in H} M_x \otimes \Bbbk_x,$$

where the M_x are $\Bbbk G$-modules. The action of $\Bbbk G$ is on the first tensor and $\Bbbk[H]$ acts on the second.

In [19], Benson and Witherspoon prove the following formula for the decomposition of a tensor product of A-modules:

$$(M \otimes \Bbbk_x) \otimes (N \otimes \Bbbk_y) = (M \otimes {}^{x}N) \otimes \Bbbk_{xy}$$

on homogeneous components. Here and below for $M \in \mathsf{Mod}(\Bbbk G)$ and $x \in H$, ${}^{x}M \in \mathsf{Mod}(\Bbbk G)$ denotes the conjugate of M by the action of $x \in H \to \mathrm{Aut}(G)$. On homogeneous components, the dual of a module is given by

$$(M \otimes \Bbbk_x)^* = {}^{x}(M^*) \otimes \Bbbk_{x^{-1}}.$$

By the definition of the smash product, we have an embedding of Hopf algebras

$$\Bbbk[G] \hookrightarrow \Bbbk[G] \# \Bbbk H.$$

Hence, when we dualize to the smash coproduct, we get a Hopf algebra surjection

$$\Bbbk G \twoheadleftarrow A.$$

We will use the following notation:

(i) The cohomology rings of A and $\Bbbk G$ will be denoted by

$$R_A = \operatorname{Ext}_A^\bullet(\Bbbk, \Bbbk) \quad \text{and} \quad R_G = \operatorname{Ext}_G^\bullet(\Bbbk, \Bbbk),$$

respectively.

(ii) Denote the spaces

$$X^A = \operatorname{Proj} R_A \quad \text{and} \quad X^G = \operatorname{Proj} R_G.$$

The collections of their specialization closed subsets and all subsets will be denoted respectively by

$$\mathcal{X}^A_{sp},\ \mathcal{X}^A,\ \mathcal{X}^G_{sp},\ \mathcal{X}^G.$$

(iii) We will use the support functions on $\mathsf{StMod}(A)$ and $\mathsf{StMod}(\Bbbk G)$ from [16], as described above in Section 2.7, where the relevant ring R is taken to be R_A and R_G, respectively. Recall that they are defined using the localization and colocalization functors from Theorem 2.7.1. They take values in the sets of all subsets of the spaces of $\mathcal{X}^A$ and $\mathcal{X}^G$, respectively, and denote them:

$$W_G(-) : \mathsf{StMod}(\Bbbk G) \to \mathcal{X}^G,$$

$$\widetilde{W}_A(-) : \mathsf{StMod}(A) \to \mathcal{X}^A.$$

These support maps extend the corresponding cohomological support functions defined in Section 2.6.

(iv) The map Φ associated to the support $\widetilde{W}_A$ will be denoted by $\widetilde{\Phi}_A$. It takes thick subcategories of $\mathsf{StMod}(A)$ to subsets of $\mathcal{X}^A$.

(v) The functor $\mathsf{Mod}(\Bbbk G) \to \mathsf{Mod}(A)$ defined on objects by

$$M \mapsto M \otimes \Bbbk_e$$

will be denoted by $\mathcal{F}$.

Let p be a prime number and n be a positive integer. In [19, Example 3.3] Benson and Witherspoon proved that for $G := (\mathbb{Z}/p\mathbb{Z})^n$, $H := \mathbb{Z}/n\mathbb{Z}$ (with H cyclically permuting the factors of G) and $\Bbbk$ a field of characteristic p, the smash coproduct A admits a non-projective finite dimensional module M such that $M \otimes M$ is projective. By Corollary 3.4.5, the universal support datum map for $\mathsf{stmod}(A)$ does not satisfy the tensor product property.

Benson and Witherspoon constructed [19, Example 3.2] a smash coproduct A which has a pair of finite dimensional representations M and N with the property that $M \otimes N$ is not projective, but $N \otimes M$ is is projective. The group G is chosen to be the Klein 4-group, H is the cyclic group of order 3 whose generator cyclically permutes the non-identity elements of G, and the field $\Bbbk$ has characteristic 2. By Corollary 3.4.6, for this Hopf algebra A, the universal support datum map for $\mathsf{stmod}(A)$ does not satisfy the tensor product property either.

Lemma 7.1.1. *The functor $\mathcal{F}$ descends to a fully faithful tensor triangulated functor* For : $\mathsf{StMod}(\Bbbk G) \to \mathsf{StMod}(A)$.

Proof. By the formula for the tensor product of A-modules, $\mathcal{F}$ is monoidal, since

$$\mathcal{F}(M \otimes N) \cong \mathcal{F}(M) \otimes \mathcal{F}(N).$$

It is clear that $\mathcal{F}$ is exact, and it is fully faithful since morphisms of A-modules are the same as graded morphisms of $\Bbbk G$-modules. The functor $\mathcal{F}$ descends to a functor $\mathsf{StMod}(\Bbbk G) \to \mathsf{StMod}(A)$ because it sends projective modules to projective modules, and has the property that for each morphism f in $\mathsf{Mod}(\Bbbk G)$, if $\mathcal{F}(f)$ factors through a projective module in $\mathsf{Mod}(A)$, then f factors through a projective module in $\mathsf{Mod}(\Bbbk G)$. $\square$

Denote by For : $\mathsf{Mod}(A) \to \mathsf{Mod}(\Bbbk G)$ the forgetful functor. It is clear that it descends to a tensor triangulated functor $\mathsf{StMod}(\Bbbk G) \to \mathsf{StMod}(A)$.

Theorem 7.1.2. *For all Benson–Witherspoon Hopf algebras A the following hold:*

(a) *There is a canonical isomorphism $R_G \cong R_A$. (Denote $R := R_G \cong R_A$).*

(b) *If C and Q are $\Bbbk G$-modules, there is an isomorphism of R-modules*

$$\mathrm{Ext}^\bullet_G(C, Q) \cong \mathrm{Ext}^\bullet_A(\mathcal{F}(C), \mathcal{F}(Q)),$$

and A satisfies the (fg) condition, see Definition 2.5.7.

(c) *For an A-module N,*

$$\widetilde{W}_A(N) = W_G(\mathrm{For}(N)).$$

(d) *For an A-module Q,*

$$\widetilde{\Phi}_A(\langle Q \rangle) = H \cdot W_G(\mathrm{For}(Q)).$$

Proof. For (a) and (b), suppose

$$0 \to \mathcal{F}(Q) \to N_1 \to \ldots \to N_i \to \mathcal{F}(C) \to 0$$

is an exact sequence representing an element of $\mathrm{Ext}^\bullet_A(\mathcal{F}(C), \mathcal{F}(Q))$. Then we claim it is equivalent to an exact sequence which is supported only at the identity component. To do this, we may just note that the natural maps give an equivalence of extensions:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & \mathcal{F}(Q) & \longrightarrow & N_1 & \longrightarrow & \dots & \longrightarrow & N_i & \longrightarrow & \mathcal{F}(C) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}(Q) & \longrightarrow & (N_1)_e & \longrightarrow & \dots & \longrightarrow & (N_i)_e & \longrightarrow & \mathcal{F}(C) & \longrightarrow & 0 \end{array}$$

This gives a vector space isomorphism

$$\mathrm{Ext}^\bullet_A(\mathcal{F}(Q), \mathcal{F}(C)) \cong \mathrm{Ext}^\bullet_G(Q, C).$$

This isomorphism is compatible with the actions of R_A and R_G because $\mathcal{F}$ is a monoidal functor. This decomposition allows us to conclude (fg) for A, since it is well-known that this assumption holds for $\Bbbk G$.

For (c), write $N = \bigoplus_{z\in H} N_z \otimes \Bbbk_z$ with $N_z \in \mathsf{mod}(\Bbbk G)$. Note that by [16, Theorem 5.2],

$$\widetilde{W}_A(N) = \bigcup_{C \text{ compact}} \min \mathrm{Hom}^\bullet_{\mathsf{StMod}(A)}(C, N),$$

where, for an R_A-module L, $\min L$ refers to the minimal primes in the support of L.

Hence,

$$\begin{aligned} \widetilde{W}_A(N) &= \bigcup_{C\in \mathsf{mod}(\Bbbk G), z\in H} \min \mathrm{Hom}^\bullet_{\mathsf{StMod}(A)}(C\otimes \Bbbk_z, N) \\ &= \bigcup_{C,z} \min \mathrm{Hom}^\bullet_{\mathsf{StMod}(A)}(C\otimes \Bbbk_z, N_z\otimes \Bbbk_z) \\ &= \bigcup_{C,z} \min \mathrm{Hom}^\bullet_{\mathsf{StMod}(A)}(C\otimes \Bbbk_e, (N_z\otimes \Bbbk_z)\otimes(\Bbbk\otimes\Bbbk_z)^*) \\ &= \bigcup_{C,z} \min \mathrm{Hom}^\bullet_{\mathsf{StMod}(A)}(C\otimes \Bbbk_e, N_z\otimes \Bbbk_e) \\ &= \bigcup_{C,z} \min \mathrm{Hom}^\bullet_{\mathsf{StMod}(\Bbbk G)}(C, N_z) = \bigcup_{z} W_G(N_z). \end{aligned}$$

The second to last equality follows from the fact that for $i > 0$,

$$\mathrm{Hom}^i_{\mathsf{StMod}(A)}(C\otimes \Bbbk_e, N_z\otimes \Bbbk_e) \cong \mathrm{Ext}^i_A(C\otimes \Bbbk_e, N_z\otimes \Bbbk_e)$$

by [28, Proposition 2.6.2], which is isomorphic to $\mathrm{Ext}^i_G(C, N_z)$ by (2). Additionally, for $i = 0$ we have

$$\mathrm{Hom}_{\mathsf{StMod}(A)}(C \otimes \Bbbk_e, N_z \otimes \Bbbk_e) \cong \mathrm{Hom}_{\mathsf{StMod}(\Bbbk G)}(C, N_z)$$

since the functor $\mathcal{F}$ is fully faithful.

For (d), we have

$$\begin{aligned}
\widetilde{\Phi}_A(\langle Q\rangle) &= \bigcup_{M,N,x,y,z} \widetilde{W}_A((M_x \otimes \Bbbk_x) \otimes (Q_z \otimes \Bbbk_z) \otimes (N_y \otimes \Bbbk_y)) \\
&= \bigcup_{M,N,x,y,z} \widetilde{W}_A((M_x \otimes^x Q_z \otimes^{xz} N_y) \otimes \Bbbk_{xzy}) \\
&= \bigcup_{M,N,x,y,z} W_G(M_x \otimes^x Q_z \otimes^{xz} N_y) \\
&= \bigcup_{M,N,x,y,z} (W_G(M_x) \cap W_G({}^x Q_z) \cap W_G({}^{xz} N_y)) \\
&= \bigcup_{x,z} x.W_G(Q_z) = H \cdot (W_G(\mathrm{For}(Q))).
\end{aligned}$$

$\square$

Corollary 7.1.3. *For all Benson–Witherspoon Hopf algebras A,*

$$\widetilde{W}_A : \mathsf{StMod}(A) \to \mathcal{X}(\mathrm{Proj}(R_A))$$

is an extended weak support datum on $\mathsf{StMod}(A)$ *satisfying the Faithfulness Property* (4.2).

Proof. The fact that $\widetilde{W}_A$ satisfies conditions (a)–(d) in Definition 4.1.8 follows from Theorem 7.1.2(c) and the fact that W_G is a support datum for $\mathsf{StMod}(\Bbbk G)$. For condition (e) in Definition 4.1.8, we need to verify the property

$$\widetilde{\Phi}_A(\langle M\rangle \otimes \langle N\rangle) = \widetilde{\Phi}_A(\langle M\rangle) \cap \widetilde{\Phi}_A(\langle N\rangle).$$

This follows as both sides are equal to

$$[H \cdot W_G(\mathrm{For}(M))] \cap [H \cdot W_G(\mathrm{For}(N))]$$

by Theorem 7.1.2(d).

To check the faithfulness of $\widetilde{W}_A$, assume that $M = \oplus_{x\in H} M_x \otimes \Bbbk_x \in \mathsf{Mod}(A)$ is such that $\widetilde{\Phi}_A(\langle M\rangle) = \varnothing$. Applying Theorem 7.1.2(d), gives that $H \cdot W_G(\mathrm{For}(M)) = \varnothing$. By the faithfulness of $\widetilde{W}_G$, $\mathrm{For}(M) = \oplus_{x\in H} M_x$ is a projective $\Bbbk G$-module, and thus, M_x are projective $\Bbbk G$-modules for all $x \in H$. This implies that M is a projective A-module. $\square$

In order to explicitly describe the Balmer spectrum of $\mathsf{stmod}(A)$, we must produce a weak support datum having the Faithfulness and Realization Properties (4.2)–(4.3). To get the Realization Property (4.3), we will need to consider a new support datum built from $\widetilde{W}_A$. Denote

$$X_H = H\text{-}\,\mathrm{Proj}(R_A),$$

the space of nonzero homogeneous H-prime ideals of A in the sense of Lorenz [69], i.e. nonzero H-invariant homogeneous ideals P of R_A that have the property $IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$ for all H-invariant homogeneous ideals I, J of R_A. X_H is a Zariski space by the argument in [21, Section 2.3]. The space of H-orbits in $\mathrm{Proj}(R_A)$ will be denoted by

$$\widetilde{X}_H = H\backslash\,\mathrm{Proj}(R_A).$$

By [69, Section 1.3], there are maps

$$\begin{tikzcd}
 & & \mathfrak{p} \arrow[dddll, mapsto] \arrow[dddrr, mapsto] & & \\
 & & X_A \arrow[dl, twoheadrightarrow, "\pi_1"'] \arrow[dr, twoheadrightarrow, "\pi"] & & \\
 & \widetilde{X}_H \arrow[rr, twoheadrightarrow, "\pi_2"] & & X_H & \\
H \cdot \mathfrak{p} \arrow[rrrr, mapsto] & & & & \bigcap_{h \in H} h \cdot \mathfrak{p}
\end{tikzcd}$$

and the topologies on $\widetilde{X}_H$ and X_H are defined to be the final topologies with respect to the surjections from X_A.

Denote

$$W_A = \pi \circ \widetilde{W}_A : \mathsf{StMod}(A) \to \mathcal{X}(X_H).$$

Denote by Φ_A the associated map Φ_{W_A} map given by (4.1).

Lemma 7.1.4. *For all Benson–Witherspoon Hopf algebras A, W_A is a weak support datum satisfying the Faithfulness and Realization Properties* (4.2)–(4.3)*.*

Proof. Since $\widetilde{W}_A$ is a weak support datum satisfying the Faithfulness Property, the same is true for $W_A = \pi \circ \widetilde{W}_A$.

Because X_H is equipped with the final topology with respect to π, and the preimage of $W_A(Q) = \pi(\widetilde{W}_A(Q))$ is $\widetilde{W}_A(Q)$, which is closed, we have that $W_A(Q)$ is closed in X_H.

Let $Y \subseteq X_H$ be closed. Then $\pi^{-1}(Y)$ is a closed H-stable subset of X_A. This implies there is some Q with

$$W_G(Q) = \pi^{-1}(Y).$$

Since $\pi^{-1}(Y)$ is H-stable, using Theorem 7.1.2, we may check

$$\begin{aligned}\Phi_A(\langle \mathcal{F}(Q)\rangle) &= \pi \circ \widetilde{\Phi}_A(\langle \mathcal{F}(Q)\rangle) = \pi(H \cdot W_G(Q)) \\ &= \pi(H \cdot \pi^{-1}(Y)) = \pi(\pi^{-1}(Y)) = Y.\end{aligned}$$

Hence, $W_A(-)$ also satisfies the realizability property. $\square$

Applying Theorem 4.3.4 we obtain:

Theorem 7.1.5. *Let $A = \mathrm{Hom}_{\Bbbk}(\Bbbk[G]\#\Bbbk H, \Bbbk)$ where G and H are finite groups with H acting on G and $\Bbbk$ is a base field of positive characteristic dividing the order of G. Let R_A be the cohomology ring of A, i.e. $R_A = \mathrm{Ext}^{\bullet}_A(\Bbbk, \Bbbk)$. The following hold:*

(a) *There exists a bijection*

$$\{\textit{thick two-sided ideals of } \mathsf{stmod}(A)\} \underset{\Theta_A}{\overset{\Phi_A}{\rightleftarrows}} \{\textit{specialization closed sets of } H\text{-}\mathrm{Proj}(R_A)\},$$

where Θ_A is the map given by (4.6) for the weak support datum W_A.

(b) *There exists a homeomorphism $f : H\text{-}\mathrm{Proj}(R_A) \to \mathrm{Spc}(\mathsf{stmod}(A))$.*

7.2. Drinfeld doubles of Benson-Witherspoon smash coproducts

We now consider the Drinfeld doubles of the Benson-Witherspoon smash coproducts described in the previous section, as an application of the general methods developed in Chapter 5.

In this section, just as in the previous, A will denote the Benson-Witherspoon Hopf algebra as defined above corresponding to groups G and H, where H acts on G by group automorphisms, and $\Bbbk$ a field of characteristic dividing the order of G.

Theorem 7.2.1. *Let $\mathbf{C}$ the category $\mathsf{mod}(A)$, and $\mathsf{Z}(\mathbf{C})$ the category $\mathsf{mod}(D(A))$ for the Drinfeld double $D(A)$ of A. Then:*

(a) *The continuous map* $f : \operatorname{Spc} \mathsf{st}(\mathbf{C}) \to \operatorname{Spc} \mathsf{st}(\mathsf{Z}(\mathbf{C}))$ *constructed in Section 5.1 is injective.*

(b) *The map* $\Phi_W \circ \Theta_W$ *constructed in Section 5.1 is equal to the identity, as a map* $\mathcal{X}_{sp}(\operatorname{Spc} \mathsf{st}(\mathbf{C})) \to \mathcal{X}_{sp}(\operatorname{Spc} \mathsf{st}(\mathbf{C}))$.

Remark 7.2.2. We note that if $\mathbf{C}$ was braided, then both (1) and (2) would follow directly from Theorem 5.3.6. However, since A is not a quasitriangular Hopf algebra; the category of A-modules is not generally braided.

Theorem 7.2.1 will be proven by first showing the following intermediary lemma.

Lemma 7.2.3. *Suppose* $\mathbf{I}$ *and* $\mathbf{J}$ *are thick ideals of* $\mathsf{st}(\mathbf{C})$ *such that*

$$\{X \in \mathsf{st}(\mathsf{Z}(\mathbf{C})) : \overline{F}(X) \in \mathbf{I}\} = \{X \in \mathsf{st}(\mathsf{Z}(\mathbf{C})) : \overline{F}(X) \in \mathbf{J}\}.$$

Then $\mathbf{I} = \mathbf{J}$.

Proof. Suppose $\mathbf{I}$ and $\mathbf{J}$ are thick ideals satisfying the condition above. Since $\mathbf{I}$ and $\mathbf{J}$ are thick, it is enough to show that the indecomposable objects in $\mathbf{I}$ are equal to the indecomposable objects in $\mathbf{J}$. Suppose $M_x \otimes \Bbbk_x$ is an object in $\mathbf{I}$, where $x \in H$ and M_x is a G-module (recall that all A-modules are direct sums of modules of this form). Then the module

$$(M_x \otimes \Bbbk_x) \otimes (\Bbbk \otimes \Bbbk_{x^{-1}}) \cong M_x \otimes \Bbbk_{\mathrm{id}}$$

is in $\mathbf{I}$. We also then have

$$(\Bbbk \otimes \Bbbk_y) \otimes (M_x \otimes \Bbbk_{\mathrm{id}}) \otimes (\Bbbk \otimes \Bbbk_{y^{-1}}) \cong {}^{y}M_x \otimes \Bbbk_{\mathrm{id}}$$

is an object of $\mathbf{I}$ as well. The ideal $\mathbf{I}$ then contains the direct sum

$$\hat{M} := \bigoplus_{y \in H} {}^{y}M_x \otimes \Bbbk_{\mathrm{id}}.$$

We claim that $\hat{M}$ is in the image of $\overline{F}$; in other words, $\hat{M}$ has a half-braiding which allows it to be lifted to the Drinfeld center. To see this, consider an A-module $N_z \otimes \Bbbk_z$. We observe that

$$\hat{M} \otimes (N_z \otimes \Bbbk_z) \cong \bigoplus_{y \in H} ({}^{y}M_x \otimes N_z) \otimes \Bbbk_z$$

$$(N_z \otimes \Bbbk_z) \otimes \hat{M} \cong \bigoplus_{y \in H} (N_z \otimes^{zy} M_x) \otimes \Bbbk_z$$

Since $\Bbbk G$ is itself cocommutative (and thus ${}^{y}M_x \otimes N_z \cong N_z \otimes^{y} M_x$ in a natural way), this formula can be used to observe a natural isomorphism $\hat{M} \otimes - \cong - \otimes \hat{M}$. This isomorphism satisfies the half-braiding condition, and so $\hat{M}$ is in the image of $\overline{F}$.

Since $\mathbf{I}$ and $\mathbf{J}$ are assumed to agree on their intersections with the image of $\overline{F}$, we can conclude that $\hat{M}$ is in $\mathbf{J}$ as well. But then its summand $M_x \otimes \Bbbk_{\mathrm{id}}$, and then

$$(M_x \otimes \Bbbk_{\mathrm{id}}) \otimes (\Bbbk \otimes \Bbbk_x) \cong M_x \otimes \Bbbk_x$$

is also an object of $\mathbf{J}$. Note that we have proven generally that $M_x \otimes \Bbbk_x$ is in any thick ideal if and only if $\hat{M}$, as constructed above, is in that ideal. Thus, the objects of $\mathbf{I}$ are a subset of the objects of $\mathbf{J}$, and by symmetry the ideals are equal. $\square$

We can now prove Theorem 7.2.1, as a consequence of Lemma 7.2.3:

Proof. The map f is defined by

$$f(\mathbf{P}) = \{X : \overline{F}(X) \in \mathbf{P}\}$$

for a given prime ideal $\mathbf{P}$ in $\operatorname{Spc} \mathsf{st}(\mathbf{C})$. But Lemma 7.2.3 has shown that if $\mathbf{P}$ and $\mathbf{Q}$ are two prime ideals with $f(\mathbf{P}) = f(\mathbf{Q})$, then since $\mathbf{P}$ and $\mathbf{Q}$ are more generally examples of thick ideals, we have $\mathbf{P} = \mathbf{Q}$. Hence, f is injective, showing (a).

For (b), let S be an arbitrary specialization-closed set in $\operatorname{Spc} \mathsf{st}(\mathbf{C})$, in other words, a (possibly infinite) union $S = \bigcup_{i \in I} S_i$ where each S_i is a closed set. Recall that by construction, it is automatic that $\Phi_w(\Theta_W(S)) \subseteq S$ (the details are included above in the proof of Theorem 5.3.6). To show the opposite containment, we note that by the classification of thick ideals and Balmer spectrum of $\mathsf{st}(\mathbf{C})$ as given in Theorem 7.1.5, each of the closed sets S_i in $\operatorname{Spc} \mathsf{st}(\mathbf{C})$ can be written as $V_{\mathbf{C}}(M_i)$ for some $M_i \in \mathsf{st}(\mathbf{C})$. By the proof of Lemma 7.2.3, M_i is in a thick ideal if and only if $\hat{M}_i$ is in that thick ideal, for $\hat{M}_i$ as constructed in that proof; this implies that $V_{\mathbf{C}}(M_i) = V_{\mathbf{C}}(\hat{M}_i)$. And now we recall $\hat{M}_i$ is in the image of $\overline{F}$; in other words, we can pick an object X_i in $\mathsf{st}(\mathsf{Z}(\mathbf{C}))$ with $\overline{F}(X_i) = \hat{M}_i$. Since

$$W(X_i) = V_{\mathbf{C}}(\overline{F}(X_i)) = V_{\mathbf{C}}(\hat{M}_i) = V_{\mathbf{C}}(M_i) = S_i \subseteq S,$$

we have $X_i \in \Theta_W(S)$ by definition. Hence, we now have

$$\begin{aligned} \Phi_W(\Theta_W(S)) &\supseteq \bigcup_{i \in I} W(X_i) \\ &= \bigcup_{i \in I} S_i \\ &= S. \end{aligned}$$

Since we have both containments, we can conclude that $\Phi_W(\Theta_W(S)) = S$ for any specialization-closed set S in $\operatorname{Spc} \mathsf{st}(\mathbf{C})$. $\square$

Chapter 8. Small Quantum Borels

8.1. Preliminaries

Let $\mathcal{R}$ be an irreducible root system of rank n. Let ℓ be a positive integer and ζ be a primitive ℓth root of unity.

We begin by introducing a general construction of the small quantum group for a Borel algebra that generalizes the well-known construction using group like elements arising from the root lattice. All of these will be finite dimensional Hopf algebras. For a given $\mathcal{R}$, let X be the corresponding weight lattice and $\mathcal{R}^+$ be a set of positive roots. Denote by $\{\alpha_1, \ldots, \alpha_n\}$ the base of simple roots for $\mathcal{R}$ corresponding to $\mathcal{R}^+$ and by $\{d_1, \ldots, d_n\}$ the collection of relatively prime positive integers that symmetrizes the corresponding Cartan matrix. Denote by $\langle -, - \rangle$ the Weyl group invariant nondegenerate symmetric inner product on the Euclidean space $\mathfrak{t}^*_{\mathbb{R}}$ spanned by $\mathcal{R}$, normalized by $\langle \beta, \beta \rangle = 2$ for short roots β. In terms of this form, the integers d_i are given by $d_i = \langle \alpha_i, \alpha_i \rangle / 2$. Let $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ be the corresponding coroots thought of as elements of $\mathfrak{t}^*_{\mathbb{R}}$ by setting

$$\alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle} = \frac{\alpha_i}{d_i}.$$

Choose a $\mathbb{Z}$-lattice, Γ, with $\mathbb{Z}\mathcal{R} \subseteq \Gamma \subseteq X$. Such a lattice Γ has rank n. Let $\{\mu_1, \ldots, \mu_n\}$ be a $\mathbb{Z}$-basis for Γ.

Let $u_\zeta(\mathfrak{b})$ be the small quantum group as described in [12, Section 2.2]. Then $u_\zeta(\mathfrak{b}) = u_\zeta(\mathfrak{u}) \# u_\zeta(\mathfrak{t})$ where $u_\zeta(\mathfrak{u})$ is generated by the root vectors $\{E_\beta \mid \beta \in \mathcal{R}^+\}$ satisfying $E_\beta^\ell = 0$ and $u_\zeta(\mathfrak{t})$ is a Hopf algebra isomorphic to the group algebra of $\mathbb{Z}\mathcal{R}/(\ell\mathbb{Z}\mathcal{R})$ over $\mathbb{C}$, realized as

$$u_\zeta(\mathfrak{t}) = \mathbb{C}[K_{\alpha_1}^{\pm 1}, \ldots, K_{\alpha_n}^{\pm 1}]/(K_{\alpha_i}^\ell - 1, 1 \leq i \leq n)$$

where K_{α_i} are group like elements. The relations in $u_\zeta(\mathfrak{b})$ defining the smash product are

$$K_{\alpha_i} E_\beta K_{\alpha_i}^{-1} = \zeta^{\langle \beta, \alpha_i \rangle} E_\beta \tag{8.1}$$

for $\beta \in \mathcal{R}^+$.

We can consider the following generalization of the small quantum group for the Borel subalgebra. Given a lattice Γ with $\mathbb{Z}\mathcal{R} \subseteq \Gamma \subseteq X$ as above, define its sublattice

$$\Gamma' := \{\nu \in \Gamma \mid \langle \nu, \mathcal{R} \rangle \subseteq \ell\mathbb{Z}\}.$$

Obviously, $\Gamma' \supseteq \ell\Gamma$, so Γ/Γ' is a factor group of $\Gamma/\ell\Gamma \cong (\mathbb{Z}/\ell\mathbb{Z})^n$. Denote the canonical projection

$$\Gamma \twoheadrightarrow \Gamma/\Gamma' \quad \text{by} \quad \mu \mapsto \overline{\mu}. \tag{8.2}$$

Let

$$u_{\zeta,\Gamma}(\mathfrak{t}) \ \text{ denote the group algebra of } \Gamma/\Gamma' \text{ over } \mathbb{C}. \tag{8.3}$$

For $\mu \in \Gamma/\Gamma'$ denote by K_μ the element of $u_{\zeta,\Gamma}(\mathfrak{t})$ corresponding to μ. Consider the Hopf algebra

$$u_{\zeta,\Gamma}(\mathfrak{b}) = u_\zeta(\mathfrak{u}) \# u_{\zeta,\Gamma}(\mathfrak{t})$$

with relations

$$K_\mu E_\alpha K_\mu^{-1} = \zeta^{\langle \alpha, \mu_0 \rangle} E_\alpha \quad \text{for} \quad \mu \in \Gamma/\Gamma', \alpha \in \mathcal{R}^+, \tag{8.4}$$

where $\mu_0 \in \Gamma$ is a preimage of μ. By the definition of the lattice Γ', the right hand side does not depend on the choice of preimage. The coproduct of the generators E_{α_i} is given by

$$\Delta(E_{\alpha_i}) = E_{\alpha_i} \otimes 1 + K_{\overline{\alpha_i}} \otimes E_{\alpha_i} \tag{8.5}$$

for $1 \leq i \leq n$. The antipode is given by $S(E_{\alpha_i}) = -K_{\overline{\alpha_i}}^{-1} E_{\alpha_i}$.

In all of the above definitions, the lattice Γ' can be replaced with any sublattice of Γ'. The motivation for the use of the full lattice Γ' is that this makes $u_{\zeta,\Gamma}(\mathfrak{b})$ small in the sense that the only group-like central elements of $u_{\zeta,\Gamma}(\mathfrak{b})$ are the scalars.

Remark 8.1.1. Consider two lattices Γ_1 and Γ_2 such that $\mathbb{Z}\mathcal{R} \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq X$. Then $\Gamma_1' = \Gamma_1 \cap \Gamma_2'$. Hence, we have a Hopf algebra embedding

$$u_{\zeta,\Gamma_1}(\mathfrak{b}) \hookrightarrow u_{\zeta,\Gamma_2}(\mathfrak{b}) \quad \text{given by} \quad K_{\mu+\Gamma_1'} \mapsto K_{\mu+\Gamma_2'}, E_\alpha \mapsto E_\alpha$$

for $\mu \in \Gamma_1$, $\alpha \in \mathcal{R}^+$.

8.2. Assumptions on ℓ

For the remainder of this section we will employ one of the following assumptions in the statements of our results where ζ is an ℓth root of unity.

Assumption 8.2.1. *Let ℓ be a positive integer such that*

(a) *ℓ is odd;*

(b) *If $\mathcal{R}$ is of type G_2 then $3 \nmid \ell$;*

(c) *If $\mathcal{R}$ is of type A_1 then $\ell \geq 3$, otherwise $\ell > 3$.*

Conditions (a)-(b) in Assumption 8.2.1 are equivalent to saying that ℓ is an odd positive integer which is coprime to $\{d_1, \ldots, d_n\}$.

Assumption 8.2.2. *Let ℓ be a positive integer such that*

(a) *ℓ is odd;*

(b) *If $\mathcal{R}$ is of type G_2 then $3 \nmid \ell$;*

(c) *$\ell > h$ where h is the Coxeter number for $\mathcal{R}$.*

Note that if ℓ satisfies Assumption 8.2.2 then ℓ satisfies Assumption 8.2.1.

The group of group-like elements of $u_{\zeta,\Gamma}(\mathfrak{t})$ is isomorphic to Γ/Γ'. Next we explicitly describe this finite abelian group.

Proposition 8.2.3. *(a)* *If ℓ is coprime to $\{d_1, \dots, d_n\}$, then*

$$\Gamma' = \Gamma \cap \ell X.$$

That is, $\Gamma/\Gamma' \cong \Gamma/(\Gamma \cap \ell X)$.

(b) *If ℓ is coprime to $\{d_1, \dots, d_n\}$ and $|X/\Gamma|$, then*

$$\Gamma' = \ell\Gamma.$$

That is, $\Gamma/\Gamma' \cong \Gamma/(\ell\Gamma) \cong (\mathbb{Z}/\ell\mathbb{Z})^n$.

Proof. (a) Let $\nu = \sum m_i\omega_i \in \Gamma \subseteq X$ for some $m_i \in \mathbb{Z}$. Then $\nu \in \Gamma' \Leftrightarrow$

$$\begin{aligned}
&\langle \nu, \alpha_i \rangle \in \ell\mathbb{Z}, \quad \forall 1 \le i \le n \Leftrightarrow \\
&m_i d_i \in \ell\mathbb{Z}, \quad \forall 1 \le i \le n \Leftrightarrow \\
&m_i \in \ell\mathbb{Z}, \quad \forall 1 \le i \le n \Leftrightarrow \\
&\nu \in \Gamma \cap \ell X.
\end{aligned}$$

(b) In view of part (a), we have to prove that under the assumptions in part (b), $\Gamma \cap \ell X = \ell\Gamma$. Clearly,

$$\Gamma \cap \ell X \supseteq \ell\Gamma.$$

For the opposite inclusion, take $\nu \in \Gamma \cap \ell X$. Then the order of $\nu/\ell + \Gamma$ in X/Γ divides ℓ. Since ℓ is coprime to the order of the group X/Γ, the order of $\nu/\ell + \Gamma$ equals 1. Therefore $\nu/\ell \in \Gamma$, and thus, $\nu \in \ell\Gamma$. Hence, $\Gamma \cap \ell X = \ell\Gamma$. $\square$

Example 8.2.4. The standard notion of a small quantum Borel subalgebra $u_\zeta(\mathfrak{b})$ is recovered from the above one as follows. Proposition 8.2.3(b), applied for the root lattice $\Gamma = \mathbb{Z}\mathcal{R}$, implies that, if ℓ is coprime to $\{d_1, \ldots, d_n\}$ and $|X/\mathbb{Z}\mathcal{R}|$, then

$$u_{\zeta,\mathbb{Z}\mathcal{R}}(\mathfrak{b}) \cong u_\zeta(\mathfrak{b}).$$

Note that both aforementioned algebras are defined for general values of ℓ, but become isomorphic under the coprimeness conditions.

8.3. Automorphisms, representations and cohomology

We first set notational conventions.

Denote the character group of Γ/Γ' by

$$\widehat{\Gamma/\Gamma'}.$$

By abuse of notation, for $\lambda \in \widehat{\Gamma/\Gamma'}$ we denote by the same symbol the one dimensional representation of $u_{\zeta,\Gamma}(\mathfrak{b})$ given by

$$K_\mu \mapsto \lambda(\mu), \quad E_\alpha \mapsto 0, \quad \forall \mu \in \Gamma/\Gamma', \alpha \in \mathcal{R}^+.$$

For each $\lambda \in \widehat{\Gamma/\Gamma'}$, one can define an automorphism, γ_λ of $u_{\zeta,\Gamma}(\mathfrak{b})$ as follows:

$$\gamma_\lambda(E_\alpha) = \lambda(\overline{\alpha})E_\alpha, \quad \gamma_\lambda(K_\mu) = K_\mu, \quad \forall \mu \in \Gamma/\Gamma', \alpha \in \mathcal{R}^+.$$

Denote the subgroup $\Pi = \{\gamma_\lambda : \ \lambda \in \widehat{\Gamma/\Gamma'}\} \subseteq \mathrm{Aut}(u_{\zeta,\Gamma}(\mathfrak{b}))$. For any $u_{\zeta,\Gamma}(\mathfrak{b})$-module, Q, the automorphism γ_λ can be used to define a new module structure on it called the twist: Q^{γ_λ}. The underlying vector space of Q^{γ_λ} is still Q with the action given by $x.m = \gamma_\lambda(x)m$ for all $x \in u_{\zeta,\Gamma}(\mathfrak{b})$ and $m \in Q^{\gamma_\lambda}$.

Let $R = \mathrm{H}^\bullet(u_{\zeta,\Gamma}(\mathfrak{b}), \mathbb{C})$ be the cohomology ring of $u_{\zeta,\Gamma}(\mathfrak{b})$. An automorphism in Π acts on the cohomology ring by taking an n-fold extension of $\mathbb{C}$ with $\mathbb{C}$ and twisting each module in the n-fold extension to produce a new n-fold extension. This provides an action of the group Π on the ring R. The following proposition summarizes properties of the automorphisms in Π and how they interact with representations and the cohomology.

Proposition 8.3.1. *Let $u_{\zeta,\Gamma}(\mathfrak{b})$ be the small quantum group for the Borel subalgebra and $R = \mathrm{H}^\bullet(u_{\zeta,\Gamma}(\mathfrak{b}), \mathbb{C})$ be the cohomology ring.*

(a) The irreducible representations for $u_{\zeta,\Gamma}(\mathfrak{b})$ are one-dimensional and are precisely the representations λ for $\lambda \in \widehat{\Gamma/\Gamma'}$.

(b) For any $u_{\zeta,\Gamma}(\mathfrak{b})$-module, Q, and $\lambda \in \widehat{\Gamma/\Gamma'}$ one has

$$\lambda \otimes Q \otimes \lambda^{-1} \cong Q^{\gamma_\lambda}.$$

(c) The action of Π on R is trivial.

(d) The action of Π on $\mathrm{Proj}(R)$ is trivial.

Proof. (a) The relations $E_\alpha^\ell = 0$ for $\alpha \in \mathcal{R}^+$ imply that all root vectors E_β are in the radical of the finite dimensional algebra $u_{\zeta,\Gamma}(\mathfrak{b})$ and so they act by 0 on every irreducible representation of $u_{\zeta,\Gamma}(\mathfrak{b})$. Hence, every irreducible representations of $u_{\zeta,\Gamma}(\mathfrak{b})$ is an irreducible representation of $u_{\zeta,\Gamma}(\mathfrak{t})$, which is the group algebra of Γ/Γ', so the irreducible representation of $u_{\zeta,\Gamma}(\mathfrak{t})$ are precisely the representations λ for $\lambda \in \widehat{\Gamma/\Gamma'}$.

(b) The isomorphism follows from the coproduct formula (8.5) and the fact that the set $\{K_\mu, E_{\alpha_i} \mid \mu \in \Gamma, i = 1, \ldots, n\}$ generates the algebra $u_{\zeta,\Gamma}(\mathfrak{b})$.

(c and d) Note that (d) follows immediately from (c). So to finish the proof we show that the action of Π on the cohomology ring R is trivial.

By using the Lyndon-Hochschild-Serre (LHS) spectral sequence and the fact that the representations for $u_{\zeta,\Gamma}(\mathfrak{t})$ are completely reducible (because $u_{\zeta,\Gamma}(\mathfrak{t})$ is isomorphic to the group algebra over $\mathbb{C}$ of a finite group), it follows that $R = \mathrm{H}^{\bullet}(u_{\zeta}(\mathfrak{u}), \mathbb{C})^{u_{\zeta,\Gamma}(\mathfrak{t})}$ with respect to the action (8.4) (cf. [46, Theorem 2.5]). Consequently, for every weight $\nu \in \mathbb{Z}\mathcal{R}$ of R

$$\langle \nu, \Gamma \rangle \subseteq \ell\mathbb{Z} \Rightarrow \langle \nu, \mathcal{R} \rangle \subseteq \ell\mathbb{Z} \Rightarrow \nu \in \mathbb{Z}\mathcal{R} \cap \Gamma' \Rightarrow \overline{\nu} = 0.$$

Let $f \in R$ be of weight ν. The automorphism $\gamma_\lambda \in \Pi$ acts on f by

$$\gamma_\lambda(f) = \lambda(\overline{\nu})f = f,$$

which proves the triviality of the Π-action on R. $\square$

8.4. Finite generation

In order to verify the finite generation conditions on the cohomology, we state the following result from [12, Proposition 5.6.3] on the cohomology for $u_{\zeta}(\mathfrak{u})$.

Theorem 8.4.1. *Let ℓ satisfy Assumption 8.2.1, and ζ be an ℓth root of unity. There exists a polynomial ring $S^{\bullet}(\mathfrak{u}^*)$ such that the following holds:*

(a) $\mathrm{H}^{\bullet}(u_{\zeta}(\mathfrak{u}), \mathbb{C})$ *is finitely generated over* $S^{\bullet}(\mathfrak{u}^*)$*;*

(b) $\mathrm{H}^{\bullet}(u_{\zeta}(\mathfrak{u}), \mathbb{C})$ *is a finitely generated* $\mathbb{C}$*-algebra.*

Theorem 8.4.1 allows us to consider the issue of finite generation of cohomology for $u_{\zeta,\Gamma}(\mathfrak{b})$. The filtration in [12, Section 2.9] on $u_{\zeta}(\mathfrak{u})$ that induces the grading as in [12, Lemma 5.6.1] is stable under the action of K_{μ_i}, $i = 1, 2 \ldots, n$. Consequently, there exists a spectral sequence

$$E_1^{i,j} = \mathrm{H}^{i+j}(\mathrm{gr}\ u_{\zeta}(\mathfrak{u}), \mathbb{C})_{(i)} \Rightarrow \mathrm{H}^{i+j}(u_{\zeta}(\mathfrak{u}), \mathbb{C}) \tag{8.6}$$

such that

$$\mathrm{H}^{n}(\mathrm{gr}\ u_{\zeta}(\mathfrak{u}), \mathbb{C}) \cong \bigoplus_{2a+b=n} S^{a}(\mathfrak{u}^*)^{[1]} \otimes \Lambda_{\zeta}^{b}.$$

Here $S^\bullet(\mathfrak{u}^*)^{[1]}$ is the symmetric algebra on $\mathfrak{u}^*$ (the dual of $\mathfrak{u}$, the [1] indicates that $u_\zeta(\mathfrak{t})$ acts trivially) and Λ^b_ζ is a deformation of the exterior algebra on $\mathfrak{u}^*$ with generators and relations defined in [12, Section 2.9]. In the proof of Theorem 8.4.1 (given in [12, Proposition 5.6.3]), it is shown that under the assumptions on ℓ, $d_r(S^\bullet(\mathfrak{u}^*)^{[1]}) = 0$ for $r \geq 1$ where d_r is the differential on the E_r-page of the spectral sequence (8.6). One can then conclude part (a) of Theorem 8.4.1.

Since $u_\zeta(\mathfrak{u})$ is normal in $u_{\zeta,\Gamma}(\mathfrak{b})$ (cf. [12, Section 2.8]) with quotient $u_{\zeta,\Gamma}(\mathfrak{t})$, and the filtration is stable under $u_{\zeta,\Gamma}(\mathfrak{t})$, it follows that $u_{\zeta,\Gamma}(\mathfrak{t})$ acts on the spectral sequence (8.6). Furthermore, one can verify that $u_{\zeta,\Gamma}(\mathfrak{t})$ acts trivially on $S^\bullet(\mathfrak{u}^*)^{[1]}$.

Since finite-dimensional representations for $u_{\zeta,\Gamma}(\mathfrak{t})$ are completely reducible, the fixed point functor $(-)^{u_{\zeta,\Gamma}(\mathfrak{t})}$ is exact. By using the LHS spectral sequence and the exactness, one shows that

$$\mathrm{H}^\bullet(u_{\zeta,\Gamma}(\mathfrak{b}), \mathbb{C}) \cong \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), \mathbb{C})^{u_{\zeta,\Gamma}(\mathfrak{t})}.$$

Moreover, the fixed point functor can be applied to get a spectral sequence:

$$E_1^{i,j} = [\mathrm{H}^{i+j}(\mathrm{gr}\ u_\zeta(\mathfrak{u}), \mathbb{C})_{(i)}]^{u_{\zeta,\Gamma}(\mathfrak{t})} \Rightarrow \mathrm{H}^{i+j}(u_\zeta(\mathfrak{b}), \mathbb{C}). \tag{8.7}$$

We can now verify the requisite finite generation assumptions on the cohomology for $u_{\zeta,\Gamma}(\mathfrak{b})$.

Theorem 8.4.2. *Let ℓ satisfy Assumption 8.2.1, ζ be an ℓth root of unity, and $u_{\zeta,\Gamma}(\mathfrak{b})$ be a small quantum group for a Borel subalgebra. Then*

(a) $\mathrm{H}^\bullet(u_{\zeta,\Gamma}(\mathfrak{b}), \mathbb{C})$ *is a finitely generated $\mathbb{C}$-algebra;*

(b) *For any finite-dimensional $u_{\zeta,\Gamma}(\mathfrak{b})$-module, M, $\mathrm{H}^\bullet(u_{\zeta,\Gamma}(\mathfrak{b}), M)$ is finitely generated over $\mathrm{H}^\bullet(u_{\zeta,\Gamma}(\mathfrak{b}), \mathbb{C})$.*

Proof. (a) Let $R := \mathrm{H}^\bullet(u_{\zeta,\Gamma}(\mathfrak{b}), \mathbb{C})$. From Theorem 8.4.1(a), and the spectral sequence (8.7), we have polynomial ring $S := S^\bullet(\mathfrak{u}^*)^{[1]}$ with $d_r(S) = 0$ for $r \geq 1$. Consequently, R finitely generated over S. This shows (a).

(b) By using induction on the composition length of M and the long exact sequence in cohomology one can reduce the statement to showing that $\mathrm{H}^\bullet(u_{\zeta,\Gamma}(\mathfrak{b}), M)$ is finitely generated over R for M a simple $u_{\zeta,\Gamma}(\mathfrak{b})$-module.

The simple $u_{\zeta,\Gamma}(\mathfrak{b})$-modules are one-dimensional and indexed by $\lambda \in \widehat{\Gamma/\Gamma'}$. By using the LHS spectral sequence, one has

$$\mathrm{H}^\bullet(u_{\zeta,\Gamma}(\mathfrak{b}), \lambda) \cong \mathrm{Hom}_{u_{\zeta,\Gamma}(\mathfrak{t})}(-\lambda, \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), \mathbb{C})) = A_\lambda.$$

Now S acts on $\mathrm{H}^\bullet(u_{\zeta,\Gamma}(\mathfrak{b}), \lambda)$ and thus acts on A_λ. This action is compatible with the action on $T = \mathrm{H}^\bullet(u_\zeta(\mathfrak{u}), \mathbb{C})$. We have $T \cong \oplus_{\lambda \in \widehat{\Gamma/\Gamma'}} A_\lambda$, and by Theorem 8.4.1, T is finitely generated over S. Consequently, A_λ is finitely generated over S, thus finitely generated over R. □

8.5. Calculation of the cohomology ring

In this section we calculate the cohomology ring $R := \mathrm{H}^\bullet(u_{\zeta,\Gamma}(\mathfrak{b}), \mathbb{C})$ for $\ell > h$. We will need the following fact proved by Andersen and Jantzen [3, §2.2 statement (2)].

Lemma 8.5.1. *Let $\mathcal{R}$ be an irreducible root system. For every weight λ of $\Lambda^\bullet(\mathfrak{u}^*)$ and simple root α_i,*

$$|\langle \lambda, \alpha_i^\vee \rangle + 1| \leq h - 1,$$

where h is the Coxeter number for $\mathcal{R}$.

The following theorem provides a natural generalization to the fundamental result of Ginzburg and Kumar [46, Theorem 2.5].

Theorem 8.5.2. *Let ℓ satisfy Assumption 8.2.2 (in particular, $\ell > h$), ζ be an ℓth root of unity, and $u_{\zeta,\Gamma}(\mathfrak{b})$ be a small quantum group for a Borel subalgebra. Then*

(a) $\quad \mathrm{H}^{2\bullet}(u_{\zeta,\Gamma}(\mathfrak{b}),\mathbb{C}) \cong S^{\bullet}(\mathfrak{u}^*)^{[1]}$;

(b) $\quad \mathrm{H}^{2\bullet+1}(u_{\zeta,\Gamma}(\mathfrak{b}),\mathbb{C}) = 0$.

Proof. Consider the spectral sequence (8.7) and

$$\mathrm{H}^n(\mathrm{gr}\ u_\zeta(\mathfrak{u}),\mathbb{C})^{u_{\zeta,\Gamma}(\mathfrak{t})} \cong \bigoplus_{2a+b=n} S^a(\mathfrak{u}^*)^{[1]} \otimes [\Lambda_\zeta^b]^{u_{\zeta,\Gamma}(\mathfrak{t})}.$$

The $u_{\zeta,\Gamma}(\mathfrak{t})$-weights of Λ_ζ^b come from the $\mathfrak{t}$-weights of $\Lambda^\bullet(\mathfrak{u}^*)$. If λ is a weight of $\Lambda^\bullet(\mathfrak{u}^*)$ corresponding to an element in $[\Lambda_\zeta^b]^{u_{\zeta,\Gamma}(\mathfrak{t})}$, then $\langle\lambda,\Gamma\rangle \subseteq \ell\mathbb{Z}$. Therefore $\langle\lambda,\alpha_i\rangle \in \ell\mathbb{Z}$ for all $1 \le i \le n$. For each simple root α_i of $\mathcal{R}$ we have

$$\langle\lambda,\alpha_i^\vee\rangle = \frac{1}{d_i}\langle\lambda,\alpha_i\rangle.$$

Since $\langle\lambda,\alpha_i^\vee\rangle$ is an integer, $\langle\lambda,\alpha_i\rangle \in \ell\mathbb{Z}$ and $\gcd(\ell,d_i) = 1$, we have that that $\langle\lambda,\alpha_i^\vee\rangle$ is a multiple of ℓ. Lemma 8.5.1 gives that

$$|\langle\lambda,\alpha_i^\vee\rangle| \le h < \ell.$$

The combination of the two facts implies that $\langle\lambda,\alpha_i^\vee\rangle = 0$ for all simple roots α_i. Thus $\lambda = 0$ and

$$[\Lambda_\zeta^b]^{u_{\zeta,\Gamma}(\mathfrak{t})} \cong \begin{cases} 0 & \text{if } b > 0 \\ \mathbb{C} & \text{if } b = 0. \end{cases} \tag{8.8}$$

Consequently, the $E_1^{i,j}$-term of the spectral sequence only contains terms of the form $S^a(\mathfrak{u}^*)^{[1]}$ where $2a = i + j$. From Theorem 8.4.2, $d_r(S^\bullet(\mathfrak{u}^*)^{[1]}) = 0$ for $r \ge 1$. Thus, the spectral sequence (8.7) collapses and yields (a) and (b). $\square$

8.6. Classification of tensor ideals

Let $\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))$ be the stable module category of finitely generated $u_{\zeta,\Gamma}(\mathfrak{b})$-modules. The stable module category for all $u_{\zeta,\Gamma}(\mathfrak{b})$-modules will be denoted by $\mathsf{StMod}(u_{\zeta,\Gamma}(\mathfrak{b}))$. Recall that the category $\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))$ is a monoidal triangulated category. The goal of this section will be to describe the thick tensor ideals in $\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))$ and its Balmer spectrum.

Let $R := \mathrm{H}^\bullet(u_{\zeta,\Gamma}(\mathfrak{b}), \mathbb{C})$ be the cohomology ring for the small quantum group $u_{\zeta,\Gamma}(\mathfrak{b})$. In Theorem 8.4.2(a), it was shown that R is a a finitely generated $\mathbb{C}$-algebra. Therefore, $Y = \mathrm{Proj}(R)$, the space of (nontrivial) homogeneous prime ideals of R, is a Noetherian topological space. In fact, Y is a Zariski space.

For brevity, the set of subsets, closed subsets, and specialization-closed subsets of Y will be denoted respectively by $\mathcal{X}, \mathcal{X}_{cl}$, and $\mathcal{X}_{sp}$. Let $W(-)$ be the cohomological support $\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b})) \to \mathcal{X}_{cl}$, as defined in Section 2.6. This extends to a support map $\mathsf{StMod}(u_{\zeta,\Gamma}(\mathfrak{b})) \to \mathcal{X}_{sp}$ by [16] as discussed in Section 2.7, which we will also denote by $W(-)$.

Let

$$\Phi = \Phi_W : \{\text{thick right ideals of } \mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))\} \to \mathcal{X}$$

be the map given by (4.1). Note that it takes values in $\mathcal{X}_{sp}$ because $W(M) \in \mathcal{X}_{cl}$ for all $M \in \mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))$. On the other hand, we can define an assignment

$$\Theta : \mathcal{X}_{sp} \to \{\text{thick right ideals of } \mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))\}$$

by

$$\Theta(Z) = \{M \in \mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b})) \mid W(M) \subseteq Z\} \quad \text{for} \quad Z \in \mathcal{X}_{sp}.$$

We can now state the theorem that classifies thick ideals in $\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))$.

Theorem 8.6.1. *Let $u_{\zeta,\Gamma}(\mathfrak{b})$ be the small quantum group for the Borel subalgebra for an arbitrary finite dimensional complex simple Lie algebra. Assume that ℓ satisfies Assumption 8.2.2 (in particular, $\ell > h$), which implies that $R \cong S^{\bullet}(\mathfrak{u}^*)$.*

(a) *The above Φ and Θ are mutually inverse bijections*

$$\{\textit{thick right ideals of } \mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))\} \underset{\Theta}{\overset{\Phi}{\rightleftarrows}} \{\textit{specialization closed sets of } \mathrm{Proj}(R)\}.$$

(b) *Every thick right ideal of $\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))$ is two-sided.*

(c) *There exists a homeomorphism $f : \mathrm{Proj}(R) \to \mathrm{Spc}(\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b})))$.*

For the proof of the theorem we will need the following auxiliary lemma

Lemma 8.6.2. *In the setting of Theorem 8.6.1, for every finite dimensional $u_{\zeta,\Gamma}(\mathfrak{b})$-module Q and its dual Q^*,*

$$W(Q) = W(Q^*).$$

Proof. Every object of $\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))$ is rigid. Recall that by the proof of Lemma 3.1.3, if Q is a finite dimensional $u_{\zeta,\Gamma}(\mathfrak{b})$-module, then Q is a summand of $Q \otimes Q^* \otimes Q$. So,

$$W(Q) \subseteq W(Q \otimes Q^* \otimes Q).$$

Since Q has a composition series by subquotients isomorphic to the one dimensional modules $\lambda \in \widehat{\Gamma/\Gamma'}$,

$$W(Q \otimes Q^* \otimes Q) = \bigcup_{\lambda \in \widehat{\Gamma/\Gamma'}} W(\lambda \otimes Q^* \otimes Q).$$

The cohomological support W is automatically a quasi support datum. Applying this fact and Proposition 8.3.1 (b-c), we obtain that

$$W(\lambda \otimes Q^* \otimes Q) \subseteq W((Q^*)^{\gamma_\lambda} \otimes \lambda \otimes Q) \subseteq W((Q^*)^{\gamma_\lambda}) = W(Q^*)$$

for all $\lambda \in \widehat{\Gamma/\Gamma'}$. Combining the above inclusions gives $W(Q) \subseteq W(Q^*)$. Since the square of the antipode of $u_{\zeta,\Gamma}(\mathfrak{b})$ is an inner automorphism, $Q^{**} \cong Q$. Interchanging the roles of Q and Q^* gives $W(Q^*) \subseteq W(Q)$. Hence, $W(Q) = W(Q^*)$. $\square$

Proof of Theorem 8.6.1. (a) This statement follows by Theorem 4.4.3. The (fg) assumption is established in Theorem 8.4.2. The arguments in [22, Section 7.4], together with Lemma 8.6.2, verify Assumption 4.4.2.

We will prove (b) and (c) by an analogous argument to Theorem 4.3.4. As noted earlier, the cohomological support W is a quasi support datum and satisfies Assumption 4.4.2. One now needs to verify that W satisfies:

(Realization) If V is a closed set in Y, then there exists a compact object M with $\Phi(\langle M\rangle) = V$.

We compute:

$$\begin{aligned}
\Phi(\langle M\rangle) &= \bigcup_{C,D\in\mathbf{K}^c} W(C\otimes M\otimes D)\\
&= \bigcup_{C\in\mathbf{K}^c} W(C\otimes M)\\
&= \bigcup_{\lambda\in\widehat{\Gamma/\Gamma'}} W(\lambda\otimes M)\\
&= \bigcup_{\lambda\in\widehat{\Gamma/\Gamma'}} W(\lambda\otimes M\otimes\lambda^{-1})\\
&= \bigcup_{\lambda\in\widehat{\Gamma/\Gamma'}} W(M^{\gamma_\lambda})\\
&= \Pi\cdot W(M)\\
&= W(M).
\end{aligned}$$

The second and fourth equalities follow from the fact that W is a quasi support datum, the fourth since

$$W(\lambda \otimes M) \subseteq W(\lambda \otimes M \otimes \lambda^{-1}) \subseteq W(\lambda \otimes M \otimes \lambda^{-1} \otimes \lambda) = W(\lambda \otimes M).$$

The third, fifth, and seventh equalities follow from Proposition 8.3.1, parts (a), (b), and (d) respectively. Since $\Phi(\langle M \rangle) = W(M)$ and every closed set of $\operatorname{Proj} R$ may be realized as $W(M)$ for some compact M, W satisfies the Realization Property.

Analogously to the proof of Theorem 4.3.4, the condiitons that W is a quasi support datum satisfying Assumption 4.4.2 and the Realization Property allow one to conclude that there exists an order-preserving bijection:

$$\{\text{thick two-sided ideals of } \mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))\} \underset{\Theta}{\overset{\Phi}{\rightleftarrows}} \{\text{specialization closed sets of } \operatorname{Proj}(R)\}.$$

Since we already know by (a) that Φ induces a bijection between the thick right ideals of $\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))$ and specialization closed sets of $\operatorname{Proj}(R)$, it follows immediately that every thick right ideal is two-sided.

In order to obtain part (c), we must show that Φ is a weak support datum. Let $\mathbf{I}$ and $\mathbf{J}$ be two thick ideals of $\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))$. We claim that $\langle \mathbf{I} \otimes \mathbf{J} \rangle = \mathbf{I} \cap \mathbf{J}$. It is clear that $\langle \mathbf{I} \otimes \mathbf{J} \rangle \subseteq \mathbf{I} \cap \mathbf{J}$, by definition. Both thick ideals $\langle \mathbf{I} \otimes \mathbf{J} \rangle$ and $\mathbf{I} \cap \mathbf{J}$ of $\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))$ are semiprime, by Proposition 3.3.3. In other words,

$$\begin{aligned}\langle \mathbf{I} \otimes \mathbf{J} \rangle =& \bigcap \{\mathbf{P} \in \operatorname{Spc}(\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))) : \langle \mathbf{I} \otimes \mathbf{J} \rangle \subseteq \mathbf{P}\} \\ =& \bigcap \{\mathbf{P} \in \operatorname{Spc}(\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))) : \mathbf{I} \subseteq \mathbf{P}\} \cap \\ & \bigcap \{\mathbf{P} \in \operatorname{Spc}(\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))) : \mathbf{J} \subseteq \mathbf{P}\},\end{aligned}$$

and

$$\mathbf{I} \cap \mathbf{J} = \bigcap \{\mathbf{P} \in \mathrm{Spc}(\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))) : \mathbf{I} \cap \mathbf{J} \subseteq \mathbf{P}\}.$$

Then it is clear that $\mathbf{I} \cap \mathbf{J} \subseteq \langle \mathbf{I} \otimes \mathbf{J} \rangle$, since each prime ideal containing either $\mathbf{I}$ or $\mathbf{J}$ must necessarily contain $\mathbf{I} \cap \mathbf{J}$. Therefore $\mathbf{I} \cap \mathbf{J} = \langle \mathbf{I} \otimes \mathbf{J} \rangle$. By (a), Φ gives an order-preserving bijection between thick two-sided ideals of $\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))$ and specialization closed sets of $\mathrm{Proj}(R)$, which shows that

$$\begin{aligned} \Phi(\langle \mathbf{I} \otimes \mathbf{J} \rangle) &= \Phi(\mathbf{I} \cap \mathbf{J}) \\ &= \Phi(\mathbf{I}) \cap \Phi(\mathbf{J}). \end{aligned}$$

Therefore W is a weak support datum, and Theorem 4.3.4 gives part (c). $\square$

8.7. The tensor product property for the cohomological support

In this section we illustrate Theorem 3.4.3. We prove that the cohomological support maps for all small quantum Borel algebras associated to arbitrary complex simple Lie algebras and arbitrary choices of group-like elements have the tensor product property. This was conjectured by Negron and Pevtsova [85] and proved by them in the type A case.

Theorem 8.7.1. *Let $u_{\zeta,\Gamma}(\mathfrak{b})$ be the small quantum group for the Borel subalgebra of an arbitrary finite dimensional complex simple Lie algebra and a lattice $\mathbb{Z}\mathcal{R} \subseteq \Gamma \subseteq X$. Assume that ℓ satisfies Assumption 8.2.2 (in particular, $\ell > h$). Then the following hold:*

(a) All prime ideals of $\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))$ are completely prime.

(b) The cohomological support

$$W(-) : \mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b})) \to \mathcal{X}_{cl}(\mathrm{Proj}(\mathrm{H}^{\bullet}(u_{\zeta,\Gamma}(\mathfrak{b}), \mathbb{C})))$$

has the tensor product property $W(A \otimes B) = W(A) \cap W(B)$ *for all* $A, B \in \mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))$.

Proof. Part (a) of the theorem follows by combining Theorems 3.4.3 and 8.7.1(a).

(b) Recall the universal support datum

$$V : \mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b})) \to \mathcal{X}_{cl}(\mathrm{Spc}(\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b}))))$$

discussed in Section 4.1. It follows from Theorem 3.4.2 and part (a) of this theorem that V has the tensor product property.

In the proof of Theorem 8.6.1 it was shown that W is a weak support datum. By Theorem 4.3.4, there exists a homeomorphism

$$f : \mathrm{Proj}(\mathrm{H}^\bullet(u_{\zeta,\Gamma}(\mathfrak{b}), \mathbb{C})) \to \mathrm{Spc}(\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b})))$$

satisfying $\Phi_W(\langle M \rangle) = f^{-1}(V(M))$ for all $M \in \mathsf{stmod}(u_\zeta(\mathfrak{b}))$. Applying Theorem 8.6.1(b), (3.6) and the fact that W is a quasi support datum, we obtain

$$W(M) \subseteq \Phi(\langle M \rangle) = \Phi(\langle M \rangle_{\mathrm{r}}) \subseteq W(M)$$

for all $M \in \mathsf{stmod}(u_\zeta(\mathfrak{b}))$. Therefore,

$$W(M) = \Phi(\langle M \rangle) = f^{-1}(V(M)), \quad \forall M \in \mathsf{stmod}(u_\zeta(\mathfrak{b})).$$

Now Theorem 3.4.3, the continuity of f and the fact that the universal support datum V has the tensor product property give

$$\begin{aligned} W(A \otimes B) &= f^{-1}(V(A \otimes B)) = f^{-1}(V(A) \cap V(B)) \\ &= f^{-1}(V(A)) \cap f^{-1}(V(B)) = W(A) \cap W(B) \end{aligned}$$

for all $A, B \in \mathsf{stmod}(u_\zeta(\mathfrak{b}))$. $\square$

Example 8.2.4 and Theorem 8.7.1 imply the following:

Corollary 8.7.2. *Let $u_\zeta(\mathfrak{b})$ be the standard small quantum group for the Borel subalgebra of an arbitrary finite dimensional complex simple Lie algebra. Assume that ℓ satisfies Assumption 8.2.2 and that ℓ is coprime to $|X/\mathbb{Z}\mathcal{R}|$. Then the following hold:*

(a) *All prime ideals of $\mathsf{stmod}(u_\zeta(\mathfrak{b}))$ are completely prime.*

(b) *The cohomological support*

$$W(-) : \mathsf{stmod}(u_\zeta(\mathfrak{b})) \to \mathcal{X}_{cl}(\mathrm{Proj}(\mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C})))$$

has the tensor product property $W(A \otimes B) = W(A) \cap W(B)$ for all $A, B \in \mathsf{stmod}(u_\zeta(\mathfrak{b}))$.

Remark 8.7.3. Assume that ℓ satisfies Assumption 8.2.2 and that ℓ is coprime to $|X/\mathbb{Z}\mathcal{R}|$. Then by Proposition 8.2.3(b), the small quantum Borel subalgebra $u_{\zeta,\Gamma}(\mathfrak{b})$ is based off the group algebra of the lattice $\Gamma/\ell\Gamma$, cf. (8.3). Therefore, the statements in parts (a) and (b) of Theorem 8.7.1 hold for the version of a small quantum Borel subalgebra based off the group algebra of the lattice $\Gamma/\ell\Gamma$.

8.8. The Negron–Pevtsova small quantum Borel algebras

In [84, 85] Negron and Pevtsova considered a different version of small quantum Borel subalgebras. For a lattice, Γ, with $\mathbb{Z}\mathcal{R} \subseteq \Gamma \subseteq X$, set

$$\Gamma^\perp := \{\nu \in \Gamma \mid \langle \nu, \Gamma \rangle \subseteq \ell\mathbb{Z}\}.$$

Denote the canonical projection

$$\Gamma \twoheadrightarrow \Gamma/\Gamma^\perp \quad \text{by} \quad \mu \mapsto \overline{\overline{\mu}}.$$

Let

$$\widetilde{u}_{\zeta,\Gamma}(\mathfrak{t}) \text{ denote the group algebra of } \Gamma/\Gamma^{\perp} \text{ over } \mathbb{C}.$$

For $\mu \in \Gamma/\Gamma^{\perp}$ denote by K_μ the corresponding element of $u_{\zeta,\Gamma}(\mathfrak{t})$. Following [84, 85], define the Hopf algebra

$$\widetilde{u}_{\zeta,\Gamma}(\mathfrak{b}) = u_\zeta(\mathfrak{u}) \# \widetilde{u}_{\zeta,\Gamma}(\mathfrak{t})$$

with relations

$$K_\mu E_\alpha K_\mu^{-1} = \zeta^{\langle \alpha, \mu_0 \rangle} E_\alpha \quad \text{for} \quad \mu \in \Gamma/\Gamma', \alpha \in \mathcal{R}^+,$$

where $\mu_0 \in \Gamma$ is a preimage of μ. By the definition of the lattice $\Gamma^{\perp}$, the right hand side does not depend on the choice of preimage. The coproduct of the generators E_{α_i} is given by

$$\Delta(E_{\alpha_i}) = E_{\alpha_i} \otimes 1 + K_{\overline{\overline{\alpha_i}}} \otimes E_{\alpha_i} \tag{8.9}$$

for $1 \le i \le n$. The antipode is given by $S(E_{\alpha_i}) = -K_{\overline{\overline{\alpha_i}}}^{-1} E_{\alpha_i}$.

Clearly, $\Gamma' \supseteq \Gamma^{\perp}$ and the elements

$$\{K_\mu \mid \mu \in \Gamma'/\Gamma^{\perp}\}$$

are in the center of $\widetilde{u}_{\zeta,\Gamma}(\mathfrak{b})$. In other words, $\widetilde{u}_{\zeta,\Gamma}(\mathfrak{b})$ has a larger center than $u_{\zeta,\Gamma}(\mathfrak{b})$.

By abuse of notation we will denote by $\mu \mapsto \overline{\mu}$ the canonical projection $\Gamma/\Gamma^{\perp} \twoheadrightarrow \Gamma/\Gamma'$, recall (8.2). We have the surjective Hopf algebra homomorphism

$$\widetilde{u}_{\zeta,\Gamma}(\mathfrak{t}) \twoheadrightarrow u_{\zeta,\Gamma}(\mathfrak{t})$$

given by $K_\mu \mapsto K_{\overline{\mu}}$ for $\mu \in \Gamma/\Gamma^{\perp}$ and $E_\alpha \mapsto E_\alpha$ for $\alpha \in \mathcal{R}^+$. Its kernel is the ideal generated by the central elements

$$\{K_\mu - 1 \mid \mu \in \Gamma'/\Gamma^{\perp}\}.$$

Let d be the minimal positive integer such that the restriction of $\langle -,- \rangle$ to Γ takes values in $\mathbb{Z}/d$. Choose a primitive $(d\ell)$th root of unity ξ such that $\zeta = \xi^d$. Consider the symmetric (multiplicative) bicharacter

$$\chi : \Gamma/\Gamma^{\perp} \times \Gamma/\Gamma^{\perp} \to \mathbb{C}^{\times} \quad \text{given by} \quad \chi(\mu, \nu) := \xi^{\langle \mu, \nu \rangle} \text{ for } \mu, \nu \in \Gamma/\Gamma^{\perp},$$

where μ_0 and ν_0 are preimages of μ and ν in Γ. By the definition of $\Gamma^{\perp}$, the bicharacter is well-defined and nondegenerate. It induces the isomorphism

$$\varphi : \Gamma/\Gamma^{\perp} \xrightarrow{\cong} \widehat{\Gamma/\Gamma^{\perp}} \quad \text{given by} \quad \varphi(\mu) := \chi(\mu, -) \text{ for } \mu \in \Gamma/\Gamma^{\perp}. \tag{8.10}$$

Similarly to the discussion for $u_{\zeta,\Gamma}(\mathfrak{t})$, for $\lambda \in \widehat{\Gamma/\Gamma^{\perp}}$ define the one dimensional representation of $\widetilde{u}_{\zeta,\Gamma}(\mathfrak{t})$

$$K_{\mu} \mapsto \lambda(\mu), \quad E_{\alpha} \mapsto 0, \quad \forall \mu \in \Gamma/\Gamma^{\perp}, \alpha \in \mathcal{R}^{+}.$$

The irreducible representations of $\widetilde{u}_{\zeta,\Gamma}(\mathfrak{t})$ are one-dimensional and are indexed by $\widehat{\Gamma/\Gamma^{\perp}}$. We have a much simplified version of Proposition 8.3.1 for the algebras $\widetilde{u}_{\zeta,\Gamma}(\mathfrak{t})$, which was originally proved in [84]:

Proposition 8.8.1. *(a)* *The irreducible representations for $\widetilde{u}_{\zeta,\Gamma}(\mathfrak{b})$ are one-dimensional and are precisely the representations λ for $\lambda \in \widehat{\Gamma/\Gamma^{\perp}}$.*

(b) *For any $\widetilde{u}_{\zeta,\Gamma}(\mathfrak{b})$-module, Q, and $\lambda \in \widehat{\Gamma/\Gamma^{\perp}}$ one has*

$$\lambda \otimes Q \otimes \lambda^{-1} \cong Q.$$

Part (a) is proved in the same way as Proposition 8.3.1(a). Part (b) follows at once by combining the following two facts:

(i) For any $\widetilde{u}_{\zeta,\Gamma}(\mathfrak{b})$-module, Q, and $\lambda \in \widehat{\Gamma/\Gamma^{\perp}}$, $\lambda \otimes Q \otimes \lambda^{-1} \cong Q^{\gamma''_{\lambda}}$ where, γ''_{λ} is the automorphism of $\widetilde{u}_{\zeta,\Gamma}(\mathfrak{b})$ given by

$$\gamma''_\lambda(E_\alpha) = \lambda(\overline{\overline{\alpha}})E_\alpha, \quad \gamma_\lambda(K_\mu) = K_\mu, \quad \forall \mu \in \Gamma/\Gamma^\perp, \alpha \in \mathcal{R}^+$$

(this follows from (8.9));

(ii) γ''_λ equals the an inner automorphism $x \mapsto K_{\varphi^{-1}(\mu)} x K^{-1}_{\varphi^{-1}(\mu)}$ (this follows from (8.10)).

From this point further the proofs of Theorems 8.5.2, 8.7.1, and 8.7.1, extend mutatis mutandis from the family of algebras $u_{\zeta,\Gamma}(\mathfrak{b})$ to the family of algebras $\widetilde{u}_{\zeta,\Gamma}(\mathfrak{b})$. Furthermore, there is a simplification in the proof of the analog of Theorem 8.7.1: on the third line of the long display $\lambda \otimes M \otimes \lambda^{-1} \cong M$ and the rest of the equalities in the display can be omitted. This proves the following:

Theorem 8.8.2. *Let $\widetilde{u}_{\zeta,\Gamma}(\mathfrak{b})$ be the version of the small quantum group for the Borel subalgebra of an arbitrary finite dimensional complex simple Lie algebra and a lattice $\mathbb{Z}\mathcal{R} \subseteq \Gamma \subseteq X$ defined in [84]. Assume that ℓ satisfies Assumption 8.2.2. Then the following hold:*

(a) $\mathrm{H}^{2\bullet+1}(\widetilde{u}_{\zeta,\Gamma}(\mathfrak{b}), \mathbb{C}) = 0$ *and* $R := \mathrm{H}^{2\bullet}(\widetilde{u}_{\zeta,\Gamma}(\mathfrak{b}), \mathbb{C}) \cong S^\bullet(\mathfrak{u}^*)^{[1]}$.

(b) There exist two mutually inverse bijections

$$\{\textit{thick right ideals of } \mathsf{stmod}(\widetilde{u}_{\zeta,\Gamma}(\mathfrak{b}))\} \underset{\Theta}{\overset{\Phi}{\rightleftarrows}} \{\textit{specialization closed sets of } \mathrm{Proj}(R)\},$$

where Φ and Θ are given by

$$\Phi(\mathbf{I}) := \bigcup_{A \in \mathbf{I}} W(A)$$

for the cohomological support $W : \mathsf{stmod}(\widetilde{u}_{\zeta,\Gamma}(\mathfrak{b})) \to \mathcal{X}_{cl}(\mathrm{Proj}(R))$ and

$$\Theta(Z) := \{M \in \mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b})) \mid W(M) \subseteq Z\} \quad \textit{for} \quad Z \in \mathcal{X}_{sp}(\mathrm{Proj}(R)).$$

(c) *Every thick right ideal of* $\mathsf{stmod}(\widetilde{u}_{\zeta,\Gamma}(\mathfrak{b}))$ *is two-sided.*

(d) *There exists a homeomorphism* $\mathrm{Proj}(R) \cong \mathrm{Spc}(\mathsf{stmod}(u_{\zeta,\Gamma}(\mathfrak{b})))$.

(e) *All prime ideals of* $\mathsf{stmod}(\widetilde{u}_{\zeta,\Gamma}(\mathfrak{b}))$ *are completely prime.*

(f) *The cohomological support*

$$W(-) : \mathsf{stmod}(\widetilde{u}_{\zeta,\Gamma}(\mathfrak{b})) \to \mathcal{X}_{cl}(\mathrm{Proj}\, R)$$

has the tensor product property $W(A \otimes B) = W(A) \cap W(B)$ *for all* $A, B \in \mathsf{stmod}(\widetilde{u}_{\zeta,\Gamma}(\mathfrak{b}))$.

There is a further simplification in the proof of part (c) of the theorem compared to that of Theorem 8.6.1(b). Since the algebras $\widetilde{u}_{\zeta,\Gamma}(\mathfrak{b})$ satisfy the property in Proposition 8.8.1(b), part (c) of the theorem also follows directly from this property.

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Vita

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