1993

Phase Transitions and Flux Distributions of SU(2) Lattice Gauge Theory.

Yingcai Peng
Louisiana State University and Agricultural & Mechanical College

Follow this and additional works at: https://digitalcommons.lsu.edu/gradschool_disstheses

Recommended Citation
https://digitalcommons.lsu.edu/gradschool_disstheses/5589

This Dissertation is brought to you for free and open access by the Graduate School at LSU Digital Commons. It has been accepted for inclusion in LSU Historical Dissertations and Theses by an authorized administrator of LSU Digital Commons. For more information, please contact gradetd@lsu.edu.
INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI

University Microfilms International
A Bell & Howell Information Company
300 North Zeeb Road. Ann Arbor, MI 48106-1346 USA
313-761-4700 800-521-0600
Phase transitions and flux distributions of $SU(2)$ lattice gauge theory

Peng, Yingcai, Ph.D.
The Louisiana State University and Agricultural and Mechanical Col., 1993
PHASE TRANSITIONS AND FLUX DISTRIBUTIONS OF SU(2) LATTICE GAUGE THEORY

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

The Department of Physics and Astronomy

by

Yingcai Peng
B.S., Tsinghua University, Beijing, 1986
M.S., Louisiana State University, 1987
August 1993
ACKNOWLEDGEMENTS

I am indebted to my advisor, Professor Richard W. Haymaker for his continuous patient advisory and valuable guidance during my dissertation research. Without his helps this work would not have been possible. I am also happy to thank Profs. A.R.P. Rau, J. Callaway, W. Metcalf, L.H. Chan and D. Koester for their introducing me atomic physics, solid state physics, high energy physics, field theory and general relativity theory.

It is a pleasure to thank Drs. H. Chen, V. Singh, C.H. Lee, L. Zhang and K. Yee, and my friends H. Zheng, X. Zhang, C. Bahri, V. Poduri, J. Vijapurkar, L. Bi for many helps and interesting discussions. I am grateful to Prof. R.W. Haymaker and the Physics Department for the financial support. I also want to extend my appreciations to all members of the Department for their kindness and helps throughout my graduate study period.

Finally I want to thank Monika Lee and Hortensia T. Valdes for their many helps in using computing facilities. This dissertation was also supported in part by the U.S. Department of Energy.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Acknowledgements</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of Tables</td>
<td>v</td>
</tr>
<tr>
<td>List of Figures</td>
<td>vii</td>
</tr>
<tr>
<td>Abstract</td>
<td>xi</td>
</tr>
</tbody>
</table>

## Chapter

1. Introduction .................................................. 1

2. Lattice Gauge Theory Basics ............................ 14
   2.1 Introduction of LGT ................................. 14
   2.2 Interpretation of Some Physical Quantities ..... 20
   2.3 The Scaling Relation $a(\beta)$ of $SU(2)$ LGT ... 23
   2.4 Transfer Matrix and Temperatures on Lattices ... 25

3. Calculation Methods and Numerical Results ........... 29
   3.1 Calculation Methods of LGT ......................... 29
   3.2 Numerical Results ..................................... 31

4. Finite Temperature Phase Transition .................. 39
   4.1 Basic Theory .......................................... 39
   4.2 Data Analysis ......................................... 43

5. Flux Distributions .......................................... 53
   5.1 Motivations ............................................. 53
   5.2 Flux Distributions at Finite Temperatures ....... 57
   5.3 Implementation to Finite Volumes .................. 69

6.1 Review and Generalizations of Michael Sum Rules ............. 74
6.2 Analysis of Flux Data ..................................................... 86
SUMMARY ..................................................................................... 93
REFERENCES .................................................................................. 95
APPENDICES
A MONTE CARLO METHODS IN LGT ......................................... 98
  A.1 Metropolis Algorithm ....................................................... 99
  A.2 Heat Bath Algorithm ....................................................... 104
  A.3 Over-Relaxation Algorithm ............................................. 104
B PROOF OF A RELATION ......................................................... 110
VITA .......................................................................................... 112
# LIST OF TABLES

<table>
<thead>
<tr>
<th>TABLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>The correspondence of the lattice spacing $a$ and the coupling constant $\beta$ for the $SU(2)$ LGT.</td>
</tr>
<tr>
<td>2</td>
<td>The statistics of data on the lattice $4 \cdot 11^2 \cdot 65$.</td>
</tr>
<tr>
<td>3</td>
<td>The measured values of $&lt;</td>
</tr>
<tr>
<td>4</td>
<td>The raw string tension data $\kappa$ extracted from the correlation $&lt;P(0)P^t(z)&gt;_c$ at different distance $z$, they were measured on the lattice $4 \cdot 11^2 \cdot 65$ with $\beta = 2.25$.</td>
</tr>
<tr>
<td>5</td>
<td>The string tension data $\kappa$ measured in lattice units on lattices of the size $4 \cdot 9^2 \cdot 65$, $4 \cdot 11^2 \cdot 65$, $6 \cdot 7^2 \cdot 65$ and $6 \cdot 11^2 \cdot 37$.</td>
</tr>
<tr>
<td>6</td>
<td>The string tension data $\kappa$ measured in physical units on lattices of the size $4 \cdot 9^2 \cdot 65$, $4 \cdot 11^2 \cdot 65$, $6 \cdot 7^2 \cdot 65$ and $6 \cdot 11^2 \cdot 37$.</td>
</tr>
<tr>
<td>7</td>
<td>The typical measured flux density values (errors) of $\rho_E$ and $\rho_A$ in the region of $\beta &gt; \beta_c$ on the edges of the transverse plane midway between the $qq$, as shown in Fig. 5.2. The flux data are in the physical unit $Gev/fm^3$.</td>
</tr>
<tr>
<td>8</td>
<td>The correspondence of the two parameters $z_g$ and $z_K$.</td>
</tr>
<tr>
<td>9</td>
<td>The values of the string tension data and $z_K$ on lattices of size $4 \cdot N_s^2 \cdot 64$ with $N_s = 4, 6, 8$ and $\beta = 2.36, 2.38$ and $2.41$. The data are quoted from ref.[18].</td>
</tr>
<tr>
<td>Section</td>
<td>Description</td>
</tr>
<tr>
<td>---------</td>
<td>-------------</td>
</tr>
<tr>
<td>10</td>
<td>Our string tension data $\kappa$ and the values of the $z_\kappa$ parameter on lattices of the size $4 \cdot N_s^2 \cdot 65$ with $N_s = 5$, 7, 9, 11 and $\beta = 2.25$ and 2.28. .................................................. 72</td>
</tr>
<tr>
<td>11</td>
<td>The center slice flux energies and action of a $q\bar{q}$ pair at various separations $r/a$ in the confined phase, which were calculated from the flux data on the lattice $4 \cdot 11^2 \cdot 65$ with $\beta = 2.25$ ($\beta_c \approx 2.30$). The data are in the physical unit $\text{Gev/fm}$. ......................... 88</td>
</tr>
<tr>
<td>12</td>
<td>The comparison of the predictions of generalized Michael sum rules and the data of center slice action $\sigma_A$. .......................... 89</td>
</tr>
<tr>
<td>13</td>
<td>The $\sigma_A$ data in the unconfined phase ($\beta &lt; \beta_c$), which were measured on lattices, $4 \cdot 9^2 \cdot 65$ and $4 \cdot 11^2 \cdot 65$. The data are in the physical unit $\text{Gev/fm}$. .................................................. 92</td>
</tr>
</tbody>
</table>
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(a) Screening of electric charge as a consequence of vacuum polarisation of QED. (b) Antiscreening of color charge as a consequence of vacuum polarisation of QCD. 7</td>
</tr>
<tr>
<td>2</td>
<td>The conjectured $q \bar{q}$ flux distribution in the confined phase, which has the string-like flux tube form. 9</td>
</tr>
<tr>
<td>3</td>
<td>The properties of QCD can be shown diagrammally by the 'phase diagram' on the plane of $T$ and $1/V$. 12</td>
</tr>
<tr>
<td>4</td>
<td>(a). The link variable $U_\mu$; (b). The plaquette variable $U_P$. 17</td>
</tr>
<tr>
<td>5</td>
<td>(a) The Wilson loop $W$ on the $z$-$t$ plane, (b) The Polyakov loop $P$ along the time direction. 22</td>
</tr>
<tr>
<td>6</td>
<td>Monte Carlo data for $\langle</td>
</tr>
<tr>
<td>7</td>
<td>Monte Carlo data of $\langle P(0)P(r) \rangle$ vs. $r/a$ in the confined phase, calculated on the lattice of size $4 \cdot 11^2 \cdot 65$ with $\beta = 2.25$ (squares), $\beta = 2.28$ (triangles) and $\beta = 2.29$ (diamonds). (a). linear axies, (b). logarithmic y axes and linear x axes. 46</td>
</tr>
<tr>
<td>8</td>
<td>Monte Carlo data of $\langle P(0)P(r) \rangle$ vs. $r$ in the unconfined phase, calculated on the lattice $4 \cdot 9^2 \cdot 65$ with $\beta = 2.36$ (squares) and $\beta = 2.40$ (triangles). (a). linear axies, (b). logarithmic y axes and linear x axes. 47</td>
</tr>
</tbody>
</table>
The plot of $\kappa$ vs. $T$ near the transition point $T_c$. The data were calculated on lattices, $4 \cdot 9^2 \cdot 65$ (circles), $4 \cdot 11^2 \cdot 65$ (squares) and $6 \cdot 7^2 \cdot 65$ (triangles). The solid line is the fitting function in Eq. (4.12) with $\alpha = 0.35 \pm 0.04$. The string tension $\kappa$ is in the physical unit $\text{Gev/fm}$. .................................................. 50

$\kappa$ vs. $\beta$ for $N_t = 4$, the transition point $\beta_c \sim 2.2985$. The data were calculated on lattices, $4 \cdot 9^2 \cdot 65$ (squares), $4 \cdot 11^2 \cdot 65$ (triangles). The solid line is the fitting function in Eq. (4.14) with $\delta = 0.22 \pm 0.03$. The string tension $\kappa$ is in the physical unit $\text{Gev/fm}$. ........... 52

The $q\bar{q}$ color sources in a long rectangular box. The flux lines between the $q\bar{q}$ pair are confined by side walls and have a flux tube form. .................................................. 56

The energy density $\rho_E$ flux distributions in the region of $\beta > \beta_c$. The flux data were measured on lattices $4 \cdot N_s^2 \cdot 65$ with $\beta = 2.40$ for various spatial sizes, (a) $N_s = 5$, (b) $N_s = 7$, (c) $N_s = 9$, and (d) $N_s = 11$. The data are in the physical unit $\text{Gev/fm}^3$. ............... 59

The action density $\rho_A$ flux distributions in the region $\beta > \beta_c$, with the $q\bar{q}$ separations, (a) $r = 3a$, (b) $r = 4a$, (c) $r = 5a$ and (d) $r = 6a$. The data were measured on lattices $4 \cdot 5^2 \cdot 65$ with $\beta = 2.40$, and in the physical unit $\text{Gev/fm}^3$. ....................... 61

The center slice energy $\sigma_E$ and action $\sigma_A$ versus the $q\bar{q}$ separation $r$ in the region of $\beta > \beta_c$, (a) $\sigma_E$ vs. $r$, (b) $\sigma_A$ vs. $r$. The data were measured on lattices $4 \cdot N_s^2 \cdot 65$ with $\beta = 2.36$ for various spatial sizes, $N_s = 5$ (circles), $N_s = 7$ (squares), $N_s = 9$ (triangles), $N_s = 11$ (diamonds). The data are in the physical unit $\text{Gev/fm}$. ............... 62
15 The plot of $\sigma_E$ in the region of $\beta > \beta_c$ versus the transverse size $N_s$ at large $q\bar{q}$ separation $r = 6a$. The solid line is the Coulomb behavior $(N_s a)^{-2}$, the dashed line is the inverse quartic behavior $(N_s a)^{-4}$, both are normalized to the data at $N_s = 5$.  

16 The action density $\rho_A$ flux distributions in the region of $\beta < \beta_c$ for various separations (a) $r = 3a$, (b) $r = 4a$, (c) $r = 5a$ and (d) $r = 6a$. The flux data were measured on lattices $4 \cdot 11^2 \cdot 65$ with $\beta = 2.25$, and measured in the physical unit $GeV/fm^3$.  

17 The center slice energy $\sigma_E$ and action $\sigma_A$ versus the $q\bar{q}$ separation $r$ in the region of $\beta < \beta_c$, (a) $\sigma_E$ vs. $r$, (b) $\sigma_A$ vs. $r$. The data were measured on lattices $4 \cdot N_s^2 \cdot 65$ with $\beta = 2.28$ for various spatial size, $N_s = 5$ (circles), $N_s = 7$ (squares), $N_s = 9$ (triangles), $N_s = 11$ (diamonds); and are in the physical unit $GeV/fm$.  

18 The Wilson loop $W$ of the temporal size $na$ and the spatial size $r$. The plaquette $P_1$ is outside the Wilson loop $W$, and the plaquette $P_2$ is inside the Wilson loop $W$.  

19 The plot of the predictions of $\sigma_A$ vs. $\beta$ of Eq.(6.46) (solid lines), and Eq.(6.51) (dashed lines). The two solid lines represent the upper and lower limits of the predictions of Eq.(6.46). The data of $\sigma_A$ were measured from lattices $4 \cdot 9^2 \cdot 65$ (squares) and $4 \cdot 11^2 \cdot 65$ (triangles), which have the physical unit $GeV/fm$.  

20 The weight function $p(x)$ vs. $x$, which is in the interval $[a, b]$. $x_i$ is some accepted $x$ value.
The locus of the minimum of the action $S(U)$ in the phase space of the link variable $U$. Here $\omega_U$ is the phase factor of the link value $U$ to be updated, $\omega'$ is the phase factor of the trial value $U'$, and $\omega_g$ is the phase factor of the element $g_0$ which minimizes the action $S$. ................................................................. 106

One 'staple' $M_{jkli}$ that couples with the link $U_{ij}$, where $M_{jkli} = U_{jk}U_{kl}U_{li}$. ................................................................. 108
The strong interactions between quarks are believed to be described by *Quantum Chromodynamics* (QCD), which is a non-abelian $SU(3)$ gauge theory. It is known that QCD undergoes a deconfining phase transition at very high temperatures, that is, at low temperatures QCD is in a confined phase, at sufficient high temperatures it is in a unconfined phase. Also, quark confinement is believed to be due to the string formation.

In this dissertation we studied $SU(2)$ gauge theory using numerical methods of LGT, which will provide some insights about the properties of QCD because $SU(2)$ is similar to $SU(3)$. We measured the flux distributions of a $q\bar{q}$ pair at various temperatures in different volumes. We find that in the limit of infinite volumes the flux distribution is different in the two phases. In the confined phase strong evidence is found for the string formation, however, in the unconfined phase there is no string formation. On the other hand, in the limit of zero temperature and finite volumes we find clear signal for string formation in the large volume region, however, the string tension measured in intermediate volumes is due to finite volume effects, there is no intrinsic string formation.

The color flux energies (action) of the $q\bar{q}$ pair are described by Michael sum rules, The original Michael sum rules deal with a static $q\bar{q}$ pair at zero temperature in infinite volumes. To check these sum rules with our flux data at finite temperatures, we present a complete derivation for the sum rules, then generalize them to account for finite temperature effects. We find that our flux data are consistent with the prediction of generalized sum rules.
Our study elucidate the rich structures of QCD, and provide evidences for the quark confinement and string formation. This supports the belief that QCD is a correct theory for strong interactions, and quark confinement can be explained by QCD.
CHAPTER 1

INTRODUCTION

There is strong evidence that particles, such as nucleons and mesons, are made of quarks, although no free isolated quark has yet been found\[1\]. One way out of this dilemma is to conjecture that quarks can not exist in the free state, but only exist in bound states. For example, nucleons are bound states of three quarks \(qqq\), mesons are those of a quark and an antiquark \(q\bar{q}\). This is known as ‘quark confinement’. The interactions between quarks are the strong interactions, which are believed to be described by quantum chromodynamics (QCD). According to QCD the interactions between quarks become weak at short distances, however, with the increase of the distance it becomes strong. This is called ‘asymptotic freedom’\[2\]. It was also proposed that QCD undergoes a deconfining phase transition at very high temperatures\[3, 4\], that is, quarks are confined at low temperatures, however, at sufficiently high temperatures they are in the unconfined state. These results show that there are rich structures of QCD to be studied. In this thesis we will consider QCD in finite volumes and at finite temperatures, to study the quark confinement problem and other properties of QCD in different phases using numerical methods of lattice gauge theory (LGT). To motivate this we first review the introduction of the quark model and QCD theory, and explain why we need a numerical approach to this problem. Then we will summarise our main results and explain the organization of this thesis.
Particles are classified according to their interactions, those with strong interactions are called hadrons. The hadron fermions are called baryons, such as $p$, $n$ and $\Lambda$, which have spin $\frac{1}{2}$. The hadron bosons are called mesons, such as $\pi$ and $K$, which have spin 0. By the early 1960s hundreds of particles and resonant states belong to the hadron category have been discovered. To solve the problem of the large number of “elementary particles”, people began to suggest that all hadrons have a common underlying substructure, in a manner analogous to the atomic theory explanation of the profusion of elements in the periodic table.

In 1964, Gell-Mann [5] and independently Zweig [6], proposed that hadrons were composed of fractionally charged sub-particles called quarks. All baryons are bound states of three quarks ($qqq$), and all mesons are composed of a quark and an antiquarks ($q\bar{q}$). At that time they proposed that there are three kinds (or flavors) of quarks, named $u$, $d$ and $s$. Later more flavors of quarks were discovered. According to this model a proton is composed of two ‘up’ quarks and a ‘down’ quarks, ($uud$), and a $\pi^+$ meson is ($u\bar{d}$), other hadrons have similar quark compositions. This quark model works very successfully in explaining why the vast number of hadrons can be arranged into some supermultiplets, and in predicting the hadron masses. Also the prediction of the ratio of neutron and proton magnetic moments from this quark model are in excellent agreement with experimental results [1].

Strong evidence for the quark model came from experiments carried out in the 1960’s [7]. High energy electrons (7-17 Gev) were made to collide with protons, and ‘deep inelastic’ events were selected, i.e. $ep \rightarrow eX$ where $X$ indicates all final states. These events involve a large amount of momentum
and energy transfer to the proton. It was observed that some incident electrons were scattered at much larger angles than expected if the proton was a continuous charge distribution. This shows that the electron is scattered from pointlike constituents of the proton. The situation is similar to the Rutherford’s atomic scattering experiment, which uncovered the nuclear structure of atoms. Detailed analysis of the experiments strongly suggest that nucleons contain pointlike constituents, which have spin $\frac{1}{2}$ and fractionally charges.

The success of the quark model spurred a number of experiments to search for free quarks [1, 8]. However, no evidence is seen for the existence of free quarks in terrestrial conditions. This supports the idea of quark confinement.

So far all experimental evidence supports the quark model. However, a problem arises when one considers the $\Delta^{++}$ particle, which is composed of three $u$ quarks, and has spin $J = \frac{3}{2}$ with zero orbital angular momentum, $L = 0$. This is a system with three identical spin $\frac{1}{2}$ fermions $u$. The total wave function of the system is

$$\psi(\Delta^{++}) = \psi_{\text{spin}}\psi_{\text{flavor}}\psi_{\text{space}}$$

$$= (\uparrow\uparrow\uparrow)(uuu)\psi_{\text{space}}. \quad (1.1)$$

One can immediately see that the spin part and flavor part of the wave function, $\psi_{\text{spin}}$ and $\psi_{\text{flavor}}$, are symmetric under the interchange of any two quarks. Since the orbital angular momentum $L$ is zero, the space part $\psi_{\text{space}}$ must also be symmetric. So the overall wave function $\psi(\Delta^{++})$ is symmetric. This appears to violate the Pauli exclusion principle, which requires that a composite state of identical fermions should have an antisymmetric wave function.
To solve this problem one can assume that quarks possess another degree of freedom, described by a new quantum number called color, which can take three values, say $R$ (red), $B$ (blue) and $G$ (green). Then one can construct a color wave function $\psi_{\text{color}}$ which is antisymmetric under quark interchange. The overall wave function $\psi(\Delta^{++})$ in Eq. (1.1) should be replaced by

$$\psi(\Delta^{++}) = (\uparrow\uparrow\uparrow)(uuu)\psi_{\text{space}}\psi_{\text{color}}, \quad (1.2)$$

with

$$\psi_{\text{color}} = \frac{1}{\sqrt{6}}(RGB + GBR + BRG - RBG - BGR - GRB). \quad (1.3)$$

where $\psi_{\text{color}}$ is antisymmetric under quark interchange. Then the problem of quark statistics is solved with the introduction of the color quantum number.

There also exist some other rather convincing experimental evidence for the color hypothesis, which are discussed in ref. [1]. What is very interesting is that $\psi_{\text{color}}$ in Eq. (1.3) is a singlet representation of the $SU(3)$ group, i.e. it is invariant under permutation of color variables. By extension, it was found that all hadrons are color singlets. Quarks carry the color indices, $R$, $B$ or $G$, which form a triplet representation of the $SU(3)$ group. This seems to imply that only color singlets are not confined, and quark confinement is simply color confinement.

If color is confined, there must be a force, or a field, associated with color, so that any particles with color would be confined by the force, then only "colorless" (or color singlet) particles can exist freely. It is believed that the strong interactions between quarks are due to this color force, and this force can be described by quantum chromodynamics (khroma=color), or QCD. The QCD theory says that like electric and weak charges, the color charge is also associated with a gauge field whose quanta mediate the strong interactions.
between quarks. Since there are three color charges, i.e. $R$, $B$ or $G$, which form a triplet representation of a $SU(3)$ group, the color gauge field preserves the local $SU(3)$ gauge symmetry. This is different from quantum electrodynamics (QED), which has a local $U(1)$ gauge symmetry because there is only one electric charge. The quanta of the color field are called gluons, which have $J^P = 1^-$ and $m = 0$. There are eight gluons, forming an 8 dimensional representation (the octet) of the $SU(3)$ color gauge group. Since gluons themselves carry color charges, they can couple to each other.

Because of the gluon-gluon coupling, QCD has different features from QED. According to QED a real electron is surrounded by clouds of virtual $e^+e^-$ pairs due to vacuum polarisation. This leads to the screening of the electric charge. So when one measures the charge of the electron by the Coulomb force on a test charge, the effective electric charge measured by the test charge at large distances is less than what it is at small distances, as shown in Fig. 1(a) below. The situation in QCD is similar, a quark with color charge is surrounded by a cloud of virtual $q\bar{q}$ pairs and gluons. But there is an additional feature that gluons couple to themselves, which makes a crucial difference. The effect is to make a $R$ color charge (say) attract preferentially other $R$ charges (not $B$ or $G$) to itself. This differs from QED and results in the antiscreening of color charges. So the effective color charge of the quark measured by a test color charge at large distances is greater than that at small distances, as shown in Fig. 1(b) below. This peculiar property of QCD is known as asymptotic freedom. From this behaviour one can see that the coupling between quarks becomes strong with the increase of their separation, which supports the quark confinement hypothesis.
The asymptotic freedom of QCD has been confirmed by the deep inelastic scattering experiments [1]. In these experiments the nucleons were found to be a loosely bound collection of quarks and gluons. At high energies (corresponding to short distances) the coupling between quarks becomes weak, and the higher the energy, the weaker the coupling. This agrees with asymptotic freedom.

The usual analytical approach to solve field theories is perturbation theory, which involves an expansion in the coupling constant. This method is extremely successful in calculating properties of QED at low energies (i.e. large distances), because the dimensionless coupling, $\alpha = e^2/(4\pi\hbar c)$, of QED is very small ($\ll 1$) in this case. However, to study the low-energy properties of QCD this method can not be used, because the coupling constant of QCD is large at low energies. One must search for some nonperturbative methods.

The most productive nonperturbative approach is lattice gauge theory (LGT), proposed by K. Wilson in 1974 [9]. This is a formulation of quantum field theory on a Euclidean spacetime grid, which in our study is a four-dimensional hypercube. We will discuss the details of LGT in next chapter. One advantage of LGT is that one can avoid the infinity problems arising in a continuum field theory. Another motivation for LGT is that it takes advantage of the deep connections between quantum field theories and statistical mechanics. Thus many methods in statistical mechanics, such as Monte Carlo techniques and high temperature expansions, can be applied to LGT. Now by using these techniques one can study the properties of QCD even in the strong coupling region.
Figure 1. (a) Screening of electric charge as a consequence of vacuum polarisation of QED. (b) Antiscreening of color charge as a consequence of vacuum polarisation of QCD.
Although some results of QCD support the quark confinement hypothesis, no proof of this hypothesis from QCD has been found yet. To understand how quarks can be permanently confined, it was conjectured that the color field lines between quarks have a string-like flux tube form, as shown in Fig. 2 below. And the energy stored in the flux tube per unit length is a constant, \( \kappa \), called the string tension. Consider the quark-antiquark bound state \((q\bar{q})\), the meson. If string formation occurs, the potential energy \( V(r) \) of the \( q\bar{q} \) pair would increases linearly with the \( q\bar{q} \) separation \( r \), \( V(r) \sim \kappa r \). This results in the permanent confinement of the \( q\bar{q} \) pair.

By using LGT techniques one can calculate the potential energy \( V(r) \) and the flux distributions of a \( q\bar{q} \) pair. Some results have shown clear evidence that in the limit of large volumes and zero temperature the potential energy \( V(r) \) is linear for large \( r \) [10], and the flux lines form a string-like flux tube between the \( q\bar{q} \) pair[11].

In 1978 A.M. Polyakov[3] and L. Susskind[4] proposed independently that QCD undergoes a phase transition at a very high temperatures \( (T_c) \). For \( T < T_c \), quarks and gluons are in confined bound states; however, For \( T > T_c \), they are in a plasma-like state. This has also been confirmed by many LGT results[12, 13]. The study of QCD at high temperatures is of many important significances. High energy heavy ion collisions can produce hadronic matters at high temperatures, if the colliding energy is high enough one expects that the quark-gluon plasma would form in the process. Such experiments could provide further tests of QCD as being the correct theory describing the strong interactions. Also, such hadronic matter at very high temperatures and densities was surely present in the early stages of the big-bang universe, and
Figure 2. The conjectured $q\bar{q}$ flux distribution in the confined phase, which has the string-like flux tube form.
may be encountered in neutron stars. Hence it is important to study QCD at very high temperatures for constructing models of the universe and neutron stars.

Another way to study QCD is to take advantage of the asymptotic freedom character of QCD. As we discussed above, the coupling between quarks is weak at small distances, this opens the possibility to solve QCD analytically. M. Luscher considered the $SU(N)$ gauge theory ($SU(3)$ for QCD) in small volumes[14], and derived an effective hamiltonian for the theory. Subsequently the lowest energy levels of $SU(2)$[15] and $SU(3)$[16] gauge theories in small volumes were computed by using Luscher's hamiltonian. He also introduced a dimensionless variable[15],

$$z_g = m(0^+)L,$$

(1.4)

to characterize the physical size of the volume. Where $m(0^+)$ is the lowest glueball mass, which is the energy gap or the inverse of the correlation length; and $L$ is the linear size of the box. The small volumes are defined as those with $z_g \leq 1$. In this domain the string tension is found to be zero [15, 18], which is consistent with asymptotic freedom. To compare with physics of real world, one must extend these results to large volumes. Later P. van Baal and J. Koller[17, 18] generalized Luscher's analytical calculations to the intermediate volumes, i.e. $1 \leq z_g \leq 5$, in which LGT calculations are also accessible. B.A. Berg and A.H. Billoire [19] carried out a thorough LGT study in the intermediate volume region. Their results are in good agreement with the analytical results of van Baal and Koller. In this domain they did not find evidence for string formation, although they did find a string tension. It is suspected that the string tension is not due to the intrinsic string formation,
but to finite volume effects. Intrinsic string formation is expected to occur in large volumes, i.e. $z_g \geq 5$.

Now we can summarize the properties of QCD clearly on the plane of the temperature ($T$) and the inverse volume ($1/V$), as shown in Fig. 3. This figure takes liberties, because there is no true phase transition in finite volumes. This figure shows that in the infinite volume limit ($V \to \infty$), QCD has a phase transition at the finite temperature, $T_c$. At zero temperature the system has no string formation in small and intermediate volumes, and the string formation is expected to occur in large volumes. In this figure for cases of finite volume 'confinement' only means string formation, 'unconfinement' only represents that there is no string formation.

Pure $SU(2)$ gauge theory contains already essential features of full QCD, but it is much simpler to study than $SU(3)$, and takes much less computer time for Monte Carlo simulations. So in this work we shall only consider the pure $SU(2)$ gauge theory, which corresponds to the case of static quark sources. We wish to answer the question of the volume at which the intrinsic string formation occurs in the zero temperature limit, and understand the properties of QCD in both phases in the infinite volume limit. We measured the flux distributions of a static $q\bar{q}$ pair in various volumes and temperatures, which covers the finite temperature phase transition region, and the region from intermediate volumes to large volumes in the zero temperature limit. Our results provide evidence that string formation occurs in the confined phase, but does not occur in the unconfined phase. We then measured the string tension $\kappa$ at various temperatures in the confined phase, and studied its behaviour with temperature. For cases of finite volumes our results verified the expectation
Figure 3. The properties of QCD can be shown diagrammally by the 'phase diagram' on the plane of $T$ and $1/V$. 
that the string tension in intermediate volumes are just due to finite volume
effects, and we also found some evidences for string formation in large volumes.

For a $q\bar{q}$ pair the color flux energies are related to the potential energy
$V(r)$ by some identities called Michael sum rules[20]. To analyse our flux data
measured on finite lattices using the sum rules, we generalized these sum rules
to account for the finite size effects. Our flux data support this generalization.

This dissertation is organized as follows. Chapter 2 gives a brief intro-
ductive review of LGT and some basic concepts. The implementations and
interpretations of LGT will be discussed. In Chapter 3 we describe the calcula-
tion methods on lattices, and the measured quantities and their interpretations.
Then we present our numerical results. In Chapter 4 we discuss some results of
the finite temperature phase transition from our LGT calculations, In Chap-
ter 5 we then present detailed analysis of the $q\bar{q}$ flux distributions measured on
various lattices. In Chapter 6 we shall review the derivation of Michael sum
rules, and generalize these sum rules to include finite size effects. Then we
apply the sum rules to analyse our flux data. Finally, we present the summary
and conclusions. A detailed discussion of Monte Carlo methods used in LGT
calculations are presented in Appendix A. Also the proof of a relation used in
the derivation of Michael sum rules is given in Appendix B.
CHAPTER 2

LATTICE GAUGE THEORY BASICS

In this chapter we review the introduction of lattice gauge theory, and discuss in general terms how a lattice calculation is performed and interpreted.

2.1 Introduction of LGT

In four-dimensional Euclidean space the action of the full QCD theory is given by

\[ S = S_G + S_F = \frac{1}{2} \int d^4x \text{Tr}(F_{\mu\nu}F_{\mu\nu}) + \int d^4x \bar{\psi}(x)(\gamma_\mu D_\mu + M)\psi(x). \]  

(2.1)

where \( S_G \) represents the gauge field part of the action, and \( S_F \) the fermion field (quarks) part. the color field strength tensor is defined as

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu], \]

(2.2)

with the gauge field \( A_\mu \) which can be expanded in terms of the generators \( \tau_i \) of the \( SU(N) \) gauge group, i.e.

\[ A_\mu(x) = \frac{1}{2}\tau_i A^{i}_\mu \quad (i = 1, 2, \ldots, n_g). \]

(2.3)

For QCD the gauge group is \( SU(3) \) which has eight generators, i.e. \( n_g = 8 \). The fermion field \( \psi(x) \) describes the quarks of the theory, \( D_\mu = \partial_\mu + igA_\mu \), \( M \) is the mass matrix for quarks.
In the static quark limit ($M \to \infty$), one can show that the effective action of the theory is only the gauge field part [10],

$$S = S_G = \frac{1}{2} \int d^4x Tr(F_{\mu \nu} F_{\mu \nu}).$$  \hspace{1cm} (2.4)

Fortunately, the fundamental problems of QCD, such as quark confinement, can be studied without the introduction of dynamical quarks. It can also be argued that the static quark approximation is good to describe real heavy quarks such as the $b$-quark. In the following we shall only consider the theory in this limit.

In field theory all physical information about the system can be extracted from a correlation function, e.g. $\langle C \rangle$, where $C$ represents a product of operators. The correlations can be calculated from the following path integral expression,

$$\langle C \rangle = \frac{\int [DA] C e^{-S}}{\int [DA] e^{-S}}.$$  \hspace{1cm} (2.5)

As we mentioned before, the renormalized coupling constant of QCD becomes strong in the low energy region because of asymptotic freedom. One cannot calculate the quantity in Eq. (2.5) analytically in this case. However, LGT opens the way to solve the problem using numerical methods.

Let us now turn to the lattice formulation of quantum field theory [9, 21, 10]. In this formalism spacetime is represented by a discrete, hypercubical grid. The gauge fields, which mediate the interactions between quarks, are defined on the links (strictly speaking, gauge group elements exist on links), while matter fields like quarks exist on the sites of the lattice. On a finite lattice the system has a finite number of degrees of freedom, which enable us to do numerical calculations. Then the lattice spacing is supposed to be
taken to zero, and the LGT calculated results approach those in the continuous Euclidean space.

For the pure $SU(N)$ LGT the fundamental variable is the link variable, which is the group element associated with the link, i.e. $U_\mu(n)$, as shown in Fig. 4 (a). The link variable can be written as,

$$U_\mu(n) = \exp(iB_\mu(n)),$$

with $B_\mu(n) = \frac{1}{2}ag\tau_iA^i_\mu(n) \ (i = 1, 2, \cdots, n_g)$, and $a$ the lattice spacing. The index $n$ denotes the site of the link, $\mu$ is the direction of the link. If the link takes the opposite direction $-\mu$, it is associated with the inverse group element, $U^{-1}_\mu(n)$, that is,

$$U_{-\mu}(n + \mu) \equiv U^{-1}_\mu(n).$$

Any physical gauge field theory should be locally gauge invariant, that is, physical quantities are invariant under local gauge transformations. At each lattice site we define an arbitrary group element,

$$g(n) = \exp\left[-i\frac{1}{2}\tau_i\chi_i(n)\right],$$

where $\chi_i(n)$ is a function of the lattice site $n$, and $i = 1, 2, \cdots, n_g$. The local gauge transformation for the link variable $U_\mu(n)$ (gauge field) is defined as

$$U_\mu(n) \to U'_\mu(n) = g(n)U_\mu(n)g^{-1}(n + \mu).$$

Then in LGT the simplest quantity which is locally gauge invariant is the plaquette variable, which is the product of link variables $U_\mu(n)$ around the plaquette, i.e.
Figure 4: (a). The link variable $U_\mu$; (b). The plaquette variable $U_P$. 
\[ U_P(n, \mu, \nu) = U_\mu(n)U_\nu(n + \nu)U_{-\mu}(n + \mu + \nu)U_{-\nu}(n + \nu) \]
\[ = U_\mu(n)U_\nu(n + \mu)U_{-\mu}(n + \nu)U_{-\nu}(n) \]  
(2.10)

as shown in Fig. 4 (b). So in LGT the physical quantities of local gauge invariance can be built from the plaquette variables \( U_P \).

The standard Wilson action of the \( SU(N) \) LGT is constructed from the plaquette variables as [21],

\[ S = \beta \sum_{n,P} \left[ 1 - \frac{1}{N} Re(Tr U_P(n, \mu, \nu)) \right], \]

(2.11)

with the sum over all lattice sites \( n \) and the plaquette orientations \( P \), and \( \beta = 2N/g^2 \), which is analogous to \( 1/kT \) in statistical mechanics. In the following we shall show that in the naive continuum limit, \( a \to 0 \), this action becomes that of the continuous \( SU(N) \) gauge theory in Eq. (2.4).

In the limit of \( a \to 0 \), the field variables \( B_\mu(n + \nu) \) and \( B_\nu(n + \mu) \) associated with the link variables \( U_\mu(n + \nu) \) and \( U_\nu(n + \mu) \) can be Taylor expanded as

\[ B_\nu(n + \mu) \approx B_\nu(n) + a \partial_\mu B_\nu(n) + O(a^2), \]
\[ B_\mu(n + \nu) \approx B_\mu(n) + a \partial_\nu B_\mu(n) + O(a^2). \]

(2.12)

Then the plaquette variable becomes,

\[ U_P(n, \mu, \nu) = U_\mu(n)U_\nu(n + \mu)U_{-\mu}(n + \nu)U_{-\nu}(n), \]
\[ \approx \exp[iB_\mu(n)]\exp[i(B_\nu(n) + a\partial_\mu B_\nu(n))] \]
\[ \times \exp[-i(B_\mu(n) + a\partial_\nu B_\mu(n))]\exp[-iB_\nu(n)]. \]

(2.13)
After using the operator identity,

\[ e^A e^B = \exp(A + B + \frac{1}{2}[A, B] + \cdots), \]  

we get

\[ U_P(n, \mu, \nu) \approx \exp(ia(\partial_\mu B_\nu - \partial_\nu B_\mu) - [B_\mu, B_\nu]) \]

\[ \approx \exp(ia^2 g(\partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu])) \]

\[ \approx \exp[ia^2 g F_{\mu\nu}(n)]. \]  

(2.15)

Therefore, one has

\[ \text{Tr}(U_P(n, \mu, \nu)) \approx \text{Tr}(\exp[ia^2 g F_{\mu\nu}(n)]) \]

\[ \approx \text{Tr}(1 + ia^2 g F_{\mu\nu}(n) - \frac{1}{2}a^4 g^2 F_{\mu\nu}^2(n) + \cdots) \]

\[ \approx N - \frac{1}{2}a^4 g^2 \text{Tr}(F_{\mu\nu}^2(n)) + \cdots. \]  

(2.16)

So the Wilson action in Eq. (2.11) becomes,

\[ S \approx \beta \sum_{n,P} \frac{1}{2N} a^4 g^2 \text{Re} \text{Tr}(F_{\mu\nu}^2(n)). \]  

(2.17)

If we apply the replacement to Eq. (2.17),

\[ \sum_{n,P} \rightarrow \int \frac{d^4 x}{a^4} \sum_{\mu,\nu}, \]  

(2.18)

we get

\[ S \approx \sum_P \int d^4 x \text{Tr}(F_{\mu\nu}^2), \]

\[ = \frac{1}{2} \sum_{\mu,\nu} \int d^4 x \text{Tr}(F_{\mu\nu} F_{\mu\nu}). \]  

(2.19)

This is just the action $S$ in Eq. (2.4), which is the gauge field action of QCD in the continuous Euclidean space.
We now have a Euclidean statistical mechanical system in the canonical ensemble with partition function

\[ Z = \int [dU] e^{-S(U)} = \int \prod_{n,\mu} [dU_\mu(n)] e^{-S(U)}. \]  

(2.20)

This is a multi-dimensional integration over all link variables \( U_\mu(n) \) on the lattice. As the gauge groups are compact, the group volumes are finite, such integrals are well defined and have simple invariance properties. With this definition the path integral formalism of Eq. (2.5) can be transformed into the following form. If \( H \) is some physical quantity, which is a local gauge invariant function of the link variables \( U_\mu(n) \), then the expectation value can be calculated as

\[ < H > = Z^{-1} \int (dU) H(U) e^{-S(U)}. \]  

(2.21)

This expression can be calculated by Monte Carlo simulation methods in LGT, which will be discussed in Appendix A. In the following I proceed to discuss the interpretations of a few important physical quantities in LGT.

### 2.2 Interpretation of Some Physical Quantities

(1). The Wilson Loop \( W \)

Static quark sources were first introduced by Wilson[9] in his original formulation of LGT, via the Wilson loop. To understand this[22], consider a \( q\bar{q} \) pair to be created at the time \( t = 0 \), and kept at a distance \( R \) for a time \( T \), then brought together and annihilated. The world line of such a \( q\bar{q} \) pair would form a closed loop in spacetime. The Euclidean amplitude for this process is
given by the matrix element,

\[ <i|e^{-HT}|f>, \]  

(2.22)

where |i>, |f> represent the initial and final states, i.e. the q\bar{q} pair a distance \(R\) apart, and \(H\) is the Hamiltonian. The transition amplitude can be written as the path integral form similar to Eq. (2.5),

\[ <i|e^{-HT}|f> = \frac{\int [DA_\mu] \exp[-S + ig \int d^4x A_\mu J_\mu]}{\int [DA_\mu] e^{-S}}, \]  

(2.23)

where \(J_\mu\) is an external current density describing the world lines of the q\bar{q} pair. For a point source current loop the term \(A_\mu J_\mu d^4x\) becomes \(A_\mu dx_\mu\). Since |i>, |f> are identical and because this is a static process, the above equation reduces to,

\[ \exp(-E(R)T) = <trP \exp[ig \oint A_\mu dx_\mu]> \]  

(2.24)

where \(E(R)\) is the color field energy of the q\bar{q} pair[20], and \(P\) stands for path ordering. The quantity on the R.H.S. of Eq. (2.24) is called the Wilson loop correlation function. On the lattice the Wilson loop can be defined as the product of link variables \(U\) along a closed path[22],

\[ W = \frac{1}{N} tr \prod_C U_\mu(n), \]  

(2.25)

where \(N\) is for \(SU(N)\). In Figure 5 (a) we show such a Wilson loop on the z-t plane of a lattice. From Eq. (2.24) one can see that the expectation of the Wilson loop is associated with the energy \(E(R)\) of a static q\bar{q} pair,

\[ <W> \sim e^{-TE(R)}. \]  

(2.26)

The expectation \(<W>\) is given by Eq. (2.21) in LGT, which can be calculated by numerical methods and other techniques.
Figure 5. (a) The Wilson loop $W$ on the $z$-$t$ plane, (b) The Polyakov loop $P$ along the time direction.
(2). The Polyakov Loop $P$

In LGT the Polyakov loop along the time direction is defined to be the product of all links along a path in that direction. Since the lattice has periodic boundary conditions, the path forms a closed loop, that is,

$$P(x) = \frac{1}{N} tr \prod_{\tau=1}^{N_t} U_t(x, \tau),$$

(2.27)

where $N$ is for $SU(N)$, and $N_t$ is the lattice size in the time direction, as shown in Fig. 5 (b). As we discussed above, a time directed line on the lattice represents a color charge source (or a quark), thus one can show that the expectation value of $P(x)$ is associated with the free energy $F_q$ of an isolated quark $q$ in the limit of infinite volume\[23],

$$< P(x) > \sim e^{-L t F_q},$$

(2.28)

where $L_t = N_t a$ is the temporal size of the lattice in physical units.

For two Polyakov loops $P(0)$ and $P(\vec{r})$, the correlation of them, $< P(0)P^\dagger(\vec{r}) >$, is related to the free energy $V(\vec{r})$ of a $q\bar{q}$ pair,

$$< P(0)P^\dagger(\vec{r}) > \sim e^{-L t V(\vec{r})},$$

(2.29)

where $P^\dagger(\vec{r})$ is the hermitian conjugate of $P(\vec{r})$, and $\vec{r}$ is the separation of the $q\bar{q}$. Usually, $V(\vec{r})$ is identified to be the potential energy of the $q\bar{q}$ pair.

2.3 The Scaling Relation $a(\beta)$ Of $SU(2)$ LGT

In LGT physical quantities are calculated in unit of the lattice spacing $a$. To obtain values of physical quantities in physical units one needs to know the lattice spacing $a$ in physical units. In $SU(N)$ LGT there exists a scaling
relation $a(\beta)$ between the lattice spacing $a$ and the lattice coupling constant $\beta = \frac{2N}{g^2}$ in the continuum limit ($a \to 0$)[21],

$$a = \Lambda_L^{-1}(\frac{2N}{\beta}b_0)^{-b_1/(2b_0^2)}exp(-\frac{\beta}{4Nb_0})(1 + O(g^2)),$$  \hspace{1cm} (2.30)

with $b_0 = \frac{1}{16\pi^2}\frac{11N}{3}$, and $b_1 = (\frac{1}{16\pi^2})^2\frac{24N^2}{3}$, and $\Lambda_L$ is the lattice scaling constant, which can be determined from experimental data.

This relation was derived from the requirement that physical quantities should approach their physical values in the continuum limit ($a \to 0$)[21]. From the scaling relation $a(\beta)$ in Eq. (2.30) one can see that as the lattice spacing $a \to 0$, $\beta \to \infty$. This implies that the coupling constant $g \to 0$ at small length scales for the $SU(N)$ gauge theory, which agrees with asymptotic freedom.

For $SU(2)$ LGT Eq. (2.30) becomes,

$$a = \Lambda_L^{-1}(\frac{6}{11}\pi^2\beta)^\frac{33}{111}exp(-\frac{3}{11}\pi^2\beta)(1 + O(g^2)).$$  \hspace{1cm} (2.31)

To calculate the lattice scaling constant $\Lambda_L^{-1}$, we follow refs.[25, 26] to choose a scale such that the string tension $\kappa_0$ at zero temperature ($T = 0$) in infinite volume is given by

$$\sqrt{\kappa_0} = 0.44 \text{ Gev},$$  \hspace{1cm} (2.32)

which can be determined from the Regge trajectory[27]. Here we take the string tension data from the real world as that of $SU(2)$ gauge theory for definiteness. From the data in refs.[24, 25] one can extract the following relation between the lattice spacing $a$ and the lattice coupling constant $\beta$, as shown in Table 1, which is consistent with the similar result in ref.[28]. The values of $\Lambda_L^{-1}$ were calculated from the scaling relation in Eq. (2.31) by substituting the values of $a$ and $\beta$. 
Table 1. The correspondence of the lattice spacing $a$ and the coupling constant $\beta$ for the $SU(2)$ LGT.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$a(\beta)$ fm</th>
<th>$\Lambda_L^{-1}$ fm</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.22</td>
<td>0.1981 (61)</td>
<td>27.41 (84)</td>
</tr>
<tr>
<td>2.30</td>
<td>0.1616 (47)</td>
<td>27.32 (79)</td>
</tr>
<tr>
<td>2.40</td>
<td>0.1210 (5)</td>
<td>26.30 (11)</td>
</tr>
<tr>
<td>2.50</td>
<td>0.0843 (8)</td>
<td>23.57 (22)</td>
</tr>
</tbody>
</table>

From Table 1 one can see that the values of $\Lambda_L^{-1}$ changes slowly with $\beta$ in the region, $2.22 \leq \beta \leq 2.5$. This implies that the perturbative asymptotic scaling relation in Eq. (2.31) is not exactly valid in this region. To study the non-perturbative physics and obtain the relation $a(\beta)$ for other $\beta$ values in this region, we use the function in Eq. (2.31) to fit the data of Table 1, with $\Lambda_L^{-1}$ considered as a function of $\beta$. For simplicity, we chose a quadratic function to fit the $\Lambda_L^{-1}$ values. The result is,

$$a(\beta) = \Lambda_L^{-1}(\beta) \left( \frac{6}{11} \pi^2 \beta \right)^{\frac{1}{3}} \exp \left[ -\frac{3}{11} \pi^2 \beta \right] = \Lambda_L^{-1}(\beta) f(\beta),$$  \hspace{1cm} (2.33)

with

$$\Lambda_L^{-1}(\beta) = d_1 + d_2 \beta + d_3 \beta^2 \text{ fm},$$  \hspace{1cm} (2.34)

where the coefficients $d_1 = 59.37 \pm 0.86$, $d_2 = -5.96 \pm 0.39$ and $d_3 = -3.28 \pm 0.17$. Using this scaling relation $a(\beta)$ one can transform the calculated values of physical quantities in $SU(2)$ LGT from the lattice units to physical units.

2.4 Transfer Matrix and Temperatures on Lattices

The path integral formulation of LGT reveals deep connections between field theories and statistical mechanics. Starting with the path integral formulation one can construct another formulation of LGT, i.e. the transfer
matrix formalism[21, 22], which resembles Hamiltonian quantum mechanics and exposes some of the physical properties of LGT quite simply.

For simplicity, let us introduce the transfer matrix formalism by first considering a one-dimensional quantum mechanical problem, a particle of mass $m$ moving in the potential $V(r)$. In Euclidean space the Lagrangian of this particle is

$$\begin{align*}
L(x, \dot{x}) &= \frac{1}{2} m \dot{x}^2 + V(x),
\end{align*}$$

(2.35)

and the action is

$$S = \int d\tau L(x(\tau), \dot{x}(\tau)).$$

(2.36)

In the path integral formalism the path integral corresponding to the partition function of the system is

$$Z = \int [dx(\tau)] e^{-S}.$$  

(2.37)

To calculate the above path integral one makes the time axis discrete and consider the total length to be finite (after calculations we then take the limit of continuum and infinite length,), that is, one introduces a one-dimensional time like lattice of size $N_t$,

$$\tau_n = na \quad (n = 0, 1, 2, \cdots, N_t),$$

(2.38)

where $a$ is the lattice spacing, and we assume that the lattice has the periodic boundary condition, i.e. $\tau_0 = \tau_{N_t}$. Then the action of Eq. (2.36) can be reduced to a sum

$$S = a \sum_{i=0}^{N_t-1} \left[\frac{1}{2} m \left(\frac{x_{i+1} - x_i}{a}\right)^2 + V(x_i)\right],$$

(2.39)

and the partition function of Eq. (2.37) can be written as

$$Z = \int (\Pi_i dx_i) e^{-S}.$$
\[ = \int (\prod_i dx_i) < x_{N_t} | T | x_{N_t-1} > \cdots < x_1 | T | x_0 >, \tag{2.40} \]

with \( < x' | T | x > = \exp[-\frac{m}{2a}(x'-x)^2 - \frac{g}{2}(V(x') + V(x))] \). The operator \( T \) is called the transfer matrix operator. From the completeness relation of the basis states \( \{|x_i>\} \), i.e.

\[ \int dx_i |x_i><x_i|=1, \tag{2.41} \]

and the periodic boundary condition, \( |x_{N_t}> = |x_0> \), one can rewrite the partition function of Eq. (2.40) in the compact form,

\[ Z = \int dx_0 < x_0 | T^N | x_0 > = Tr(T^{N_t}). \tag{2.42} \]

The key to the transfer matrix formalism of \( SU(N) \) LGT in four-dimensional Euclidean lattices is to note that the plaquettes in the action of Eq. (2.11) can be classified into two types, the time-like plaquettes with a link along the time direction, and the space-like plaquettes without a time link. The time-like plaquettes are special, we can reorganize the plaquettes in the action so that the partition function of this theory in Eq. (2.20) can be written in the form[22]

\[ Z = \int <\{U_{N_t}\}|T|\{U_{N_t-1}\}>[\prod dU_{N_t-1}] \cdots <\{U_1\}|T|\{U_0\}>[\prod dU_0] = Tr(T^{N_t}), \tag{2.43} \]

where \( \{U_i\} \) represents all link variables at a given time \( \tau_i \). In Eq. (2.43) we have used the completeness relation and the periodic boundary condition of the basis states \( |\{U_i\}> \). Clearly, the transfer matrix \( T \) evolves the system one link in the time direction, which is similar to the Hamiltonian operator \( \hat{H} \) in quantum mechanics.
Consider a four-dimensional Euclidean lattice of the size $N_t \times N_x \times N_y \times N_z$, if one chooses the direction of $N_t$ to be the time direction, then one can write the transfer matrix $T$ in this direction as $[19]$, $T = T(N_x, N_y, N_z)$. And the eigenstates and eigenvalues of $T$ are defined by means of $$T|n> = \lambda_n|n>,$$ (2.44) with the eigenvalue $\lambda_n = e^{-m_n}$, and the eigenstate $|n> (n \geq 0)$, the state $|0>$ denotes the vacuum. Where $m_n$ is the energy difference between the eigenstate $|n>$ and the vacuum, then one has, $\lambda_0 = 1$.

With the definition of transfer matrix $T$ we can rewrite the partition function $Z$ of Eq. (2.43) as

$$Z = Tr(T^{N_t}) = \sum_n e^{-m_n a N_t},$$ (2.45)

and the expectation of a physical quantity $<H>$ in Eq. (2.21) becomes

$$<H> = Z^{-1} \sum_n <n|H|n> e^{-m_n a N_t}.$$ (2.46)

Comparing this equation with the expectation of a physical quantity in statistical mechanics, one can see that the quantity $1/(N_t a)$ plays the role of the temperature. More strictly one can show that in LGT the temperature $T$ is related to the inverse temporal size of the lattice[29], i.e.

$$T = \frac{1}{N_t a},$$ (2.47)

with $N_t$ to be temporal size of the lattice, if the spacial sizes of the lattice $N_x, N_y, N_z$ are all large compared to $N_t$. 
CHAPTER 3

CALCULATION METHODS AND NUMERICAL RESULTS

In this chapter we shall discuss the calculation methods in LGT, and describe our measurements and present the raw data.

3.1 Calculation Methods of LGT

To perform a numerical study of the \( SU(N) \) LGT, we choose a four-dimensional Euclidean lattice of the size, \( N_t \times N_x \times N_y \times N_z \). As we discussed before the fundamental degree of freedom is the link variable. The configuration of the lattice system is determined by the values of each link. Since the lattice is finite, we impose boundary conditions, usually, periodic boundary conditions are applied. However, for convenience in programming we choose our lattices with helical boundary conditions[30]. We characterize the lattice sites by a set of integer coordinates \((x,y,z,t)\), which correspond to \( N_x, N_y, N_z, N_t \) respectively. Then the helical boundary conditions implies that if we start along the \( N_x \) direction, after going through \( N_x \) sites, say, the link \( U_x(N_x + 1, 1, 1, 1) \) is equal to the link \( U_x(1, 2, 1, 1) \), but not \( U_x(1, 1, 1, 1) \) as it is for periodic boundary conditions. Similar results hold for other links. With this boundary condition a site at the integer coordinates \((x,y,z,t)\) can be uniquely associated with the integer,

\[
j = x + N_x((y - 1) + N_y((z - 1) + N_z(t - 1))). \quad (3.1)
\]
To update the lattice configuration, the links in any given direction were updated in a checkerboard style. In order to get a four-dimensional "checkerboard" three dimensions must be of odd sizes, and the fourth even. All links emanating in a positive direction from odd sites are updated first followed by those from even sites[30]. We chose \( N_x, N_y, N_z \) odd, and \( N_t \) even. The lattice is truly periodic only in the \( N_t \) direction. We use the Metropolis algorithm alternated with the over-relaxation algorithm to update lattice configurations. The details of these algorithms will be discussed in Appendix A.

The lattice system is a statistical system. Before we make measurements, we must thermalize this system. The expectation of a quantity \( H \) in LGT, such as that in Eq. (2.21), can be calculated by the lattice average. That is, in a particular thermalized configuration we make one measurement, say \( H_1 \), then we update the lattice configuration with a number of sweeps and make another measurement, \( H_2 \). By repeating this process many times (e.g. \( N_m \)) we can obtain the lattice average,

\[
< H > = \frac{1}{N_m} \sum_{i=1}^{N_m} H_i. \tag{3.2}
\]

The statistical error of this measurement can be calculated as

\[
\Delta H = \sqrt{\frac{< H^2 > - < H >^2}{N_m}}. \tag{3.3}
\]

where \( < H^2 > \) can be calculated in the same manner as \( < H > \) in Eq. (3.2).
3.2 Numerical Results

In our study of the $SU(2)$ LGT we used the standard Wilson action defined in Eq. (2.11), i.e.

$$S(U) = \beta \sum_P (1 - \frac{1}{2} \text{Tr} U_P),$$  \hspace{1cm} (3.4)

where $\beta = 4/g^2$ with $g$ to be the lattice coupling constant, and $U_P$ is the plaquette variable defined in Eq. (2.10).

To study the finite temperature phase transition and the physics in finite volumes at zero temperature (which will be explained in detail in next Chapter), we follow ref. [19] to choose our lattices of the geometry $N_t \cdot N_s^2 \cdot N_z$, with $N_t < N_s \ll N_z$. In our measurements we choose $N_t = 4$ and 6, $N_s = 5, 7, 9, 11$, and $N_z = 65$ in most cases, $N_z = 37$ in a few cases with $N_t = 6$. The lattice coupling constant $\beta$ is in the range $2.25 \leq \beta \leq 2.40$ for $N_t = 4$, and $2.30 \leq \beta \leq 2.50$ for $N_t = 6$. We updated the lattice configurations by using the standard Metropolis algorithm alternated with overrelaxation methods. We typically thermalized the lattice configurations for 4000 sweeps, and made one measurement every 10 sweeps. The total number of measurements for each data set is about 2000. The actual number may vary by a small amount in each case. Table 2 gives an overview of the typical statistics of our measurements, which were performed on the lattice $4 \times 11^2 \times 65$. The label 'ovr' in the table represents the overrelaxation updating, and 'met' means Metropolis updating, 'mhit' means using multihit techniques, $N_{\text{therm}}$ is the number of sweeps dropped for thermalization, $N_{\text{btw}}$ is the number of sweeps between two measurements, $N_{\text{mea.}}$ means the number of total measurements. All calculations were done on LSU's IBM 3090 mainframe. In the following we proceed to discuss our measurements on lattices and present some raw data.
Table 2: The statistics of data on the lattice $4 \cdot 11^2 \cdot 65$.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>mhit</th>
<th>$N_{\text{therm}}$</th>
<th>$N_{\text{btw}}$</th>
<th>$N_{\text{mea.}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.25</td>
<td>$N$</td>
<td>2000 ovr, 2000 met</td>
<td>2 ovr, 2 met</td>
<td>2100</td>
</tr>
<tr>
<td>2.25</td>
<td>$Y$</td>
<td>10000 ovr, 10000 met</td>
<td>5 ovr, 5 met</td>
<td>1800</td>
</tr>
<tr>
<td>2.28</td>
<td>$N$</td>
<td>1800 ovr, 1800 met</td>
<td>2 ovr, 2 met</td>
<td>1900</td>
</tr>
<tr>
<td>2.28</td>
<td>$Y$</td>
<td>5600 ovr, 5600 met</td>
<td>5 ovr, 5 met</td>
<td>2080</td>
</tr>
<tr>
<td>2.29</td>
<td>$N$</td>
<td>2000 ovr, 2000 met</td>
<td>2 ovr, 2 met</td>
<td>2300</td>
</tr>
<tr>
<td>2.29</td>
<td>$Y$</td>
<td>6600 ovr, 6600 met</td>
<td>5 ovr, 5 met</td>
<td>2000</td>
</tr>
<tr>
<td>2.30</td>
<td>$N$</td>
<td>2000 ovr, 2000 met</td>
<td>2 ovr, 2 met</td>
<td>2200</td>
</tr>
<tr>
<td>2.30</td>
<td>$Y$</td>
<td>6000 ovr, 6000 met</td>
<td>5 ovr, 5 met</td>
<td>2000</td>
</tr>
<tr>
<td>2.34</td>
<td>$N$</td>
<td>2000 ovr, 2000 met</td>
<td>2 ovr, 2 met</td>
<td>2300</td>
</tr>
<tr>
<td>2.34</td>
<td>$Y$</td>
<td>6000 ovr, 6000 met</td>
<td>5 ovr, 5 met</td>
<td>2000</td>
</tr>
<tr>
<td>2.36</td>
<td>$N$</td>
<td>2000 ovr, 2000 met</td>
<td>2 ovr, 2 met</td>
<td>2300</td>
</tr>
<tr>
<td>2.36</td>
<td>$Y$</td>
<td>6600 ovr, 6600 met</td>
<td>5 ovr, 5 met</td>
<td>2000</td>
</tr>
<tr>
<td>2.40</td>
<td>$N$</td>
<td>1800 ovr, 1800 met</td>
<td>2 ovr, 2 met</td>
<td>2200</td>
</tr>
<tr>
<td>2.40</td>
<td>$Y$</td>
<td>6400 ovr, 6400 met</td>
<td>5 ovr, 5 met</td>
<td>2000</td>
</tr>
</tbody>
</table>

(1). The Order Parameter $< |P| >$

On each lattice we measured the expectation of the modulus of the Polyakov loop $< |P| >$, which plays the role of the order parameter for the finite temperature phase transition on lattices [13, 23], where $P$ is closed in the time direction, as defined in Eq. (2.27). We shall discuss this further in the next chapter. In Table 3 we show the measured values of $< |P| >$ on lattices $4 \cdot 9^2 \cdot 65$, $4 \cdot 11^2 \cdot 65$, $6 \cdot 7^2 \cdot 65$ and $6 \cdot 11^2 \cdot 37$. Here all data were measured by using multihit techniques.

(2). The Correlation $< P(0)P^\dagger(z) >$ and String Tension $\kappa$

We also measured the correlation function $< P(0)P^\dagger(z) >$, which enables us to study the $q\bar{q}$ potential $V(r)$ at various temperatures. In the
Table 3. The measured values of $< |P| >$ on lattices $4 \cdot 9^2 \cdot 65$, $4 \cdot 11^2 \cdot 65$, $6 \cdot 7^2 \cdot 65$ and $6 \cdot 11^2 \cdot 37$.

<table>
<thead>
<tr>
<th>$N_t = 4$</th>
<th>$4 \cdot 9^2 \cdot 65$</th>
<th>$4 \cdot 11^2 \cdot 65$</th>
<th>$N_t = 6$</th>
<th>$6 \cdot 7^2 \cdot 65$</th>
<th>$6 \cdot 11^2 \cdot 37$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$&lt;</td>
<td>P</td>
<td>&gt;$</td>
<td>$&lt;</td>
<td>P</td>
</tr>
<tr>
<td>2.25</td>
<td>0.02638 (44)</td>
<td>0.02124 (38)</td>
<td>2.30</td>
<td>0.01269 (20)</td>
<td>0.01074 (18)</td>
</tr>
<tr>
<td>2.28</td>
<td>0.04009 (62)</td>
<td>0.03746 (56)</td>
<td>2.36</td>
<td>0.02214 (34)</td>
<td>0.01930 (30)</td>
</tr>
<tr>
<td>2.29</td>
<td>0.04655 (78)</td>
<td>0.04684 (74)</td>
<td>2.40</td>
<td>0.03921 (58)</td>
<td>0.03142 (58)</td>
</tr>
<tr>
<td>2.30</td>
<td>0.04443 (80)</td>
<td>0.04856 (78)</td>
<td>2.42</td>
<td>0.03476 (56)</td>
<td>0.05433 (76)</td>
</tr>
<tr>
<td>2.32</td>
<td>0.08365 (126)</td>
<td>0.17156 (160)</td>
<td>2.45</td>
<td>0.04326 (72)</td>
<td>0.08517 (94)</td>
</tr>
<tr>
<td>2.34</td>
<td>0.23603 (36)</td>
<td>0.23772 (32)</td>
<td>2.50</td>
<td>0.08406 (90)</td>
<td>0.14275 (48)</td>
</tr>
<tr>
<td>2.36</td>
<td>0.27094 (30)</td>
<td>0.27182 (26)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We chose the correlation along the longest extent of the lattice, $N_z$.

As we mentioned in Chap. 1, in the confined phase the potential $V(r)$ of a $q \bar{q}$ pair is linear. The proportionality constant is the string tension $\kappa$, which can be extracted from the correlation, $< P(0)P^t(z) >$, as follows[19],

$$< P(0)P^t(z) >_c = \text{const.} (e^{-L_1\kappa z} + e^{-L_1\kappa(L_z - z)}),$$  \hspace{1cm} (3.5)

where on the L.H.S. of this equation the connected correlation $< P(0)P^t(z) >_c$ is defined as, $< P(0)P^t(z) > - < P >^2$. So the dependence on the coordinates $t, x, y$ can be suppressed, and one can extract the effective string tension. On the R.H.S. the new term $e^{-L_1\kappa(L_z - z)}$ accounts for the finite size effects along the correlation direction $N_z$, where $L_z = N_z a$.

From Eq. (3.5) we can calculate the string tension $\kappa$ from the measured correlations $< P(0)P^t(z) >_c$. In fact, one can obtain a string tension data at each separation $z$, as $z$ is large enough. Table 4 gives an example of the measured data for various separations $z$. To get an estimate of the asymptotic
Table 4. The raw string tension data $\kappa$ extracted from the correlation $< P(0)P^\dagger(z) >_c$ at different distance $z$, they were measured on the lattice $4 \cdot 11^2 \cdot 65$ with $\beta = 2.25$.

<table>
<thead>
<tr>
<th>$z/a$</th>
<th>$\sqrt{\kappa a}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.429 (3)</td>
</tr>
<tr>
<td>4</td>
<td>0.379 (7)</td>
</tr>
<tr>
<td>5</td>
<td>0.347 (12)</td>
</tr>
<tr>
<td>6</td>
<td>0.328 (20)</td>
</tr>
<tr>
<td>7</td>
<td>0.313 (32)</td>
</tr>
<tr>
<td>8</td>
<td>0.307 (48)</td>
</tr>
<tr>
<td>9</td>
<td>0.324 (68)</td>
</tr>
<tr>
<td>10</td>
<td>0.327 (103)</td>
</tr>
</tbody>
</table>

string tension ($z \to \infty$), we need to choose a data at appropriately large $z$, because as $z$ becomes too large the data have very large uncertainties.

We find the measured string tension data are consistent within errors over a “relevant distance” range. Usually, the end of this range can be determined as following[19], if the data at a distance $z$ is larger than its preceding one, i.e. the data at the distance $z - 1$, then $z - 1$ is a natural choice of the last relevant distance. For example, in Table 4 the last relevant distance is $8a$. The reason for such a choice is that the increase of data at large distances is in contradiction with general principles and here reflects that the inaccuracy of the data has become larger than the correction effects we would like to see. We then choose one of the first string tension data (and its error bar) in the “relevant distance” range as estimate of the asymptotic string tension $\kappa (z \to \infty)$. In this range we usually choose the data with relatively small error bar. For instance, from Table 4 we extract the asymptotic string tension to be, $\sqrt{\kappa a} = 0.313 (32)$, on the lattice $4 \cdot 11^2 \cdot 65$ with $\beta = 2.25$. 
**Table 5.** The string tension data $\kappa$ measured in lattice units on lattices of the size $4 \cdot 9^2 \cdot 65$, $4 \cdot 11^2 \cdot 65$, $6 \cdot 7^2 \cdot 65$ and $6 \cdot 11^2 \cdot 37$.

<table>
<thead>
<tr>
<th>$N_t = 4$</th>
<th>$4 \cdot 9^2 \cdot 65$</th>
<th>$4 \cdot 11^2 \cdot 65$</th>
<th>$N_t = 6$</th>
<th>$6 \cdot 7^2 \cdot 65$</th>
<th>$6 \cdot 11^2 \cdot 37$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$\sqrt{\kappa a}$</td>
<td>$\sqrt{\kappa a}$</td>
<td>$\beta$</td>
<td>$\sqrt{\kappa a}$</td>
<td>$\sqrt{\kappa a}$</td>
</tr>
<tr>
<td>2.25</td>
<td>0.301 (38)</td>
<td>0.313 (32)</td>
<td>2.30</td>
<td>0.310 (48)</td>
<td>0.377 (27)</td>
</tr>
<tr>
<td>2.28</td>
<td>0.242 (33)</td>
<td>0.243 (29)</td>
<td>2.36</td>
<td>0.248 (21)</td>
<td>0.284 (13)</td>
</tr>
<tr>
<td>2.29</td>
<td>0.224 (17)</td>
<td>0.181 (33)</td>
<td>2.40</td>
<td>0.172 (30)</td>
<td>0.217 (17)</td>
</tr>
<tr>
<td>2.30</td>
<td>0.216 (19)</td>
<td>0.203 (17)</td>
<td>2.42</td>
<td>0.200 (17)</td>
<td>0.177 (41)</td>
</tr>
</tbody>
</table>

In Table 5 we list the extracted asymptotic string tension data $\kappa$ measured in lattice units on lattices of the size $4 \cdot 9^2 \cdot 65$, $4 \cdot 11^2 \cdot 65$, $6 \cdot 7^2 \cdot 65$ and $6 \cdot 11^2 \cdot 37$. Again the multihit techniques were used.

In this table some string tension data were measured in the transition region, e.g. $\beta = 2.30$ for $(N_t = 4)$ (we shall explain this further in Chapter 4), in this case we assume Eq. (3.5) is still valid. Since our lattices have the geometry, $N_z \gg N_s > N_t$, we find that finite size effects are small as $N_s/N_t \geq 2$ in the limit of $N_z \to \infty$. The detailed analysis of finite size effects is presented in Chapter 5. The results also agree with those of refs.[19, 31]. Therefore, we expect that the data measured on lattices $4 \cdot 9^2 \cdot 65$ and $4 \cdot 11^2 \cdot 65$ can be viewed as the string tension in the large volume limit. However, the data from the lattice $6 \cdot 7^2 \cdot 65$ might have some large finite size effects because $N_s = 7$ is not large enough (for $N_t = 6$). For data measured on the lattice $6 \cdot 11^2 \cdot 37$ the transverse size $N_s = 11$ may be large enough, but some large finite size effects may arise due to the relatively small $N_z$ (i.e. $N_z = 37$). In the next Chapter we will discuss these finite size effects further.
(3). $q\bar{q}$ Flux Distributions

To measure the flux distributions of a $q\bar{q}$ pair, we calculated the quantity\cite{24, 32},

$$f_{\mu\nu}(\vec{r}, \vec{x}) = \frac{\beta}{\alpha^4} \left[ \frac{< P(0) P^\dagger(\vec{r}) \Box_{\mu\nu} >}{< P(0) P^\dagger(\vec{r}) >} - < \Box_{\mu\nu} > \right], \quad (3.6)$$

where the Polyakov loop $P(\vec{r})$ is defined in Eq. (2.27), with $N_t$ chosen to be the time direction. In this case the Polyakov loops $P$ and $P^\dagger$ represent the $q\bar{q}$ sources. The plaquette is, $\Box_{\mu\nu} = \frac{1}{2} Tr(U_P)$, with $U_P$ defined in Eq. (2.10), which plays the role of a 'test charge'. The plaquette has 6 different orientations, $(\mu, \nu) = (2, 3), (1, 3), (1, 2), (1, 4), (2, 4), (3, 4)$.

Using Eq. (2.16) in the continuum limit ($a \to 0$), the plaquette $\Box_{\mu\nu}$ becomes,

$$\Box_{\mu\nu}(\vec{x}, 0) = \frac{1}{2} tr U_P(\vec{x}, 0),$$

$$\approx \frac{1}{2} \left( 2 - \frac{1}{2} a^4 g^2 Tr[F_{\mu\nu}^2] \right),$$

$$\approx \frac{1}{2} \left( 2 - \frac{1}{4} a^4 g^2 \sum_i (F_{\mu\nu}^i)^2 \right), \quad (3.7)$$

where $F_{\mu\nu}^i$ are the color components of the field strength tensor, $F_{\mu\nu}$, defined in Eq. (2.2) with the color index $i = 1, 2, 3$ for $SU(2)$. Therefore, Eq. (3.6) becomes,

$$f_{\mu\nu} \xrightarrow{a \to 0} - \frac{1}{2} \sum_i (F_{\mu\nu}^i)^2 >_{q\bar{q} - vac} \quad (3.8)$$

with $vac$ representing the vacuum. The R.H.S. of this equation is just the energy density of the color field $F_{\mu\nu}$ for the $q\bar{q}$ sources relative to the vacuum.

To decrease statistical noise and save CPU time three enhancements of the measurements were used in our calculations.
The first enhancement is that we measure the quantity \[32\]

\[ f'_{\mu\nu}(\vec{r}, \vec{x}) = \frac{\beta}{a^4} \left[ \frac{\langle P(0)P^t(\vec{r})\square_{\mu\nu}(\vec{x}) \rangle - \langle P(0)P^t(\vec{r})\square_{\mu\nu}(\vec{x}_R) \rangle}{\langle P(0)P^t(\vec{r}) \rangle} \right], \tag{3.9} \]

as the flux distribution instead of Eq. (3.6), where the reference point \(\vec{x}_R\) was chosen to be far from the \(q\bar{q}\) sources. This replacement does not change the measured average due to the cluster decomposition theorem \[32\]. Nevertheless the main fluctuations of the quantity \(PP^t\square\), which are caused by the \(PP^t\) itself, cancel when the difference in Eq. (3.9) is computed configuration by configuration.

The second enhancement is to use the multihit technique \[32\] to improve the statistical accuracy. That is, we make the replacement for Polyakov loops in Eq. (3.9),

\[ P(\vec{r}) \rightarrow \bar{P}(\vec{r}) = \frac{1}{2} Tr \prod_{\tau=1}^{N_t} \bar{U}_t(\vec{r}, \tau), \tag{3.10} \]

where \(\bar{U}_t(\vec{r}, \tau)\) is given by

\[ \bar{U}_t = \frac{\int U_t \exp[\beta tr(U_tX_t^\dagger + h.c.)]dU_t}{\int \exp[\beta tr(U_tX_t^\dagger + h.c.)]dU_t} = X_t^\dagger \frac{I_2(\beta \lambda)}{\lambda I_1(\beta \lambda)}, \tag{3.11} \]

with \(X_t^\dagger\) to be the neighborhood of \(U_t\), i.e. \(U_tX_t^\dagger = \sum \square_p\), the sum extends over all plaquettes containing \(U_t\). And \(\lambda = \sqrt{det(X_t)}\), \(I_1\) and \(I_2\) are the modified Bessel functions.

The third enhancement was to use the vectorized fast Fourier transforms \[33\] in order to compute the convolutions implied by Eq. (3.9). In this way we have reduced the measurement time which was the major fraction of the required CPU time.
The six components of $f^\mu_\nu$ in Eq. (3.9) correspond to the components of the chromo-electric and chromo-magnetic fields $(\vec{E},\vec{B})$ in Minkowski space, i.e.

\[
\begin{align*}
  f_{23} & \to -\frac{1}{2} < \sum_i (B_{1i}^2) > = -\frac{1}{2} < B_1^2 >, \\
  f_{13} & \to -\frac{1}{2} < \sum_i (B_{2i}^2) > = -\frac{1}{2} < B_2^2 >, \\
  f_{12} & \to -\frac{1}{2} < \sum_i (B_{3i}^2) > = -\frac{1}{2} < B_3^2 >, \\
  f_{14} & \to \frac{1}{2} < \sum_i (E_{1i}^2) > = \frac{1}{2} < E_1^2 >, \\
  f_{24} & \to \frac{1}{2} < \sum_i (E_{2i}^2) > = \frac{1}{2} < E_2^2 >, \\
  f_{34} & \to \frac{1}{2} < \sum_i (E_{3i}^2) > = \frac{1}{2} < E_3^2 >. 
\end{align*}
\]

with $i = 1, 2, 3$ for $SU(2)$.

The total electric and magnetic energy densities are defined as

\[
\begin{align*}
  \rho_{el} &= \frac{1}{2} < \mathcal{E}^2 > = \frac{1}{2} [ < E_1^2 > + < E_2^2 > + < E_3^2 > ], \\
  \rho_{ma} &= \frac{1}{2} < \mathcal{B}^2 > = \frac{1}{2} [ < B_1^2 > + < B_2^2 > + < B_3^2 > ].
\end{align*}
\]

In the following we will concentrate on studying the total energy and action densities $\rho_E$ and $\rho_A$, which are the combinations of $\rho_{el}$ and $\rho_{ma}$. The total energy density is

\[
\rho_E = \rho_{el} + \rho_{ma},
\]

The total action density is

\[
\rho_A = \rho_{el} - \rho_{ma}.
\]

We measured the flux distributions $f^\mu_\nu$ on various lattices, which cover the regions of both confined and unconfined phases. In Chapter 5 we will present the analysis our flux data.
CHAPTER 4

FINITE TEMPERATURE PHASE TRANSITION

In this chapter we shall review some properties of the finite temperature phase transition for the $SU(2)$ gauge theory. Then we discuss some results of our data.

4.1 Basic Theory

The finite temperature physics of $SU(N)$ gauge theory can be studied very conveniently on the lattice. On a four-dimensional Euclidean lattice of the size, $N_t \cdot N_s^2 \cdot N_z$, the temperature $T$ is related to the temporal extent of the lattice, as we described in Section 2.4. If one chooses $N_t$ as the time direction, then the temperature can be defined by Eq. (2.47), i.e.

$$ T = \frac{1}{N_t a} $$

(4.1)

where $a$ is the lattice spacing, which is a function of the lattice coupling constant $\beta$.

It is known for pure $SU(N)$ gauge theory that the expectation value of the Polyakov loop $< P >$ plays the role of an order parameter[13] for the finite temperature phase transition. Here the Polyakov loop $P$ is closed in the time direction, as defined in Eq. (2.27). That is, in the infinite volume limit one has
in the confined phase, $T < T_c$, $< P > = 0$;

in the unconfined phase, $T > T_c$, $< P > \neq 0$;

because of the spontaneous breaking of the global $Z_N$ symmetry\[23\], where $T_c$ is the transition temperature. The global $Z_N$ symmetry can be described as follows. Under the transformation of the Polyakov loop,

$$P(\vec{r}) \rightarrow e^{i2\pi k/N} P(\vec{r})$$ \hspace{1cm} (4.2)

the action of the $SU(N)$ gauge theory is invariant. Here $e^{i2\pi k/N}$ is an element of the $Z_N$ group, with $k = 0, 1, \ldots, N - 1$. This symmetry is spontaneously broken if $< P > \neq 0$.

The expectation value $< P >$ is associated with the free energy $F_q$ of an isolated quark $q$ in the infinite volume limit, as we discussed in section 2.2, the relation is given by Eq. (2.28), i.e.

$$| < P > | \sim e^{-L_t F_q}$$ \hspace{1cm} (4.3)

where $L_t = N_t a$, which is the inverse temperature. Further, the correlation function $< P(0) P^\dagger(r) >$ is related to the potential energy $V(r)$ of a $q\bar{q}$ pair, as shown in Eq. (2.29), i.e.

$$< P(0) P^\dagger(r) > = const. exp[-L_t V(r)]$$ \hspace{1cm} (4.4)

In the confined phase, $T < T_c$, one expects Eq. (4.4) becomes as $r \rightarrow \infty[12]$,

$$\lim_{r \rightarrow \infty} < P(0) P^\dagger(r) > = \lim_{r \rightarrow \infty} const. exp[-L_t kr] = < P >^2 = 0$$ \hspace{1cm} (4.5)
because of $< P > = 0$ in the confined phase, where $\kappa$ is the string tension. Eq. (4.5) implies that the $q\bar{q}$ potential in the confined phase is linear, e.g. $V(r) \sim \kappa r$. This causes quark confinement in the confined phase.

In the unconfined phase, $T > T_c$, one has that for $SU(2)$ gauge theory Eq. (4.4) becomes as $r \to \infty$[12],

$$
\lim_{r \to \infty} < P(0)P^\dagger(r) > = \lim_{r \to \infty} \text{const.}[1 + L_t \frac{3}{\beta_4 4\pi r} \exp(-mr)]
\approx < P >^2 \neq 0,
$$

for $< P > \neq 0$ in the unconfined phase. Eq. (4.6) implies that the $q\bar{q}$ potential in the unconfined phase is a screened Coulomb potential, e.g. $V(r) \sim \frac{e^{-mr}}{r}$ with $m^{-1}$ the Debye screening length. To the lowest order of perturbation theory one has[12]

$$
m^2 = \frac{2}{3} g^2 T^2.
$$

Ref.[34] gives a detailed discussion about the $q\bar{q}$ potential in the unconfined phase. We shall not discuss it in the following.

The transition temperature $T_c$ can be determined by Monte Carlo calculations. For $SU(2)$ one recent result is[35, 36],

$$
\beta_c = 2.2985 \pm 0.0006. \quad \text{(for } N_t = 4)\quad (4.8)
$$

This transition point $\beta_c$ was calculated on large lattices (e.g. $4 \times 26^3$), on which finite size effects are small, the result can be considered to be the transition point in the infinite volume limit.

By using the scaling relation in Eqs. (2.33) and (2.34) one can calculate the transition temperature $T_c$ in physical units from Eqs. (4.1) and (4.8).
The result is,

\[ T_c \Lambda L^{-1} = \frac{1}{N_t f(\beta_c)}, \]
\[ = 42.11 \pm 0.06. \]  \hspace{1cm} (4.9)

This is consistent with the result of ref.[19]. If we substitute the value of \( \Lambda L^{-1}(\beta_c) \) in Eq. (2.34), we can obtain the transition temperature \( T_c \) in physical units,

\[ T_c = 1.487 \pm 0.140 \ (1/\text{fm}), \]  \hspace{1cm} (4.10)

which corresponds to \( T_c = 293 \pm 28 \ Mev \). This result was calculated in the static quark approximation. If we consider the effects of dynamical quarks, this result will change[10].

From the scaling relation \( a(\beta) \) of Eqs. (2.33) and (2.34) one can also estimate the transition point \( \beta_c \) on lattices of any temporal size \( N_t \) for \( SU(2) \) gauge theory, if we assume the transition temperature \( T_c \) in Eq. (4.9) is independent of the lattice size, for example, the estimated value of \( \beta_c \) for \( N_t = 6 \) and 8 are

\[ \beta_c \simeq 2.42 \hspace{1cm} \text{(for} \ N_t = 6), \]
\[ \beta_c \simeq 2.50 \hspace{1cm} \text{(for} \ N_t = 8). \]  \hspace{1cm} (4.11)

In the following we proceed to discuss some results in our study of the finite temperature phase transition on lattices.
4.2 Data Analysis

(1). Results of The Order Parameter \(< |P| >\)

As we discussed above, the expectation \(< P >\) plays the role of an order parameter in the infinite volume limit. However, on a finite lattice this quantity is always zero if the computation time is taken to be infinite, because the system would flip between two ordered states, and the values of the Polyakov loop \(P\) would flip sign after some iterations. Therefore, one chooses the expectation value of the modulus of the Polyakov loop \(< |P| >\) as the "order parameter" on finite lattices[13, 23].

The measured data of \(< |P| >\) are listed in Table 3. We then plotted \(< |P| >\) versus \(\beta\) in Fig. 6, which shows that there is a rapid increase of \(< |P| >\) at \(\beta \approx 2.30\) for \(N_t = 4\), and another rapid increase at \(\beta \approx 2.42\) for \(N_t = 6\). This implies that a phase transition occurs at \(\beta_c \approx 2.30\) for \(N_t = 4\), or \(\beta_c \approx 2.42\) for \(N_t = 6\), in the infinite volume limit. These results agree with Eqs. (4.8) and (4.11). For cases of \(N_t = 4\) the data measured on the lattice \(4 \cdot 9^2 \cdot 65\) agree very well with those from \(4 \cdot 11^2 \cdot 65\), both approach the infinite volume limit. However, for cases of \(N_t = 6\) the data from the lattice \(6 \cdot 7^2 \cdot 65\) have some discrepancies with the data from \(6 \cdot 11^2 \cdot 37\). This may be due to finite-volume effects, as discussed in ref. [35], because in these cases \(N_s\) (or \(N_z\)) are not large enough to approach the infinite volume limit.

(2). Results of The Correlation \(< P(0)P^\dagger(z) >\)

Since the correlation \(< P(0)P^\dagger(z) >\) is related to the \(q\bar{q}\) potential energy \(V(r)\), as shown in Eq. (4.4), we then measured the correlation on each
Figure 6. Monte Carlo data for $\langle |P| \rangle$ vs. $\beta$, calculated from lattices of size $4 \cdot 9^2 \cdot 65$ (circles), $4 \cdot 11^2 \cdot 65$ (squares), $6 \cdot 7^2 \cdot 65$ (triangles) and $6 \cdot 11^2 \cdot 37$ (diamonds).
lattice. Some typical results are shown in Figs. 7 and 8. Fig. 7 shows $< P(0)P^\dagger(r) >$ versus $r$ in the confined phase, Fig. 7 (a) shows the results in a linear plot. Fig. 7 (b) was drawn as a logarithmic plot. Fig. 8 is similar to Fig. 7, but in the unconfined phase. From Fig. 7 one can see that the correlation $< P(0)P^\dagger(r) > \to 0$ as $r \to \infty$, and $\ln < P(0)P^\dagger(r) >$ is linear with $r$. This implies that the $q\bar{q}$ potential $V(r)$ is linear in the confined phase, as predicted by Eq. (4.5). The slope of the straight lines in Fig. 7 (b) corresponds to the string tension $\kappa$, which shows that the string tension decreases with $\beta$ (or temperature). Fig. 8 shows that in the unconfined phase the correlation $< P(0)P^\dagger(r) > \not\to 0$ as $r \to \infty$, which is consistent with the prediction of Eq. (4.6).

(3). Behaviour of $\kappa$ with Temperature

The measured string tension data are listed in Table 5, which are in lattice units. We can use the scaling relation of of Eqs. (2.33) and (2.34) to transform the string tension data from lattice units to physical units. The results are shown in Table 6. From this table one can see that the string tension $\kappa$ decreases with the temperature $T$ (or $\beta$) in the confined phase. If we assume that $\kappa$ is a continuous function of $T$ (or $\beta$), one has

$$\frac{\partial \kappa}{\partial T} < 0 \quad \text{for } T < T_c,$$

$$\frac{\partial \kappa}{\partial \beta} < 0 \quad \text{for } \beta < \beta_c.$$  \hspace{1cm} (4.12)

This behavior is also confirmed by some recent works [37, 38]. In the following we proceed to study this behaviour of $\kappa$ in further detail.
Figure 7. Monte Carlo data of $\langle P(0)P(r) \rangle$ vs. $r/a$ in the confined phase, calculated on the lattice of size $4 \cdot 11^2 \cdot 65$ with $\beta = 2.25$ (squares), $\beta = 2.28$ (triangles) and $\beta = 2.29$ (diamonds). (a) linear axes, (b) logarithmic y axes and linear x axes.
Figure 8. Monte Carlo data of $< P(0)P(r) >$ vs. $r$ in the unconfined phase, calculated on the lattice $4 \cdot 9^2 \cdot 65$ with $\beta = 2.36$ (squares) and $\beta = 2.40$ (triangles). (a) linear axes, (b) logarithmic y axes and linear x axes.
Table 6. The string tension data $\kappa$ measured in physical units on lattices of the size $4 \cdot 9^2 \cdot 65$, $4 \cdot 11^2 \cdot 65$, $6 \cdot 7^2 \cdot 65$ and $6 \cdot 11^2 \cdot 37$.

<table>
<thead>
<tr>
<th>$N_t = 4$</th>
<th>$4 \cdot 9^2 \cdot 65$</th>
<th>$4 \cdot 11^2 \cdot 65$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$T\Lambda_L^{-1}$</td>
<td>$T$ (1/fm)</td>
</tr>
<tr>
<td>2.25</td>
<td>37.29</td>
<td>1.271</td>
</tr>
<tr>
<td>2.28</td>
<td>40.20</td>
<td>1.401</td>
</tr>
<tr>
<td>2.29</td>
<td>41.22</td>
<td>1.447</td>
</tr>
<tr>
<td>2.30</td>
<td>42.27</td>
<td>1.494</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$N_t = 6$</th>
<th>$6 \cdot 7^2 \cdot 65$</th>
<th>$6 \cdot 11^2 \cdot 37$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$T\Lambda_L^{-1}$</td>
<td>$T$ (1/fm)</td>
</tr>
<tr>
<td>2.30</td>
<td>28.18</td>
<td>0.996</td>
</tr>
<tr>
<td>2.36</td>
<td>32.76</td>
<td>1.213</td>
</tr>
<tr>
<td>2.40</td>
<td>36.23</td>
<td>1.386</td>
</tr>
<tr>
<td>2.42</td>
<td>38.10</td>
<td>1.482</td>
</tr>
</tbody>
</table>

Since the string tension $\kappa$ decreases with the temperature $T$ in the confined phase and vanishes in the unconfined phase, we expect that in the confined phase $\kappa$ approach zero as $T \to T_c$. For simplicity, we then use the simple function with power behavior to fit the string tension near $T_c$, that is,

$$\kappa(T) = \kappa_0 (1 - \frac{T}{T_c})^\alpha \quad (\text{for } T < T_c).$$

(4.13)

This is a simplified form of the fitting function used in ref. [38]. Here the constant $\kappa_0$ corresponds to the string tension $\kappa$ in the limit of zero temperature and infinite volumes, which is given by Eq. (2.32), i.e. $\kappa_0 \approx 0.981 \text{ Gev/fm}$. The transition temperature $T_c$ is given by Eq. (4.10) in physical units.

As we discussed in section 3.2, the string tension data measured on lattices $6 \cdot 7^2 \cdot 65$ and $6 \cdot 11^2 \cdot 37$ may have large finite size effects. By viewing the data in Table 6 we find that the string tension data from the lattice $6 \cdot 7^2 \cdot 65$ are consistent with the data from lattices $4 \cdot 9^2 \cdot 65$ and $4 \cdot 11^2 \cdot 65$, on which finite size effects can be neglected[39]. However, data from the lattice $6 \cdot 11^2 \cdot 37$ have
large discrepancies from the above data. This shows that finite size effects on string tension data are sensitive to the longitudinal size $N_z$, but not as much to the transverse size $N_s$.

We then use the string tension data from lattices of small finite size effects (i.e. $4 \cdot 11^2 \cdot 65$ and $6 \cdot 7^2 \cdot 65$), to fit the function in Eq. (4.13). The only fitting parameter is the exponential index $\alpha$, the best fit result is,

$$\alpha = 0.35 \pm 0.04.$$  \hfill (4.14)

Fig. 9 shows this fitting function and the measured string tension data of small finite size effects. In this figure one can see that a few data point very close to the transition point $T_c$ have large discrepancies from the fitting function, because fluctuations are large in the transition region. We then excluded these data at, $T \approx 1.48 \ (1/\text{fm})$, from our fitting process. Also all other data above $T_c$ were excluded because they would correspond to complex values for the fitting function. Below the transition region our data agree very well with the conjectured behavior in Eq. (4.13).

To get the relation between $\kappa$ and $\beta$ with fixed $N_t$, we also fit the string tension data with the following function similar to Eq. (4.13),

$$\kappa(\beta) = \kappa_0 (1 - \frac{\beta}{\beta_c})^\delta \quad \text{(for $\beta < \beta_c$).} \quad \hfill (4.15)$$

We then choose the transition point $\beta_c = 2.2985 \pm 0.0006$ for $N_t = 4$ from Eq. (4.8), and $\beta_c = 2.42 \pm 0.01$ for $N_t = 6$ from Eq. (4.11). We find that the best fit for the exponential index $\delta$ in Eq. (4.15) is

$$\delta = 0.22 \pm 0.03 \quad \text{(for $N_t = 4$),}$$

$$\delta = 0.14 \pm 0.04 \quad \text{(for $N_t = 6$).} \quad \hfill (4.16)$$
Figure 9. The plot of $\kappa$ vs. $T$ near the transition point $T_c$. The data were calculated on lattices, $4 \times 9^2 \times 65$ (circles), $4 \times 11^2 \times 65$ (squares) and $6 \times 7^2 \times 65$ (triangles). The solid line is the fitting function in Eq. (4.12) with $\alpha = 0.35 \pm 0.04$. The string tension $\kappa$ is in the physical unit GeV/fm.
In Fig. 10 we plot the fitting function of Eq. (4.15) and our measured string tension data for \( N_t = 4 \). In this figure the two data measured at \( \beta = 2.30 \) were excluded from fitting, which are just above the transition point, \( \beta_c = 2.2985 \). Again the data agree well with the fitting function. In Chapter 6 we shall use these relations to study Michael sum rules.
Figure 10. $\kappa$ vs. $\beta$ for $N_t = 4$, the transition point $\beta_c \sim 2.2985$. The data were calculated on lattices, $4 \times 9^2 \times 65$ (squares), $4 \times 11^2 \times 65$ (triangles). The solid line is the fitting function in Eq. (4.14) with $\delta = 0.22 \pm 0.03$. The string tension $\kappa$ is in the physical unit $\text{Gev/fm}$. 
CHAPTER 5

FLUX DISTRIBUTIONS

5.1 Motivations

As we discussed in Chapter 1, there are two complementary ways to obtain approximate solutions of strong coupling QCD. One is to take 3-space to be a finite volume torus and obtain semi-analytic solutions[14, 15, 17] of an effective Hamiltonian. The second is via lattice simulations. Confinement can be studied by each method in limited domains. In small physical volumes the fields are very rigid and the problem can be treated using a variational method applied to a small number of dynamical variables. In this domain the string tension is found to be zero. This result is consistent with asymptotic freedom since only short wavelength modes occur. At intermediate volumes there is a clear signal for string tension and further that it is a consequence of a tunneling amplitude between the vacua that are degenerate for small volumes. In this domain, lattice methods are also accessible and are in good agreement for quantities such as glueball masses and string tension. Lattice calculations can take over to study larger volumes where semi-analytic methods become prohibitive.

The existence of string tension in finite volumes does not imply confinement. Clearly if the volume is not large enough to allow the fields to spread out, the finite box itself may be responsible for the linearly rising potential energy between quarks. Global studies have left open the question of the volume
at which intrinsic confinement takes over [19]. In this Chapter we look at a local quantity, the flux tube profile between static quarks as a function of physical volume in SU(2) lattice gauge theory in order to elucidate this question.

M. Lüscher studied QCD in a small box, $z_g \leq 1$, with periodic boundary conditions, where the parameter $z_g$ is defined in Eq. (1.4). He derived a low-energy effective Hamiltonian for SU(N) gauge theory in small volumes [14] (i.e. $z_g \leq 1$). Subsequently the lowest energy levels of SU(2) [15] and SU(3) [16] gauge theories in small volumes were computed by using this Hamiltonian. van Baal and Koller then found that the crucial tunneling between degenerate vacua can be obtained by imposing appropriate nonperturbative boundary conditions on the Raleigh-Ritz trial wave functions [17]. They extended the calculation of the SU(2) glueball masses up to $z_g \approx 5.0$.

Berg and Billoire [19] carried out a thorough study of glueball masses, electric flux states, and string tension for intermediate volumes (i.e. $1 \leq z_g \leq 5$) on lattices. They provided a detailed comparison between their numerical results and the analytic results of van Baal and Koller. They chose lattice sizes $N_t \times N_s^2 \times N_z$, with

$$N_t \leq N_s \ll N_z. \quad (5.1)$$

By identifying $N_z$ to be the time extent, one can simulate the zero temperature finite volume, $N_t N_s^2 a^3$, field theory. As we discussed in Sec. 2.4, in this case the temperature can be defined as

$$T_B = \frac{1}{N_s a}, \quad (5.2)$$

where $T_B$ is called box temperature in ref. [19].

Physically, the choice of the time direction is related to interpreting Polyakov loop correlations as the $q\bar{q}$ potential; then the time direction is the
one in which the Polyakov loop closes. In our study we follow Berg and Billoire to choose Polyakov loops closed in the $N_t$ direction and their correlations measured along the $N_z$ direction. Then the physical temperature is defined to be

$$T_p = \frac{1}{N_t a}.$$  

where $T_p$ is same as the definition of $T$ in Eqs. (2.47) and (4.1). In the following we shall keep using the physical temperature ($T_p$) interpretation, so we shall drop the subscript $p$, i.e. $T = T_p$. In the last section of this chapter we shall relate this to the box temperature ($T_B$) interpretation.

After a complete study in the intermediate volume region ($1 < z_g < 5$), Berg and Billoire concluded that they did not find evidence for string formation in this region, but it is expected to occur in larger volumes. The question is how to understand the string tension measured in intermediate volumes. Is the string tension just due to finite size effects? Since the string tension was calculated from correlations of Polyakov loops closed in the $N_t$ direction, this can be considered as measuring the potential energy of a $q\bar{q}$ pair in a long rectangular box, with the volume $V = N_s^2 N_z a^3$ and at the finite temperature $T = 1/N_t a$, as shown in Fig. 11. As we know, at high temperatures, $T > T_c$, where $T_c$ is the deconfining temperature, the $q\bar{q}$ is unconfined in the infinite volume limit ($N_s, N_z \to \infty$). However, as the transverse size $N_s$ is made small, i.e. $r/N_s a \geq 1$, one expects the side walls of the box would squeeze the flux lines of the $q\bar{q}$ pair to form a tube although there is no intrinsic string formation in this case, as displayed in Fig. 11. In the following we will present our studies about the $q\bar{q}$ flux distributions which would support the above description.
Figure 11. The $q\bar{q}$ color sources in a long rectangular box. The flux lines between the $q\bar{q}$ pair are confined by side walls and have a flux tube form.
5.2 Flux Distributions at Finite Temperatures

By calculating the quantity $f'_{\mu\nu}$ in Eq. (3.9) we measured the $q\bar{q}$ flux distributions on lattices $N_t \cdot N_s^2 \cdot N_z$ with $N_t = 4$, $N_s = 5, 7, 9$ and $11$, and $N_c = 65$. We then transformed the flux data from lattice units to physical units by using the scaling relation of Eqs. (2.33) and (2.34). As we discussed in Chapter 4, $SU(2)$ gauge theory has a deconfining phase transition at the critical point, $\beta_c \approx 2.30$ for $N_t = 4$, in the infinite volume limit, i.e. $N_s, N_z \to \infty$. For $\beta < \beta_c$, the system is in the confined phase, otherwise, it is in the unconfined phase.

Since our lattices satisfy, $N_s \gg N_t, N_s$, as $N_s/N_t$ gets large, we expect the system to approach the infinite volume limit and one should see the two phases. However, for $N_s$ small, that is, $N_s$ satisfying the condition,

$$N_s \simeq N_t, \quad \text{and} \quad N_s a/r \leq 1,$$

the $q\bar{q}$ pair is in a finite box and one expects finite volume effects to be large.

In the following we will study the flux distributions with $\beta > \beta_c$ and $\beta < \beta_c$ in various spacial volumes respectively.

(1). Flux Distributions with $\beta > \beta_c$

In this case the $q\bar{q}$ system approaches the unconfined phase as the volume becomes large ($N_s \to \infty$), we expect that there is no string formation. However, for a small volume one expects finite volume effects to be large.

In Fig. 12 we show some typical results of the energy density $\rho_E$ distributions in the region of $\beta > \beta_c$ ($\beta = 2.40$), where $\rho_E$ is defined in Eq. (3.9). They are the flux distributions on the transverse plane midway between the $q\bar{q}$ sources with fixed separation $r = 4a$, and they were measured on lattices of 4
different spatial sizes $N_s=5, 7, 9, 11$. For $N_s$ small (i.e. $N_s=5, 7$) the transverse plane is the whole lattice. For large $N_s$ ($N_s=9, 11$) the data are truncated on the margins. The signal is lost in the noise beyond the region shown.

From Fig. 12 we can compare the flux distributions in boxes of different transverse size $N_s$. One can clearly see for $N_s$ small the flux density $\rho_E$ at the edges of each plane have large values, as shown in Figs. 12 (a) and 12 (b). This implies that finite volume effects are significant when $N_s$ is small because in small volumes flux lines of the $q\bar{q}$ sources would be squeezed dramatically by side walls of the box, so the values of $\rho_E$ at edges are large in this case. As we increase the transverse size $N_s$ finite volume effects become smaller, the flux density $\rho_E$ at edges decrease rapidly to zero, as shown in Figs. 12 (c) and 12 (d).

Here we want to emphasize that the values of $\rho_E$ at the edges of each plane are not due to the reference point in Eq. (3.9). In our flux measurement we choose the reference point $\vec{x}_R$ far from the $q\bar{q}$ sources, and we find the reference value $\langle P(0)P^\dagger(\vec{r})\Box_{\mu\nu}(\vec{x}_R) \rangle$ in Eq. (3.9) is consistent with the product $\langle P(0)P^\dagger(\vec{r}) \rangle <\Box_{\mu\nu}>$ within errors.

In Table 7 we list the typical values of flux densities $\rho_E$, $\rho_A$ at edges of each plane in Fig. 12 and their corresponding errors. From this table one can explicitly see that the values $\rho_E$ and $\rho_A$ at edges decrease rapidly with $N_s$, as we observed from Fig. 12. However, we also notice that even in cases of large $N_s$ (i.e. $N_s=9, 11$) the values of $\rho_A$ shown are non-zero within errors. This may be caused by a number of factors. The edge of the plane is not the boundary of the lattice for cases of $N_s=9, 11$, the flux values may in fact vanish on the boundary of the lattice, but we did not calculate them there for
Figure 12. The energy density $\rho_E$ flux distributions in the region of $\beta > \beta_c$. The flux data were measured on lattices $4 \cdot N_s^2 \cdot 65$ with $\beta = 2.40$ for various spatial sizes, (a) $N_s = 5$, (b) $N_s = 7$, (c) $N_s = 9$, and (d) $N_s = 11$. The data are in the physical unit $Gev/fm^3$. 
Table 7. The typical measured flux density values (errors) of $\rho_E$ and $\rho_A$ in the region of $\beta > \beta_c$ on the edges of the transverse plane midway between the $q\bar{q}$, as shown in Fig. 5.2. The flux data are in the physical unit GeV/fm$^3$.

<table>
<thead>
<tr>
<th>$\beta = 2.40$</th>
<th>$N_s$</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_E$</td>
<td>2.5 (5)</td>
<td>0.86 (24)</td>
<td>0.11 (18)</td>
<td>0.31 (17)</td>
<td></td>
</tr>
<tr>
<td>$\rho_A$</td>
<td>17.0 (6)</td>
<td>2.9 (3)</td>
<td>1.5 (2)</td>
<td>1.0 (2)</td>
<td></td>
</tr>
</tbody>
</table>

By practical reasons. Also $N_s$ is perhaps not large enough (even for $N_s = 11$), so there are still some small finite volume effects. Finally our error bars were only calculated from statistical error, the actual errors may be larger due to systematic errors.

To see the behavior of the flux distributions changing with the $q\bar{q}$ separation $r$, in Fig. 13 we show the $\rho_A$ distribution on the transverse plane for 4 different separations, $r = 3a$, $4a$, $5a$ and $6a$. The flux data were measured on the lattice $4 \cdot 5^2 \cdot 65$ with $\beta = 2.40$. From this figure one can see the peak values of $\rho_A$ decrease rapidly with the increase of $r$. At large $r$ (i.e. $r = 6a$) the peak of $\rho_A$ almost vanishes, the flux density on the plane approaches a uniform distribution.

We then calculated the center slice energy $\sigma_E$ and action $\sigma_A$ from our flux data, which are the energy and action stored in the transverse slice of unit thickness midway between the $q\bar{q}$ pair. The results are shown in Fig. 14. In this figure we plot the behaviors of $\sigma_E$ versus $r$ and $\sigma_A$ versus $r$ respectively for $\beta = 2.36$. This shows that for $N_s$ small $\sigma_E$ and $\sigma_A$ do not decrease to zero as $r$ increases. However, for $N_s$ large (e.g. $N_s = 9, 11$), $\sigma_E$ and $\sigma_A$ decrease rapidly with $r$ and become very small at large $q\bar{q}$ separations (i.e. $r = 6a$).
Figure 13. The action density $\rho_A$ flux distributions in the region $\beta > \beta_c$, with the $q\bar{q}$ separations, (a) $r = 3a$, (b) $r = 4a$, (c) $r = 5a$ and (d) $r = 6a$. The data were measured on lattices $4 \cdot 5^2 \cdot 65$ with $\beta = 2.40$, and in the physical unit $Gev/fm^3$. 
Figure 14. The center slice energy $\sigma_E$ and action $\sigma_A$ versus the $q\bar{q}$ separation $r$ in the region of $\beta > \beta_c$, (a) $\sigma_E$ vs. $r$, (b) $\sigma_A$ vs. $r$. The data were measured on lattices $4 \cdot N_s^2 \cdot 65$ with $\beta = 2.36$ for various spatial sizes, $N_s = 5$ (circles), $N_s = 7$ (squares), $N_s = 9$ (triangles), $N_s = 11$ (diamonds). The data are in the physical unit Gev/fm.
(2). Analysis Of Finite Volume Effects For Cases $\beta > \beta_c$

To see how the finite volume effects influence $q\bar{q}$ flux distributions in a finite box, let us consider an electrostatic charge pair $+e$, $-e$ enclosed in a similar long rectangular box such as that of Fig. 11. The interaction between charges is the Coulomb interaction $V(r) \sim 1/r$. As the charge separation $r$ becomes very large, i.e. $r/N_s a \to \infty$, one can assume that the electric field $\mathcal{E}$ on the middle transverse plane is uniform, and can be written as

$$\mathcal{E} = \frac{\Phi}{(N_s a)^2},$$

where $\Phi$ is electric flux through the transverse plane, which is a constant for the Coulomb interaction, and $(N_s a)$ is the transverse size of the box. So the total electric field energy on the transverse plane is

$$(\sigma_E)_C = \frac{1}{2} \mathcal{E}^2 (N_s a)^2 \sim \frac{1}{(N_s a)^2}.$$  \hspace{1cm} (5.6)

This shows that at large charge separations the center slice energy $(\sigma_E)_C$ decreases with the behavior $(N_s a)^{-2}$ as the transverse size $N_s$ increases, and $(\sigma_E)_C$ vanishes as $N_s \to \infty$, where the label 'C' denotes a Coulomb interaction.

Now let us return to the $q\bar{q}$ pair in the box with $\beta > \beta_c$. As we discussed in the above section, the system approaches the unconfined phase for large $N_s$. In this phase the $q\bar{q}$ interaction is a screened Coulomb interaction $V(r) \sim e^{-mr}/r$ with $m$ the screening mass[12], as we discussed in Sec. 4.1. Then if the $q\bar{q}$ pair is put in a box such as that in Fig. 11, the flux $\Phi$ through the transverse plane is not a constant, but would decrease with $r$ and $N_s$. So we expect that in this case the center slice energy $\sigma_E$ for a $q\bar{q}$ pair would decay faster with $N_s$ than $(\sigma_E)_C$, which has the inverse square behavior, $(\sigma_E)_C \sim (N_s a)^{-2}$, for $r \to \infty$. In Fig. 15 we plot our $\sigma_E$ data versus the transverse size.
$N_s$ for large $q\bar{q}$ separation $r$. Our data are compared with the Coulomb behavior $(N_s a)^{-2}$ and for reference purposes the inverse quartic behavior $(N_s a)^{-4}$. The data were measured on lattices $4 \cdot N_s^2 \cdot 65$ with $N_s = 5, 7, 9, 11$ and $\beta = 2.40$.

From Fig. 15 one can see that for large $r$ our $\sigma_E$ data appear to decay faster with $N_s$ than the Coulomb behavior $(N_s a)^{-2}$ as expected. This shows us that the $q\bar{q}$ interaction in the unconfined phase at least contains a term that decays faster than the Coulomb interaction, such as the screened Coulomb interaction, although the data are not good enough to determine the screening mass.

In conclusion, our flux data in the region of $\beta > \beta_c$ show that in the unconfined phase there is no string formation. For a small transverse size $N_s$, the finite volume effects are large, the flux lines between a $q\bar{q}$ would be squeezed by side walls of the box significantly. This would result in a finite string tension.

(3). Flux Distributions with $\beta < \beta_c$

For $\beta < \beta_c$ the $q\bar{q}$ system approaches the confined phase as the volume becomes large ($N_s \to \infty$). We expect string formation would occur in this phase. To see this we need to study behaviors of the flux distributions as a function of the $q\bar{q}$ separation $r$.

In Fig. 16 we show the action density $\rho_A$ distribution changing with the $q\bar{q}$ separation $r$. The flux data were measured on the lattice $4 \cdot 11^2 \cdot 65$ with $\beta = 2.25$. Since $N_s$ is large ($N_s=11$) we deleted the margins of the transverse cross section of the lattice, as we did in Fig. 12. This figure should be compared with Fig. 13, which is for $\beta > \beta_c$. One can see Fig. 16 shows
Figure 15. The plot of $\sigma_F$ in the region of $\beta > \beta_c$ versus the transverse size $N_s$ at large $q\bar{q}$ separation $r = 6a$. The solid line is the Coulomb behavior $(N_s a)^{-2}$, the dashed line is the inverse quartic behavior $(N_s a)^{-4}$, both are normalized to the data at $N_s = 5$. 
significantly different behavior from Fig. 13. In Fig. 16 the peak values of \( \rho_A \) on the plane approach a finite value as \( r \) becomes large. Even at large \( r \) (i.e. \( r = 6a \)) this peak still exists. However, in Fig. 13 the peak of the \( \rho_A \) distribution almost disappears at \( r = 6a \). So the flux distribution in Fig. 16 implies that intrinsic string formation occurs in the region of \( \beta < \beta_c \). This string formation is not due to finite volume effects because these effects are small at large volumes (i.e. \( N_b = 9, 11 \)), as we discussed in Fig. 12 and Table 7.

We also calculated the center slice energy and action \( \sigma_E, \sigma_A \) from our flux data in the region \( \beta < \beta_c \). If string formation occurs in the confined phase, for \( N_s \) large both \( \sigma_E \) and \( \sigma_A \) should approach some finite non-zero constants when \( r \to \infty \). In Fig. 17 we plot the behaviors of \( \sigma_E \) versus \( r \) and \( \sigma_A \) versus \( r \) respectively for \( \beta = 2.28 \). The data were measured on lattices of various spatial sizes \( N_b = 5, 7, 9, 11 \). From this figure one can see that in all cases \( \sigma_E \) and \( \sigma_A \) do not decrease with \( r \). For each fixed \( N_s \) the values of \( \sigma_E \) and \( \sigma_A \) are almost constant as \( r \) increases. Further, for large \( N_s \) (\( N_s = 9, 11 \)), where finite volume effects are small, both \( \sigma_E \) and \( \sigma_A \) keep as finite non-zero constants as \( r \) becomes large. This behavior is totally different from that of Fig. 14, which is in the region of \( \beta > \beta_c \). So Fig. 17 also implies that string formation occurs. In this figure we also notice that fluctuations of the data are large compared to the unconfined data. This is a typical behavior because confinement corresponds to disorder.

In conclusion, our flux data in the region of \( \beta < \beta_c \) provide evidence for intrinsic string formation in the confined phase. This string formation is not due to finite volume effects because these effects are small as the volume becomes large (i.e. \( N_s = 9, 11 \)).
Figure 16. The action density $\rho_A$ flux distributions in the region of $\beta < \beta_c$ for various separations (a) $r = 3a$, (b) $r = 4a$, (c) $r = 5a$ and (d) $r = 6a$. The flux data were measured on lattices $4 \cdot 11^2 \cdot 65$ with $\beta = 2.25$, and measured in the physical unit Gev/fm$^3$. 
Figure 17. The center slice energy $\sigma_E$ and action $\sigma_A$ versus the $q\bar{q}$ separation $r$ in the region of $\beta < \beta_c$, (a) $\sigma_E$ vs. $r$, (b) $\sigma_A$ vs. $r$. The data were measured on lattices $4 \cdot N_s^2 \cdot 65$ with $\beta = 2.28$ for various spatial size, $N_s = 5$ (circles), $N_s = 7$ (squares), $N_s = 9$ (triangles), $N_s = 11$ (diamonds); and are in the physical unit $\text{Gev/fm}$. 
5.3 Implementation to Finite Volumes

As we discussed in section 5.1, we have two ways to interpret the LGT calculations. One way is to identify the shortest extent $N_t$ of lattices to be the temporal size. This is a convenient way to study the LGT system. In previous sections we have discussed our flux data in this way. However, to compare with analytical results in finite volumes at zero temperatures, one can use another way to interpret LGT results, that is, the longest extent $N_z$ of lattices is chosen as the time direction, so that the temperature is as low as possible. In this case Polyakov loops closed in the $N_t$ (or $N_s$) direction are no longer viewed as quark sources, they are considered to be spatial operators. Quantities such as, the glueball mass and string tension, can be calculated from Polyakov loop correlations along the time direction ($N_z$). No matter which way we choose, quantities calculated in LGT are the same, the mathematics of the two ways are equivalent. In this section we shall look at the system in terms of this second interpretation, and reinterpret the results of previous sections.

If we choose $N_z$ as the time direction, we are studying $SU(2)$ gauge theory in the volume $U = N_t \cdot N_z^2 \cdot a^3$ at near zero temperature $T_B = 1/N_z a$ because $N_z$ is large. According to 't Hooft[40] the color field flux in a finite volume is quantized. For a unit of flux along the $N_t$ direction in the volume $U$ the flux energy can be written as

$$E_1 = \kappa N_t a. \tag{5.7}$$

where the string tension $\kappa$ can also be measured from Eq. (3.5) in LGT[19]. For convenience, instead of $z_g = m(0^+)N_s a$, as defined in Eq. (1.4) we shall use another parameter, $z_\kappa = \sqrt{\kappa L} = \sqrt{\kappa N_z a}$, to characterize the physical size of the volume, where $\kappa$ corresponds to $\kappa_t$ of ref.[19]. J. Koller and P. van Baal
used $z_g$ in their analytical calculations[17], so that they could easily compare with the Monte Carlo results of LGT. B.A. Berg and A.H. Billoire used both $z_g$ and $z_\kappa$ parameters and other $z$ parameters in analyses of their Monte Carlo data for convenience[19]. The parameter $z_\kappa$ is equivalent to $z_g$. For example, from the data of ref.[19] one can see that $z_g \approx 1$ corresponds to $z_\kappa \approx 0.24$, and $z_g \approx 5$ corresponds to $z_\kappa \approx 1.3$. In Table 8 we show the correspondence of $z_g$ and $z_\kappa$.

(1). Results in Intermediate Volumes ($0.24 < z_\kappa < 1.3$)

In this part we will show that our lattices with $\beta > \beta_c$ belong to the intermediate volume region. Since in this region our string tension data have large error bars, we just simply use the string tension data of ref.[19] to show that our lattices with $\beta > \beta_c$ satisfy $0.24 < z_\kappa < 1.3$. The data of ref.[19] are given in Table 9, which have high statistical accuracies.

From Table 9 we can see the values of string tension $\sqrt{\kappa a}$ and the parameter $z_\kappa$ decrease with the increase of $N_s$ and $\beta$. Our flux data were measured on lattices of the size $4 \cdot N_s^2 \cdot 65$ with $N_s = 5, 7, 9, 11$ and $\beta = 2.36$ and 2.40, which are similar to the lattices in Table 9. We can estimate our string tension and values of $z_\kappa$ from Table 9. For example, the estimated string tension for the lattice $4 \cdot 5^2 \cdot 65$ with $\beta = 2.36$ could be interpolated from the string
Table 9. The values of the string tension data and \( z_\kappa \) on lattices of size \( 4 \cdot N_s^2 \cdot 64 \) with \( N_s = 4, 6, 8 \) and \( \beta = 2.36, 2.38 \) and 2.41. The data are quoted from ref.[18].

\[
\begin{array}{|c|c|c|c|}
\hline
\beta & 4 \cdot N_s^2 \cdot 64 & \sqrt{\kappa a} & z_\kappa = N_s \sqrt{\kappa a} \\
\hline
2.36 & 4 & 0.2475 (18) & 0.99 (1) \\
& 6 & 0.1593 (39) & 0.96 (2) \\
& 8 & 0.1091 (39) & 0.87 (3) \\
2.38 & 4 & 0.2424 (23) & 0.97 (1) \\
& 6 & 0.1357 (44) & 0.81 (4) \\
& 8 & 0.0732 (81) & 0.59 (6) \\
2.41 & 4 & 0.2264 (14) & 0.91 (1) \\
& 6 & 0.1314 (51) & 0.79 (3) \\
\hline
\end{array}
\]

tension measured on lattices \( 4 \cdot N_s^2 \cdot 64 \) with \( N_s = 4, 6 \) and \( \beta = 2.36 \). From Table 9 one can see that for \( N_s = 5 \) the string tension satisfies, \( 0.1593 < \sqrt{\kappa a} < 0.2475 \), and \( 0.80 < z_\kappa = \sqrt{\kappa a} N_s < 1.24 \), which is in the intermediate volume region, \( 0.24 < z_\kappa < 1.3 \), as shown in Table 8. By comparing our lattices with Table 9 in this way, we find that our lattices with \( \beta = 2.36 \) and 2.40 for \( N_s = 5, 7, 9, 11 \) are all in the intermediate volume region, with the “box temperature” approaching zero, i.e. \( T_B \to 0 \).

As we discussed in section 5.2, for \( \beta > \beta_c \) there is no intrinsic string formation. However, since finite volume effects are large when the transverse size \( N_s \) is small, this results in the observed string tension in these cases. As \( N_s \) becomes larger, finite volume effects becomes smaller, the string tension becomes smaller. This is confirmed by Table 9. Since the lattices with \( \beta > \beta_c \) in our study all belong to the intermediate volume region, in these cases we find there is no intrinsic string formation, and the string tension is due to finite volume effects. In general, we expect that the results apply to the whole intermediate volume region in the zero temperature limit \( (T_B \to 0) \).
Table 10. Our string tension data $\kappa$ and the values of the $z_\kappa$ parameter on lattices of the size $4 \cdot N_s^2 \cdot 65$ with $N_s = 5, 7, 9, 11$ and $\beta = 2.25$ and 2.28.

<table>
<thead>
<tr>
<th>$N_s$</th>
<th>$\beta$</th>
<th>$\sqrt{\kappa a}$</th>
<th>$z_\kappa = N_s \sqrt{\kappa a}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2.25</td>
<td>0.300 (22)</td>
<td>1.5 (1)</td>
</tr>
<tr>
<td></td>
<td>2.28</td>
<td>0.251 (34)</td>
<td>1.3 (2)</td>
</tr>
<tr>
<td>7</td>
<td>2.25</td>
<td>0.288 (31)</td>
<td>2.2 (2)</td>
</tr>
<tr>
<td></td>
<td>2.28</td>
<td>0.258 (31)</td>
<td>1.8 (2)</td>
</tr>
<tr>
<td>9</td>
<td>2.25</td>
<td>0.262 (84)</td>
<td>2.4 (8)</td>
</tr>
<tr>
<td></td>
<td>2.28</td>
<td>0.226 (67)</td>
<td>2.0 (6)</td>
</tr>
<tr>
<td>11</td>
<td>2.25</td>
<td>0.307 (48)</td>
<td>3.4 (5)</td>
</tr>
<tr>
<td></td>
<td>2.28</td>
<td>0.232 (64)</td>
<td>2.6 (7)</td>
</tr>
</tbody>
</table>

(2). Results in Large Volumes ($z_\kappa > 1.3$)

For $\beta < \beta_c$ we can easily extract the string tension, which is given in Table 10. In this table we also show the values of $z_\kappa$ for lattices with $\beta < \beta_c$. One can see that most lattices listed in Table 10 satisfy $z_\kappa > 1.3$, with one case in the critical region, $z_\kappa \approx 1.3$ for $\beta = 2.28$ and $N_s = 5$. From Table 8 we know that lattices satisfying the condition, $z_\kappa > 1.3$, belong to the large volume region in the zero temperature limit ($T_B \to 0$). So our lattices with $\beta < \beta_c$, as listed in Table 10, are in this region.

As we discussed in section 5.2, a system with $\beta < \beta_c$ is in the confined phase in the infinite volume limit. In this case we have shown that intrinsic string formation occurs. As the transverse size $N_s$ is made small (e.g. $N_s = 5$), we also observed some finite volume effects, as shown in Fig. 16; however, it does not change the nature of $q\bar{q}$ confinement. We have shown that the lattices with $\beta < \beta_c$ in our study all belong to the large volume region in the zero temperature limit ($T_B \to 0$). From our study we have found evidence
for intrinsic string formation in these cases, which is not due to finite volume effects. We expect that the intrinsic string formation occurs in the whole large volume region.
As we discussed in last chapter, we measured the $q\bar{q}$ flux distributions at various temperatures. The flux energies are related to the potential energy $V(r)$ of the $q\bar{q}$ system by Michael sum rules\cite{20}. To make our discussions clear, we shall briefly review the derivation of these sum rules and present our supplements and generalizations. Then we compare our flux data with the predictions from these sum rules.

6.1 Review and Generalizations of Michael Sum Rules

For a static $q\bar{q}$ system at the spatial separation $r$ Michael sum rules states that, in the limit of zero temperatures, infinite volumes and the lattice spacing $a \to 0$, the electric and magnetic flux energies of a $q\bar{q}$ system in Minkowski space are\cite{20},

$$E_{el}(r) = \sum_o \frac{1}{2} a^2 < \mathcal{E}^2 > = -\frac{1}{2} V(r)\left[\frac{\partial \ln a}{\partial \ln \beta} - 1\right] - \frac{1}{2a} [\beta \frac{\partial f}{\partial \beta} - f],$$  \hspace{1cm} (6.1)

$$E_{ma}(r) = \sum_o \frac{1}{2} a^2 < \mathcal{B}^2 > = \frac{1}{2} V(r)\left[\frac{\partial \ln a}{\partial \ln \beta} + 1\right] + \frac{1}{2a} [\beta \frac{\partial f}{\partial \beta} + f].$$  \hspace{1cm} (6.2)

where the sums extends over the whole space around the $q\bar{q}$, $V(r)$ is the $q\bar{q}$ potential energy, and $f/a$ is the self energy of the $q\bar{q}$ sources. $E_{el}(r)$ and $E_{ma}(r)$ denote the electric and magnetic parts of the flux energy, $< \mathcal{E}^2 >$ and $< \mathcal{B}^2 >$ are given by Eq. (3.13).
(1). The Action Sum Rules

To derive Eqs. (6.1) and (6.2), it is convenient to work in the transfer matrix formalism of LGT. We choose the temporal gauge[22], that is, all temporal links are trivial, $U_t(n) = e^{igA_t(n)} = 1$. One remark about the temporal gauge of a finite lattice is that along a time axis one cannot choose all temporal links $U_t = 1$, because of the restriction that Polyakov loop closed in the time direction is gauge invariant, which can have values other than the trivial one, $P(\vec{r}) = 1$. Therefore, on a finite lattice one can choose all temporal links to be trivial, except one link on each time axis, which can be chosen far from the operators under considerations.

Let's consider a Wilson loop $W$ of the temporal size $na$ and the spatial size $r$, as shown in Fig. 18. The time directed pathes represent the $q\bar{q}$ sources, the space directed pathes $P_r(0)$ and $P_r(n)$ create and annihilate the color field of the $q\bar{q}$ system from the vacuum. In the temporal gauge the expectation of the Wilson loop becomes,

$$< W > = < P_r(0)P_r(n) >,$$

where the expectation is evaluated in the partition function form, as defined in Eq. (2.21), i.e.,

$$< W > = \frac{\int d[U] e^{-\beta S'} W}{\int d[U] e^{-\beta S'}},$$

with $\beta S' = S$, the action given by Eq. (3.4) for $SU(2)$, i.e. $S' = \sum(1 - \Box)$, with $\Box$ defined in Eq. (3.6), the sum is over all plaquettes on the lattice.

Using the transfer matrix approach[22, 41], as we discussed in section 2.4, and assuming a discrete spectrum for the lowest eigenstates of the
Figure 18. The Wilson loop $W$ of the temporal size $na$ and the spatial size $r$. The plaquette $P_1$ is outside the Wilson loop $W$, and the plaquette $P_2$ is inside the Wilson loop $W$. 
transfer matrix, one can evaluate Eq. (6.3) as

\[
< W > = \frac{1}{Z} tr(P_r(0)T_{qq}^n P_r(n)T_{N_t-n}^{N_t})
\]

\[
= \frac{1}{\sum_\alpha \lambda_{N_t}^\alpha} \sum_{\mu,\nu} < \mu | P_r | \nu, r > \lambda_{\nu}^\alpha(r) < \nu, r | P_r | \mu > \lambda_{\mu}^{N_t-n}
\]

\[
= \frac{1}{\sum_\alpha \lambda_{N_t}^\alpha} \sum_{\mu,\nu} d_{\mu\nu} d_{\nu\mu} \lambda_{\nu}^\alpha(r) \lambda_{\mu}^{N_t-n},
\] (6.5)

where \( T_{qq} \) is the transfer matrix projected into the \( q\bar{q} \) sector of the Hilbert space, \( T \) is the transfer matrix without the \( q\bar{q} \). \( |\nu, r> \) and \( |\mu> \) are the eigenstates of \( T_{qq} \) and \( T \) respectively, and \( \lambda_{\nu}(r), \lambda_{\mu} \) are the corresponding eigenvalues, with \( \lambda_{\nu}(r) = e^{-aE_{\nu}(r)} \) and \( \lambda_{\mu} = e^{-aE_{\mu}} \). The coefficient \( d_{\mu\nu} =< \mu | P_r | \nu, r > \). In the limit of \( N_t \to \infty \), the partition function \( Z = \lambda_{N_t}^{N_t} \), and in Eq. (6.5) the dominant contributions correspond to \( \mu = 0 \), that is,

\[
< W > = \sum_{\nu} d_{0\nu}^2 \left( \frac{\lambda_{\nu}(r)}{\lambda_0} \right)^n, \quad \text{(as } N_t \to \infty \text{) (6.6)}
\]

If the temporal size \( n \) of the Wilson loop \( W \) is very large \( (n \to \infty) \), the dominant term in Eq. (5.5) is given by \( \nu = 0 \),

\[
< W > = d_{00}^2 \left( \frac{\lambda_0(r)}{\lambda_0} \right)^n = d_{00}^2 e^{-na(E_0(r)-E_0)}, \quad \text{(as } N_t, n \to \infty \text{) (6.7)}
\]

where the energy of the vacuum \( E_0 \) is usually chosen to be zero, in the following we will take this choice which implies \( \lambda_0 = 1 \).

Now we consider the \( \beta \)-derivative of Eq. (6.7), that is,

\[
\frac{\partial < W >}{\partial \beta} = - < WS' > + < W > < S' >= \frac{\partial}{\partial \beta} [d_{00}^2 e^{-naE_0(r)}],
\] (6.8)

where we have taken \( E_0 = 0 \), and \( S' \) is defined after Eq. (6.4).

Consider a plaquette \( \square(m) \) outside the Wilson loop \( W \) in the time direction (i.e., \( 0 < n < m \)), e.g., the plaquette \( P_1 \) in Fig. 18, where we draw a
plaquette with a time extension. In the limit of infinite large $n$ and infinitesimal lattice spacing, $a \rightarrow 0$, one can neglect the time extension of $P_1$. The contribution of $P_1$ to the L.H.S. of Eq. (6.8) is $< W \square(m) > - < W > < \square(m) >$. As $\square(m)$ is far from $W$ (i.e. $m-n \gg 1$), one has $< W \square(m) > \approx < W > < \square(m) >$, then the contribution vanishes, $< W \square(m) > - < W > < \square(m) > \rightarrow 0$. However, when $\square(m)$ is close to $W$ ($m \approx n$), one expects that this contribution does not vanish. Now let us show this explicitly.

$$< W \square(m) > - < W > < \square(m) > \quad (\text{for } 0 < n < m)$$

$$= \frac{1}{Z} \text{tr}(P_r(0)T_{qq}^nP_r(n)T^{m-n} \square(m)T^{N_t-m}) - < W > < \square(m) >$$

$$\xrightarrow{N_t \rightarrow \infty} \sum_{\mu, \nu} d_{0\mu} \left( \frac{\lambda_\mu(r)}{\lambda_0} \right)^n d_{\mu\nu} \left( \frac{\lambda_\nu}{\lambda_0} \right)^{m-n} < \nu | \square(m) | 0 >$$

$$- \sum_{\mu} d_{0\mu}^2 \left( \frac{\lambda_\mu(r)}{\lambda_0} \right)^n < 0 | \square(m) | 0 >,$$  \quad (6.9)

where in the last step we have used Eq. (6.6)

From Eq. (6.9) one can see that as $m-n \gg 1$, the dominant term of $< W \square(m) >$ corresponds to $\nu = 0$, which would be cancelled by the product $< W > < \square(m) >$, so one has $< W \square(m) > - < W > < \square(m) > \approx 0$ as $m-n \gg 1$.

However, for $m \approx n$, the term of $\nu = 0$ of the quantity, $< W \square(m) >$, in Eq. (6.9) is cancelled by the product $< W > < \square(m) >$. The major contribution comes from the term of $\nu = 1$. In the limit of $n \rightarrow \infty$ Eq. (6.9) becomes,

$$< W \square(m) > - < W > < \square(m) > \quad (\text{for } 0 < n < m)$$

$$\approx \sum_{\mu} d_{0\mu} \left( \frac{\lambda_\mu(r)}{\lambda_0} \right)^n d_{\mu\nu} \left( \frac{\lambda_\nu}{\lambda_0} \right)^{m-n} < 1 | \square(m) | 0 >$$

$$\xrightarrow{n \rightarrow \infty} e^{-naE_0(r)}d_{00}d_{01} \lambda_1^{m-n} < 1 | \square(m) | 0 >.$$  \quad (6.10)
where in the last step we have used the fact that $\lambda_0 = 1$, and the dominant term is $\mu = 0$ as $n \to \infty$. Eq. (6.10) implies that when the plaquette $\Box(m)$ is close to the Wilson loop in the time direction, it gives a contribution of the order $e^{-naE_0(r)}$, because the coefficients $d_{00}$, $d_{01}$ and $\lambda_i^{m-n}$ are of order of unity in this case.

Finally for the plaquette $\Box(m)$ inside the Wilson loop $W$ in the time direction (i.e. $0 < m < n$), e.g., the plaquette $P_2$ in Fig. 18, its contribution to the L.H.S. of Eq. (6.8) is,

$$< W \Box(m) > - < W > < \Box(m) > \quad (0 < m < n)$$

$$= \frac{1}{Z} tr(P_r(0)T_{\Box}^n m(m)T_{\Box}^{n-m}P_r(n)T^{N_t-n}) - < W > < \Box(m) >$$

$$\longrightarrow \sum_{\mu, \nu} d_{0,\mu} d_{0,\nu} \left( \frac{\lambda_\mu(r)}{\lambda_0} \right)^m \left( \frac{\lambda_\nu(r)}{\lambda_0} \right)^{n-m} < \mu, r | \Box(m) | \nu, r > - < W > < \Box(m) > . \quad (6.11)$$

In the limit of $n \to \infty$, and for cases that both $m$ and $n - m$ are large, the dominant term of Eq. (6.11) is given by $\mu = \nu = 0$, that is,

$$< W \Box(m) > - < W > < \Box(m) > \quad (0 < m < n)$$

$$\approx d_{00}^2 \left( \frac{\lambda_0(r)}{\lambda_0} \right)^n \left[ < 0, r | \Box(m) | 0, r > - < 0 | \Box(m) | 0 > \right]$$

$$= d_{00}^2 \left( \frac{\lambda_0(r)}{\lambda_0} \right)^n (\Box(m))_{r-0}, \quad (6.12)$$

where we denote $(\Box(m))_{r-0} = \left[ < 0, r | \Box(m) | 0, r > - < 0 | \Box(m) | 0 > \right]$, and Eq. (6.7) is used.

From the above discussion we can conclude that for plaquettes outside the Wilson loop $W$ in the time direction, their contributions to the L.H.S. of Eq. (6.8) can be neglected as they are far enough from $W$. For the Wilson loop of large temporal size, i.e., $n \to \infty$, the major contribution to the L.H.S. of
Eq. (6.8) comes from plaquettes inside $W$. The contribution of one plaquette inside $W$ is given by Eq. (6.12). Summing over contributions from all such plaquettes gives the dominant term of the L.H.S. in Eq. (6.8), that is,

$$
\frac{\partial <W>}{\partial \beta} = -<WS'> + <W><S'>
$$

$$
= nd_{00}^2 \left( \frac{\lambda_0(r)}{\lambda_0} \right)^n \left( \sum_s \Box(m) \right)_{r=0},
$$

(6.13)

where the sum $\sum_s$ is over all plaquettes in the spatial volume at one fixed time value. The factor of $n$ comes from summing equal contributions from plaquettes on each time slice. The dominant term on the R.H.S. of Eq. (6.8) is, as $n \rightarrow \infty$,

$$
\frac{\partial}{\partial \beta} [a^2 e^{-naE_0(r)}] = -nd_{00}^2 e^{-naE_0(r)} \frac{\partial [aE_0(r)]}{\partial \beta}.
$$

(6.14)

Collecting Eqs. (6.8) and (6.13), (6.14) yields,

$$
- \frac{\partial aE_0(r)}{\partial \beta} = <r,0|\sum_s \Box|r,0> - <0|\sum_s \Box|0> = (\sum_s \Box)_{r=0},
$$

(6.15)

where the R.H.S. can be measured by calculating the quantity $<W\Box>/<W>$ in LGT. In the continuum limit, $a \rightarrow 0$, the R.H.S. of Eq. (6.15) becomes,

$$
(\sum_s \Box)_{r=0} = \frac{1}{\beta} \sum_s \frac{1}{2} a^4 [<E^2>_{r=0} - <B^2>_{r=0}] = \frac{a}{\beta} A,
$$

(6.16)

where $E$ and $B$ are the color electric and magnetic fields in Minkowski space, $A$ is the integration of the action density over the spatial volume.

On the L.H.S. of Eq. (6.15) if we write the $q\bar{q}$ color field energy $E_0(r)$ in the form [20],

$$
E_0(r) = V(r) + f(\beta)/a,
$$

(6.17)
where $V(r)$ is the $q\bar{q}$ potential energy, $f/a$ is the self-energy. Then we obtain the following relation from Eq. (6.15) in the continuum limit,

$$A = -\frac{\beta}{a} \frac{\partial a E_0(r)}{\partial \beta} = -\frac{\beta}{a} \frac{\partial a V(r)}{\partial \beta} - \frac{\beta f}{a} \frac{\partial f}{\partial \beta}. \quad (6.18)$$

This is just the Michael's action sum rule, which can be obtained by subtracting Eq. (6.2) from Eq. (6.1) because of $A = E_{el} - E_{ma}$, if we assume that $V(r)$ scales and hence is independent of $\beta$.

(2). Energy Sum Rules and Their Generalizations

To derive the energy sum rules in Eqs. (6.1) and (6.2) we need to study the color electric and color magnetic fields separately. Let us consider an asymmetric lattice with the time-spacing $a_t$ and the spatial-spacing $a_s (= a)$, and the asymmetry is $\xi = a_s/a_t$ [20]. The action for $SU(N)$ LGT becomes,

$$S_A = -\beta_t \sum \square_t - \beta_s \sum \square_s + \text{const}. \quad (6.19)$$

where $\square_t$ is a plaquette with a time extent and $\square_s$ is a space-like plaquette. In the continuum limit ($a_s, a_t \to 0$) the action must become the classical action in Eq. (2.4), that is,

$$S_A \longrightarrow \frac{1}{4} \int d^4x \langle F_{\mu\nu}^c \rangle^2. \quad (6.20)$$

In this limit the time-like plaquette becomes

$$\beta_t \square_t = \beta_t \square_{j4} = \beta_t \frac{1}{N} tr(e^{ia_t a_s G_{j4}}) \approx \beta_t (1 - \frac{1}{4N} a_t^2 a_s^2 g_t^2 (F_{j4}^c)^2), \quad (6.21)$$

where $N$ is for $SU(N)$. $j = 1, 2, 3$ and $c$ is the colour index. To get the correct continuum action of Eq. (6.20) we should require that in Eq. (6.21),

$$\frac{\beta_t}{2N} a_t^2 a_s^2 g_t^2 = a_s a_t^3,$$
then we obtain the relation between $\beta_t$ and $g_t$.

$$\beta_t = \frac{2N}{a_t^2} \frac{a_s}{g_t^2} = \frac{2N}{g_t^2} \xi. \quad (6.22)$$

Similarly one can obtain the relation between $\beta_s$ and $g_s$,

$$\beta_s = \frac{2N}{a_t^2} \frac{a_s}{g_s^2} = \frac{2N}{g_s^2} \xi^{-1}. \quad (6.23)$$

In the weak coupling limit, $\beta_s$, $\beta_t$ can be expanded in terms of the coupling $\beta$ of the corresponding symmetric lattice [20, 42],

$$\beta_s \xi = \beta(a) + 2N c_s(\xi) + O(\beta^{-1}), \quad (6.24)$$
$$\beta_t \xi^{-1} = \beta(a) + 2N c_t(\xi) + O(\beta^{-1}); \quad (6.25)$$

where the coefficients $c_s(\xi)$ and $c_t(\xi)$ satisfy the conditions [20, 43],

$$c_s(\xi)|_{\xi=1} = c_t(\xi)|_{\xi=1} = 0,$$
$$\frac{\partial}{\partial \xi} [c_s(\xi) + c_t(\xi)]|_{\xi=1} = (c'_s + c'_t)|_{\xi=1} = b_0 = 11N/48\pi^2, \quad (6.26)$$

and $c'_s$ ($c'_t$) denotes the $\xi$-derivative of $c_s$ ($c_t$).

With these definitions we can now study the electric and magnetic fields of the $q\bar{q}$ sources. If we consider a Wilson loop $W$ of the size $na_t$ in the time direction and $r$ in the space direction on the asymmetric lattice, we can obtain a similar result as Eq. (6.7),

$$< W > = d_{00}^2 e^{-n a_t E_0(r)}, \quad (as \ N_t, n \to \infty ) \quad (6.27)$$

where $E_0(r)$ is the ground-state energy of the $q\bar{q}$ pair, and $< W >$ is defined by the partition function formalism in Eq. (6.4), with the action $\beta S'$ replaced by $S_A$ in Eq. (6.19).
By taking the derivatives of Eq. (6.27) with respect to $\beta_s$ and $\beta_t$ respectively one can get the following relations similar to Eq. (6.15),

\[
-\frac{\partial a_t E_0(r)}{\partial \beta_t} = (\sum_s \Box_t)_{r=0},
\]

\[
-\frac{\partial a_t E_0(r)}{\partial \beta_s} = (\sum_s \Box_s)_{r=0}.
\]

(6.28)  

(6.29)

where the sum $\sum_s$ is over the whole spacial volume as before. The R.H.S. of Eqs. (6.28) and (6.29) become the total color electric and color magnetic energies of the $q\bar{q}$ pair respectively in the continuum limit ($a_s, a_t \to 0$). On the symmetric lattice ($\xi = 1$) one has,

\[
\lim_{a \to 0} (\sum_s \Box_s)_{r=0} = \frac{1}{\beta} \sum_s \frac{1}{2} a^4 < E^2 >_{r=0},
\]

(6.30)

\[
\lim_{a \to 0} (\sum_s \Box_t)_{r=0} = -\frac{1}{\beta} \sum_s \frac{1}{2} a^4 < B^2 >_{r=0};
\]

(6.31)

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields in Minkowski space.

To evaluate the L.H.S. of Eqs. (6.28) and (6.29) we need to resort following relations which relate the quantities on the asymmetric lattice with the quantities of the equivalent symmetric lattice,

\[
\left( \frac{\partial F(\beta_s, \beta_t)}{\partial \beta_t} \right)_{\xi=1} = \frac{1}{2} \left( \frac{\partial F}{\partial \beta} + \frac{1}{\beta} \frac{\partial F}{\partial \xi} \right)_{\xi=1},
\]

(6.32)

\[
\left( \frac{\partial F(\beta_s, \beta_t)}{\partial \beta_s} \right)_{\xi=1} = \frac{1}{2} \left( \frac{\partial F}{\partial \beta} - \frac{1}{\beta} \frac{\partial F}{\partial \xi} \right)_{\xi=1};
\]

(6.33)

where $F(\beta_s, \beta_t)$ is a function of $\beta_s, \beta_t$. From Eqs. (6.24) and (6.25) one can see that $\beta_s, \beta_t$ and $\beta, \xi$ are two equivalent sets of variables, that is, $F(\beta_s, \beta_t) = F(\beta_s(\beta, \xi), \beta_t(\beta, \xi))$. The proof of the above relations will be given in Appendix B.

Since we can write the energy $E_0(r)$ of Eqs. (6.32) and (6.33) in the form, $E_0(r) = V(r) + f(\beta)/a$, from Eq. (6.17), then the L.H.S. of Eq. (6.28)
becomes,
\[
\left( \frac{\partial a_t E_0(r)}{\partial \beta_t} \right)_{\xi=1} = \left( \frac{\partial a_t V(r)}{\partial \beta_t} \right)_{\xi=1} + \left( \frac{\partial (f/\xi)}{\partial \beta_t} \right)_{\xi=1}.
\] (6.34)

Applying Eq. (6.32) to the above equation, and using the fact that the potential
\(V(r)\) is independent of \(\beta\) in the limit of infinite lattice sizes and infinitesimal
lattice spacing \((a \to 0)\), one can obtain,
\[
\left( \frac{\partial a_t V(r)}{\partial \beta_t} \right)_{\xi=1} = \frac{1}{2} \left( \frac{\partial a_t V(r)}{\partial \beta} + \frac{\partial a_t V(r)}{\partial \xi} \right)_{\xi=1}
= \frac{1}{2} V(r) \left( \frac{\partial a}{\partial \beta} - \frac{a}{\beta} \right),
\] (6.35)

where we have used \((\frac{\partial a_t}{\partial \beta})_{\xi=1} = \frac{\partial a}{\partial \beta}\) and \((\frac{\partial a_t}{\partial \xi})_{\xi=1} = -a\). Also the second term
on the L.H.S of Eq. (6.34), \((\frac{\partial (f/\xi)}{\partial \beta_t})_{\xi=1}\) in Eq. (6.34), can be evaluated as,
\[
\left( \frac{\partial (f/\xi)}{\partial \beta_t} \right)_{\xi=1} = \frac{1}{2} \left( \frac{\partial f(\beta)}{\partial \beta} - \frac{f(\beta)}{\beta} \right).
\] (6.36)

where the self-energy \(f(\beta)/a\) depends on \(\beta\) and \(a\). Substituting Eqs. (6.35)
and (6.36) into Eq. (6.34) yields
\[
\left( \frac{\partial a_t E_0(r)}{\partial \beta_t} \right)_{\xi=1} = \frac{1}{2} V(r) \left( \frac{\partial a}{\partial \beta} - \frac{a}{\beta} \right) + \frac{1}{2} \left( \frac{\partial f(\beta)}{\partial \beta} - \frac{f(\beta)}{\beta} \right).
\] (6.37)

Similarly, one can also evaluate the L.H.S. of Eq. (6.29) by applying
Eq. (6.33), the result is,
\[
\left( \frac{\partial a_t E_0(r)}{\partial \beta_s} \right)_{\xi=1} = \frac{1}{2} V(r) \left( \frac{\partial a}{\partial \beta} + \frac{a}{\beta} \right) + \frac{1}{2} \left( \frac{\partial f(\beta)}{\partial \beta} + \frac{f(\beta)}{\beta} \right).
\] (6.38)

From Eqs. (6.28), (6.29), (6.30), (6.31) and (6.37), (6.38) one can get the sum
rules in Eqs. (6.1) and (6.2).

Now we proceed to consider the generalization of these sum rules to
the finite temperature case. In the limit of finite temperature \((T)\) and infinite
volume the potential \( V(r) \) would be a function of \( r \) and \( T \) (or \( \beta \)) because of \( T = 1/N_t a(\beta) \). For example, in the confined phase \( V(r) \sim \kappa r \), with the string tension \( \kappa = \kappa(\beta) \) for fixed lattice size \( N_t \), as discussed in section 4.2. In this case one can write, \( V(r) = V(r, \beta) \). Then Eq. (6.35) should be rewritten as,

\[
\left( \frac{\partial a_i V(r)}{\partial \beta_i} \right)_{\xi=1} = \frac{1}{2} \left( \frac{\partial a V(r, \beta)}{\partial \beta} \right) - \frac{a}{\beta} V(r, \beta).
\]  

(6.39)

Similarly one has,

\[
\left( \frac{\partial a_i V(r)}{\partial \beta_i} \right)_{\xi=1} = \frac{1}{2} \left( \frac{\partial a V(r, \beta)}{\partial \beta} + \frac{a}{\beta} V(r, \beta) \right).
\]

(6.40)

The self-energy part \( f(\beta)/a \) does not change in this case. Therefore, at finite temperature Michael sum rules in Eqs. (6.1) and (6.2) should be modified as following [44],

\[
E_{el}(r) = -\frac{1}{2} \left( \frac{\beta}{a} \frac{\partial [a V(r)]}{\partial \beta} - V(r) \right) - \frac{1}{2a} \left( \beta \frac{\partial f}{\partial \beta} - f \right),
\]

(6.41)

\[
E_{ma}(r) = \frac{1}{2} \left( \frac{\beta}{a} \frac{\partial [a V(r)]}{\partial \beta} + V(r) \right) + \frac{1}{2a} \left( \beta \frac{\partial f}{\partial \beta} + f \right).
\]

(6.42)

These modified sum rules can be obtained from Eqs. (6.1) and (6.2) by the replacement,

\[
V(r) \frac{\partial a}{\partial \beta} \rightarrow \frac{\partial [a V(r)]}{\partial \beta}.
\]

(6.43)

We expect that the modified sum rules would account for the finite temperature effects on the \( q\bar{q} \) system. Although Eqs. (6.41) and (6.42) are derived by considering the Wilson loop representation of the \( q\bar{q} \) pair, we expect that they can be applied in describing the flux distributions \( f'_{\mu\nu} \) in Eq. (3.9), which involves the Polyakov loops, because both the Wilson loop, \( W \), and Polyakov loops, \( P(0)P^t(r) \), represent a \( q\bar{q} \) pair. In the following we shall use our \( SU(2) \) flux data in the confined phase to check the modified Michael sum rules in Eqs. (6.41) and (6.42).
6.2 Analysis of Flux Data

In this part we shall compare our flux data with the predictions of the modified Michael sum rules in Eqs. (6.41) and (6.42).

As we mentioned in section 3.2, we used the multihit technique[32] in the flux measurements. The flux data is only good as the test charge (plaque-ttte) is away from the $q\bar{q}$ sources (Polyakov loops). Thus we only consider the flux data on the middle transverse slice between the $q\bar{q}$ pair at large separations (i.e. $r \geq 3a$).

In the confined phase the string formation is expected to occur. As the separation $r$ of a $q\bar{q}$ pair is large enough, the flux energy stored in the flux tube of unit length between the $q\bar{q}$ should equal to the string tension $\kappa$. In this case one can obtain the color electric and magnetic energy of the flux tube per unit length from Eqs. (6.41) and (6.42),

$$
\sigma_{el} = \frac{dE_{el}(r)}{dr} = -\frac{1}{2} \left( \frac{\beta}{a} \frac{\partial [a\kappa]}{\partial \beta} - \kappa \right),
$$

$$
\sigma_{ma} = \frac{dE_{ma}(r)}{dr} = \frac{1}{2} \left( \frac{\beta}{a} \frac{\partial [a\kappa]}{\partial \beta} + \kappa \right).
$$

The corresponding flux action is

$$
\sigma_A = \sigma_{el} - \sigma_{ma} = \frac{\beta}{a} \frac{\partial [a\kappa]}{\partial \beta},
$$

which is the integration of the action density on the transverse plane. And the corresponding flux energy is

$$
\sigma_E = \sigma_{el} + \sigma_{ma} = \frac{dV(r)}{dr} = \kappa(\beta).
$$

which agrees with our expectation for string formation.

Eq. (6.46) can also be obtained from the action sum rule of Eq. (6.18). The R.H.S. of Eq. (6.46) is, $\frac{\beta}{a} \frac{\partial [a\kappa]}{\partial \beta} = \kappa \frac{\partial \eta_{\alpha\theta}}{\partial \eta_{\beta\gamma}} + \beta \frac{\partial \kappa}{\partial \beta}$. The first term can be
estimated from the scaling relation of Eqs. (2.33) and (2.34), e.g. \( \frac{\partial \sigma_{ma}}{\partial \ln \beta} \sim -8 \) for \( \beta \sim 2.4 \), which is consistent with the flux measurement results of ref.[28]. The second term \( \beta \frac{\partial \kappa}{\partial \beta} \) can also be estimated from the fitting function in Eq. (4.15), and it is negative because \( \kappa \) decreases with \( \beta \), as shown in Eq. (4.12). So Eq. (6.46) implies that \( \sigma_A \gg \sigma_E = \kappa \) for cases we consider. Also Eqs. (6.44) and (6.45) show that the electric part of the flux energy \( \sigma_{el} \) is positive, but the magnetic part \( \sigma_{ma} \) negative, their magnitudes are of the same order with \( \sigma_{el} \) slightly larger. These predictions can be summarized as below,

\[
\begin{align*}
\sigma_{el} & > 0 \quad \text{and} \quad \sigma_{ma} < 0, \\
\sigma_E & = \sigma_{el} + \sigma_{ma} = \kappa > 0, \\
\sigma_A & \gg \sigma_E, \quad (6.48)
\end{align*}
\]

where these flux energies (action) are defined in the confined phase.

From our flux data we can calculate the values of \( \sigma_{el}, \sigma_{ma}, \sigma_E \) and \( \sigma_A \), which are the energies (action) stored in the center slice between the \( q\bar{q} \) pair. One typical result is shown in Table 11. One can see that the measured data agree with Eq. (6.48) qualitatively. However, when we compare \( \sigma_E \) data with the measured string tension, i.e. \( \kappa = 0.499 \pm 0.102 \ GeV/fm \) for \( \beta = 2.25 \) on the lattice \( 4 \cdot 11^2 \cdot 65 \), as shown in Table 6, we find that the \( \sigma_E \) data are not good enough to make conclusions. This may be due to the fact that \( \sigma_E \) is obtained from the strong cancellation between two terms (i.e. \( \sigma_{el} \) and \( \sigma_{ma} \))[20], so it is difficult to measure \( \sigma_E \) accurately.

In the following we proceed to study the prediction of the action sum rule in Eq. (6.46). As we mentioned above, the R.H.S. of Eq. (6.46) contains two terms, the first term, \( \kappa \frac{\partial \sigma_{ma}}{\partial \ln \beta} \) was predicted by the original Michael sum.
Table 11. The center slice flux energies and action of a $q\bar{q}$ pair at various separations $r/a$ in the confined phase, which were calculated from the flux data on the lattice $4 \cdot 11^2 \cdot 65$ with $\beta = 2.25$ ($\beta_c \approx 2.30$). The data are in the physical unit Gev/fm.

<table>
<thead>
<tr>
<th>$r/a$</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{el}$</td>
<td>4.54 (8)</td>
<td>5.06 (16)</td>
<td>5.41 (24)</td>
</tr>
<tr>
<td>$\sigma_{ma}$</td>
<td>-3.26 (9)</td>
<td>-3.65 (14)</td>
<td>-3.96 (27)</td>
</tr>
<tr>
<td>$\sigma_{E}$</td>
<td>1.28 (12)</td>
<td>1.41 (22)</td>
<td>1.44 (36)</td>
</tr>
<tr>
<td>$\sigma_{A}$</td>
<td>7.80 (12)</td>
<td>8.70 (22)</td>
<td>9.37 (36)</td>
</tr>
</tbody>
</table>

rules in Eqs. (6.1) and (6.2), the second one, $\beta \frac{\partial \sigma_A}{\partial \beta}$, is a new term, which is only predicted by the modified sum rules of Eqs. (6.41) and (6.42). We expect that the second term describes finite temperature effects.

From the scaling relation $a(\beta)$ of Eq. (2.33) one can estimate the quantity,

$$ -\frac{\partial \ln a}{\partial \ln \beta} = -\frac{51}{121} + \frac{3\pi^2}{11} \beta + \frac{d_2 + 2d_3\beta}{\Lambda_{\Lambda}^{-1}(\beta)} \beta \quad (6.49) $$

with $\Lambda_{\Lambda}^{-1}(\beta)$ given by Eq. (2.34). And by using the fitting function $\kappa(\beta)$ in Eq. (4.15) one can calculate the derivative,

$$ \frac{\partial \kappa}{\partial \beta} = -\frac{\kappa_0}{\beta_c} \delta (1 - \frac{\beta}{\beta_c})^{\delta - 1}, \quad (6.50) $$

with the constants $\kappa_0$ and $\delta$ given by Eqs. (4.13) and (4.16).

By similar considerations that lead to Eq. (6.46), one can also obtain the prediction from the original Michael sum rules in Eqs. (6.1) (6.2). The result is

$$(\sigma_A)_0 = -\kappa \frac{\partial \ln a}{\partial \ln \beta}, \quad (6.51) $$

which is just the first term of R.H.S. in Eq. (6.46).
Table 12. The comparison of the predictions of generalized Michael sum rules and the data of center slice action $\sigma_A$.

<table>
<thead>
<tr>
<th>$N_t = 4$</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$\kappa(\beta)$</td>
<td>$-\frac{\partial a}{\partial \ln \beta}$</td>
<td>$-\frac{\partial \kappa}{\partial \beta}$</td>
<td>$-\frac{\partial \ln a}{\partial \ln \beta}$</td>
<td>$\sigma_A$</td>
<td>$\sigma_A$</td>
</tr>
<tr>
<td>2.25</td>
<td>0.42</td>
<td>3.03 (37)</td>
<td>4 (1)</td>
<td>7 (2)</td>
<td>12.18 (23)</td>
<td>8.70 (22)</td>
</tr>
<tr>
<td>2.28</td>
<td>0.34</td>
<td>2.50 (39)</td>
<td>9 (3)</td>
<td>12 (3)</td>
<td>10.95 (14)</td>
<td>9.97 (13)</td>
</tr>
<tr>
<td>2.29</td>
<td>0.29</td>
<td>2.12 (40)</td>
<td>17 (7)</td>
<td>19 (7)</td>
<td>10.44 (12)</td>
<td>9.39 (10)</td>
</tr>
</tbody>
</table>

$N_t = 6$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\kappa(\beta)$</th>
<th>$-\frac{\partial a}{\partial \ln \beta}$</th>
<th>$-\frac{\partial \kappa}{\partial \beta}$</th>
<th>$-\frac{\partial \ln a}{\partial \ln \beta}$</th>
<th>$\sigma_A$</th>
<th>$\sigma_A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.30</td>
<td>0.64</td>
<td>4.8 (5)</td>
<td>1.7 (6)</td>
<td>7 (1)</td>
<td>9.07 (89)</td>
<td>8.86 (1.08)</td>
</tr>
<tr>
<td>2.36</td>
<td>0.58</td>
<td>4.6 (6)</td>
<td>3 (1)</td>
<td>8 (2)</td>
<td>8.26 (36)</td>
<td>11.55 (52)</td>
</tr>
<tr>
<td>2.40</td>
<td>0.50</td>
<td>4.0 (7)</td>
<td>8 (4)</td>
<td>12 (4)</td>
<td>8.82 (28)</td>
<td>9.73 (35)</td>
</tr>
<tr>
<td>2.42</td>
<td>0.33</td>
<td>3 (1)</td>
<td>–</td>
<td>–</td>
<td>13.29 (25)</td>
<td>9.37 (25)</td>
</tr>
</tbody>
</table>

In Table 12 we list the predictions of Eq. (6.46) obtained by substituting Eqs. (6.49) and (6.50), and the measured $\sigma_A$ data in the confined phase. The data were measured on lattices $4 \cdot 9^2 \cdot 65$, $4 \cdot 11^2 \cdot 65$, $6 \cdot 7^2 \cdot 65$ and $6 \cdot 11^2 \cdot 37$. The quantities $\kappa(\beta)$, $\beta \frac{\partial \kappa}{\partial \beta}$, $\kappa \frac{\partial \ln a}{\partial \ln \beta}$ and $\sigma_A$ have the physical unit $Gev/fm$. The values of string tension $\kappa(\beta)$ were estimated from Eq. (4.15). In the transition region (i.e. $\beta \sim 2.3$ for $N_t = 4$) no stable prediction can be obtained. Since in the confined phase the value of $\sigma_A$ should not change with the $qq$ separation $r$ in the limit of $r \to \infty$, we then choose the $\sigma_A$ data at a moderate value of $r$ as the asymptotic value of $\sigma_A (r \to \infty)$, because error bars are large at very large $r$. In Table 12 most $\sigma_A$ data were calculated from the flux measurements of $q\bar{q}$ pair at $r = 4a$, in few cases we choose the data at $r = 3a$ or $5a$, depending on the quality of data.

In Fig. 19 we plot the predictions of Eqs. (6.46) and (6.51) respectively, and compare them with the $\sigma_A$ data for the case of $N_t = 4$. From this figure we can see that our measured data are consistent with the prediction of
Eq. (6.46), but have large difference from the predictions of Eq. (6.51). Especially, in the transition region ($\beta \approx \beta_c$) Eq. (6.51) predicts that $\sigma_A$ approaches zero as $\beta \rightarrow \beta_c$. However, our $\sigma_A$ data have large values in this region, which agree with the predictions of Eq. (6.46). As the temperature decreases ($T \rightarrow 0$ or $\beta \rightarrow 0$), both predictions coincide. However, it is difficult to obtain data with clear signal in the small $\beta$ region (i.e., $\beta < 2.25$).

We also notice that in Fig. 19 our $\sigma_A$ data in the confined region ($\beta < \beta_c$) have some discrepancies with the prediction of Eq. (6.46). This may be due to finite size effects and finite lattice spacing effects of lattices, and to the large fluctuations in the confined phase, because confinement corresponds to disorder.

To compare the behaviors of $\sigma_A$ in both phases, in Fig. 19 we also plot the $\sigma_A$ data in the unconfined region (i.e., $\beta > \beta_c$). These data were measured at large $q\bar{q}$ separations, i.e., $r = 6a$, which still have clear signals, as shown in Table 13. Since in the unconfined phase there is no string formation, one expects that $\sigma_A$ vanishes at large $r$. From this figure one can see that near the transition point (i.e. $\beta \sim 2.30$), our $\sigma_A$ data in the unconfined region decrease rapidly with $\beta$, and becomes very small beyond the transition region. This agrees with the expectation. The fact that our $\sigma_A$ data in the unconfined region do not vanish may be due to the following factors, in the transition region finite-size effects are large, beyond this region the contribution from the self-energy of the $q\bar{q}$ pair still exists, because the $q\bar{q}$ separation, $r = 6a$, is not large enough to approach the asymptotic region.

In conclusion our flux data are consistent with the prediction of the generalized sum rules, which shows explicitly that the $q\bar{q}$ flux distribution has
Figure 19. The plot of the predictions of $\sigma_A$ vs. $\beta$ of Eq.(6.46) (solid lines), and Eq.(6.51) (dashed lines). The two solid lines represent the upper and lower limits of the predictions of Eq.(6.46). The data of $\sigma_A$ were measured from lattices $4 \cdot 9^2 \cdot 65$ (squares) and $4 \cdot 11^2 \cdot 65$ (triangles), which have the physical unit $\text{Gev/fm}$. 
Table 13. The $\sigma_A$ data in the unconfined phase ($\beta < \beta_c$), which were measured on lattices, $4 \cdot 9^2 \cdot 65$ and $4 \cdot 11^2 \cdot 65$. The data are in the physical unit GeV/fm.

$$N_t = 4$$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>2.30</th>
<th>2.32</th>
<th>2.34</th>
<th>2.36</th>
<th>2.40</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\sigma_A)_{4 \cdot 9^2 \cdot 65}$</td>
<td>11.77 (11)</td>
<td>10.40 (9)</td>
<td></td>
<td>1.78 (4)</td>
<td>1.28 (4)</td>
</tr>
<tr>
<td>$(\sigma_A)_{4 \cdot 11^2 \cdot 65}$</td>
<td>11.03</td>
<td></td>
<td>3.81 (4)</td>
<td>1.77 (3)</td>
<td>1.32 (4)</td>
</tr>
</tbody>
</table>

Different behaviours in the two phases. In the confined phase the asymptotic value of the center slice action, $\sigma_A (r \to \infty)$ has large values, however, across the transition region $\sigma_A (r \to \infty)$ becomes very small in the unconfined phase. This agrees with our expectation that string formation occurs in the confined phase, but disappears in the unconfined phase.
SUMMARY

In this dissertation research we investigated pure $SU(2)$ gauge theory systematically using LGT method. This theory possesses the essential features of QCD which is a $SU(3)$ gauge theory, but it is much simpler for studies. It is known that QCD has different properties at different temperatures and different volumes. In this work we studied $SU(2)$ gauge theory in two limits by measuring the flux distributions of a $q\bar{q}$ pair. One is the large volume and finite temperature limit, another is the zero temperature and finite volume limit.

In the limit of infinite volume and finite temperatures we studied some general properties of the finite temperature deconfining phase transition. We find clear evidence that the string formation occurs in the confined phase, but not occur in the unconfined phase. The behavior of string tension $\kappa$ with temperature is also studied, our data agree very well with the fitting function,

$$\kappa(T) = \kappa_0 (1 - \frac{T}{T_c})^\alpha \quad \text{(for } T < T_c).$$

By choosing the time direction of lattices appropriately we can access the limit of zero temperature and finite volumes. In this limit we find that the string tension in intermediate volumes is due to finite size effects, there is no intrinsic string formation. However, in the large volume region our results support the expectation that the intrinsic string formation occurs. This elucidates the question raised by the global LGT calculations.
To check Michael sum rules with the flux data at finite temperatures, a complete derivation of the sum rules are presented, and a generalization of the sum rules is suggested to account for the finite temperature effects. We find that our flux data are consistent with the prediction of the generalized sum rules.

Our study clarified many properties of QCD, and provide strong numerical evidence for the quark confinement and the string formation. This supports the belief that QCD is the correct theory describing strong interactions, and it can explain quark confinement. This numerical study paves the way for further studies of QCD and final analytical proof of quark confinement in QCD.

Our results also confirmed that quarks are unconfined in the deconfining phase at high temperatures due to the disappearance of string formation. It is believed that quarks and gluons form the quark-gluon plasma at high temperatures, which is a new form of matter. Experiments involving high energy collisions of heavy nuclei were performed at CERN[50], which have shown some possible signals for the quark-gluon plasma. To get an unambiguous evidence, more detailed experimental information and higher energies are required.

So far we only studied the pure $SU(2)$ gauge theory model. To reach the real physics, we need to consider $SU(3)$ gauge theory with dynamical quarks. In the future we plan to investigate QCD in the following aspects, full QCD with dynamical quarks, the mechanism of quark confinement and properties of deconfining phase at high temperatures. We expect that by these studies strong interactions between quarks can be understood thoroughly.
REFERENCES


APPENDIX A

MONTE CARLO METHODS IN LGT

The lattice formulation reduces the Feynman path formula for the gauge theory into a multiple ordinary integral. For a finite lattice one can evaluate the partition function in Eq. (2.20) numerically. However, the dimensionality of the multiple integral is usually very large. For example, consider a $10^4$ size lattice, a size fairly typical for numerical work. Such a system has 40,000 link variables (in 4-dimensional space). This high multidimensionality of the integrals makes conventional mesh techniques for numerical calculations of integrals impractical. One must seek some other methods.

Monte Carlo simulation method is just such a technique dealing with multiple integrals with high dimensionalities, which is now the dominant method in LGT calculations. In LGT physical quantities are evaluated from the expression in Eq. (2.21), which is an average over configurations with the weight of the Boltzmann factor $e^{-S(C)}$, where $C$ represents the configurations of the system. The basic idea of Monte Carlo method is by using some algorithm one can choose a small number of configurations which are typical of thermal equilibrium in the statistical analog, then the averages of physical quantities, such as that in Eq. (2.21), can be evaluated from the set of configurations. In the following I shall briefly review some Monte Carlo simulation methods used in LGT.
A.1 Metropolis Algorithm

Since the goal of Monte Carlo simulation is to find, in a stochastic manner, a set of configurations with a probability distribution given by the Boltzmann factor,

\[ P_{eq}(C) \approx e^{-S(C)}. \]  \hspace{1cm} (A.1)

Let’s suppose in a simulation process the system be in a configuration \( C \), then one choose a new configuration \( C' \) as a possible update to \( C \). \( C' \) can be taken with an arbitrary trial probability \( P_{T,C}(C') \). The trial change \( C' \) may depend on the previous state \( C \).

In Metropolis algorithm one accepts \( C' \) with the conditional probability

\[ P_{acc} = \min \left[ 1, \frac{P_{T,C'}(C)e^{-S(C')}}{P_{T,C}(C')e^{-S(C)}} \right]. \]  \hspace{1cm} (A.2)

Usually \( P_{T,C'}(C) = P_{T,C}(C') \). So one has

\[ P_{acc} = \min \left[ 1, \frac{e^{-S(C')}}{e^{-S(C)}} \right]. \]  \hspace{1cm} (A.3)

To understand how to realize the Metropolis algorithm, let’s consider a simple example, the 1-dimensional integral as shown below[45].

\[ \langle f(x) \rangle = \frac{\int_a^b f(x)p(x)dx}{\int_a^b p(x)dx}, \]  \hspace{1cm} (A.4)

with \((x \in [a, b])\). This is the average of the one variable function \( f(x) \) with respect to the weight function \( p(x) \), as shown in Figure 20.

The Metropolis procedures used in calculating the average \( \langle f \rangle \) in Eq. (A.4) are given below.

(1). Choose an initial value \( x_1 (x_1 \in [a, b]) \) arbitrarily.

(2). Choose a trial value \( x_t \), usually \( x_t \) is chosen from a small interval around \( x_1 \) randomly, that is, \( x_t \in [x_1 - \delta, x_1 + \delta] \), where the step size \( \delta \) is
Figure 20. The weight function $p(x)$ vs. $x$, which is in the interval $[a, b]$. $x_i$ is some accepted $x$ value.
adjusted properly so that the acceptance probability of the trial value $x_t$ is about 1/3 to 1/2.

(3). Calculate the ratio $w = p(x_t)/p(x_i)$.

if $w \geq 1$, accept the trial value $x_t$, that is, let $x_2 = x_t$.

if $w < 1$, call a uniform random number $r$ in the interval $[0,1]$.

(4). If $w < 1$, then compare $w$ with $r$.

if $r < w$, accept $x_t$, let $x_2 = x_t$.

if $r > w$, reject $x_t$.

(5). Repeat steps (2) – (4) many times, finally one can obtain a set of accepted $x$ values, such as

$$\{x_1, x_2, \cdots, x_K\}.$$ 

where $K$ is a large integer.

We can shown that the set of $x$ values has the required probability distribution $p(x)$\cite{22}. So we have the average of Eq. (A.4) can be calculated as,

$$< f(x) > = \frac{\int_a^b f(x)p(x)dx}{\int_a^b p(x)dx} = \lim_{K \to \infty} \sum_{i=1}^{K} f(x_i).$$ (A.5)

To apply the above procedures to the calculations in LGT, one notices that in LGT the configurations of the lattice are much more complicated than that of the above example. In LGT a particular configuration of the lattice is determined by the value of each link variable $U$, which is an element of the gauge group. We can denote the configuration of a lattice as $C = \{U_\mu(n)\}$, with $n$ running through the sites of the lattice, and $\mu = 1, \cdots, 4$, the four directions of the links.

The Monte Carlo Simulation in LGT is to obtain a set of configurations, $\{C_1, C_2, \cdots, C_K\}$, with the correct Boltzmann distribution $e^{-S(U)}$, where
$S(U)$ is the action of the LGT as defined in Eq. (3.4). To obtain a new accepted configuration $C_{i+1}$ from the old configuration $C_i$, one must update all link variables of the lattice, usually we call this process to be one sweep.

The procedures of the conventional Metropolis algorithm in LGT are given below.

(1). **Updating link variable $U$**

Suppose one need to update the link variable $U$, one choose a trial value,

$$U' = UH,$$

where $H$ is a random element of the gauge group, and is chosen with a probability $P(H)$. We require that the probability for its inverse $H^{-1}$ is the same, that is,

$$P(H) = P(H^{-1}).$$

Usually the element $H$ is randomly selected from a table, Eq. (A.7) is insured by having the table contain the inverse of each of its elements. Eq (A.7) implies that the trial probability in Eq. (A.2) satisfies

$$P_{T,U}(U') = P_{T,U'}(U).$$

So the acceptance probability for the trial value $U'$ is

$$P_{acc} = \min[1, e^{-(S' - S)}],$$

where $S' = S(U')$ is the action with the trial value $U'$ assigned to the link to be updated, and $S = S(U)$ is the action with the link of the old value $U$. This can be realized by following the analogous procedures of steps (3) and (4) of
the above example for the calculation of \( < f(x) > \), that is, we can use the procedure below.

Calculate the ratio

\[
    w = e^{-S'}/e^{-S} = e^{-(S'-S)}. \tag{A.10}
\]

If \( w \geq 1 \), we accept the trial value \( U' \). If \( w < 1 \), we conditionally accept \( U' \) by calling a uniform random number \( r \) in the interval \([0,1]\); if \( r < w \) we accept \( U' \), otherwise, we reject it.

(2). Updating a configuration \( C \)

To update a configuration \( C \) of the lattice, we apply the above procedures to each link of the lattice. After we finish updating one link \( U \), we then go to update the next link until we finish updating all links of the lattice. This gives us a new accepted configuration \( C' \) of the lattice.

(3). Generate a set of configurations

If we repeat the above steps many times, we obtain a set of accepted configurations,

\[ \{C_1, C_2, \cdots, C_K \}. \]

We can also show that these states satisfy the correct Boltzmann distribution \( e^{-S(C)} \)[22]. Therefore, the average of a physical quantity \( < H > \) in Eq. (2.21) can be calculated as

\[
    < H > = Z^{-1} \int (dU)H(U)e^{-S(U)} = \lim_{K \to \infty} \frac{1}{K} \sum_{i=1}^{K} H(C_i). \tag{A.11}
\]
A.2 Heat Bath Algorithm

This algorithm is a kind of variation of general Metropolis algorithm discussed above. In this method when one updates a link variable $U$, one takes the trial element $U'$ with a probability of the Boltzmann factor,

$$P_{T,U}(U') \propto e^{-S(U')},$$

(A.12)

with all other links fixed.

In this case the ratio of trial probabilities satisfies the condition,

$$\frac{P_{T,U}(U)}{P_{T,U}(U')} = \frac{e^{-S(U)}}{e^{-S(U')}}.$$

(A.13)

Then according to Eq. (A.2) the acceptance probability $P_{acc.}$ of the trial element $U'$ is just

$$P_{acc.} = 1.$$

(A.14)

Therefore, in the heat bath algorithm the trial change is always accepted. The procedures of this algorithm have been given by ref. [21] in details.

A.3 Over-Relaxation Algorithm

This method is also a variation of general Metropolis algorithm discussed above. The basic idea of the over-relaxation was first introduced by S.L. Adler[46] and C. Whitmer[47]. The generalization of this idea to LGT was done by F.R. Brown, et al.[48] and M. Creutz[49].

The idea of the over-relaxation method in LGT can be described as following[49].

When one updates a link variable $U$ of the lattice, one picks a trial change $U'$ for the link $U$, such that in a region of the phase space of the link
variable $U$ the trial element $U'$ is as far as possible from the old value $U$ while not paying a severe energy penalty. This is done by approximately locating the locus of the minimum energy (or action) for the link variable $U$, and then selecting the trial element $U'$ on the "opposite" side of the old value $U$, as shown in Fig. 21.

In Fig. 21 we show one locus of the minimum of the action $S(U(\omega))$ in the phase space of $U$, where $\omega$ is the phase factor of the link variable $U(\omega) = \exp(i\tau_j \omega_j) \ (j = 1, 2, \cdots, n_g)$, as defined in Eq. (2.6). For simplicity we assume that all the phase factors $\omega_j$ are fixed except one of them, i.e. $\omega_k = \omega$. Then the link value to be updated can be rewritten as $U = \text{const.} \exp(i\tau_k \omega_U)$, and the action $S$ is just a function of $\omega$, $S = S(\omega)$. Near the minimum point $\omega = \omega_g$ the action $S(\omega)$ can be approximated by a quadratic function of $(\omega - \omega_g)$, i.e.,

$$S(\omega) \approx c_1(\omega - \omega_g)^2 + c_2(\omega - \omega_g) + c_3,$$  

(A.15)

with $c_1, c_2, c_3$ the coefficients.

Suppose $\omega = \omega_U$ be the phase factor of the link value $U$ to be updated, then one chooses the trial element $U'$ such that its phase factor $\omega'$ is on the "opposite" side of $\omega_U$ about the minimum point $\omega = \omega_g$ of the action, that is,

$$\omega' \approx \omega_g + (\omega_g - \omega_U) = 2\omega_g - \omega_U,$$  

(A.16)

as shown in Fig. 21.

So the trial element $U'$ can be chosen as

$$U' = \text{const.} e^{i\tau_k \omega'} \approx \text{const.} e^{i\tau_k (2\omega_g - \omega_U)} \approx g_0 U^{-1} g_0,$$  

(A.17)

with $U^{-1} = \text{const.} \exp(-i\tau_k \omega_U)$ and $g_0 = \text{const.} \exp(i\tau_k \omega_g)$. So $U^{-1}$ is the inverse of the old element $U$, $g_0$ is the element corresponding to the minimum of the action.
Figure 21. The locus of the minimum of the action $S(U)$ in the phase space of the link variable $U$. Here $\omega_U$ is the phase factor of the link value $U$ to be updated, $\omega'$ is the phase factor of the trial value $U'$, and $\omega_g$ is the phase factor of the element $g_0$ which minimizes the action $S$. 
Now we can write down the procedures for the over-relaxation method in LGT.

(1). To update a link $U$, one choose an element $g_0$ which approximately minimize the action $S(U)$ with all other links fixed, then one constructs the trial element,

$$U' = g_0 U^{-1} g_0.$$  \hfill (A.18)

Then one uses the Metropolis procedures in Eq. (A.10) to decide if the trial element $U'$ is accepted or rejected.

(2). Repeat the above step for all links of the lattice to finish one sweep. If one repeats many sweeps, one can obtain a set of accepted configurations with the correct Boltzmann distribution.

To be more specific about the selection of the element $g_0$, let's consider the action in Eq. (3.4). We can rewrite it in the form,

$$S(U) = S_0 - \beta / N Re[Tr(U M)],$$  \hfill (A.19)

where the second term contains the link $U$ under consideration, while the first term is the remaining part of the action, $M$ is the sum of the six staples coupling to $U$, as shown in Fig. 22.

A natural choice of $g_0$ is

$$g_0 = Proj.(M^{-1}),$$  \hfill (A.20)

which is the inverse of a group element obtained by projecting $M$ onto the gauge group. In particular, for $U(1)$ and $SU(2)$ LGT, the matrix $M$ is proportional to a group element, say $g_M$. Then we choose

$$g_0 = g_M^{-1}.$$  \hfill (A.21)
Figure 22. One 'staple' $M_{jkl}$ that couples with the link $U_{ij}$, where $M_{jkl} = U_{jk}U_{kl}U_{li}$. 
and the trial element \( U' = g_0 U^{-1} g_0 \). In this case one can easily show that

\[
S(U') = S(U), \tag{A.22}
\]

because of the relations,

\[
Tr(U' g_M) = Tr(g_M^{-1} U^{-1} g_M^{-1} g_M) = Tr(g_M^{-1} U^{-1}), \tag{A.23}
\]

and

\[
Re[Tr(U g_M)] = Re[Tr(g_M^{-1} U^{-1})]. \tag{A.24}
\]

Again one has that the trial probability of choosing \( U' \) equals that of choosing \( U \), that is, \( P_{T,U'}(U') = P_{T,U}(U) \), because \( U = g_0 U^{-1} g_0 \) and \( U_t = g_0 U^{-1} g_0 \). Therefore, from Eq. (A.2) one has the acceptance probability for the trial element \( U' \) is just

\[
P_{\text{acc.}} = 1. \tag{A.25}
\]

The trial change is always accepted in the over-relaxation method of \( U(1) \) and \( SU(2) \) LGT.
APPENDIX B

PROOF OF A RELATION

To prove Eqs. (6.32) and (6.33), we notice that Eqs. (6.24) and (6.25) implies that $\beta_s$, $\beta_t$, and $\beta$, $\xi$ are two equivalent sets of variables, that is, $F(\beta_s, \beta_t) = F(\beta_s(\beta, \xi), \beta_t(\beta, \xi))$, then we take the partial derivatives of $F$ with respect to $\beta$ and $\xi$ respectively, one has

$$
\left( \frac{\partial F}{\partial \beta} \right)_\xi = \left( \frac{\partial F}{\partial \beta_t} \right)_{\beta_s} \left( \frac{\partial \beta_t}{\partial \beta} \right)_\xi + \left( \frac{\partial F}{\partial \beta_s} \right)_{\beta_t} \left( \frac{\partial \beta_s}{\partial \beta} \right)_\xi, \tag{B.1}
$$

$$
\left( \frac{\partial F}{\partial \xi} \right)_\beta = \left( \frac{\partial F}{\partial \beta_t} \right)_{\beta_s} \left( \frac{\partial \beta_t}{\partial \xi} \right)_\beta + \left( \frac{\partial F}{\partial \beta_s} \right)_{\beta_t} \left( \frac{\partial \beta_s}{\partial \xi} \right)_\beta. \tag{B.2}
$$

From Eqs. (6.24) and (6.25) one can get that if $\beta$ is large enough ($\beta \to \infty$),

$$
\left( \frac{\partial \beta_t}{\partial \beta} \right)_{\xi=1} = \left( \frac{\partial \beta_s}{\partial \beta} \right)_{\xi=1} = 1; \\
\left( \frac{\partial \beta_t}{\partial \xi} \right)_{\xi=1} = -\beta + 2Nc'_t|_{\xi=1}, \\
\left( \frac{\partial \beta_s}{\partial \xi} \right)_{\xi=1} = \beta + 2Nc'_s|_{\xi=1}. \tag{B.3}
$$

where $N$ denotes for $SU(N)$, and $c'$ the $\xi$-derivative of $c$. Substituting Eqs. (B.3) into Eqs. (B.1) and (B.2) yields

$$
\left( \frac{\partial F}{\partial \beta} \right)_{\xi=1} = \left( \frac{\partial F}{\partial \beta_t} \right)_{\xi=1} + \left( \frac{\partial F}{\partial \beta_s} \right)_{\xi=1}, \tag{B.4}
$$

$$
\left( \frac{\partial F}{\partial \xi} \right)_{\xi=1} = \left[ \frac{\partial F}{\partial \beta_t}(\beta + 2Nc'_t) + \frac{\partial F}{\partial \beta_s}(-\beta + 2Nc'_s) \right]_{\xi=1}. \tag{B.5}
$$
F. Karsch has studied the coefficients $c_s$ and $c_t[43]$. His results show that the derivatives $c'_t|_{\xi=1}$, $c'_t|_{\xi=1}$ vanish for sufficient large lattice coupling $\beta$. For example, as $\beta \geq 2.2$, the values of $c'_t|_{\xi=1}$ and $c'_t|_{\xi=1}$ of $SU(2)$ LGT are about 0.1, which is much less than the values of $\beta$. So in the case of large $\beta$ Eqs. (B.4) and (B.5) become

$$
\left( \frac{\partial F}{\partial \beta} \right)_{\xi=1} = \left( \frac{\partial F}{\partial \beta_t} \right)_{\xi=1} + \left( \frac{\partial F}{\partial \beta_s} \right)_{\xi=1},
$$

(B.6)

$$
\left( \frac{\partial F}{\partial \xi} \right)_{\xi=1} = \beta \left( \frac{\partial F}{\partial \beta_t} \right)_{\xi=1} - \beta \left( \frac{\partial F}{\partial \beta_s} \right)_{\xi=1}.
$$

(B.7)

After carrying out some simple algebra, we can obtain Eqs. (6.32) and (6.33).
VITA

The author was born in August 22, 1964 in Chongqing, China. He attended Nankai Middle School in Chongqing, then did undergraduate work in Physics at Tsinghua University, Beijing, China, and got a BS degree there in 1986. He came to Louisiana State University in Fall, 1986, and got his MS degree in Physics in 1987. He then joined the Lattice Gauge Theory group of Professor Haymaker to work for his PhD in 1988.
DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Yingcai Peng

Major Field: Physics

Title of Dissertation: Phase Transitions and Flux Distributions of SU(2) Lattice Gauge Theory

Approved:

[Signature]
Major Professor and Chairman

[Signature]
Dean of the Graduate School

EXAMINING COMMITTEE:

[Signature]
Joseph Callaway

[Signature]
Lai-Hui Chan

[Signature]
Detlev Voeste

[Signature]
Robert

[Signature]
Rajiv K. Kalia

[Signature]
Peter E. Conner

Date of Examination:

April 6, 1993