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Stress-strain analysis of adhesive-bonded composite single-lap joints under various loadings

Yang, Chihdar, Ph.D.

The Louisiana State University and Agricultural and Mechanical Col., 1993
STRESS-STRAIN ANALYSIS OF ADHESIVE-BONDED COMPOSITE SINGLE-LAP JOINTS UNDER VARIOUS LOADINGS

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

The Department of Mechanical Engineering

by

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ABSTRACT

The objective of this study is to develop mathematical relations for stress and strain distributions of adhesive-bonded single-lap joints under cylindrical bending and tension. Based on the Theory of Mechanics of Composite Materials and Anisotropic Laminated Plate Theory, elastic models are proposed to predict the stress-strain distributions of the laminates and the adhesive under cylindrical bending and tension. A simplified elastic-plastic model is also recommended for the case of tension loading. For each case, the Laminated Anisotropic Plate Theory is first used in the derivation of the governing equations of the two bonded laminates. The entire coupled system is then obtained through assuming the peel stress between the two laminates. With the Fourier series and appropriate boundary conditions, the solutions of the system are obtained. In this analytical study, the effects due to the transverse shear deformation as well as the coupling effects of external tension and bending of an asymmetric laminate are included.

These developed elastic models are compared to the finite element models. An existing finite element analysis code, "ALGOR," is used as a comparison with these developed elastic models. Results from the developed model for tension are also compared with Goland and Reissner and Hart-Smith's theories.
Based on the developed models, the effects of the overlay length and laminate properties on the maximum adherend and adhesive stresses under both cylindrical bending and tension are evaluated.
CHAPTER 1
INTRODUCTION

Fiber-reinforced plastics is a group of materials which consist of two major components -- fibers and matrix. Usually, the fiber serves as the load carrying member, while the matrix is used to keep the shape of the object and distribute load. With the introduction of high performance fibers such as carbon, boron, and kevlar, also with some new and improved matrix materials, advanced composites have established themselves as engineering structural materials. The high strength/weight and high stiffness/weight ratio of advanced composites, their excellent fatigue resistance, and high internal damping properties have made advanced composites take the place of many traditional materials such as steel, aluminum, and other alloys in more and more applications. Furthermore, the most important characteristic of composite materials is the anisotropic nature in their mechanical and thermal properties. This comes from the difference of properties between the fiber and the matrix. With proper orientation and stacking sequence of fiber layers when the composites are being manufactured, they can be reinforced in any specific direction instead of in all directions. For example, a unidirectional composite may be very strong and stiff in the fiber direction, its transverse and shear properties may be very much poorer. Thus, a composite can be highly anisotropic in respect of both stiffness and strength,
and this not only can minimize the weight and dimension of the structure, but also provide more flexibility in design.

Ideally, a structure would be designed without joints, since joints could be a source of weakness and/or excess weight. Limitations on component size imposed by manufacturing processes, and the requirement of inspection, accessibility, repair and transportation/assembly, mean that some load-carrying joints are inevitable in all large structures.

Adhesive-bonded joints have been widely used for composite materials as a necessary alternative to conventional mechanical joint designs. The primary limitations of such designs arise from machining difficulties and subsequent damage to the laminate following these operations. This highlights the needs for research in the field of joint design and analysis. The purpose of the present study is to provide elastic models on adhesive-bonded single-lap joints under cylindrical bending and tension. The theory developed correlates the adherends as being limited to orthotropic laminates, especially unidirectional or cross-ply composite laminates. Plane-strain condition is assumed for both loadings during the present study for wide joints. Governing equations are first derived based on the Theory of Mechanics of Composite Materials and the Anisotropic Laminated Plate Theory with the small deformation assumption. After determining the appropriate boundary conditions, solutions for the stress and strain distributions of the above mentioned joints under the corresponding loading
conditions are then obtained through numerical calculations, while in the recommended elastic-plastic model of the joints under tension, numerical iterations are performed in order to determine the plastic region. By utilizing an available finite element software, finite element analyses are conducted to verify the developed elastic models.

With the predicted stress and strain distributions of the laminates, the adhesive effects of joint parameters such as material properties and overlay length on joint performance are studied.
CHAPTER 2
PREVIOUS WORK

Many different methods have been used to analyze the stress and strain behaviors of adhesive-bonded lap joints. Some excellent review papers in the literature can be found. The early analysis work for isotropic adherends prior to 1961 was reviewed by Kutscha [1], and those analyses from 1961 to 1969 were reviewed by Kutscha and Hofer [2]. A review of the theoretical work, including classical and finite element methods related to all aspects of adhesive bonded joints in composite materials has been provided by Matthews [3]. In 1989, Vinson [4] summarized the published work dealing with the adhesive bonding of polymer matrix composite structures. It has been shown that, in order to get a solution, it is inevitable that some simplification be made, the correspondence between the theoretical and experimental results depending critically on which factors are omitted from the analysis.

Through these extensive review publications, it was found that most of the work done in this field was concentrated on the analysis of joints under tension. Very rarely have papers regarding lap joints under bending been published. Among the few bending studies is Yang, Pang, and Griffin’s [5] work on double-lap composite joints under cantilevered bending in which they proposed a strain gap model to describe the stress-strain behavior. Also, Wah [6,7] (1973, 1976) studied scarf joints with isotropic adherends in
bending and single-lap joints with anisotropic adherends in cylindrical bending. The anisotropic adherends used during Wah's derivation, however, are restricted to symmetric laminates. In other words, the bending and stretching terms of the laminates are uncoupled. However, one of the special properties of a composite laminate is the coupled bending and tension caused by the asymmetric stacking sequence. In order to better utilize composite materials, this unique property needs to be addressed and studied.

In general, the publications regarding lap joint under tension can be separated into two categories: (1) Joints with Isotropic Adherends and (2) Joints with Anisotropic Adherends. They are discussed in the following two sections.

2.1 Joints with Isotropic Adherends

The basic theoretical treatment of bonded joints in metals was based on the classical analytical methods of continuum mechanics. The simplest analysis on single-lap joint under tension loading considers the adherends to be rigid and the adhesive to deform only in shear. This is shown in Fig. 1. If the length is $l$, and the load per unit width is $P$, then the shear stress $\tau$ is given by:

$$\tau = \frac{P}{l}$$
Volkersen [8] introduced the phenomenon called differential shear (or shear lag). This assumes continuity of the adhesive adherend interface, and the uniformly sheared parallelograms of adhesive shown in Fig. 1 become distorted to the shapes given in Fig. 2.

In Volkersen's analysis, it is assumed that the adhesive deforms only in shear, while the adherend deforms only in tension.

The effects due to the rotation of the adherends were first taken into account by Goland and Reissner [9]. They introduced a factor, $k_m$, which relates the bending moment on the adherend at the end of the overlay, $M_o$, to the in-plane loading by the relationship:
where \( t \) and \( \eta \) are the thickness of the adherend and the adhesive, respectively. \( k_m \) is a coefficient which depends upon the load, the joint dimensions, and the physical properties of the adherends as

\[
k_m = \frac{\cosh u_2 c}{\cosh u_2 c + 2\sqrt{2}\sinh u_2 c}
\]

where

\[
u_2 = \frac{1}{t}\sqrt{\frac{3p(1-v^2)}{2tE}}
\]

\( c \) is one half of the overlay length; \( v \) and \( E \) are the Poisson’s ratio and the longitudinal Young’s modulus of the adherends, respectively.

As the load is increased the overlay rotates, bringing the line of action of the load closer to the centerline of the adherends, as shown in Fig. 3.

\[
M_o = k_m \frac{P(t+\eta)}{2}
\]

**Figure 3** A Geometrical Representation of the Goland and Reissner Bending Moment Factor. (a) Undeformed Joint \((k_m=1)\); (b) Deformed Joint \((k_m<1)\)
While the basic approach of Goland and Reissner theory was based on beam theory, or rather, on cylindrically bent-plate theory, which treated the overlay section as a beam of twice as thick as the adherend. Their work was examined photoelastically by McLaern, et al. [10] who found that their general conclusions are correct in so far as they deduce tensile and tearing stresses increasing toward the joint edge, these increases being reduced by the bending under load. Sharpe and Muha [11] measured the shear stress in the epoxy adhesive layer of a plexiglas single-lap joint model by monitoring the fringe pattern generated by a laser beam incident on single wires imbedded on each side of the bond layer. The experimental stresses were compared with those predicted by analytical, numerical, and finite element solutions. Predictions of the Goland and Reissner theory were found to agree well with the results. In 1971, Erdogan and Ratwani [12] studied the stress distribution in bonded stepped joint with one isotropic plate and one orthotropic plate without considering the bending effect. As a limiting case, the solution for bonded plates with a smoothly tapered joint is given. Hart-Smith [13-17] has published a series of papers regarding single-lap, double-lap, scarf, and stepped-lap joints involving a continuum model in which the
adherends are isotropic or anisotropic elastic, and the adhesive is modeled as elastic, elastic-plastic, or bielastic. Basically, the classical plate theory was adopted during Hart-Smith's derivation. The $k_m$ in Eq. (2) introduced by Hart-Smith can be written as

$$k_m = \frac{1 + \frac{\xi^2 \lambda^2}{32 (\lambda')^4} \left[ 1 + \frac{2 \lambda' c}{3} \frac{2 \lambda' c}{\tanh(2 \lambda' c)} \right]}{1 + \xi \left[ \frac{3 \lambda^2 \xi^2}{32 k_b (\lambda')^4} \left[ 1 + \frac{(2 \lambda' c)^2}{3} \frac{2 \lambda' c}{\tanh(2 \lambda' c)} \right] \right]}$$  \hspace{1cm} (4)

where

$$\xi = \frac{P}{D}$$  \hspace{1cm} (5)

$$\lambda = \sqrt{\frac{2G}{E \eta}}$$  \hspace{1cm} (6)

$$k_b = \frac{D}{E \lambda^3} \frac{12(1 - v^2)}{12(1 - v^2)}$$  \hspace{1cm} (7)

$$\lambda' = \frac{1 + 3(1 - v^2)}{4} \lambda k_b$$  \hspace{1cm} (8)

$D$ stands for the bending rigidity of the adherends; $G$ is the shear modulus of the adhesive.

The effects of transverse shear deformation, which has been shown to be important when span-to-depth ratio is small [18-21], however, was not
included in either Goland and Reissner and Hart-Smith’s theories. Moreover, edge effects were neglected and adhesive stress were assumed constant through the thickness in most of the analyses.

2.2 Joints with Anisotropic Adherends

In the case of anisotropic adherends, elasticity solution is difficult to obtain. The most common theories utilized in the literature are Mechanics of Composite Materials and Laminated Plate Theory. Most of the analyses neglect the effect of transverse shear deformation and the edge effects, which was extensively discussed by Grimes and Greimann [22] and Spilker [23]. The major differences between isotropic and anisotropic adherends are

1. The coupling effect between bending and middle-plane extension of the asymmetric adherend,

2. Effect of transverse shear deformation is more significant for laminated adherends when overlay length is small, and

3. Interlaminar failure of the laminated adherends may happen before adhesive failure.

In 1969, Whitney [24,25] obtained a closed form solution for anti-symmetric cross-ply and angle-ply laminates under transverse loading by expanding the load in a double Fourier series. It has been shown that the coupling between bending and middle-plane extension of an unbalanced
orthotropic composite laminate can increase maximum deflections by as much as 300% compared to analogous in which coupling is neglected.

The importance of shear deformation of anisotropic laminated plate also has been brought into attention [26-30]. Pagano [31,32] investigated the limitation of classical plate theory (CPT) by comparing solutions of several specific boundary value problems after this theory to the corresponding theory of elasticity solutions. As would be expected, CPT underestimates the plate deflection and gives a very poor estimate for relatively low span-to-depth ratio. This shows the necessity of incorporating the influence of shear deformation in the case of small span-to-depth ratio. For higher span-to-depth ratio, the exact solution approaches the CPT result asymptotically. In order to correlate the shear deformation, higher-order [26,33-37] and refined anisotropic plate theories [38] have been proposed extensively.

Although Hart-Smith's derivation did not correlate the coupling between bending and middle-plane extension of an asymmetric plate and did not include the effects of transverse shear deformation, the maximum adhesive and adherend stresses calculated using Hart-Smith's analysis was found to agree with those calculated using a finite element technique by Long [39]. Long also compared the analytical results with the experimental results of ARALL-1, which consists of thin aluminum alloy sheets alternating with aramid fiber/epoxy prepreg layers, single- and double-lap joints. Allred and Guess [40] conducted an experimental and finite element analysis on the
adhesive-bonded fin-hub joint subjected to bending by the use of an existing finite element computer code, SAAS IIa. As expected, the maximum shear stress of the adhesive and the maximum normal stress of the adherends were found to locate on the edge of overlay. In 1973, Wah [6] studied the stress distribution in a bonded single-lap joint under cylindrical bending by the use of the laminate constitutive equations. In his approach, the adherends were assumed to be symmetric, therefore the bending and stretching terms are uncoupled. Transverse shear effects were also neglected by Wah.
CHAPTER 3
SINGLE-LAP JOINTS UNDER CYLINDRICAL BENDING

Figure 4 shows the configuration of a single-lap joint under pure bending. The coordinate system and the symbols of joint dimensions used in the derivation are defined in Fig. 5. Based on the first-order laminated plate theory, the displacement field of the upper and lower laminates, $u$ in the $x$-direction and $w$ in the $z$-direction, can be written as

$$u = u^0(x) + z\psi(x)$$  \hspace{1cm} (9)

$$w = w(x)$$  \hspace{1cm} (10)

where the superscript $^0$ represents the parameter for the middle-plane element, and $\psi$ is its corresponding bending slope. By substituting Eq. (9) into the strain-displacement relations, the normal strain $\varepsilon_x$ and shear strain $\varepsilon_{xz}$ can be expressed as
For orthotropic laminates, the stress resultant (or unit width force resultant) in \(x\)-direction, \(N_x\), and the unit width moment in \(y\)-direction, \(M_y\), are related to only the mid-plane strain and the plate curvature and not to the in-plane shear strain. Because of the assumed plane strain condition and with the positive directions defined in Fig. 5, the stress and moment resultants of the upper and lower laminates are [41]

\[
N_x^U = A_{11}^U \frac{du^U}{dx} + B_{11}^U \frac{d\psi^U}{dx}
\]

\[
N_x^L = A_{11}^L \frac{du^L}{dx} + B_{11}^L \frac{d\psi^L}{dx}
\]
\[ M_y^U = B_{11}^U \frac{du}{dx} + D_{11}^U \frac{d\psi}{dx} \]  
\[ M_y^L = B_{11}^L \frac{du}{dx} + D_{11}^L \frac{d\psi}{dx} \]

where the \([A], [B],\) and \([D]\) are the matrices of the equivalent modulus for the laminate and are defined as

\[ (A_{11}^U, B_{11}^U, D_{11}^U) = \int_{-h/2}^{h/2} Q_{11}^{(0)}(1, z_1, z_1^2)dz_1 \]  
\[ (A_{11}^L, B_{11}^L, D_{11}^L) = \int_{-h/2}^{h/2} Q_{11}^{(0)}(1, z_2, z_2^2)dz_2 \]

The \(Q_{11}^{(i)}\) represents the stiffness in \(x\)-direction of the \(i^{th}\) ply. The superscript \(U\) and superscript \(L\) denote the upper and lower laminate, respectively; \(h\) is the thickness, and \(z_1\) and \(z_2\) are measured from the middle-plane of the upper and lower laminate as shown in Fig. 5.

From the constitutive relation for transverse shear \(Q_x\), [41]

\[ Q_x = kA_{55}e_{xz} \]

where \(k\) is the shear correction factor. The \(A_{55}\) is so defined that for the upper and lower laminates,

\[ A_{55}^U = \int_{-h/2}^{h/2} Q_{55}^{(0)}dz_1 \]
where \( Q_{55}^{(i)} \) is the shear stiffness of the \( i \)th ply.

The transverse shear resultants for the upper and lower laminates can be related to the displacement fields by the substitution of Eq. (12) into Eq. (19).

\[
A_{55}^L - \int_{-h/2}^{h/2} Q_{55}^{(0)} dz_2
\]  

(21)

Where \( k^U \) and \( k^L \) denote the shear correction factors of the upper and lower adherend, respectively.

The above relations from existing theory correlate the laminate force and the moment to the displacement field by the definition of equivalent modulus matrices. The next issue is to develop new relations/models describing the behavior of the whole joint including the adhesive.

Consider a segment of the top laminate as a free body shown in Fig. 6, and by neglecting higher order terms, the equations of equilibrium can be written as

\[
\frac{dN_x^U}{dx} = -\tau
\]  

(24)
Figure 6 Free Body Diagram and Sign Convention

\[
\frac{dM_y^u}{dx} = -Q_x^u + \frac{h^u}{2} \tau \tag{25}
\]

and

\[
\frac{dQ_x^u}{dx} = -q \tag{26}
\]

where \( q \) is the peel stress between the two laminates. The shear stress in the adhesive, \( \tau \), arises from both the relative displacement between the bottom surface of the upper laminate and the top surface of the lower laminate and from the first order derivative of the vertical deflection. By assuming that the shear stress is uniform throughout the thickness of the adhesive and by utilizing the average value of the slopes of the two laminates, the adhesive shear stress can be obtained as

\[
\tau = \frac{G}{\eta} (u^L - u^U) + \frac{G}{\eta} \left( \frac{h^L}{2} \psi^L + \frac{h^U}{2} \psi^U \right) - \frac{G}{\eta} \left( \frac{dw^L}{dx} + \frac{dw^U}{dx} \right) \tag{27}
\]

where \( G \) and \( \eta \) are the shear modulus and the thickness of the adhesive.
The same equilibrium conditions used for Eqs. (24), (25), and (26) can also be applied to the lower laminate. Combining these equilibrium equations with the constitutive relation in Eqs. (13) - (16), (19) and (20), the governing equations for the joint as a system can be described by

\[
\frac{A_{11}^U d^2 u^U}{dx^2} + \frac{B_{11}^L d^2 \psi^L}{dx^2} = \frac{-G}{\eta} \left( u^{ol} - u^U \right)
- \frac{G(h^L \psi^L + h^U \psi^U)}{2} - \frac{G}{dx} \left( \frac{dw^L}{dx} + \frac{dw^U}{dx} \right)
\]

(28)

\[
\frac{A_{11}^L d^2 u^L}{dx^2} + \frac{B_{11}^L d^2 \psi^L}{dx^2} = \frac{G}{\eta} \left( u^{ol} - u^U \right)
+ \frac{G(h^L \psi^L + h^U \psi^U)}{2} - \frac{G}{dx} \left( \frac{dw^L}{dx} + \frac{dw^U}{dx} \right)
\]

(29)

\[
\frac{B_{11}^U d^2 u^U}{dx^2} + \frac{D_{11}^U d^2 \psi^U}{dx^2} - k_u^U A_{55}(\psi^U + \frac{dw^U}{dx}) = \frac{G h^U}{2} (u^{ol} - u^U)
+ \frac{G(h^L \psi^L + h^U \psi^U)}{2} - \frac{G}{dx} \left( \frac{dw^L}{dx} + \frac{dw^U}{dx} \right)
\]

(30)

\[
\frac{B_{11}^L d^2 u^L}{dx^2} + \frac{D_{11}^L d^2 \psi^L}{dx^2} - k_l^L A_{55}(\psi^L + \frac{dw^L}{dx}) = \frac{G h^L}{2} (u^{ol} - u^U)
+ \frac{G(h^L \psi^L + h^U \psi^U)}{2} - \frac{G}{dx} \left( \frac{dw^L}{dx} + \frac{dw^U}{dx} \right)
\]

(31)

\[
k_u^U A_{55}(\frac{d\psi^U}{dx} + \frac{d^2 w^U}{dx^2}) = q
\]

(32)
Equations (28) - (33) are six coupled second-order ordinary differential equations with 6 variables: \( u^U, u^L, \Psi^U, \Psi^L, w^U, \) and \( w^L \). In order to obtain the homogeneous solutions of these variables, the characteristic equation needs to be solved.

\[
k^L A_{33}^L \left( \frac{d\psi^L}{dx} + \frac{d^2\omega^L}{dx^2} \right) = -q \tag{33}
\]

Equation (34) can be reduced to

\[
k^U k^L A_{33}^U A_{33}^L (Q_1 D^2 + Q_2) D^{10} = 0 \tag{35}
\]

where

\[
Q_1 = (A_{11}^U D_{11}^U - B_{11}^U) (A_{11}^L D_{11}^L - B_{11}^L) \tag{36}
\]

\[
Q_2 = -G \left[ \left( \frac{h^U}{4} + \frac{h^U}{4} \right) A_{11}^U + \left( \frac{h^U}{2} + \frac{h^U}{2} \right) B_{11}^U + \frac{1}{\eta} D_{11}^U \right] \left( A_{11}^L D_{11}^L - B_{11}^L \right)

- G \left[ \left( \frac{h^L}{4} + \frac{h^L}{4} \right) A_{11}^L + \left( \frac{h^L}{2} + \frac{h^L}{2} \right) B_{11}^L + \frac{1}{\eta} D_{11}^L \right] \left( A_{11}^U D_{11}^U - B_{11}^U \right) \tag{37}
\]
Letting

\[ a^2 = -\frac{Q_2}{Q_1} \]  

(38)

the eigenfunctions for the system of ordinary differential equations (Eqs. (28) - (33)) are then

\[ 1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, \cosh \alpha x, \sinh \alpha x \]

Because the peel stress \( q \) is expected to be continuous within the joint and has a non-zero value at \( x=0 \), it can be represented by the right half of an even function \( p \) which is defined from \( x=-l \) to \( x=l \), where \( l \) is the overlay length of the joint. The Fourier series expansion of function \( p \) contains only cosine terms since \( p(-x)=p(x) \). The peel stress \( q \) can then be represented as a \((n+1)\)-term Fourier cosine series with \( n+1 \) coefficients, \( d_o, ..., d_n \), as

\[ q = \sum_{i=0}^{n} d_i \cos \frac{i\pi x}{l} \]  

(39)

By defining \( v_i \) as the variables needed to be solved,

\[ v_1 = u^o, \quad v_2 = u^L, \quad v_3 = \psi^u, \quad v_4 = \psi^L, \quad v_5 = w^U, \quad v_6 = w^L \]

By combining the homogeneous and particular solutions, \( v_i \) can be written as

\[ v_j = \sum_{i=0}^{9} a_{ji} x^i + b_{ji} \cosh \alpha x + \sum_{i=0}^{n} c_{ji} \sin \frac{i\pi x}{l} \]  

(40)
where the sine and cosine terms represent the particular solutions.

Because the highest order of $D$ in Eq. (35) is 12, there are only 12 independent coefficients with which all $a_{ij}$ and $b_{ij}$ can be determined. Also, $c_{ji}$ can be obtained as functions of $d_i$ through the governing equations (Eqs. (28) - (33)). There are only ten independent coefficients of the sixty coefficients in $a$'s. The selection of the ten independent coefficients and the relation between the ten and the other coefficients can be seen in Appendix A.

The $(n+1)$ undetermined coefficients, $d_0$, ..., $d_n$, together with the 12 independent coefficients in $a$'s and $b$'s, result in a total of $(n+13)$ unknowns. In addition to the expression of peel stress in Eq. (39), $q$ can also be related to the difference of vertical deflections between the upper and lower laminates. With the expressions for $w^u$ and $w^l$ in Eq. (41), $q$ can be written as

$$q = \frac{E}{\eta} (w^u - w^l)$$

$$- \frac{E}{\eta} \sum_{i=0}^{9} (a_{ij} - a_{6j}) x^i + \frac{E}{\eta} (b_{51} - b_{61}) \cosh \alpha x$$

$$+ \frac{E}{\eta} (b_{52} - b_{62}) \sinh \alpha x + \frac{E}{\eta} \sum_{i=0}^{n} (c_{3i} - c_{6i}) \cos \frac{i \pi x}{l}$$

$$j=5,6 \quad (41)$$
From the above equation and the orthogonal properties of Fourier series, the following \((n+1)\) equations can be obtained:

\[
\begin{align*}
\frac{d_r}{\eta l} & = \frac{2E}{\eta l} \left[ \int_0^l \sum_{m=0}^9 (a_{5m} - a_{6m}) x^m \cos \frac{r \pi x}{l} \, dx \right. \\
& \quad + (b_{51} - b_{61}) \int_0^l \cosh ax \cos \frac{r \pi x}{l} \, dx + (b_{52} - b_{62}) \int_0^l \sinh ax \cos \frac{r \pi x}{l} \, dx \\
& \quad + (c_{5r} - c_{6r}) \frac{l}{2} \bigg] \quad r=0,1,2,\ldots,n
\end{align*}
\] (43)

The remaining 12 equations result from the assumption of appropriate boundary conditions. Since the stress and strain are related to the derivatives of the variables \(u^*, \psi, \) or \(w,\) the datum of these variables is irrelevant. Therefore, for convenience, these variables for the upper laminate can be set to zero at \(x=0.\)

\[
\begin{align*}
v_1(0) &= 0 \\
v_3(0) &= 0 \\
v_5(0) &= 0
\end{align*}
\] (44)

Under pure bending, the axial force resultant must be zero at the edge of the overlay, so

\[
\begin{align*}
N_x^u(0) &= A_{11} \frac{du}{dx} + B_{11} \frac{dv_3}{dx} = 0 \\
N_x^u(l) &= A_{11} \frac{du}{dx} + B_{11} \frac{dv_3}{dx} = 0 \\
N_x^L(0) &= A_{11} \frac{dv_1}{dx} + B_{11} \frac{dv_3}{dx} = 0
\end{align*}
\] (45)
The right hand side of Eqs. (28) and (29), which govern the axial force resultants on the upper and lower laminates, have the same magnitude but opposite sign. Once the three conditions above are satisfied, \( N_x^L(l) \) will automatically be zero. This is why \( N_x^L(l) = 0 \) is not an independent boundary condition. The applied moment \( M_o \) is taken by the upper laminate at \( x = 0 \) and by the lower laminate at \( x = l \), so

\[
\begin{align*}
M_y^U(0) &= B_{11} \frac{d v_1(0)}{d x} + D_{11} \frac{d v_3(0)}{d x} - M_o \\
M_y^L(l) &= B_{11} \frac{d v_1(l)}{d x} + D_{11} \frac{d v_3(l)}{d x} - 0 \\
M_y^U(0) &= B_{11} \frac{d v_2(0)}{d x} + D_{11} \frac{d v_4(0)}{d x} - 0 \\
M_y^L(l) &= B_{11} \frac{d v_2(l)}{d x} + D_{11} \frac{d v_4(l)}{d x} - M_o
\end{align*}
\]  

(46)

The transverse shear force resultants of the upper and lower laminates should be zero at the edge of the overlay. By applying the same argument for the axial force resultant and because the integration of the peel stress must be zero, only two independent boundary conditions are available. They can be written as

\[
\begin{align*}
Q_x^U(0) &= -k \frac{A_{55} v_3(0)}{u} + \frac{d v_3(0)}{d x} = 0 \\
Q_x^L(0) &= -k \frac{A_{55} v_4(0)}{u} + \frac{d v_4(0)}{d x} = 0
\end{align*}
\]  

(47)
With the twelve boundary conditions listed along with the \((n+1)\) equations from Eq. (43), the entire solution may be determined.
CHAPTER 4
SINGLE-LAP JOINTS UNDER TENSION

Figures 7 (a) and (b) show the configuration of a single-lap joint under tension before and after deformation, respectively. The tensile loading, shown as \( P \), represents a loading per unit width. The definitions of the coordinate system and the symbols of joint dimensions used in this Chapter are shown in Fig. 7 (a). The displacement field of the two adherends are defined in Eqs. (9) and (10) in Chapter 3.

With the same approach used in Chapter 3, the stress resultant in \( x \)-direction \( N_x \), the unit width moment in \( y \)-direction \( M_y \), and the transverse shear stress resultant \( Q_x \) can be obtained as functions of \( u^\circ \), \( \psi \), and \( w \). These relations regarding the two laminates can be referred to Eqs. (13) - (16) and (22) - (23).
Because of the varying laminate loading conditions, it is convenient to separate the construction into three sections as shown in Fig. 7. The mechanical behavior of the adherend (adherends) in each section is discussed separately in the following sections. In the following sections, the subscript 1, 2, and 3 of the displacement fields $u^U$, $u^L$, $\psi^U$, $\psi^L$, $w^U$, and $w^L$ denote the sections in which they are located.

(a) Section One

In order to balance the two loading forces applied to the joints, there must be an oblique angle $\theta$ from the two forces to the central axes of the two adherends. This can be seen in Fig 7.

\[ \theta = \frac{h^U + h^L + \eta}{2 \frac{l_1 + l_2 + l}{l_1 + l_2 + l}} \]  

(48)

where $h^U$, $h^L$ are the thickness of the upper and lower adherends, and $l_1$, $l_2$ are the lengths of the upper and lower adherends outside the overlay, respectively. The overlay length is represented by $l$, and the adhesive thickness is represented by $\eta$. The bending moment of the upper adherend, $M_{yl}^U$, which is induced by the tilted applied force, can be related to the oblique angle and the transverse displacement as

\[ M_{yl}^U = P(\theta x_1 + w_1^U) \]  

(49)
where \( x_1 \) is defined from the left edge of the upper laminate as shown in Fig. 7. Assume that the slope \((d\psi_1^U/dx_1)\) of the upper laminate is small; neglecting the higher order terms results in the axial stress resultant at each cross-section of the upper adherend as

\[
N_{x1}^U = P \tag{50}
\]

With the adopted sign convention, as shown in Fig. 6, the transverse shear stress resultant can be determined from the transverse component of the applied force as

\[
Q_{x1}^U = -P\left(\theta + \frac{d\psi_1^U}{dx_1}\right) \tag{51}
\]

Substitute the above kinematic relations into constitutive relations (Eqs. (13), (15) and (22)), and the governing equations with the three variables, \( u_1^U, \psi_1^U, \) and \( \omega_1^U \) of the upper adherend in this section are then

\[
\begin{align*}
A_1 \frac{du_1^U}{dx_1} + B_1 \frac{d\psi_1^U}{dx_1} &= -P \\
B_1 \frac{du_1^U}{dx_1} + D_1 \frac{d\psi_1^U}{dx_1} + P\omega_1^U &= -P\theta x_1 \\
k^U A_{55} \psi_1^U + (k^U A_{55}^U + P) \frac{d\omega_1^U}{dx_1} &= -P\theta
\end{align*}
\tag{52}
\]
The homogeneous solutions can be obtained by solving the characteristic equation

\[
\begin{vmatrix}
A_{11}^u & B_{11}^u & 0 \\
B_{11}^u & D_{11}^u & P \\
0 & k^u A_{55}^u & (k^u A_{55}^u + P)\alpha
\end{vmatrix} = 0
\]

(53)

Equation (53) can be satisfied by

\[
\alpha = 0 \quad \text{or} \quad \pm \sqrt{\frac{PA_{11}^u k^u A_{55}^u}{(k^u A_{55}^u + P)(A_{11}^u D_{11}^u - B_{11}^u)}}
\]

(54)

Define \( \alpha_1 \) as

\[
\alpha_1 = \sqrt{\frac{PA_{11}^u k^u A_{55}^u}{(k^u A_{55}^u + P)(A_{11}^u D_{11}^u - B_{11}^u)}}
\]

(55)

The homogeneous solutions together with the particular solutions can be written as

\[
\begin{align*}
\psi_1^u &= -a_2 - \frac{A_{11}^u}{B_{11}^u} \cosh \alpha_1 x_1 - a_3 \frac{A_{11}^u}{B_{11}^u} \sinh \alpha_1 x_1 + \psi_1^p \\
\psi_1 &= -a_2 \cosh \alpha_1 x_1 - a_3 \sinh \alpha_1 x_1 + \psi_1^p \\
\psi_1^u &= -a_2 (B_{11}^u A_{11}^u D_{11}^u) \alpha_1 \cosh \alpha_1 x_1 - a_2 (B_{11}^u A_{11}^u D_{11}^u) \alpha_1 \sinh \alpha_1 x_1 - \frac{B_{11}^u}{A_{11}^u}
\end{align*}
\]

(56)
where $a_1$, $a_2$, and $a_3$ are three independent coefficients which need to be determined from boundary conditions.

In the case of a symmetric upper adherend ($B_{11}^U=0$), solutions of Eq. (52), Eq. (56), need to be modified to

$$
\begin{align*}
&u_1^U = a_1 + \frac{P}{A_{11}} x_1 \\
&\psi_1^U = a_2 \cosh \alpha_1 x_1 + a_3 \sinh \alpha_1 x_1 + \theta \\
&w_1^U = -a_3 \frac{D_{11} \alpha_1}{P} \cosh \alpha_1 x_1 - a_2 \frac{D_{11} \alpha_1}{P} \sinh \alpha_1 x_1 - \theta x_1
\end{align*}
$$

(b) Section Two

The governing equations of the lower laminate in section two, as defined in Fig. 7 (a), are almost the same as those of the upper adherend in section one. The only difference is in the induced bending moment. Because the origin of the $x_2$ coordinate is located at the edge of the overlay instead of at the right end of the lower adherend, the moment of the lower adherend is related to the coordinate $x_2$ and the transverse displacement $w_2^L$ as

$$
M_2^L = P[\theta(l_2-x_2)-w_2^L]
$$

The governing equations are then
When the same technique used in Section One is applied, the solutions of the lower laminate in section two can be obtained as

\[
\begin{align*}
\left\{ \begin{array}{l}
A_{11}^L \frac{du_2^L}{dx_2} + B_{11}^L \frac{d\psi_2^L}{dx_2} = -P \\
B_{11}^L \frac{du_2^L}{dx_2} + D_{11}^L \frac{d\psi_2^L}{dx_2} + Pw_2^L = -P(\ell_2 - x_2) \\
k^L A_{55}^L \psi_2^L + (k^L A_{55}^L + P) \frac{dw_2^L}{dx_2} = -P\theta
\end{array} \right. \\
\end{align*}
\]  

(59)

With three undetermined coefficients \( b_1, b_2, \) and \( b_3 \) and with

\[
\begin{align*}
u_2^L &= b_1 + b_2 \cosh \alpha x_2 + b_3 \sinh \alpha x_2 + \frac{P}{A_{11}^L} x_2 \\
\psi_2^L &= -b_2 - \frac{A_{11}^L}{B_{11}^L} \cosh \alpha x_2 - b_3 - \frac{A_{11}^L}{B_{11}^L} \sinh \alpha x_2 + \theta \\
w_2^L &= -b_3 \left( \frac{A_{11}^L D_{11}^L}{B_{11}^L} \right) \frac{\alpha_2}{P} \cosh \alpha x_2 - b_2 \left( \frac{A_{11}^L D_{11}^L}{B_{11}^L} \right) \frac{\alpha_2}{P} \sinh \alpha x_2 + \theta (\ell_2 - x_2) - \frac{B_{11}^L}{A_{11}^L}
\end{align*}
\]

(60)

with \( \alpha_2 \) given by

\[
\alpha_2 = \sqrt{\frac{PA_{11}^L k^L A_{55}^L}{(k^L A_{55}^L + P)(A_{11}^L D_{11}^L - B_{11}^L)}}
\]

(61)

Again, when the lower adherend is symmetric \((B_{11}^L = 0)\), the solutions are replaced by
(c) Section Three

Section three is the overlay region. The laminate behavior of this section is exactly the same as the laminate behavior within the overlay region of the joint under pure bending. The same governing equations regarding the two laminates with the same homogeneous and particular solutions can be applied to this section. The peel stress of the adhesive $q$ is also assumed as an $(n+1)$-term Fourier cosine series with $n+1$ coefficients, $c_0, ..., c_n$, as

$$q = \sum_{i=0}^{n} c_i \cos \frac{i\pi x}{l}$$

where $l$ stands for the overlay length. By defining $\nu_i$ as the variables which need to be solved,

$$\nu_1 - u_3^{\nu L}, \; \nu_2 - u_3^{\nu L}, \; \nu_3 - \psi_3^{U}, \; \nu_4 - \psi_3^{U}, \; \nu_5 - w_3^{U}, \; \nu_6 - w_3^{L}$$

and by combining the homogeneous and particular solutions, $\nu_i$ can be written as

$$\nu_j = \sum_{i=0}^{9} d_{ij} x^i + e_{ij} \cosh \alpha_3 x + e_{ij} \sinh \alpha_3 x + f_j(x) + \sum_{i-1}^{n} g_{ij} \sin \frac{i\pi x}{l} \quad j=1,2,3,4$$
\[ v_j = \sum_{i=0}^{9} d_i x^i + e_i \cosh \alpha_3 x + e_i \sinh \alpha_3 x + f_j(x) + \sum_{i=1}^{n} g_i \cos \frac{i \pi x}{l} \quad j=5,6 \quad (65) \]

where \(d\)'s and \(e\)'s are undetermined coefficients of the homogeneous solutions and \(\alpha_3\) is the same as \(\alpha\) in Chapter 3 (Eqs. (36) - (38)). It is noted that a constant term, \(c_0\), is included in the peel stress, \(q\), which is the forcing function of the system. The constant term, \(c_0\), implies the average adhesive peel stress within the joint. Since the constant term is also one of the eigenfunctions of the system of governing equations, the particular solutions corresponding to the constant term will be an \(x\) polynomial with an order up to 10 (the highest order of the eigenfunctions plus one). If \(c_0\) happens to be 0, the particular solutions will be very much simplified. Unfortunately, it can be shown that \(c_0\) cannot be 0 by examining the portion of the upper adherend within the overlay range as a free body. The equilibrium condition of the force in the \(z_1\)-direction shows that the integration of adhesive peel stress must balance the transverse shear stress resultant on the left edge of this free body. As long as the slope of the upper adherend at the left overlay edge is not zero, the average peel stress, \(c_0\), will not be 0. Therefore, the 10th order polynomials \(f_j\) in Eqs. (64) and (65) are necessary. The approach used to determine \(f_j\) can be seen in Appendix B. The sine and cosine terms in Eqs. (64) and (65) represent the particular solutions corresponding to the other cosine terms in the peel stress.
Again, because the highest order of $\alpha$ in Eq. (35) is twelve, the homogeneous solutions, shown as the first three terms in Eqs. (64) and (65) of the system, are so related that there are only 12 independent coefficients with which all $d_{ji}$, $e_{ji}$, and $e_{j2}$ can be determined. Also, $g_{ji}$ can be obtained as functions of $c_i$ through the governing equations (Eqs. (28)-(33)). The $(n+1)$ undetermined coefficients $c_0$, ..., $c_n$ together with the 12 independent coefficients in $d's$ and $e's$ result in a total of $(n+13)$ unknowns.

In addition to the expression of peel stress in Eq. (63), $q$ can also be related to the difference of vertical deflections between the upper and lower laminates. With the expressions for $w_3^U$ and $w_3^L$ in Eq. (65), $q$ can be written as

$$q = \frac{E}{\eta} (w_3^U - w_3^L)$$

$$= \frac{E}{\eta} \sum_{i=0}^{9} (d_{5i} - d_{6i})x^i + \frac{E}{\eta} (e_{51} - e_{61}) \cosh \alpha x + \frac{E}{\eta} (e_{52} - e_{62}) \sinh \alpha x$$

$$+ [f_5(x) - f_6(x)] + \frac{E}{\eta} \sum_{i=1}^{n} (g_{5i} - g_{6i}) \cos \frac{i\pi x}{l}$$

(66)

From the above equation and from the orthogonal properties of Fourier series, the following $n$ equations can be obtained:

$$c_r = \frac{-2E}{\eta l} \left\{ \int_0^l \sum_{m=0}^{9} (d_{5m} - d_{6m}) x^m \cos \frac{r\pi x}{l} dx ight\}$$

$$+ (e_{51} - e_{61}) \int_0^l \cosh (\alpha x) \cos \left( \frac{r\pi x}{l} \right) dx + (e_{52} - e_{62}) \int_0^l \sinh (\alpha x) \cos \left( \frac{r\pi x}{l} \right) dx$$

$$+ \int_0^l [f_5(x) - f_6(x)] \cos \left( \frac{r\pi x}{l} \right) dx + (g_{5r} - g_{6r}) \frac{l}{2} \right\}$$

$$r = 1, 2, 3, ..., n$$
As previously discussed, the constant term of the peel stress, \( c_0 \), is equal to the average peel stress over the overlay range. The total peel force should balance the transverse shear stress resultant of the upper adherend at \( x_1 = l_1 \) and of the lower adherend at \( x_2 = 0 \). If the two adherends are not identical, different shear stress resultants at the locations of each adherend mentioned above will be expected. In order to compromise for this discrepancy which is due to the neglecting of higher order terms, an average of the shear stress resultants of the two adherends at the two overlay ends is used to balance the total peel force. The equation regarding \( c_0 \) is then

\[
\frac{Q_{x3}'(0)+Q_{x3}'(l)}{2} = \frac{2E}{\eta l} \left\{ \int_0^l \sum_{m=1}^9 (d_{5m} - d_{6m})x^m dx \right. \\
+ (e_{51} - e_{61}) \int_0^l \cosh(\alpha_2 x) dx + (e_{52} - e_{62}) \int_0^l \sinh(\alpha_2 x) dx \\
+ \left. \int_0^l [f_3(x) - f_6(x)] dx \right\}
\]

(68)

A total of \((n+1)\) relations can be found in Eqs. (67) and (68). The remaining 12 equations can be obtained by assuming appropriate boundary conditions.

Combining the aggregate solutions of the two adherends in all the three sections leads to a total of 18 undetermined coefficients. This is to say, eighteen boundary conditions (or equations) are needed to solve the whole problem. Two boundary conditions can be obtained when starting from the very left end of the structure by assuming a hinged end at this edge of the upper adherend and by assuming a reference point for mid-plane displacement:
At the junction of sections one and three, because of the continuity of the upper laminate and the force equilibrium,

\[ w_1^u(0) = 0 \]  \hspace{1cm} (69)

\[ u_1^{ou}(0) = 0 \] \hspace{1cm} (70)

Because the left end of the lower adherend is a free surface,

\[ M_{y3}^L(0) = 0 \]  \hspace{1cm} (77)

\[ N_{x3}^L(0) = 0 \] \hspace{1cm} (78)

Also, the right end of the upper adherend is a free surface:

\[ M_{y3}(l) = 0 \] \hspace{1cm} (79)
At the junction between sections three and two, from continuity and force equilibrium

\[ u_3^{ol}(l) - u_2^{ol}(0) \]  \hspace{1cm} (80)

\[ \psi_3^L(l) - \psi_2^L(0) \]  \hspace{1cm} (81)

\[ w_3^L(l) - w_2^L(0) \]  \hspace{1cm} (82)

\[ M_{33}^L(l) - P[\theta l_2 + w_2^L(0)] \]  \hspace{1cm} (83)

\[ N_{x3}^L(l) = P \]  \hspace{1cm} (84)

\[ Q_{x3}^L(l) = -P[\theta + \frac{dw_2^L(0)}{dx}] \]  \hspace{1cm} (85)

The last boundary condition is at the very right end of the lower adherend where a roller support is assumed:

\[ w_2^L(l_2) = 0 \]  \hspace{1cm} (86)

It is noted that at the edges of the overlay, four conditions on bending moment, three conditions on axial stress resultant, and only two conditions on transverse shear stress resultant are utilized. Intuitively, four conditions on each stress resultant of these edges are needed. However, the right hand side of Eqs. (28) and (29) which governs the axial stress resultants on both the adherends has the same magnitude but opposite direction. Once the three conditions (Eqs. (75), (78), and (84)) are satisfied, \( N_{x3}^U(l) \) will
automatically be zero. While the same argument can be seen in Eqs. (32) and (33) together with the assumption that the total peel force balances the transverse stress resultant at the edges, there are only two independent boundary conditions that can be posed (Eqs. (76) and (85)).
Since finite element analysis is not the main purpose of this research but an additional approach to verify the developed models, the existing finite element software ALGOR will be utilized to fulfill this purpose. A four node 2-D solid elasticity element with linear displacement distribution was used in this analysis. As shown in Fig. 8, mesh is refined at the end of the overlay to account for the large strain gradient. A total of 1,456 nodes and 1,350 2-Dimension elements were generated in order to perform this analysis.
Figure 8 Finite Element Mesh
CHAPTER 6

RESULTS AND DISCUSSIONS

6.1 Single-Lap Joints under Cylindrical Bending

Most of the previous research regarding single-lap joints does not correlate the coupling of the axial tension and the induced bending which is caused by the asymmetry of the laminate. In other words, the difference between the analyses of composite joints and those of isotropic adherends was only the expression of the adherend bending rigidity. In order to demonstrate the application of this developed model, a generalized example with an asymmetric laminate as one of the adherends is given. In this illustration, T300/5208 (Graphite/Epoxy) with ply thickness 0.25 mm was used for both upper and lower adherends. The upper laminate consists of 16 plies with orientation and sequence $[90_4/0_4/90_4/0_4]_T$, while the lower laminate consists of 12 plies with orientation and sequence $[0_4/90_2]_S$. The engineering constants of T300/5208 are $E_x=181$ GPa, $E_y=10.3$ GPa, $E_z=7.17$ GPa, and $v_x=0.28$ [43].

For the case of plane strain, the mechanical constants of the two laminates per unit width are listed in Table 1.

Table 1 Laminate Constants of Sample Joint

<table>
<thead>
<tr>
<th></th>
<th>$A_{11}$ (MN)</th>
<th>$B_{11}$ (kNm)</th>
<th>$D_{11}$ (Nm²)</th>
<th>$A_{55}$ (MN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upper Laminate</td>
<td>384</td>
<td>-171</td>
<td>512</td>
<td>28.6</td>
</tr>
<tr>
<td>Lower Laminate</td>
<td>374</td>
<td>0</td>
<td>394</td>
<td>21.5</td>
</tr>
</tbody>
</table>
Each laminate was 0.1 m long. The overlay length of the joint was 0.05 m, and the applied bending moment was -2 Nm/m. The adhesive was assumed to be Metlbond 408 [44] (Adams and Wake, 1984) with the following properties:

\[ E = 0.96 \text{ GPa}, \quad G = 0.34 \text{ GPa}, \quad \eta = 0.1 \text{ mm} \]

The shear correction factor \( k \) in Eq. (11) was introduced by Reissner [19] and Mindlin [45] for isotropic plate. For anisotropic plates, the choice of the shear correction factors is not trivial. The value of the factor has been shown to depend on both laminate materials and stacking sequence. Some mathematical expressions for the shear correction factor have also been provided (Chow [46]; Whitney [27]; Whitney, [47]; Whitney, [30]; Reissner, [48]; Bert [49]; Chou and Carleone, [50]; Dharmarajan and McCutchen, [51]; Miller and Adams, [52]; Murthy, [53]). Values of 2/3 and 5/6 were adopted by Whitney and Pagano [28], and their results on the cross-ply laminate under bending were shown to be close to the exact solutions. In the first example above, both values of 2/3 and 5/6 were used as \( k^U \) and \( k^L \), and the results are shown in Figs. 9 - 13. It was found that the adhesive peel stress, adhesive shear stress, and laminate axial stress resultant distributions based on these two different \( k \)'s are almost identical, while the laminate bending moment and the shear stress resultant distributions were found to show a small deviation.
Figure 9 Adhesive Peel Stress Distribution
Figure 10 Adhesive Shear Stress Distribution
Figure 11 Bending Moment Distributions of the Laminates
Figure 12 Axial Stress Resultants of the Laminates
Figure 13 Transverse Shear Stress Resultants of the Laminates
With this specified joining system, the peel and shear stress distribution of the adhesive from the developed model are shown in Figs. 9 and 10, respectively. In order to justify the developed model, a finite element analysis was conducted by the use of ALGOR finite element software (Algor Inc., [54]). A four node 2-D solid elasticity element with linear stress field and plane strain was used in this analysis. The finite element results are also superposed in Figs. 9 and 10. It can be seen that there is almost no distinction between these two methodologies. While finite element methods can provide results solely through computer calculations, the current model provides a closed-form solution of the entire system.

As expected, the peel stress is concentrated on the edges and is almost zero elsewhere. The shear stress distribution of the adhesive is smoother than that of the peel stress, while both have their maximum values at the edges.

The moment distributions of the upper and lower laminates are shown in Fig. 11. It is noted that the total moment of the two laminates is not equal to the applied moment, -2 Nm, except at the edges. This is explained by the existence of an axial stress resultant, \( N_x \), in each laminate, as shown in Fig. 12. However, these stress resultants in the two laminates will ultimately result in a couple which will ensure that the total moment is equal to the applied moment, -2 Nm. Figure 13 shows the transverse shear stress resultants in the upper and lower laminates. From Figs. 11 and 13, it can be
seen that both the moments and transverse shear of the laminates are concentrated near the edges of the overlay. However, the axial stress resultants have their maximum values for about one third of the overlay length at the center.

In order to compare the effects of laminate asymmetry using this developed model, several joints made of different combinations of two laminates with similar $A_{11}$, $D_{11}$, and $A_{55}$, but different $B_{11}$ were investigated. Again, T300/5208 (Graphite/Epoxy) with ply thickness 0.25 mm was used as laminate material. The first laminate was a symmetric laminate consisting of 12 plies with orientation and sequence $[0/90_2/0/90/0]_s$. The properties of this laminate are

$$A_{11} = 288 \text{ MN}, \quad B_{11} = 0 \text{ kNm}, \quad D_{11} = 222 \text{ Nm}^2, \quad A_{55} = 21.5 \text{ MN},$$

The other laminate was an asymmetric laminate with orientation and sequence of $[0_6/90_6]_T$. The properties of this laminate are

$$A_{11} = 288 \text{ MN}, \quad B_{11} = 193 \text{ kNm}, \quad D_{11} = 216 \text{ Nm}^2, \quad A_{55} = 21.5 \text{ MN},$$

This laminate is more rigid near the top surface. When under bending, the strain on the top surface is expected to be greater than the strain on the bottom surface. If this laminate is flipped over, it becomes $[90_6/0_6]_T$. The properties are then

$$A_{11} = 288 \text{ MN}, \quad B_{11} = -193 \text{ kNm}, \quad D_{11} = 216 \text{ Nm}^2, \quad A_{55} = 21.5 \text{ MN},$$
The first joint was made of two of the first laminates, and the other joints were made of the second laminate with different combinations. The properties of these five joints are listed in the following table.

<table>
<thead>
<tr>
<th>Joint No.</th>
<th>Upper Laminate</th>
<th>Lower Laminate</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. 1</td>
<td>288 0 222 21.5</td>
<td>288 0 222 21.5</td>
</tr>
<tr>
<td>No. 2</td>
<td>288 -193 216 21.5</td>
<td>288 193 216 21.5</td>
</tr>
<tr>
<td>No. 3</td>
<td>288 193 216 21.5</td>
<td>288 -193 216 21.5</td>
</tr>
<tr>
<td>No. 4</td>
<td>288 -193 216 21.5</td>
<td>288 -193 216 21.5</td>
</tr>
<tr>
<td>No. 5</td>
<td>288 193 216 21.5</td>
<td>288 193 216 21.5</td>
</tr>
</tbody>
</table>

The overlay length was 0.05 m; the bending moment was -2 Nm/m; a value of 5/6 was adopted for $k$.

The adhesive peel stress and the shear stress distributions of these five joints are shown in Figs. 14 and 15. It can be seen that Joint No. 1, which was made of two symmetric laminates, has the best performance in both adhesive peel stress and shear stress. The maximum peel and/or the maximum shear stresses are believed to be the most critical criteria for joint
Figure 14 Comparison of Adhesive Peel Stress Distribution Between Several Joints
Figure 15 Comparison of Adhesive Shear Stress Distribution Between Several Joints
strength. From Figs. 14 and 15, the joint with symmetric laminates as both adherends has up to 50% reduction of the maximum peel and shear stress when compared to the joints with the same joint geometry but asymmetric adherends. Even when the joints with symmetric adherends have better performance, it is still important to have a theory to predict the response of the asymmetric adherends. While the existing theories are restricted to symmetric laminates, the current developed theory can cover both symmetric and asymmetric laminates.

Even though most of the stresses are concentrated at the overlay edges, this does not imply that a shorter overlay length will result in a more efficient joint. The same joint as in the first example with varying overlay length was investigated to determine the effects of the overlay length on the maximum adhesive stresses. Figure 16 is a plot of maximum peel and shear stresses of adhesive versus overlay length based on the proposed model. Apparently, from the plots, there exists an optimal overlay length for minimizing the adhesive peel stress at the overlay edges. The optimal overlay length, for this case, is 0.02 m. The optimal overlay length for minimum the adhesive shear stress, however, is not evident. If the peel stress is the most critical, the optimal design of this joint should have an overlay length of approximately 0.02 m.
Figure 16 Maximum Peel and Shear Stresses versus Overlay Length
6.2 Single-Lap Joints under Tension

Again, in this illustration, T300/5208 (Graphite/Epoxy) with ply thickness 0.25 mm was used for both upper and lower adherends. The upper laminate consists of 16 plies with orientation and sequence \([90\_4/0\_4/90\_4/0\_4]_T\), while the lower laminate consists of 12 plies with orientation and sequence \([0\_4/90\_2]_S\). The engineering constants of T300/5208 are \(E_x = 181\) GPa, \(E_y = 10.3\) GPa, \(E_s = 7.17\) GPa, and \(v_s = 0.28\) (Tsai [42]). For the case of plane strain, the mechanical constants of the two laminates per unit width are as those listed in Table 1. Again, each laminate was of 0.1 m length. The overlay length of the joint was 0.05 m, and the applied tensile load was 1,000 N/m.

The adhesive is assumed to be Metlbond 408 (Adams and Wake [43]) with the following properties:

\[ E = 0.96 \text{ GPa}, \quad G = 0.34 \text{ GPa}, \quad \eta = 0.1 \text{ mm} \]

A value of \(5/6\) was chosen to simulate both \(k^U\) and \(k^L\).

With this specified joining system, the peel and shear stress distributions of the adhesive from the developed model are shown in Figs. 17 and 18, respectively. Results of finite element analysis using the FEA code "Algor" are also superposed in Figs. 17 and 18. The same mesh used in bending loading was used in this current tension loading. It can be seen that the results from the developed model correlate the results from FEA model very well. At each overlay edge, there exists a free surface at the longitudinal direction for one of the adherends. In order to maintain the
Figure 17  Adhesive Peel Stress Distribution
Figure 18 Adhesive Shear Stress Distribution
moment equilibrium of the discontinued laminate, a large adhesive peel stress is expected. This can be seen from Fig. 17 as the peel stress concentrates on the edges and is almost zero elsewhere. Moreover, the larger adhesive peel stress is expected at the edge where the laminate with larger bending rigidity ends and this can also be seen from the plots in Fig. 17. The large stress gradient across the adhesive thickness near the overlay edges may be the reason for the deviation of the adhesive shear stress distribution between the analytical and finite element models.

As shown in Fig. 7 (b), after deformation, the bending moment distributions of the adherends are related to both the transverse deflection and the original configuration. Based on the model developed in this study, as shown in Fig. 19, the bending moment of each adherend has its maximum value at one overlay edge and has a value of zero at the other edge because of the free-end condition. The axial stress resultants of the upper and lower laminates are shown in Fig. 20. It can be seen that the upper adherend and lower adherend take the entire loading at the left and right edge, respectively. The total of the stress resultants of the two adherends at every cross-section equals the applied load. Figure 21 shows the transverse shear stress resultants in the upper and lower laminates. From Figs. 19 and 21, it can be seen that both the moments and the transverse shear of the laminates are concentrated near the edges of the overlay.
Figure 19 Bending Moment Distributions of the Adherends
Figure 20 Axial Stress Resultants of the Adherends
Figure 21 Transverse Shear Stress Resultants of the Adherends
Failure of composite joints can be seen mostly in three modes: (1) adherend longitudinal tensile or compressive failure, (2) adherend interlaminar or adhesive peel failure, and (3) adherend interlaminar or adhesive shear failure.

The adherend longitudinal tensile and the compressive stresses come from a combination of longitudinal stress resultant and bending moment. It can be seen from Figs. 19 and 20 that both the maximum bending moment and the maximum longitudinal stress resultant occur at the edges of the overlay. The adherend bending moments at the edge of the overlay are then the most critical for joint strength based on the adherend longitudinal failure mode.

In order to compare the results of the present study with the papers of both Goland and Reissner [9] and Hart-Smith [17], a specific case was investigated. Because the earlier theories did not correlate the coupling behavior of the external tension and the induced bending moment, both adherends have symmetric stacking sequence and are identical. The lower adherend in the previous example was used in this case as both the adherends. By defining the total length as the sum of \( l_1, l_2, \) and \( l \), joints of two different total lengths, 0.25 m and 0.75 m, were used in this case. \( l \) is also assumed to be equal to \( l_2 \) in both joints. The applied load was 650,000 N/m. The eccentricity factor \( k_c \) and the non-dimensionalized overlay \( \xi_c \) are adopted as in Hart-Smith’s paper. They are defined as
where \( M_o \) is the bending moment at both edges of the overlay. Because the two adherends are identical, \( h^U \) is equal to \( h^L \), \( D_{11}^U \) is equal to \( D_{11}^L \) and the bending moments are the same at both edges. Figure 22 shows the plots of eccentricity versus overlay length and non-dimensionalized overlay \( \xi_c \). Intuitively, \( M_o \) will approach zero when the overlay length approaches the total length. However, their theories failed to show this phenomenon because of their assumption of

\[
\sinh \xi L_1 = \cosh \xi L_1 = \frac{1}{2} e^{\xi L_1} \quad \sinh \xi L_2 = \cosh \xi L_2 = \frac{1}{2} e^{\xi L_2}
\]

This assumption restricts their theories to joints with long adherend, small bending rigidities, and large applied loads. For example, in the current loading condition, their theories cannot be applied to the joints with \( L_1 \) and \( L_2 \) less than 0.04 m. If the load is 10,000 N/m, Eq. (73) cannot be satisfied unless \( L_1 \) and \( L_2 \) are both greater than 0.25 m. The two curves for the two different total lengths from the developed model show a similarity with small
**Figure 22** Eccentricity Factor versus Overlay Length and Non-Dimensionalized Overlay Length
overlay length. Investigating more cases led to the conclusion that when Eq. (73) is satisfied, the joint can be considered as long adherends with large load, and in the case of long adherends with large load, the eccentricity factor is very similar even for joints with the same overlay length but different total lengths. This is reasonable because when the total length is beyond certain value, the effects of $\Theta$ become less significant.

The second failure mode occurs either at the adhesive or at the adherend close to the glue line. The maximum adhesive peel stress can be used as a criterion for both adhesive and interlaminar peel failure. Figure 23 shows the relation between the overlay length and the maximum adhesive peel stress which occurs at the edges of the overlay. A small value of peel stress can be found at an overlay length of 0.1 m. Moreover, it can be seen that the two joints with the two different total lengths have the same maximum peel stress up to an overlay length of 0.12 m. This result shows a useful design criterion and also indicates a weakness of long overlay lengths.

Figure 24 shows plots of maximum adhesive shear stress versus overlay length, where the maximum shear stress is also located at the overlay edges. Again, joints with different total lengths have the same effects of overlay length on the maximum shear stress. The joint efficiency was defined as the ratio of the average shear stress to the maximum shear stress. As expected, it can be seen that a smaller maximum shear stress and a smaller
Figure 23 Maximum Adhesive Peel Stress versus Overlay Length
Figure 24 Maximum Adhesive Shear Stress versus Overlay Length
joint efficiency are accompanied with a longer overlay length. However after a certain length, no further reduction of maximum shear stress can be achieved. An optimal overlay length can be determined by combining the results from Figs. 22 - 24.

One of the advantages of the present investigation is that it covers the coupling behavior between bending and tension of an asymmetric laminate. Although there exists some difficulty when manufacturing asymmetric thermal set composite laminate, the use of asymmetric laminates can provide more flexibility in design. A comparison between two joints is given to show the advantages. The same material, T300/5208, used in the first example is adopted. In order to show the effects of laminate asymmetry, two laminates with similar $A_{11}$, $D_{11}$, and $A_{55}$ but different $B_{11}$ were chosen as the adherends. The first joint consists of two identical, symmetric adherends with orientation and sequence $[0/90_2/0/90/0]_T$. The second joint is made of two laminates with more zero-degree fiber reinforcement near the glue line. The orientation and sequence of the laminate used for the second joint was $[0_9/90_6]_T$. Quantitatively, the properties of the laminates per unit width are listed as follows:

**Table 3 Constants of Joint No. 1 (with symmetric adherends)**

<table>
<thead>
<tr>
<th></th>
<th>$A_{11}$ (MN)</th>
<th>$B_{11}$ (kNm)</th>
<th>$D_{11}$ (Nm²)</th>
<th>$A_{55}$ (MN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upper Laminate</td>
<td>288</td>
<td>0</td>
<td>222</td>
<td>21.5</td>
</tr>
<tr>
<td>Lower Laminate</td>
<td>288</td>
<td>0</td>
<td>222</td>
<td>21.5</td>
</tr>
</tbody>
</table>
Table 4  Constants of Joint No. 2 (with asymmetric adherends)

<table>
<thead>
<tr>
<th></th>
<th>( A_{11} ) (MN)</th>
<th>( B_{11} ) (kNm)</th>
<th>( D_{11} ) (Nm²)</th>
<th>( A_{55} ) (MN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upper Laminate</td>
<td>288</td>
<td>-193</td>
<td>216</td>
<td>21.5</td>
</tr>
<tr>
<td>Lower Laminate</td>
<td>288</td>
<td>193</td>
<td>216</td>
<td>21.5</td>
</tr>
</tbody>
</table>

The results can be seen from Figs. 25, 26, and 27. In Fig. 25, the bending moments on the overlay edges are shown to be the same on both the joints consisting of symmetric and asymmetric adherends. However, except at the end points, the plots show greater moment distribution of the joint with asymmetric adherends. Figure 26 shows the plots of adhesive peel stress distributions of the two joints. As expected, the stresses on the edges, which are the most critical on the joint strength, are dramatically reduced when asymmetric adherends are used. The same results can be seen for the adhesive shear stress (Fig. 27). Although an asymmetric laminate may warp during the curing process, it can provide design flexibility and property improvement of composite structures.
Figure 25 Bending Moment Distributions of Joints with Symmetric and Asymmetric Adherends
Figure 26 Adhesive Peel Stress Distributions of Joints with Symmetric and Asymmetric Adherends
Figure 27 Adhesive Shear Stress Distributions of Joints with Symmetric and Asymmetric Adherends

\[ l_1 = 0.05\text{m}; \quad l_2 = 0.05\text{m}; \quad l = 0.05\text{m} \]
LOAD = 1,000N/m

- JOINT No. 1 (SYMMETRIC ADHERENDS)
- JOINT No. 2 (ASYMMETRIC ADHERENDS)

- MAXIMUM OF JOINT No. 1
- MAXIMUM OF JOINT No. 2
CHAPTER 7

FURTHER STUDY:
AN ELASTIC-PLASTIC MODEL FOR SINGLE-LAP
JOINTS UNDER TENSION

Realistically, the adhesive of the joint will reach its plastic region before failure. The yield criterion of the adhesive can be obtained based on either von Mises cylindrical criterion or the yield criterion for the amorphous polymer proposed by Raghava et al. [42]. From the theory of plasticity, the yield stress of peel is coupled with the yield stress of shear. Numerical methods with large amount of iterations to increase the loading from elastic deformation to plastic deformation can be used to obtained the stress distributions. However, due to the complicated relation between the coupled peel and shear stresses, the analytical solution of the plastic analysis is not feasible. In order to incorporate the plastic deformation of the adhesive, Hart-Smith [17] proposed an elastic-plastic model in which the adhesive is assumed to be elastic-plastic in shear and elastic in peel. Because the composite laminate is much weaker in interlaminar tension than the adhesive is in peel, the adhesive is assumed to be elastic in peel. Which says that in most cases the laminate would fail due to interlaminar tension before the adhesive reaches its plastic region. However, the adhesive shear in plastic deformation can spread before the joint fails. Moreover, the peel stress is concentrated on the edges of the overlay and almost zero elsewhere. The
contribution of the peel stress to the behavior of the whole structure is not significant. Based on the above argument, the yield stress for shear is assumed to be constant through the plastic region.

As shown in Fig. 28, the joint is divided into five sections for easy interpretation. Sections 1 and 5 are the parts of the upper and lower adherends outside the overlay region, respectively. Sections 2 and 4 are the two sections within the overlay region where the adhesive shear stress reaches its yielding stress. Section 3 is located at the middle of the joint where adhesive stresses are within the elastic range. Each section has its length denoted as $l_1, l_2, ..., l_5$.

![Figure 28 Adhesive-Bonded Single-Lap Joint under Tension (Elastic-Plastic Model)](image)

Because of the varying laminate loading conditions in the five joint sections, the solution procedure is described separately in the following sections.

(a) **Section One**

This section covers the upper laminate outside the overlay region. The same loading situation as in the Section One of the elastic model discussed
in Chapter 4 is applied to this section. The oblique angle $\theta$ from the two forces to the central axes of the two adherends is related the geometry of the structure as

$$\theta = \frac{h^U + h^L}{2 \frac{l_1 + l_2 + l_3 + l_4 + l_5}{l_1 + l_2 + l_3 + l_4 + l_5}}$$

(91)

where $h^U$, $h^L$ are the thickness of the upper and lower adherends, and $l_1$, $l_2$, $l_3$, $l_4$, and $l_5$ are the lengths of joint sections 1, 2, 3, 4, and 5, respectively. The same solutions of the three variables $u^{oU}$, $\Psi^U$, and $w^U$ in Section One of the elastic model are adopted.

$$u^{oU} = a_1 + a_2 \cosh \alpha x_1 + a_3 \sinh \alpha x_1 + \frac{P}{A_{11}} x_1$$

$$\psi^U = -a_2 \frac{A_{11}^U}{B_{11}^U} \cosh \alpha x_1 - a_3 \frac{A_{11}^U}{B_{11}^U} \sinh \alpha x_1 + \theta$$

$$w^U = -a_3 \frac{A_{11}^U D_{11}^U}{B_{11}^U P} \cosh \alpha x_1 - a_2 \frac{A_{11}^U D_{11}^U}{B_{11}^U P} \sinh \alpha x_1 - \theta x_1 \frac{B_{11}^U}{A_{11}^U}$$

(56)

where $a_1$, $a_2$, and $a_3$ are three independent coefficients which need to be determined from boundary conditions.

In the case of a symmetric upper adherend ($B_{11}^U = 0$), solutions of Eq. (52) need to be modified to
(b) Section Two

This section is within the overlay region having the adhesive shear stress in the plastic region. Consider a segment of the top laminate as a free body as shown in Fig. 3. Neglecting of higher order terms, the equations of equilibrium can be written as

\[
\begin{align*}
\frac{dN_{x2}^U}{dx} &= -\tau_p \quad (92) \\
\frac{dM_{x2}^U}{dx} &= Q_{x2}^U + \frac{h^U}{2} - \tau_p \quad (93)
\end{align*}
\]

and

\[
\frac{dQ_{x2}^U}{dx} = q \quad (94)
\]

where \( q \) is the adhesive peel stress between the two laminates. The peel stress arise from the difference of vertical deflections between the upper and lower adherends and can be written as

\[
q = \frac{E}{\eta} (w_2^U - w_2^L) \quad (95)
\]
The same equilibrium conditions used for Eqs. (92), (93), and (94) can also be applied to the lower laminate. Combining the force equilibrium conditions and the constitutive relations, Eqs. (13), ..., (16), (22) and (23), the system of equations governing the displacement field of the two adherends can be obtained as

\[
\begin{align*}
A_{11} \frac{d^2 u_2^U}{dx_2^2} + B_{11} \frac{d^2 \psi_2^U}{dx_2^2} &= -\tau_p \\
A_{11} \frac{d^2 u_2^{OL}}{dx_2^2} + B_{11} \frac{d^2 \psi_2^L}{dx_2^2} &= -\tau_p \\
B_{11} \frac{d^2 u_2^{SU}}{dx_2^2} + D_{11} \frac{d^2 \psi_2^L}{dx_2^2} - k A_{55} \left( \frac{d^2 w_2^U}{dx_2^2} + \frac{d^2 w_2^{SU}}{dx_2^2} \right) &= \frac{h^U}{2} \tau_p \\
B_{11} \frac{d^2 u_2^{OL}}{dx_2^2} + D_{11} \frac{d^2 \psi_2^L}{dx_2^2} - k A_{55} \left( \frac{d^2 w_2^L}{dx_2^2} + \frac{d^2 w_2^{OL}}{dx_2^2} \right) &= \frac{h^L}{2} \tau_p \\
k A_{55} \left( \frac{d \psi_2^U}{dx_2} + \frac{d^2 w_2^U}{dx_2^2} \right) - \frac{E}{\eta} (w_2^U - w_2^L) &= 0 \\
k A_{55} \left( \frac{d \psi_2^L}{dx_2} + \frac{d^2 w_2^L}{dx_2^2} \right) + \frac{E}{\eta} (w_2^U - w_2^L) &= 0
\end{align*}
\]  

(96)

The above system includes six second-order coupled ordinary differential equations with six variables, \( u_2^{SU}, u_2^{OL}, \psi_2^U, \psi_2^L, w_2^U, \) and \( w_2^L \). Again, the solutions can be obtained by solving the following characteristic equation.

\[
(R_1 \alpha^4 + R_2 \alpha^2 + R_3) \alpha^8 = 0
\]  

(97)

where
The corresponding eigenfunctions are (see Appendix C)

\begin{align*}
R_1 &= k^2 u k A_{33} A_{33} (D_{11} - \frac{B_{11}^U}{A_{11}}) (D_{11}^L - \frac{B_{11}^L}{A_{11}}) \\
R_2 &= -\frac{E}{\eta} (k^2 u k A_{33} A_{33} (D_{11} - \frac{B_{11}^U}{A_{11}}) (D_{11}^L - \frac{B_{11}^L}{A_{11}})) \\
R_3 &= -\frac{E}{\eta} (k^2 u k A_{33} A_{33} [(D_{11} - \frac{B_{11}^U}{A_{11}}) + (D_{11}^L - \frac{B_{11}^L}{A_{11}})])
\end{align*}

1, \ x_2, \ x_2^2, \ x_2^3, \ \cosh \alpha_{21} x_2, \ \sinh \alpha_{21} x_2, \ \cosh \alpha_{22} x_2, \ \sinh \alpha_{22} x_2

where \( \alpha_{21} \) and \( \alpha_{22} \) are the two positive non-zero roots of Eq. (97). The homogeneous solutions, which are linear combinations of the above eigenfunctions, together with the particular solutions provide the solutions of Eq. (96).

There are twelve independent coefficients to decide from the appropriate boundary conditions.

(c) Section Three

As in section two, section three is within the overlay region. However, the shear stress in this section is elastic. With the same approach of assuming an \( (n+1) \)-term Fourier cosine series as the peel stress, the governing equations and solutions in this section are identical to those in Section Three in the elastic model. A total of \( (n+1) \) relations can be used to
solve for the n+1 unknown coefficients of the Fourier cosine series. The remaining twelve equations can be obtained by assuming appropriate boundary conditions.

(d) Section Four

In this section, the governing equations are identical to those in section two. Twelve independent coefficients are needed to obtain the solutions in this section.

(e) Section Five

The governing equations of the lower laminate in section five are almost the same as those of the upper adherend in section one. The only difference is at the induced bending moment. Because the origin of the $x_5$ coordinate is located at the edge of the overlay instead of at the right end of the lower adherend, the moment of the lower adherend is related to the coordinate $x_5$ and the transverse displacement $w_{5}^L$ as

$$M_5^L = P\left[\theta(l_5 - x_5) - w_{5}^L\right] \quad (101)$$

The governing equations are then

$$\begin{align*}
A_{11}^L \frac{du_5^L}{dx_5} + B_{11}^L \frac{d\psi_5^L}{dx_5} &= -P \\
B_{11}^L \frac{du_5^L}{dx_3} + D_{11}^L \frac{d\psi_5^L}{dx_3} &= -Pw_5^L + P\theta(l_5 - x_5) \\
k^L A_{55}^L \frac{d\psi_5^L}{dx_5} + (k^L A_{55}^L + P) \frac{dw_5^L}{dx_5} &= -P\theta
\end{align*} \quad (102)$$
When the same technique used in Section One is applied, the solutions of the lower laminate in section five can be obtained as

\[
\begin{align*}
\mathbf{u}^{\text{ol}}_5 &= b_1 + b_2 \cosh \alpha x_5 + b_3 \sinh \alpha x_5 + \frac{P}{A_{11}^{\text{L}}} x_5 \\
\psi_5^L &= -b_2 A_{11}^{\text{L}} \cosh \alpha x_5 + b_3 A_{11}^{\text{L}} \sinh \alpha x_5 + \theta \\
\psi_5^L &= -b_3 (B_{11}^{\text{L}} + A_{11}^{\text{L}} D_{11}^{\text{L}}) \alpha_2 \cosh \alpha x_5 - b_2 (B_{11}^{\text{L}} - A_{11}^{\text{L}} D_{11}^{\text{L}}) \alpha_2 \sinh \alpha x_5 + \theta (l_2 - x_5) - \frac{B_{11}^{\text{L}}}{A_{11}^{\text{L}}}
\end{align*}
\]

(103)

with the three undetermined coefficients \(b_1\), \(b_2\), and \(b_3\) and with

\[
\alpha_2 = \sqrt[3]{\frac{PA_{11}^{\text{L}} k_5 A_{55}^{\text{L}}}{(k_5 A_{55}^{\text{L}} + P)(A_{11}^{\text{L}} D_{11}^{\text{L}} - B_{11}^{\text{L}})}}
\]

(104)

Again, when the lower adherend is symmetric \((B_{11}^{\text{L}} = 0)\), the solutions are replaced by

\[
\begin{align*}
\mathbf{u}^{\text{ol}}_5 &= b_1 + \frac{P}{A_{11}^{\text{L}}} x_5 \\
\psi_5^L &= b_2 \cosh \alpha x_5 + b_3 \sinh \alpha x_5 + \theta \\
\psi_5^L &= -b_3 \frac{D_{11}^{\text{L}} \alpha_2}{P} \cosh \alpha x_5 - b_2 \frac{D_{11}^{\text{L}} \alpha_2}{P} \sinh \alpha x_5 + \theta (l_2 - x_5)
\end{align*}
\]

(105)

Combining the aggregate solutions of the two adherends in all the five sections leads to a total of 42 undetermined coefficients; therefore, forty two boundary conditions (or equations) are needed to solve the whole problem. Two boundary conditions can be obtained when starting from the very left
end of the structure by assuming a hinged end at this edge of the upper adherend and by assuming a reference point for mid-plane displacement:

\[ w_1' (0) = 0 \]  \hspace{1cm} (106)

\[ u_1^0' (0) = 0 \]  \hspace{1cm} (107)

At the junction of sections one and two, because of the continuity of the upper laminate and the force equilibrium,

\[ u_1^0' (l_1) = u_2^0' (0) \]  \hspace{1cm} (108)

\[ \psi_1' (l_1) = \psi_2' (0) \]  \hspace{1cm} (109)

\[ w_1' (l_1) = w_2' (0) \]  \hspace{1cm} (110)

\[ -P[\theta l_1 + w_1' (l_1)] - M_{y2}' (0) \]  \hspace{1cm} (111)

\[ P = N_{y2}' (0) \]  \hspace{1cm} (112)

\[ -P[\theta + \frac{d w_1' (l_1)}{dx_1}] - Q_{z2}' (0) \]  \hspace{1cm} (113)

Because the left end of the lower adherend is a free surface,

\[ M_{y2}' (0) = 0 \]  \hspace{1cm} (114)
At the junctions between sections two and three and between sections three and four, the displacement fields of both the upper and lower adherend should be continuous. The adherends' stress resultants at these two interfaces should be balanced as well. These criteria result in a total of 24 equations regarding $u^O$, $u^L$, $\Psi^U$, $\Psi^L$, $w^U$, $w^L$, $M_y^U$, $M_y^L$, $N_x^U$, $N_x^L$, $Q_x^U$, $Q_x^L$.

The right end of the upper adherend is a free surface:

$$M^U_{yf}(l_4) = 0 \quad (116)$$

At the junction between sections four and five, from continuity and force equilibrium

$$u_4^L(l_4) = u_5^L(0) \quad (117)$$

$$\Psi_4^L(l_4) = \Psi_5^L(0) \quad (118)$$

$$w_4^L(l_4) = w_5^L(0) \quad (119)$$

$$M^L_{yf}(l_4) = P[\theta I_5 + w_5^L(l_4)] \quad (120)$$

$$N^L_{xf}(l_4) = P \quad (121)$$

$$Q^L_{xf}(l_4) = -P[\theta + \frac{dw_5^L(l_4)}{dx_5}] \quad (122)$$
The last boundary condition is at the very right end of the lower adherend where a roller support is assumed:

\[ w_2^L(l_4) = 0 \]  \hspace{1cm} (123)

It is noted that at the edges of the overlay, four conditions on bending moment, three conditions on axial stress resultant, and only two conditions on transverse shear stress resultant are utilized. Intuitively, four conditions on each stress resultant of these edges are needed. However, the adhesive shear stress, which affects the adherend normal stress resultants, occurs at both adherends with the same magnitude and opposite direction. Once the three conditions (Eqs. (112), (115), and (121)) are satisfied, \( N_{x4}(l_4) \) will automatically be zero. While the same argument can be seen in adherend transverse shear stress resultants, there are only two independent boundary conditions that can be posed (Eqs. (113) and (122)).

In order to present the results of this model, again, T300/5208 (Graphite/Epoxy) with ply thickness 0.25 mm was used for both the upper and lower adherends of the sample joint. The constants for the upper and lower laminates are listed in Table 1. Each laminate was of 0.1 m long. The overlay length of the joint was 0.05 m. The adhesive is assumed to have the following properties:

\[ E=0.96 \text{ GPa}, \ G=0.34 \text{ GPa}, \ \tau_p=12.6 \text{ MPa}, \ \text{and} \ \eta=0.1 \text{ mm} \]

In order to obtain the lengths of the two plastic regions, iterations are performed to insure the continuity of the shear stress at the junctions
between the elastic and plastic regions. With this specified joint system, the peel and shear stress distributions of the adhesive from the developed model are shown in Figs. 29 and 30, respectively. The results from the elastic model (previous section) are superposed on the results from the developed elastic-plastic model. At each overlay edge, there exists a free surface at the longitudinal direction for one of the adherends. In order to maintain the moment equilibrium of the discontinued laminate, a large adhesive peel stress is expected. This can be seen from Fig. 29 as the peel stress is concentrated on the edges and is almost zero elsewhere. It can also be seen from Fig. 29 that the peel stress distributions from both the elastic model and the developed elastic-plastic model are almost identical. As described in the Introduction, if the joint would fail because of the maximum peel stress at the edges of the overlay, it should have failed before the adhesive reaches its yielding condition. Because once the adhesive yielding occurs, the peel stress would decrease. The non-zero peel stress covers very little of the overlay, so the assumption of constant yielding shear stress within the plastic region is verified. Moreover, the larger adhesive peel stress is expected at the edge where the laminate with larger bending rigidity ends and this can also be seen from the plots in Fig. 29. It can be seen that the results from both elastic the and elastic-plastic models are almost identical. This also implies that the yielding shear stress has little effects on the peel stress distribution.
Figure 29 Adhesive Peel Stress Distribution

Load = 400 kN/m

- Elastic-Plastic Model
- Elastic Model
Figure 30 Adhesive Shear Stress Distribution
As shown in Fig. 7 (b), after deformation, the bending moment distributions of the adherends are related to both the transverse deflection and the original configuration. Based on the model developed, as shown in Fig. 31, the bending moment of each adherend has its maximum value at one overlay edge and has a value of zero at the other edge because of the free-end condition. Figure 31 also shows the bending moment distribution obtained from the elastic model. The same maximum bending moment can be seen from Fig. 31 for both models.

Based on the developed elastic-plastic model the peel stress distributions of three different loadings are shown in Fig. 32. The maximum values of the peel stresses on the edges are close to linear to the loadings, while the zero value regions are very consistent. The shear stress distributions of the three loadings are shown in Fig. 33 where the length of the plastic regions and the minimum shear stress at the middle show a highly nonlinear behavior. The shear strain distributions, as shown in Fig. 34, also depict the nonlinear behavior. Figures 35 - 37 represent the distributions of transverse shear stress resultants, normal stress resultants, and bending moment under the three different loadings, respectively. It is worth noting that in Fig. 36 the normal stress resultants have linear distributions with respect to location within each corresponding plastic region. This is expected from the constant shear stress within the plastic region.
Figure 31 Bending Moment Distributions of the Adherends
Figure 32 Adhesive Peel Stress Distributions under Three Different Loadings
Figure 33 Adhesive Shear Stress Distributions under Three Different Loadings
Figure 34 Adhesive Shear Strain Distributions under Three Different Loadings
Figure 35 Adherend Transverse Shear Stress Resultant
Figure 36 Adherend Normal Stress Resultant
Figure 37 Adherend Bending Moment
The failure of adhesive can be seen either from the maximum peel stress or the maximum shear strain at the edges of the overlay. However, in most cases the laminate would fail due to interlaminar tension before the adhesive peel stress reaches values in its plastic region. The adhesive shear failure would happen when its plastic region spreads over the entire overlay or it reaches the maximum allowable value at either of the overlay edges. By assuming the maximum allowable adhesive shear strain $\gamma_f$ be three times the yielding strain $\gamma_p$, Fig. 38 shows effects of overlay length on the joint strength. In this figure, the trend can be seen; as the overlay length increases, it would increase the strength of the joint. Figure 39 shows that the increase of overlay length also reduces the maximum adhesive peel stress at the edges. From Fig. 40 it can be seen that the joints with overlay length less than 0.02 m would have yielding by shear stress over the entire overlay before the edge strain reaches the maximum allowable strain. By defining the joint efficiency as the ratio of the length of the plastic region to the entire overlay length, Fig 41 shows the decrease of joint efficiency with the increase of overlay length.
Figure 38 Joint Strength versus Overlay Length
Figure 39 Edge Peel Stress versus Overlay Length
Figure 40 Edge Shear Strain versus Overlay Length
Figure 41 Joint Efficiency versus Overlay Length
CHAPTER 8

CONCLUSIONS

Mathematical models were developed to predict the stress and strain distributions of adhesive-bonded single-lap joints under bending and tensile loadings. Finite element analysis has been performed to confirm the developed elastic model for both loadings. The results have been found to correlate well with those from the finite element method. An elastic-plastic model has also been recommended to predict the development of the shear yielding of the adhesive. All the three models developed correlate the shear deformation which is important to laminated composites.

The adhesive peak stresses were located and their values were also determined. Examples of joints consisted of the same composite material, but different stacking sequences were given to show the effects of laminate asymmetry. With suitable failure criterion, the predicted peak stresses can be used to determine the joint strength. The effects of joint length on adhesive stresses of the elastic model on two different loadings are provided. Based on the recommended elastic-plastic model, the strength of joints with different overlay lengths under tensile loading are studied by assuming the maximum allowable adhesive shear strain.
REFERENCES


APPENDIX A

HOMOGENEOUS SOLUTIONS
(ELASTIC MODEL)

This section describes the homogeneous solutions of Eqs. (28)-(33). The six 9th order \( x \)-polynomials for the six variables, \( u^o_U, u^o_L, \ldots, \) and \( w^L \), include 60 coefficients as the following table (also can be seen in Eqs. (40) and (41)).

<table>
<thead>
<tr>
<th>Table A1</th>
<th>Coefficients of the Homogeneous Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>( u^o_U )</td>
<td>( a_{10} )</td>
</tr>
<tr>
<td>( u^o_L )</td>
<td>( a_{20} )</td>
</tr>
<tr>
<td>( \psi^U )</td>
<td>( a_{31} )</td>
</tr>
<tr>
<td>( \psi^L )</td>
<td>( a_{40} )</td>
</tr>
<tr>
<td>( w^U )</td>
<td>( a_{50} )</td>
</tr>
<tr>
<td>( w^L )</td>
<td>( a_{60} )</td>
</tr>
</tbody>
</table>

The sixty variables are defined as \( a_{ij} \) as shown in the above table. Because the maximum order of the eigenfunctions is nine, there are only ten independent coefficients. Other 50 coefficients can be solved from the 10 independent coefficients.
Because of the complexity of the system, the selection of the 10 independent coefficients is not trivial. If \( a_{10}, a_{20}, \ldots, a_{60}, \) and \( a_{11}, a_{21}, \ldots, a_{61} \) are chosen as 12 independent coefficients (two of them are actually related to others), in order to determine the other 48 coefficients, 48 equations and the assumed values of the 12 independent coefficients are necessary.

From Eqs. (28) and (29), 18 equations can be obtained as follows:

\[
\begin{align*}
\frac{G}{\eta}a_{1n} + A_{11}v(n+1)(n+2)a_{1(n+2)} + \frac{G}{\eta}a_{2n} + \frac{Gh_1}{2\eta} - a_{3n} \\
+ B_{11}v(n+1)(n+2)a_{3(n+2)} + \frac{Gh_2}{2\eta}a_{4n} - \frac{G}{2}(n+1)a_{5(n+1)} \\
- \frac{G}{2}(n+1)a_{6(n+1)} = 0 \\
n=0,1,\ldots,6
\end{align*}
\]

\[
\begin{align*}
\frac{G}{\eta}a_{1n} - \frac{G}{\eta}a_{2n} + A_{11}L(n+1)(n+2)a_{2(n+2)} - \frac{Gh_1}{2\eta}a_{3n} \\
- \frac{Gh_2}{2\eta}a_{4n} + B_{11}L(n+1)(n+2)a_{4(n+2)} + \frac{G}{2}(n+1)a_{5(n+1)} \\
+ \frac{G}{2}(n+1)a_{6(n+1)} = 0 \\
n=0,1,\ldots,6
\end{align*}
\]

and
From Eqs. (30) and (31), another 14 equations are determined as:

\[
\begin{align*}
\frac{Gh_1}{2\eta}a_{1n} + B_{11}u_{(n+1)(n+2)}a_{1(n+2)} - \frac{Gh_1}{2\eta}a_{2n} &= \\
\frac{Gh_1^2}{4\eta} + k_{A_{55}}u_{3n} + D_{11}u_{(n+1)(n+2)}a_{3(n+2)} &= \\
\frac{Gh_1}{4\eta}a_{4n} + \frac{Gh_1}{4} - k_{A_{55}}u_{(n+1)}a_{5(n+1)} &= \\
\frac{Gh_1}{4}a_{6(n+1)} &= 0 \\
& \quad n=0,1,\ldots,6 \\
\end{align*}
\]

Equations (32) and (33) result in the following 16 relations:

\[
\begin{align*}
\frac{Gh_2}{2\eta}a_{1n} - \frac{Gh_2}{2\eta}a_{2n} + B_{11}L_{(n+1)(n+2)}a_{2(n+2)} &= \\
\frac{Gh_2}{4\eta}a_{3n} - \frac{Gh_2^2}{4\eta} + k_{A_{55}}L_{a4n} &= \\
D_{11}L_{(n+1)(n+2)}a_{4(n+2)} + \frac{Gh_2}{4}a_{5(n+1)} &= \\
+ \frac{Gh_2}{4} - k_{A_{55}}L_{(n+1)}a_{6(n+1)} &= 0 \\
& \quad n=0,1,\ldots,6 \\
\end{align*}
\]

Equations (32) and (33) result in the following 16 relations:

\[
\begin{align*}
ku_{A_{55}}u_{(n+1)}a_{3(n+1)} + k_{A_{55}}u_{(n+1)(n+2)}a_{5(n+2)} &= 0 \\
& \quad n=1,2,\ldots,8 \\
\end{align*}
\]
The 48 unknown coefficients can be obtained by solving the above 48 equations with the assumed value of the 12 independent coefficients. However, in the system, there are actually only 10 independent coefficients. The following two equations are added to the system of governing equations in order to solve for the two extra independent coefficients.

\[ k^L A_{55}^L(n+1)a_{4(n+1)} + k^L A_{55}^L(n+1)(n+2)a_{6(n+2)} = 0 \quad n=1,2,\ldots,8 \quad (A7) \]

The 48 unknown coefficients can be obtained by solving the above 48 equations with the assumed value of the 12 independent coefficients. However, in the system, there are actually only 10 independent coefficients. The following two equations are added to the system of governing equations in order to solve for the two extra independent coefficients.

\[
\begin{align*}
\frac{G}{\eta} a_{17} + \frac{G}{\eta} a_{27} + \frac{G \eta}{2} a_{37} + \frac{G \eta}{2} a_{47} & - \frac{9G}{2} a_{58} - \frac{9G}{2} a_{68} = 0 \\
\frac{G}{\eta} a_{18} + \frac{G}{\eta} a_{28} + \frac{G \eta}{2} a_{38} + \frac{G \eta}{2} a_{48} & - \frac{9G}{2} a_{59} - \frac{9G}{2} a_{69} = 0
\end{align*}
\]

(A8)
APPENDIX B

PARTICULAR SOLUTIONS FOR THE CONSTANT FORCING TERM (JOINT SECTION THREE ELASTIC MODEL UNDER TENSION)

This section describes the particular solutions of Eqs. (28)-(32) with respect to the constant forcing term in $q$. Because the highest order of the homogeneous solutions of the system of equations is 9, the particular solutions corresponding to the constant term are 10th order $x$-polynomials. The six 10th order $x$-polynomials for the six variables, $u_3^{oU}$, $u_3^{oL}$, ..., and $w_3^L$, have 66 coefficients as the following table.

**Table B1** Particular Solutions Corresponding to the Constant Term

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>x</th>
<th>$x^2$</th>
<th>$x^3$</th>
<th>$x^4$</th>
<th>$x^5$</th>
<th>$x^6$</th>
<th>$x^7$</th>
<th>$x^8$</th>
<th>$x^9$</th>
<th>$x^{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_3^{oU}$</td>
<td>$f_{10}$</td>
<td>$f_{11}$</td>
<td>$f_{12}$</td>
<td>$f_{13}$</td>
<td>$f_{14}$</td>
<td>$f_{15}$</td>
<td>$f_{16}$</td>
<td>$f_{17}$</td>
<td>$f_{18}$</td>
<td>$f_{19}$</td>
<td>$f_{20}$</td>
</tr>
<tr>
<td>$u_3^{oL}$</td>
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<td>$f_{21}$</td>
<td>$f_{22}$</td>
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<td>$f_{26}$</td>
<td>$f_{27}$</td>
<td>$f_{28}$</td>
<td>$f_{29}$</td>
<td>$f_{30}$</td>
</tr>
<tr>
<td>$\psi_3^U$</td>
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<td>$f_{31}$</td>
<td>$f_{32}$</td>
<td>$f_{33}$</td>
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<tr>
<td>$\psi_3^L$</td>
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<td>$f_{46}$</td>
<td>$f_{47}$</td>
<td>$f_{48}$</td>
<td>$f_{49}$</td>
<td>$f_{50}$</td>
</tr>
<tr>
<td>$w_3^U$</td>
<td>$f_{50}$</td>
<td>$f_{51}$</td>
<td>$f_{52}$</td>
<td>$f_{53}$</td>
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<td>$f_{57}$</td>
<td>$f_{58}$</td>
<td>$f_{59}$</td>
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</tr>
<tr>
<td>$w_3^L$</td>
<td>$f_{60}$</td>
<td>$f_{61}$</td>
<td>$f_{62}$</td>
<td>$f_{63}$</td>
<td>$f_{64}$</td>
<td>$f_{65}$</td>
<td>$f_{66}$</td>
<td>$f_{67}$</td>
<td>$f_{68}$</td>
<td>$f_{69}$</td>
<td>$f_{70}$</td>
</tr>
</tbody>
</table>

The above table can be referred to Eqs. (64) and (65). Where
\[ f_j = \sum_{i=0}^{t-10} f_j x^i \quad j=1,2,...,6 \]  

However, because \( f_{10}, f_{20}, ..., f_{60}, \) and \( f_{11}, f_{21}, ..., f_{61} \) are covered by the 12 independent coefficients of the homogeneous solutions, their values can be assigned arbitrarily before the coefficients of the homogeneous solutions are determined. Because the constant term of the forcing function \( q \) can be found only in Eqs. (32) and (33), the only coefficients which are directly related to the constant term of \( q \) are \( f_{31}, f_{41}, f_{52}, \) and \( f_{62} \). The possible non-zero coefficients are those which are related \( f_{31}, f_{41}, f_{52}, \) and \( f_{62} \). After this arrangement, the possible non-zero coefficients can be listed in Table B2.

There is a total of 27 undetermined coefficients.

Based on the governing equations, Eqs. (28)-(33), there are 6 equations for each of \( r_1 - r_7, r_4 - r_{13}, r_{10} - r_{19}, \) and \( r_{16} - r_{25} \). Three more equations can be found from Eqs. (28)-(33) which govern \( r_{22}, r_{23}, ..., r_{27} \). A total of 27 equations can be found to solve for the 27 unknowns, \( r_1, r_2, ..., r_{27} \).
Table B2 Non-Zero Coefficients

| $u_3^{oU}$ | $r_1$ | $r_4$ | $r_{10}$ | $r_{16}$ | $r_{22}$ |
| $u_3^{oL}$ | $r_3$ | $r_{11}$ | $r_{17}$ | $r_{23}$ |
| $\psi_3^U$ | $r_6$ | $r_{12}$ | $r_{18}$ | $r_{24}$ |
| $\psi_3^L$ | $r_7$ | $r_{13}$ | $r_{19}$ | $r_{25}$ |
| $w_3^{oU}$ | $r_2$ | $r_8$ | $r_{14}$ | $r_{20}$ | $r_{26}$ |
| $w_3^{oL}$ | $r_3$ | $r_9$ | $r_{15}$ | $r_{21}$ | $r_{27}$ |
The characteristic equation for the system of equations in Eq. (96) shows eight multiple roots of zero. Supposedly, the homogeneous solutions should cover x-polynomials through the 7th order. However, after substituting $u_2^U$ and $u_2^L$ with $\psi_2^U$ and $\psi_2^L$ from the first two equations of Eq. (96), the system of equations can be written as

\begin{align*}
(D_{11}^U - \frac{B_{11}^U}{A_{11}^U}) \frac{d^2 \psi_2^U}{dx_2^2} - k U A_{55}^U \psi_2^U & - k U A_{55}^U \frac{dw_2^U}{dx_2} = 0 \quad (C1) \\
(D_{11}^L - \frac{B_{11}^L}{A_{11}^L}) \frac{d^2 \psi_2^L}{dx_2^2} - k L A_{55}^L \psi_2^L & - k L A_{55}^L \frac{dw_2^L}{dx_2} = 0 \quad (C2) \\
k U A_{55}^U \frac{d \psi_2^U}{dx_2} + k U A_{55}^U \frac{d^2 w_2^U}{dx_2^2} - \frac{E}{\eta} \frac{w_2^U}{\eta} + \frac{E}{\eta} w_2^L = 0 \quad (C3) \\
k L A_{55}^L \frac{d \psi_2^L}{dx_2} + \frac{E}{\eta} w_2^L & + k L A_{55}^L \frac{d^2 w_2^L}{dx_2^2} - \frac{E}{\eta} w_2^L = 0 \quad (C4)
\end{align*}

This reduces four zero roots of the characteristic equation (Eq. (97)). The new characteristic equation for the above four equations becomes
with $R_1$, $R_2$, and $R_3$ defined in Eqs. (98)-(100).

The homogeneous solutions of $\psi_2^U$, $\psi_2^L$, $w_2^U$, and $w_2^L$ then include $x$-polynomials up to the third order. After examining the relations between the coefficients of the four polynomials from the above four equations, the non-zero coefficients can be shown as

<table>
<thead>
<tr>
<th>Table C1. Homogeneous Solutions of Eq. (96)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
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<tr>
<td>$\psi_2^U$</td>
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<tr>
<td>$\psi_2^L$</td>
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<tr>
<td>$w_2^U$</td>
</tr>
<tr>
<td>$w_2^L$</td>
</tr>
</tbody>
</table>

Where $s_1$, $s_2$, ..., $s_4$ are the four independent coefficients. $r_1$, $r_2$, $r_3$, and $r_4$ are related to the four independent coefficients as

\[
 r_2 = -\frac{1}{2}s_4 \quad (C6)
\]
The replacement of $u_2^U$ and $u_2^L$ in Eq. (96) reduces the order of the characteristic equation by four. Four independent coefficients can be found from both of $u_2^U$ and $u_2^L$. By combining the relations with $\psi_2^U$ and $\psi_2^L$, $u_2^U$ and $u_2^L$ can be written as

$$u_2^U = s_5 + s_6 x_2 - \frac{B^U_{11}}{A_{11}} r_4 x_2^2$$

$$u_2^L = s_7 + s_8 x_2 - \frac{B^L_{11}}{A_{11}} r_4 x_2^2$$

A total of 8 independent coefficients corresponding to the x-polynomials as a part of the homogeneous solutions for Eq. (96) needs to be decided from the boundary conditions.
APPENDIX D

PARTICULAR SOLUTIONS
(JOINT SECTION TWO, ELASTIC-PLASTIC MODEL)

Based on the results from Appendix C, the particular solutions of Eq. (96) corresponding to the forcing term $\tau_p$ must be $x$-polynomials with highest order of four. After examining Eq. (96) with all the coefficients, the non-zero coefficients are listed as the follows

<table>
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<tr>
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<td>$r_1$</td>
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<tr>
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<td>$r_1$</td>
<td>$r_2$</td>
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with the following relations

$$-k U A_{55} U_1 r_1 + 6 \left( \frac{B_{11} U^2}{A_{11} U} - D_{11} U_2 \right) r_2 - \left( \frac{h U}{2} \frac{B_{11} U}{A_{11} U} \right) \tau_p$$

(D1)

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\[-k^L A_{35}^L r_1 + 6 \left( \frac{B_{11}^L}{A_{11}^L} - D_{11}^L \right) r_2 - \left( \frac{h^L}{2} \frac{B_{11}^L}{A_{11}^L} \right) \tau_p \]  \hspace{1cm} (D2)

\[6B_{11}^U r_2 - 2A_{11}^U r_3 = \tau_p \]  \hspace{1cm} (D3)

\[6B_{11}^L r_2 - 2A_{11}^L r_4 = -\tau_p \]  \hspace{1cm} (D4)

and

\[3r_2 + r_5 = 0 \]  \hspace{1cm} (D5)
VITA

Chihdar Yang was born in Taipei, Taiwan, Republic of China in 1962. He obtained his B.S. in Civil Engineering and M.S. in Applied Mechanics from National Taiwan University. He was an university rugby team member during the undergraduate and graduate years at National Taiwan University. He served as the team captain in his senior year and also received the National Outstanding College Rugby Player award in the same year. He published his Master Thesis, "Optimal Attitude Control of Asymmetric Spacevehicles by Internal Reaction Wheels," in Proceedings 11th National Conference on Theoretical and Applied Mechanics, Taiwan, 1987. He came to the U.S. and entered the Ph.D. program of Mechanical Engineering Department at Louisiana State University in 1989. During his Ph.D. study, he published 9 technical papers (5 appeared in 1993) and 2 technical reports. He received the Best Research Proposal Award from the Society of Plastics Engineers' Plastics Analysis Division in 1991.
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Major Field: Mechanical Engineering

Title of Dissertation: Stress-Strain Analysis of Adhesive-Bonded Composite Single-Lap Joints under Various Loadings

Approved:

Su-Seng Pang
Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

[Signatures]

Date of Examination:

April 27, 1993